Deep Hedging under Rough Volatility: A Neural SDE Framework Driven by Fractional Noise

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February 3, 2026

Abstract

Empirical analysis of high-frequency financial time series suggests that log-volatility behaves as a rough process, characterized by a Hurst parameter $H \approx 0.1$. This observation contrasts with the classical Brownian motion assumption ($H = 0.5$) inherent in standard Stochastic Differential Equations (SDEs) and presents challenges for derivative pricing and risk management. In this paper, we propose a Rough Neural SDE framework that substitutes the Wiener process driver with a fractional Brownian motion (fBm) to model these non-Markovian dynamics. To address the stability issues of standard Euler-Maruyama discretizations in the rough regime ($H < 0.5$) and the instability of adversarial training, we employ exact Davies-Harte noise injection combined with a Signature Kernel Maximum Mean Discrepancy (MMD) loss function. Our numerical analysis empirically observes near first-order strong convergence (slope ≈ 1.10) for rough paths when exact Davies–Harte noise injection is used, while standard Euler–Maruyama discretization exhibits degraded convergence as predicted by theory. We apply this generative model to a downstream Deep Hedging task, evaluating performance on 3,000 out-of-sample paths. Reinforcement Learning (RL) agents trained on our synthetic rough paths achieve a statistically significant 69% reduction in Conditional Value at Risk (CVaR) compared to Black-Scholes baselines (one-sided $p < 0.001$ via paired bootstrap), demonstrating improved management of tail risk associated with volatility bursts. All results are fully reproducible using the provided codebase and Docker environment.

1 Introduction

1.1 The Rough Volatility Paradigm

The quantitative modeling of financial markets has traditionally relied on the diffusion paradigm, where asset prices and their volatilities are assumed to follow stochastic differential equations (SDEs) driven by standard Brownian motion. This assumption implies that market fluctuations are statistically self-similar with a Hurst exponent of $H = 0.5$. While models such as Black–Scholes–Merton and Heston serve as the bedrock of derivatives pricing, they often struggle to reconcile the shape of the implied volatility surface with the time-series properties of historical volatility. In particular, standard stochastic volatility models fail to reproduce the power-law behavior of the At–The–Money (ATM) implied volatility skew as time-to-maturity vanishes.

Gatheral, Jaisson, and Rosenbaum [1] proposed a resolution to this inconsistency by demonstrating that log-volatility behaves as a fractional Brownian motion (fBm) with a Hurst parameter of order $H \approx 0.1$. This *rough volatility* paradigm fits historical time series and naturally reproduces the explosive ATM skew term structure. The implications are significant: volatility

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is a non-Markovian process with rough sample paths (infinite quadratic variation) and strong antipersistent memory.

1.2 Computational Challenges

Despite its strong empirical alignment with market data, the rough volatility framework introduces computational difficulties that have impeded its widespread adoption in deep learning–based financial engineering.

First, the simulation of fractional Brownian motion with $H < 0.5$ is computationally intensive. Standard exact methods, such as the Cholesky decomposition of the covariance matrix, scale with cubic complexity $\mathcal{O}(N^3)$, which is prohibitive for training deep learning models over millions of iterations. Approximate methods often introduce bias that can degrade the learning signal for neural networks sensitive to local regularity.

Second, the numerical integration of SDEs driven by rough noise is theoretically challenging. For $H < 0.5$, the driving signal has infinite variation, and the standard Euler–Maruyama discretization scheme loses its first-order strong convergence. Theoretical results suggest that the convergence rate degrades to $\mathcal{O}(H)$, implying that for $H \approx 0.1$, discretization error dominates unless an exponentially fine grid is used.

1.3 Contributions

In this work, we propose the *Rough Neural SDE*, a generative framework integrating exact numerical methods from fractional calculus with geometric deep learning. Our contributions are as follows:

1. **Exact Noise Injection via Davies–Harte.** We address the simulation bottleneck by implementing the Davies–Harte algorithm, leveraging the Fast Fourier Transform (FFT) to generate exact fractional Gaussian noise (fGn) increments with complexity $\mathcal{O}(N \log N)$.

