1 What is Game Theory?

GAME theory is concerned with how rational individuals make decisions when they are mutually interdependent. In recent years this theory has been increasingly applied to various branches of economics. Often this synthesis has significantly improved our understanding of economic issues and led to important new insights being developed. In many instances the application of game theory has transformed the way economists think about both microeconomic and macroeconomic problems. This is evidenced by the frequent use of the adjective 'new' when game theory is applied to different branches of economics. For example, it is now common for economists to refer to New Industrial and New International Economics. Both of these areas have developed as game theory has been applied to traditional disciplines within economics. Also, while not uniquely defined by their use of game theory, much of New Classical and New Keynesian Macroeconomics has incorporated game theoretical analysis. Indeed, so widespread is the use of game theory in economics that it is difficult to find an area where such an approach has not yielded new insights and challenged traditional theory. The aim of this book is to provide an introduction to basic game theory concepts and to illustrate how these have been applied to a diverse range of economic issues. In the first section of this chapter we discuss the broad characteristics of game theory. These, in turn, delimit the range of economic issues for which game theoretical analysis is applicable. In the second section we provide an outline of subsequent chapters.

1.1 Basic Assumptions of Game Theory

As stated above game theory is concerned with how rational individuals make decisions when they are interdependent. To understand this definition more fully we discuss what is meant by individualism, rationality, and mutual interdependence.

1.1.1 Individualism

It is usual to distinguish two separate branches of game theory. These are co-operative and non-cooperative game theory. Strictly speaking the previous definition of game theory only applies to non-cooperative game theory. In *non-cooperative game theory* the individuals, or players, in a game are unable to enter into binding and enforceable agreements with one another. Due to this assumption non-cooperative game theory is inherently individualistic. In contrast, *co-operative game theory* analyses situations where such agreements are possible. The focus of co-operative game theory is therefore on how groups of individuals committed to each other formulate rational decisions. This distinction does not mean that non-cooperative game theory precludes individuals working together. However, it does state that this will only happen if individuals perceive such co-operation to be in their own self-interest. From this perspective individuals work together not because they have to, but because they voluntarily choose to do so. This individualistic approach is clearly consistent with the dominant emphasis within neoclassical economics. For this reason it is non-cooperative game theory that has had the greatest impact within mainstream economics. Given this prominence we restrict ourselves in this book to economic applications of non-cooperative game theory. None the less it should be realized that in many cases we consider the co-operative and non-cooperative approaches are not clearly distinguished. For example, in many instances complex organizations such as firms, governments, and indeed countries are considered to act as individual decision-makers. Clearly this is an extreme simplification and one that ignores how decisions are formulated within these institutions. The value of such a simplification is to make the resulting models more tractable. As in other areas the skill of the economist is to select the level of aggregation most appropriate for the problem being analysed.

1.1.2 Rationality

The second characteristic of game theory is that individuals are assumed to be instrumentally rational. This means that individuals are assumed to act in their own self-interest. This presupposes that individuals are able to determine, at least probabilistically, the outcome of their actions, and have preferences over these outcomes. As with individualism this characteristic dominates neoclassical economics and its justification has been attempted in a number of ways.

The *first* justification is to argue that individuals are indeed rational. However, given the complexity of many decisions, and the amount of information that often needs to be analysed, this seems unrealistic. Indeed evidence from many experimental studies suggests that individuals are not fully rational but instead solve complex decisions by adopting simplistic rules that are generally suboptimal. A *second* justification for rationality is that due to some process of natural selection the economy eventually converges on the fully rational outcome. From this perspective the assumption of rationality is consistent with focusing on the long-run equilibrium of the economy. For example, it is argued that if firms suboptimize then the competitive process will eventually force them to leave the industry. The result of this is that in long-run equilibrium all remaining firms must be optimizing and fully rational. One problem with this argument, however, is that although such an evolutionary process may be considered relevant for competition between firms, it is not always clear how it applies in other contexts. For example, there seems to be no evolutionary process whereby rational consumers can eliminate non-rational consumers. Without such a process of selection the economy will not necessarily converge on the rational outcome. The *final* justification for rationality is that it is not intended to describe how individuals actually solve complex decisions, but rather it is only assumed that individuals act *as if* they were fully rational. Once again the assumption of rationality is used to make the resulting models more tractable. As noted by Friedman (1953) all theories must involve some simplification, as none can include all the possible features of reality. According to this positive methodology the assumption of rationality should not be dismissed merely because it is believed to be unrealistic. This is because all simplifying assumptions are necessarily unrealistic. Instead rationality should only be rejected if the results based on this assumption are found to be unhelpful. This will be true if the theory either gives rise to no relevant predictions or these predictions are falsified by empirical evidence. With this methodology a theory should be judged on its usefulness rather than on the supposed realism of its assumptions. In this book it is argued that game theory based on rationality can be extremely useful in helping us understand a diverse range of economic issues. This, however, does not imply that departures from complete rationality will not also provide useful insights and predictions. Indeed, a major theme of this book is that minor departures from full rationality are often required in order to derive meaningful results from game theory, and that further research incorporating these modifications is warranted. An alternative justification for analysing models that incorporate departures from full rationality is that as economists we may not only be interested in finding *useful* theories, but also interested in discovering theories that are true. If this is our aim, then the positivist methodology fails. If a theory's assumptions are falsified, then these need to be modified so that they conform to reality. As discussed in this book this is an ongoing process in the development of game theory and its application to economics.

1.1.3 Mutual interdependence

The final characteristic of game theory is that it considers situations where individuals are mutually interdependent. In this situation the welfare of any one individual in a game is, at least partially, determined by the actions of other players in the game.

Significantly with mutual interdependence individuals may now have the incentive to act strategically. With strategic decision-making individuals seek to anticipate the effect their own actions will have on the behaviour of others. Given this expectation each individual then determines his/her optimal response in order to achieve the most desirable outcome. In contrast to individualism and rationality this characteristic of mutual interdependence is less prominent within neoclassical economics. For example, in General Equilibrium Theory all agents are assumed to be atomistic. This ensures that the actions of agents taken in isolation have no effect on market outcomes or the welfare of others. This is assumed true for both firms and consumers. With this, and other assumptions it can be demonstrated that the competitive equilibrium is Pareto efficient. This means that no one individual can be made better off without making someone else worse off. In contrast once interdependence is introduced, so that an individual's welfare depends on the actions of others, there is the possibility of market failure and Pareto inefficiency. In this situation at least one individual can be made better off without any other agent being made worse off. The possibility of such inefficiency has been confirmed in numerous economic applications of game theory. Examples of mutual interdependence considered in this book include those between different firms, between firms and their employees, between the government of a country and the private sector, and between different governments.

1.2 Outline of Subsequent Chapters

The purpose of this book is to introduce readers to the main concepts of non-cooperative game theory, and to examine how these concepts have been applied within economics. These general aims are reflected in the structure of the book. The first two chapters focus on game theory itself, with little economic analysis except by way of illustration. In contrast Chapters 3 to 11 focus much more on various economic issues that have been analysed using game theory. In the final chapter we review the current state of game theory by discussing a number of criticisms levelled against recent models. From these criticisms we make some recommendations concerning the direction of future research.

In Chapter 2 we examine static games. These are one-off games where the players are considered to determine their actions simultaneously. In this context we examine two alternative ways in which games can be represented. These are the normal form and the extensive form. We also discuss various techniques commonly used to solve static games. These solutions correspond to predictions for what each player will do in the game, and are based on the concepts of dominance or equilibrium. It is here that we introduce the commonly used solution technique of Nash equilibrium, and discuss how this may be found for both pure and mixed strategies.

Chapter 3 continues our analysis of game theory by examining dynamic games.

These games conform more closely to real world examples where individuals and organizations repeatedly interact with each other. In these games players are often able to condition their actions on past events. This greatly enhances the set of strategies available to players. Once more this chapter discusses how dynamic games can be analysed and predictions made. In this context the key concept of credibility is introduced, and various refinements of Nash equilibrium, such as subgame perfection and sequential equilibrium, are presented.

In Chapters 4 and 5 we begin to focus more exclusively on the economic applications of game theory by considering two topics taken from industrial economics. The first is concerned with oligopoly, and examines the consequences of strategic interdependence between currently competing firms. Initially we consider one-off games between oligopolies, and discuss the now classic models of Cournot, Stackelberg, and Bertrand competition. The results from these one-off games are then contrasted with the results derived when firms are assumed to repeatedly interact. In particular, it is demonstrated that repeated interaction may enable firms to co-ordinate on the collusive outcome where joint profits are maximized. The second application taken from industrial economics is that of entry deterrence. Here the interdependence is between firms already in the market and potential entrants. To illustrate this type of strategic interdependence we examine the situation where a monopolist has the incentive to try and deter other firms from entering the market and competing against it. Initially this involves a critical discussion of Bain's (1956) Theory of Limit Pricing. Subsequent to this we present more recent models of entry deterrence based on non-cooperative game theory. These models serve to highlight the important roles of predatory pricing, precommitment, and incomplete information in oligopolistic markets.

Whilst the games considered in Chapters 4 and 5 are primarily microeconomic, Chapters 6 and 7 focus on macroeconomic games. In Chapter 6 we analyse New Classical results and give them a game theory interpretation. In the first two sections we illustrate how New Classical Macroeconomics has challenged earlier results related to the effectiveness of government policy. This discussion naturally raises the issue of time inconsistency. This occurs when the government has a short-run incentive to deviate from its long-run optimal policy. With the private sector perceiving such an incentive the final equilibrium is Pareto inefficient. In the third section of this chapter we evaluate various suggestions for how governments might avoid the problem of time inconsistency.

In Chapter 7 our attention turns to New Keynesian Macroeconomics. Specifically we examine a number of game theory models that attempt to explain the occurrence of involuntary unemployment and the effectiveness of government demand policy, whilst assuming that all agents are rational. Three separate, though related, strands of New Keynesian Macroeconomics are presented. The first focuses on efficiency wage models, where unemployment is due to real rigidities. The second examines how unemployment can result when agents do not fully adjust nominal wages and prices to an adverse demand shock. Finally, we examine models that exhibit multiple equilibria and co-ordination failure. This occurs when agents within the economy co-ordinate on a Pareto-dominated equilibrium. Significantly it is argued that this possibility may arise even in the absence of real or nominal rigidities.

Chapters 8, 9, and 10 analyse game theory models set in an international context. In Chapter 8 we examine the role of international policy co-ordination. This chapter argues that given the presence of spillover effects between countries uncoordinated policy is likely to lead to an inefficient outcome. This provides the incentive for countries to try and co-ordinate their domestic policies. Despite potential gains there are a number of problems associated with policy co-ordination. These are also discussed. Finally, this chapter seeks to assess the likely magnitude of such gains by reviewing a number of empirical studies.

Chapter 9 considers the possibility that a government may improve domestic welfare with the appropriate use of strategic trade policy. This possibility is discussed in two separate contexts. The first context is where all markets are perfect, but the country itself has some degree of market power. This occurs when the country in question is large, and gives rise to the 'optimal tariff argument'. With two or more countries pursuing such a policy, however, all countries can be made worse off. Various mechanisms for avoiding this outcome are discussed. The second context that provides some justification for strategic trade policy is when domestic industries engaged in international trade have some degree of market power. Faced with oligopolistic competition we analyse how government trade policy can be welfare enhancing and review some of the problems associated with such a policy.

The final chapter set in an international context is concerned with environmental economics. In Chapter 10 we analyse the incentives for countries to enter into international environmental agreements (IEAs). Initially we consider bilateral agreements and then analyse multilateral agreements. In each case we highlight the costs of countries failing to reach agreement over environmental control, and discuss various ways co-ordination can be achieved. In particular we examine the use of side payments between countries, the prospect of punishing countries that break environmental agreements, and how the number of countries signing IEAs might be expanded.

Chapter 11 is somewhat different from the previous chapters. In this chapter we focus on a recent branch of economics known as Experimental Economics. Instead of analysing the theoretical implications of game theory, we discuss a number of experiments that have been designed to test the predictions of this theory. This is done by testing whether they are confirmed by the behaviour of individuals in a controlled environment. Given how rapidly the literature within experimental economics is expanding we focus on studies concerned with testing three of the more important game theory concepts widely used in economics. These are the concepts of Nash equilibrium, sequential equilibrium, and the possibility of co-ordination failure in games with Pareto-ranked multiple equilibria. From a review of such experiments we conclude that the predictions of game theory often perform remarkably well. None the less it is clear that not all relevant game theory predictions are confirmed by experimental evidence. Despite the inherent problems with interpreting such results we argue that greater research is needed on how individuals seek to solve complex decisions under conditions of uncertainty, and how they seek to learn and co-ordinate appropriate strategies over time. Similar conclusions are reached in Chapter 12. In this final chapter we focus on a number of theoretical criticisms levelled at recent game theory models. These specifically relate to assumptions concerning the rationality of individuals. It is argued that models based purely on instrumental rationality are either self-defeating, incomplete, or inconsistent. However, rather than being purely destructive these criticisms can be viewed as a stimulus to further research. We predict that this will involve a re-evaluation of what is meant by rationality, and a greater emphasis given to factors such as the role of institutions, culture, and the previous experience of agents. Such research is an ongoing process, and it is confidently expected that it will lead to further fruitful applications to economics.

2
Static Game Theory

IN this chapter we will look at how static games can be represented, and examine some ways that have been suggested how they might be solved. A solution to a game is a prediction of what each player in that game will do. In static games the players make their moves in isolation without knowing what other players have done. This does not necessarily mean that all decisions are made at the same time, but rather only *as if* the decisions were made at the same time. An example of a static game is a one-off sealed bid auction. In this type of auction each player submits only one bid, without knowing what any of the other players has bid. The highest bid is then accepted as the purchase price. In contrast to static games, dynamic games have a sequence to the order of play and players observe some, if not all, of one another's moves as the game progresses. An example of a dynamic game is a so-called English auction. Here players openly bid up the price of an object. The final and highest bid is accepted as the purchase price.

2.1 Normal Form and Extensive Form Games

In non-cooperative game theory there are two ways in which a game can be represented. The first type is called a *normal form game* or *strategic form game*. The second type is called an *extensive form game*. Both are widely used in economics and we examine each in turn.

2.1.1 Normal form games

A normal form game is any game where we can identify the following three things:

(1) The players
The players in a game are the individuals who make the relevant decisions. For there to be interdependence we need to have at least two players in the game. In most of the applications we look at there will be only two players. In some games 'Nature' is consid-

ered a further player, whose function is to determine the outcome of certain random events, such as the weather or the 'type' of players in the game.

