Affine Subspace Concentration Conditions for Centered Polytopes

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joint work with Martin Henk and Christian Kipp

Outline

- Recap of the (linear) subspace concentration condition for polytopes
- Wu's affine subspace concentration condition
- Proof for general centered polytopes
- Equality case

Setting

Consider an n-polytope $P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1,\ 1 \leq i \leq m\}$

-> P contains the origin in its interior

-> We assume that the inequalities are irredundant

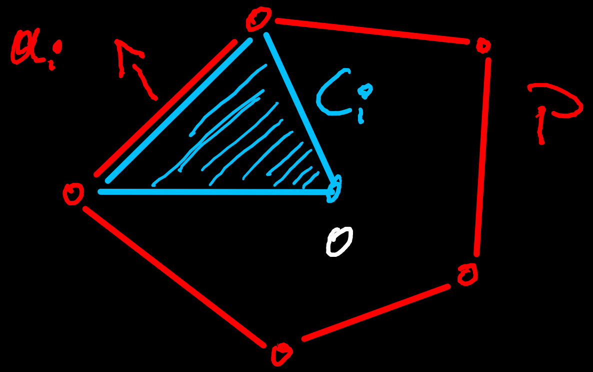
Setting 1

Consider an n-polytope $P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq 1,\ 1 \leq i \leq m\}$

- P contains the origin in its interior
- We assume that the inequalities are irredundant

Let $F_i = \{x \in P : \langle x, a_i \rangle = 1\}$ be the facet corresponding to a_i.

Let $C_i = \operatorname{conv}(\{0\} \cup F_i)$ be the cone corresponding to a_i.



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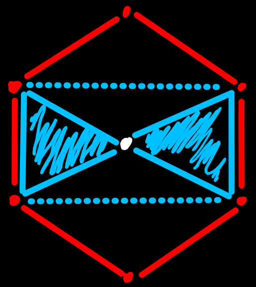

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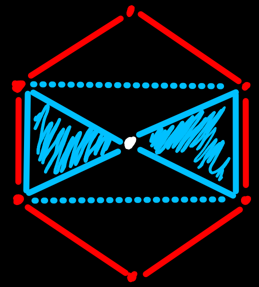

<u>In general</u>: $\leq \frac{1}{n} \operatorname{vol}(P)$. This is a consequence of

Thm (Henk, Schürmann, Wills): Let $P \subseteq \mathbb{R}^n$ be a symmetric n-polytope. For any linear d-subspace $L \subseteq \mathbb{R}^n$, we have

$$\sum_{i:\, a_i \in L} \operatorname{vol} C_i \leq \frac{d}{n} \operatorname{vol} P.$$

Symmetric Polytopes!

$-P = P = \{x : \langle x, a_i \rangle \le 1,\ 1 \le i \le m\}$, L d-subspace

Henk et al: $\sum_{i : a_i \in L} \operatorname{vol} C_i \le \frac{d}{n} \operatorname{vol} P$

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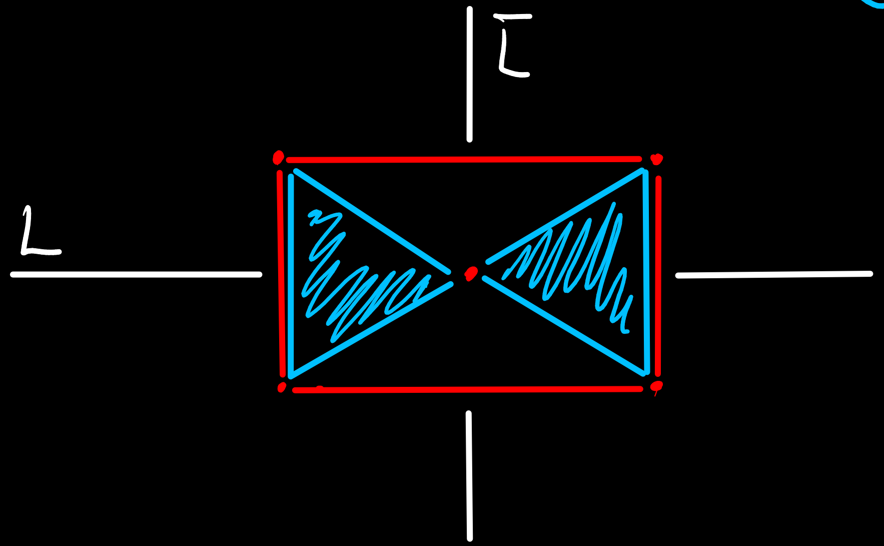
For any d-subspace $L \subseteq \mathbb{R}^n$, we have

