COL 351 : Analysis & Design of Algorithms

LECTURE 5

DIVIDE & CONQUER IV :

Counting Inversions (contd.), Matrix Multiplication and Master Theorem

July 31, 2024 | Rohit Vaish

COUNTING INVERSIONS

input: a length n array A of distinct integers

output: the number of inversions of A

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inversions: pairs (i, j) of array indices with $i < j$ and $A[i] > A[j]$.

COUNTING INVERSIONS

Brute force algorithm: check every pair of indices (i, j)

$\Theta(n^2)$

COUNTING INVERSIONS

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$\Theta(n^2)$

Can we do better?

Yes! $O(n \log n)$ algorithm via divide-and-conquer.

COUNTING INVERSIONS

Call an inversion (i, j) where $i < j$

left inversion if $i, j \leq n/2$

right inversion if $i, j > n/2$

split inversion if $i \leq \frac{n}{2} < j$

COUNTING INVERSIONS

Call an inversion (i, j) where $i < j$

left inversion if $i, j \leq n/2$

right inversion if $i, j > n/2$

Compute these recursively

split inversion if $i \leq \frac{n}{2} < j$ ← Compute these in "combine" step

HIGH-LEVEL ALGORITHM

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s := count split inversions of A

return $\ell + r + s$

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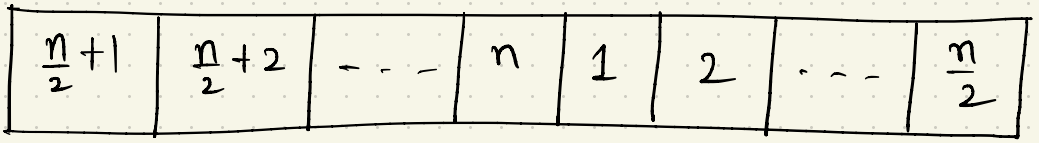
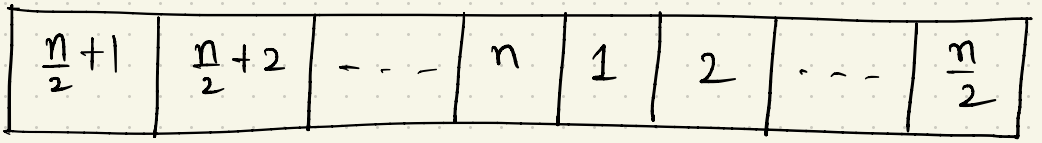
A = 

split inversions = ?

$A =$

$\frac{n}{2}+1$	$\frac{n}{2}+2$	...	n	1	2	...	$\frac{n}{2}$

split inversions $= n^2/4$

A = 

split inversions = $n^2/4$

Possible to compute split inversions in O(n) time ?

suffices for O(n log n) time overall

Piggyback on Merge Sort

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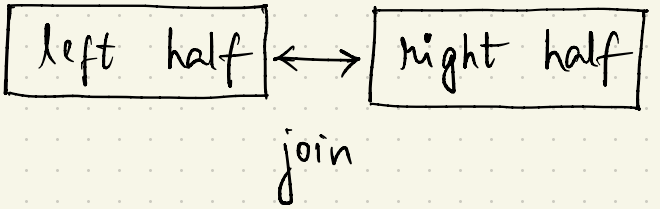
Concatenation!



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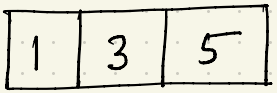
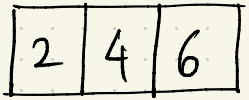
Consider merging 

and 

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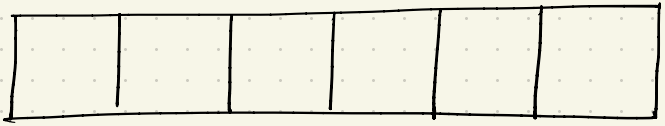
Output 