(2) The strategies available to each player
A strategy is a complete description of how a player could play a game. This does not necessarily just list the player's alternative actions. Instead it describes how the player's actions are dependent on what he or she observes other players in the game to have done. For example, if I am thinking about selling my car, then my actions are limited to selling it or keeping it. My chosen strategy, however, tells me how these possible actions are dependent on what other people do. If someone offers me £5,000 or more for my car, I will certainly sell it. If they offer me less than £5,000 I will keep the car. In dynamic games such as this a player's strategy set will be much larger than his or her possible actions. In static games, however, the two are the same. This is because in static games decisions are taken in isolation and so players cannot make their actions dependent on what other players do. In the example where I try to sell my car, this would correspond to the very strange game where I have to accept or reject someone's offer without knowing what it is! In this case my strategies are the same as my actions: to sell or not to sell. (In this discussion we have ignored the possibility of players adopting mixed strategies. These are discussed later in this chapter.)

(3) The pay-offs
A pay-off is what a player will receive at the end of the game contingent upon the actions of all the players in the game. A normal form game shows the pay-offs for every player, except Nature, for every possible combination of available strategies. These are then represented in the form of a matrix or matrices. The pay-offs are defined so that the players in the game always prefer higher to smaller pay-offs. For example, the pay-offs may correspond to monetary rewards, such as profits, or the utility each player obtains at the end of the game. Players are said to be rational when they seek to maximize their pay-off. Players who do not have this objective are said to be irrational, because they are not acting in their own self-interest.

To make the ideas discussed more specific we will look at one well-known static game called 'the *Prisoners' Dilemma*'. In this game the police have arrested two suspects of a crime. However, they lack sufficient evidence to convict either of them unless at least one of them confesses. The police hold the two suspects in separate cells and explain the consequences of their possible actions. If neither confesses, then both will be convicted of a minor offence and sentenced to one month in prison. If both confess, they will be sent to prison for six months. Finally, if only one of them confesses, then that prisoner will be released immediately while the other one will be sentenced to nine months in prison—six months for the crime and a further three months for obstructing the course of justice.

The above description of the game satisfies the three requirements of a normal form game. We have two players, each of whom has two strategies (which in this static game are the same as the prisoners' actions, to confess or not confess), and pay-offs for each possible combination of strategies. The normal form for this game is shown in Fig. 2.1. The pay-offs are shown as the negative number of months in prison for each outcome and for each prisoner. This assumes that each suspect, if rational, seeks to minimize the amount of time spent in prison. By convention the first pay-off listed in each cell refers to the row player, prisoner 1, and the second pay-off refers to the column player, prisoner 2.

		Prisoner 2	
		Confess	Don't confess
Prisoner 1	Confess	−6,−6	0,−9
	Don't confess	−9,0	−1,−1

Fig. 2.1 The Prisoners' Dilemma in Normal Form

2.1.2 Extensive form games

In extensive form games greater attention is placed on the *timing* of the decisions to be made, as well as on the amount of *information* available to each player when a decision has to be made. This type of game is represented not with a matrix but with a decision, or game, tree. The extensive form for the prisoners' dilemma is shown in Fig. 2.2.

Starting at the left of the diagram the open circle represents the first decision to be made in the game. It is labelled 1 to show that it is prisoner 1 that makes this decision. The branches coming out of this initial node represent the actions available to the player at that point in the game. Prisoner 1 can either confess to the crime or not confess. At the end of these branches there is a node representing prisoner 2's decision. Again this prisoner can either confess to the crime or not confess, as given by the branches coming from his decision nodes. However, prisoner 2 makes this decision without knowing what prisoner 1 has done. This is shown by joining prisoner 2's decision nodes with a dotted line. This dotted line shows that the connected nodes are in the same information set. This means that prisoner 2 is unable to distinguish which of the two nodes he is at, at the time this decision is made. This is because he does not know if prisoner 1 has confessed or not confessed to the crime. Finally, at the end of the game we have the pay-offs for each player. These are again dependent on what each pris-

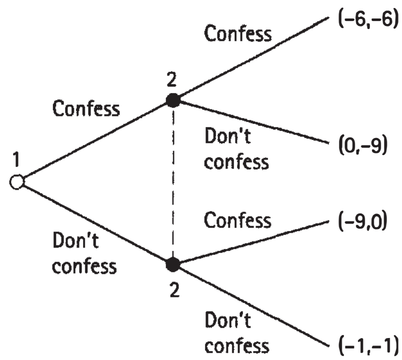


Fig. 2.2 The Prisoners' Dilemma Game in Extensive Form

oner has done in the game, and they are listed in the order of the players in the game, i.e. prisoner 1's pay-off is first, and prisoner 2's pay-off is second.

Generalizing from Fig. 2.2 we can state that extensive form games have the following four elements in common:

Nodes This is a position in the game where one of the players must make a decision. The first position, called the initial node, is an open dot, all the rest are filled in. Each node is labelled so as to identify who is making the decision.

Branches These represent the alternative choices that the person faces, and so correspond to available actions.

Vectors These represent the pay-offs for each player, with the pay-offs listed in the order of players. When we reach a pay-off vector the game ends. When these pay-off vectors are common knowledge the game is said to be one of complete information. (Information is *common knowledge* if it is known by all players, and each player knows it is known by all players, and each player knows that it is known that all players know it, and so on *ad infinitum.*) If, however, players are unsure of the pay-offs other players can receive, then it is an incomplete information game.

Information Sets When two or more nodes are joined together by a dashed line this means that the player whose decision it is does not know which node he or she is at. When this occurs the game is characterized as one of imperfect information. When each decision node is its own information set the game is said to be one of perfect information, as all players know the outcome of previous decisions.

A fundamental assumption of game theory is that the structure of the game is common knowledge. This places three specific requirements on information sets. The *first* is that players always remember whether they have moved previously in the game. This does not, however, mean that they always remember what decision they previously made, only that a decision was made. The *second* requirement is that nodes in the same information set have the same player moving. The *final* condition is that nodes in the same information set have the same possible actions coming from them. If this were not true, players could differentiate between the nodes by examining the available actions. Again generalizing from Fig. 2.2 we can state one further requirement that is always satisfied for extensive form games :

> **Each node has at least one branch pointing out of it (some action is available to the player) and at most one branch pointing into it. (The initial node has no branch pointing to it.)**

This means that at whatever node we begin at there is only one possible path back to the initial node and we never cycle back to the node we started from. For this reason extensive form games always look like trees. From the initial node we always branch out and a branch never grows back into itself.

We have now seen that there are two different ways of representing the same game, either as a normal form game or as an extensive form game. The normal form gives the minimum amount of information necessary to describe a game. It lists the players, the strategies available to each player, and the resulting pay-offs to each player. The extensive form gives additional details about the game concerning the timing of the decisions to be made and the amount of information available to each player when each decision has to be made. Clearly the two forms are closely related and we can state the following two results:

> **For every extensive form game there is one and only one corresponding normal form game.**

> **For every normal form game there are, in general, several corresponding extensive form games.**

The reason for this lack of one-to-one correspondence between a normal form game and an extensive form game is that, as described above, the extensive form game

EXERCISE 2.1

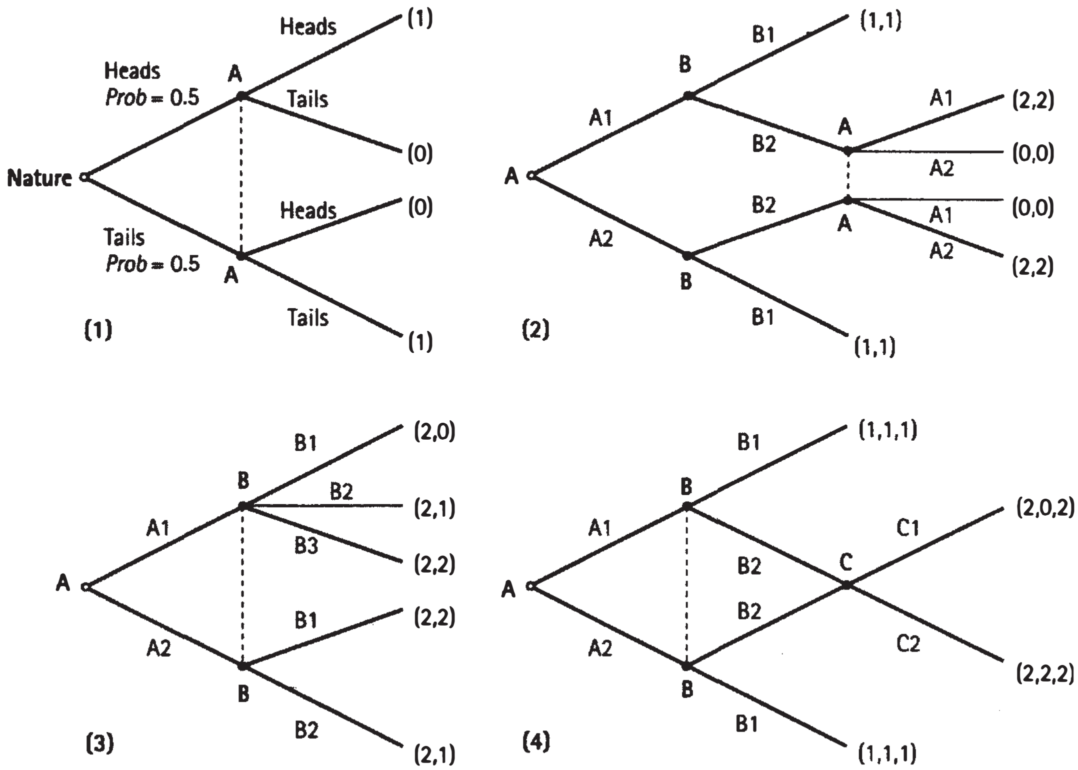
Depict the following situation as both a normal form game and an extensive form game:

Two rival firms are thinking of launching a similar product at the same time. If both firms launch the product, then they will each make a profit of £40,000. If only one firm launches its product, then it can act as a monopolist and will make a profit of £100,000. If either firm decides not to launch the product that firm makes a loss of £50,000, due to costs already incurred in developing the product.

includes additional information. This implies that different extensive forms can be drawn from the same normal form game, depending on what is assumed about these additional details of the game.

EXERCISE 2.2.

Interpret the following diagrams and discuss whether they represent valid extensive form games.



2.2 Solution Techniques for Solving Static Games

As stated at the beginning of this chapter a solution to a game is a prediction of what each player in that game will do. This may be a very precise prediction, where the solution gives one optimal strategy for each player. When this occurs the solution is said to be unique. However, it is often the case that the solution to a particular game is less precise, even to the extent that none of the available strategies are ruled out. As may be expected many different solution techniques have been proposed for different types of games. For static games two broad solution techniques have been applied. The first set of solution techniques rely on the concept of *dominance*. Here the solution to a game is determined by attempting to rule out strategies that a rational person would never play. Arguments based on dominance seek to answer the question 'What strategies would a rational player never play?' The second set of solution techniques is based on the concept of *equilibrium*. In non-cooperative games an equilibrium occurs when none of the players, acting individually, has an incentive to deviate from the predicted solution. With these solution techniques a game is solved by answering the question 'What properties does a solution need to have for it to be an equilibrium?'

In the following section we examine various dominance techniques that can be applied to static games, and two equilibrium concepts. In subsequent chapters further equilibrium concepts that are commonly used in game theory will be presented and discussed.

2.2.1 Strict dominance

A strategy is said to be strictly dominated if another strategy always gives improved pay-offs whatever the other players in the game do. This solution technique makes the seemingly reasonable assumption that a rational player will never play a strictly dominated strategy. If players knowingly play a strictly dominated strategy, they cannot be maximizing their expected pay-off, given their beliefs about what other players will do. In this sense a player who plays a strictly dominated strategy is said to be irrational. Applying the principle of strict dominance rules out this type of irrational behaviour. To illustrate this technique we use it to solve the prisoners' dilemma game. In applying the principle of strict dominance we examine each player in turn and exclude all those strategies that are strictly dominated. This process may rule out all but one strategy for each player. This is true for the prisoners' dilemma game, and so this technique produces a unique solution for this game.

Consider first the dilemma facing prisoner 1. Should she confess or should she remain quiet hoping the other prisoner does the same. The principle of strict dominance argues that prisoner 1 should confess. The reason for this is that whatever prisoner 2 decides to do prisoner 1 is always better off confessing. This means not confessing is strictly dominated and so it seems reasonable to suppose it will not be played. The same logic applies equally to prisoner 2 and so strict dominance predicts that he will also confess. The solution to this game based on strict dominance is that both prisoners confess even though both would be better off if neither confessed. As at least one of the players in this game can, with a different outcome, be made better off without the other player being made worse off this solution is said to be Pareto inefficient. (In fact if neither player confesses, both would be better off.) This is a very common feature of many games used in economics, and it will be illustrated in many contexts throughout this book.

It should be noted here that the cause of Pareto inefficiency is not that the players cannot communicate, but rather that they cannot commit themselves to the Pareto-efficient outcome. Even if both prisoners agreed before being arrested that neither of them will confess, once in custody it is in their individual self-interest to do the opposite. This illustrates the difference between non-cooperative and co-operative game theory. In co-operative game theory the two prisoners could enter into a binding and enforceable agreement not to confess and so be made better off. This is not possible in non-cooperative game theory.

EXERCISE 2.3

Solve the previous product launch game described in Exercise 2.1 using the principle of strict dominance.

2.2.2 Weak dominance

A strategy is said to be weakly dominated if another strategy makes the person better off in some situations and leaves them indifferent in all others. Again it seems reasonable to assume that a rational player will not play a weakly dominated strategy, as he or she could do at least as well, and possibly even better, by playing the dominant strategy. Consider the normal form game shown in Fig. 2.3. In this game there are two players each with two possible strategies. Player 1 can move either 'up' or 'down', and player 2 can move either 'left' or 'right'. The pay-offs are given in the matrix, where the first figure is the pay-off for player 1 and the second figure is the pay-off for player 2. For this game none of the available strategies is ruled out using the principle of strict dominance. This is because no strategy makes that player worse off in all circumstances. For example, if player 1 plays 'up', then player 2 is indifferent between 'left' and 'right'. Similarly if player 2 plays 'left', player 1 is indifferent between 'up' and 'down'. Although we cannot appeal to the principle of strict dominance to rule out any of the available strategies, we can apply the principle of weak dominance.