$$\sum_{i : a_i \in L} \operatorname{vol} C_i \le \frac{d}{n} \operatorname{vol} P,$$

with equality, iff $\{a_1, \dots, a_m\} \subseteq L \cup \bar{L}$,

where $\bar{L}$ is a complementary subspace to L.

Linear Subspace Concentration Condition (LSC)



The (even) log-Minkowski problem

Consider a finite set of normal vectors $\{\pm u_1, \dots, \pm u_m\} \subseteq S^{n-1}$

Assign a virtual cone volume $\gamma_i > 0$ to each u_i.

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Böröczky, Lutwak, Yang, Zhang: This condition is also sufficient !

What are the necessary / sufficient conditions in the non-symmetric case ?

Centered Polytopes

$P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1, \ 1 \leq i \leq m\}$

$c(P) := \frac{1}{\mathrm{vol}(P)} \int_P x \, dx$ is the centroid of P.

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Thm (Henk, Linke): Let $P \subseteq \mathbb{R}^n$ be a centered n-polytope.
Then (LSC) holds, i.e.,

for any linear d-space $L \subseteq \mathbb{R}^n$,
we have $\sum_{i : u_i \in L} \mathrm{vol}\, C_i \le d/n \ \mathrm{vol}\, P$,
with equality, iff $\{a_1, \ldots, a_m\} \subseteq L \cup \bar{L}$,
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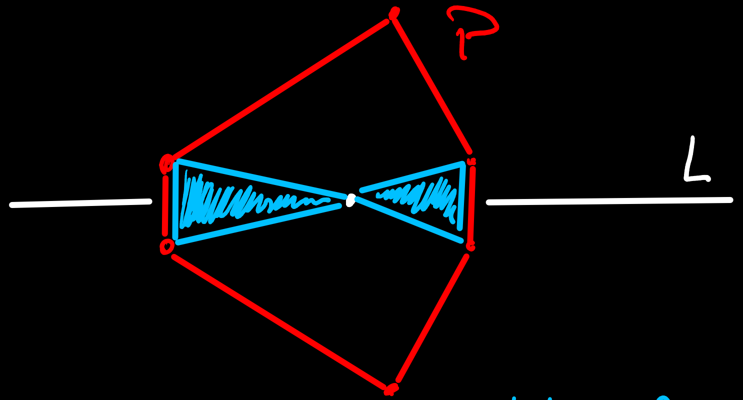


This is a necessary condition for the log-Minkowski problem

Affine Subspace Concentration!

Hering, Nill, Süß found an interpretation of (LSC) in toric geometry.

From this, Wu derived:

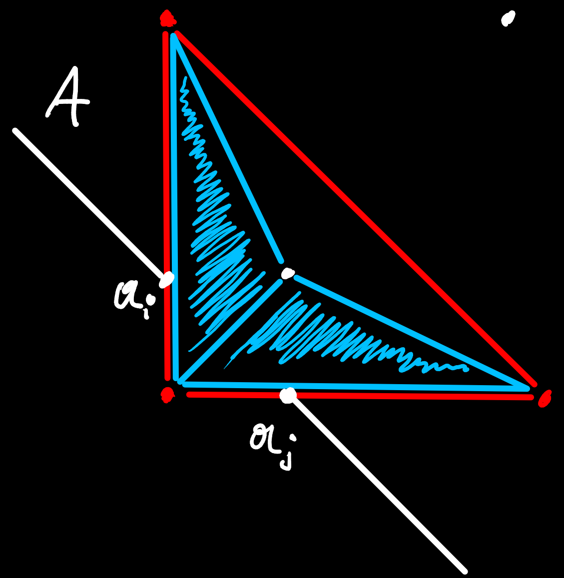
rare!

