Tensor Valuations on Lattice Polygons Beyond Ehrhart Coefficients

Modern Valuation Theory, Jena

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Work in Progress w/ Monika Ludwig & Martin Rubey

EHRHART THEORY

$\mathcal{P}(\mathbb{Z}^n) = \{\text{LATTICE POLYTOPES IN } \mathbb{R}^n\}$

EHRHART'S THEOREM: LET $P \in \mathcal{P}(\mathbb{Z}^n)$. THERE EXIST $L_0(P), \dots, L_n(P) \in \mathbb{R}$ WITH

$$L(kP) := \#(kP \cap \mathbb{Z}^n) = \sum_{i=0}^{n} L_i(P) k^i, \qquad \forall k \in \mathbb{Z}_{>0}$$

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- L_i IS $GL_n(\mathbb{Z})$-INVARIANT: $L_i(\phi P) = L_i(P)$, $\forall P \in \mathcal{P}(\mathbb{Z}^n)$, $\phi \in GL_n(\mathbb{Z})$.
- L_i IS $\mathbb{Z}^n$-INVARIANT: $L_i(P+y) = L_i(P)$, $\forall P \in \mathcal{P}(\mathbb{Z}^n)$, $y \in \mathbb{Z}^n$.
- L_i IS i-HOMOGENEOUS: $L_i(kP) = k^i L_i(P)$, $\forall P \in \mathcal{P}(\mathbb{Z}^n)$, $k \in \mathbb{Z}_{>0}$.

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BETKE + KNESER: LET $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ BE A $GL_n(\mathbb{Z})$- & $\mathbb{Z}^n$-INVARIANT VALUATION

$\Longleftrightarrow \quad Z = c_0 L_0 + \dots + c_n L_n$, FOR CERTAIN $c_i \in \mathbb{R}$

EHRHART TENSORS

$\mathbb{T}^r = \{$ SYMMETRIC RANK-r TENSORS ON $\mathbb{R}^n$ $\}$

$\cong \{$ SYMMETRIC r-LINEAR MAPS $(\mathbb{R}^n)^r \longrightarrow \mathbb{R}$ $\}$

$$A \odot B(v_1, \ldots, v_r) = \frac{1}{r!} \sum_{\sigma} A \otimes B(v_{\sigma_1}, \ldots, v_{\sigma_r}), \quad A \in \mathbb{T}^s, B \in \mathbb{T}^{r-s}$$

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LUDWIG + SILVERSTEIN / KHOVANSKIĬ + PUKHILOV: THERE ARE FUNCTIONALS

$L_i^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ WITH

$$L^r(kP) = \sum_{i=1}^{n+r} L_i^r(P) k^i, \quad \forall P \in \mathcal{P}(\mathbb{Z}^n), \ k \in \mathbb{Z}_{>0}.$$

Ehrhart tensor coefficients

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- ▷ L_i^r is $\mathbb{Z}^n$-covariant: $L_i^r(P+y) = L_i^r(P) + \sum_{j=1}^{r} Z^{r-j}(P) \odot \frac{y^{\odot j}}{j!}$, for certain $Z^{r-j} : \mathcal{P}(\mathbb{Z}^n) \longrightarrow \mathbb{T}^{r-j}$ (in fact $Z^{r-j} = L_{i-j}^{r-j}$)

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BERG + JOCHEMKO + SILVERSTEIN STUDIED THE h^*-COEFFICIENTS OF $L^r(kP)$

CLASSIFICATION OF TENSOR VALUATIONS

IN THE CONTINUOUS CASE: $M^r(P) = \frac{1}{r!} \int_P x^{\otimes r} \, dx$

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MINKOWSKI TENSORS

STEINER FORMULA:

$$M^r(P+sB^n) = \sum_{j=0}^{n+r} s^{n+r-j}\,\mathrm{vol}(B^{n+r-j}) \sum_{k\geq 0} \Phi_{j-r+k}^{r-k,k}(P)$$

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ALESKER: $Z: \mathcal{K}^n \to \mathbb{T}^r$ IS A CONTINUOUS, $O(n)$-EQUIVARIANT, $\mathbb{R}^n$-COVARIANT VALUATION

