A Unimodular Theory of Reduced & Complete Convex Bodies

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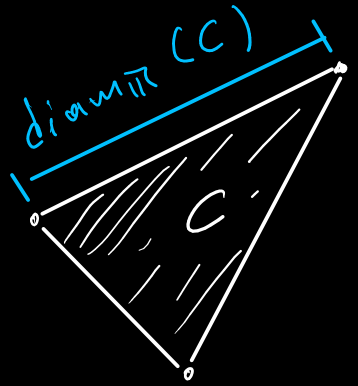
Joint work with Giulia Codenotti

Basics

- Convex bodies $\hat{=}$ convex & compact $C \subseteq \mathbb{R}^d$ w/ non-empty interior

- Euclidean diameter:

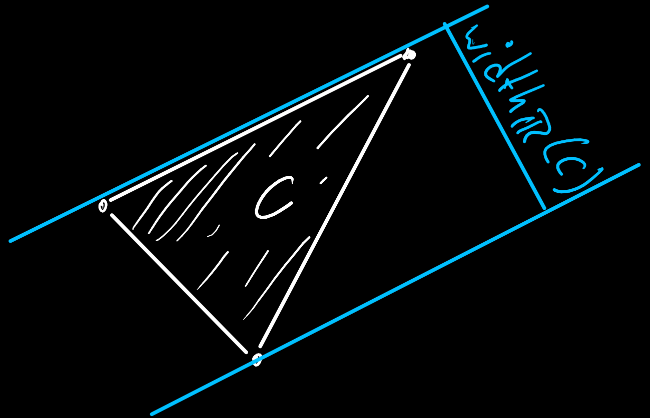
$$\mathrm{diam}_{\mathbb{R}}(C) = \max\{\mathrm{vol}_1(I) : I \subseteq C \text{ line segment}\}$$



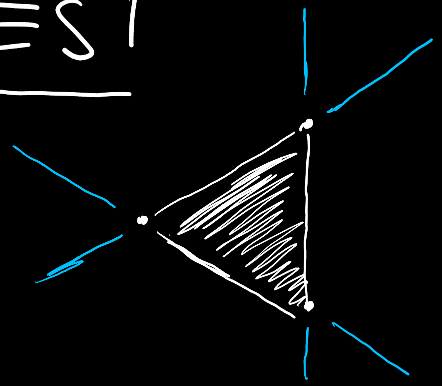
- Euclidean width:

$$\mathrm{width}_{\mathbb{R}}(C) = \min_{u \in S^{d-1}} \max_{a,b \in C} u \cdot (a-b)$$

$= $ minimum distance of two parallel supporting hyperplanes



REDUCED & COMPLETE CONVEX BODIES!



- $C \subseteq \mathbb{R}^d$ is reduced, if

  $\forall\, \tilde{C} \subsetneq C$ convex body : $\text{width}_{\mathbb{R}}(\tilde{C}) < \text{width}_{\mathbb{R}}(C)$

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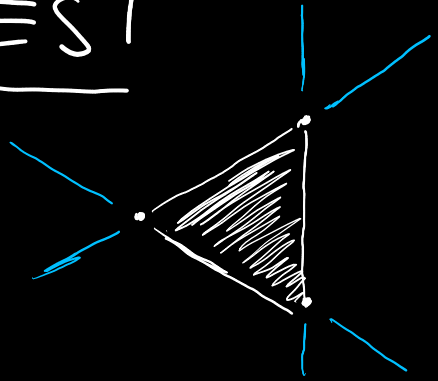

- $C \subseteq \mathbb{R}^d$ is complete, if

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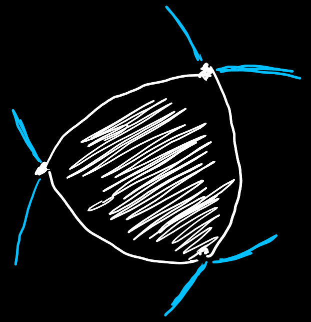

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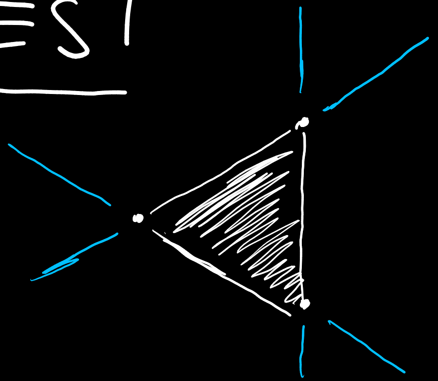

