Stochastic Calculus

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The goal of these notes is to study Stochastic Quantization, and to take the shortest path to getting there.

In the context of ODEs we've seen success in treating autonomous quasi-linear equations which take the form

$$x^{(n)}(t) = F(x^{(0)}(t), \cdots, x^{(n-1)}(t)).$$

where $x : [0, \infty) \to \mathbb{R}^d$ can be thought of as the configuration of a physical system. In what follows, we will think of x as the position of some particle.

Let's specialize to the following 1-st order ODE:

$$x'(t) = V(x(t)) \tag{1}$$

where $V : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector field. Since we are working in a flat space, there is no difference between the space in which position and velocity vectors live, however this changes in general in Riemannian geometry for instance and should be kept in mind.

The dynamics induced by (1) are deterministic. We can view the time evolution of the system as an iterative scheme where at each time t the position $x(t)$ updates the velocity $x'(t) = V(x(t))$ at that time, which then in turn drives the energy further. But what if we were trying to model phenomena where at each time t, there were random effects at play which biased the velocity $V(x(t))$ to some $V(x(t)) + \xi(t)$, where ξ is supposed to reflect the random kicks to the particle which disturb the velocity. This is where we start our study of SDEs – Stochastic Differential equations.

The above suggests looking (more generally) at

$$X'(t) = V(X(t)) + B(X(t))\xi(t), \tag{2}$$

which at this point only formally makes sense. Here, ξ denotes *white noise* and $B : \mathbb{R}^d \to \mathbb{R}^d$ is the (current position-dependent) bias which controls how the noise disturbs the velocity.