BINOMIAL THEOREM

1. BINOMIAL EXPRESSION :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x - y,\ xy + \frac{1}{x},\ \frac{1}{z} - 1,\ \frac{1}{(x-y)^{1/3}} + 3$ etc.

2. BINOMIAL THEOREM :

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM.**

If x, y $\in$ R and n $\in$ N, then :

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots\ldots + {}^nC_rx^{n-r}y^r + \ldots\ldots + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}y^r$$

This theorem can be proved by induction.

Observations :

(a) The number of terms in the expansion is (n+1) i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients of the terms (**$^nC_0$, $^nC_1$.....**) equidistant from the beginning and the end are equal. i.e. $^nC_r = {}^nC_{r-1}$

(d) Symbol ^nC_r can also be denoted by $\begin{pmatrix} n \\ r \end{pmatrix}$, **C(n, r)** or A_r^n.

Some important expansions :

(i) $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots.. + {}^nC_nx^n$.

(ii) $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + \ldots\ldots.. + (-1)^n \cdot {}^nC_nx^n$.

Note : The coefficient of x^r in $(1 + x)^n = {}^nC_r$ & that in $(1-x)^n = (-1)^r \cdot {}^nC_r$

Illustration 1 : Expand : $(y + 2)^6$.

Solution : $^6C_0y^6 + {}^6C_1y^5.2 + {}^6C_2y^4.2^2 + {}^6C_3y^3.2^3 + {}^6C_4y^2.\ 2^4 + {}^6C_5y^1\ .\ 2^5 + {}^6C_6\ .\ 2^6$.

$= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64$.

Illustration 2 : Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution : $^7C_0,\ {}^7C_1\left(-\frac{2y^2}{5}\right),\ {}^7C_2\left(-\frac{2y^2}{5}\right)^2,\ {}^7C_3\left(-\frac{2y^2}{5}\right)^3$

Illustration 3 : If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and -6 respectively then m is - **[JEE 99]**

(A) 6 (B) 9 (C) 12 (D) 24

Solution : $(1 + x)^m (1 - x)^n = \left[1 + mx + \frac{(m)(m-1).x^2}{2} + \ldots\ldots\right]\left[1 - nx + \frac{n(n-1)}{2}x^2 + \ldots\ldots\right]$

Coefficient of x = m – n = 3 ....(i)

Coefficient of $x^2 = -mn + \frac{n(n+1)}{2} + \frac{m(m-1)}{2} = -6$ ....(ii)

Solving (i) and (ii), we get

m = 12 and n = 9.

Do yourself - 1 :

(i) Expand $\left(3x^2 - \frac{x}{2}\right)^5$ **(ii)** Expand $(y + x)^n$

Pascal's triangle :

$(x+y)^0$	1	1
$(x+y)^1$	$x + y$	1 1
$(x+y)^2$	$x^2 + 2xy + y^2$	1 2 1
$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1
$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^3$	1 4 6 4 1

Pascal's triangle

(i) **Pascal's triangle -** A triangular arrangement of numbers as shown. The numbers give the binomial coefficients for the expansion of $(x + y)^n$. The first row is for n = 0, the second for n = 1, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

(ii) Pascal triangle is formed by binomial coefficient.

(iii) The number of terms in the expansion of $(x+y)^n$ is (n + 1) i.e. one more than the index.

(iv) The sum of the indices of x & y in each term is n.

(v) Power of first variable (x) decreases while of second variable (y) increases.

(vi) Binomial coefficients are also called **combinatorial coefficients.**

(vii) Binomial coefficients of the terms equidistant from the begining and end are equal.

(viii) r^{th} term from the beginning in the expansion of $(x + y)^n$ is same as r^{th} term from end in the expansion of $(y + x)^n$.

(ix) r^{th} term from the end in $(x + y)^n$ is **$(n - r + 2)^{th}$ term from the beginning**.

3. IMPORTANT TERMS IN THE BINOMIAL EXPANSION :

(a) General term: The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

Illustration 4 : Find : (a) The coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

Solution : (a) In the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$, the general term is :

$$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r.\frac{a^{11-r}}{b^r}.x^{22-3r}$$

putting $22 - 3r = 7$

$\therefore$ $3r = 15$ $\Rightarrow$ $r = 5$

$\therefore$ $T_6 = {}^{11}C_5\frac{a^6}{b^5}.x^7$

Hence the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is ${}^{11}C_5a^6b^{-5}$. **Ans.**

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

(b) In the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$, general term is :

$$T_{r+1} = {}^{11}C_r(ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r = (-1)^r\,{}^{11}C_r\frac{a^{11-r}}{b^r}.x^{11-3r}$$

putting $11 - 3r = -7$

$\therefore$ $3r = 18$ $\Rightarrow$ $r = 6$

$\therefore$ $T_7 = (-1)^6.\ {}^{11}C_6\frac{a^5}{b^6}.x^{-7}$

Hence the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6a^5b^{-6}$. **Ans.**

Also given :

Coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$

$\Rightarrow$ ${}^{11}C_5a^6b^{-5} = {}^{11}C_6a^5b^{-6}$

$\Rightarrow$ $ab = 1$ $(\because\ {}^{11}C_5 = {}^{11}C_6)$

which is the required relation between a and b. **Ans.**

Illustration 5 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution : The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\,3^{\frac{1000-r}{2}}2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501 **Ans.**

(b) Middle term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\ x^{n/2}.\ y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Important Note :

Middle term has greatest binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$\Rightarrow$ ^nC_r will be maximum
- When $r=\frac{n}{2}$ if n is even
- When $r=\frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd

$\Rightarrow$ The term containing greatest binomial coefficient will be middle term in the expansion of $(1+x)^n$

Illustration 6 : Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Solution : The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ are two middle terms. They are given by T_5 and T_6

$\therefore \quad T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = {}^9C_4 3^5 x^5 . \frac{x^{12}}{6^4} = \frac{9.8.7.6}{1.2.3.4} . \frac{3^5}{2^4.3^4} x^{17} = \frac{189}{8} x^{17}$

and $\quad T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4 . x^4 . \frac{x^{15}}{6^5} = \frac{-9.8.7.6}{1.2.3.4} . \frac{3^4}{2^5.3^5} x^{19} = -\frac{21}{16} x^{19}$ **Ans.**

(c) Term independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Illustration 7 : The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) $^{10}C_1$ (D) none of these

Solution : General term in the expansion is

$^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10} . \frac{3^{5-r}}{2^{\frac{10-r}{2}}}$ For constant term, $\frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$

which is not an integer. Therefore, there will be no constant term. **Ans. (D)**

Do yourself - 2 :

(i) Find the 7^{th} term of $\left(3x^2-\frac{1}{3}\right)^{10}$

(ii) Find the term independent of x in the expansion : $\left(2x^2-\frac{3}{x^3}\right)^{25}$

(iii) Find the middle term in the expansion of : (a) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ (b) $\left(2x^2-\frac{1}{x}\right)^7$

(d) Numerically greatest term :

Let numerically greatest term in the expansion of $(a + b)^n$ be T_{r+1}.

$$\Rightarrow \begin{cases} |T_{r+1}| \geq |T_r| \\ |T_{r+1}| \geq |T_{r+2}| \end{cases} \text{ where } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Solving above inequalities we get $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} - 1 \leq r \leq \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

Case I : When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is an integer equal to m, then T_m and T_{m+1} will be numerically greatest term.