According to the principle of weak dominance player 1 will never play 'down' and so this can be ruled out. Similarly player 2 will never play 'right', and so this can also be ruled out. This leaves only one remaining strategy for each player. The predicted outcome is that player 1 will move 'up' and player 2 will move 'left'. Again this is a Pareto-inefficient solution. This is because the outcome 'down/left' makes player 2 better off

		Player 2	
		Left	Right
Player 1	Up	7,2	5,2
	Down	7,4	2,0

Fig. 2.3 An Application of Weak Dominance

and player 1 no worse off. The reason player 1 does not switch to playing 'down', even though this leads to a Pareto improvement, is that it entails greater risk for this player. If player 2 were to play 'right', then player 1 is definitely worse off moving 'down' instead of 'up'. This element of avoiding unnecessary risk is reflected in the principle of weak dominance.

2.2.3 Iterated strict dominance

Iterated strict dominance assumes that strict dominance can be applied successively to different players in a game. For example, if one player rules out a particular strategy, because it is strictly dominated by another, then it is assumed other players recognize this and that they also believe the other player will not play this dominated strategy. This in turn may lead them to exclude dominated strategies, and so on. In this way it may be possible to exclude all but one strategy for each player, and so make a unique prediction for the game being analysed. Consider the game shown in Fig. 2.4.

In this game player 1 has two possible strategies, 'up' and 'down', and player 2 has three possible strategies, 'left', 'middle', and 'right'. Initially neither 'up' nor 'down' is strictly dominated by the other for player 1. However, for player 2 'right' is strictly dom-

		Player 2		
		Left	Middle	Right
Player 1	Up	1,0	1,2	0,1
	Down	0,3	0,1	2,0

Fig. 2.4 An Application of Iterated Strict Dominance

inated by 'middle'. Appealing to strict dominance we can reason that player 2 will never play 'right'. If player 1 also knows that player 2 is rational and will not play 'right', then 'up' now strictly dominates 'down' for player 1. Iterated strict dominance now predicts that 'down' will not be played. Finally, if player 2 knows that player 1 will never move 'down', then iterated strict dominance predicts that player 2 will play 'middle'. The unique solution to this game based on successive or iterated strict dominance is therefore 'up/middle'.

2.2.4 Iterated weak dominance

The final dominance technique is iterated weak dominance. This is the same as iterated strict dominance except here it is weak dominance that is applied successively to different players in the game. Again it is possible that this technique can produce a unique solution to a particular game.

One problem with iterated weak dominance, which is not shared by iterated strict dominance, is that the predicted solution can depend on the order in which players' strategies are eliminated. This is true for the game shown in Fig. 2.5. If we start by applying weak dominance to player 1, then we predict that the players will choose the unique solution 'up/middle'. If we first apply weak dominance to player 2, then all we can conclude is that player 2 will not play 'right'. Clearly the order in which we apply weak dominance significantly affects the predicted outcome of the game. Unfortunately for most games this choice is totally arbitrary.

		Player 2		
		Left	Middle	Right
Player 1	Up	10,0	5,1	4,-2
	Down	10,1	5,0	1,-1

Fig. 2.5 An Application of Iterated Weak Dominance

It should be noted that in applying iterated-dominance arguments we are assuming a stronger version of rationality than we did with mere dominance. With dominance we assumed that rational players will not play dominated strategies. With iterated dominance we assume that rational players will not play dominated strategies, and also that players assume that other players are rational and will not do this. For iterated dominance to predict accurately people must not only be rational but assume that others are rational as well, and this requirement needs to be strengthened with each iteration. (For example, I need to assume that you believe that I believe that you believe that I am rational, and so on. When this sequence of reasoning continues *ad infinitum* we have the frequently used assumption of *common knowledge of rationality*.) As the number of iterations becomes large these additional assumptions become increasingly more dubious. An example of a game where the principle of iterated strict dominance is taken to extreme lengths is Rosenthal's (1981) centipede game. This dynamic game is discussed at the end of Chapter 3.

If a game yields a unique solution by applying either strict, weak, or iterated dominance, then that game is said to be dominance solvable. The main problem with all these solution techniques is that often they give very imprecise predictions about a game. Consider the game shown in Fig. 2.6. In this game arguments based on dominance lead to the very imprecise prediction that anything can happen! If a more specific solution to this type of game is needed, then a stronger solution technique must be applied. This leads us on to solution techniques based not on dominance but on the concept of equilibrium.

		Player 2		
		Left	Middle	Right
Player 1	Up	0,4	4,0	5,3
	Centre	4,0	0,4	5,3
	Down	3,5	3,5	6,6

Fig. 2.6 An Illustration of the Problem with Dominance Techniques

2.2.5 Nash equilibrium

As stated in the introduction to this section arguments based on dominance ask the question 'What strategies would a rational player never play?' In contrast the concept of Nash equilibrium is motivated by the question 'What properties must an equilibrium have?' The answer to this question from John Nash (1951), based on much earlier work by Cournot (1838), was that in equilibrium each player's chosen strategy is optimal given that every other player chooses the equilibrium strategy. If this were not the case, then at least one player would wish to choose a different strategy and so we could not be in an equilibrium. Again this concept seeks to apply the economist's assumption that individuals are rational in the sense that they seek to maximize their own self-interest.

Finding the Nash equilibrium for any game involves two stages. *First*, we identify each player's optimal strategy in response to what the other players might do. This involves working through each player in turn and determining their optimal strategies. This is done for every combination of strategies by the other players. *Second*, a Nash equilibrium is identified when all players are playing their optimal strategies simultaneously.

Strictly speaking, the above methodology only identifies pure-strategy Nash equilibria. It does not identify mixed-strategy Nash equilibria. A pure-strategy equilibrium is where each player plays one specific strategy. A mixed-strategy equilibrium is where at least one player in the game randomizes over some or all of their pure strategies. This means that players place a probability distribution over alternative strategies. For example, players might decide to play each of two available pure strategies with a probability of 0.5, and never play any other strategy. A pure strategy is therefore a restricted mixed strategy with a probability of one given to the chosen strategy, and zero to all the others. The concept of mixed-strategy Nash equilibrium is discussed later in this section.

To illustrate the two-stage methodology for finding a (pure-strategy) Nash equilibrium we apply it to the prisoners' dilemma game. This is shown in Fig. 2.7.

		Prisoner 2	
		Confess	Don't confess
Prisoner 1	Confess	-6,-6	0,-9
	Don't confess	-9,0	-1,-1

Fig. 2.7 The Nash Equilibrium of the Prisoners' Dilemma Game

Stage One.

We first need to identify the optimal strategies for each prisoner, dependent upon what the other prisoner might do. If prisoner 1 expects prisoner 2 to confess, then prisoner 1's best strategy is also to confess (–6 is better than –9). This is shown in Fig. 2.7 by underlining this pay-off element for prisoner 1 in the cell corresponding to both prisoners confessing. If prisoner 1 expects prisoner 2 not to confess, then prisoner 1's best strategy is still to confess (this time 0 is better than –1). Again we show this by underlining this pay-off element for prisoner 1. The same analysis is undertaken for prisoner 2 and his best strategy pay-offs are underlined.

Stage Two.

Next we determine whether a Nash equilibrium exists by examining the occurrence of the previously identified optimal strategies. If all the pay-offs in a cell are underlined, then that cell corresponds to a Nash equilibrium. This is true by definition, since in a Nash equilibrium all players are playing their optimal strategy given that other players also play their optimal strategies. In the prisoners' dilemma game only one cell has all its elements underlined. This corresponds to both prisoners confessing, and so this is the unique Nash equilibrium for this game.

This prediction for the prisoners' dilemma game is the same as that derived using strict dominance. In fact it is always true that a unique strict dominance solution is the unique Nash equilibrium. The reverse of this statement is, however, not always true. A unique Nash equilibrium is not always a unique strict dominant solution. In this sense the Nash equilibrium is a stronger solution concept than strict dominance. For this reason the Nash equilibrium concept may predict a unique solution to a game where strict dominance does not. This is illustrated in the game used previously to demonstrate that a game may not be dominance solvable. This game shown in Fig. 2.6 is reproduced in Fig. 2.8. As stated before, arguments based on dominance applied to this game predict that anything can happen. Using the two-stage methodology of finding a (pure-strategy) Nash equilibrium, however, yields the unique prediction that player 1 will choose 'down' and player 2 will choose 'right'. The concept of Nash equilibrium may therefore be particularly useful when dominance arguments do not provide a unique solution.

One important result from game theory is that for any finite game (i.e. games with a finite number of players and strategies) there always exists at least one Nash equilibrium. Before thinking that this result means that we can always make a definite predic-

		Player 2		
		Left	Middle	Right
	Top	$0,\underline{4}$	$\underline{4},0$	$5,3$
Player 1	Centre	$\underline{4},0$	$0,\underline{4}$	$5,3$
	Bottom	$3,5$	$3,5$	$\underline{6},\underline{6}$

Fig. 2.8 A Further Application of Nash Equilibrium

tion about what people will do in any game the following two qualifications need to be stated.

First, the above result is only true if we include mixed strategies, as well as pure strategies. This means that we cannot always state for certain what all players in a game will do, but instead we may only be able to give the probabilities for various outcomes occurring. This possibility is discussed below.

Second, the above result does not rule out the possibility of multiple Nash equilibria. Indeed, many games do exhibit multiple Nash equilibria. With multiple equilibria the problem is how to select one equilibrium from many. In answer to this question

EXERCISE 2.4.

State whether the following games have unique pure strategy solutions, and if so what they are and how they can be found.

(1)

Player 2

Player 1		Left	Middle	Right
	Up	4, 3	2, 7	0, 4
	Down	5, 5	5, −1	−4, −2

(2)

Player 2

Player 1		Left	Middle	Right
	Up	4, 10	3, 0	1, 3
	Down	0, 0	2, 10	1, 3

(3)

Player 2

Player 1		Left	Middle	Right
	Up	10, 10	4, 3	7, 2
	Down	5, 6	8, 10	6, 12

numerous refinements of Nash equilibrium have been proposed to try and restrict the set of possible equilibria. Some of these refinements are discussed in later chapters.

2.2.6 Mixed strategy Nash equilibrium

To illustrate that there may be multiple Nash equilibria to a particular game, and also the idea of mixed strategies, we look at another classic game called the '*Battle of the Sexes*'. In this game a husband and wife are trying to decide where to go for an evening out. Whilst apart they must choose either to go to a boxing match, or to the ballet. Both players would rather go anywhere together, but given this the man prefers the boxing and the woman the ballet. (This game was proposed in the 1950s, which partly explains its stereotypical views.) These preferences are represented in the normal form game shown in Fig. 2.9.

		Husband	
		Go to boxing	Go to ballet
Wife	Go to boxing	$\underline{1},\underline{2}$	0,0
	Go to ballet	0,0	$\underline{2},\underline{1}$

Fig. 2.9 The Battle of the Sexes Games in Normal Form

Applying the two-stage method of identifying a pure-strategy Nash equilibrium we can see that the above game has two such equilibria. These are that either both will go to the boxing or both will go to the ballet. This means that each person will go wherever they think the other person will go. This is not very helpful, as it tells neither player what the other person is likely to do. As there is no unique pure-strategy Nash equilibrium neither player can confidently predict what the other person will do. Playing a mixed strategy is a response to this uncertainty. A mixed strategy is when a player randomizes over some or all of his or her available pure strategies. This means that the player places a probability distribution over their alternative strategies. A mixed-strategy equilibrium is where at least one player plays a mixed strategy and no one has the incentive to deviate unilaterally from that position.

The key feature of a mixed-strategy Nash equilibrium is that every pure strategy played as part of the mixed strategy has the same expected value. If this were not true, a player would play the strategy that yields the highest expected value to the exclusion of all others. This means the initial situation could not have been an equilibrium. Here we show how to identify the mixed-strategy Nash equilibrium for the battle of the sexes game.

Let $Prob(\text{boxing})_H$ be the probability that the husband goes to the boxing match, and $Prob(\text{boxing})_W$ the probability that the wife goes to the boxing match. Similarly let $Prob(\text{ballet})_H$ be the probability that the man goes to the ballet, and $Prob(\text{ballet})_W$ the probability that the woman goes to the ballet. As these are the only two alternatives it must be true that $Prob(\text{boxing}) + Prob(\text{ballet}) = 1$ for both the husband and wife. Given these probabilities we can calculate the expected value of each person's possible action.

From the normal form game the expected pay-off value for the wife if she chooses to go to the boxing match is given as

$$\pi(\text{boxing})_W = Prob(\text{boxing})_H(1) + Prob(\text{ballet})_H(0)$$
$$= Prob(\text{boxing})_H.$$

Similarly the expected pay-off value if she goes to the ballet is

$$\pi(\text{ballet})_W = Prob(\text{boxing})_H(0) + Prob(\text{ballet})_H(2)$$
$$= 2Prob(\text{ballet})_H.$$

In equilibrium the expected value of these two strategies must be the same and so we get

$$\pi(\text{boxing})_W = \pi(\text{ballet})_W$$
$$\therefore\ Prob(\text{boxing})_H = 2\ Prob(\text{ballet})_H$$
$$\therefore\ 1 - Prob(\text{ballet})_H = 2\ Prob(\text{ballet})_H$$
$$\therefore\ 1 = 3\ Prob(\text{ballet})_H$$
$$\therefore\ Prob(\text{ballet})_H = \tfrac{1}{3} \text{ and } Prob(\text{boxing})_H = \tfrac{2}{3}.$$

This means that in the mixed-strategy equilibrium the husband will go to the ballet with a 1/3 probability and the boxing with a 2/3 probability. We can perform the same calculations for the husband's expected pay-off and derive the similar result that in equilibrium his wife will go to the ballet with a probability of 2/3 and the boxing with a probability of 1/3. With these individual probabilities we can calculate that they will both go to the boxing with a probability of 2/9, both go to the ballet with a probability of 2/9, and go to separate events with a probability of 5/9.

This combination of mixed strategies constitutes a third Nash equilibrium for this game. Intuitively this seems the most reasonable Nash equilibrium of the three, as it explicitly takes into account the inherent uncertainty in the game. It should be noted that playing a mixed strategy does not mean that players flip a coin or roll a dice to make their decisions. Rather playing a mixed strategy is a rational response to uncertainty about what other players will do.