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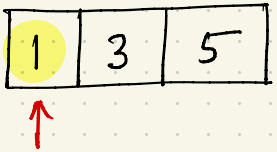
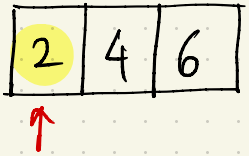
Consider merging 

and 

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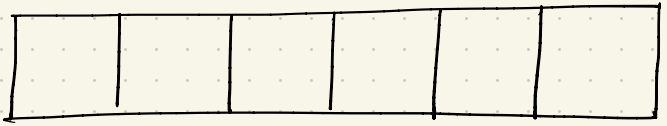
Output 

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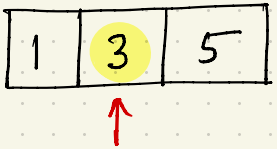
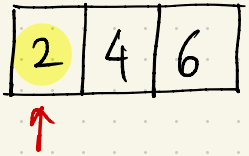
Consider merging 

and 

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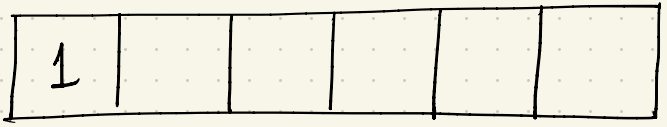
Output 

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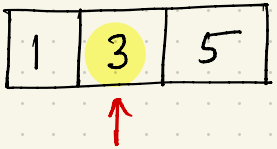
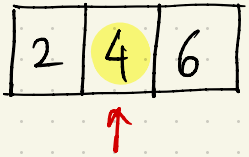
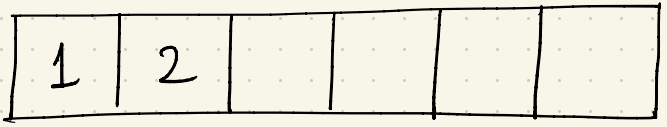
Consider merging 

and 

Output 

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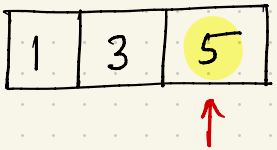
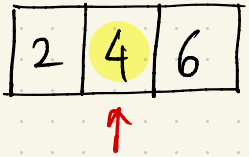
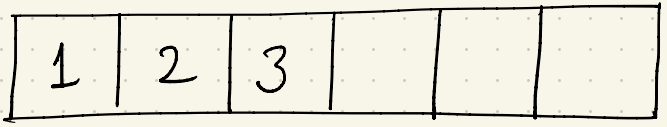
Consider merging 

and 

Output 

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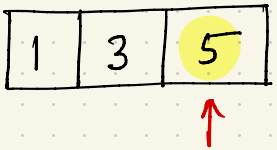
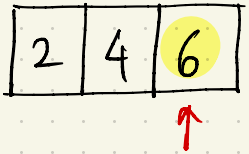
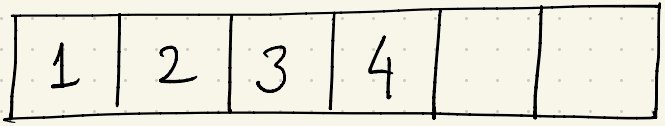
Consider merging 

and 

Output 

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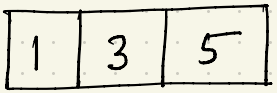
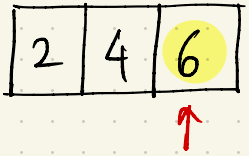
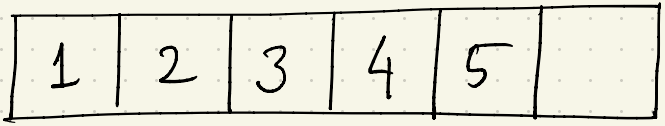
Consider merging 

and 

Output 

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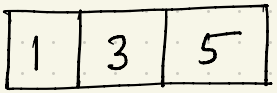
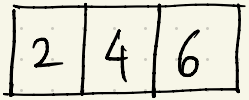
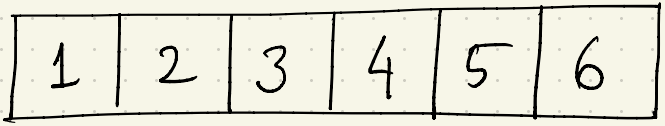
Consider merging  and  .