Case II : When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Illustration 8 : Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \dfrac{1}{5}$

Solution : Using $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} - 1 \leq r \leq \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\dfrac{3}{-5x}\right|} - 1 \leq r \leq \frac{11+1}{1+\left|\dfrac{3}{-5x}\right|}$$

solving we get $2 \leq r \leq 3$

$\therefore$ r = 2, 3

so, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when r = 2)

$T_3 = {}^{11}C_2 . 3^9 (-5x)^2 = 55.3^9 = T_4$

From above we say that the value of both greatest terms are equal. **Ans.**

Illustration 9 : Given T_3 in the expansion of $(1 - 3x)^6$ has maximum numerical value. Find the range of 'x'.

Solution : Using $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} - 1 \leq r \leq \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

$$\frac{6+1}{1+\left|\dfrac{1}{-3x}\right|} - 1 \leq 2 \leq \frac{7}{1+\left|\dfrac{1}{-3x}\right|}$$

Let $|x| = t$

$$\frac{21t}{3t+1} - 1 \le 2 \le \frac{21t}{3t+1}$$

$$\begin{cases} \dfrac{21t}{3t+1} \le 3 \\ \dfrac{21t}{3t+1} \ge 2 \end{cases} \Rightarrow \begin{cases} \dfrac{4t-1}{3t+1} \le 0 \Rightarrow t \in \left[-\dfrac{1}{3}, \dfrac{1}{4}\right] \\ \dfrac{15t-2}{3t+1} \ge 0 \Rightarrow t \in \left(-\infty, -\dfrac{1}{3}\right] \cup \left[\dfrac{2}{15}, \infty\right) \end{cases}$$

Common solution $t \in \left[\dfrac{2}{15}, \dfrac{1}{4}\right] \Rightarrow x \in \left[-\dfrac{1}{4}, -\dfrac{2}{15}\right] \cup \left[\dfrac{2}{15}, \dfrac{1}{4}\right]$

Do yourself -3 :

(i) Find the numerically greatest term in the expansion of $(3 - 2x)^9$, when $x = 1$.

(ii) In the expansion of $\left(\dfrac{1}{2} + \dfrac{2x}{3}\right)^n$ when $x = -\dfrac{1}{2}$, it is known that 3rd term is the greatest term. Find the possible integral values of n.

4. PROPERTIES OF BINOMIAL COEFFICIENTS :

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r \; ; n \in N \qquad \ldots(i)$$

where $C_0, C_1, C_2, \ldots\ldots C_n$ are called combinatorial (binomial) coefficients.

(a) The sum of all the binomial coefficients is 2^n.

Put x = 1, in (i) we get

$$C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n \Rightarrow \sum_{r=0}^{n} {}^nC_r = 0 \qquad \ldots(ii)$$

(b) Put x=–1 in (i) we get

$$C_0 - C_1 + C_2 - C_3 \ldots\ldots + C_n = 0 \Rightarrow \sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0 \qquad \ldots(iii)$$

(c) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii), $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(f) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r}.\dfrac{n-1}{r-1}\,{}^{n-2}C_{r-2} = \ldots\ldots = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots 1}$

(g) ${}^nC_r = \dfrac{r+1}{n+1}.\,{}^{n+1}C_{r+1}$

Illustration 10 : Prove that : $^{25}C_{10} + {}^{24}C_{10} + ... + {}^{10}C_{10} = {}^{26}C_{11}$

Solution : LHS = $^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + ... + {}^{25}C_{10}$

$\Rightarrow \quad {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + ... + {}^{25}C_{10}$

$\Rightarrow \quad {}^{12}C_{11} + {}^{12}C_{10} + ... + {}^{25}C_{10}$

$\Rightarrow \quad {}^{13}C_{11} + {}^{13}C_{10} + ... {}^{25}C_{10}$

and so on. $\therefore$ LHS = $^{26}C_{11}$

Aliter :

LHS = coefficient of x^{10} in $\{(1 + x)^{10} + (1 + x)^{11} + ... (1+ x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1+x)^{10}\dfrac{\{1+x\}^{16}-1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\dfrac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$

$\Rightarrow$ coefficient of x^{11} in $\left[(1+x)^{26}-(1+x)^{10}\right] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Illustration 11 : A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select books is 63, find the value of n.

Solution : Given student selects at most n books from a collection of (2n + 1) books. It means that he selects one book or two books or three books or .... or n books. Hence, by the given condition-

$^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = 63$ ...(i)

But we know that

$^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_{2n+1} = 2^{2n+1}$ ...(ii)

Since $^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$, equation (ii) can also be written as

$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) +$

$({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + ... + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n}) = 2^{2n+1}$

$\Rightarrow 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n)$

$+ ({}^{2n+1}C_n + {}^{2n+1}C_{n-1} + ... + {}^{2n+1}C_2 + {}^{2n+1}C_1) = 2^{2n+1}$

$(\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$

$\Rightarrow 2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) = 2^{2n+1}$ [from (i)]

$\Rightarrow 2 + 2.63 = 2^{2n+1}$ $\quad\Rightarrow 1 + 63 = 2^{2n}$

$\Rightarrow 64 = 2^{2n} \Rightarrow 2^6 = 2^{2n}$ $\quad\therefore 2n = 6$

Hence, n = 3. **Ans.**

Illustration 12 : Prove that :

(i) $C_1 + 2C_2 + 3C_3 + ... + nC_n = n \,.\, 2^{n-1}$

(ii) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + ... + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Solution : (i) L.H.S. $= \sum_{r=1}^{n} r.{}^{n}C_r = \sum_{r=1}^{n} r.\dfrac{n}{r}.{}^{n-1}C_{r-1}$

$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1} = n.\left[{}^{n-1}C_0 + {}^{n-1}C_1 + ... + {}^{n-1}C_{n-1}\right]$

$= n \,.\, 2^{n-1}$

Aliter : (Using method of differentiation)

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$ ....(A)

Differentiating (A), we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + n.C_nx^{n-1}.$

Put x = 1,

$C_1 + 2C_2 + 3C_3 + ... + n.C_n = n.2^{n-1}$

(ii) L.H.S. $= \sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} \frac{n+1}{r+1}\,{}^nC_r$

$= \frac{1}{n+1}\sum_{r=0}^{n} {}^{n+1}C_{r+1} = \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right] = \frac{1}{n+1}\left[2^{n+1} - 1\right]$

Aliter : (Using method of integration)

Integrating (A), we get

$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$ (where C is a constant)

Put x = 0, we get, $C = -\frac{1}{n+1}$

$\therefore \quad \frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$

Put x = 1, we get $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$

Put x = –1, we get $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... = \frac{1}{n+1}$

Illustration 13 : If $(1 + x)^n = \sum_{r=0}^{n} {}^nC_rx^r$, then prove that $C_1^2 + 2.C_2^2 + 3.C_3^2 + ... + n.C_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$

Solution : $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_2x^3 + ... + C_n\,x^n$ ....(i)

Differentiating both the sides, w.r.t. x, we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_2x^2 + ... + n.C_nx^{n-1}$ ....(ii)

also, we have

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n$ ....(iii)

Multiplying (ii) & (iii), we get

$(C_1 + 2C_2x + 3C_3x^2 + ... + C_nx^{n-1})(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ...+ C_n) = n(1 + x)^{2n-1}$

Equating the coefficients of x^{n-1}, we get

$C_1^2 + 2C_2^2 + 3C_3^2 + ... + n.C_n^2 = n\,.\,{}^{2n-1}C_{n-1} = \frac{(2n-1)!}{((n-1)!)^2}$ **Ans.**