$\iff$ $Z \in \mathrm{span}\{ Q^l \odot \Phi_k^{m,s} : 2l+m+s = r \}$

METRIC TENSOR $Q(x,y) = x\cdot y$, $x,y \in \mathbb{R}^n$

Low Ranks

Write $Z : \mathcal{P}(\mathbb{Z}^n) \xrightarrow{u} \mathbb{T}^r$ for a valuation Z that's $GL_n(\mathbb{Z})$-equivariant & $\mathbb{Z}^n$-covariant

$$Z(\phi P)(v_1, \dots, v_r) = Z(P)(\phi^t v_1, \dots, \phi^t v_r)$$

$$Z(P+y) = \sum_{j=0}^{r} Z^{r-j}(P) \odot \frac{y^{\odot j}}{j!}$$

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KHOVANSKIĬ + PUKHLIKOV / LUDWIG + SILVERSTEIN: LET $Z : \mathcal{P}(\mathbb{Z}^n) \xrightarrow{u} \mathbb{T}^r$.

$\Rightarrow$ $Z(kP) = \sum_{i=0}^{n+r} Z_i(P) k^i$, WHERE $Z_i : \mathcal{P}(\mathbb{Z}^n) \xrightarrow{u} \mathbb{T}^r$ IS i-HOMOGENEOUS

FOR $r > 0$, WE HAVE $Z_0 = 0$.

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Ludwig + Silverstein: Let $Z : \mathcal{P}(\mathbb{Z}^n) \xrightarrow{u} \mathbb{T}^r$, $r \leq 8$.

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- THE CONDITION $r \le 8$ IS NECESSARY TO PROVE THAT L_1^r IS THE "ONLY" 1-HOMOGENEOUS VALUATION $\mathcal{P}(\mathbb{Z}^2) \xrightarrow{u} \mathbb{T}^r$, IF $r \ge 3$ IS ODD
- FOR $r \ge 9$ COUNTEREXAMPLES EXIST

ODD RANKS, $n=2$

GOAL: CHARACTERIZE 1-HOM' VALUATIONS

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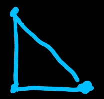
$\mathrm{conv}\{0,e_1,e_2\}$

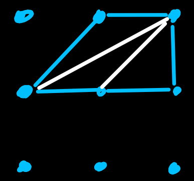
BY $GL_2(\mathbb{Z})$-EQUIVARIANCE: Z IS UNIQUELY DETERMINED BY $Z(T_2)$



WE HAVE $Z(P)=\sum_{T\in\mathcal{T}} Z(T_2)\cdot\phi_T^t$, $A\cdot\psi(v_1\dots v_r):=A(\psi v_1,\dots,\psi v_r)$

WHERE $\mathcal{T}$ IS A UNIMODULAR TRIANGULATION OF P

AND $\phi_T\in GL_2(\mathbb{Z})$ S.TH. $\phi_T T_2+y=T$ FOR SOME $y\in\mathbb{Z}^2$



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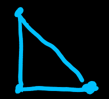

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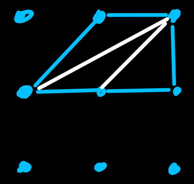
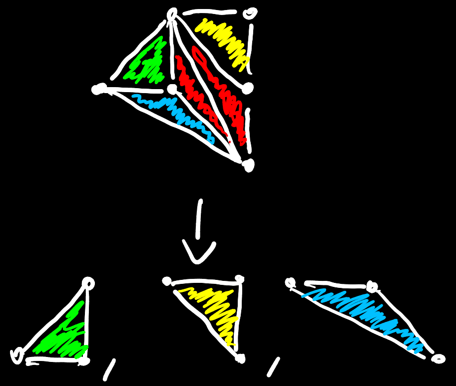


▷ WHICH VALUES FOR $Z(T_2)$ YIELD A WELL-DEFINED VALUATION?