Output 

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Claim: Let A be an array, and let L and R be sorted versions of first and second halves of A.

The elements $x \in L$ and $y \in R$ form a split inversion

if and only if

merge (L, R) copies y to the output before x.

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Proof: Since $x \in L$ and $y \in R$

(x, y) split inversion $\iff$ $y < x$ $\iff$ y copied before x.

ALGORITHM FOR COUNTING INVERSIONS

input: an array A of n distinct integers

output: the number of inversions of A

if $n \leq 1$ return 0

else l := recursively count inversions on left half of A

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return $l + r + s$

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(B, s) := (merge and) count split inversions of A (using L and R)

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How to implement this in linear time?

RECALL MERGE SUBROUTINE

```
initialize i := 1
initialize j := 1
for k := 1 to n
        if L[i] < R[j] then
                B[k] := L[i]
                increment i by 1
        else
                B[k] := R[j]
                increment j by 1

return B
```

MERGE-AND-COUNT-SPLIT-INVERSIONS

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return B
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MERGE-AND-COUNT-SPLIT-INVERSIONS

initialize $s := 0$
initialize $i := 1$
initialize $j := 1$
for $k := 1$ to n
    if $L[i] < R[j]$ then
        $B[k] := L[i]$
        increment i by 1
    else
        $B[k] := R[j]$
        increment j by 1
        $s := s + \underbrace{\left(\frac{n}{2} - i + 1\right)}_{\#\text{ left in } L}$
return (B, s)

Theorem: For every input array A of length $n \geq 1$, our algorithm correctly computes the number of inversions of A and runs in $O(n \log n)$ time.

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Proof: Exercise

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input: Two $n \times n$ matrices X and Y of integers

Output: the product $Z = X \cdot Y$ where $z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$

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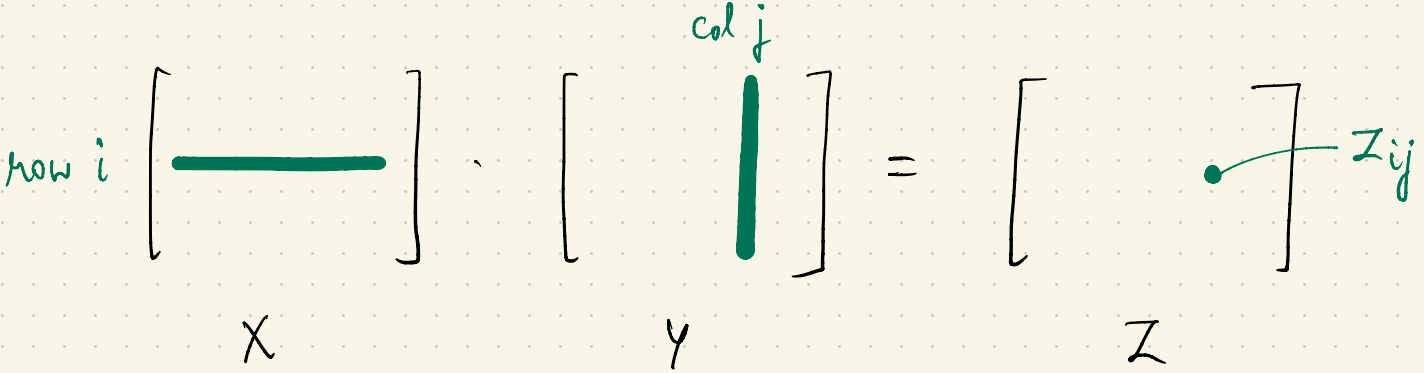