Thm (Wu): Let $P \subseteq \mathbb{R}^n$ be a centered reflexive smooth lattice polytope.

For any affine d-space $A \subseteq \mathbb{R}^n$, we have

$$\sum_{i:\, a_i \in A} \operatorname{vol} C_i \leq \frac{d+1}{n+1} \operatorname{vol}(P)$$

with equality, iff $\{a_1, \dots, a_m\} \subseteq A \cup \bar{A}$, for some complementary affine space $\bar{A}$.



Affine Subspace Concentration

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Affine Subspace Concentration Condition (ASC)

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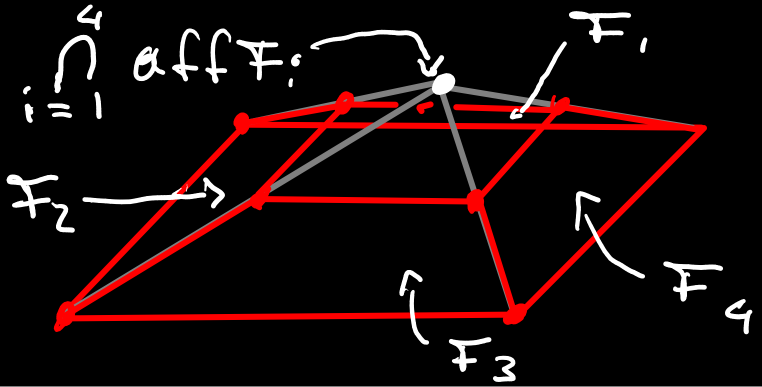

- Normalization of the a_i's is critical

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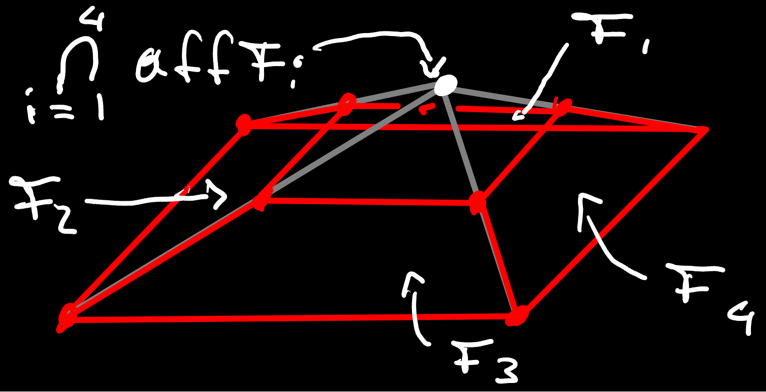

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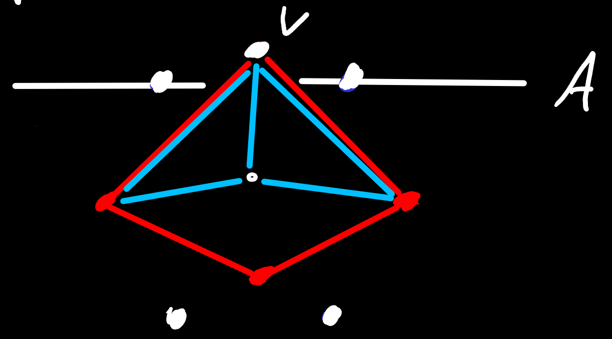
$\Rightarrow$ For any $(n-d-1)$-face $G \leq P$, $\{a_i : G \leq F_i\}$ is d-dimensional

A special case

P centered n-polytope, $v \in P$ vertex of P.

Let $A = \operatorname{aff}\{a_i : v \in F_i\}$

This is a hyperplane.



