Q3

$A_{m \times n}$ $\quad m \leq n$ $\quad P = A^T A \quad {}_{n \times n}$

$Q = A A^T \quad {}_{m \times m}$

(a)

$$y^T P y = y^T A^T A y$$
$$= (Ay)^T (Ay)$$
$$= \|Ay\|^2 \geq 0$$

$$z^T Q z = z^T A A^T z$$
$$= (A^T z)^T A^T z$$
$$= \|A^T z\|^2 \geq 0$$

$\Rightarrow$ P & Q are both positive semi definite

For psd matrices, eigen values are non negative since let's say ~~a~~ negative eigen value λ exists with eigenvector v

$$Pv = \lambda v$$

$$v^T P v = v^T \lambda v = \lambda \|v\|^2$$

$\downarrow \geq 0$ by proof above

$$\lambda \|v\|^2 \geq 0 \Rightarrow \lambda \geq 0$$

This is a contradiction.

(b) u is eigen vector of P with eig value λ

~~Att~~ $\Rightarrow$ $Pu = \lambda u$

$$A^T A u = \lambda u \qquad (1)$$

~~Q~~ $Q\,Au$

$$= (AA^T) Au = A (A^T A u)$$

$$Q\,Au = \lambda Au$$

$\Rightarrow$ Au is eigen vector of Q with eigen value λ.

v is eig vector of Q with eigen value μ

$$\Rightarrow Qv = \mu v$$

$$AA^T v = \mu v$$

$$P A^T v = A^T AA^T v = A^T \mu v = \mu A^T v$$

$$\Rightarrow P(A^T v) = \mu (A^T v)$$

$\Rightarrow$ $A^T v$ is an eigen vec of P with eigen value μ

u is $n \times 1$ vector

v is $m \times 1$ vector

(c) v_i is eig vec of Q. Let say with eigen value c_i

$$u_i \triangleq \frac{A^T v_i}{\|A^T v_i\|_2}$$

$c_i \geq 0$
(proved in (a))

To prove

$$\exists\, \gamma_i \text{ s.t } A u_i = \gamma_i v_i, \ \lambda_i \geq 0$$

$$A u_i = \frac{A A^T v_i}{\|A^T v_i\|_2} = \frac{Q v_i}{\|A^T v_i\|_2} = \frac{c_i v_i}{\|A^T v_i\|_2}$$

$$c_i \geq 0 \Rightarrow \frac{c_i}{\|A^T v_i\|_2} \geq 0$$

Let

$$\lambda_i \triangleq \frac{c_i}{\|A^T v_i\|_2}$$

$$\Rightarrow A u_i = \lambda_i v_i \quad \text{with } \lambda_i = \frac{c_i}{\|A^T v_i\|_2} \geq 0$$

Hence proved.

(d) From (c) we have

$$V_i = \text{eigen-vec}(Q)$$

$$u_i = \frac{A^T v_i}{\|A^T v_i\|_2}$$

$$A u_i = r_i v_i$$

further, $u_j^T u_i = 0$ if $j \neq i$ (proof in the problem statement itself)

$v_j^T v_i = 0$ " "

$$u_i^T u_i = \frac{(A^T v_i)^T (A^T v_i)}{\|A^T v_i\|^2} = 1$$

$v_i^T v_i = 1$ (assume norm of the eigen vector is one since eigen value can be adjusted accordingly)

$U = [v_1 | v_2 | \quad \cdots \quad | v_m]$ $m \times m$

$V = [u_1 | u_2 | u_3 \quad \cdots \quad | u_m]$ $n \times m$

Consider $U^T A V$

$$= \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \vdots \\ v_m^T \end{bmatrix} A \, [u_1 | u_2 \quad u_m]$$

Using $A u_i = r_i v_i$

C

$$U^T A V = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} \left[\gamma_1 v_1 \mid \gamma_2 v_2 \mid \cdots \gamma_m v_m \right]$$

$$= \begin{bmatrix} \gamma_1 v_1^T v_1 & \gamma_2 v_1^T v_2 & \cdots \\ \vdots & \ddots & \\ & & \ddots \end{bmatrix} = \left[\gamma_j v_i^T v_j \right]_{m \times m}$$

$\downarrow$

diagonal matrix

$\Gamma_{m \times m}$

since $v_j^T v_i = 0$

& if $j \neq i$

$v_i^T v_i = I$

so γ_i along the diagonals

So,

$$U^T A V = \Gamma$$

$$\underbrace{U U^T}_{I_{m \times m}} A \underbrace{V V^T}_{I_{n \times n}} = U \Gamma V^T$$

$$A = U \Gamma V^T$$

<u>Hence proved</u>.