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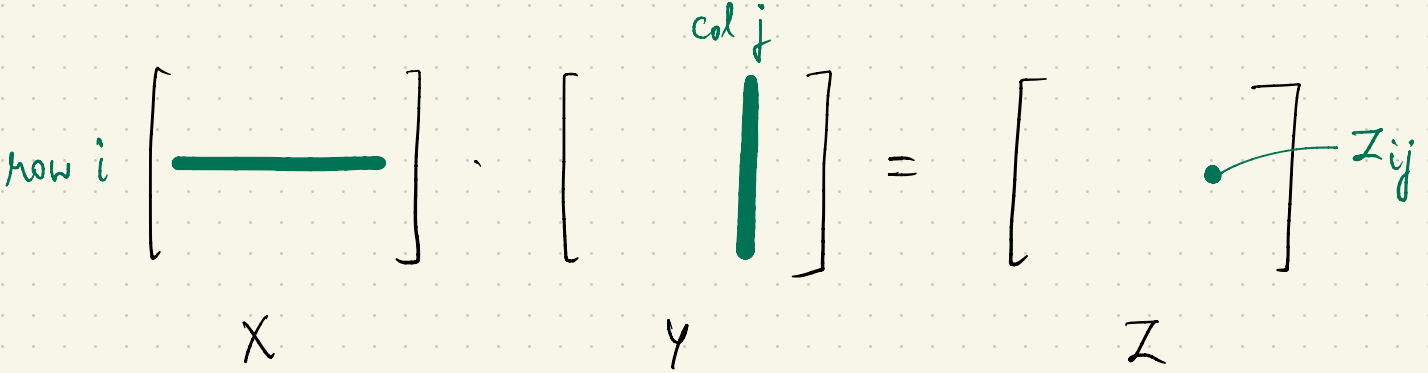

input size: $O(n^2)$

MATRIX MULTIPLICATION

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

X Y Z

MATRIX MULTIPLICATION

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \\ & \end{bmatrix}$$

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Running time for $n \times n$ matrices : ?

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Running time for $n \times n$ matrices : $O(n^3)$

assuming multiplication/addition of two numbers takes $O(1)$ time

MATRIX MULTIPLICATION

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{X} \cdot \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}}_{Z}$$

Running time for $n \times n$ matrices : $O(n^3)$

Can we do better?

APPLYING DIVIDE & CONQUER

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$$X = \left[\begin{array}{c|c} L & R \end{array}\right]_{n \times n}$$

non-square matrices

APPLYING DIVIDE & CONQUER

$$X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{n \times n}$$

A, B, C, D are $\frac{n}{2} \times \frac{n}{2}$ matrices

(assuming n is even)

APPLYING DIVIDE & CONQUER

$$X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{n \times n} \qquad Y = \left[\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right]_{n \times n}$$

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Then, $$X \cdot Y = \begin{bmatrix} A \cdot E + B \cdot G & A \cdot F + B \cdot H \\ C \cdot E + D \cdot G & C \cdot F + D \cdot H \end{bmatrix}$$

verify!

Recursive Algorithm

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RECURSIVE ALGORITHM

Step 1 :- Recursively compute the eight matrix products

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Is there a way to save a recursive call?

STRASSEN'S ALGORITHM

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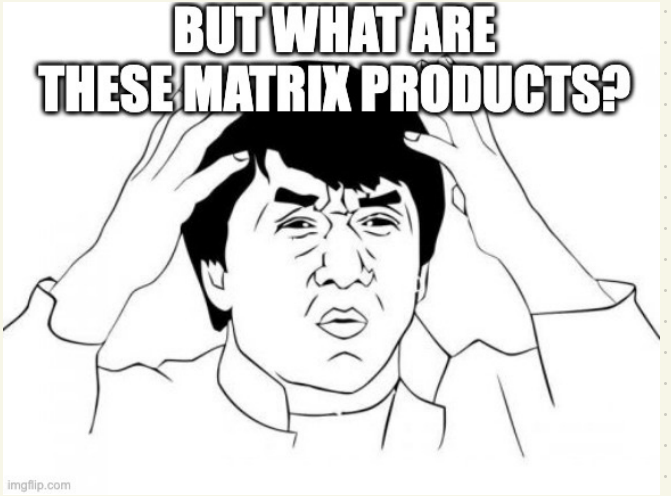