ASC: $\displaystyle\sum_{i : a_i \in A} \operatorname{vol} C_i \le \frac{n}{n+1} \operatorname{vol} P$

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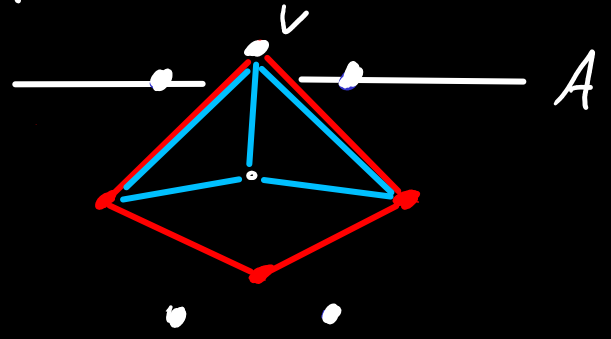

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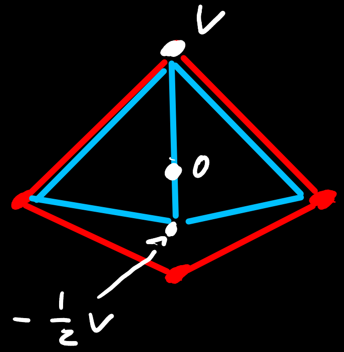
P centered $\Rightarrow -\frac{1}{n} v \in P$



Consider $\overline{C}_i = \operatorname{conv}(F_i \cup \{-\frac{1}{n}v\}) \subseteq P$.

We have $\operatorname{vol} \overline{C}_i = \frac{n+1}{n} \operatorname{vol} C_i$

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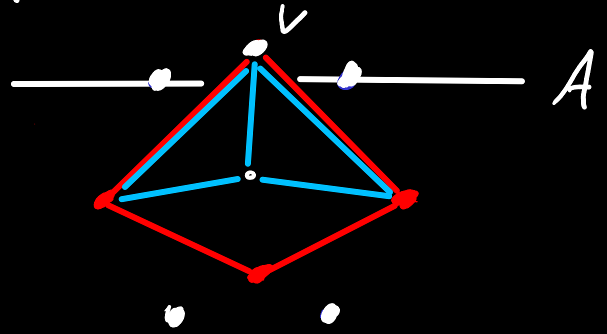

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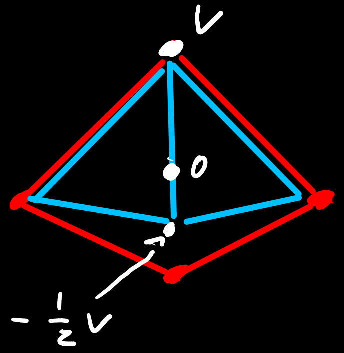

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$\displaystyle\Rightarrow \mathrm{vol}\, P \geq \sum_{i: a_i \in A} \mathrm{vol}\, \overline{C_i} \geq \frac{n+1}{n} \sum_{i: a_i \in A} \mathrm{vol}\, C_i$

Arbitrary centered polytopes!

Thm (F, Henk, Kipp): Let $P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1,\ 1 \leq i \leq m\}$ be a centered polytope. For any affine d-subspace $A \subseteq \mathbb{R}^n$, we have

$$\sum_{i : a_i \in A} \operatorname{vol} C_i \leq \frac{d+1}{n+1} \operatorname{vol} P.$$

Proof uses a limit argument for $n \to \infty$

We loose control over the equality case.

Pyramids

$$P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \le 1,\ 1 \le i \le m \}$$

Consider $P^{(1)} = \operatorname{conv}\left(P \times \{1\} \cup \{-n e_{n+1}\} \right) \subseteq \mathbb{R}^{n+1}$

$$= \{ x \in \mathbb{R}^{n+1} : x_{n+1} \le 1,\ \langle a_i^{(1)}, x \rangle \le 1,\ 1 \le i \le m \},$$

where

$$a_i^{(1)} = \begin{pmatrix} \frac{n+2}{n+1} a_i \\ -\frac{1}{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$$

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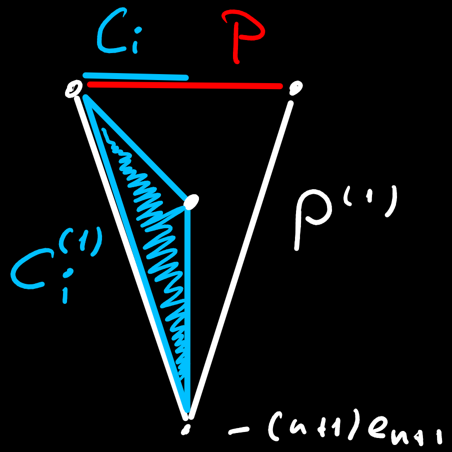