INDEPENDENT OF $\mathcal{T}$ AND ϕ_T

ODDITY OF A POLYGON

- LET $P \in \mathcal{P}(\mathbb{Z}^2)$ AND $\mathcal{T}$ A UNIMODULAR TRIANGULATION OF P
- CANCEL TRANSLATIVE COPIES OF T AND $-T$ IN $\mathcal{T}$:



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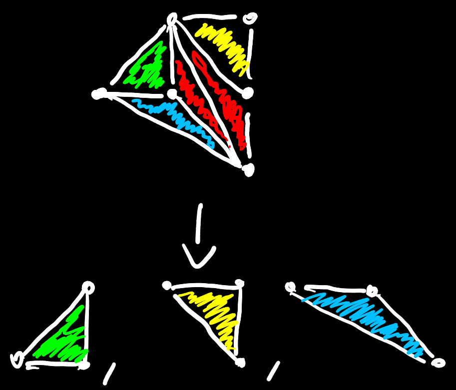

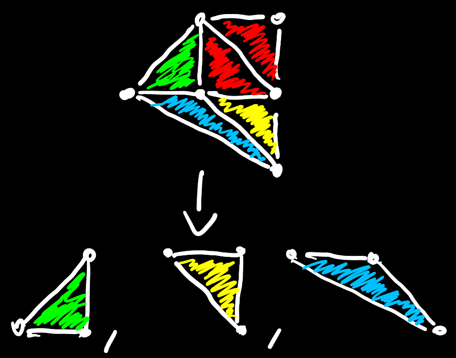

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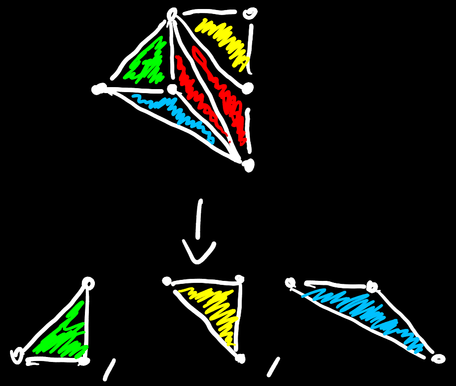

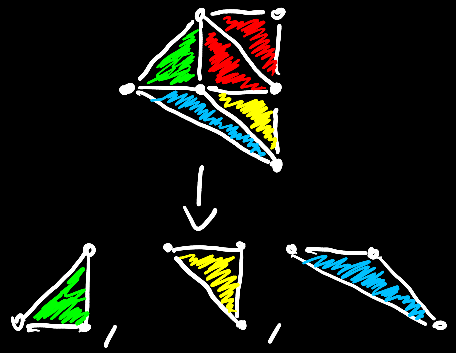

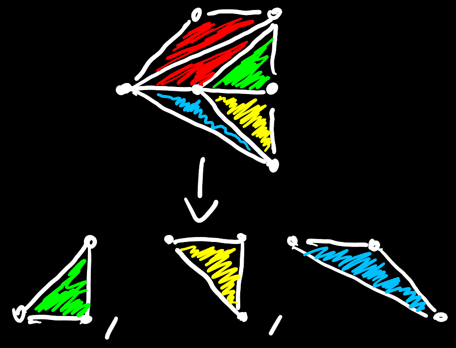

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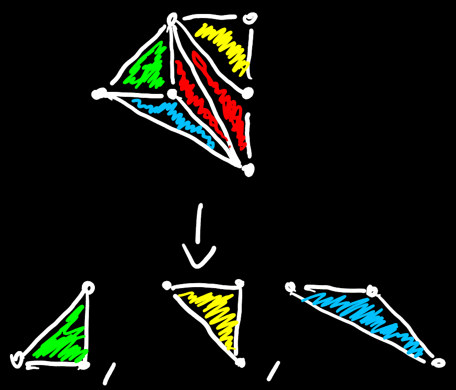
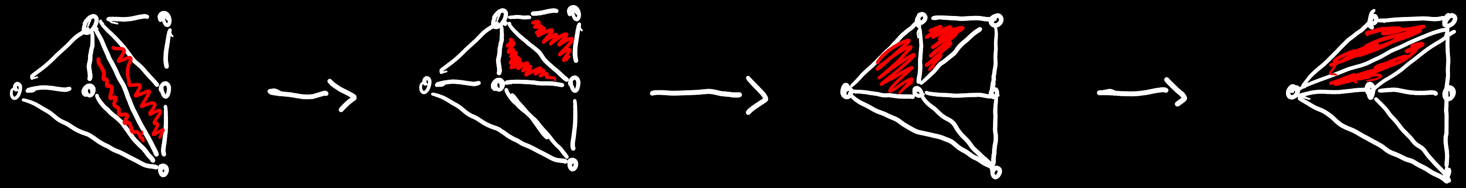


- THE REMAINING TRIANGLES ARE THE SAME FOR EACH TRIANGULATION!