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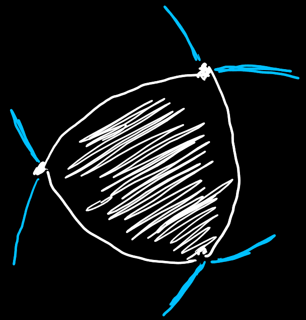

FACT: C is complete, iff

$$S^{d-1} \to \mathbb{R}, \quad u \mapsto \max_{a,b \in C} u \cdot (a-b) \quad \text{is constant}$$

$\rightsquigarrow$ "bodies of constant width"

Reduced & Complete Convex Bodies, II

- Reduced Bodies are extremal in the "isominwidth - inequality", which asks for the maximum of $\mathrm{width}_{\mathbb{R}}(C)$ among convex $C \subseteq \mathbb{R}^d$ with $\mathrm{vol}(C) \leq 1$.
  - ↳ Pál: In $\mathbb{R}^2$, the regular triangle is extremal.
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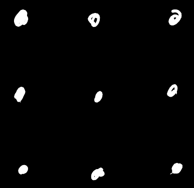
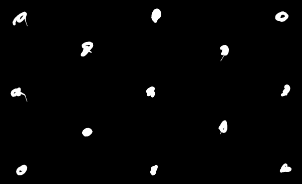
- Reduced / Complete polytopes were also studied in non-Euclidean settings:
  - → Spaces of constant curvature (Bezdek, Böröczky & Sagmeister, Lassak, ...)
  - → Minkowski spaces (González Merino et al., Groemer, Martini, ...)

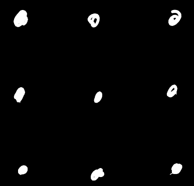
THE DISCRETE SETTING I

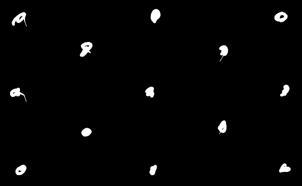




- Lattices $\hat{=}$ discrete subgroups $\Lambda \subseteq \mathbb{R}^d$
- Dual lattice: $\Lambda^* = \{ y \in \mathbb{R}^d : x \cdot y \in \mathbb{Z}, \forall x \in \Lambda \}$

The Discrete Setting!





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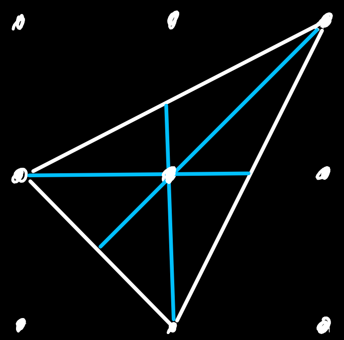
- Lattice segments: $I = [a,b] \subseteq \mathbb{R}^d$ s.th. $\dim(\Lambda \cap \operatorname{span}(a-b)) = 1$.

  $\mathrm{Vol}_1(I) = \dfrac{\mathrm{vol}_1(I)}{|v|}$, where v generates $\Lambda \cap \operatorname{span}(a-b)$

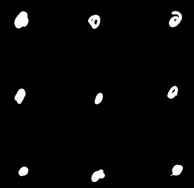
  $\mathrm{diam}_\Lambda(C) = \max\{ \mathrm{Vol}_1(I) : I \subseteq C \text{ lattice segment} \}$

  ↑ lattice diameter

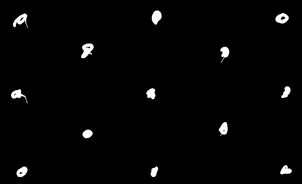


The Discrete Setting!

$\mathbb{Z}^2$



$A_2 \cong \mathbb{R}^3 \cap \mathbb{1}^{\perp}$



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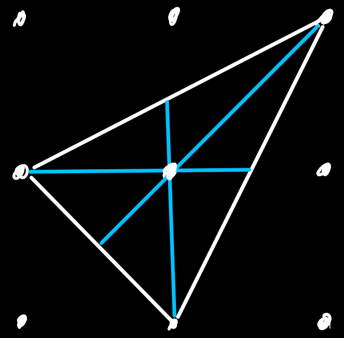
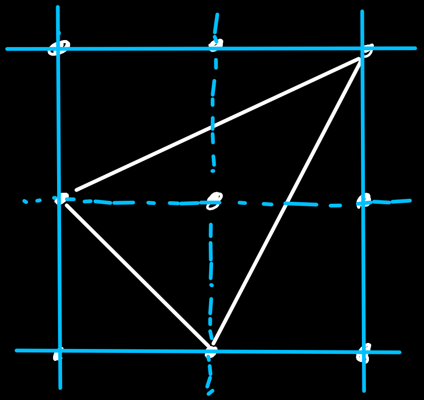