Observe : i) $\mathrm{vol}_{n+1}(P^{(1)}) = \frac{n+2}{n+1} \mathrm{vol}_n(P)$

ii) $P^{(1)}$ is centered

iii) $\mathrm{vol}_{n+1}(C_i^{(1)}) = \mathrm{vol}_n(C_i)$

Pyramids

$A \subseteq \mathbb{R}^n$ affine d-subspace

$P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1,\ 1 \leq i \leq m\}$ centered

$P^{(1)} = \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq 1,\ \langle a_i^{(1)}, x \rangle \leq 1,\ 1 \leq i \leq m\}$ centered

$\leadsto \mathrm{vol}_{n+1}(P^{(1)}) = \frac{n+2}{n+1} \mathrm{vol}_n(P)$ and $\mathrm{vol}_{n+1}(C_i^{(1)}) = \mathrm{vol}_n(C_i)$

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Wlog, let $A = \operatorname{aff}\{a_i : i \in I\}$ for some $I \subseteq \{1, \dots, m\}$

Define $L^{(1)} = \operatorname{lin}\{a_i^{(1)} : i \in I\} \leadsto$ linear $(d+1)$-space

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$$\Rightarrow \sum_{i \in I} \mathrm{vol}\, C_i = \sum_{i \in I} \mathrm{vol}\, C_i^{(1)} \le \frac{d+1}{n+1} \cdot \mathrm{vol}(P^{(1)}) = \frac{d+1}{n+1}\,\frac{n+2}{n+1}\,\mathrm{vol}\, P$$

Linear SC for $P^{(1)}$ and $L^{(1)}$

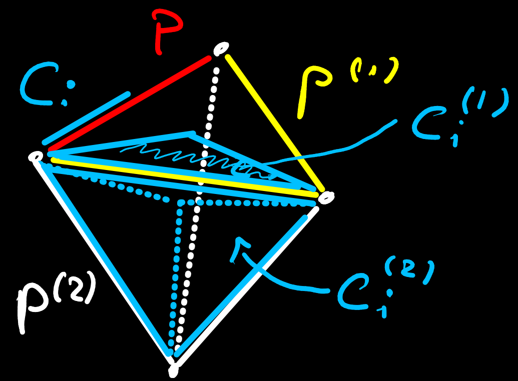
Pyramids over pyramids

→ Iterate the procedure to obtain centered pyramids $P^{(j)} \subseteq \mathbb{R}^{n+j}$ with

i) $\mathrm{vol}_{n+j}(P^{(j)}) = \frac{n+j+1}{n+j}\,\mathrm{vol}_{n+j-1}(P^{(j-1)}) = \dots = \frac{n+j+1}{n+1}\,\mathrm{vol}_n(P)$

ii) $\mathrm{vol}_{n+j}(C_i^{(j)}) = \dots = \mathrm{vol}_n(C_i)$

Cone wrt $a_i^{(j)}$ in $P^{(j)}$



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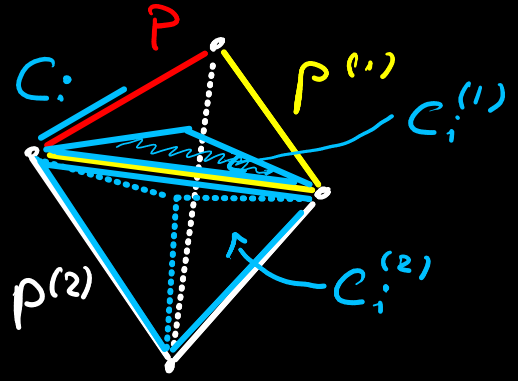

