COL 351 : Analysis & Design of Algorithms

Lecture 3

Divide & Conquer II :

Merge Sort (contd.) and Asymptotic Analysis

July 26, 2024 | Rohit Vaish

ANNOUNCEMENTS

Tutorial Sheet 1 is available.

Tutorials start on Monday, July 29

Groups will be announced by Saturday (July 27)

Integer Multiplication

input: two n digit numbers x and y

output: the product $x \cdot y$

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Grade-school multiplication $\leq 9n^2$ basic operations

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Recursive algorithm (4 calls) ?

Karatsuba algorithm (3 calls) ?

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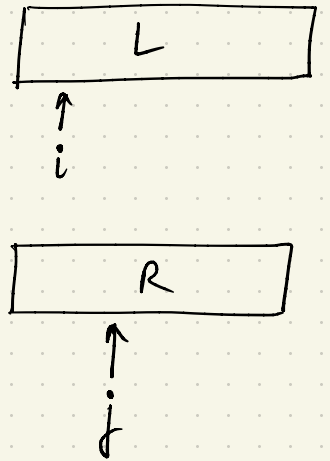

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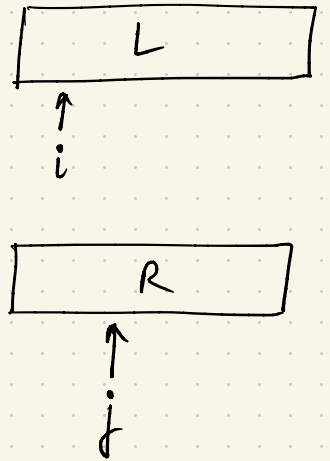

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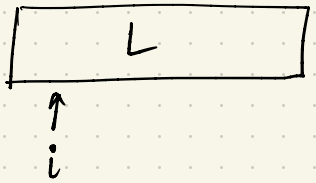
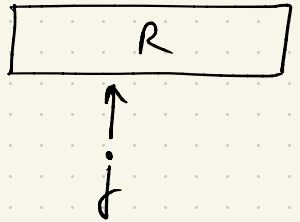
if $L[i] < R[j]$ then





else

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initialize i := 1
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for k := 1 to n
        if L[i] < R[j] then
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                increment i by 1
        else
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                increment j by 1
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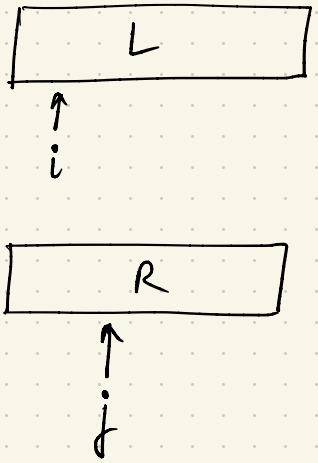

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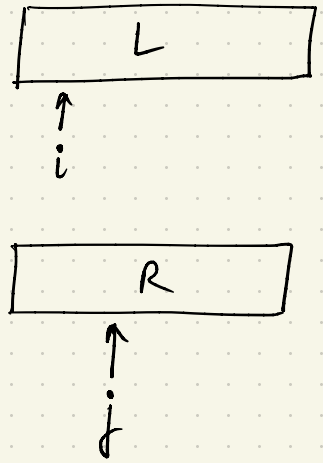

ignoring end cases when i or j falls off

MERGE SORT RUNNING TIME

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- commonly, any pairwise comparison
- for now, a line-by-line implementation using a debugger

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Let's start by analyzing MERGE subroutine.

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MERGE SUBROUTINE

input : sorted arrays L and R of length $n/2$ each

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assume n is even

initialize $i := 1$

initialize $j := 1$

} 2 operations

for $k := 1$ to n

    if $L[i] < R[j]$ then

        $B[k] := L[i]$

        increment i by 1

    else

        $B[k] := R[j]$

        increment j by 1

MERGE SUBROUTINE

input : sorted arrays L and R of length $n/2$ each

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for $k := 1$ to n executes n times

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if L[i] < R[j] then
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initialize i := 1
initialize j := 1                    } 2 operations
for k := 1 to n                      executes n times
        if L[i] < R[j] then          }
                B[k] := L[i]         }  ?? operations
                increment i by 1     }
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initialize i := 1
initialize j := 1                    } 2 operations
for k := 1 to n                      executes n times
        if L[i] < R[j] then          } 3+1 = 4 operations
                B[k] := L[i]            (+1 = incrementing k)
                increment i by 1
        else
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        else                               }
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$4n + 2$ operations

MERGE SUBROUTINE

Lemma: Running time of MERGE subroutine on input arrays of length $n/2$ each is $\leq 4n + 2$.

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(So we can plug in other values of l later)

MERGE SORT RUNNING TIME

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Theorem: For every input array of length $n \geq 1$, Merge Sort performs at most $6n \log_2 n + 6n$ operations.

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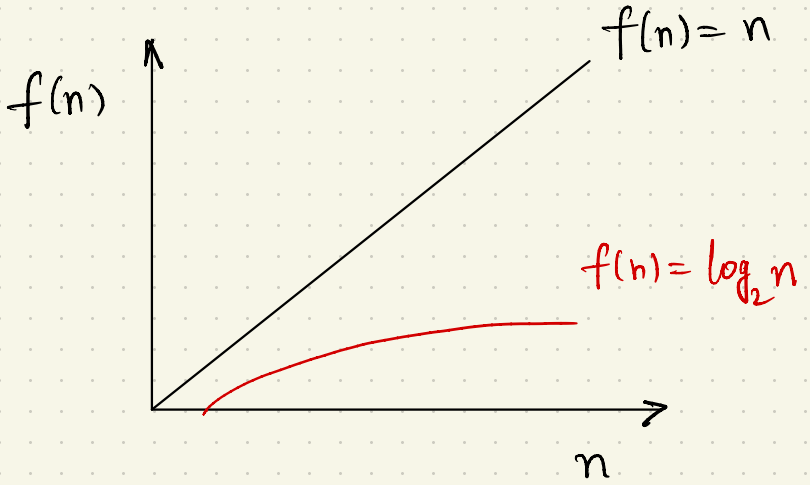

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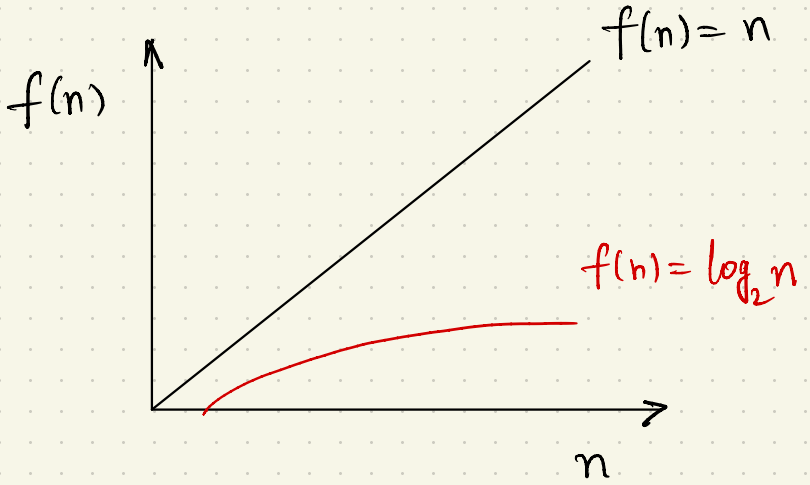

$\log_2$: how many "divide by 2" operations are needed to bring the answer to ≤ 1.