<u>PROOF</u>: TRANSFORM UNIMODULAR TRIANGULATIONS WITH PARALLELOGRAM FLIPS



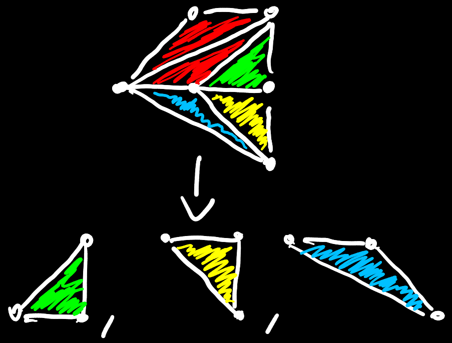
ODDITY OF A POLYGON

UP TO TRANSLATION !

FORMALLY : LET $\mathbb{R}^\Delta = \text{span}\{ e_T : T \text{ UNIMOD' TRIANGLE} \}$

$V^\Delta = \mathbb{R}^\Delta / \text{span}\{ e_T + e_{-T} : T \text{ UNIMOD TRIANGLE}\}$

$\pi^\Delta : \mathbb{R}^\Delta \longrightarrow V^\Delta$ CANONICAL PROJECTION



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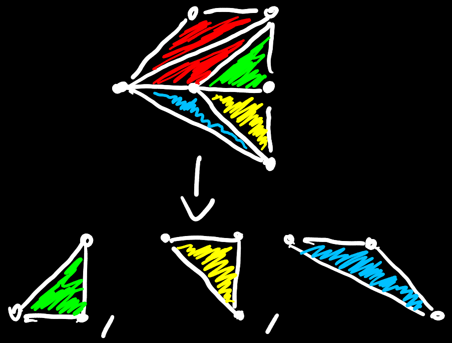

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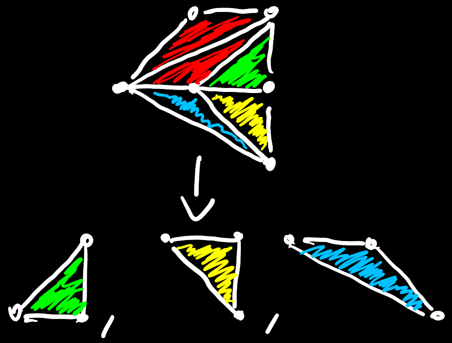
$\Rightarrow \quad \text{odd}(P) := \pi^\Delta \left(\sum_{T \in \mathcal{T}} e_T \right)$ IS A WELL-DEFINED VALUATION

$(\text{odd}(Q) := 0, \ \forall Q . \dim Q < 2)$

ODDITY OF A POLYGON

UP TO TRANSLATION !



FORMALLY : LET $\mathbb{R}^{\Delta} = \text{span}\{ e_T : T \text{ UNIMOD' TRIANGLE} \}$

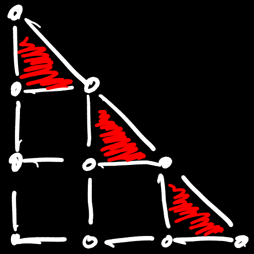
$V^{\Delta} = \mathbb{R}^{\Delta} / \text{span}\{ e_T + e_{-T} : T \text{ UNIMOD' TRIANGLE} \}$

$\pi^{\Delta} : \mathbb{R}^{\Delta} \longrightarrow V^{\Delta}$ CANONICAL PROJECTION

$\Rightarrow$ $\text{odd}(P) := \pi^{\Delta}\Big(\sum_{T \in \mathcal{T}} e_T \Big)$ IS A WELL-DEFINED VALUATION