Lattice width: $$\operatorname{width}_\Lambda(C) = \min_{y \in \Lambda^*} \underbrace{\max_{a,b \in C} y \cdot (a-b)}$$

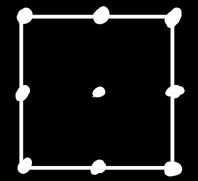
$$\approx \#\left(\begin{array}{l} \text{lattice planes orthogonal to } y \\ \text{that intersect } C \end{array} \right)$$



LATTICE REDUCED / COMPLETE CONVEX BODIES I

- C is lattice reduced w.r.t. Λ, if

  $\forall \tilde{C} \subsetneq C$ convex body: $\mathrm{width}_\Lambda(\tilde{C}) < \mathrm{width}_\Lambda(C)$



- C is lattice complete w.r.t. Λ, if

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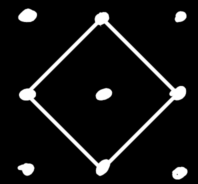

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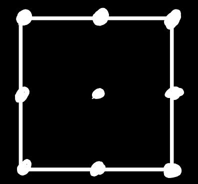

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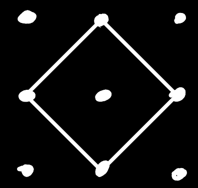
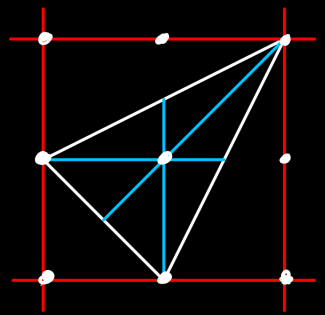


- C is reduced / complete w.r.t. Λ
  $\Rightarrow$ AC is reduced / complete w.r.t. $A\Lambda$, $\forall A \in GL_d(\mathbb{R})$

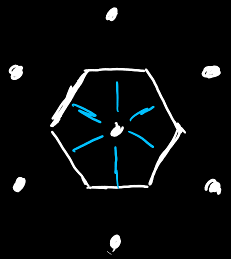
- C is reduced / complete w.r.t. $\mathbb{Z}^d$
  $\Rightarrow$ UC is reduced / complete w.r.t. $\mathbb{Z}^d$, $\forall U \in GL_d(\mathbb{Z})$

EXAMPLES

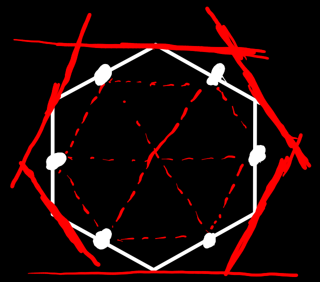
- $S_d = \operatorname{conv}\{\mathbb{1}, -e_1, \dots, -e_d\}$ is reduced <u>and</u> complete w.r.t. $\mathbb{Z}^d$.



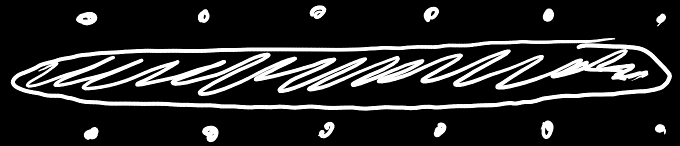
- For any lattice Λ, its Voronoi cell
$$V_\Lambda = \{ x \in \mathbb{R}^d : |x| \le |x - a|, \ \forall a \in \Lambda \}$$
is complete w.r.t. Λ



- $(V_\Lambda)^*$ is reduced w.r.t. Λ^*



THE FLATNESS CONSTANT I



- $C \subseteq \mathbb{R}^d$ is "hollow", if $\operatorname{int} C \cap \Lambda = \emptyset$

  ↳ Hollow bodies can have arbitrary large volume, but:

<u>THM</u> (Khinchine '48): $C \subseteq \mathbb{R}^d$ hollow $\Rightarrow$ $\operatorname{width}_\Lambda(C) \leq \operatorname{Flt}(d)$,

where $\operatorname{Flt}(d)$ is independent of C (Flatness constant)

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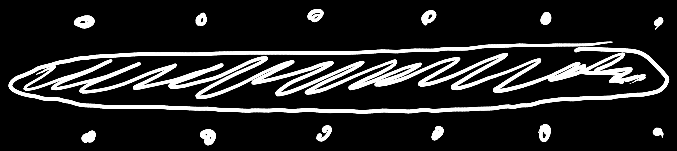

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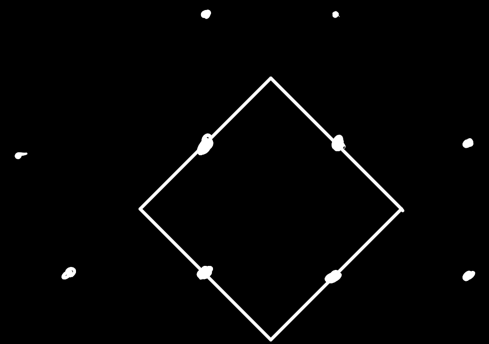
- How do realizers of $\mathrm{Flt}(d)$ look like?

<u>Lovász</u>: Suffices to consider inclusion-maximal hollow bodies

$\hookrightarrow$ Polytopes with $\leq 2^d$ facets.



The Flatness Constant, II

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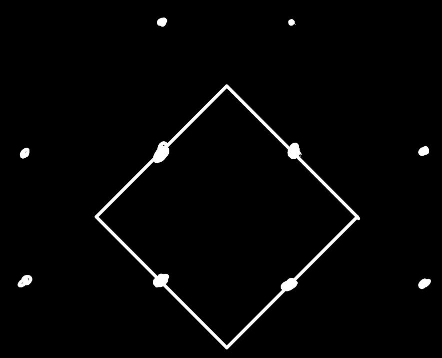

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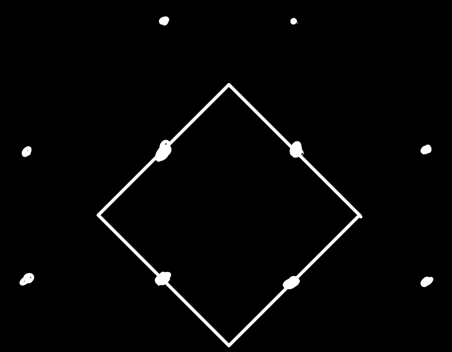

Alternatively:

THM (Codenotti, F. '23+): For any convex body $C \subseteq \mathbb{R}^d$, there's a reduced body $R \subseteq C$ with $\mathrm{width}_\Lambda(R) = \mathrm{width}_\Lambda(C)$.

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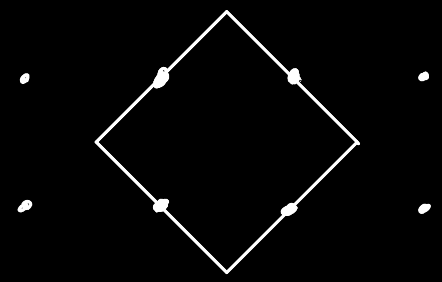

Lovász: $\mathrm{Flt}(d)$ is attained by C inclusion-maximal hollow
$\Rightarrow$ C is a polytope with $\leq 2^d$ facets.

Alternatively:

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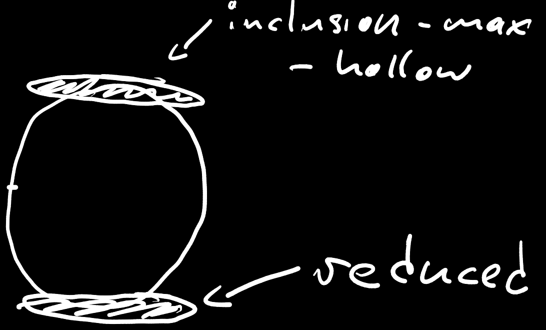

<u>THM</u> (Codenotti, F. '23+): Let $S \subseteq \mathbb{R}^d$ be a local maximum of $\mathrm{width}_\Lambda : \{S \subseteq \mathbb{R}^d \text{ hollow } d\text{-simplex}\} \to \mathbb{R}$.
Then, S is Λ-reduced.