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STRASSEN'S ALGORITHM

$P_1 = A \cdot (F - H)$

$P_2 = (A + B) \cdot H$

$P_3 = (C + D) \cdot E$

$P_4 = D \cdot (G - E)$

$P_5 = (A + D) \cdot (E + H)$

$P_6 = (B - D) \cdot (G + H)$

$P_7 = (A - C) \cdot (E + F)$

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Claim: $$X \cdot Y = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

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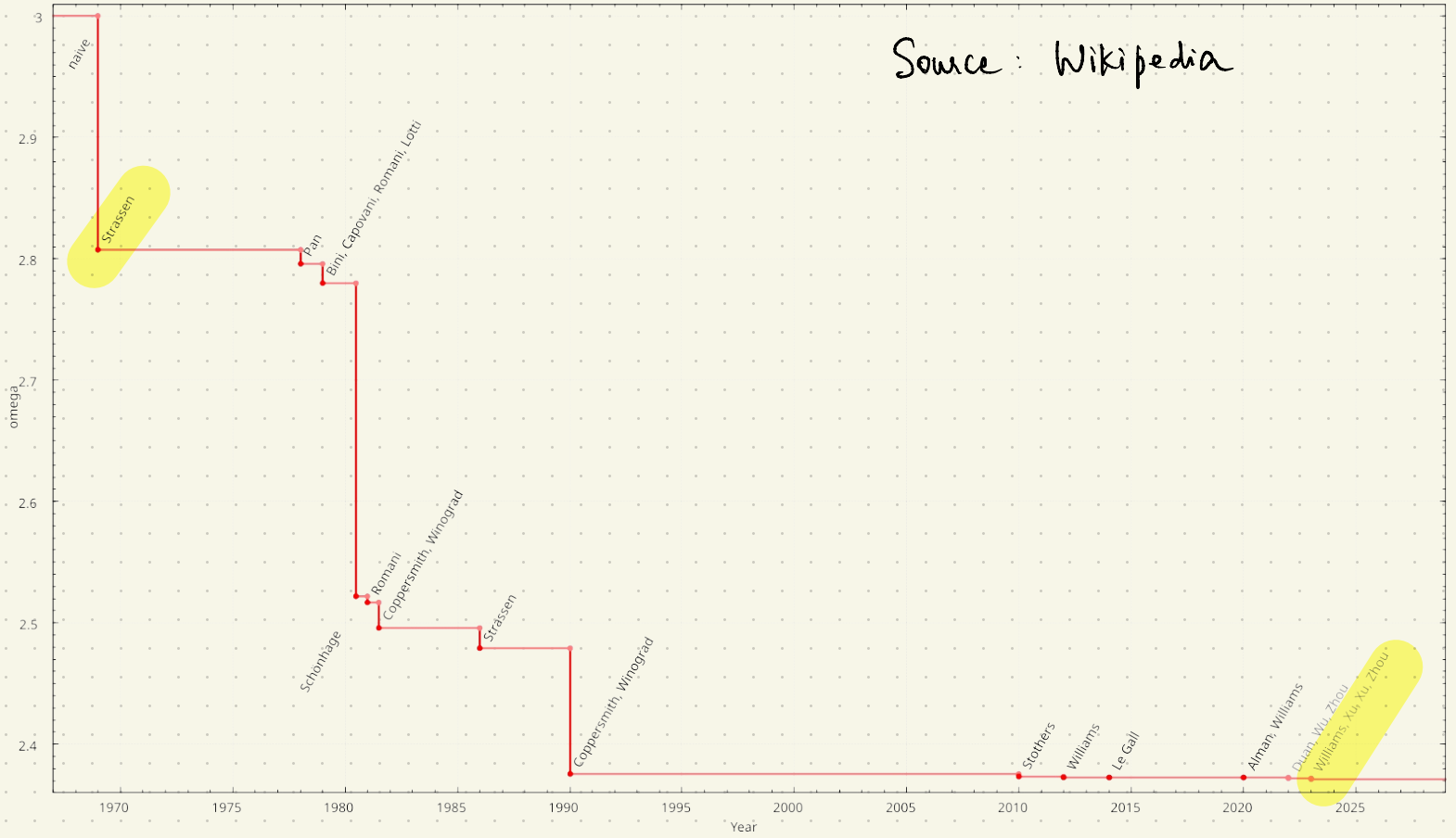
Strassen's algorithm saves a recursive call!

