Q2 $\mathcal{X} = \{x_1, x_2 \dots x_N\}$

$x_i \in \mathbb{R}^d$, average $= \bar{x}$

Proved in class

$\vec{e}$ that minimizes $\sum_{i=1}^{N} \| x_i - \bar{x} - (e \cdot (x_i - \bar{x}))\, e \|_2^2$

is obtained by maximizing $e^t C e$.

for $f \perp e$ $\cancel{f^t e = 0}$ $(f^t e = 0)$

to maximize $f^t C f$

f needs to ~~have~~ be the eigen vector with second highest value.

maximize $f^t C f$, subject to the constraints

(i) $f^t f = 1$ (direction)

(ii) $f^t e = 0$

$\Rightarrow$ use lagrange multipliers

$$L(f, \lambda_1, \lambda_2) = f^t C f - \lambda_1 (f^t f - 1) - \lambda_2 f^t e$$

$$\downarrow$$

$$\frac{\partial L}{\partial f} = 0$$

$$2Cf - \lambda_1 2f - \lambda_2 e = 0 \quad - (1)$$

~~left multiply with~~

Left multiply eqn (1) with e^t

$$2e^tCf - 2e^tf\lambda_1 - \lambda_2 e^te = 0$$

$\because$ e is an eigen vector of C ; $Ce = ke$, $e^tC = ke^t$ (C is cov matrix & symmetrical)

$$\underbrace{2 \cdot ke^tf - 2e^tf\lambda_1}_{=0} = \lambda_2$$

$= 0$ $\because e^tf = f^te = 0$ ($f \perp^r e$)

$\Rightarrow \lambda_2 = 0$

Substituting $\lambda_2 = 0$ back in equation (1)

$$2Cf - \lambda_1 2f = 0$$

$$Cf = \lambda_1 f \quad \text{and} \quad f^tCf = \lambda_1$$

$\Rightarrow$ f is an eigen vector of C with eigen value $= \lambda_1$

$\Rightarrow$ We have $\lambda_1 = f^tCf$, which we want to maximize. The highest eigen value already corresponds to e, so the optimal f for our constraints is the eigen vector corresponding to the second highest eigen value of C. (next highest eigen value exists since rank (C) >2 and it is lower than the eigen value for e since eigen values are given to be distinct)

Thus, for f perpendicular to e, to maximize f^tCf, f has to be an eigen vec of C with second highest eigen value.