One curious aspect of a mixed-strategy equilibrium is that because each of the chosen pure strategies in the mixed strategy has the same expected pay-off value, each player is indifferent as to which strategy he or she actually plays. A mixed-strategy equilibrium is, therefore, said to be a weak equilibrium because none of the players is made worse off if they abandon their mixed strategy, and play any one of the pure strategy components of their mixed strategy. This feature of a mixed-strategy Nash equilibrium has caused its application within economics to be controversial. In particular this solution technique has been criticized as imposing unacceptable constraints on players' beliefs. Some of these criticisms are discussed in Chapter 12.

EXERCISE 2.5.

Draw the normal form game for the following game and identify both the pure- and mixed-strategy equilibria. In the mixed-strategy Nash equilibrium determine each firm's expected profit level if it enters the market.

There are two firms that are considering entering a new market, and must make their decision without knowing what the other firm has done. Unfortunately the market is only big enough to support one of the two firms. If both firms enter the market, then they will each make a loss of £10m. If only one firm enters the market, that firm will earn a profit of £50m., and the other firm will just break even.

2.3 Conclusions

Static games are where players make decisions in isolation. Each decision is made without knowing what the other players have done. These games can be represented as either normal or extensive form games. Normal form games give the minimum amount of information necessary to describe a game. They list the players in the game, the strategies available to each player, and the pay-offs dependent on the outcome of the game. Extensive form games give additional details on the timing of decisions and the amount of information players have when making these decisions. Static games are predominantly represented as normal form games. This is because in such games the amount of information available to players does not vary within the game, and the timing of decisions has no effect on players' choices. In the next chapter we examine dynamic games where the timing of decisions and information constraints critically determine the outcome of the game.

In attempting to predict the outcome of static games various solution techniques have been suggested. These are either based on the concept of dominance or equilibrium. These solution techniques try and predict what rational players will do in specified games. Sometimes they yield a definite prediction of what each player will do. Often, however, the solution is less precise. These solution techniques can also be applied to dynamic games, but as we will see in the next chapter additional assumptions are typically needed so that reasonable predications are generated.

2.4 Solutions to Exercises

Exercise 2.1

The normal and extensive forms for this static game are shown in Figs. 2.10 and 2.11 respectively:

		Firm B	
		Launch	Don't launch
Firm A	Launch	£40,000 £40,000	£100,000 −£50,000
	Don't launch	−£50,000 £100,000	−£50,000 −£50,000

Fig. 2.10

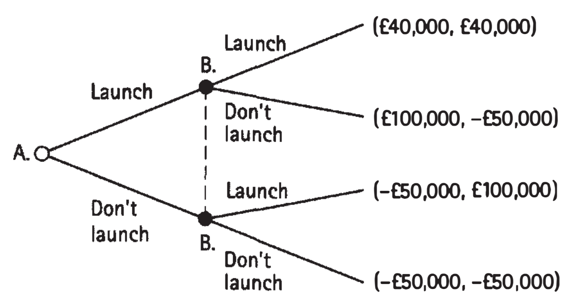


Fig. 2.11

Exercise 2.2.

(1) This is a one-player static game against nature with imperfect information. Nature determines the outcome of the toss of an unbiased coin. Without knowing whether the outcome is heads or tails, player A calls either heads or tails. If the call is correct, the player wins a pay-off of 1. If the call is wrong, the player receives nothing. This diagram is a valid extensive form game. In such games we assume that players simply maximize their expected pay-off. In particular there are no strategic considerations in one-player games. For this reason this book only analyses games with two or more (rational) players.

(2) This is a dynamic game with imperfect recall. Player A initially decides between A1 and A2. This is observed by player B who then decides between B1 and B2. If B1 is chosen, the game ends. If B2 is chosen, player A moves again, playing either A1 or A2. Significantly these two final decision nodes are in the same information set, which means that player A does know which one she is at. However, the only difference in the paths to these nodes is player A's initial move. This means that player A must have forgotten what her first move was! This is a

valid extensive form game, and indeed some economic models have assumed that agents have imperfect recall. In this book, however, we limit ourselves to games where all players have perfect recall. This means that players do not forget any information that has been previously revealed to them.

(3) This is not a valid extensive form game as it entails a logical contradiction. In the diagram player B's decision nodes are in the same information set, which means that they cannot be distinguished. However, at the decision node following A1 there are three possible actions, while at the node following A2 there are only two options. Player B must know the actions available to him and so based on this information he will be able to distinguish between his decision nodes. This contradicts the fact they are shown as being in the same information set. To avoid such logical contradictions it is required that the set of possible actions from nodes in the same information set must be identical.

(4) This is not a valid extensive form game, as it violates one of the previous assumptions. This is the requirement that each node has at most one branch pointing to it. This is not true for player C's decision node. The reason this assumption is made is to guarantee a unique path from any decision node back to the initial node. (This is important for the application of backward induction discussed in the next chapter.) This diagram does not satisfy this feature, as there are two possible paths back to the initial node from player C's decision node.

Exercise 2.3

Strict dominance predicts that both firms will launch their respective products because this gives each firm a higher pay-off whatever the other firm does.

Exercise 2.4

(1) The unique pure-strategy equilibrium is 'down/left'. This is both a Nash equilibrium and an iterated strict dominant solution. The process of elimination for the dominant solution is 'right', 'up', 'middle'.

(2) The unique pure-strategy Nash equilibrium this time is 'up/left', and is both a Nash equilibrium and an iterated weak dominant solution. The process of elimination in the latter case is 'down', 'middle', 'right'.

(3) This game is not dominance solvable, but 'up/left' is a Nash equilibrium.

Exercise 2.5

The normal form for this static entry game is given in Fig. 2.12. Using the two-stage method for finding a pure-strategy Nash equilibrium we can see that there are two such equilibria. Both involve one firm entering the market, and the other firm staying out.

We can determine the mixed-strategy Nash equilibrium in the following way. Let $Prob(\text{enter})_1$ and $Prob(\text{enter})_2$ be the probabilities of firm 1 and firm 2 entering the market respectively. And let $Prob(\text{stay out})_1$ and $Prob(\text{stay out})_2$ be the probabilities of the two firms staying out of the market.

Expected profits for firm 1 if it enters the market are therefore

$$\pi(\text{enter})_1 = -10\,.Prob(\text{enter})_2 + 50\,.Prob(\text{stay out})_2$$

		Firm 2	
		Enters	Stays out
Firm 1	Enters	-£10m., -£10m.	£50m., 0
	Stays out	0, £50m.	0, 0

Fig. 2.12

and its expected profit if it stays out of the market is 0. In equilibrium these expected values must equal each other and so we get

$$-10 .$$

$$\therefore 50 . Prob(\text{stay out})_2 = 10 .$$

$$\therefore 5 .$$

$$\therefore 5 . Prob(\text{stay out})_2 = 1 - Prob(\text{stay out})_2$$

$$\therefore 6 . Prob(\text{stay out})_2 = 1$$

$$\therefore$$

We could do the same calculations to find the same probabilities of firm 1 entering and staying out of the market.

Substituting these probabilities back into the equation for $\pi(\text{enter})_1$ we can find the expected value for firm 1 of entering the market.

$$\pi(\text{enter})_1 = -10 . Prob(\text{enter})_2 + 50 . Prob(\text{stay out})_2$$

$$\therefore \pi(\text{enter})_1 = -10 . \tfrac{5}{6} + 50 . \tfrac{1}{6}$$

$$\therefore \pi(\text{enter})_1 = 0.$$

The same result holds for firm 2. In the mixed-strategy Nash equilibrium the expected value for both firms if they enter the market is zero. This could have been found by noting that this equals the expected value of not entering which equals zero.

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Static Game Theory

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3
Dynamic Game Theory

IN the previous chapter we focused on static games. However, for many important economic applications we need to think of the game as being played over a number of time-periods, making it dynamic. A game can be dynamic for two reasons. *First*, the interaction between players may itself be inherently dynamic. In this situation players are able to observe the actions of other players before deciding upon their optimal response. In contrast, static games are ones where we can think of players making their moves simultaneously. *Second*, a game is dynamic if a one-off game is repeated a number of times, and players observe the outcome of previous games before playing later games. In section 3.1 we consider one-off dynamic games, and in section 3.2 we analyse repeated games.

3.1 Dynamic One-Off Games

An essential feature of all dynamic games is that some of the players can condition their optimal actions on what other players have done in the past. This greatly enhances the strategies available to such players in that these are no longer equivalent to their possible actions. To illustrate this we examine the following two-period dynamic entry game, which is a modified version of the static game used in Exercise 2.4.

There are two firms (A and B) that are considering whether or not to enter a new market. Unfortunately the market is only big enough to support one of the two firms. If both firms enter the market, then they will both make a loss of £10m. If only one firm enters the market, that firm will earn a profit of £50m., and the other firm will just break even. To make this game dynamic we assume that firm B observes whether firm A has entered the market before it decides what to do. This game can be represented by the extensive form diagram shown in Fig. 3.1.

In time-period 1 firm A makes its decision. This is observed by firm B which decides to enter or stay out of the market in period 2. In this extensive form game firm B's decision nodes are separate information sets. (If they were in the same information set, they would be connected by a dashed line.) This means that firm B observes firm A's action before making its own decision. If the two firms were to make their moves

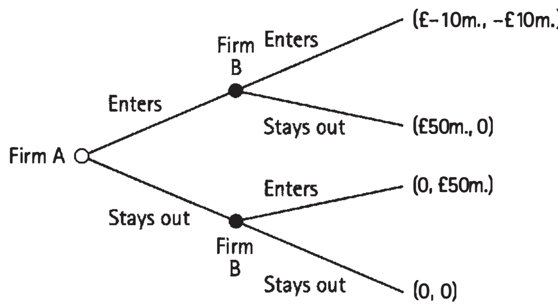


Fig. 3.1 The Dynamic Entry Game in Extensive Form

simultaneously, then firm B would have only two strategies. These would be either to enter or stay out of the market. However, because firm B initially observes firm A's decision it can make its decision conditional upon what firm A does. As firm A has two possible actions, and so does firm B, this means that firm B has four (2 × 2) strategies. We can list these as

> Always enter the market whatever firm A does.
> Always stay out of the market whatever firm A does.
> Do the same as firm A.
> Do the opposite of firm A.

Recognizing that firm B now has these four strategies we can represent the above game in normal form. This is shown in Fig. 3.2. Having converted the extensive form game into a normal form game we can apply the two-stage method for finding pure-strategy Nash equilibria, as explained in the previous chapter. *First*, we identify what each player's optimal strategy is in response to what the other players might do. This involves working through each player in turn and determining their optimal strategies. This is illustrated in the normal form game by underlining the relevant pay-off. *Second*, a Nash equilibrium is identified when all players are playing their optimal strategies simultaneously.

As shown in Fig. 3.2 this dynamic entry game has three pure-strategy Nash equilibria. In these three situations each firm is acting rationally given its belief about what the other firm might do. Both firms are maximizing their profits dependent upon what they believe the other firm's strategy is. One way to understand these possible outcomes is to think of firm B making various threats or promises, and firm A acting accordingly. We can therefore interpret the three Nash equilibria as follows:

1. Firm B threatens always to enter the market irrespective of what firm A does. If firm A believes this threat, it will stay out of the market.
2. Firm B promises always to stay out of the market irrespective of what firm A does. If firm A believes this promise, it will certainly enter the market.

		Firm B			
		Always enters	Always stays out	Same as firm A	Opposite of firm A
Firm A	Enters	-£10m. -£10m.	£50m. 0	-£10m. -£10m.	£50m. 0
	Stays out	0 £50m.	0 0	0 0	0 £50m.

Fig. 3.2 The Dynamic Entry Game in Normal Form

3. Firm B promises always to do the opposite of what firm A does. If firm A believes this promise, it will again enter the market.

In the first two Nash equilibria firm B's actions are not conditional on what the other firm does. In the third Nash equilibrium firm B does adopt a conditional strategy. A *conditional strategy* is where one player conditions his or her actions upon the actions of at least one other player in the game. This concept is particularly important in repeated games, and is considered in more detail in the next section.

In each of the equilibria firm A is acting rationally in accordance with its beliefs. However, this analysis does not consider which of its beliefs are themselves rational. This raises the interesting question 'Could firm A not dismiss some of firm B's threats or promises as mere bluff?' This raises the important issue of *credibility*. The concept of credibility comes down to the question 'Is a threat or promise believable?' In game theory a threat or promise is only credible if it is in that player's interest to carry it out at the appropriate time. In this sense some of firm B's statements are not credible. For example, firm B may threaten always to enter the market irrespective of what firm A does, but this is not credible. It is not credible because if firm A enters the market, then it is in firm B's interest to stay out. Similarly the promise always to stay out of the market is not credible because if firm A does not enter, then it is in firm B's interest to do so. Assuming that players are rational, and that this is common knowledge, it seems reasonable to suppose that players only believe credible statements. This implies that incredible statements will have no effect on other players' behaviour. These ideas are incorporated into an alternative equilibrium concept to Nash equilibrium (or a refinement of it) called subgame perfect Nash equilibrium.

3.1.1 Subgame perfect Nash equilibrium

In many dynamic games there are multiple Nash equilibria. Often, however, these equilibria involve incredible threats or promises that are not in the interests of the players making them to carry out. The concept of subgame perfect Nash equilibrium rules out these situations by saying that a reasonable solution to a game cannot involve players believing and acting upon incredible threats or promises. More formally a subgame perfect Nash equilibrium requires that the predicted solution to a game be a Nash equilibrium in every subgame. A *subgame*, in turn, is defined as a smaller part of the whole game starting from any one node and continuing to the end of the entire game, with the qualification that no information set is subdivided. A subgame is therefore a game in its own right that may be played in the future, and is a relevant part of the overall game. By requiring that a solution to a dynamic game must be a Nash equilibrium in every subgame amounts to saying that each player must act in his or her own self-interest in every period of the game. This means that incredible threats or promises will not be believed or acted upon.

To see how this equilibrium concept is applied, we continue to examine the dynamic entry game discussed above. From the extensive form of this game, given in Fig. 3.1, we can observe that there are two subgames, one starting from each of firm B's decision nodes. For the predicted solution to be a subgame perfect Nash equilibrium it must comprise a Nash equilibrium in each of these subgames. We now consider each of the Nash equilibria identified for the entire game to see which, if any, is also a subgame perfect Nash equilibrium.