$(\text{odd}(Q) := 0, \ \forall Q . \dim Q < 2)$

F. + LUDWIG + RUBEY: i) $\text{odd}(P)$ IS $GL_2(\mathbb{Z})$-EQUIVARIANT AND 1-HOM'



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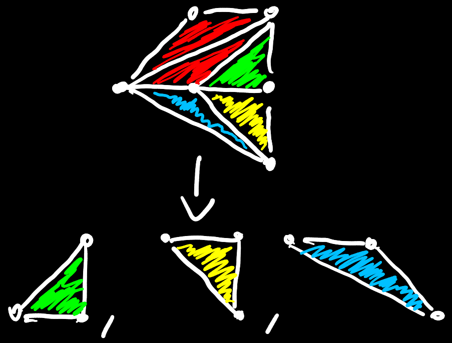

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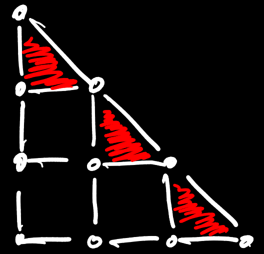
ii) $\operatorname{odd}(P) = 0 \iff P$ IS CENTRALLY SYMMETRIC



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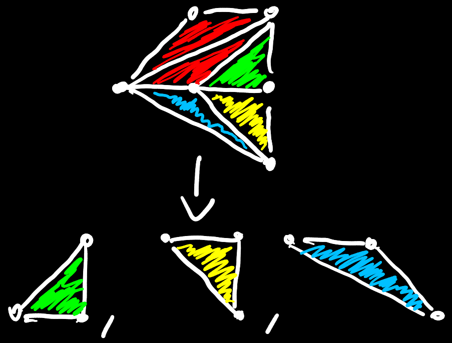

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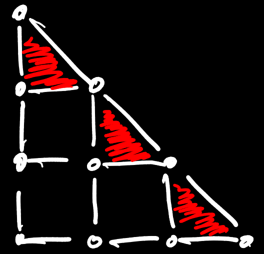

iii) ANY SIMPLE, ODD, $\mathbb{Z}^2$-INVARIANT VALUATION $Z : \mathcal{P}(\mathbb{Z}^2) \to V$

HAS THE FORM $Z = F \circ \mathrm{odd}$, FOR A LINEAR MAP $F : V^\Delta \to V$

1-HOMOGENEOUS TENSOR VALUATIONS FOR $r \geq 3$ ODD

THM (F + LUDWIG + RUBEY): LET V BE A LINEAR SPACE WITH A $GL_2(\mathbb{Z})$-REPRESENTATION ρ SUCH THAT $\rho(-id) = -id$. LET $Z: \mathcal{P}(\mathbb{Z}^2) \to V$.
THE FOLLOWING ARE EQUIVALENT

i) Z IS A $GL_2(\mathbb{Z})$-EQUIVARIANT, $\mathbb{Z}^2$-INVARIANT, ODD VALUATION

ii) $Z = F \circ \mathrm{odd}$, WHERE $F: V^\Delta \to V$ IS $GL_2(\mathbb{Z})$-EQUIVARIANT

iii) $Z = \sum_{T \in \mathcal{T}} \rho(\phi_T) v$, WHERE $\phi_T \in GL_2(\mathbb{Z})$ SUCH THAT $\phi_T T_2 + y = T$

AND $v \in V^G$ IS AN INVARIANT OF

$G := \mathrm{Stab}(T_2 + \mathbb{Z}^2) \leq GL_2(\mathbb{Z})$

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COROLLARY: FOR $r \geq 3$ ODD, THE 1-HOM' VALUATIONS $Z : \mathcal{P}(\mathbb{Z}^2) \xrightarrow{u} \mathbb{T}^r$ ARE GIVEN BY $Z_A(P) = \sum_{T \in \mathcal{T}} A \bullet \phi_T^t$, $A \in (\mathbb{T}^r)^G$

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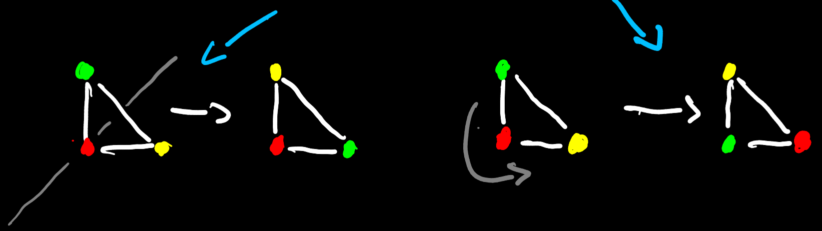