MATRIX MULTIPLICATION



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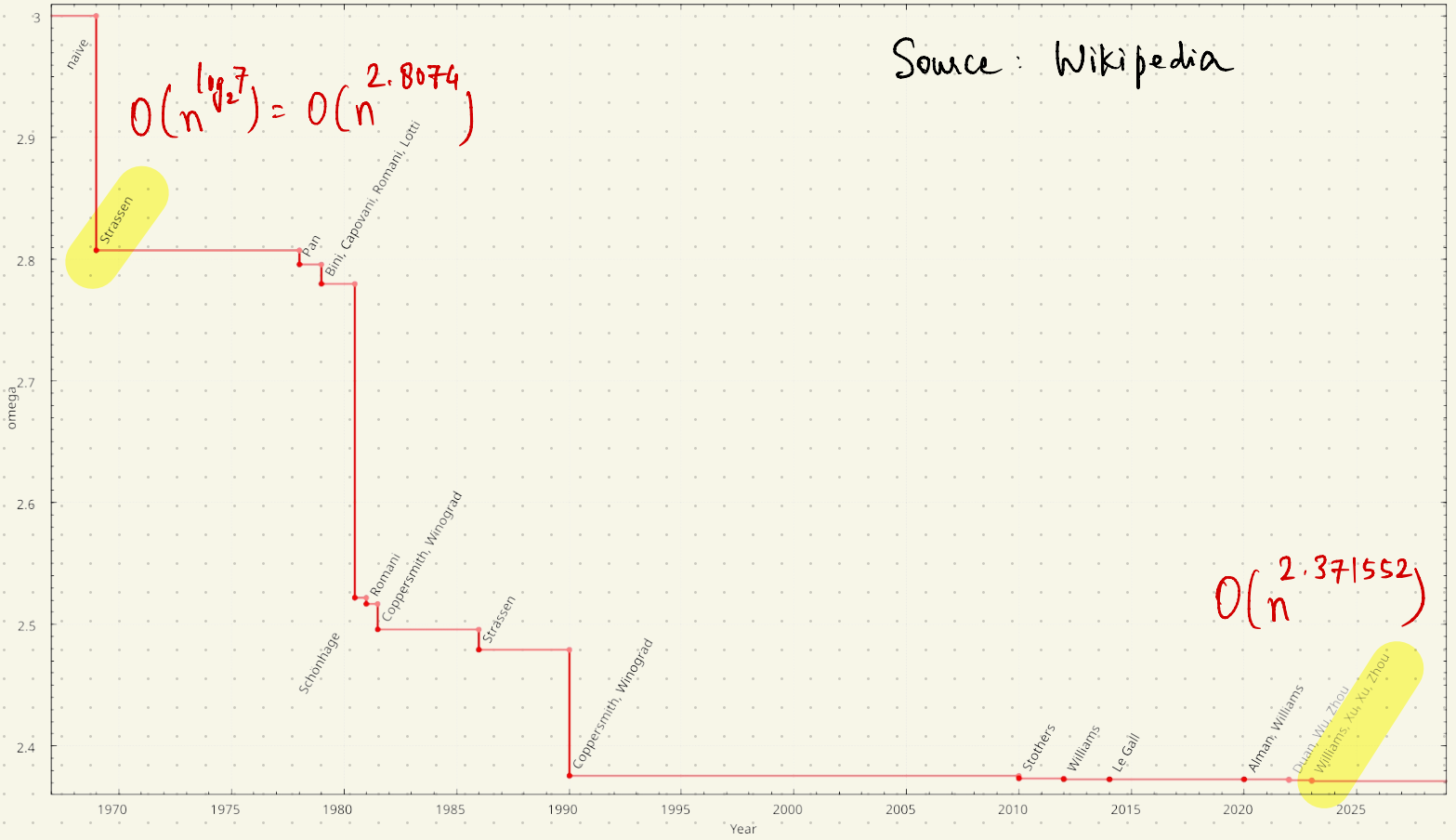