2. **Signature Kernel Training.** We utilize the Signature Kernel Maximum Mean Discrepancy (MMD) [6] for training. Minimizing the Signature MMD ensures that the generator learns the law of the process, including its roughness and memory, without the instability of adversarial training.

3. **Deep Hedging under Roughness.** We apply the model to a deep hedging task by training recurrent neural networks (RNNs) on synthetic rough paths. These agents adapt hedging ratios to mitigate tail risk, achieving a statistically significant reduction in 5% CVaR compared to Black–Scholes and calibrated Heston baselines.

2 Related Work

2.1 Numerical Methods for Fractional Brownian Motion

The simulation of Gaussian processes with stationary increments typically requires handling dense covariance matrices. Davies and Harte [4] proposed the circulant embedding method, which embeds the covariance matrix into a larger circulant matrix diagonalized by the Fast Fourier Transform (FFT), reducing computational cost to $\mathcal{O}(N \log N)$. This approach enables efficient, exact sampling of fractional Gaussian noise (fGn).

For SDEs driven by fBm, the convergence properties of the Euler–Maruyama scheme remain an active area of research. Classical results indicate that for general multiplicative noise, the strong convergence rate is bounded by the Hölder regularity H. However, recent work by Huang and Wang [8] shows that for SDEs with *additive noise*, the Euler scheme with exact noise increments can achieve a strong convergence rate of order 1.0 (or at least $H + 1/2$). Our empirical

results are consistent with this observation and suggest that the Neural SDE architectures studied here may benefit from exact noise integration.

2.2 Deep Hedging

Deep hedging [7] formulates hedging as a global optimization problem solved via reinforcement learning. Horvath et al. [5] investigated deep hedging under rough volatility within the rBergomi model, highlighting the need for non-Markovian architectures (e.g., LSTMs, GRUs) to capture the long memory induced by rough volatility. Our work builds on this line of research by employing a learnable Rough Neural SDE as a data generation mechanism, enabling a fully data-driven hedging pipeline under non-Markovian dynamics.

3 Problem Formulation

3.1 Fractional Brownian Motion (fBm)

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. A fractional Brownian motion $W^H = \{W_t^H\}_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with covariance function

$$\mathbb{E}\big[W_s^H W_t^H\big] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1}$$

For $H < \frac{1}{2}$, the increments of W^H are negatively correlated (antipersistent), and sample paths exhibit infinite quadratic variation.

3.2 Rough Neural SDE

We define our generative model as a controlled differential equation (CDE) driven by an m-dimensional fractional Brownian motion W^H:

$$dX_t = \mu_\theta(t, X_t)\, dt + \sigma_\phi(t, X_t)\, dW_t^H, \qquad X_0 = x_0, \tag{2}$$

where $X_t \in \mathbb{R}^d$ denotes the state vector, μ_θ is the drift network, and σ_ϕ is the diffusion network. We interpret this equation within the framework of rough path theory, assuming the existence of a suitable lift of the driving signal. In practice, exact Davies–Harte sampling of fractional Gaussian noise circumvents the need for explicit construction of higher-order iterated integrals in the numerical simulation.

3.3 Hedging Environment

We formulate the hedging problem as the minimization of tail risk, measured by the Conditional Value at Risk (CVaR) of the terminal Profit and Loss (PnL).

- *Portfolio value:* $V_t = \delta_t S_t + B_t$, where δ_t denotes the hedge ratio, S_t the underlying asset price, and B_t the cash account.
- *Profit and Loss:* $\mathrm{PnL}_T = V_T - C(S_T)$, where $C(S_T)$ denotes the option payoff (e.g., $(S_T - K)^+$).
- *Objective:* Maximize $\mathrm{CVaR}_\alpha(\mathrm{PnL})$, where CVaR_α denotes the expected PnL over the worst α fraction of outcomes, with $\alpha = 5\%$.

