Section 15.3
Constrained Optimization:
Lagrange Multipliers

- Often times in the real world when we would like to optimize something (either minimize or maximize) we are subject to a constraint
- For example, maximizing profit is constrained by a budget
- Minimizing costs is constrained by the amount works must be paid
- Maximizing the time you spend doing homework is constrained by the amount of sleep you need
- And so on…

- Suppose we want to minimize the function

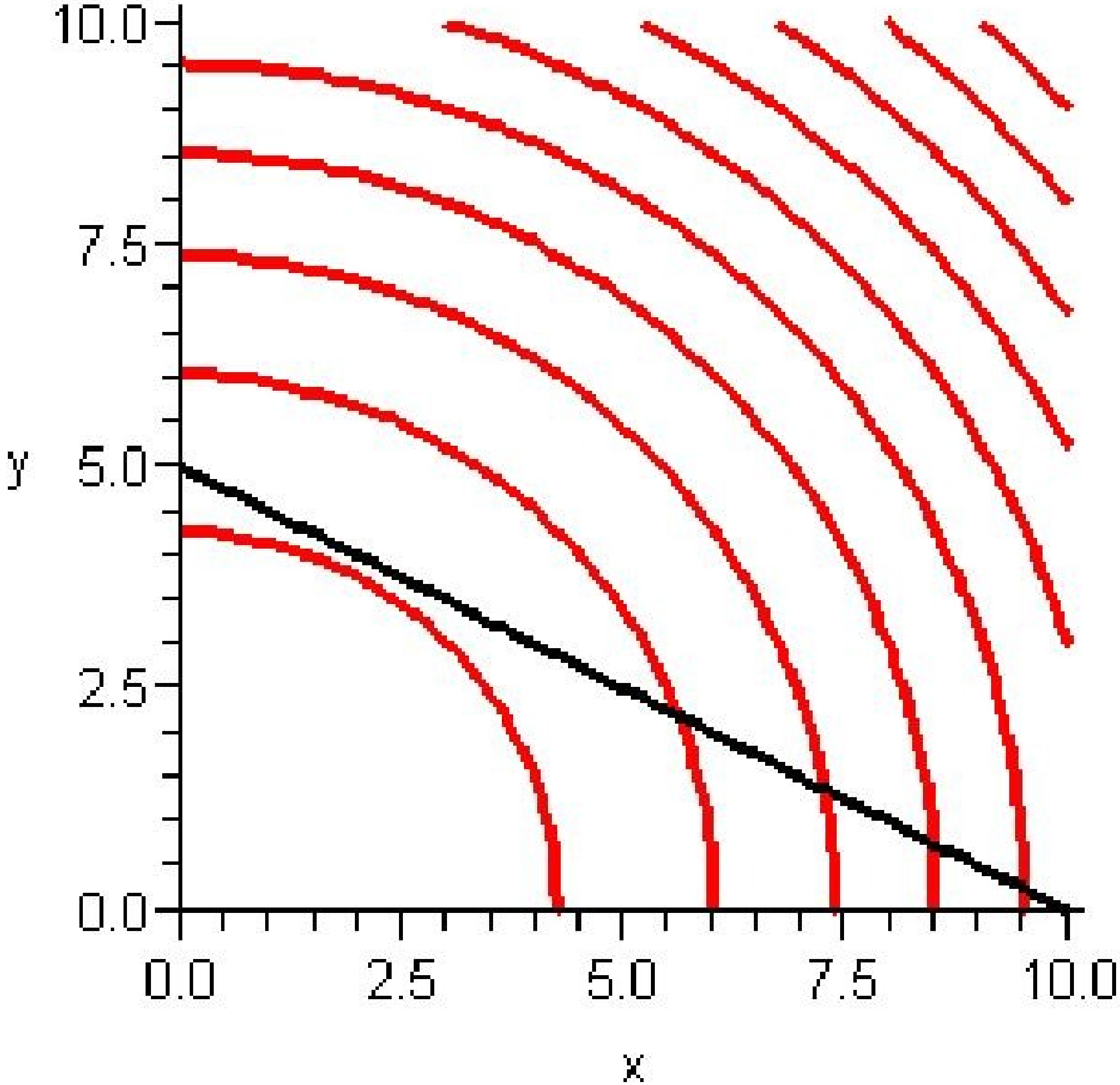
$$f(x, y) = x^2 + y^2$$

subject to the constraint

$$g(x, y) = x + 2y = 10$$

- We refer to *f* as the **objective function** and *g* as the **constraint equation**
- What we will do is consider graphs of the level curves of *f* as well as the constraint equation

Graph of the Level Curves of f
and the Constraint Equation g = 10



- So we would like to have the level curve that is tangent to the constraint equation
  - This point gives us the smallest value of our function that is also a point on our constraint equation
- We can consider $g(x,y) = 10$ as a single level curve of $g(x,y)$
  - The value of this is that the gradient of each at their point of intersection should be pointing in the same direction
- So what we want is

$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = 10$$

- Let's work through this problem

Lagrange Multipliers

- Let f and g have continuous partial derivatives. To find the extremum of $f(x,y)$ subject to the constraint $g(x,y) = C$ we only need to solve the 3x3 system

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = C \end{cases}$$

- λ is called the **Lagrange Multiplier**
- If $g(x,y) = C$ is bounded (and closed) then we are guaranteed both a global min and global max

Interpretation of Lagrange Multiplier, λ

- Suppose (a,b) was an extremum found by using Lagrange Multipliers to optimize $z = f(x,y)$ subject to $g(x,y) = C$
- We can consider $a = a(C)$ and $b = b(C)$ as functions of C because as we change C, our point will change as well
- Now $z_0 = f(a,b)$ where (a,b) is a solution to

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \end{cases}$$

- Since $a = a(C)$ and $b = b(C)$, $z_0 = f(a,b)$ is a function of C

$$\frac{dz_0}{dC} = f_x(a,b)\frac{da}{dC} + f_y(a,b)\frac{db}{dC}$$

$$= \lambda g_x(a,b)\frac{da}{dC} + \lambda g_y(a,b)\frac{db}{dC}$$

$$= \lambda\left[g_x(a,b)\frac{da}{dC} + g_y(a,b)\frac{db}{dC}\right]$$

$$= \lambda\left[\frac{dg}{dC}(a,b)\right] = \lambda\frac{dC}{dC} = \lambda \text{ b/c } g(a,b) = C$$

- So we have $$\frac{dz_0}{dC} = \lambda$$

- Thus λ is the change in the local extreme value of f per unit increase in C

Examples

- Optimize the following functions with the given constraint

$$f(x, y) = 3x - 2y, \quad x^2 + 2y^2 = 44$$

$$g(x, y) = x^2 - xy + y^2, \quad x^2 + y^2 = 1$$

- We will look at both of these in maple