MASTER METHOD

Black-box for determining the running time of recursive algorithms

RECALL: INTEGER MULTIPLICATION

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Grade-school multiplication $\Theta(n^2)$

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Recursive algorithm

$$\left[x \cdot y = 10^{n} ac + 10^{n/2}(ad + bc) + bd \right]$$

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$T(n)$

maximum number of operations the algorithm needs to multiply two n-digit numbers

RECALL : INTEGER MULTIPLICATION

Grade-school multiplication $\Theta(n^2)$

Recursive algorithm $T(n) \leq 4 \cdot T(n/2) + O(n)$

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Recursive algorithm $T(n) \leq 4 \cdot T(n/2) + O(n)$

$$[x \cdot y = 10^{n} ac + 10^{n/2}(ad + bc) + bd]$$

Karatsuba algorithm $T(n) \leq 3 \cdot T(n/2) + O(n)$

$$[(a+b)(c+d) - ac - bd = ad + bc]$$

MASTER METHOD

Black-box for solving recurrences

Key assumption: All subproblems have the same size.

for unbalanced subproblems and more: Akra-Bazzi method

STANDARD RECURRENCE FORMAT

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Base case:

General case:

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- b : factor by which input size shrinks (> 1)
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independent of n

- a : number of recursive calls (≥ 1)
- b : factor by which input size shrinks (> 1)
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Theorem: If $T(n) \leq a\,T\left(\frac{n}{b}\right) + O(n^d)$, then

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only upper bounds

Examples

EXAMPLES

Merge Sort

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$$T(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(n)$$

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$a = 2, b = 2, d = 1$

(Case 1)

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$O(n \log n)$

BINARY SEARCH

input: a sorted array A of length n and a target T

output: index of T in A (if exists)

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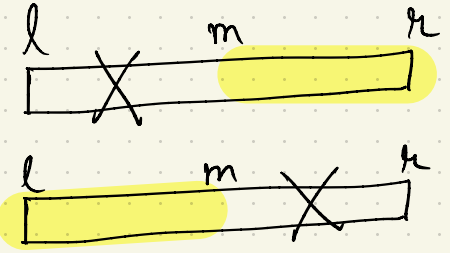


BINARY SEARCH

input: a sorted array A of length n and a target T

output: index of T in A (if exists)

initialize $\ell := 0$

initialize $r := n-1$

initialize $m := \left\lfloor \frac{\ell + r}{2} \right\rfloor$

if $A[m] < T$

recurse with $\ell := m+1$

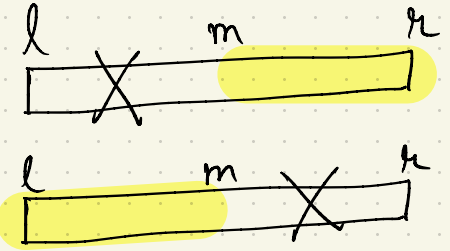
else if $A[m] > T$

recurse with $r := m-1$

else

return m

return No



EXAMPLES

Merge Sort

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$a = 2, b = 2, d = 1$

(Case 1)

$O(n \log n)$

Binary Search

$T(n) \leq T\left(\frac{n}{2}\right) + O(1)$

$a = 1, b = 2, d = 0$

(Case 1)

$O(\log n)$

EXAMPLES

Recursive integer mult^n

$T(n) \leq 4T\left(\frac{n}{2}\right) + O(n)$

$a = 4, b = 2, d = 1$

(Case 3)

$O\left(n^{\log_2 4}\right) = O(n^2)$

same as grade-school algorithm

Karatsuba mult^n

$T(n) \leq 3T\left(\frac{n}{2}\right) + O(n)$

$a = 3, b = 2, d = 0$

(Case 3)

$O\left(n^{\log_2 3}\right) = O(n^{1.59})$

faster!

Strassen's algorithm

$T(n) \leq 7T\left(\frac{n}{2}\right) + O(n)$

$a = 7, b = 2, d = 1$

(Case 3)

$O\left(n^{\log_2 7}\right) = O(n^{2.81})$