Integral Calculus over Millenia

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0   Summary

Bringmann and Cao consider the following Stochastic Abelian-Higgs SPDE:

$$\begin{cases} \partial_t A & = -\,\mathbf{D}_A^* \, F_A - B(\mathbf{D}_A\,\phi, \phi) + \xi \\ \partial_t \phi & = -\,\mathbf{D}_A^* \, \mathbf{D}_A\,\phi - |\phi|^{q-1}\phi + \zeta \end{cases} \tag{1}$$

where

$$\begin{aligned} A &: [0, \infty) \times \mathbb{T}^2 \to \mathbb{R}^2 \\ \phi &: [0, \infty) \times \mathbb{T}^2 \to \mathbb{C} \\ B &: \mathbb{C} \times \mathbb{C} \to \mathbb{R} \text{ a certain bi-linear map} \\ \mathbf{D}_A &: \text{ covariant derivative} \\ F_A &: \text{ curvature tensor} \\ \xi, \zeta &: \text{ space-time white noise} \\ q &\geq 3 \text{ an odd integer.} \end{aligned}$$

Momentarily, so that everything is well defined, consider the following *smooth state-space* for (A, ϕ) at fixed time:

$$\mathcal{S} := C^\infty(\mathbb{T}^2 \to \mathbb{R}^2) \times C^\infty(\mathbb{T}^2 \to \mathbb{C}). \tag{2}$$

The *group of gauge transformations* is given by

$$\mathcal{G} := \{g \in C^\infty(\mathbb{R}^2 \to \mathbb{R}) : \nabla g \text{ and } e^{-ig} \text{ are periodic}\}, \tag{3}$$

which is an additive group with addition defined pointwise.

Now $\mathcal{G}$ defines a group action on $\mathcal{S}$ defined by

$$(A, \phi) \mapsto (A^g, \phi^g) := (A + \nabla g, e^{-ig}\phi). \tag{4}$$

The physical system under consideration can be in any state (A, ϕ). It turns out that all observables of this physical system remain invariant under action of $\mathcal{G}$ on the state. That

is, for any state (A, ϕ), all elements of the gauge orbit $\{(A^g, \phi^g) : g \in \mathcal{G}\}$ are physically indistinguishable. Therefore, the group action given by (4) is also called the *gauge symmetry* of the physical system, and we call any two states in the same gauge orbit *gauge equivalent*.

For example, the energy of the system in state (A, ϕ) is given by

$$E(A, \phi) := \int_{\mathbb{T}^2} \left(\frac{|F_A|^2}{4} + \frac{|\mathbf{D}_A \phi|^2}{2} + \frac{|\phi|^{q+1}}{q+1} \right). \tag{5}$$

To understand energy, $\forall (A, \phi) \in \mathcal{S}$, we first define the covariant derivatives $(\mathbf{D}_A^j \phi)_{1 \leq j \leq 2}$ and curvature tensor $(F_A^{jk})_{1 \leq j,k \leq 2}$ by

$$\mathbf{D}_A^j \phi := \partial^j \phi + i A^j \phi, \tag{6}$$

$$F_A^{jk} := \partial^j A^k - \partial^k A^j. \tag{7}$$

Now in (5), $|F_A|^2 = F_{A,jk} F_A^{jk}$ and $|\mathbf{D}_A \phi|^2 = \overline{\mathbf{D}_{A,j} \phi} \mathbf{D}_A^j \phi$ where we follow the Einstein summation convention. For any $g \in \mathcal{G}, (A, \phi) \in \mathcal{S}$, it then follows that

$$
\begin{aligned}
\mathbf{D}_{A^g}^j \phi^g &= \partial^j \phi^g + i A^{g,j} \phi^g \\
&= \partial^j (e^{-ig} \phi) + i(A^j + \partial^j g) e^{-ig} \phi \\
&= e^{-ig} \partial^j \phi - \cancel{i e^{-ig} \partial^j g \phi} + \cancel{i e^{-ig} \partial^j g \phi} + i e^{-ig} A^j \phi \\
&= e^{-ig} [\partial^j \phi + i A^j \phi] \\
&= e^{-ig} \mathbf{D}_A^j \phi \\
&= (\mathbf{D}_A^j \phi)^g.
\end{aligned}
$$