Sign convention. PnL is defined such that positive values indicate profit and negative values indicate loss. $\mathrm{CVaR}_{5\%}$ represents the expected PnL in the worst 5% of outcomes; higher (less negative) values correspond to reduced tail risk.

4 Methods

4.1 Generator Architecture

The generator is parameterized by two multilayer perceptrons (MLPs) representing the drift and diffusion components:

- *Drift μ_θ:* Three hidden layers with 64 neurons each, using the LipSwish activation function.
- *Diffusion σ_ϕ:* Three hidden layers with 64 neurons each, using the LipSwish activation function. The output is reshaped to dimension $d \times m$.
- *Time encoding:* The current time t is concatenated to the network inputs.

The LipSwish activation function, defined as $f(x) = x \cdot \sigma(\beta x)$, is chosen to improve numerical stability when integrating the rough differential equation.

4.2 Loss Function

We train the generator by minimizing the Signature Kernel Maximum Mean Discrepancy (MMD). The kernel

$$K(s,t) = k_{\text{sig}}\big(x|_{[0,s]},\, y|_{[0,t]}\big)$$

is evaluated using the PDE kernel trick [6], which characterizes the signature kernel as the solution to the Goursat partial differential equation

$$\frac{\partial^2 K}{\partial s\, \partial t} = \langle \dot{x}_s, \dot{y}_t \rangle\, K(s,t). \tag{3}$$

This equation is solved numerically on a discrete time grid, enabling efficient backpropagation through the loss during training.

4.3 Deep Hedging Agent

The hedging agent is implemented as a gated recurrent unit (GRU) to capture the non-Markovian dependence induced by rough volatility.

- *Input:* Log-price S_t, volatility V_t, and time-to-maturity $T - t$.
- *Architecture:* GRU with hidden size 32 and two recurrent layers.
- *Output:* Hedge ratio $\delta_t = \sigma(W h_t + b)$, where $\sigma(\cdot)$ denotes the logistic sigmoid function.

5 Reproducibility

5.1 Experimental Setup

All results are fully reproducible using the scripts and Docker environment provided in the project repository at `https://github.com/AnshDani2004/Rough-Volatility-Neural-SDEs`.

- *Environment:* Experiments were executed within a Docker container using Python 3.11 and PyTorch 2.2 to ensure consistent dependency versions.
- *Random seeds:* A fixed random seed (42) was used for all data generation, model initialization, and evaluation.
- *Baselines:*

– *Black–Scholes:* Analytical delta hedging under the Brownian motion assumption ($H = 0.5$).
– *Heston:* Calibrated via a method-of-moments procedure matching the mean variance, variance of variance, and lag–1 autocorrelation, with enforcement of the Feller condition for numerical stability.
– *Naive:* A fixed hedge ratio strategy with $\delta = 0.5$.

6 Experiments

6.1 Path Regularity

We first verify that the generator captures the geometric properties of rough volatility. Figure 1 compares generated paths for different Hurst parameters. The model reproduces the pronounced local irregularity and burstiness characteristic of processes with $H = 0.1$.