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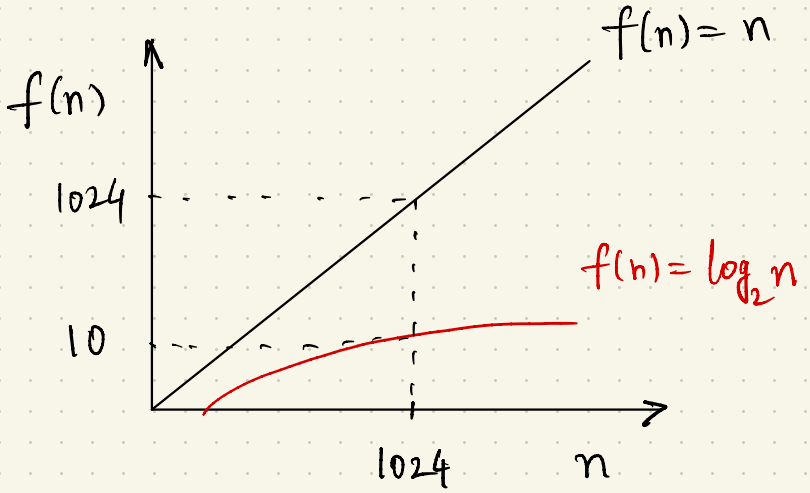

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Proof via recursion tree

MERGE SORT RUNNING TIME

Proof: (assuming n is power of 2)

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level 0 outermost call

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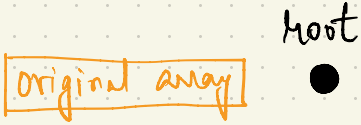

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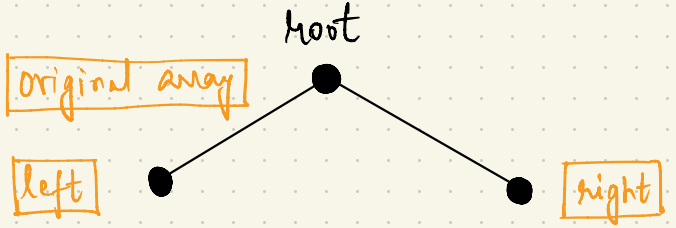

level 0 outermost call

level 1 first recursive calls

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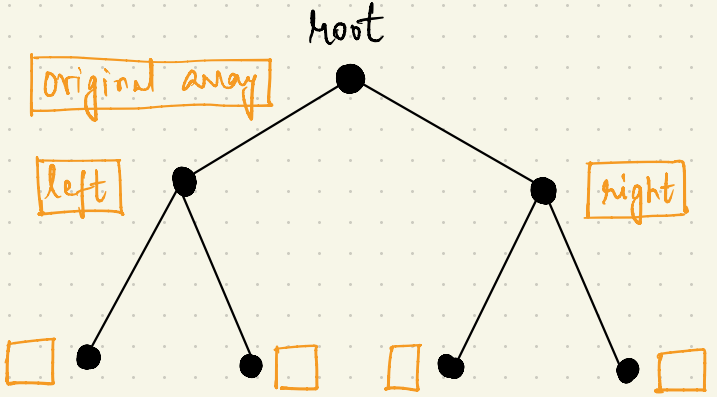

level 0 outermost call

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level 2

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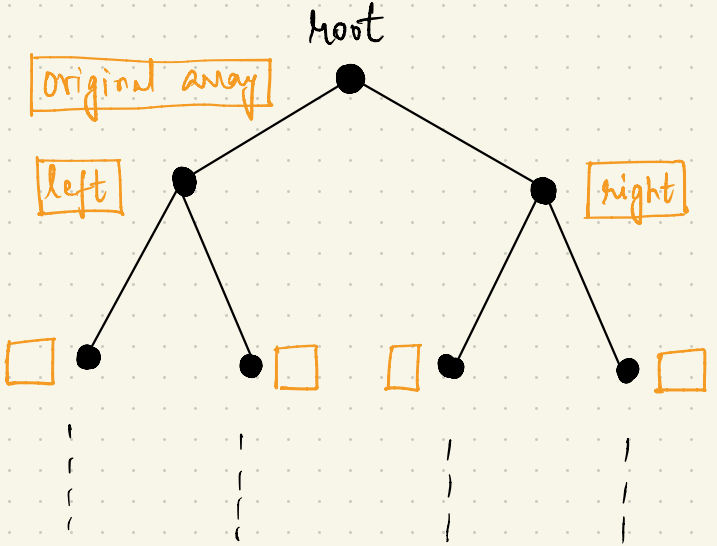

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level 2

leaves (single-element arrays)

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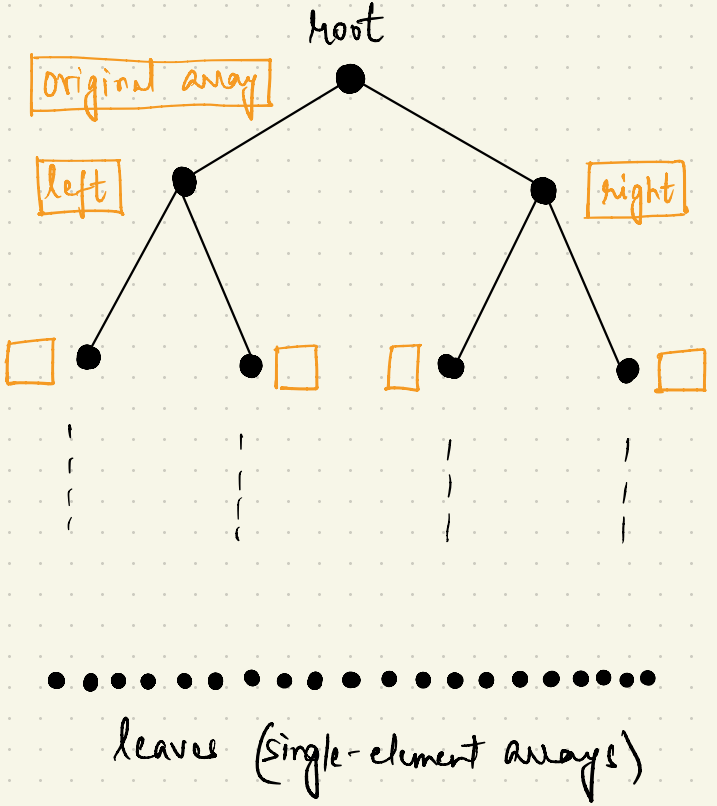

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level 2

level ??

