
2-D SSM: A General Spatial Layer for Visual Transformers

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Abstract

A central objective in computer vision is to design models with appropriate 2-D inductive bias. Desiderata for 2D inductive bias include two-dimensional position awareness, dynamic spatial locality, and translation and permutation invariance. To address these goals, we leverage an expressive variation of the multidimensional State Space Model (SSM). Our approach introduces efficient parameterization, accelerated computation, and a suitable normalization scheme. Empirically, we observe that incorporating our layer at the beginning of each transformer block of Vision Transformers (ViT) significantly enhances performance for multiple ViT backbones and across datasets. The new layer is effective even with a negligible amount of additional parameters and inference time. Ablation studies and visualizations demonstrate that the layer has a strong 2-D inductive bias. For example, vision transformers equipped with our layer exhibit effective performance even without positional encoding.¹

1 Introduction

Incorporating image-specific inductive bias into computer vision networks could play a crucial role in their success, by shaping the hypothesis space in a way that fits image data and improves generalization. Common ingredients of image-specific inductive bias include two-dimensional neighborhood structure, locality, translation equivariance and invariance, and extraction of hierarchical features. Traditionally, it was injected into the model through the backbone architecture. However, more recently, it has been modeled as part of the data. For example, two-dimensional neighborhood structures are typically expressed in one of two ways: (i) Vision Transformers [10] use 1-D positional encoding [45], which is considered weak inductive bias. (ii) ConvNets employ 2-D kernels, which provide strong priors on the underlying image structure [43].

Most ConvNets employ relatively small filters in the convolution layers, and the balance between local and global features is handled by increasing the receptive field with depth. However, other kernel sizes can be beneficial. For example, ConvNeXt improved ResNet by 0.7% on Imagenet, by only increasing its kernel size from 3×3 to 7×7 [33]. More generally, using fixed-size filters limits the type of dependencies the layer can capture.

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¹The implementation of the method is available at https://github.com/ethanbar11/ssm_2d

The objective of this study is to develop a new layer that is adept at integrating both local and global spatial features, with a particular focus on a two-dimensional neighborhood structure. We accomplish this by building on recent developments in 1-D SSM-based layers, which are renowned for capturing various types of dependencies. By extending this 1-D concept to 2-D, our layer is deeply rooted in control theory, and much like its predecessors, maintains a strong bias towards position awareness and parameter efficiency.

Our main contribution is the 2D-SSM layer, which is a new spatial layer based on Roesser's model for multidimensional state space [27]. We show that simple design choices, such as diagonalization and normalization, can make the layer numerically stable and efficiently computed without recurrence using a 2-D convolution (left panel of Fig. 1). Our layer has some unique properties, including: (i) A strong inductive bias towards two-dimensional neighborhood and locality, which stems from the multi-dimensional recurrent rule. As far as we know, this novel concept does not appear in other layers, (ii) The new layer can capture unrestricted controllable context. The SSM parameters of the layer can be focused on short or long, horizontal, vertical, or diagonal dependencies (middle panel of Fig. 1). (iii) The layer is parameter-efficient and can express kernels of any length via 8 scalars. Visualization and ablation studies demonstrate these key aspects of our layers. Finally, the layer is well-grounded in control theory, and further theoretical analysis shows that it generalizes S4ND [37] and proves its greater expressiveness.

Empirically, we showed that our layer can be used as a general-purpose booster for vision transformers (the schematic architecture is illustrated in the right panel of Fig. 1), with negligible additional parameters and computation at inference. Furthermore, it appears that our 2D-SSM surpasses standard methods, such as incorporating positional encoding, in effectively integrating positional bias into Vision Transformers.

2 Background and Notations

Framing Our research delves into two emerging research domains. The first domain focuses on the development of multi-axes global convolution techniques. Although 1-D (long) global convolution has shown promise in 1-D sequence modeling, leveraging methods such as SSM [6, 17, 19, 20, 36] or other recent approaches [13],[38],[31], its applicability and effectiveness in modern computer vision tasks remain uncertain. Our work aims to explore and highlight the potential of these techniques in this domain, by extending them into 2-D.

The second domain investigates the synergistic combination of attention and SSM in 1-D modeling across various domains[35, 39, 25, 49, 6, 36]. For example, the SSM-based H3 [6] outperforms GPT-Neo-2.7B [2] (as well as other transformers of the same size) with only 2 attention layers. However, the question of whether these components are complementarity in 2-D modeling remains unanswered. We provide empirical evidence supporting the complementary nature of these components.

State Space Model (SSM) The state space model maps an input scalar function $u(t) : \mathbb{R} \rightarrow \mathbb{R}$ to a N-D latent state $x(t) \in \mathbb{R}^N$ before projecting it to an output signal $y(t) : \mathbb{R} \rightarrow \mathbb{R}$:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \quad (1)$$

The use of SSMs is widespread across numerous scientific disciplines and is closely associated with latent state models, such as Hidden Markov Models. There is a well-known connection between linear time-invariant SSMs, such as 1 and continuous convolution, thus allowing efficient training using the aforementioned equation as a discrete convolution. The S4 [17] and LSSL [19] layers leveraged the SSM as a black-box representation in deep sequence modeling, with learned parameters A, B, C, and D, and achieved strong results on several tasks, especially ones that require handling long-range dependencies, such as the Long Range Arena (LRA) [41], audio generation [15], and long-text processing [36, 16, 6].

The underlying reasons for the suitability of S4 and SSMs for modeling long sequences were recently analyzed [31, 13]. It was found that (i) employing global kernels with decaying structures, and (ii) using regularized kernels, are both critical design choices in this area.