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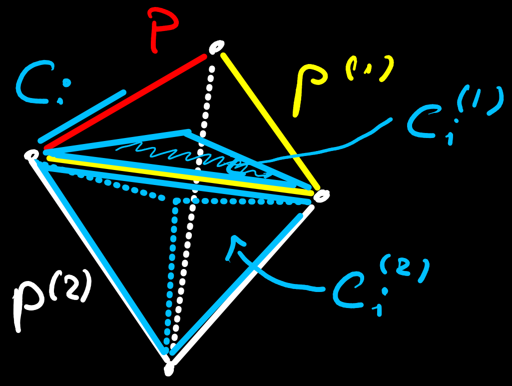

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$$\Rightarrow \sum_{i\in I} \mathrm{vol}_n(C_i) = \sum_{i \in I} \mathrm{vol}_{n+j}\left(C_i^{(j)}\right) \leq \frac{d+1}{n+j} \cdot \mathrm{vol}_{n+j}\left(P^{(j)}\right) = \frac{d+1}{n+1} \cdot \underbrace{\frac{n+j+1}{n+j}\,\mathrm{vol}_n(P)}_{\xrightarrow{(j\to\infty)}\ 1}$$

(LSC) in $\mathbb{R}^{n+j}$ for $P^{(j)}$ & $L^{(j)}$

Equality in subspace concentration conditions!

P centered. A, L d-spaces
(affine / linear)

Henk, Linke:

$$\sum_{i: a_i \in L} \operatorname{vol} C_i = \frac{d}{n} \operatorname{vol} P,$$

iff $\{a_1, \dots, a_m\} \subseteq L \cup \bar{L}$, for some complementary linear $(n-d)$-space $\bar{L}$.

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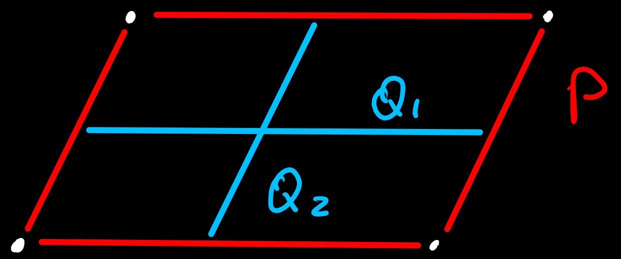

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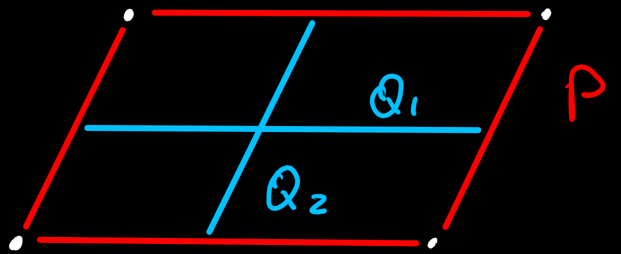

Wu: (for P reflexive & smooth)

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iff $\{a_1, \dots, a_m\} \subseteq A \cup \bar{A}$, for some complementary affine $(n-d-1)$-space $\bar{A}$.

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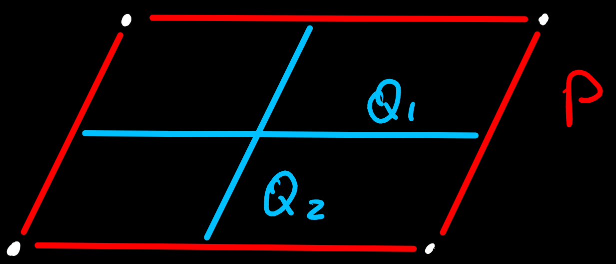

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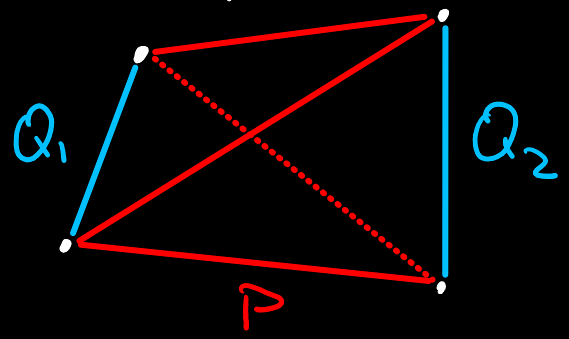

Equality in subspace concentration conditions!