PROPERTIES OF REDUCED / COMPLETE BODIES I

- Both reduced and complete bodies are polytopes.

REDUCED	COMPLETE
→ At most $2 \cdot (2^d - 1)$ vertices	→ At most $2 \cdot (2^d - 1)$ facets

Properties of Reduced / Complete Bodies!

- Both reduced and complete bodies are polytopes.

Reduced	Complete
→ At most $2\cdot(2^d-1)$ vertices	→ At most $2(2^d-1)$ facets

- How many independent width / diameter realizing directions are there?

→ $\dim\{y \in \Lambda^* \text{ width direction}\} \geq \mathcal{O}(\log d)$	→ $\dim\{v \in \Lambda \text{ diameter direction}\} \geq \mathcal{O}(\log d)$

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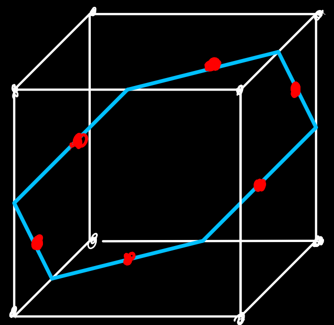
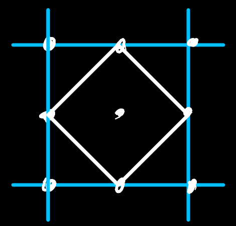
- Consider $V_{A_d^*} \cong$ regular d-permutahedron



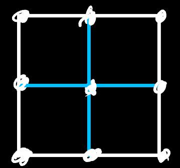
- This is a complete polytope w.r.t. A_d^*, it has $2 \cdot (2^d - 1)$ facets.
- Lifting the normals of $V_{A_d^*}$ into $\mathbb{R}^{2^d-1}$ gives a complete polytope with d independent diameter directions

DUALITY (?) I



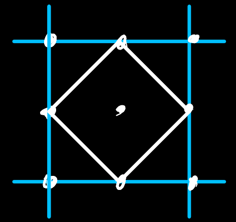
PROPOSITION (Codenotti, F, '23+): Let P be origin-symmetric.

$$P \text{ is complete w.r.t. } \Lambda \iff P^* \text{ is reduced w.r.t. } \Lambda^*$$



Moreover, the diameter directions of P are the width directions of P^*.

Duality (?) I



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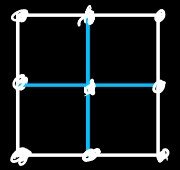
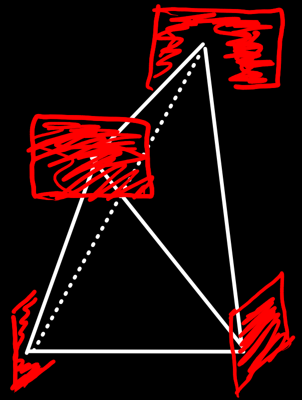


Moreover, the diameter directions of P are the width directions of P^*.

$\rightsquigarrow$ Is there a "natural" duality in the general case?



Proposition (Codenotti, F., '23+): Let $S \subseteq \mathbb{R}^d$ be a complete simplex.
Then, $\dim\{\text{diameter directions of } S\} = d$.

The dual statement for reduced simplices is false.

INCLUSION (?) I

- Recall that, in the Euclidean case, "complete" is stronger than "reduced".

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<u>THM</u> (Codenotti, F. '23⁺): Any $\mathbb{H}^2$-complete triangle is also $\mathbb{H}^2$-reduced.

- This does <u>not</u> extend to polygons, or d-simplices.

INCLUSION (?)

- Recall that, in the Euclidean case, "complete" is stronger than "reduced".

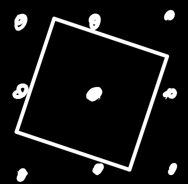
<u>THM</u> (Codenotti, F., '23+): Any $\mathbb{Z}^2$-complete triangle is also $\mathbb{Z}^2$-reduced.

- This does <u>not</u> extend to polygons or d-simplices.

How do "reduced" and "complete" interact?

How do polytopes look like that are reduced <u>and</u> complete?

<u>Examples</u>: $\mathrm{conv}\{-\mathbb{1}, e_1, \dots, e_d\}$, Voronoi cell of A_2, "skew squares",



some exceptional 3- and 4-simplices

→ Are there symmetric examples for $d \geq 3$?

THANK YOU FOR YOUR ATTENTION!