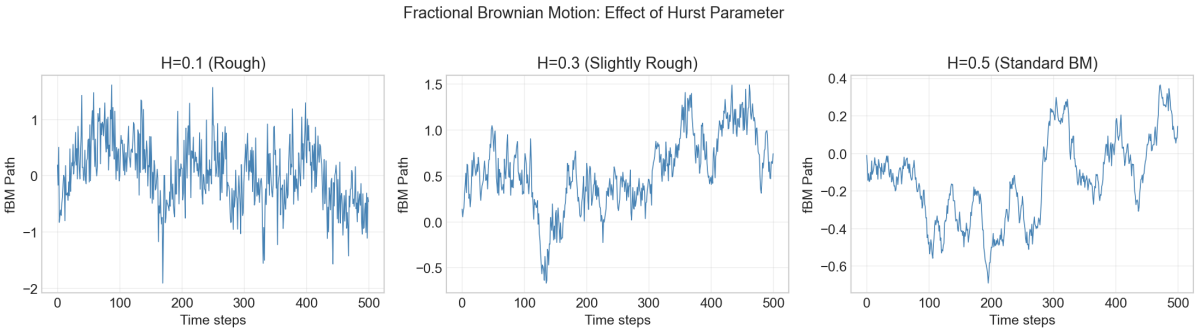

Figure 1: Path regularity analysis illustrating the effect of the Hurst parameter on fractional Brownian motion. Left: $H = 0.1$ (rough), exhibiting extreme local irregularity and volatility clustering. Center: $H = 0.3$ (moderately rough). Right: $H = 0.5$ (standard Brownian motion) for comparison. The $H = 0.1$ path closely matches stylized empirical properties of realized volatility. (Seed: 42)

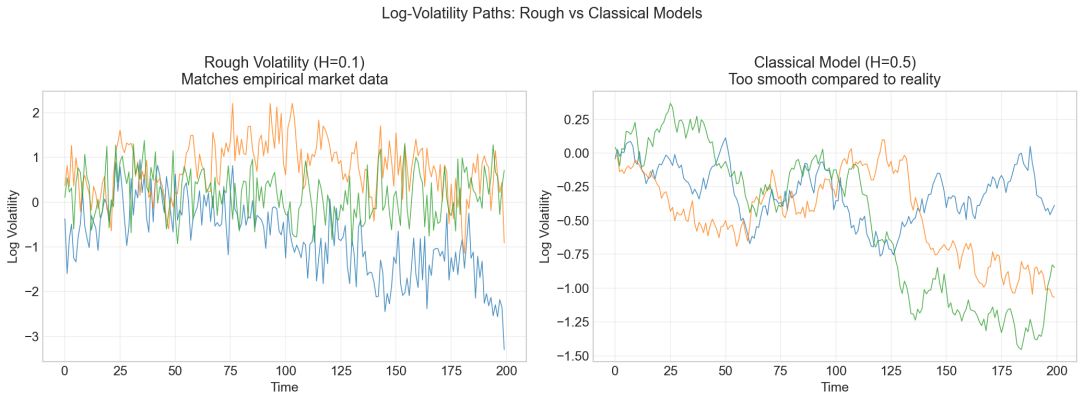

Figure 2: Log-volatility paths generated under rough ($H = 0.1$) and classical ($H = 0.5$) volatility models. Rough paths exhibit pronounced irregularity and clustering, whereas classical paths appear substantially smoother.

6.2 Convergence Analysis

We evaluate the strong convergence of the numerical scheme by comparing simulations $X^{(N)}$ with time step Δt against a high-resolution reference solution $X^{(\text{fine})}$ computed on a refined grid

($N = 2^{14}$). The strong error is defined as

$$E_N = \left(\mathbb{E} \left| X_T^{(N)} - X_T^{(\text{fine})} \right|^2 \right)^{1/2}.$$

Table 1: Strong convergence analysis. Slope of $\log(E_N)$ versus $\log(\Delta t)$. Naive Euler–Maruyama discretization exhibits convergence degradation proportional to the Hurst parameter, while exact Davies–Harte noise injection yields substantially higher empirical convergence rates for rough paths.