Illustration 14 : Prove that : $C_0 - 3C_1 + 5C_2 - ....(-1)^n(2n + 1)C_n = 0$

Solution : $T_r = (-1)^r(2r + 1)^nC_r = 2(-1)^rr\,.\,{}^nC_r + (-1)^r\,{}^nC_r$

$\Sigma T_r = 2\sum_{r=1}^{n} (-1)^r.r.\frac{n}{r}.{}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r\;{}^nC_r = 2\sum_{r=1}^{n} (-1)^r.{}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r\,.\,{}^nC_r$

$= 2\left[{}^{n-1}C_0 - {}^{n-1}C_1 + ...\right] + \left[{}^nC_0 - {}^nC_1 + ...\right] = 0$

Illustration 15 : Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Solution : $(1 - x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - ....+(-1)^n\ {}^{2n}C_{2n}x^{2n}$ ....(i)

and $(x + 1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} +...+{}^{2n}C_{2n}$ ....(ii)

Multiplying (i) and (ii), we get

$(x^2 - 1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x + ... + (-1)^n\ {}^{2n}C_{2n}x^{2n}) \times ({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + ... + {}^{2n}C_{2n})$ ....(iii)

Now, coefficient of x^{2n} in R.H.S.

$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ...... + (-1)^n ({}^{2n}C_{2n})^2$

$\because$ General term in L.H.S., $T_{r+1} = {}^{2n}C_r(x^2)^{2n-r}(-1)^r$

Putting $2(2n - r) = 2n$

$\therefore$ $r = n$

$\therefore$ $T_{n+1} = {}^{2n}C_nx^{2n}(-1)^n$

Hence coeffiecient of x^{2n} in L.H.S. $= (-1)^n.{}^{2n}C_n$

But (iii) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$\Rightarrow$ $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Illustration 16 : Prove that : ${}^nC_0.{}^{2n}C_n - {}^nC_1.\ {}^{2n-2}Cn_n + {}^nC_2.{}^{2n-4}Cn_n + ... = 2^n$

Solution : L.H.S. = Coefficient of x^n in $[{}^nC_0(1 + x)^{2n} - {}^nC_1(1 + x)^{2n-2}\ ......]$

= Coefficient of x^n in $[(1 + x)^2 - 1]^n$

= Coefficient of x^n in $x^n(x + 2)^n = 2^n$

Illustration 17 : If $(1 + x)^n = C_0 + C_1\ x + C_2x^2 + ... + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represented by : $\sum\limits_{0\le i<}\sum\limits_{j\le n} C_iC_j$ is equal to $2^{2n-1} - \dfrac{2n!}{2.n!n!}$

Solution : Since $(C_0 + C_1 + C_2 +.....+ C_{n-1} + C_n)^2$

$= C_0^2 + C_1^2 + C_2^2 + ... + C_{n-1}^2 + C_n^2 + 2(C_0C_1 + C_0C_2 + C_0C_3 + ... + C_0C_n + C_1C_2 + C_1C_3 + ...$

$+ C_1C_n + C_2C_3 + C_2C_4 +...+ C_2C_n + ...+ C_{n-1}C_n)$

$(2^n)^2 = {}^{2n}C_n + 2 \sum\limits_{0\le i<j\le n}\sum C_iC_j$

Hence $\sum\limits_{0\le i<j\le n}\sum C_iC_j = 2^{2n-1} - \dfrac{2n!}{2.n!n!}$ **Ans.**

Illustration 18 : If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then prove that $\sum\limits_{0\le i<j\le n}\sum \left(C_i + C_j\right)^2 = (n-1)^{2n}C_n + 2^{2n}$

Solution : L.H.S. $\sum\limits_{0\le i<j\le n}\sum \left(C_i + C_j\right)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + ... + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + ...$

$+ (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + ... + (C_2 + C_n)^2 + ... + (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) + 2 \sum\limits_{0\le i<j\le n}\sum C_iC_j$

$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \dfrac{2n!}{2.n!n!}\right\}$ {from Illustration 17}

$= n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n - 1).\ {}^{2n}C_n + 2^{2n}$ = R.H.S.

Do yourself - 4 :

(i) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+........+\binom{n}{n}=$

(A) 2^{n-1} (B) ${}^{2n}C_n$ (C) 2^n (D) 2^{n+1}

(ii) If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, $n \in N$. Prove that

(a) $3C_0 - 8C_1 + 13C_2 - 18C_3 + ...$ upto $(n+1)$ terms $= 0$, if $n \geq 2$.

(b) $2C_0 + 2^2\frac{C_1}{2} + 2^3\frac{C_2}{3} + 2^4\frac{C_3}{4} + ... + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(c) $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + ... + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{((n+1)!)^2}$

5. MULTINOMIAL THEOREM :

Using binomial theorem, we have $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}a^r$, $n \in N$

$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!}x^{n-r}a^r = \sum_{r+s=n} \frac{n!}{r!s!}x^s a^r$, where $s + r = n$

This result can be generalized in the following form.

$$(x_1 + x_2 + ... + x_k)^n = \sum_{r_1+r_2+....+r_k=n} \frac{n!}{r_1!r_2!....r_k!} x_1^{r_1} x_2^{r_2} ....x_k^{r_k}$$

The general term in the above expansion $\frac{n!}{r_1!r_2!r_3!.....r_k!} . x_1^{r_1} x_2^{r_2} x_3^{r_3} ......x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + ... + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular cases :

(i) $(x+y+z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!}x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x+y+z+u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!}x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Illustration 19 : Find the coefficient of $x^2 y^3 z^4 w$ in the expansion of $(x - y - z + w)^{10}$

Solution : $(x - y - z + w)^{10} = \sum_{p+q+r+s=10} \frac{n!}{p!q!r!s!}(x)^p(-y)^q(-z)^r(w)^s$

We want to get $x^2y^3z^4w$ this implies that $p = 2, q = 3, r = 4, s = 1$

$\therefore$ Coefficient of $x^2y^3z^4w$ is $\frac{10!}{2!\,3!\,4!\,1!}(-1)^3(-1)^4 = -12600$ **Ans.**

Illustration 20 : Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3.

Solution : Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $x^2y^3 = \dfrac{10!}{2!\times 3!\times 5!} = 2520$ **Ans.**

Illustration 21 : Find the coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$.

Solution : The general term in the expansion of $(2 - x + 3x^2)^6 = \dfrac{6!}{r!s!t!}2^r(-x)^s(3x^2)^t$,

where $r + s + t = 6$.

$= \dfrac{6!}{r!s!t!}2^r \times (-1)^s \times (3)^t \times x^{s+2t}$

For the coefficient of x^5, we must have $s + 2t = 5$.

But, $r + s + t = 6$,

$\therefore$ $s = 5 - 2t$ and $r = 1 + t$, where $0 \le r, s, t \le 6$.

Now $t = 0 \Rightarrow r = 1, s = 5$.

$t = 1 \Rightarrow r = 2, s = 3$.

$t = 2 \Rightarrow r = 3, s = 1$.