P centered, A, L d-spaces (affine / linear)

Henk, Linke:

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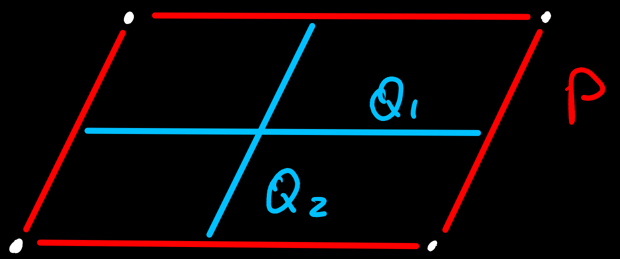

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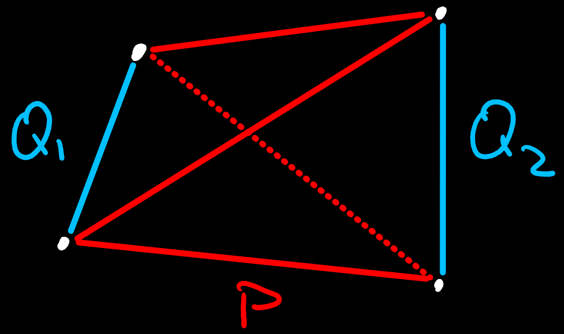

A simple polytope is a join, iff it is a simplex!

Back to the special case!

Wlog $\mathrm{vol}(P) = 1$

Let P be centered and $H = \mathrm{aff}\{a_i : v \in F_i\}$ for some vertex $v \in P$.

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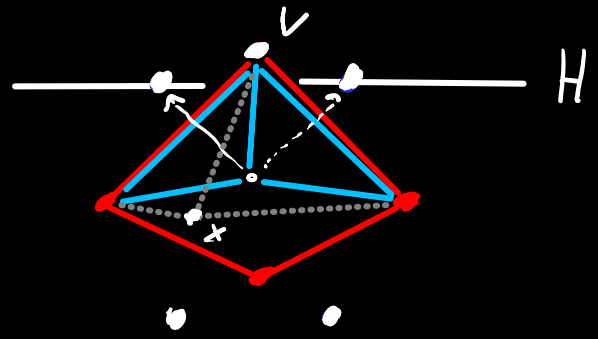
affine function



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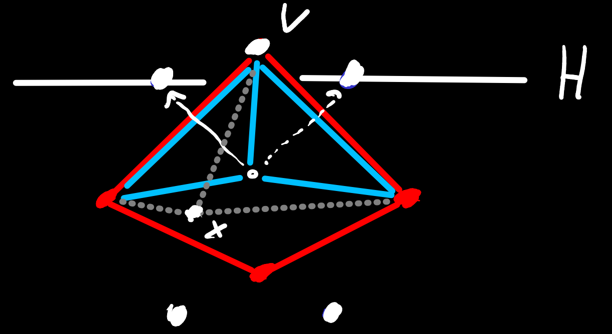

$$= 1 - \int_0^1 \mathrm{vol}\Big(\underbrace{P \cap \overbrace{\{f \leq t\}}^{\text{half-space}}}_{=:\, P_t}\Big) \, dt$$

<u>Observe</u>: $P_t = P \;\; \forall t \geq \max(P)$ and $P_0 = \{v\}$ by definition of v & H

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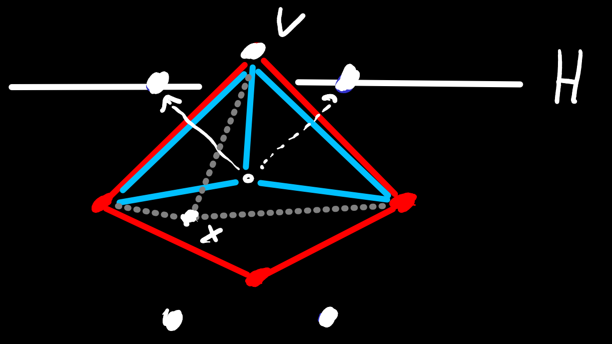