1. In the first Nash equilibrium firm B threatens always to enter the market irrespective of what firm A does. This strategy is, however, only a Nash equilibrium for one of the two subgames. It is optimal in the subgame beginning after firm A has stayed out but not in the one where firm A has entered. If firm A enters the market, it is not in firm B's interest to carry out the threat and so it will not enter. This threat is not credible and so should not be believed by firm A. This Nash equilibrium is not subgame perfect.
2. In the second Nash equilibrium firm B promises always to stay out of the market irrespective of what firm A does. Again this is only a Nash equilibrium for one of the two subgames. It is optimal in the subgame beginning after firm A has entered but not in the one when firm A has stayed out . If firm A stays out of the market, it is not in the interest of firm B to keep its promise, and so it will enter. This promise is not credible, and so should not be believed by firm A. Once more this Nash equilibrium is not subgame perfect.
3. The third Nash equilibrium is characterized by firm B promising to do the opposite of whatever firm A does. This is a Nash equilibrium for both subgames. If firm A enters, it is optimal for firm B to stay out, and if firm A stays out, it is optimal for firm B to enter. This is a credible promise because it is always in firm B's interest to carry it out at the appropriate time in the future. This promise is therefore believable, and with this belief it is rational for firm A to enter the market.

The only subgame perfect Nash equilibrium for this game is that firm A will enter and firm B will stay out. This seems entirely reasonable given that firm A has the ability to enter the market first. Once firm B observes this decision it will not want to enter the market, and so firm A maintains its monopoly position. The way we solved this dynamic game has been rather time-consuming. This was done so that the concept of subgame perfection may be better understood. Fortunately, there is often a quicker way of finding the subgame perfect Nash equilibrium of a dynamic game. This is by using the principle of backward induction.

3.1.2 Backward induction

Backward induction is the principle of iterated strict dominance applied to dynamic games in extensive form. However, this principle involves ruling out the actions, rather than strategies, that players would not play because other actions give higher pay-offs. In applying this principle to dynamic games we start with the last period first and work backwards through successive nodes until we reach the beginning of the game. Assuming perfect and complete information, and that no player is indifferent between two possible actions at any point in the game, then this method will give a unique prediction which is the subgame perfect Nash equilibrium. Once more this principle is illustrated using the dynamic entry game examined above. The extensive form for this game is reproduced in Fig. 3.3.

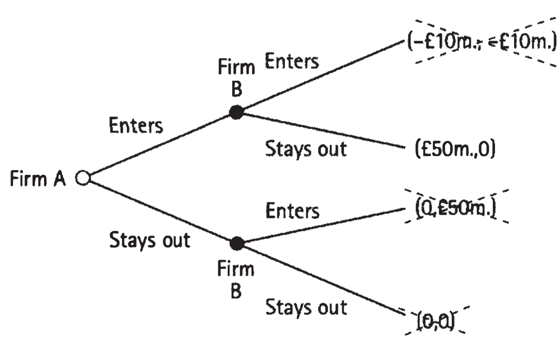
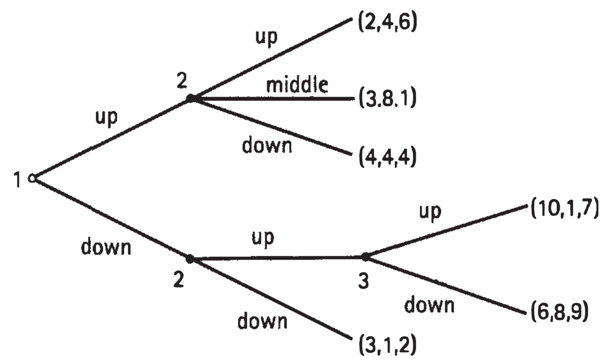


Fig. 3.3 The Dyanmic Entry Game and Backward Induction

Starting with the last period of the game first, we have two nodes. At each of these nodes firm B decides whether or not to enter the market based on what firm A has already done. At the first node firm A has already entered and so firm B will either make a loss of £10m. if it enters, or break even if it stays out. In this situation firm B will stay out, and so we can rule out the possibility of both firms entering. This is shown by crossing out the corresponding pay-off vector (–£10m., –£10m.). At the second node firm A has not entered the market, and so firm B will earn either £50m. if it enters or nothing if it stays out. In this situation firm B will enter the market, and we can rule out the possibility of both firms staying out. Once more we cross out the corresponding pay-off vector (0, 0). We can now move back to the preceding nodes, which in this game is the initial node. Here firm A decides whether or not to enter. However, if firm A assumes that firm B is rational, then it knows the game will never reach the previously excluded strategies and pay-offs. Firm A can reason therefore that it will either receive £50m. if it enters or nothing if it stays out. Given this reasoning we can rule out the possibility that firm A will stay out of the market, and so cross out the corresponding pay-off vector (0, £50m.). This leaves only one pay-off vector remaining, corresponding to firm A entering the market and firm B staying out. This, as shown before, is the subgame perfect Nash equilibrium.

EXERCISE 3.1

Using the principle of backward induction find the subgame perfect Nash equilibrium for the following three-player extensive form game. State all the assumptions you make.

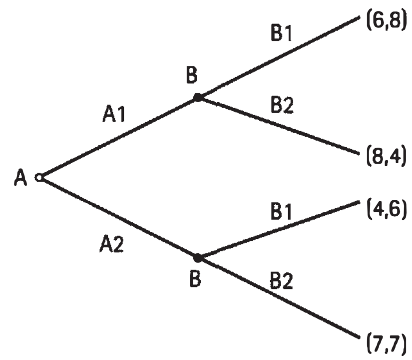


3.2 Repeated Games

The previous section considered one-off dynamic games. However, many games that economists are interested in involve repeated interaction between agents. Intuitively it would seem that repeated interaction between agents should have an effect on the predicted outcome of the game. For example, it would seem reasonable to assume that in repeated games players have more opportunity to learn to co-ordinate their actions so as to avoid the prisoners' dilemma discussed in the previous chapter. This section focuses on this issue and examines how repetition affects the predicted outcome of a game, and examines the conditions under which non-cooperative collusion is possible. This is undertaken in a number of contexts. *First*, we examine games where the one-off

EXERCISE 3.2

Convert the following extensive form game into a normal form game, and identify the Nash equilibria and subgame perfect Nash equilibrium. Finally, what is the Nash equilibrium if both players make their moves simultaneously?



game (called the stage game) is repeated an infinite number of times. *Second*, we look at games where the stage game is repeated only a finite number of times and where the one-off game has a unique Nash equilibrium. In this context we discuss the so called 'paradox of backward induction'. This paradox is that while co-ordination may be possible in infinitely repeated games it is not possible in certain situations with finitely repeated games. This is true no matter how large the number of repetitions. *Finally*, we examine ways in which this paradox of backward induction may be avoided.

3.2.1 Infinitely repeated games

To illustrate some of the issues involved with repeated games we will consider the following situation between two competing firms. Assume that two firms, A and B, dominate a specific market. The marketing departments of both companies have discovered that increased advertising expenditure, all other things remaining the same, has a positive effect on that firm's sales. However, total sales also depends negatively on how much the other company spends on advertising. This is because it adversely affects the market share of the first company. If we assume there are only two levels of advertising (high or low expenditure) which each firm can carry out, then the pay-off matrix in terms of profits (£m. per year) for the two companies is given in Fig. 3.4.

From this normal form game both firms are better off if they each maintain low advertising expenditure compared to both incurring high advertising costs. This is

		Firm 2	
		High	Low
Firm 1	High	4,4	6,3
	Low	3,6	5,5

Fig. 3.4 The Advertising Game in Normal Form

because if both firms increase their advertising expenditure simultaneously, then market shares are unaffected, but due to increased advertising costs profits fall. However, each firm has an incentive to try and increase its level of advertising above its competitor's level as this increases market share and overall profits.

If this game were played only once, then there is a unique Nash equilibrium with both firms incurring high advertising costs. This is a prisoners'-dilemma-type game as both firms would be made better off if they both had low-cost advertising campaigns. The problem for the firms is how they can co-ordinate their actions on this Pareto-dominant outcome without the use of legally enforceable contracts. In the one-off game this would not seem to be possible as there is a clear incentive to increase advertising expenditure. However, if the interaction between the firms is infinitely repeated, then it is possible for the two firms to co-ordinate their actions on the Pareto-optimal outcome. This will occur if both firms adopt an appropriate conditional strategy and do not discount the future too much.

As stated previously a conditional strategy is where a player conditions what he or she does on the actions of at least one other player in the game. This allows the possibility for one or more players, in effect, to punish other players if they deviate from the Pareto-efficient outcome. This is known as a *punishment strategy*. If the prospect of punishment is sufficiently severe, players will be deterred from deviation. In this way the Pareto-efficient outcome can be maintained indefinitely. Once more the issue of credibility is important. For example, threatened punishment of deviant behaviour will only maintain the co-operative outcome if the threat is credible. This will only happen if it is in the interests of the person threatening punishment to exact it when deviation is observed. This implies that a punishment strategy will only be effective in maintaining the co-operative solution if it is part of a subgame perfect Nash equilibrium for the entire game. To illustrate how these ideas can work in practice consider the following specific punishment strategy for the previous advertising game.

> **Each firm starts off with low-cost advertising, and this is maintained provided the other firm has always done the same in previous periods. If, however, the other firm has undertaken a high-cost advertising campaign in the past, then the firm undertakes a high-cost advertising campaign thereafter.**

This particular type of punishment strategy is frequently used in infinitely repeated games and is known as a *trigger strategy*.

A trigger strategy is where the actions of one player in a game causes other players permanently to switch to another course of action. The above trigger strategy implies an infinite punishment period if either of the firms incurs high-cost advertising. Once one firm increases its level of advertising, the other firm does the same ever after. This rules out the possibility of ever returning to the Pareto-efficient outcome. A firm that undertakes a high-cost advertising campaign will see profits rise from £5m. to £6m. in the first year of deviation, but then fall to at most only £4m. per year thereafter. For this trigger strategy to maintain the Pareto-efficient outcome two conditions must be satisfied. *First*, the punishment itself must be credible. *Second*, the promise to maintain low-cost advertising, given the prospect of future punishment, must also be credible. We consider each of these issues in turn.

With the above trigger strategy the threat of punishment is credible because if one firm switches to high-cost advertising, then it is rational for the other firm to also switch to high-cost advertising. This punishment strategy is credible because it corresponds to the Nash equilibrium of the stage game. Playing the Nash equilibrium is always a credible strategy because, by definition, it is the optimal response to the other players' expected strategies. The remaining issue is whether the promise to continue with low-cost advertising is also credible. Assuming that firms attempt to maximize total discounted profits then the co-operative outcome, where both firms have low-cost advertising campaigns, will be maintained indefinitely if the present value of co-operation is greater than the present value of deviating. This will be the case if firms do not discount the future too much. This is demonstrated as follows.

As this is an infinitely repeated game we will have to assume that future pay-offs are discounted, so as to obtain a present value of future profits. Let $\delta = 1/(1 + r)$ equal each firm's rate of discount, where r is the rate of interest or the firm's rate of time preference. This represents the fact that a pound received today is worth more than a pound received in the future, because it can be invested at the rate of interest r. The further in the future a pound is received the less is its present value. With this rate of discount the present value of maintaining a low-cost advertising campaign, $PV(\text{low})$, is equal to

$$PV(\text{low}) = 5 + 5\delta + 5\delta^2 + \ldots$$

$$\therefore\ \delta\, PV(\text{low}) = 5\delta + 5\delta^2 + 5\delta^3 + \ldots$$

$$\therefore\ (1 - \delta)\, PV(\text{low}) = 5$$

$$\therefore\ PV(\text{low}) = \frac{5}{1 - \delta}.$$

Alternatively the present value of deviating from this co-operative outcome and engaging in a high-cost advertising campaign, $PV(\text{high})$, is equal to

$$PV(\text{high}) = 6 + 4\delta + 4\delta^2 + \ldots$$
$$\therefore \delta PV(\text{high}) = 6\delta + 4\delta^2 + 4\delta^3 + \ldots$$
$$\therefore (1-\delta)PV(\text{high}) = 6 + 4\delta - 6\delta$$
$$\therefore (1-\delta)PV(\text{high}) = 6(1-\delta) + 4\delta$$
$$\therefore PV(\text{high}) = 6 + \frac{4\delta}{1-\delta}.$$

Therefore the co-operative outcome will be maintained indefinitely if

$$PV(\text{low}) \geq PV(\text{high})$$
$$\therefore \frac{5}{1-\delta} \geq 6 + \frac{4\delta}{1-\delta}$$
$$\therefore \delta \geq 1/2.$$

This tells us that with infinite interaction, and the given trigger strategy, both firms will maintain low-cost advertising if their rate of discount is greater than one-half. Given that this condition is satisfied, this means that the promise to continue with low-cost advertising is credible. With both the threat of punishment and the promise to maintain low advertising being credible this corresponds to a subgame perfect Nash equilibrium for this game. (This outcome, however, is now only one of many subgame perfect equilibria. For example, another subgame perfect equilibrium is where both firms have high-cost advertising every period. The problem now becomes how firms co-ordinate upon one of the many equilibria. This problem is discussed in later chapters.) If the firms' rate of discount is less than one-half, then each firm will immediately deviate to high-cost advertising. The co-operative outcome cannot be maintained with the assumed trigger strategy because the future threat of punishment is not sufficient to deter deviation. This is because the firms place too great a weight on current profits, and not enough on future profits. The promise to maintain low-cost advertising is not credible, and so both firms will undertake high-cost advertising campaigns.

3.2.2 Finitely repeated games

In the previous section on infinitely repeated games it was shown that players may be able to maintain a non-cooperative collusive outcome, (co-operative outcome for short), which is different from the Nash equilibrium of the stage game. This was shown to be the case if the players adopt an appropriate punishment strategy and do not discount the future too much. Here we examine under what conditions this result continues to hold in the context of finitely repeated games.

EXERCISE 3.3

(1) Identify the Nash equilibrium of the following normal form game if it is played only once.

Player 2

Player 1		Left	Right
	Up	1, 1	5, 0
	Down	0, 5	4, 4

(2) Demonstrate that the Pareto-efficient solution of 'down/right' in the previous game can be a subgame perfect Nash equilibrium if the game is infinitely repeated and both players adopt the following trigger strategy:

Play 'right/down' initially or if this has always been played in the past. Otherwise play 'up/left'.

(3) How does the necessary condition on the value of the discount factor change from (2), if the players now adopt the following alternative punishment strategy for the previously infinitely repeated game?