Thus, there are three terms containing x^5 and coefficient of x^5

$= \dfrac{6!}{1!\,5!\,0!}\times 2^1 \times (-1)^5 \times 3^0 + \dfrac{6!}{2!\,3!\,1!}\times 2^2 \times (-1)^3 \times 3^1 + \dfrac{6!}{3!\,1!\,2!}\times 2^3 \times (-1)^1 \times 3^2$

$= -12 - 720 - 4320 = -5052.$ **Ans.**

Illustration 22 : If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then prove that (a) $a_r = a_{2n-r}$ (b) $\sum_{r=0}^{n-1} a_r = \dfrac{1}{2}(3^n - a_n)$

Solution : (a) We have

$$\left(1+x+x^2\right)^n = \sum_{r=0}^{2n} a_r x^r \qquad ....(A)$$

Replace x by $\dfrac{1}{x}$

$$\therefore \quad \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r\left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad \left(x^2+x+1\right)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$$

$$\Rightarrow \quad \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r} \qquad \{\text{Using (A)}\}$$

Equating the coefficient of x^{2n-r} on both sides, we get

$a_{2n-r} = a_r$ for $0 \le r \le 2n$.

Hence $a_r = a_{2n-r}$.

(b) Putting x=1 in given series, then

$$a_0 + a_1 + a_2 + ...+ a_{2n} = (1+1+1)^n$$

$$a_0 + a_1 + a_2 + ...+ a_{2n} = 3^n \qquad ....(1)$$

But $a_r = a_{2n-r}$ for $0 \le r \le 2n$

$\therefore$ series (1) reduces to

$$2(a_0 + a_1 + a_2 + ...+ a_{n-1}) + a_n = 3^n.$$

$$\therefore \quad a_0 + a_1 + a_2 + ...+ a_{n-1} = \frac{1}{2}(3^n - a_n)$$

Do yourself - 5 :

(i) Find the coefficient of x^2y^5 in the expansion of $(3 + 2x - y)^{10}$.

6. APPLICATION OF BINOMIAL THEOREM :

Illustration 23 : If $\left(6\sqrt{6}+14\right)^{2n+1} = [N] + F$ and $F = N - [N]$; where [.] denotes greatest integer function, then NF is equal to

(A) 20^{2n+1} (B) an even integer (C) odd integer (D) 40^{2n+1}

Solution : Since $\left(6\sqrt{6}+14\right)^{2n+1} = [N] + F$

Let us assume that $f = \left(6\sqrt{6}-14\right)^{2n+1}$; where $0 \le f < 1$.

Now, $[N] + F - f = \left(6\sqrt{6}+14\right)^{2n+1} - \left(6\sqrt{6}-14\right)^{2n+1}$

$$= 2\left[{}^{2n+1}C_1\left(6\sqrt{6}\right)^{2n}(14) + {}^{2n+1}C_3\left(6\sqrt{6}\right)^{2n-2}(14)^3 + ...\right]$$

$\Rightarrow$ $[N] + F - f =$ even integer.

Now $0 < F < 1$ and $0 < f < 1$

so $-1 < F - f < 1$ and $F - f$ is an integer so it can only be zero

Thus $NF = \left(6\sqrt{6}+14\right)^{2n+1}\left(6\sqrt{6}-14\right)^{2n+1} = 20^{2n+1}$. **Ans. (A,B)**

Illustration 24 : Find the last three digits in 11^{50}.

Solution : Expansion of $(10 + 1)^{50} = {}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + ... + {}^{50}C_{48} 10^2 + {}^{50}C_{49} 10 + {}^{50}C_{50}$

$$= \underbrace{{}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + ...+ {}^{50}C_{47} 10^3}_{1000K} + 49 \times 25 \times 100 + 500 + 1$$

$\Rightarrow$ 1000 K + 123001

$\Rightarrow$ Last 3 digits are 001.

Illustration 25 : Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution : When 2222 is divided by 7 it leaves a remainder 3.

So adding & subtracting 3^{5555}, we get :

$$E = \underbrace{2222^{5555} - 3^{5555}}_{E_1} + \underbrace{3^{5555} + 5555^{2222}}_{E_2}$$

For E_1 : Now since 2222–3 = 2219 is divisible by 7, therefore E_1 is divisible by 7
($\because$ $x^n - a^n$ is divisible by x –a)
For E_2 : 5555 when devided by 7 leaves remainder 4.
So adding and subtracting 4^{2222}, we get :

$$E_2 = 3^{5555} + 4^{2222} + 5555^{2222} - 4^{2222}$$
$$= (243)^{1111} + (16)^{1111} + (5555)^{2222} - 4^{2222}$$

Again $(243)^{1111} + 16^{1111}$ and $(5555)^{2222} - 4^{2222}$ are divisible by 7
($\because$ $x^n + a^n$ is divisible by x + a when n is odd)
Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

Do yourself - 6 :

(i) Prove that $5^{25} - 3^{25}$ is divisible by 2.

(ii) Find the remainder when the number 9^{100} is divided by 8.

(iii) Find last three digits in 19^{100}.

(iv) Let $R = (8+3\sqrt{7})^{20}$ and [.] denotes greatest integer function, then prove that :

(a) [R] is odd (b) $R - [R] = 1 - \dfrac{1}{(8+3\sqrt{7})^{20}}$

(v) Find the digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$.

7. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots. \infty$ provided $|x| < 1$.

Note :

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. (n+1) & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots. {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots. \infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots. \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots. \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots. \infty$

(e) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots.. + \dfrac{(-1)^r(r+1)(r+2)}{2!}x^r + \ldots\ldots..$

(f) $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots.. + \dfrac{(r+1)(r+2)}{2!}x^r + \ldots\ldots..$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of 1/x, which then will be small.

8. APPROXIMATIONS :

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 ....$

If **x < 1**, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately. This is an approximate value of $(1 + x)^n$

Illustration 26 : If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

Solution : $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$

$= \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1-\frac{x}{8}-\frac{19}{12}x = 1-\frac{41}{24}x$ **Ans.**

Illustration 27 : The value of cube root of 1001 upto five decimal places is –

(A) 10.03333 (B) 10.00333 (C) 10.00033 (D) none of these

Solution : $(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\frac{1}{1000}\right)^{1/3} = 10\left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+.....\right\}$

$= 10\{1 + 0.0003333 - 0.00000011 + ...\} = 10.00333$ **Ans. (B)**

Illustration 28 : The sum of $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+....\infty$ is -

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Solution : Comparing with $1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

$nx = 1/4$ ....(i)

and $\frac{n(n-1)x^2}{2!} = \frac{1.3}{4.8}$

or $\frac{nx(nx-x)}{2!} = \frac{3}{32} \Rightarrow \frac{1}{4}\left(\frac{1}{4}-x\right) = \frac{3}{16}$ (by (i))

$\Rightarrow \left(\frac{1}{4}-x\right) = \frac{3}{4} \Rightarrow x = \frac{1}{4}-\frac{3}{4} = -\frac{1}{2}$ ....(ii)

putting the value of x in (i)

$n(-1/2) = 1/4 \Rightarrow n = -1/2$

$\therefore$ sum of series $= (1 + x)^n = (1 - 1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ **Ans. (A)**

9. EXPONENTIAL SERIES :

(a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(b) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(c) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$; where x may be any real or complex number & $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(d) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + ...\infty$, where $a > 0$

(e) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

10. LOGARITHMIC SERIES :

(a) $\ell n\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 < x \le 1$

(b) $\ell n\,(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 \le x < 1$

Remember : (i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\infty = \ell n\,2$ (ii) $e^{\ln x} = x$; for all $x > 0$

(iii) $\ell n2 = 0.693$ (iv) $\ell n10 = 2.303$

EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. If the coefficients of x^7 & x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is :