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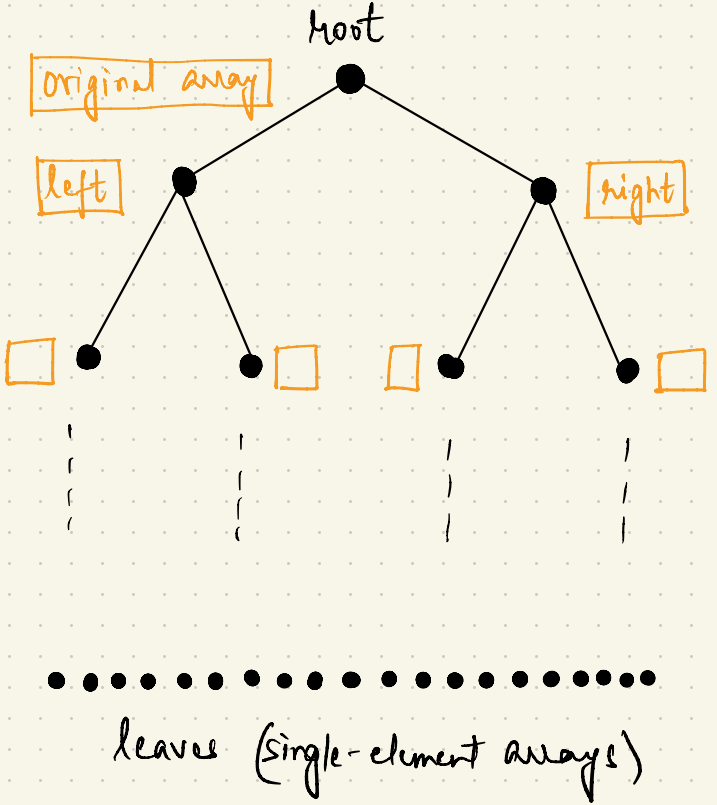

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level 2

level $\log_2 n$

MERGE SORT RUNNING TIME

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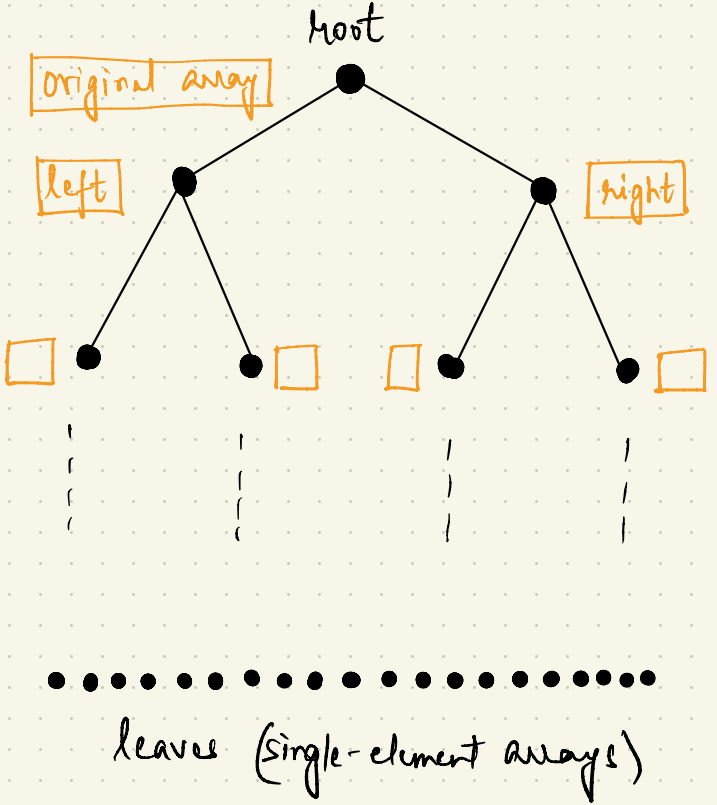
$\log_2 n + 1$ levels in total



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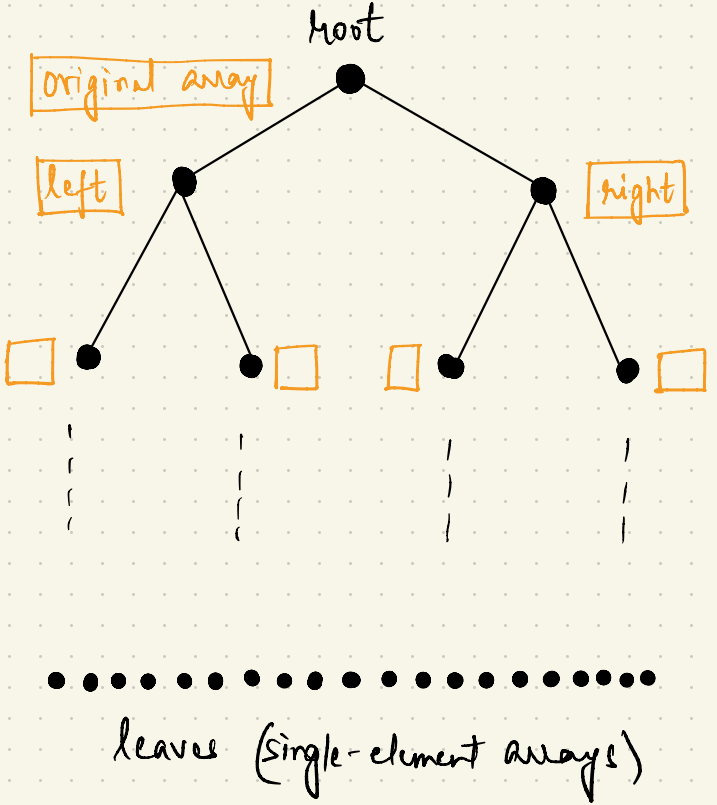

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At level $j \in \{0, 1, \ldots, \log_2 n\}$

?? subproblems

each of size ??