Play 'right/down' initially or if the outcomes 'right/down' or 'up/left' occurred in the previous period. Otherwise play 'up/left' for one period.

Explain this result.

The paradox of backward induction

One result obtained from applying the logic of backward induction to finitely repeated games is that if the one-off stage game has a unique Nash equilibrium, then the subgame perfect Nash equilibrium for the entire game is this Nash equilibrium played in every time-period. This is true however large the number of repetitions.

To understand this result consider the following argument. Suppose that a game having a unique Nash equilibrium is played a prespecified finite number of times. To find the subgame perfect Nash equilibrium for this game we start with the last period first. As the last period is just the one-off stage game itself, the predicted outcome in this period is the unique Nash equilibrium of the stage game. Now consider the penultimate period. Players using the principle of backward induction know that in the last period the Nash equilibrium will be played irrespective of what happens this period. This implies there is no credible threat of future punishment that could induce a player to play other than the unique Nash equilibrium in this penultimate period. All players know this and so again the Nash equilibrium is played. This argument can be applied to all preceding periods until we reach the first period, where again the unique Nash equilibrium is played. The subgame perfect Nash equilibrium for the entire game is simply the Nash equilibrium of the stage game played in every period. This argument implies that a non-cooperative collusive outcome is not possible.

This result can be illustrated using the advertising game described above, by assuming that it is only played for two years. In the second year we just have the stage game itself and so the predicted solution is that both firms will incur high advertising costs, and receive an annual profit of £4m. With the outcome in the second period fully determined the Nash equilibrium for the first period is again that both firms will have high-cost advertising campaigns. Similar analysis could have been undertaken for any number of finite repetitions giving the same result that the unique stage game Nash equilibrium will be played in every time-period. The subgame perfect solution, therefore, is that both firms will have high-cost advertising in both periods.

This general result is known as the paradox of backward induction. It is a paradox because of its stark contrast with infinitely repeated games. No matter how many finite number of times we repeat the stage game we never get the same result as if it were infinitely repeated. There is a discontinuity between infinitely repeated games and finitely repeated games, even if the number of repetitions is very large. This is counter-intuitive. It is also considered paradoxical because with many repetitions it seems reasonable to assume that players will find some way of co-ordinating on the Pareto-efficient outcome, at least in early periods of the game.

The reason for this paradox is that a finite game is qualitatively different from an infinite game. In a finite game the structure of the remaining game changes over time, as we approach the final period. In an infinite game this is not the case. Instead its structure always remains the same wherever we are in the game. In such a game there is no end-point from which to begin the logic of backward induction. A number of ways have been suggested in which the paradox of backward induction may be overcome. These include the introduction of bounded rationality, multiple Nash equilibria in the stage game, uncertainty about the future, and uncertainty about other players in the game. These are each examined below.

Bounded rationality

One suggested way of avoiding the paradox of backward induction is to allow people to be rational but only within certain limits. This is called bounded, or near, rationality. One specific suggestion as to how this possibility might be introduced into games is by Radner (1980). Radner allows players to play suboptimal strategies as long as the payoff per period is within epsilon, $\varepsilon \geq 0$, of their optimal strategy. This is called an ε-best reply. An ε-equilibrium is correspondingly when all players play ε-best replies. If the number of repetitions is large enough, then playing the co-operative outcome, even if it is not a subgame perfect Nash equilibrium, can still be an ε-equilibrium given appropriate trigger strategies. This is demonstrated for the repeated advertising game described above. Assume that both firms adopt the punishment strategy given when considering infinite repetitions, but now the game is only played a finite number of times. If we assume that there is no discounting, then the pay-off for continuing to play according to this strategy, if the other firm undertakes high-cost advertising, is $3 + 4(t-1)$, where t is the remaining number of periods the firms interact. The pay-off for initiating a high-cost advertising campaign is at most $6 + 4(t-1)$. Deviating from the punishment strategy therefore yields a net benefit of 3. This is equal to $3/t$ per period. If players are boundedly rational as defined by Radner, then the co-operative outcome is an ε-equilibrium if $\varepsilon > 3/t$. This will be satisfied for any value of ε, provided the remaining number of periods is great enough. Given a large number of repetitions co-operation, which in this example means low-cost advertising, will be observed in the initial periods of the game.

Although interesting to analyse the effects of bounded rationality in repeated games, it is not clear that Radner's suggestion is the best way of doing this. For example, Friedman (1986) argues that bounded rationality might imply that people only calculate optimal strategies for a limited number of periods. If this is true, then the game becomes shorter, and the result of backward induction more likely. Furthermore if people do not fully rationalize we should consider why. If it is due to calculation costs, then these costs should be added to the structure of the game itself. This is an area of ongoing research.

Multiple Nash equilibria

The paradox of backward induction can be avoided if there are multiple Nash equilibrium in the stage game. With multiple Nash equilibria there is no unique prediction concerning the last period of play. This may give the players in the game a credible threat concerning future play which induces other players to play the co-operative solution. This is illustrated for the hypothetical game shown in Fig. 3.5 which we assume is played twice.

		Player 2		
		Left	Middle	Right
Player 1	Up	$\underline{1},\underline{1}$	$\underline{5},0$	0,0
	Centre	$0,\underline{5}$	4,4	0,0
	Down	0,0	0,0	$\underline{3},\underline{3}$

Fig. 3.5 A Normal Form Stage Game with Multiple Nash Equilibria

This one-off stage game has two Nash equilibria 'up/left' and 'right/down'. These are both Pareto inefficient. If the players could co-ordinate on 'middle/centre', then both players would be better off. In the repeated game suppose that both players adopt the following punishment strategy:

> In the first **period play 'middle/centre'. In the second period play 'right/down' if 'middle/centre' was the outcome in the first round, otherwise play 'up/left'.**

Assuming the two periods are sufficiently close to each other we can ignore any discounting of pay-offs and so the matrix for the entire game is now that shown in Fig. 3.6.

		Player 2		
		Left	Middle	Right
	Up	2,2	6,1	1,1
Player 1	Centre	1,6	7,7	1,1
	Down	1,1	1,1	4,4

Fig. 3.6 The Pay-off Matrix with No Discounting and Appropriate Punishment Strategy

The choices shown in this matrix correspond to each player's first-period move, dependent on them adopting the previous punishment strategy. The pay-offs in this matrix were obtained by adding 3 to the player's second-round pay-off, when 'middle/centre' is the first-round outcome, but only 1 to all other first-round outcomes. This game now has three Nash equilibria, the previous two and now 'middle/centre'. Playing 'middle/centre' in the first round and 'right/down' in the second is a subgame perfect Nash equilibrium. Players thus avoid the paradox of backward induction, and achieve Pareto efficiency in the first period.

Uncertainty about the future

Another way of avoiding the paradox of backward induction is to introduce uncertainty about when the game might end. One way of doing this is to suppose there is a constant probability that the game will end after any one period. In this situation, although the game is finite, the exact timing of when the game will end is unknown. The implication of this is that, as with an infinitely repeated game, the structure of the remaining game does not change the more periods are played. As no one period can be classified as the last period of the game, we have no certain point from which to begin the process of backward induction and so the paradox is avoided. Backward induction is, therefore, only applicable to games which have a definite known end. If the final period is indefinite, credible threats and promises can be made that result in non-cooperative collusion in every time-period. This analysis is the same as with the infinite game except that the rate of discount, δ, has to be redefined. Instead of this depending only on the rate of interest, r, it will also depend on the probability that the game will end after any time-period. In effect players discount the future more heavily, as now there is a positive probability that future returns will not be received as the game will have ended by then. The rate of discount is now equal to

$$\delta = \frac{1 - Prob}{1 + r}$$

where *Prob* is the probability that the game ends at the end of any one period.

Uncertainty about other players

So far we have only considered games of complete information. A game is said to have *complete* information if all players' pay-off functions are common knowledge. This means that everyone knows everyones' pay-off function, and everyone knows everyone knows this, and so on *ad infinitum*. In contrast *incomplete* information means that players are not sure what the other players in the game are like, as defined by their pay-off functions, or what they know about other players. (Incomplete information should not be confused with imperfect information. Imperfect information is where at least one player does not know all the moves other players have made in either past or current periods. We have already examined games of imperfect information, indeed static games with more than one player are always, by definition, imperfect information games.) Clearly in many real-world situations people do not have complete information about everyone they interact with. For example, I may know that you are rational, but I may not know that you know that I am rational. This lack of common knowledge can greatly add to the complexity of a game, and the possible strategies that players might adopt. This is particularly true in repeated games.

To introduce some of these issues we examine an amended version of the advertising game previously analysed. Suppose that the game is played only once, but now Firm 1 can be one of two types (type A or type B) represented by different pay-offs. Firm 1 knows which type it is but Firm 2 does not. This is therefore a game of incomplete information as Firm 2 does not know the type of firm it is competing against. This situation can be illustrated by two separate extensive form games, one for each type of Firm 1. We assume that the pay-offs are as shown in Fig. 3.7 where L represents low-cost advertising and H high-cost advertising.

There are two changes to the firms' pay offs as compared to the previous advertising game represented in Fig. 3.4.

1. Firm 1 has the same pay-offs as before if it is type A, but type B receives £4m. less per year if it undertakes a high-cost advertising campaign. This can reflect either a

genuine cost differential between the two types of firm, or a difference in preferences, i.e. type B might have moral objections against high-powered advertising.

2. Firm 2 now receives £2m. per year less than before if it undertakes a high-cost advertising campaign, and firm 1 has a low-cost campaign. This might, for example, be due to a production constraint that precludes the firm benefiting from higher demand for its goods. Its other pay-offs remain the same. The effect of this change, and the reason it was made, is that Firm 2 no longer has the strictly dominant strategy of high-cost advertising. This implies that its behaviour can depend on which type of competitor it thinks it is playing against. This is the result we wish to show.

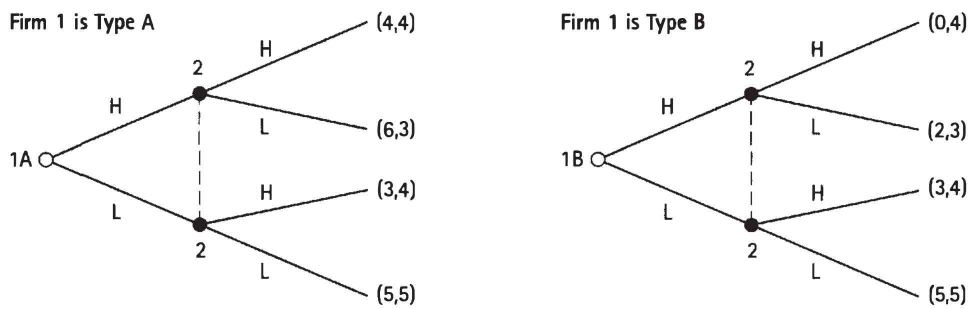


Fig. 3.7 The Modified Advertising Game with Incomplete Information

One immediate problem of solving games with incomplete information is that the techniques so far discussed cannot be directly applied to these situations. This is because they all make the fundamental assumption that players know which game they are playing. This is not the case if players are unsure what their opponents are like. As Fig. 3.7 illustrates Firm 2 does not know which of the two extensive form games it is playing. Fortunately Harsanyi (1967,1968) showed that games with incomplete information, can be transformed into games of complete but imperfect information. As we have already examined such games the previous solution techniques can be applied to these transformed games. The transformation is done by assuming that Nature determines each player's type, and hence pay-off function, according to a probability distribution. This probability distribution is assumed to be common knowledge. Each player is assumed to know his or her own type but not always the type of other players in the game. This is now a game of complete but imperfect information as not all players observe Nature's initial moves. If we assume that Nature assigns Type A to Firm 1 with probability *Prob* and Type B with probability 1 – *Prob*, then this situation can now be represented by just the one extensive form game. This is shown in Fig. 3.8.

As this is a game of complete information we can use the solution techniques used for solving static games. This specific game can be solved using the principle of iterated strict dominance which gives us a unique Nash equilibrium. If Firm 1 is of Type A, then

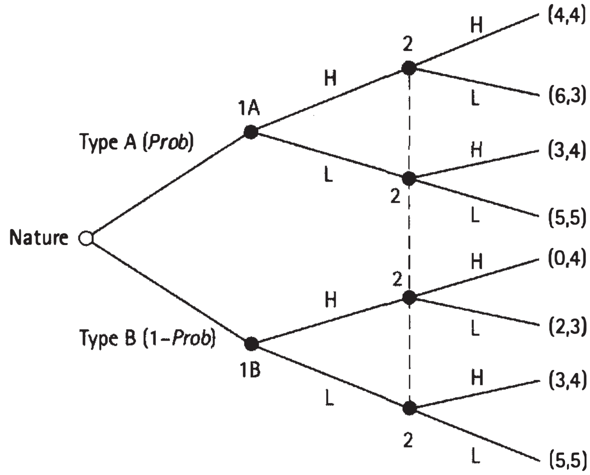


Fig. 3.8 The Modified Advertising Game with Imperfect Information

having a high-cost advertising campaign strictly dominates a low-cost campaign. If, however, it is of Type B, then low-cost advertising dominates. If Firm B assumes Firm 1 is rational, whatever its type, then it knows that it will see Firm 1 incurring a high-cost campaign with probability *Prob*, and a low-cost campaign with probability 1 – *Prob*. With these probabilities it can calculate its own expected profit conditional on its level of advertising. If Firm 2 decides on high-cost advertising then its expected profit level is

$$(\Pi_B \mid H) = 4Prob + 4(1 - Prob).$$

If it decides on low-cost advertising then its expected profit level is

$$(\Pi_B \mid L) = 3Prob + 5(1 - Prob).$$

Assuming Firm 2 wishes to maximize expected profit it will undertake a high-cost advertising campaign if $(\Pi_B \mid H) > (\Pi_B \mid L)$. This gives us the following condition:

$$4Prob + 4 - 4Prob > 3Prob + 5 - 5Prob$$

$$\therefore\ Prob > 1/2.$$

Firm 2 will play H if *Prob* > 1/2 and will play L if *Prob* < 1/2. If *Prob* = 1/2, then it is indifferent between these two options. The optimal strategy for Firm 2, therefore, depends on the probability that its competitor is a particular type. In this game the firms will achieve a Pareto-efficient solution, with both firms playing L and receiving £5m., if Firm 1 is of Type B and *Prob* < 1/2.