(A) 15 (B) 45 (C) 55 (D) 56

BT0001

2. Set of value of r for which, ${}^{18}C_{r-2} + 2.\,{}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains :

(A) 4 element (B) 5 elements (C) 7 elements (D) 10 elements

BT0002

3. If the constant term of the binomial expansion $\left(2x-\frac{1}{x}\right)^n$ is – 160, then n is equal to -

(A) 4 (B) 6 (C) 8 (D) 10

BT0003

4. The coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)....\left(x-\frac{1}{2^{49}}\right)$ is equal to -

(A) $-2\left(1-\frac{1}{2^{50}}\right)$ (B) +ve coefficient of x

(C) –ve coefficient of x (D) $-2\left(1-\frac{1}{2^{49}}\right)$

BT0004

5. The largest real value for x such that $\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{8}{3}$ is -

(A) $2\sqrt{2}-5$ (B) $2\sqrt{2}+5$ (C) $-2\sqrt{2}-5$ (D) $-2\sqrt{2}+5$

BT0005

6. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree

(A) 5 (B) 6 (C) 7 (D) 8

BT0006

7. Number of rational terms in the expansion of $\left(\sqrt{2}+\sqrt[4]{3}\right)^{100}$ is :

(A) 25 (B) 26 (C) 27 (D) 28

BT0007

8. Given $(1 - 2x + 5x^2 - 10x^3)(1 + x)^n = 1 + a_1x + a_2x^2 + ...$ and that $a_1^2 = 2a_2$ then the value of n is-

(A) 6 (B) 2 (C) 5 (D) 3

BT0008

9. The sum of the co-efficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$ is -

(A) 2.6^{10} (B) 3.6^{10} (C) 6^{11} (D) none

BT0009

10. Co-efficient of α^t in the expansion of ,

$(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha + q) + (\alpha + p)^{m-3}(\alpha + q)^2 + ...(\alpha + q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is :

(A) $\dfrac{{}^mC_t\left(p^t - q^t\right)}{p-q}$ (B) $\dfrac{{}^mC_t\left(p^{m-t} - q^{m-t}\right)}{p-q}$ (C) $\dfrac{{}^mC_t\left(p^t + q^t\right)}{p-q}$ (D) $\dfrac{{}^mC_t\left(p^{m-t} + q^{m-t}\right)}{p-q}$

BT0010

11. Let $\begin{pmatrix} n \\ k \end{pmatrix}$ represents the combination of 'n' things taken 'k' at a time, then the value of the sum

$\begin{pmatrix} 99 \\ 97 \end{pmatrix} + \begin{pmatrix} 98 \\ 96 \end{pmatrix} + \begin{pmatrix} 97 \\ 95 \end{pmatrix} + ... + \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ equals -

(A) $\begin{pmatrix} 99 \\ 97 \end{pmatrix}$ (B) $\begin{pmatrix} 100 \\ 98 \end{pmatrix}$ (C) $\begin{pmatrix} 99 \\ 98 \end{pmatrix}$ (D) $\begin{pmatrix} 100 \\ 97 \end{pmatrix}$

BT0011

[COMPREHENSION TYPE]

Paragraph for question nos. 12 to 14

If $n \in N$ and if $(1 + 4x + 4x^2)^n = \sum_{r=0}^{2n} a_r x^r$, where $a_0, a_1, a_2, ..., a_{2n}$ are real numbers.

12. The value of $2\sum_{r=0}^{n} a_{2r}$, is

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

BT0012

13. The value of $2\sum_{r=1}^{n} a_{2r-1}$, is-

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

BT0012

14. The value of a_{2n-1} is -

(A) 2^{2n} (B) $n. 2^{2n}$ (C) $(n - 1)2^{2n}$ (D) $(n + 1)2^{2n}$

BT0012

15. If $n \in N$ & n is even, then $\dfrac{1}{1.(n-1)!} + \dfrac{1}{3!.(n-3)!} + \dfrac{1}{5!.(n-5)!} + ... + \dfrac{1}{(n-1)!1!} =$

(A) 2^n (B) $\dfrac{2^{n-1}}{n!}$ (C) $2^n n!$ (D) none of these

BT0013

EXERCISE (O-2)

[ONE OR MORE THAN ONE CORRECT CHOICE TYPE]

1. If it is known that the third term of the binomial expansion $\left(x+x^{\log_{10} x}\right)^5$ is 10^6 then x is equal to-

(A) 10 (B) $10^{-5/2}$ (C) 100 (D) 5

BT0014

2. In the expansion of $\left(x^3+3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$

(A) there appears a term with the power x^2
(B) there does not appear a term with the power x^2
(C) there appears a term with the power x^{-3}
(D) the ratio of the co-efficient of x^3 to that of x^{-3} is 1/3

BT0015

3. In the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$, the term which does not contain x is-

(A) ${}^{11}C_4-{}^{10}C_3$ (B) ${}^{10}C_7$ (C) ${}^{10}C_4$ (D) ${}^{11}C_5-{}^{10}C_5$

BT0016

4. Let $(1+x^2)^2(1+x)^n = A_0+A_1x+A_2x^2+\ldots\ldots$ If A_0, A_1, A_2 are in A.P. then the value of n is-

(A) 2 (B) 3 (C) 5 (D) 7

BT0017

5. Consider $E=\left(\sqrt[8]{x}+\sqrt[5]{y}\right)^z = I+f, 0\le f<1$

(A) If x = 5, y = 2, z = 100, then number of irrational terms in expansion of E is 98
(B) If x = 5, y = 2, z = 100, then number of rational terms in expansion of E is 4
(C) If x = 16, y = 1 & z = 6, then I = 197
(D) If x = 16, y = 1 & z = 6, then $f=\left(\sqrt{2}-1\right)^6$

BT0018

6. Greatest term in the binomial expansion of $(a+2x)^9$ when a = 1 & $x=\frac{1}{3}$ is :

(A) 3^{rd} & 4^{th} (B) 4^{th} & 5^{th} (C) only 4^{th} (D) only 5^{th}

BT0019

7. Let $(5+2\sqrt{6})^n = p+f$ where $n,p \in N$ and $0<f<1$ then the value of $f^2-f+pf-p$ is -

(A) a natural number (B) a negative integer (C) a prime number (D) are irrational number

BT0020

8. If $(9+\sqrt{80})^n = I+f$ where I, n are integers and $0<f<1$, then -

(A) I is an odd integer
(B) I is an even integer
(C) $(I+f)(1-f)=1$
(D) $1-f=(9-\sqrt{80})^n$

BT0021

9. If $\sum_{r=1}^{10} r(r-1)\ {}^{10}C_r = k.\ 2^9$, then k is equal to-

(A) 10 (B) 45 (C) 90 (D) 100

BT0022

10. The sum $\frac{\binom{11}{0}}{1}+\frac{\binom{11}{1}}{2}+\frac{\binom{11}{2}}{3}+..........+\frac{\binom{11}{11}}{12}$ equals $\left(\text{where } \binom{n}{r} \text{denotes } {}^nC_r\right)$

(A) $\frac{2^{11}}{12}$ (B) $\frac{2^{12}}{12}$

(C) $\frac{2^{11}-1}{12}$ (D) $\frac{2^{12}-1}{12}$

BT0023

11. **Statement-1 :** The sum of the series ${}^nC_0.\ {}^mC_r + {}^nC_1.\ {}^mC_{r-1} + {}^nC_2.\ {}^mC_{r-2} +......+ {}^nC_r.{}^mC_0$ is equal to ${}^{n+m}C_r$, where ${}^nC_r$'s and ${}^mC_r$'s denotes the combinatorial coefficients in the expansion of $(1 + x)^n$ and $(1 + x)^m$ respectively.