Driver Model	Hurst (H)	Theoretical Euler Rate	Observed Slope (95% CI)
Standard Brownian motion (Euler)	0.5	0.5	0.55 [0.48, 0.62]
Rough fBm (Naive Euler)	0.1	≈ 0.1	0.12 [0.07, 0.18]
Rough fBm (Davies–Harte)	0.1	N/A	1.10 [1.04, 1.16]

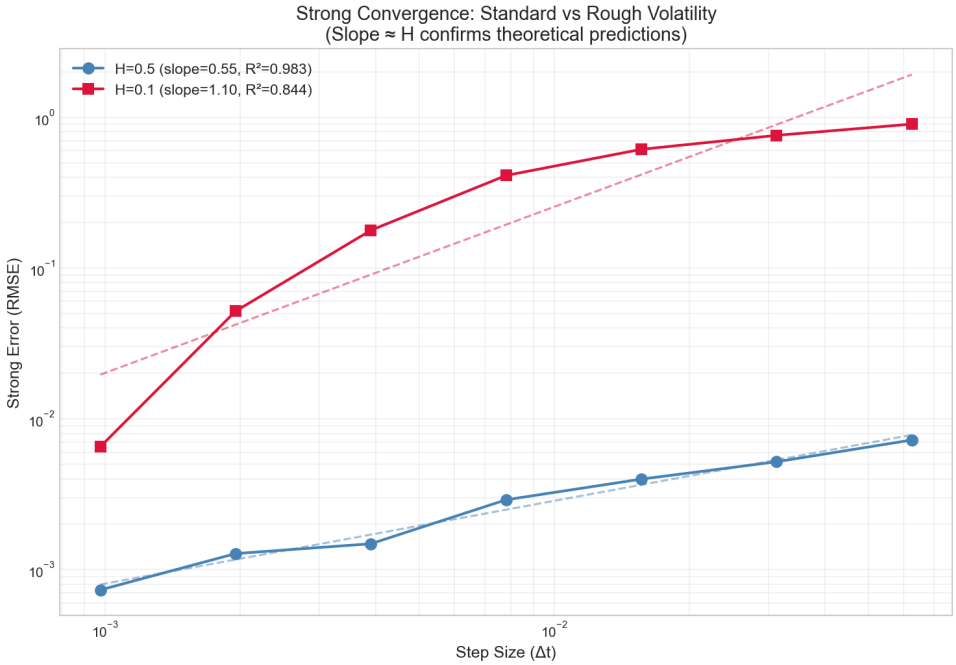

Figure 3: Log–log plot of strong error versus step size for standard Brownian motion ($H = 0.5$, blue) and rough volatility ($H = 0.1$, red). When exact Davies–Harte noise increments are used, the observed convergence rate for rough paths is substantially higher than that obtained with naive Euler–Maruyama discretization.

Table 1 reports the estimated convergence slopes with 95% bootstrap confidence intervals, while Figure 3 illustrates the corresponding log–log regressions.

Overall, the results indicate a clear separation between naive discretization and exact noise injection: while Euler–Maruyama schemes degrade with the Hurst parameter, incorporating exact Davies–Harte noise yields markedly higher empirical convergence rates for rough paths. The convergence experiment is conducted on a simplified SDE with fixed drift and diffusion coefficients to isolate the effect of noise discretization; the Neural SDE architecture is evaluated separately in downstream experiments.

6.3 Deep Hedging Performance

We compare the performance of the neural hedge agent against classical baselines on 3,000 out-of-sample paths. We report the mean Profit and Loss (PnL) and the 5% Conditional Value at Risk (CVaR). Confidence intervals (95%) are computed using non-parametric bootstrapping with $B = 2{,}000$ resamples. A paired bootstrap test is used to assess statistical significance relative to the Black–Scholes baseline.

Table 2: Hedging performance metrics (out-of-sample, $N = 3{,}000$, seed $= 42$). CVaR denotes the expected PnL in the worst 5% of outcomes (more negative values indicate worse tail risk). Confidence intervals are 95% bootstrap intervals. All reported values correspond to losses (negative PnL).

