Unimodular Valuations Beyond Ehrhart

Workshop on Geometric & Algebraic Combinatorics, Santander

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VALUATIONS ON LATTICE POLYTOPES

- $\mathcal{P}(\mathbb{Z}^n) = \{$LATTICE POLYTOPES IN $\mathbb{R}^n\}$
- A FUNCTION $Z : \mathcal{P}(\mathbb{Z}^n) \to G$ (SOME ABELIAN GROUP) IS A VALUATION, IF

$$Z(P \cup Q) = Z(P) + Z(Q) - Z(P \cap Q), \quad \forall P, Q, P \cap Q, P \cup Q \in \mathcal{P}(\mathbb{Z}^n)$$

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EXAMPLES:
- $\mathrm{Vol}(P)$ IS A VALUATION. IT'S SIMPLE, I.E. $\mathrm{Vol}(P) = 0$ IF $\dim P < n$.
- $L^0(P) := \sum_{v \in P \cap \mathbb{Z}^n} 1$ IS A VALUATION

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- LET $x_1, \dots, x_n$ BE THE COORDINATES OF $(\mathbb{R}^n)^*$
- IDENTIFY $v \in \mathbb{R}^n$ WITH $v(x_1, \dots, x_n) := v_1 x_1 + \dots + v_n x_n$

MORE EXAMPLES: • $E(P) := \int_P e^{v(x)} dv \in \mathbb{R}[[x_1, \dots, x_n]]$

• $L(P) := \sum_{v \in P \cap \mathbb{Z}^n} e^{v(x)} \in \mathbb{R}[[x_1, \dots, x_n]]$ (BARVINOK / LAWRENCE)

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- $L(P) := \sum_{v \in P \cap \mathbb{Z}^n} e^{v(x)} \in \mathbb{R}[[x_1, \dots, x_n]]$ (BARVINOK / LAWRENCE)

- WRITE $L(P) = \sum_{r \geq 0} L^r(P)$, WHERE $L^r(P) \in \mathbb{R}[x_1, \dots, x_n]_r$.

  THEN $L^r(P) = \sum_{v \in P \cap \mathbb{Z}^n} \frac{v^r}{r!}$ IS A VALUATION (BÖRÖCZKY + LUDWIG)

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GOAL: CLASSIFY VALUATIONS ON LATTICE POLYTOPES!

INTERLUDE: HADWIGER'S THEOREM

THM (HADWIGER): LET $Z : \{\text{CONVEX BODIES IN } \mathbb{R}^n\} \longrightarrow \mathbb{R}$

BE A CONTINUOUS AND RIGID MOTION INVARIANT VALUATION.

THEN,

$$Z = \sum_{i=0}^{n} \lambda_i V_i \quad , \text{ FOR CERTAIN } \lambda_0, \dots, \lambda_n \in \mathbb{R}.$$

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"INTRINSIC VOLUMES"

$\Rightarrow$ SINCE $V_i(sK) = s^i V_i(K)$:

ORTH' PROJECTION

$$\text{SURFACE-AREA}(K) = \frac{1}{Vol_{n-1}(B^{n-1})} \int_{S^{n-1}} Vol_{n-1}(K \mid u^\perp) \, du$$

VOLUME OF THE (n-1) - BALL

"CAUCHY'S PROJECTION FORMULA"

Translatively Polynomial Valuations

- $Z : \mathcal{S}(\mathbb{Z}) \to \mathbb{R}[x_1, \dots, x_n]$ is translatively polynomial of degree r, if

$$Z(P+v) = \sum_{j=0}^{r} Z^{r-j}(P) \frac{v^j}{j!},$$ for certain Z^{r-j} (associated functions)

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EXAMPLE: $$L^r(P+v) = \sum_{j=0}^{r} L^{r-j}(P) \frac{v^j}{j!}$$

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THM (KHOVANSKII + PUKHLIKOV): LET $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}[x_1, \dots, x_n]$ BE A TRANSLATIVELY POLYNOMIAL VALUATION $\Rightarrow Z = \sum_{i=0}^{n+r} Z_i \leftarrow$ i-HOMOGENEOUS: $Z_i(mP) = m^i Z(P)$, $\forall m \in \mathbb{Z}_{\geq 0}$

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"EHRHART TENSOR COEFFICIENTS"

EXAMPLE: $L^r = L_0^r + \dots + L_{n+r}^r$, $L_i^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}[x_1,\dots,x_n]_r$

BERG + JOCHEMKO + SILVERSTEIN / BAJO ET AL. STUDIED h^*-COEFFICIENTS OF $L^r(mP)$

UNIMODULAR VALUATIONS

- $GL_n(\mathbb{Z})$ ACTS ON $\mathbb{R}[X_1,\dots,X_n]$ VIA $\phi f := f \circ \phi^*$
- $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}[X_1,\dots,X_n]$ IS UNIMODULAR, IF $Z(\phi P) = \phi Z(P)$, $\forall \phi \in GL_n(\mathbb{Z}), P \in \mathcal{P}(\mathbb{Z}^n)$
- $\mathrm{Val}^r(\mathbb{Z}^n) := \{ Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}[X_1,\dots,X_n]_r$ UNIMODULAR, TRANSL' POLYNOMIAL VALUATION$\}$
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THM (BETKE + KNESER): $\mathrm{Val}_i^0(\mathbb{Z}^n) = \mathrm{span}\{L_i\}$, $\forall\, 0 \le i \le n$.