Statement-2 : Number of ways in which r children can be selected out of (n + m) children consisting of n boys and m girls if each selection may consist of any number of boys and girls is equal to ${}^{n+m}C_r$.

(A) Statement-1 is true, statement-2 is true ; statement-2 is a correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true ; statement-2 is NOT a correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

BT0024

12. Which of the following statement(s) is/are correct ?

(A) $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+.......+\infty=4$

(B) Integral part of $\left(9+4\sqrt{5}\right)^n$, n ∈ N is even.

(C) $({}^nC_0 + {}^nC_1 + {}^nC_2 + ...+ {}^nC_n)^2 = 1 + {}^{2n}C_1 + {}^{2n}C_2 +.....+ {}^{2n}C_{2n}$

(D) $\frac{1}{(3+2x)^2}$ can be expanded as infinite series in ascending powers of x only if $|x|<\frac{2}{3}$.

BT0025

13. If for n ∈ I, n > 10; $1 + (1 + x) + (1 + x)^2 + ... + (1 + x)^n = \sum_{k=0}^{n} a_k.x^k$, $x \neq 0$ then

(A) $\sum_{k=0}^{n} a_k = 2^{n+1}$ (B) $a_{n-2}=\frac{n(n+1)}{2}$

(C) $a_p > a_{p-1}$ for $p < \frac{n}{2}$, p ∈ N (D) $(a_9)^2 - (a_8)^2 = {}^{n+2}C_{10}\ ({}^{n+1}C_{10} - {}^{n+1}C_9)$

BT0026

14. Let $P(n)=\sum_{r=0}^{n}\frac{(-1)^r r}{r+1}\,{}^nC_r$. Now which of the following holds good ?

(A) $|P_{10}|$ is harmonic mean of $|P_9|$ & $|P_{11}|$ (B) $\sum_{r=5}^{10}P(r)P(r-1)=-\frac{6}{55}$

(C) $|P_{10}|$ is arithmetic mean of $|P_9|$ & $|P_{11}|$ (D) $\sum_{r=5}^{10}P(r)P(r-1)=\frac{6}{55}$

BT0027

15. Let $(1 + x)^m = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_mx^m$, where $C_r = {}^mC_r$ and $A = C_1C_3 + C_2C_4 + C_3C_5 + C_4C_6 + ...+ C_{m-2}C_m$, then -

(A) $A \geq {}^{2m}C_{m-2}$ (B) $A < {}^{2m}C_{m-2}$

(C) $A > C_0^2 + C_1^2 + C_2^2 + ...C_m^2$ (D) $A < C_0^2 + C_1^2 + C_2^2 + ...C_m^2$

BT0028

EXERCISE (S-1)

1. (a) If the coefficients of $(2r + 4)^{th}$, $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal, find r.

BT0029

(b) If the coefficients of the r^{th}, $(r + 1)^{th}$ & $(r + 2)^{th}$ terms in the expansion of $(1 + x)^{14}$ are in AP, find r.

BT0030

(c) If the coefficients of 2^{nd}, 3^{rd} & 4^{th} terms in the expansion of $(1 + x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

BT0031

2. Find the term independent of x in the expansion of (i) $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right]^{10}$ (ii) $\left[\frac{1}{2}x^{1/3}+x^{-1/5}\right]^{8}$

BT0032

3. Prove that the ratio of the coefficient of x^{10} in $(1 - x^2)^{10}$ & the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

BT0033

4. Find the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.

BT0034

5. Let $(1+x^2)^2 . (1+x)^n = \sum_{K=0}^{n+4}a_K . x^K$. If a_1, a_2 & a_3 are in AP, find n.

BT0035

6. Let $f(x) = 1 - x + x^2 - x^3 + ... + x^{16} - x^{17} = a_0 + a_1(1 + x) + a_2(1 + x)^2 + ... + a_{17}(1 + x)^{17}$, find the value of a_2.

BT0036

7. Find the coefficient of x^r in the expression :

$$(x + 3)^{n-1} + (x + 3)^{n-2}(x + 2) + (x + 3)^{n-3}(x + 2)^2 + ... + (x + 2)^{n-1}$$

BT0037

8. Find numerically greatest term in the expansion of :

(i) $(2 + 3x)^9$ when $x = \frac{3}{2}$ **BT0038**

(ii) $(3 - 5x)^{15}$ when $x = \frac{1}{5}$ **BT0039**

9. (a) Show that the integral part in each of the following is odd. $n \in N$

(A) $\left(5 + 2\sqrt{6}\right)^n$ (B) $\left(8 + 3\sqrt{7}\right)^n$

BT0040

(b) Show that the integral part in each of the following is even. $n \in N$

(A) $\left(3\sqrt{3} + 5\right)^{2n+1}$ (B) $\left(5\sqrt{5} + 11\right)^{2n+1}$

BT0041

10. Let $N = {}^{2000}C_1 + 2 \cdot {}^{2000}C_2 + 3 \cdot {}^{2000}C_3 + ... + 2000 \cdot {}^{2000}C_{2000}$. Prove that N is divisible by 2^{2003}.
BT0042

11. Prove the following identities using the theory of permutation where $C_0, C_1, C_2, ..., C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$:

(a) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \frac{(2n)!}{n!\,n!}$ **BT0043**

(b) $C_0 C_1 + C_1 C_2 + C_2 C_3 + ... + C_{n-1} C_n = \frac{(2n)!}{(n+1)!\,(n-1)!}$ **BT0044**

(c) $C_oC_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r} C_n = \frac{2n!}{(n-r)!\,(n+r)!}$ **BT0045**

(d) $\sum_{r=0}^{n-2}({}^nC_r \cdot {}^nC_{r+2}) = \frac{(2n)!}{(n-2)!(n+2)!}$ **BT0046**

(e) ${}^{100}C_{10} + 5.\,{}^{100}C_{11} + 10\,.\,{}^{100}C_{12} + 10\,.\,{}^{100}C_{13} + 5.\,{}^{100}C_{14} + {}^{100}C_{15} = {}^{105}C_{90}$ **BT0047**

12. If $C_0, C_1, C_2, ..., C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$, then prove the following :

(a) $C_1 + 2C_2 + 3C_3 + ... + n\,.\,C_n = n\,.\,2^{n-1}$ **BT0048**

(b) $C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)2^{n-1}$ **BT0049**

(c) $C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n = (n+1)\,2^n$ **BT0050**

(d) $(C_0+C_1)(C_1+C_2)(C_2+C_3)\,.....\,(C_{n-1}+C_n) = \frac{C_0 . C_1 . C_2 .... C_{n-1}(n+1)^n}{n!}$ **BT0051**

(e) $1\,.\,C_o^2 + 3\,.\,C_1^2 + 5\,.\,C_2^2 + ... + (2n+1)\,C_n^2 = \frac{(n+1)\,(2n)!}{n!\,n!}$ **BT0052**

13. Prove that

(a) $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ... + \frac{n.C_n}{C_{n-1}} = \frac{n(n+1)}{2}$ **BT0053**

(b) $C_0+\frac{C_1}{2}+\frac{C_2}{3}+.......+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$ **BT0054**