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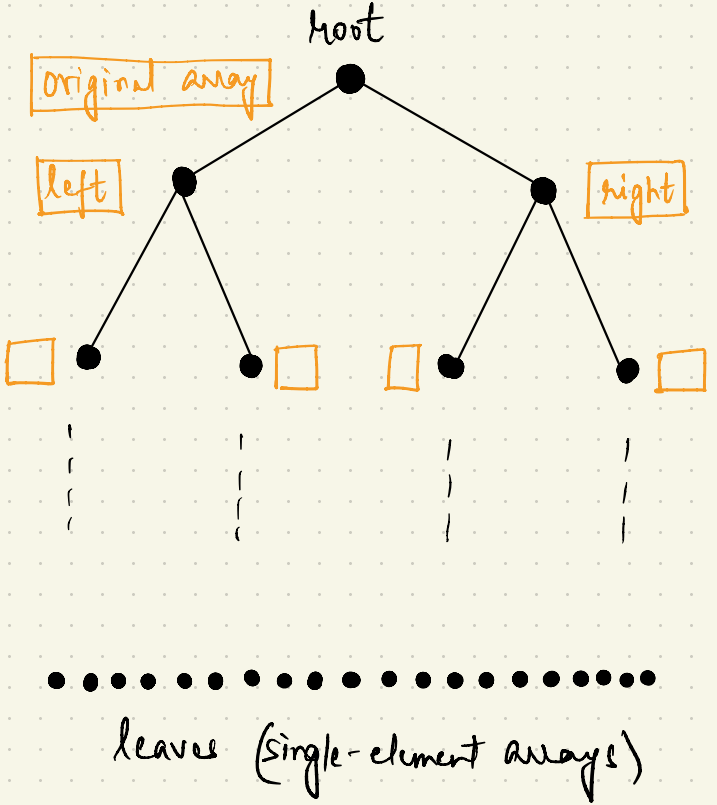

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At level $j \in \{0, 1, \ldots, \log_2 n\}$

2^j subproblems

each of size $n/2^j$

MERGE SORT RUNNING TIME

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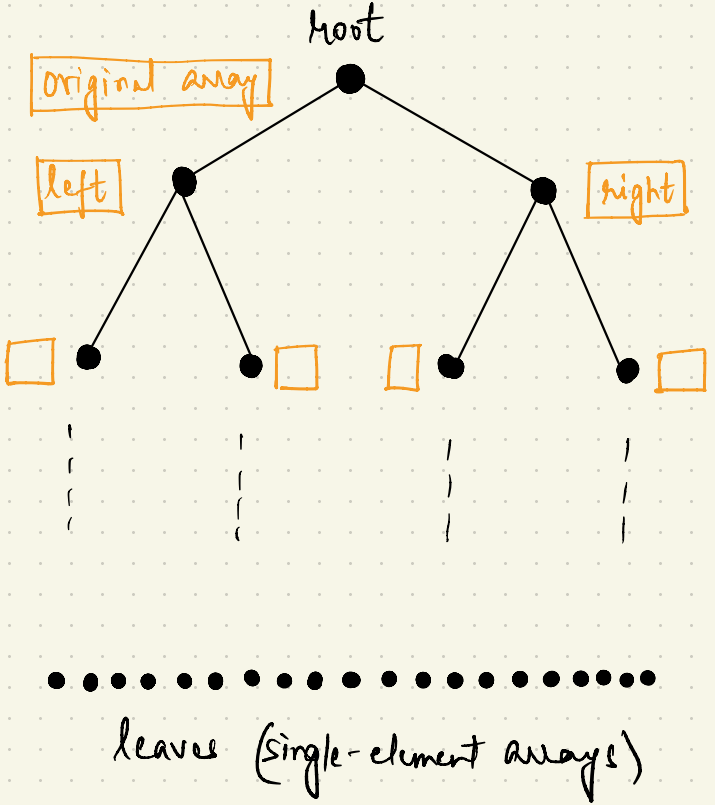

How much work does Merge Sort do at level j?

MERGE SORT : PSEUDOCODE

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```

only need to count this

MERGE SORT RUNNING TIME

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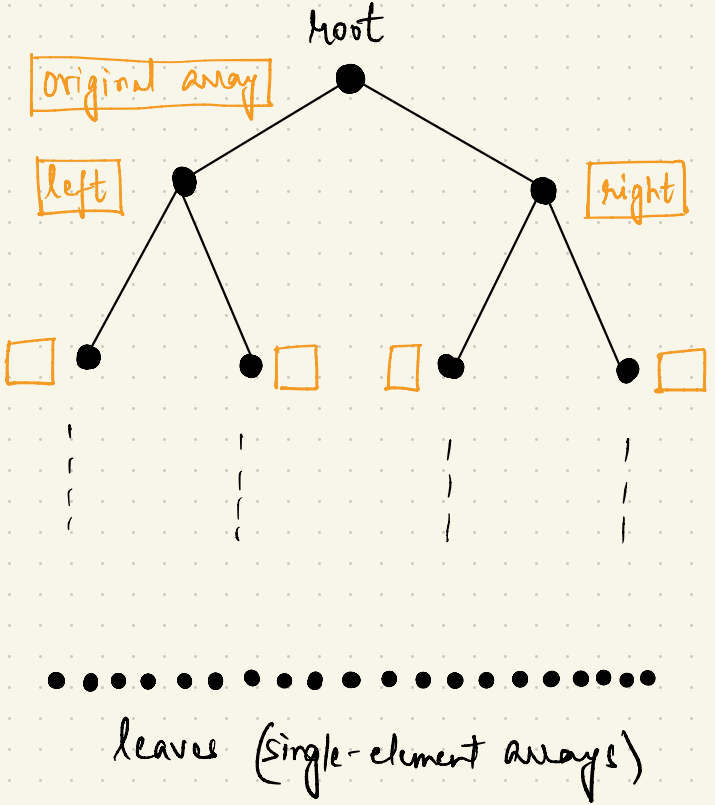

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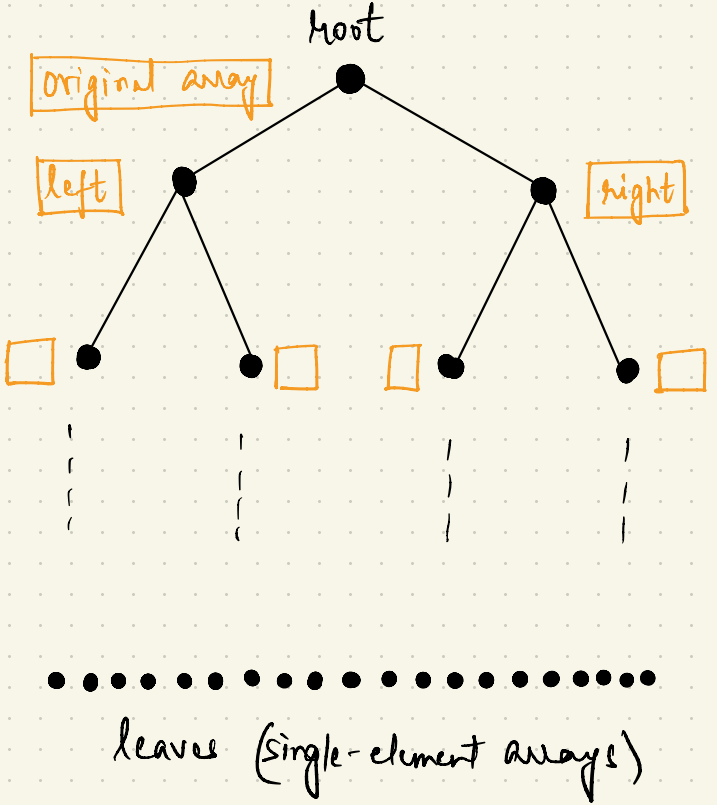