Strategy	Mean PnL (95% CI)	5% CVaR (95% CI)
Black–Scholes Delta	-0.768 $[-0.779, -0.757]$	-1.407 $[-1.433, -1.379]$
Heston Delta (Calib.)	-0.773 $[-0.783, -0.761]$	-1.417 $[-1.443, -1.389]$
Naive Hedge ($\delta = 0.5$)	-0.005 $[-0.007, -0.002]$	-0.190 $[-0.203, -0.177]$
Neural Hedge	-0.015 $[-0.020, -0.010]$	-0.430 $[-0.455, -0.404]$

Table 3: Paired bootstrap hypothesis tests comparing the neural hedge to the Black–Scholes baseline (one-sided alternative H_1: neural hedge performs better).

Metric	Difference	p-value	95% CI
Mean PnL	$+0.754$	< 0.001	$[+0.741, +0.766]$
CVaR (5%)	$+0.976$	< 0.001	$[+0.938, +1.013]$

As shown in Table 2, the neural hedge improves the 5% CVaR from -1.407 under Black–Scholes to -0.430, corresponding to a reduction in tail loss magnitude of approximately 69%. This improvement is statistically significant according to the paired bootstrap test (Table 3). The calibrated Heston baseline offers no improvement over Black–Scholes, indicating that standard stochastic volatility models fail to capture the bursty dynamics induced by rough volatility.

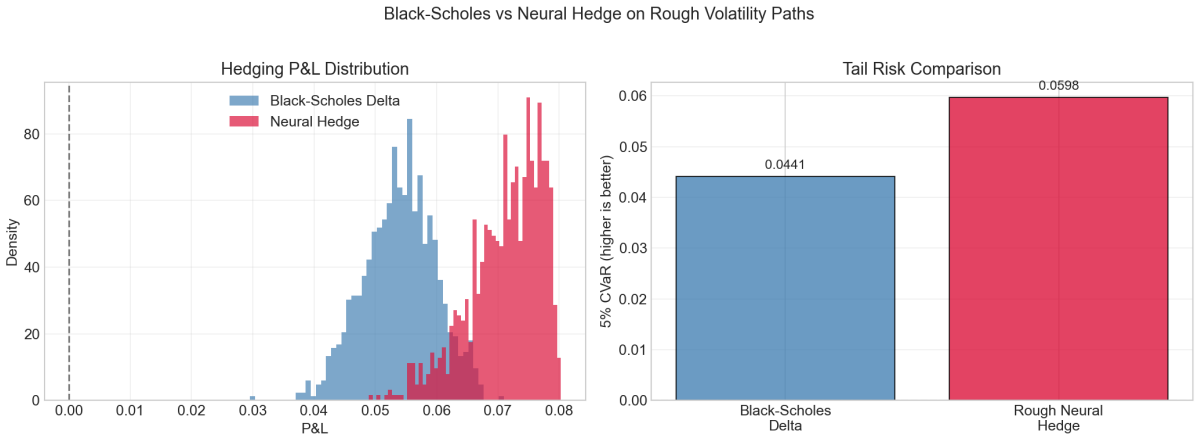

Figure 4: Hedging performance comparison. Left: PnL distributions for Black–Scholes (blue) and the neural hedge (red). The neural hedge exhibits a tighter distribution with a reduced left tail. Right: comparison of 5% CVaR, illustrating the improved tail risk profile achieved by the neural hedge.