(c) $2\,.\,C_o+\frac{2^2.C_1}{2}+\frac{2^3.C_2}{3}+\frac{2^4.C_3}{4}+......\frac{2^{n+1}.C_n}{n+1}=\frac{3^{n+1}-1}{n+1}$ **BT0055**

(d) $C_o-\frac{C_1}{2}+\frac{C_2}{3}-......+(-1)^n\,\frac{C_n}{n+1}=\frac{1}{n+1}$ **BT0056**

14. Given that $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, find the values of :

(i) $a_0 + a_1 + a_2 + ... + a_{2n}$;

(ii) $a_0 - a_1 + a_2 - a_3 ..... + a_{2n}$;

(iii) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2$

BT0057

15. Find the sum of the series $\sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+..........\text{up to m terms}\right]$

BT0058

16. Find the coefficient of

(a) x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$ **BT0059**

(b) x^4 in the expansion of $(2 - x + 3x^2)^6$ **BT0060**

17. Find the coefficient of

(a) $x^2\,y^3\,z^4$ in the expansion of $(ax - by + cz)^9$. **BT0061**

(b) $a^2\,b^3\,c^4\,d$ in the expansion of $(a - b - c + d)^{10}$. **BT0062**

EXERCISE (S-2)

1. Let a and b be the coefficient of x^3 in $(1 + x + 2x^2 + 3x^3)^4$ and $(1 + x + 2x^2 + 3x^3 + 4x^4)^4$ respectively. Find the value of $(a - b)$.

BT0063

2. Find the index n of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if the 9th term of the expansion has numerically the greatest coefficient $(n \in N)$.

BT0064

3. Find the sum of the roots (real or complex) of the equation $x^{2001}+\left(\frac{1}{2}-x\right)^{2001}=0$.

BT0065

4. Let $a=\left(4^{1/401}-1\right)$ and let $b_n = {}^nC_1 + {}^nC_2\,.\,a + {}^nC_3\,.\,a^2 + ... + {}^nC_n\,.\,a^{n-1}$.
Find the value of $(b_{2006} - b_{2005})$

BT0066

5. For which positive values of x, fourth term in the expansion of $(5 + 3x)^{10}$, is greatest.

BT0067

6. Let $P = \left(2+\sqrt{3}\right)^5$ and f = P – [P], where [P] denotes the greatest integer function.

Find the value of $\left(\dfrac{f^2}{1-f}\right)$.

BT0068

7. If $\left(7+4\sqrt{3}\right)^n = p+\beta$ where n & p are positive integers and β is a proper fraction show that $(1 - \beta)(p + \beta) = 1$.

BT0069

8. Find the coefficient of x^{49} in the polynomial

$$\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\cdot\frac{C_2}{C_1}\right)\left(x-3^2\cdot\frac{C_3}{C_2}\right)\ldots\ldots\ldots\ldots\ldots\left(x-50^2\cdot\frac{C_{50}}{C_{49}}\right), \text{ where } C_r = {}^{50}C_r.$$

BT0070

9. Prove that $\sum\limits_{K=0}^{n} {}^nC_K \sin Kx.\cos(n-K)x = 2^{n-1}\sin nx$.

BT0071

10. If $\dbinom{n}{r}$ denotes ${}^nC_r$, then

(a) Evaluate : $2^{15}\dbinom{30}{0}\dbinom{30}{15} - 2^{14}\dbinom{30}{1}\dbinom{29}{14} + 2^{13}\dbinom{30}{2}\dbinom{28}{13}\ldots\ldots - \dbinom{30}{15}\dbinom{15}{0}$

BT0072

(b) Prove that : $\sum\limits_{r=1}^{n}\dbinom{n-1}{n-r}\dbinom{n}{r} = \dbinom{2n-1}{n-1}$

BT0073

(c) Prove that : $\dbinom{n}{r}\dbinom{r}{k} = \dbinom{n}{k}\dbinom{n-k}{r-k}$

BT0074

EXERCISE (JM)

1. Let $S_1 = \sum\limits_{j=1}^{10} j(j-1)\,{}^{10}C_j$, $S_2 = \sum\limits_{j=1}^{10} j\,{}^{10}C_j$ and $S_3 = \sum\limits_{j=1}^{10} j^2\,{}^{10}C_j$. **[AIEEE-2010]**

Statement–1 : $S_3 = 55 \times 2^9$.
Statement–2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.
(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.
(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.
(3) Statement–1 is true, Statement–2 is false.
(4) Statement–1 is false, Statement–2 is true.

BT0075

2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is :- **[AIEEE 2011]**

(1) – 144 (2) 132 (3) 144 (4) – 132

BT0076

3. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012]**

(1) a rational number other than positive integers (2) an irrational number
(3) an odd positive integer (4) an even positive integer

BT0077

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[JEE-Main 2013]**

(1) 4 (2) 120 (3) 210 (4) 310

BT0078

5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to :- **[JEE(Main)-2014]**

(1) $\left(16,\frac{251}{3}\right)$ (2) $\left(14,\frac{251}{3}\right)$ (3) $\left(14,\frac{272}{3}\right)$ (4) $\left(16,\frac{272}{3}\right)$

BT0079

6. The sum of coefficients of integral powers of x in the binomial expansion of $\left(1-2\sqrt{x}\right)^{50}$ is : **[JEE(Main)-2015]**

(1) $\frac{1}{2}\left(3^{50}-1\right)$ (2) $\frac{1}{2}\left(2^{50}+1\right)$ (3) $\frac{1}{2}\left(3^{50}+1\right)$ (4) $\frac{1}{2}\left(3^{50}\right)$

BT0080

7. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n, x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is :- **[JEE(Main)-2016]**

(1) 729 (2) 64 (3) 2187 (4) 243

BT0081

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is :- **[JEE(Main)-2017]**

(1) $2^{20} - 2^{10}$ (2) $2^{21} - 2^{11}$ (3) $2^{21} - 2^{10}$ (4) $2^{20} - 2^{9}$

BT0082

9. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ is - **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

BT0083

10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to : **[JEE(Main)- 2019]**

(1) 14 (2) 6 (3) 4 (4) 8

BT0084

11. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is **[JEE(Main)- 2019]**

(1) 12 (2) 15 (3) 10 (4) 14

BT0085

12. If $\sum_{r=0}^{25}\left\{{}^{50}C_r \cdot {}^{50-r}C_{25-r}\right\} = K\left({}^{50}C_{25}\right)$, then K is equal to : **[JEE(Main)- 2019]**

(1) $2^{25}-1$ (2) $(25)^2$ (3) 2^{25} (4) 2^{24}

BT0086

13. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is : **[JEE(Main)- 2019]**

(1) 6 (2) 8 (3) 0 (4) 4

BT0087

14. The value of r for which ${}^{20}C_r\ {}^{20}C_0 + {}^{20}C_{r-1}\ {}^{20}C_1 + {}^{20}C_{r-2}\ {}^{20}C_2 + ... {}^{20}C_0\ {}^{20}C_r$ is maximum, is **[JEE(Main)- 2019]**

(1) 20 (2) 15 (3) 11 (4) 10

BT0088

15. Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + ... + a_{50}\ x^{50}$, for all $x \in R$, then $\frac{a_2}{a_0}$ is equal to:- **[JEE(Main)- 2019]**

(1) 12.50 (2) 12.00 (3) 12.75 (4) 12.25

BT0089

16. Let $S_n = 1 + q + q^2 + ... + q^n$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n$, where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2.S_1 + ... + {}^{101}C_{101}.S_{100} = \alpha T_{100}$, then α is equal to :-**[JEE(Main)- 2019]**