This gives $\overline{\mathbf{D}_{A^g,j} \phi^g} = (\overline{\mathbf{D}_{A,j} \phi})^{-g}$, and hence $|\mathbf{D}_{A^g} \phi^g|^2 = |\mathbf{D}_A \phi|^2$. Also,

$$
\begin{aligned}
F_{A^g}^{j,k} &= \partial^j A^{g,k} - \partial^k A^{g,j} \\
&= \partial^j A^k - \partial^k A^j + \partial^j \partial^k g - \partial^k \partial^j g \\
&= F_A^{j,k}
\end{aligned}
$$

which readily implies $|F_{A^g}|^2 = |F_A|^2$. Also, trivially, $|\phi^g|^{q-1} = |\phi|^{q-1}$. Thus we find that

$$\forall g \in \mathcal{G}, (A, \phi) \in \mathcal{S}, \, E(A^g, \phi^g) = E(A, \phi). \tag{8}$$

Therefore, it's advantageous to fix a gauge orbit representative in the state space which makes computations the easiest, since the physics doesn't change. For the purposes of analysing (1), the Coulomb gauge is particularly effective. It sets

$$\partial_j A^j = 0 \tag{9}$$

For this choice to be allowed, there must be a unique element in every gauge orbit satisfying (9). For any A, we have that $\partial_j A^{g,j} = \partial_j (A^j + \partial^j g) = \partial_j A^j + \Delta g$, and thus we reduce to finding $g : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla g, e^{-ig}$ are periodic and

$$\Delta g = -\nabla \cdot A.$$

This is the Poisson equation. Restricting g to the torus: $g|_{\mathbb{T}^2}$, we see that it will satisfy the above equation iff $\int_{\mathbb{T}^2} -\nabla \cdot A(x)\, dx = 0$, which is always the case by the Divergence theorem. Next we can extend the solution on the torus $g|_{\mathbb{T}^2}$ periodically to $g : \mathbb{R}^2 \to \mathbb{R}$. Then, ∇g and e^{-ig} will also be automatically periodic.

However, this choice of g is not unique – $g + h : \mathbb{R}^2 \to \mathbb{R}$ will satisfy the same conditions as long as $\Delta h = 0$ and $\nabla h, e^{-ih}$ are periodic. The following satisfies these properties:

$$\forall n \in \mathbb{Z}^2 \text{ set } h_n(x) = n \cdot x. \tag{10}$$

In fact, this basically classifies such h. To see this, note that h harmonic $\implies \partial^j h$ harmonic, which is also periodic, and therefore by Liouville's theorem $\partial^j h$ must be a constant. Thus $h(x) = a \cdot x + b$ where $a \in \mathbb{R}^2, b \in \mathbb{R}$. Next we need $e^{ih(x)} = e^{i(a \cdot x + b)}$ to be periodic, which forces $a \in \mathbb{Z}^2$. Thus any such $h : \mathbb{R}^2 \to \mathbb{R}$ will be of the form $n \cdot x + b$; $n \in \mathbb{Z}^2, b \in \mathbb{R}$.

Summarily, each gauge orbit has infinitely many states satisfying the Coulomb gauge condition. For such a state (A, ϕ), these states are precisely given by (A^h, ϕ^h) where $h = n \cdot x + b$. Since the constant term in h only affects the state by multiplying ϕ by a constant, we ignore it. The remaining h then form a subgroup which can then be identified with $\mathbb{Z}^2$:

$$(A, \phi) \mapsto (A^h, \phi^h) = (A + n, e^{-in \cdot x}\phi). \tag{11}$$

The freedom left in choosing representatives of gauge orbits after imposing a condition is called *gauge freedom*. If the condition is stringent enough to identify a unique representative (i.e., removing any gauge freedom), we call the process *gauge fixing*. In the above context, clearly the gauge freedom is discrete and finite dimensional. The former makes it hard to remove it by imposing a further condition, and the latter means it won't be that hard to control the norms we will need for showing well-posedness of (1) in each dimension of the gauge freedom. This is accomplished by considering *gauge invariant norms* defined below.