The above illustration has shown that incomplete information can lead to player's achieving a Pareto-efficient outcome. However, in this one-off game, this only happened if Firm 1 was of a type that always undertakes a low-cost advertising campaign. If the game is repeated a finite number of times, this restriction need not always apply for the outcome to be Pareto efficient. Unfortunately, solving such repeated games is far from straightforward. This is because of two added complications.

The *first* complication is that in dynamic games of incomplete information, players may be able to learn what other players are like by observing their past actions. This gives players the opportunity to try and influence other players' expectations of their type by modifying their actions. For example, consider what might happen if the above incomplete information advertising game is repeated a finite number of times. If Firm 1 can convince Firm 2 that it is Type B, then Firm 2 will incur low advertising costs and so increase Firm 1's profits. The only way that Firm 1 can convince the other firm that it is Type B is to play as a Type B firm would play. This is true even if firm 1 is actually Type A. Thus it is possible that players might seek to conceal their true identity, so as to gain a reputation for being something they are not. Gaining such a reputation can be thought of as an investment. Although obtaining a reputation for something you are not will be costly in the short-run, it brings with it the expectation of higher future returns.

The *second* complication is that players know that other players might have this incentive to conceal their true identity. This will influence how they update their probability assessment of the other player's type conditional on observing his or her actions. The other player will in turn take this into account when determining their behaviour, and so on.

Only recently have such games been explicitly solved by game theorists and applied to economic situations. The equilibrium concept often used in such games is *Bayesian Subgame Perfect Nash Equilibrium*, or Bayesian Perfect for short. This type of equilibrium satisfies two conditions:

(1) It is subgame perfect in that no incredible threats or promises are made or believed.

(2) The players update their beliefs rationally in accordance with Bayes' Theorem.

(An alternative equilibrium concept used is *sequential equilibrium*. This was developed by Kreps and Wilson (1982*b*) and is a slightly stronger condition than Bayesian perfect equilibrium with regard to the consistency of the solution. In many cases, however, the two concepts yield the same solution.)

Bayes' Theorem is explained in Appendix 3.1. The relevance of Bayesian perfect equilibrium is that even very small amounts of uncertainty concerning the type of player you are playing against can be greatly magnified in repeated games. This alters the incentives faced by players in the game, and often leads to the prediction that the Pareto-optimal solution is played in early stages of the game. In this way the paradox of backward induction is overcome. To illustrate this possibility we discuss the *centipede game* developed by Rosenthal (1981). Consider the extensive form game shown in Fig. 3.9.

In this game there are two players each with two possible moves. They can either move across (A) or down (D). The resulting pay-offs are as shown in the diagram. Solving this game by backwards induction gives the result that person 1 will immediately play down and so both players receive the same pay-off of 1. This is clearly a Pareto-inefficient outcome because if both players play across they both receive poten-

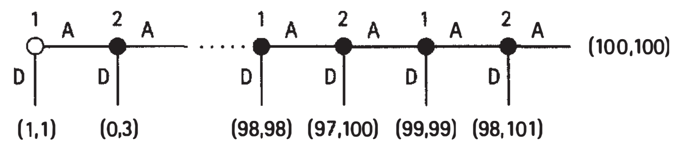


Fig. 3.9 Rosenthal's Centipede Game

tially much greater pay-offs. In this sense the game is very much like a repeated prisoner's dilemma game, where initial co-operation can make everyone better off. There are two specific points to notice about this subgame perfect prediction.

First, the prediction that the first player will immediately play down is based on 200 rounds of iterated strict dominance. In reality it is often hard to believe that players are so sure of their opponents' rationality, and that their opponents are sure of their opponents' rationality etc.

Second, what effect will player 1 have on player 2 if instead of playing downs he plays along? Player 2 now has direct evidence that player 1 is not rational. Based on this evidence, player 2 may decide that it is best to also play along, and take player 1 for a ride. In this situation we move out along the tree and both players are made better off. This reasoning suggests that it may be rational to initially pretend to be irrational! (We return to some of these issues in Chapter 12.)

Each of these points suggests that for this game the assumption of common knowledge of rationality may be inappropriate. An alternative assumption is to introduce incomplete information. With this assumption players are unsure if their opponent is rational. This has a dramatic effect on the equilibrium behaviour of a rational player. Even if there is only a very small probability that your opponent is co-operative, in that he or she always plays across, then it is rational for players to play across in the initial periods of the game. In this way each player builds up a reputation for being co-operative. It can be shown that the sequential equilibrium of this incomplete information game involves both players initially playing across, and then to randomize over their actions as they approach the end of the game. In fact the number of periods for which the players will adopt mixed strategies does not depend on the number of periods the game is played. In consequence as the number of periods increases the proportion of the game characterized as co-operative also increases. This equilibrium strategy is depicted in Fig. 3.10.

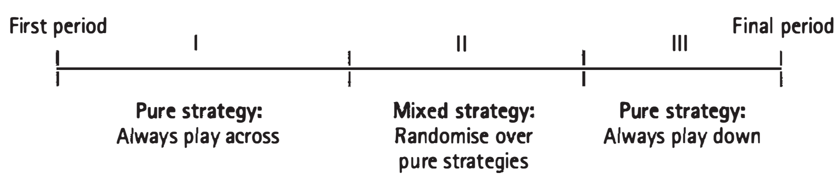


Fig. 3.10 Bayesian Subgame Perfect Nash Equilibrium for Rosenthal's Centipede Game with Incomplete Information

As players initially co-operate and play across the paradox of backward induction is partially overcome. In Appendix 3.2 we derive the sequential equilibrium for a simplified version of the centipede game. This modified version of Rosenthal's game was developed by McKelvey and Palfrey (1992), who used their simplified game to test whether experimental subjects actually played according to the sequential equilibrium hypothesis. This experiment, among others, is reviewed in Chapter 11.

EXERCISE 3.4

Assume that two players are faced with Rosenthal's centipede game. Use Bayes' theorem to calculate the players' reputation for being co-operative in the following situations if they play across.

(1) At the beginning of the game each player believes that there is a 50/50 chance that the other player is rational or co-operative. It is assumed that a co-operative player always plays across. Furthermore suppose that a rational player will play across with a probability of 0.2

(2) At their second move the players again move across. (Continue to assume that the probability that a rational player plays across remains equal to 0.2).

(3) How would the players' reputation have changed after the first move had the other player believed that rational players always play across. (Assume all other probabilities remain the same.)

(4) Finally, how would the players' reputation have changed after the first move had the other player believed that rational players never play across. (Again assume all other probabilities remain the same.)

3.3 Conclusions

Most interaction between people and organizations are not one-off events, but rather take place within the context of an ongoing relationship. This makes them dynamic. In dynamic games players observe the moves of other players before making their own optimal responses. This possibility greatly enriches the strategies that players might adopt. This chapter has examined how dynamic games can be analysed and predictions made. A key concept in all dynamic games is credibility. For a threat or promise to be credible it must be in that players interest to carry it out at the appropriate time. If a threat or promise is not credible, then it would seem reasonable to suppose that it will not be believed. Subgame perfection applies this insight to dynamic games. In games of perfect and complete information, where players are not indifferent between various actions, backward induction generates a unique prediction. This prediction is the subgame perfect Nash equilibrium.

When a game is repeated it might be supposed that the players will learn over time to co-ordinate their actions so as to avoid Pareto-inefficient outcomes. This possibility was examined in section 3.2. Initially we examined infinitely repeated games where the stage game had a unique Nash equilibrium. It was demonstrated that, provided players do not discount future returns too much, a non-cooperative collusive outcome can indeed be maintained. This result breaks down, however, if the game is only finitely repeated. This is the paradox of backward induction. As this paradox is counter-intuitive many explanations have been proposed as to how it might be avoided. This chapter examined four ways in which the paradox of backward induction is overcome. These were the introduction of bounded rationality, multiple Nash equilibria, uncertainty about the future, and incomplete information. These concepts have become central to many recent developments in game theory.

This and the previous chapter have presented many of the basic concepts needed to understand non-cooperative game theory models. All of these basic ideas are employed in subsequent chapters where we analyse, in greater depth, recent economic applications of game theory.

3.4 Solutions to Exercises

Exercise 3.1

Using the principle of backward induction we start at the end of the game and work backwards ruling out strategies that rational players would not play. If the game reaches player 3's decision node, then she receives with certainty a pay-off of 7 if she plays 'up' or 9 if she plays 'down'. Assuming this player is rational she will play 'down', and so we can exclude her strategy of 'up'.

Moving back along this branch we reach player 2's node after player 1 has moved 'down'. If we assume player 2 believes player 3 is rational then he anticipates receiving a pay-off of 8 if he moves 'up', and 1 if he moves 'down'. If player 2 is also assumed to be rational, then at this decision node he will play 'up' and we can exclude 'down' from this node.

Player 2 also has a decision node following player 1 moving 'up'. By considering player 2's certain pay-off's following this decision we can exclude 'up' and 'down' as we have already assumed this player is rational.

We now only have to solve player 1's move at the start of the game. There are now only two remaining pay-offs for the game as a whole. If we assume player 1 believes player 2 is rational then she will anticipate a pay-off of 3 if she moves 'up'. Further, if we assume that player 1 believes that player 2 believes that player 3 is rational then she will anticipate a pay-off of 6 if she moves 'down'. With these assumptions we can exclude player 1 moving 'up'.

With the above assumptions we are left with one remaining pay-off vector corresponding to the subgame perfect Nash equilibrium. This equilibrium is that player 1 moves 'down' followed by player 2 moving 'up', and finally player 3 choosing to play 'down'. We derive the same solution if we assume all players in the game are rational and that this is common knowledge. This assumption is stronger than absolutely necessary, but for convenience it is typically made when applying the principle of backward induction.

Exercise 3.2

For the extensive form game given in this exercise there are two Nash equilibria, but only one is perfect. To identify these Nash equilibria we first convert the game into a normal form game. Player A has only two strategies, A1 or A2. However, player 2 has four available strategies because he can condition his actions on what he observes player A does. Player 2's strategies are, therefore, to always play B1, to always play B2, do the same as player A, or do the opposite. This gives us the normal form game shown in Fig. 3.11.

Player B

Player A		Always B1	Always B2	Same as A	Opposite to A
	A1	6,8	8,4	6,8	8,4
	A2	4,6	7,7	7,7	4,6

Fig. 3.11

Using the two-stage procedure for finding pure-strategy Nash equilibria we can identify that 'A1/Always B1' and 'A2/Same as A' are both Nash equilibria. The concept of subgame perfect Nash equilibrium, however, rules out 'A1/Always B1' because it is not a Nash equilibrium for the subgame beginning at B's choice node following 'A2'. This is because player B threatening to always play 'B1', irrespective of what player A does, is not credible. If player A plays 'A2', then player B has the incentive to play 'B2'. The only subgame perfect equilibrium for this game is 'A2/Same as A'. This subgame perfect equilibrium could also have been found by using backward induction.

If the players take their actions simultaneously rather than sequentially, then we get the normal form game illustrated in Fig. 3.12.

In this new game we have a unique Nash equilibrium of 'A1/B1'. All that we have changed between the two games is the amount of information that player B has available to him when he makes his decision. None the less the predicted outcome and corresponding pay-offs are very different. Indeed, in this game giving player B less information, now he cannot observe player A's move, actually makes him better off!

Player B

Player A		B1	B2
	A1	6,8	8,4
	A2	4,6	7,7

Fig. 3.12

Exercise 3.3

(1) Applying the two-stage methodology for finding pure-strategy Nash equilibria it can be shown that this game has the unique Nash equilibrium of 'up/left'. It should be confirmed by the reader that there is no mixed-strategy Nash equilibrium for this prisoners' dilemma game.

(2) With infinite repetitions the co-operative outcome can result in 'right/down' being played indefinitely if the players adopt the suggested trigger strategy and players do not discount the future too much. This is demonstrated as follows.

Let the player's rate of discount be δ. The present value of always playing 'right/down' to each player is therefore

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots = \frac{4}{1-\delta}.$$

The present value of deviating to each player is

$$5 + \delta + \delta^2 + \delta^3 \ldots = 5 + \frac{\delta}{1-\delta}.$$

Thus we can predict that the two players will continue to collude as long as

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}$$

$$\therefore \delta \geq 1/4.$$

The threat of punishment is credible since having observed the other player's defection from the co-operative solution it is rational to play the static Nash equilibrium. Furthermore, if the above condition holds, then the promise to play 'right/down' is also credible, because the loss incurred as a result of deviating outweighs the possible gains.

(3) With the players adopting this alternative one-period punishment strategy, each player has the option of deviating from the Pareto-efficient outcome every other period. Indeed, if it is rational to deviate initially this is what they will do. The present value of deviating this period is therefore equal to

$$5+\delta+5\delta^2+\delta^3+\ \dots\ =\frac{5+\delta}{(1-\delta)(1+\delta)}.$$

Comparing this with the present value pay-off of not deviating we get the result that both players will continually play the Pareto-efficient outcome if

$$\frac{4}{1-\delta}\geq\frac{5+\delta}{(1-\delta)(1+\delta)}$$

$$\therefore\ \delta \geq 1/3.$$

As future punishment is less severe, lasting for only one period following each deviation, players are less likely to be deterred from deviating from the Pareto-optimal outcome. This is shown by the smaller subset of discount values which maintain Pareto efficiency.

Exercise 3.4

From Bayes' Theorem we can write

$$Prob(\mathrm{A} \mid \mathrm{B})=\frac{Prob(\mathrm{B} \mid \mathrm{A})Prob(\mathrm{A})}{Prob(\mathrm{B} \mid \mathrm{A})Prob(\mathrm{A})+Prob(\mathrm{B} \mid \text{not A})Prob(\text{not A})}.$$

where A corresponds to the statement that 'the other player is co-operative', and B to the observation that 'the other player has just played across'. The probabilities in the equation, therefore have the following interpretation.

$Prob(\mathrm{A} \mid \mathrm{B})$	= the probability that the other player is co-operative, given that he or she has just played across. This corresponds to that player's reputation for being co-operative.
$Prob(\mathrm{A})$	= the prior probability that the other player is co-operative.
$Prob(\text{not A})$	= the prior probability that the other player is rational = 1 – $Prob(\mathrm{A})$
$Prob(\mathrm{B} \mid \mathrm{A})$	= the probability that the other player will play across if he or she is co-operative. In our example this equals 1 by definition
$Prob(\mathrm{B} \mid \text{not A})$	= the probability that the other player will play across if he or she is rational.