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Work done by merge at level j

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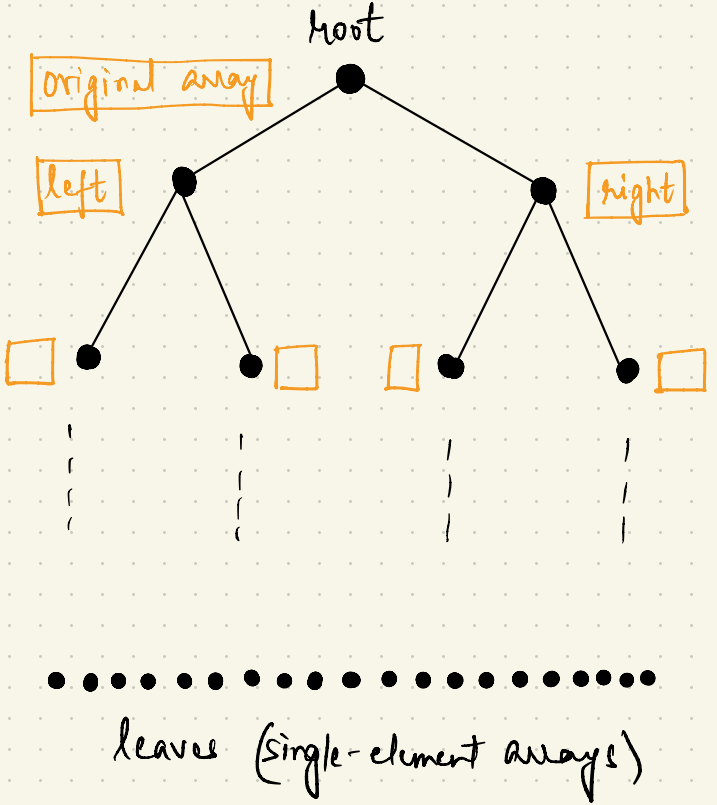

How much work does Merge Sort do at level j?

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Work done by merge at level j

$\leq 2^j$

/

subproblems

MERGE SORT RUNNING TIME

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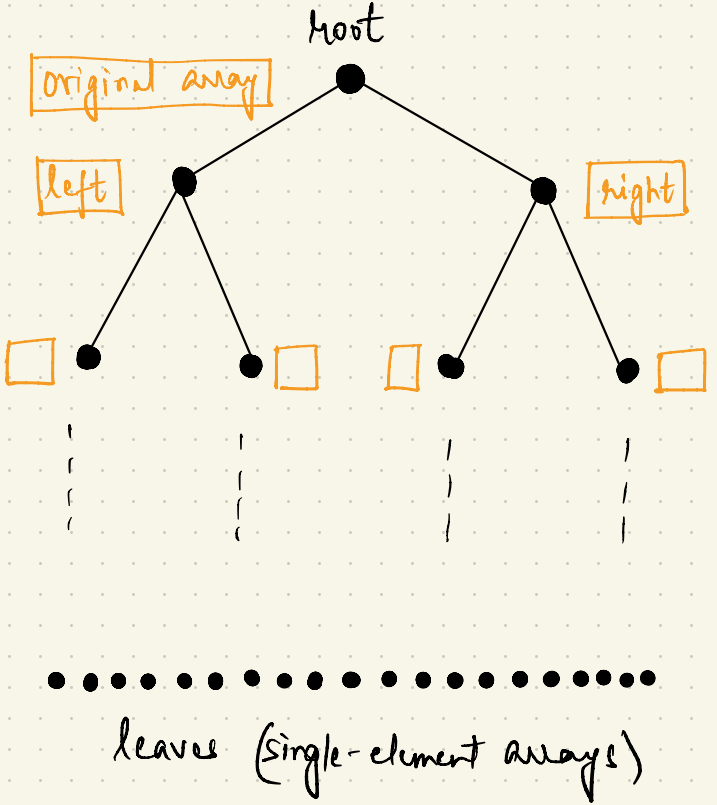

How much work does Merge Sort do at level j?

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Work done by merge at level j

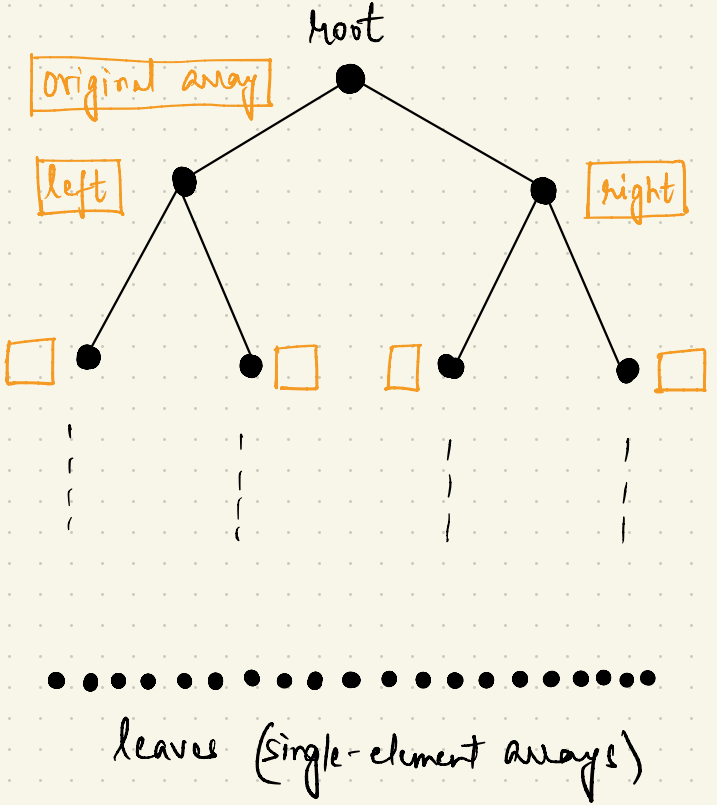
$\leq 2^j \times 6\left(n/2^j\right)$

2^j: # subproblems

$6\left(n/2^j\right)$: size $\ell = n/2^j$ (from Lemma)

MERGE SORT RUNNING TIME

Proof: (assuming n is power of 2)