Under the Coulomb gauge condition, the non-linearity in (1) exhibits a *null structure* which makes the analysis easier. We modify our state space to always satisfy the Coulomb gauge condition:

$$\mathcal{S}_C := \{(A, \phi) \in \mathcal{S} : \partial_j A^j = 0\}. \tag{12}$$

The equation now reads:

$$\begin{cases} \partial_t A &= \Delta A - P_\perp \Im(\bar{\phi}\, \mathbf{D}_A\, \phi) + P_\perp \xi \\ \partial_t \phi &= \mathbf{D}_A^j\, \mathbf{D}_{A,j}\, \phi - |\phi|^{q-1}q + \zeta \end{cases}. \tag{13}$$

As stated above, (13) is hard to make sense of due to the low spatial regularity white noise terms. Therefore, we consider a smoothened version of the equation as follows:

$$\begin{cases} \partial_t A_{\leq N} &= \Delta A_{\leq N} - P_\perp \Im(\overline{\phi_{\leq N}}\, \mathbf{D}_{A_{\leq N}}\, \phi_{\leq N}) + C_g A_{\leq N} + P_\perp \xi_{\leq N} \\ \partial_t \phi_{\leq N} &= \left(\mathbf{D}_{A_{\leq N}}^j\, \mathbf{D}_{A_{\leq N},j} + 2\sigma_{\leq N}^2\right)\phi_{\leq N} - \,:\!|\phi_{\leq N}|^{q-1}\phi\!:\, + \zeta_{\leq N} \end{cases}, \tag{14}$$

where $A_{\leq N}$ and $\phi_{\leq N}$ will be solutions corresponding to the smoothened out noise. The renormalization term $c_g A_{\leq N}$ (with $C_g = \frac{1}{8\pi}$) is added to make the limiting solution under

N gauge covariant under the discrete gauge transformation ([12](#)); it cancels a resonance in the derivative non-linearity. The other renormalization therm $2\sigma_{\leq N}^2 \phi_{\leq N}$ has been added to make the solution finite. The wick-ordered non-linearity $: |\phi_{\leq N}|^{q-1}\phi_{\leq N} :$ is considered so that the products make sense even at low regularities.

The covariant derivatives, unlike the regular derivatives, are non-commutative. From the above definitions, this can be seen as follows:

$$\begin{aligned}(\mathbf{D}_A^j\,\mathbf{D}_A^k - \mathbf{D}_A^k\,\mathbf{D}_A^j)\phi &= \mathbf{D}_A^j(\partial^k\phi + iA^k\phi) - \mathbf{D}_A^k(\partial^j\phi + iA^j\phi)\\ &= \cancel{\partial^j\partial^k\phi} + iA^j\partial^k\phi + i\partial^j(A^k\phi) - \cancel{A^jA^k\phi}\\ &\quad - \cancel{\partial^k\partial^j\phi} - iA^k\partial^j\phi - i\partial^k(A^j\phi) + \cancel{A^kA^j\phi}\end{aligned}$$

where the terms $\partial^j\partial^k\phi - \partial^k\partial^j\phi$ cancel off by interchanging order of derivatives, and the terms $-A^jA^k\phi + A^kA^j\phi$ cancel off since these are scalar functions that commute.

$$\begin{aligned}&= i\partial^j(A^k\phi) - iA^k\partial^j\phi\\ &\quad - [i\partial^k(A^j\phi) - iA^j\partial^k\phi]\end{aligned}$$

which simplifies using chain rule:

$$\begin{aligned}&= i[\partial^j A^k - \partial^k A^j]\phi\\ &= iF_A^{jk}\phi.\end{aligned}$$

From this calculation it follows that the curvature tensor captures the non-commutativity of the covariant derivatives.