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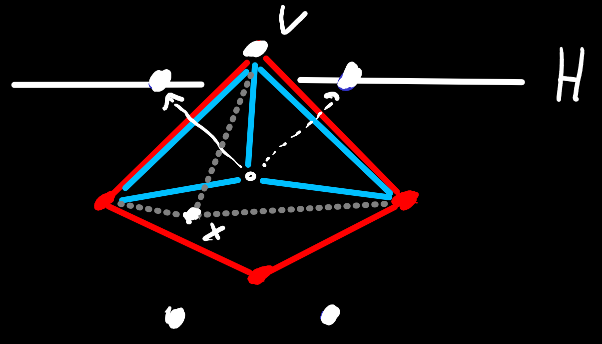

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$\Rightarrow \sum_{i: a_i \in H} \mathrm{vol}(C_i) \le 1 - (1-m) - \int_0^m \left(\frac{t}{m}\right)^n dt \le \frac{n}{n+1}$

Back to the special case

We obtain $\sum_{i : a_i \in H} \mathrm{vol}(C_i) \leq \frac{n}{n+1} \mathrm{vol}(P)$ with equality,

only if $\mathrm{vol}(P \cap \{f \leq t\}) = t^n \mathrm{vol}(P)$, $\forall t \in [0,1]$.

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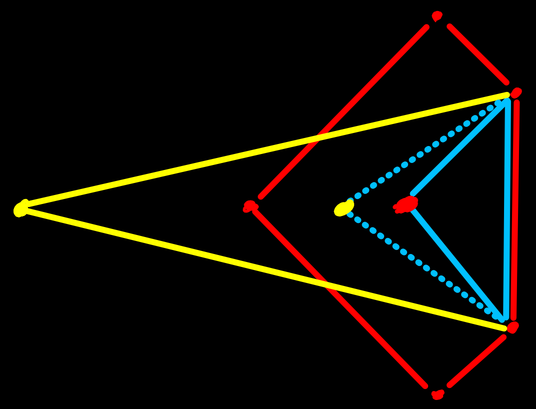
Open question: Can we generalize this argument to ...

i) k-faces instead of vertices?

ii) arbitrary hyperplanes?

Concluding Remarks I

▷ Using a 'Grünbaum-type argument', we can characterize the equality case for $d = 0$.



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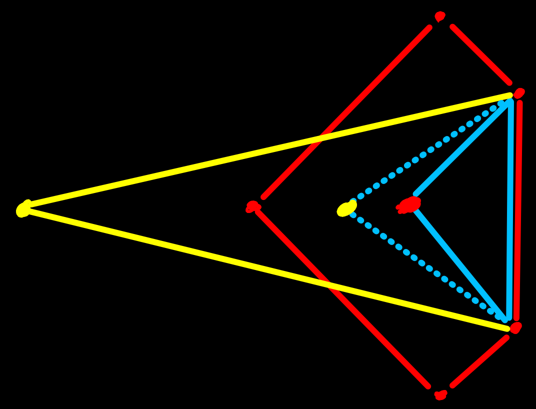
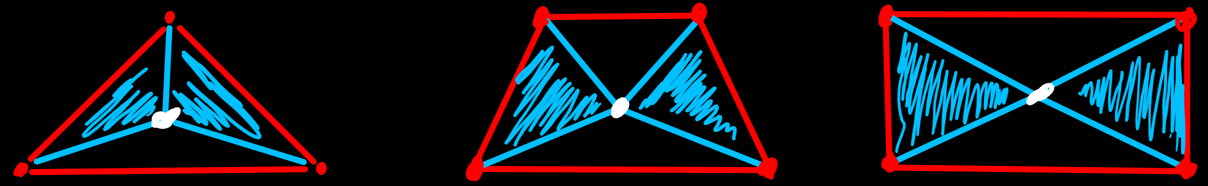


▷ If Wu's equality condition generalizes, then equality holds only for affine spaces of the form $A = \operatorname{aff}\{a_i : F \subseteq F_i\}$, where $F \subseteq P$ is a face.



→ Can one 'interpolate' between the affine and the linear case?

Thank you for your attention!