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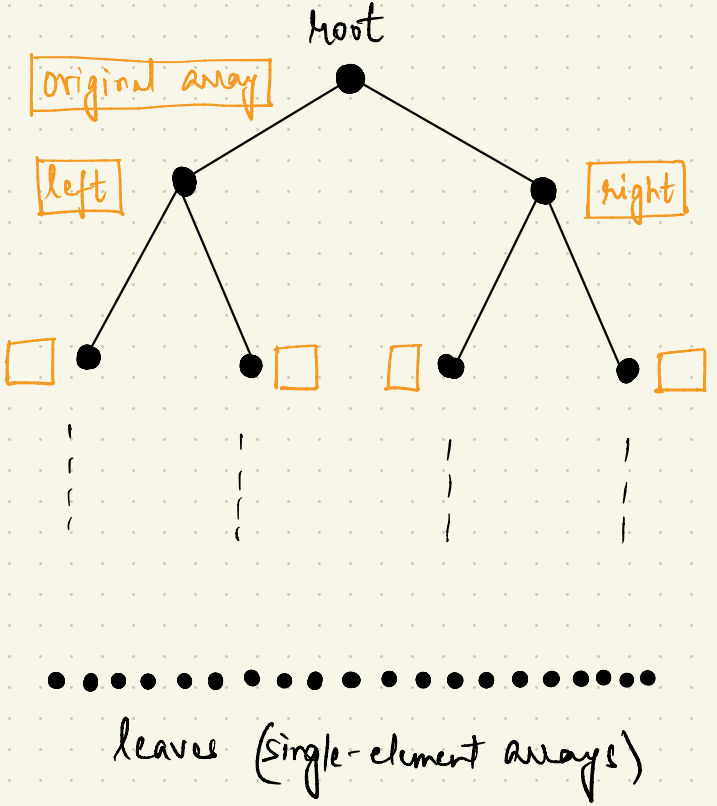

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Work done by merge at level j

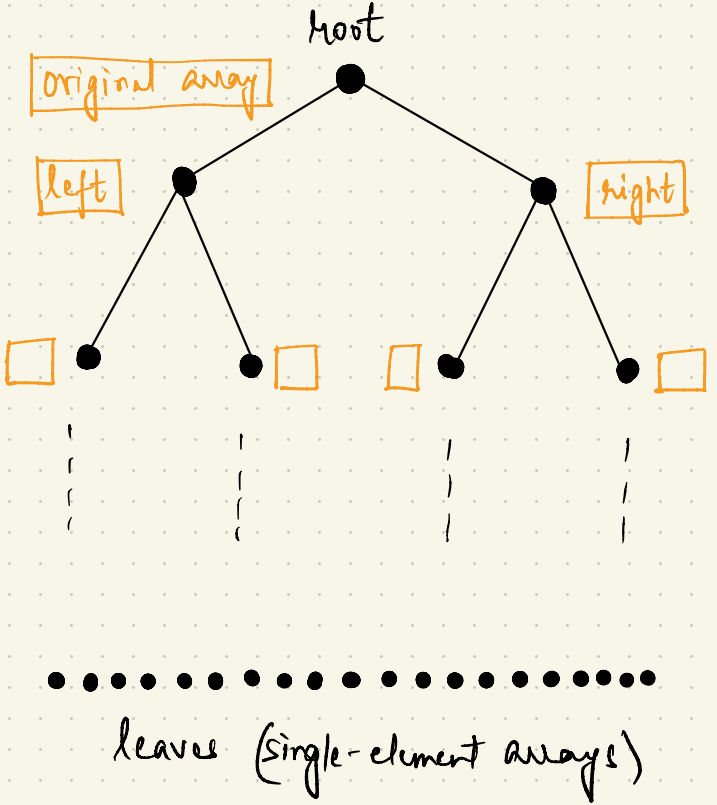
$$\leq 2^j \times 6\left(n/2^j\right)$$

$$= 6n$$

independent of j

MERGE SORT RUNNING TIME

Proof: (assuming n is power of 2)



How much work does Merge Sort do at level j?

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Work done by merge at level j

$\leq 2^j \times 6\left(n/2^j\right)$

$= 6n$

subproblems ↑ by 2X

size / subproblem ↓ by 2X

MERGE SORT RUNNING TIME

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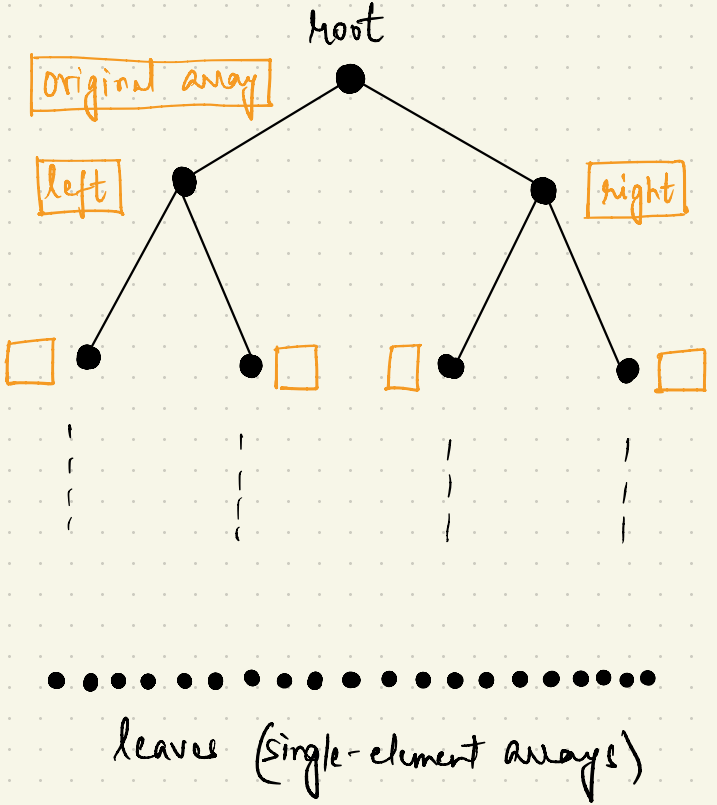

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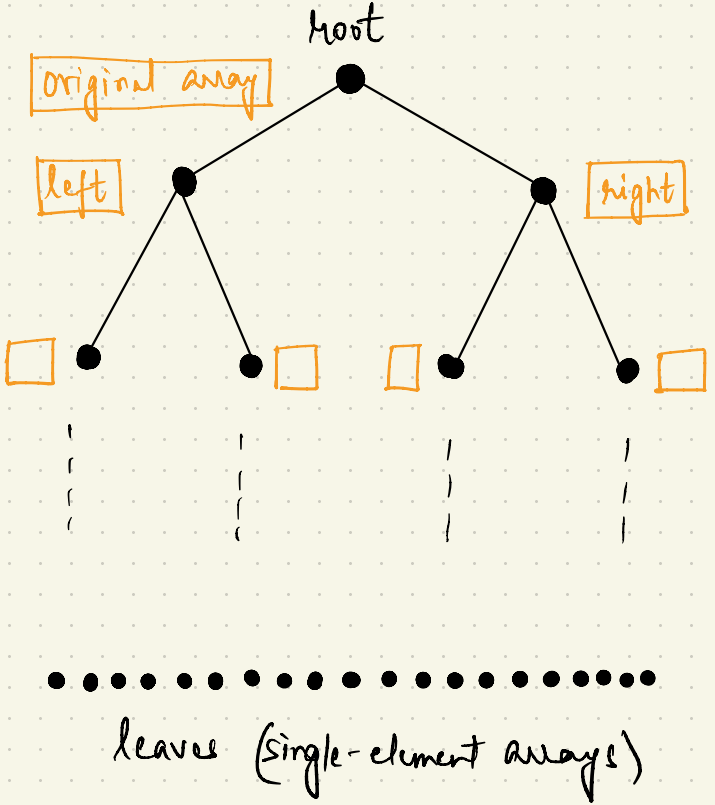
$\leq 2^j \times 6\left(n/2^j\right)$