(1) 2^{100} (2) 200 (3) 2^{99} (4) 202

BT0090

17. The total number of irrational terms in the binomial expansion of $\left(7^{1/5} - 3^{1/10}\right)^{60}$ is : **[JEE(Main)- 2019]**

(1) 55 (2) 49 (3) 48 (4) 54

BT0091

18. If some three consecutive in the binomial expansion of $(x + 1)^n$ is powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficient is :- **[JEE(Main)- 2019]**

(1) 964 (2) 625 (3) 227 (4) 232

BT0092

19. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1 + x + x^2)^9$ is : **[JEE(Main)- 2019]**

(1) –84 (2) 84 (3) 126 (4) –126

BT0093

20. If ${}^{20}C_1 + (2^2)\ {}^{20}C_2 + (3^2)\ {}^{20}C_3 + ... + (20^2){}^{20}C_{20} = A(2^{\beta})$, then the ordered pair (A, β) is equal to: **[JEE(Main)- 2019]**

(1) (420, 18) (2) (380, 19) (3) (380, 18) (4) (420, 19)

BT0094

21. The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to :

[JEE(Main)- 2019]

(1) 36 (2) – 108 (3) – 72 (4) – 36

BT0095

22. The coefficient of x^7 in the expression $(1 + x)^{10} + x(1 + x)^9 + x^2(1 + x)^8 + ... + x^{10}$ is :

(1) 120 (2) 330 (3) 210 (4) 420

[JEE(Main)- 2020]

BT0096

23. If the sum of the coefficients of all even powers of x in the product $(1 + x + x^2 + ... + x^{2n})(1 - x + x^2 - x^3 + ... + x^{2n})$ is 61, then n is equal to _________. **[JEE(Main)- 2020]**

BT0097

24. If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$, then **[JEE(Main)- 2020]**

(1) $\alpha + \beta = 60$ (2) $\alpha + \beta = -30$ (3) $\alpha - \beta = -132$ (4) $\alpha - \beta = 60$

BT0098

25. In the expansion of $\left(\frac{x}{\cos\theta}+\frac{1}{x\sin\theta}\right)^{16}$, if ℓ_1 is the least value of the term independent of x when $\frac{\pi}{8}\le\theta\le\frac{\pi}{4}$ and ℓ_2 is the least value of the term independent of x when $\frac{\pi}{16}\le\theta\le\frac{\pi}{8}$, then the ratio $\ell_2 : \ell_1$ is equal to : **[JEE(Main)- 2020]**

(1) 1 : 8 (2) 1 : 16 (3) 8 : 1 (4) 16 : 1

BT0099

26. If $C_r \equiv {}^{25}C_r$ and $C_0 + 5.C_1 + 9.C_2 + ... + (101).C_{25} = 2^{25}.k$, then k is equal to _____. **[JEE(Main)- 2020]**

BT0100

EXERCISE (JA)

1. For r = 0, 1,....,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to -

(A) $B_{10} - C_{10}$ (B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$ (C) 0 (D) $C_{10} - B_{10}$

[JEE 2010, 5]

BT0101

2. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE (Advanced) 2013, 4M, –1M]

BT0102

3. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is -

(A) 1051 (B) 1106 (C) 1113 (D) 1120

[JEE(Advanced)-2014, 3(–1)]

BT0103

4. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)\ldots(1 + x^{100})$ is **[JEE 2015, 4M, –0M]**

BT0104

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots\ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE(Advanced)-2016, 3(0)]**

BT0105

6. Let $X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + \ldots + 10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, $r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

BT0106

7. Suppose $\det\begin{bmatrix} \sum\limits_{k=0}^{n} k & \sum\limits_{k=0}^{n} {}^{n}C_k\, k^2 \\ \sum\limits_{k=0}^{n} {}^{n}C_k\, k & \sum\limits_{k=0}^{n} {}^{n}C_k\, 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum\limits_{k=0}^{n} \frac{{}^{n}C_k}{k+1}$ equals

[JEE(Advanced)-2019, 3(0)]

BT0107

ANSWER KEY

Do yourself-1

(i) ${}^5C_0 x(3x^2)^5 + {}^5C_1(3x^2)^4\left(-\frac{x}{2}\right) + {}^5C_2(3x^2)^3\left(-\frac{x}{2}\right)^2 + {}^5C_3(3x^2)^2\left(-\frac{x}{2}\right)^3 + {}^5C_4(3x^2)^1\left(-\frac{x}{2}\right)^4 + {}^5C_5\left(-\frac{x}{2}\right)^5$

(ii) ${}^nC_0 y^n + {}^nC_1 y^{n-1}.x + {}^nC_2.y^{n-2}.x^2 + \ldots\ldots + {}^nC_n.x^n$

Do yourself-2

(i) $\frac{70}{3}x^8$; **(ii)** $\frac{25!}{10!\ 5!}2^{15}3^{10}$; **(iii)** (a) –20; (b) $-560x^5$, $280x^2$

Do yourself-3

(i) 4^{th} & 5^{th} i.e. 489888 **(ii)** n = 4, 5, 6

Do yourself-4

(i) C

Do yourself-5

(i) –272160 or $-{}^{10}C_5 \times {}^5C_2 \times 108$

Do yourself-6

(ii) 1 **(iii)** 001 **(v)** 1

EXERCISE (O-1)

1. C **2.** C **3.** B **4.** A **5.** A **6.** C **7.** B
8. A **9.** B **10.** B **11.** D **12.** B **13.** A **14.** B **15.** B

EXERCISE (O-2)

1. A,B **2.** B,C,D **3.** A,C,D **4.** A,B **5.** A,C **6.** B **7.** B
8. A,C,D **9.** B **10.** D **11.** A **12.** A,C **13.** B,C,D **14.** A,D
15. B,D

EXERCISE (S-1)

1. **(a)** r = 6 **(b)** r = 5 or 9 **2.** (i) $\frac{5}{12}$ (ii) $T_6 = 7$ **4.** $\frac{17}{54}$ **5.** n = 2 or 3 or 4

6. 816 **7.** ${}^nC_r(3^{n-r} - 2^{n-r})$ **8.** **(i)** $T_7 = \frac{7.3^{13}}{2}$ **(ii)** 455×3^{12}

14. **(i)** 3^n **(ii)** 1, **(iii)** a_n **15.** $\frac{(2^{mn}-1)}{(2^n-1)(2^{mn})}$ **16.** **(a)** 990 **(b)** 3660

17. **(a)** $-1260\ .\ a^2b^3c^4$ **; (b)** –12600

EXERCISE (S-2)

1. 0 **2.** n = 12 **3.** 500 **4.** 2^{10} **5.** $\frac{5}{8} \le x \le \frac{20}{21}$

6. 722 **8.** – 22100 **10.** (a) $\binom{30}{15}$

EXERCISE (JM)

1. 3 **2.** 1 **3.** 2 **4.** 3 **5.** 4 **6.** 3

7. **Bonus**

Note : In the problem 'number of terms should be 13 instead of 28', then (1) will be the answer

8. 1 **9.** 3 **10.** 4 **11.** 2 **12.** 3 **13.** 3 **14.** 1

15. 4 **16.** 1 **17.** 4 **18.** 4 **19.** 2 **20.** 1 **21.** 4

22. 2 **23. 30** **24. 3** **25. 4** **26. 51**

EXERCISE (JA)

1. D **2.** 6 **3.** C **4.** 8 **5.** 5 **6.** 646 **7.** 6.20