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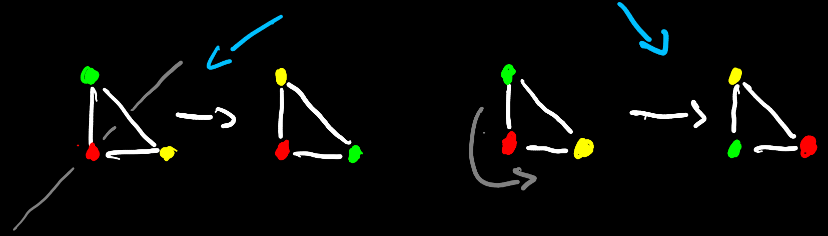

$A \mapsto A\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \dots, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$

▷ THE INVARIANT RING $\mathbb{T}^G$ IS GENERATED BY ($\mathbb{T}^r \cong \mathbb{R}[X_1, X_2]_r$)

$$p_2 := X_1^2 - X_1X_2 + X_2^2 \quad , \quad X_1^3 - \tfrac{3}{2}X_1^2X_2 - \tfrac{3}{2}X_1X_2^2 + X_2^3 =: p_3$$

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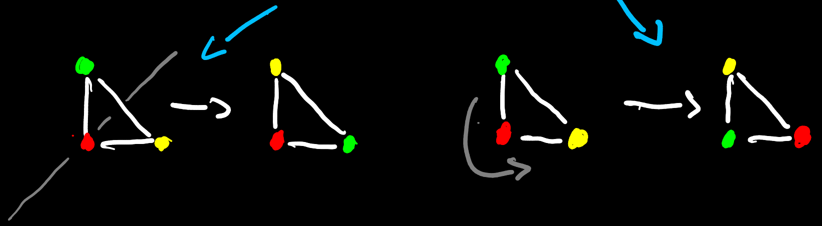

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▷ $\mathbb{R}[X_1, X_2]^G \cong \mathbb{R}[p_2, p_3]$ (CHEVALLEY)

E.G., p_3^3 AND $p_2^3 p_3$ ARE TWO INDEPENDENT INVARIANTS OF DEGREE 9

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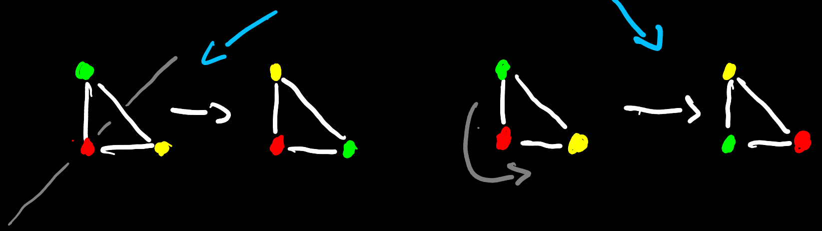

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▷ $\dim\{Z : \mathcal{P}(\mathbb{Z}^2) \xrightarrow{u} \mathbb{T}^r \text{ 1-HOM}\} = \#\{s, t \in \mathbb{Z}_{\geq 0} : 2s + 3t = r\}$

HIGHER HOMOGENEITY

- THE MISSING CASES IN THE CLASSIFICATION OF LUDWIG & SILVERSTEIN ARE i-HOMOGENEOUS $Z : \mathcal{P}(Z^2) \xrightarrow{\alpha} \Pi^r$, WHERE $r - i \geq 8$ IS EVEN.
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  a) $p = h + \overbrace{\tfrac{1}{3}(x_1 + x_2)}^{\text{CENTER OF } T_2} f$, FOR SOME $f \in \mathbb{C}[x_1, x_2]^G_{r-1}$, $h \in \mathbb{C}[x_1, x_2]^G_r$

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- A SAGEMATH - COMPUTATION LEADS TO THE CONJECTURE:

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$$\overset{??}{\shortparallel} \quad \text{(YES! FOR 3-HOM')}$$

$$\dim \{ Z : \mathcal{P}(\mathbb{C}^2) \xrightarrow{u} \mathbb{T}^{r+i} \text{ (2+i)-HOM'} \}$$

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THANK YOU!