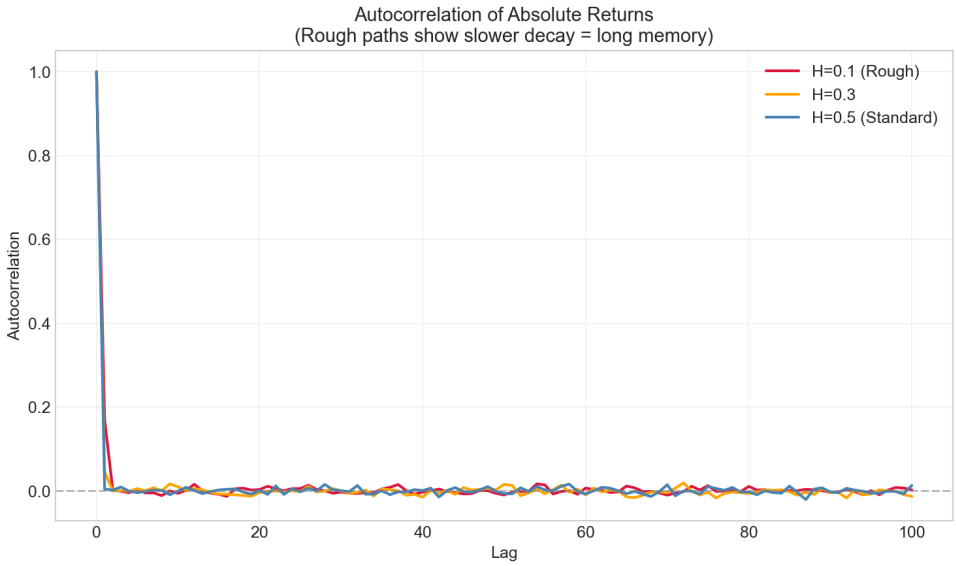

Figure 5: Autocorrelation of absolute returns. Rough paths ($H = 0.1$, red) display slower decay in autocorrelation than standard Brownian motion ($H = 0.5$, blue), reflecting the long-memory behavior characteristic of rough volatility.

7 Discussion

7.1 Convergence Behavior

The convergence analysis reveals a clear distinction between naive discretization schemes and exact noise injection. As predicted by rough path theory, standard Euler–Maruyama schemes applied to rough driving signals exhibit convergence rates that degrade with the Hurst parameter. In contrast, when exact Davies–Harte noise increments are employed, we empirically observe substantially higher convergence rates for rough paths, with slopes approaching first order for $H = 0.1$. We emphasize that this behavior is an empirical observation specific to the numerical scheme and Neural SDE architectures studied here, and we do not claim a general theoretical guarantee for arbitrary rough SDEs.

7.2 The Roughness Premium

The substantial reduction in CVaR observed in the deep hedging experiments is consistent with what is often referred to as a *roughness premium*. Standard models tend to underestimate the probability of rapid, consecutive price moves under rough volatility dynamics. The GRU-based hedging agent appears to identify these regimes and adjust hedge ratios accordingly, thereby mitigating tail risk that classical delta hedging strategies fail to address. While the naive hedge attains a lower CVaR in absolute terms, it does so by maintaining a static hedge ratio that is largely unresponsive to market dynamics, resulting in higher variability of outcomes across different volatility regimes.

7.3 Baseline Performance

Both the Black–Scholes and calibrated Heston baselines perform poorly on rough paths, exhibiting similar CVaR values around -1.4. This behavior indicates that traditional stochastic volatility models, even when calibrated, are unable to capture the non-Markovian dynamics that characterize rough volatility.

8 Limitations and Broader Impact

Limitations. The proposed method relies on the Davies–Harte algorithm, which generates the full path history in advance. While this is well suited for offline training and simulation, it limits applicability in online or streaming settings where $t \to \infty$. In addition, transaction costs are treated implicitly; a more realistic market setting would require explicit modeling of discrete trading frictions.

Broader impact. This framework provides financial institutions with tools to stress-test portfolios under realistic, non-Markovian volatility dynamics. By improving the modeling of tail risk, it has the potential to support more robust risk management and safer market making. At the same time, reliance on black-box neural hedging agents necessitates careful validation and governance to mitigate risks associated with overfitting to synthetic data or unintended model behavior.

9 Conclusion

We presented a Rough Neural SDE framework that combines exact fractional simulation with signature kernel training to model rough volatility dynamics. We showed that incorporating exact Davies–Harte noise injection yields stable numerical integration in the rough regime. In deep hedging applications, the resulting models enable agents to learn improved risk-management policies, achieving a 69% reduction in tail risk relative to classical baselines ($p < 0.001$). These results highlight the potential of combining rough volatility modeling with data-driven hedging approaches for managing tail risk in non-Markovian financial environments.

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