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$\longrightarrow$ WHAT ABOUT HIGHER DEGREES $r > 0$?

THM (LUDWIG + SILVERSTEIN): LET $1 \leq r \leq 8$, $1 \leq i \leq n$.

$$\Rightarrow \mathrm{Val}_i^r(\mathbb{Z}^n) = \mathrm{span}\{L_i^r\}$$

MOREOVER, $\dim \mathrm{Val}_1^9(\mathbb{Z}^2) \geq 2$.

POLYGONS

THM (F + LUDWIG + RUBEY): LET $p_{23}(r) = \#\{(s,t) : r = 2s + 3t\}$, $r \geq 1$.

$$\dim \mathrm{Val}_i^r(\mathbb{Z}^2) = \begin{cases} 1, & \text{IF } r \leq i \leq r+2 \text{ OR } 1 \leq i < r \text{ AND } r-i \text{ IS ODD}, \\ p_{23}(r), & \text{IF } i = 1 < r \text{ AND } r-1 \text{ IS EVEN}, \\ p_{23}\left(\frac{r-i}{2}+1\right), & \text{IF } 1 < i < r \text{ AND } r-i \text{ IS EVEN}, \\ 0 & \text{ELSE}. \end{cases}$$

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- CASE 1 IS IMPLICIT IN [LUDWIG+SILVERSTEIN].
- CASE 4 IS KHOVANSKII+PUKHLIKOV.

- IN THE CONVEX BODY SETTING, A GENERATING SYSTEM WAS FOUND BY ALESKER.

- WE FOCUS ON CASES 2&3, SO LET $r-i>0$ BE EVEN.
- ABBREVIATE $\mathrm{Val}_i^r = \mathrm{Val}_i^r(\mathbb{Z}^2)$.

1 - HOMOGENEOUS VALUATIONS

$r > 1$ ODD

▷ LET $Z \in \mathrm{Val}_1^r$ $\Rightarrow$ $Z(\triangle) \in \mathbb{R}[X,Y]_r^{\mathcal{G}}$, WHERE

$\mathrm{conv}\{0, e_1, e_2\}$

$\mathcal{G} = \{\phi \in GL_2(\mathbb{Z}) : \phi(\triangle) = \triangle + v$ FOR SOME $v \in \mathbb{Z}^2\}$. IN FACT: $\mathrm{Val}_1^r \simeq \mathbb{R}[X,Y]_r^{\mathcal{G}}$

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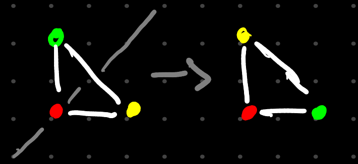
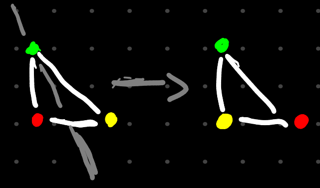
▷ $G = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \right\rangle \cong S_3$





$$\Rightarrow \mathbb{R}[X,Y]^G = \mathbb{R}\left[X^2 - XY + Y^2,\ X^3 - \tfrac{3}{2}(X^2Y + XY^2) + Y^3\right]$$

$$\Rightarrow \dim \mathrm{Val}_1^r = p_{23}(r).$$

EXPONENTIAL VALUATIONS

$d > 0$ EVEN

$$\overline{Val}_d := \left\{ \overline{Z} = \sum_{r \geq 0} Z^r \; : \; Z^r \in Val^r_{r-d} \text{ AND } \overline{Z}(P+v) = e^v \overline{Z}(P) \right\}$$

SITUATION:

$$\begin{tikzcd}[column sep=small]
 & & & \overline{Val}_d \arrow[dll, "\overline{Z} \mapsto Z^{r+d}"'] \arrow[dl] \arrow[dr] \arrow[drr] & & & \\
\cdots \arrow[r] & Val^{r+d}_d \arrow[r, "\text{ASSOC'}", "\text{FUNCT'}"'] & Val^{r+d-1}_{d-1} \arrow[r] & \cdots \arrow[r] & Val^{d+2}_2 \arrow[r] & Val^{d+1}_1 \arrow[r] & 0
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$Val^{d+2}_2 \longrightarrow \overline{Val}_d$ CAN BE REVERSED BY $\overline{Z}(\mathbb{R}^2_{\geq 0}) = \dfrac{Z_2^{d+2}(\blacksquare)}{xy} \in \mathbb{R}(x,y)_d$

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$\Rightarrow \quad \overline{Val}_d \simeq Val^{d+i}_i \,, \quad \forall i \geq 2$

FINISH BY PROVING $\dim Val^{d+2}_2 = p_{23}\left(\frac{d}{2} + 1\right)$.

DIMENSION 3

LET $r > 1$ BE ODD, $Z \in \mathrm{Val}_1^r$

- SINCE VALUATIONS IN $\mathrm{Val}_1^r(\mathbb{Z}^2)$ ARE SIMPLE:

$$Y(P) := \begin{cases} Z(P) & \dim P \leq 2 \\ \frac{1}{2} \sum\limits_{F \in \mathcal{F}_2(P)} Z(F) & \dim P = 3 \end{cases}$$

DEFINES A VALUATION IN $\mathrm{Val}_1^r(\mathbb{Z}^3)$

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THANK YOU.