$= 6n$

Total work done =

MERGE SORT RUNNING TIME

Proof: (assuming n is power of 2)



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$\leq 2^j \times 6\left(n/2^j\right)$

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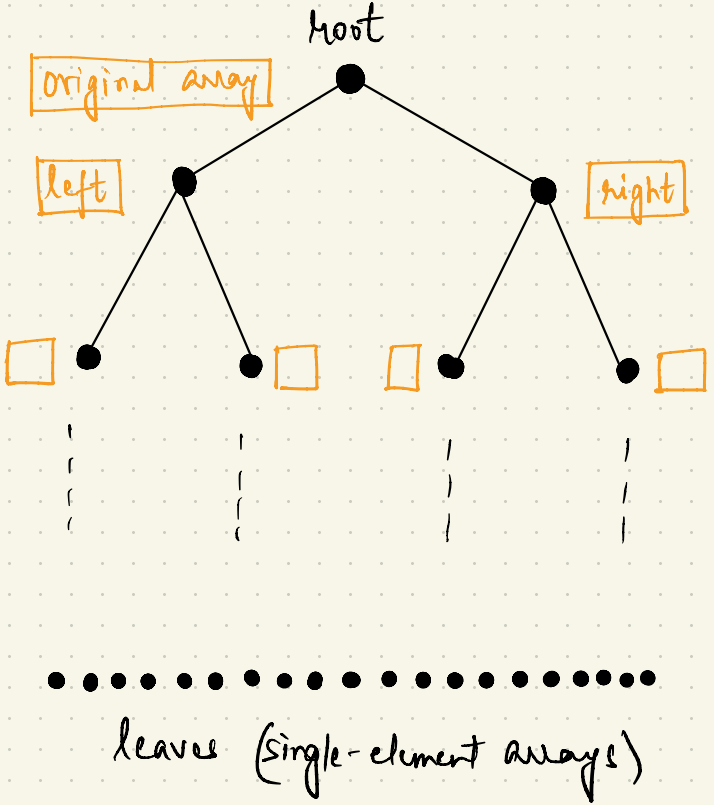
Total work done $= 6n \cdot \underbrace{(\log_2 n + 1)}_{\#\text{ levels}}$

MERGE SORT RUNNING TIME

Proof: (assuming n is power of 2)



How much work does Merge Sort do at level j?

=

Work done by merge at level j

$$\leq 2^j \times 6\left(n/2^j\right)$$

$$= 6n$$

Total work done $= 6n \cdot (\log_2 n + 1)$

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MAKING SOME ASSUMPTIONS EXPLICIT

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Worst-case (or adversarial) analysis

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Not too worried about precise constants

- used "6ℓ" instead of "$4\ell + 2$" for MERGE
- helps focus on properties of algorithms that transcend dependence on machine, programming language, architecture, etc.
- mathematically easier and no loss in predictive power

Asymptotic Analysis

ASYMPTOTIC ANALYSIS

Which running time is better : $6n \log_2 n + 6n$ or $\frac{1}{2} n^2$?

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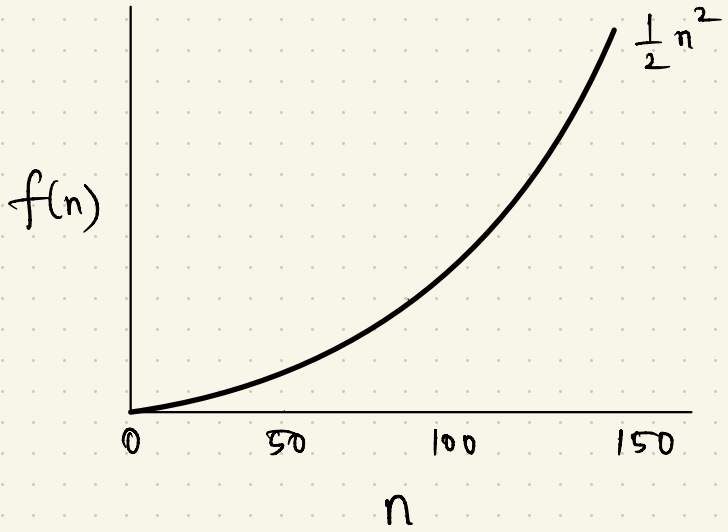

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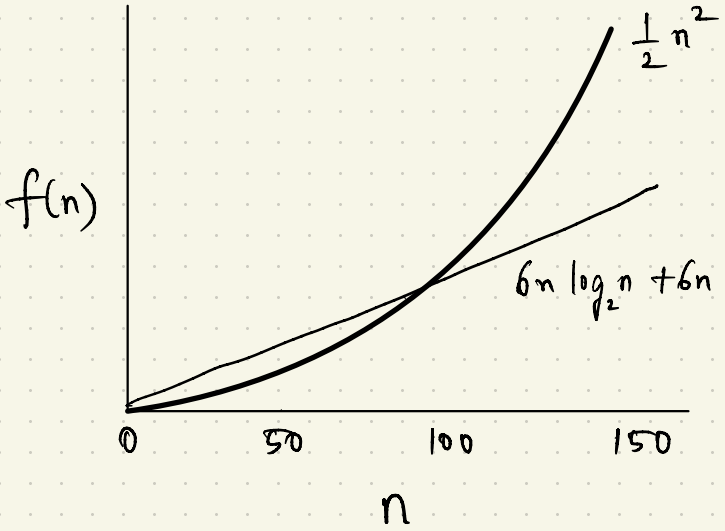

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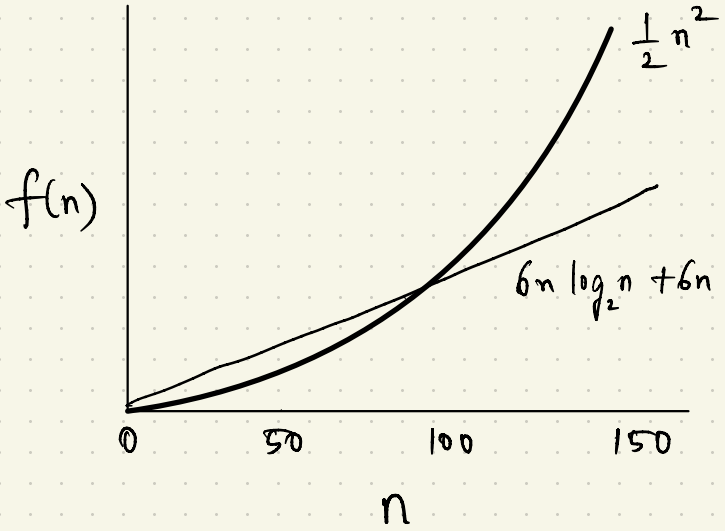