(1) From the above equation we can calculate the probability that players are co-operative if they initially play across. This equals $Prob(\mathrm{A} \mid \mathrm{B}) = (1\times 0.5)/(\,(1\times 0.5) + (0.2\times 0.5)) = 0.833$. From this we can see that the player's reputation for being co-operative has increased as a result of playing across.

(2) If the players play across with their second move, then the other player assigns an even greater probability to the belief that the player is co-operative. Using the result derived in (1) as the new prior probability that the player is co-operative we get: $Prob(\mathrm{A} \mid \mathrm{B}) = (1\times 0.833)/(\,(1\times 0.833) + (0.2\times 0.167)\,) = 0.961$.

(3) If rational players always play across then we have $Prob(\mathrm{B} \mid \text{not A}) = 1$. Substituting this into Bayes' Theorem we get $Prob(\mathrm{A} \mid \mathrm{B}) = (1\times 0.5)/(\,(1\times 0.5) + (1\times 0.5)) = 0.5$. In this situation there is no useful information learnt from observing a player's move across. This is because

both rational and co-operative players are predicted to act alike. With no new information available the players' reputations remain unchanged.

(4) In this final example we have $Prob(\mathrm{B} \mid \text{not A}) = 0$, and so $Prob(\mathrm{A} \mid \mathrm{B}) = (1 \times 0.5)/((1 \times 0.5) + (0 \times 0.5)) = 1$. In this example the players learn for certain that both are co-operative. This occurs because the prior belief was that a rational player would always play down. The only consistent explanation of observing the other player move across, therefore, is that they are co-operative.

From the results derived in (3) and (4) we can note that players' reputations can only be improved if they adopt a mixed strategy. Otherwise their reputation remains unchanged, or their identity is fully revealed.

Appendix 3.1 Bayes' Theorem

Bayes' Theorem, named after the Reverend Thomas Bayes (1702–61), shows how probabilities can be updated as additional information is received. It answers the question 'Given that event B has occurred what is the probability that event A has or will occur?' This revised probability is written as $Prob(\mathrm{A} \mid \mathrm{B})$. It is the conditional probability of event A given the occurrence of event B. For example in games of incomplete information a player can use Bayes' Theorem to update their probability assessment that another player is of a certain type, by observing what he or she does. The Theorem can be written as follows:

Suppose that event B has a non-zero probability of occurring, then for each event A_i, of which there are N possibilities,

$$Prob(\mathrm{A}_i \mid \mathrm{B}) = \frac{Prob(\mathrm{B} \mid \mathrm{A}_i) \,.\, Prob(\mathrm{A}_i)}{\sum_{i=1}^{N} Prob(\mathrm{B} \mid \mathrm{A}_i) \,.\, Prob(\mathrm{A}_i)}.$$

For example, if there are only two possible types of a certain player, A_1 and A_2, the updated probability for each type, given that this player has undertaken an action B is:

$$Prob(\mathrm{A}_1 \mid \mathrm{B}) = \frac{Prob(\mathrm{B} \mid \mathrm{A}_1) \,.\, Prob(\mathrm{A}_1)}{Prob(\mathrm{B} \mid \mathrm{A}_1) \,.\, Prob(\mathrm{A}_1) + Prob(\mathrm{B} \mid \mathrm{A}_2) \,.\, Prob(\mathrm{A}_2)}$$

and

$$Prob(\mathrm{A}_2 \mid \mathrm{B}) = \frac{Prob(\mathrm{B} \mid \mathrm{A}_2) \,.\, Prob(\mathrm{A}_2)}{Prob(\mathrm{B} \mid \mathrm{A}_1) \,.\, Prob(\mathrm{A}_1) + Prob(\mathrm{B} \mid \mathrm{A}_2) \,.\, Prob(\mathrm{A}_2)}.$$

The probabilities used on the right-hand side of these expressions are called 'prior probabilities', and are determined before event B occurs. The updated probabilities $Prob(\mathrm{A}_i \mid \mathrm{B})$ are called 'posterior probabilities'. In repeated games these posterior probabilities would then be used as the prior probabilities of $Prob(\mathrm{A}_i)$ in subsequent periods.

Proof of Bayes' Theorem

From the nature of conditional probabilities it is true that

$$Prob(\mathrm{A}_i \text{ and B}) = Prob(\mathrm{B} \mid \mathrm{A}_i) \,.\, Prob(\mathrm{A}_i) \tag{1}$$

and

$$Prob(\mathrm{A}_i \text{ and B}) = Prob(\mathrm{A}_i \mid \mathrm{B}) \,.\, Prob(\mathrm{B}). \tag{2}$$

From (2) we get

$$Prob(\mathrm{A}_i \mid \mathrm{B}) = \frac{Prob(\mathrm{A}_i \text{ and B})}{Prob(\mathrm{B})}. \tag{3}$$

Substituting in for $Prob(\mathrm{A}_i \text{ and B})$ from equation (1) gives

$$Prob(\mathrm{A}_i \mid \mathrm{B}) = \frac{Prob(\mathrm{B} \mid \mathrm{A}_i) Prob(A_i)}{Prob(\mathrm{B})}. \tag{4}$$

Multiplying through by $Prob(\mathrm{B})$ yields

$$Prob(\mathrm{B}) \,.\, Prob(\mathrm{A}_i \mid \mathrm{B}) = Prob(\mathrm{B} \mid \mathrm{A}_i) \,.\, Prob(\mathrm{A}_i). \tag{5}$$

Summing both sides over the index i gives

$$Prob(\mathrm{B}) \,.\, \sum_{i=1}^{N} Prob(A_i \mid B) = \sum_{i=1}^{N} Prob(B \mid A_i) \,.\, Prob(A_i). \tag{6}$$

Since the events $\{\mathrm{A}_1, \ldots, \mathrm{A}_N\}$ are mutually exclusive and exhaustive we know that

$$\sum_{i=1}^{N} Prob(A_i \mid B) = 1. \tag{7}$$

Substituting (7) into (6) we have

$$Prob(\mathrm{B}) = \sum_{i=1}^{N} Prob(\mathrm{B} \mid A_i) \,.\, Prob(\mathrm{A}_i). \tag{8}$$

Substituting (8) into (4) gives us Bayes' Theorem. This completes the proof.

Appendix 3.2 Sequential Equilibrium In A Modified Centipede Game

In this appendix we demonstrate how a sequential equilibrium may be identified for a modified version of Rosenthal's centipede game. This version of the game is due to McKelvey and Palfrey (1992). The extensive form of this game, assuming complete information, is shown in Fig. 3.13.

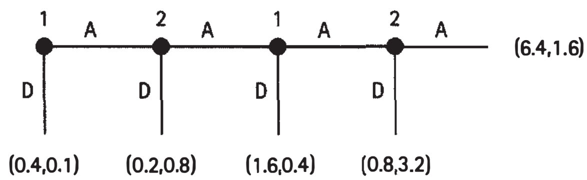


Fig. 3.13

With complete information the unique subgame perfect Nash equilibrium for this game is for player 1 to play down (D) at the first decision node. This solution can be confirmed using the principle of backward induction. With this outcome the pay-offs are 0.4 and 0.1 for players 1 and 2 respectively. Clearly this is a Pareto-inefficient outcome as both players can be made better off if they initially co-operate and play across (A). With incomplete information, however, sequential equilibrium will generally involve some degree of co-operation by both players. To illustrate this we will assume there is a small probability of either player in the game being co-operative in the sense that they always play across. This may be because they are either altruistic, and care about their opponent's final pay-off, or because they are irrational. Each player is assumed to know he or she is rational or co-operative, but not the type of the opponent. It is assumed that the prior probability that a player is co-operative is 0.05. As a 'co-operative' player always plays across we need only derive the equilibrium predictions for a 'rational' player. With this degree of uncertainty it is rational for player 1 to always play across at the first decision node, thus partially offsetting the paradox of backward induction. At the next two nodes the players will adopt a mixed strategy. If the game reaches the final node player 2 will always play down.

To derive this equilibrium we start from the last period first and work back to the beginning of the game. We will use the following notation. Let $\prod_j^k$ be the expected final pay-off for player $k = 1,2$, if the game is at decision node $j = 1, \ldots, 4$. ($j = 1$ is the first node, $j = 2$ the second etc.). This will be conditional on whether player k plays down (D) or across (A) at that node. Similarly $Prob_j^k$(A) and $Prob_j^k$(D) are the probabilities of player k, if he or she is rational, playing A or D at decision node j. Finally, r_j^k is the probability assessment by the other player that player k is 'co-operative' if the game has reached decision node j. This corresponds to player k's reputation for being 'co-operative'. Initially, by assumption, $r_1^1 = r_1^2 = r_2^2 = 0.05$. This final probability is based on the fact that player 2 can do nothing to affect his reputation prior to node 2, as this is player 2's first decision node.

4th node ($j = 4$) If the game reaches player 2's final decision node, then the expected pay-offs for that player are $\prod_4^2$ (D) = 3.2 and $\prod_4^2$ (A) = 1.6. As down strictly dominates across it is clear that if player 2 is rational he will always play down. This pure-strategy prediction implies that $Prob_4^2$ (A) = 0 and $Prob_4^2$ (D) = 1 for a rational player.

3rd node ($j = 3$) Based on the results derived for node 4 the expected pay-offs for player 1 if the game reaches node 3 are $\prod_3^1$ (D) = 1.6 and $\prod_3^1$ (A) $= 0.8\,(1 - r_3^2) + 6.4 r_3^2$. If one of these expected pay-offs is greater than the other, then player 1 will play a pure strategy at this node. If, however, these expected values are equal, then player 1 (if rational) is indifferent between playing across or down. In this situation player 1 can play a mixed strategy at this node. This will occur when $r_3^2 = 1/7$. Player 1 will only play a mixed strategy at this node if player 2 has previously enhanced his reputation for being 'co-operative'. This, in turn, can only happen if player 2 has played a mixed strategy at decision node 2. Specifically using Bayes' Theorem we can calculate the implied probability of player 2 playing down at node 2 in order to induce player 1 to play a mixed strategy at node 3. Thus

$$r_3^2 = \frac{r_2^2}{r_2^2 + [1 - Prob_2^2(\mathrm{D})](1 - r_2^2)} = 1/7$$

$$\therefore \frac{0.05}{0.05 + [1 - Prob_2^2(\mathrm{D})]0.95} = 1/7$$

$$\therefore Prob_2^2(\mathrm{D}) = 0.684.$$

$$\therefore Prob_2^2(\mathrm{A}) = 0.316.$$

Player 1 will, therefore, adopt a mixed strategy at node 3 if player 2 plays this mixed strategy at node 2.

2nd node ($j = 2$) At node 2 we can derive the following expected pay-offs for player 2.

$\Pi_2^2(\mathrm{D}) = 0.8$ and

$$\begin{aligned}\Pi_2^2(\mathrm{A}) &= 3.2[r_1^2 + (1 - r_1^2)Prob_3^1(\mathrm{A})] + 0.4(1 - r_3^1)Prob_3^1(\mathrm{D}) \\ &= 3.2 - 2.8Prob_3^1(\mathrm{D}) + 2.8r_2^1 Prob_3^1(\mathrm{D}).\end{aligned}$$

Again if either of these expected pay-offs is greater than the other, then player 2 will adopt a pure strategy. If these expected pay-offs are equal, then player 2 (if rational) can adopt a mixed strategy at this node. This will occur when $Prob_3^1(\mathrm{D}) = 6/7(1 - r_2^1)$. Given that r_2^1 is sufficiently small, player 2 will use a mixed strategy at node 2 if player 1 is anticipated to play the appropriate mixed strategy at node 3. This is similar to the prediction derived for node 3. The mixed strategies at nodes 2 and 3 are self-supporting in equilibrium.

If we assume that player 1 always plays across at node 1 (this assumption is validated below), then from Bayes' Theorem $r_2^1 = 0.05$. From the previous equation, therefore, player 2 will play a mixed strategy at node 2 if $Prob_3^1(\mathrm{D}) = 0.902$ and $Prob_3^1(\mathrm{A}) = 0.098$.

1st node ($j = 1$) In order to justify the use of mixed strategies derived above for nodes 2 and 3 all that is needed now is to demonstrate that, given these predictions, player 1 will play across at this first decision node. Again the expected pay-offs for player 1 at this node are given as $\pi_1^1(\mathrm{D}) = 0.4$ and

$$\begin{aligned}\Pi_1^1(\mathrm{A}) = {} & 0.2(1 - r_1^2)Prob_2^2(\mathrm{D}) + 1.6[r_1^2 + (1 - r_1^2)Prob_2^2(\mathrm{A})]Prob_3^1(\mathrm{D}) \\ & + 0.8[r_1^2 + (1 - r_1^2)Prob_2^2(\mathrm{A})]Prob_3^1(\mathrm{A})(1 - r_1^2) \\ & + 6.4[r_1^2 + (1 - r_1^2)Prob_2^2(\mathrm{A})]Prob_3^1(\mathrm{A}).\end{aligned}$$

Given the previous mixed-strategy predictions we can calculate that $\prod_1^1 (\mathrm{A}) = 1.036$. For player 1 playing across dominates playing down at this node, and so player 1 will always play across irrespective of their type. This pure-strategy prediction implies that $Prob_1^1 (\mathrm{A}) = 1$ and $Prob_1^1 (\mathrm{D}) = 0$ for a rational player.

This demonstrates that the use of mixed strategies at nodes 2 and 3 can form part of a sequential equilibrium. In fact it can be shown that this is the only sequential equilibrium for this game. This unique equilibrium is illustrated in Fig. 3.14.

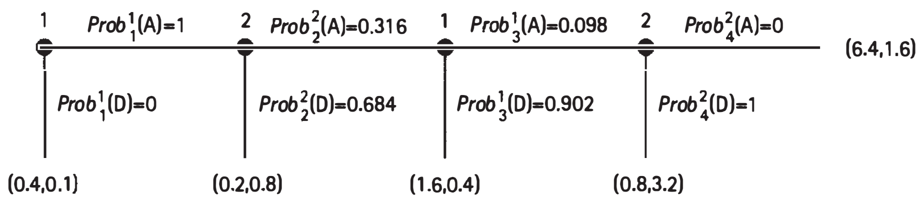


Fig. 3.14

These predictions conform to the typical pattern observed with sequential equilibrium. Initially there is a pure-strategy phase, where players co-operate. Players then adopt mixed strategies, where a player's reputation for being co-operative may be enhanced. Finally players adopt the pure strategy associated with non-cooperation.

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