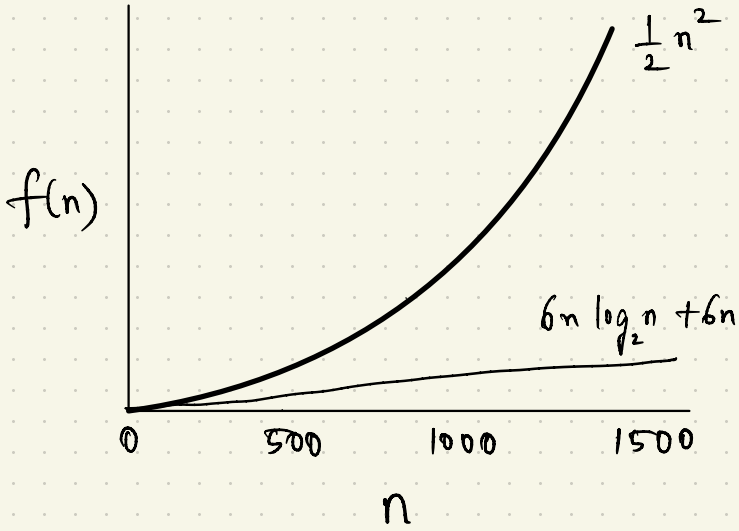

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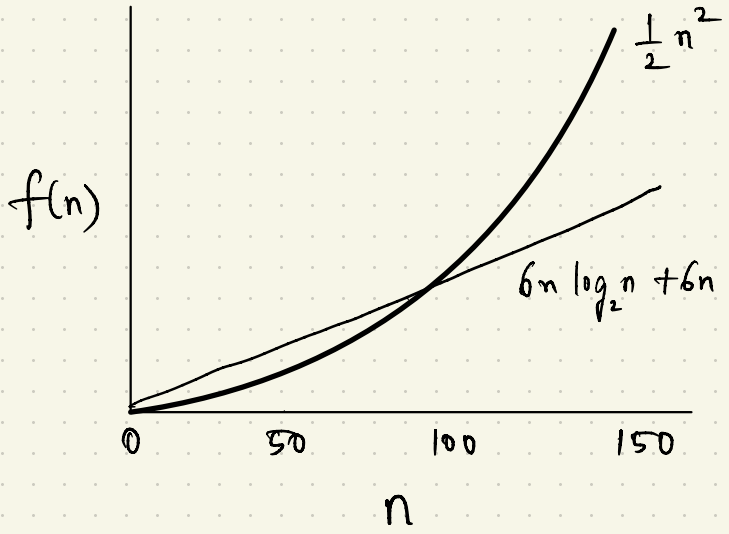

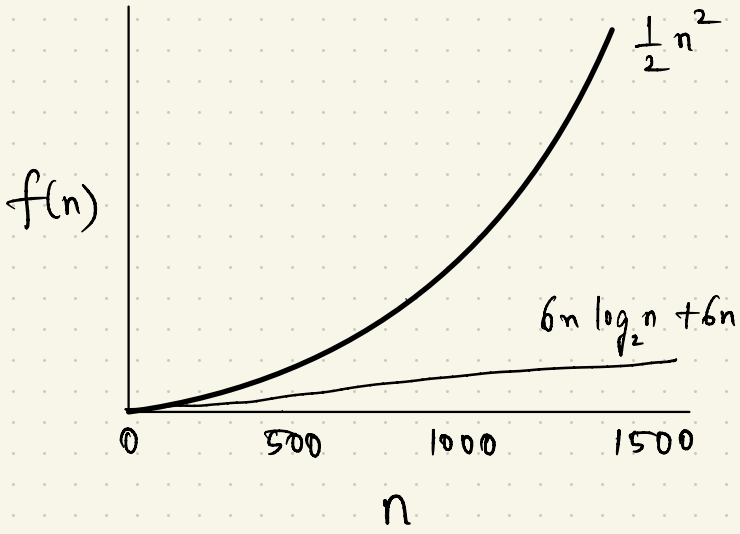

$6n \log_2 n + 6n$ is faster for sufficiently large inputs

Asymptotic Analysis

Why care about large inputs?

Asymptotic Analysis

Why care about large inputs?

- most algorithms can easily handle small inputs;
  it's the large inputs that require algorithmic cleverness

- our ambitions grow with technological progress!

Need fast algorithms for large inputs

WHAT IS A "FAST" ALGORITHM ?

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An algorithm whose worst-case running time grows slowly with input size

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polynomial time

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WHAT IS A "FAST" ALGORITHM ?

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Poly-time in theory $\Longrightarrow$ fast in practice

often but not always

Better Balance by Being Biased: A 0.8776-Approximation for Max Bisection

Per Austrin*, Siavosh Benabbas*, and Konstantinos Georgiou†

has a lot of flexibility, indicating that further improvements may be possible. We remark that, while polynomial, the running time of the algorithm is somewhat abysmal; loose estimates places it somewhere around $O\left(n^{10^{100}}\right)$; the running time of the algorithm of [RT12] is similar.

Theory Comput Syst (2014) 54:531–550
DOI 10.1007/s00224-013-9501-0

Picture-Hanging Puzzles

Erik D. Demaine · Martin L. Demaine · Yair N. Minsky · Joseph S.B. Mitchell · Ronald L. Rivest · Mihai Pătraşcu

Theorem 7 *For any $n \geq k \geq 1$, there is a picture hanging on n nails, of length $n^{c'}$ for a constant c', that falls upon the removal of any k of the nails.*

of sorting networks is under $6,100 \log_2 n$. Applying Theorem 3, we obtain a picture hanging of length $c^{6,100 \log_2 n} = n^{6,100 \log_2 c}$. Using the $c \leq 1,078$ upper bound, we obtain an upper bound of $c' \leq 6,575,800$. Using the $c \leq 256 + o(1)$ upper bound, we obtain an upper bound of $c' \leq 1,561,600 + o(1)$.

So, while this construction is polynomial, it is a rather large polynomial. For small values of n, we can use known small sorting networks to obtain somewhat reasonable constructions.

Simplex algorithm

33 languages

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From Wikipedia, the free encyclopedia

This article is about the linear programming algorithm. For the non-linear optimization heuristic, see Nelder–Mead method.

In mathematical optimization, Dantzig's **simplex algorithm** (or **simplex method**) is a popular algorithm for linear programming.[1]

Not poly-time in the worst case!