Roesser's 2D-State Space Model The attempt to extend the classical SSM for 2-D multi-axes systems was thoroughly studied in the past. [27, 11, 12, 28, 14, 22] notes a few different formulations of the problem. We employ Roesser's SSM model [27] as our discrete multi-axial model, which is the most general form of 2-axial and N-axial state-space models. As opposed to other SSMs already

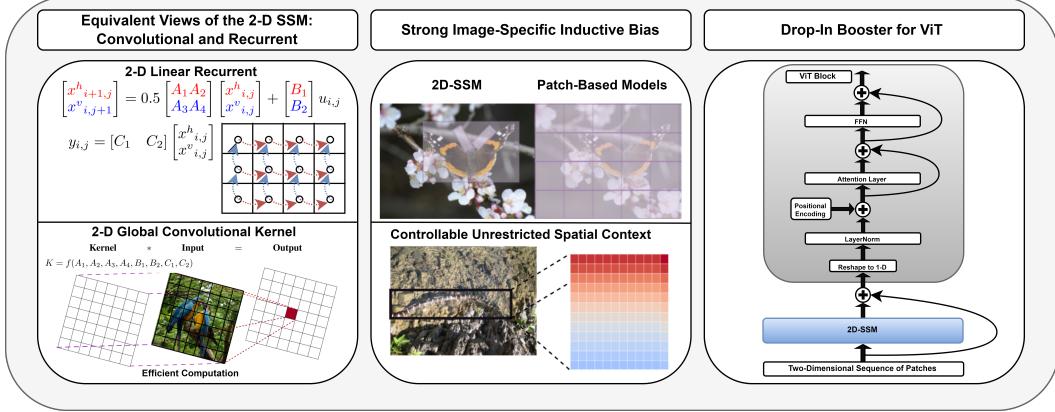


Figure 1: (Left) The 2-D SSM layer is parameterized by A, B, C, and D. It is built on top of a two-axis linear recurrent and can be efficiently computed using 2-D convolution. (Center) Since the layer is based on two-dimensional recurrence, it exhibits a strong bias toward positional awareness. The recurrent is unrestricted, allowing the layer to operate on 2-D sequences of any length. The values of A_1, A_2, A_3 , and A_4 control the layer’s focus, enabling it to capture short or long spatial dependencies in horizontal, vertical, or diagonal directions, as opposed to patch-based models. (Right) The layer can be easily integrated into ViT by applying it to the two-dimensional sequence of patches at the beginning of each transformer block.

in use in machine learning, this SSM uses M states, one per axis, rather than just one. The model in 2-D form is presented here:

$$x_{i,j} = \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix}, \quad y_{i,j} = [C_1 \quad C_2] \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix}, \quad \begin{bmatrix} x_{i,j+1}^h \\ x_{i+1,j}^v \end{bmatrix} = \begin{bmatrix} A_1 A_2 \\ A_3 A_4 \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j} \quad (2)$$

where the state $x_{i,j} \in R^{2N}$ is the concatenation of the horizontal $x_{i,j}^h \in R^N$ and vertical $x_{i,j}^v \in R^N$ states, the system matrices are $A_1, A_2, A_3, A_4 \in R^{N \times N}$ and the input and output matrices are $B_1, B_2, C_1, C_2 \in R^N$. There is also a learned parameter D that behaves as a skip-connection, omitted from now on for brevity.

2.1 Notation

The notation follows as closely as possible the notation used in the state-space layer literature [17, 20, 18]. Specifically, we use H as the number of channels, N as the state’s hidden dimension, L as the sequence length, n_{ssm} as the number of non-shared channels, and treat $A, B, C, D \in \mathbb{R}$ as the system matrices. Note that for brevity the system matrices are treated as real-valued, even though we also test a complex version of them. The number of axes is set to M .

Signal dimensions Although N -D SSM can be used in an N -Dimensional manner, since our paper focuses on using this model as a regularization method for ViT backbones and for simplifying the reading experience, we will treat it as a 2-D SSM.

$L_i \in \mathbb{R}$ is the sequence length along the i axes, $L_{tot} = L_1 * L_2$ is the total signal size, and $L_{max} = \max(L_1, L_2)$.

Kernel Notation K is the computed 2-D kernel such that $Y = U * K$, where $*$ denotes discrete convolution.

$$y_{i,j} = C_1 x_{i,j}^h + C_2 x_{i,j}^v = \sum_{0 \leq \hat{i} \leq i} \sum_{0 \leq \hat{j} \leq j} (C_1 k_{\hat{i}, \hat{j}}^h + C_2 k_{\hat{i}, \hat{j}}^v) u_{\hat{i}, \hat{j}} \quad (3)$$

2.2 Other related work

Multi-dimensional State Space Layers As far as we know, S4ND [37] is the only previous SSM-based layer that can naturally handle multidimensional data. S4ND is built on top of S4, and contains M separate instances of S4, where M is the number of axes. On the forward path, each S4 layer,

which we denote as SSM_g factors a one-dimensional local kernel k_g , and a global kernel K is computed as the outer products of those local kernels.

Vision Transformer Backbones To demonstrate the versatility and efficacy of our 2-D SSM layer as a plug-and-play component compatible with various ViTs, we evaluate its performance when integrated into the following backbone architectures: **(i) ViT** The original ViT that employs self-attention on a 1-D sequence of patches; it used a learnable 1-D position encoding. **(ii) Swin** The Swin Transformer [32] refines ViT by incorporating hierarchical structure and local connections within windows. It employs a shifted windowing scheme and stage-wise processing to efficiently capture global context, which enhances its performance across various vision tasks. **(iii) Mega** The Mega [35] model introduced a single-head gated attention mechanism enhanced with an exponential moving average, which imparts position-aware local dependencies to the attention mechanism. This approach addresses the limitations of the Transformer’s attention mechanism, such as weak inductive bias. Mega demonstrates superior performance across a variety of tasks, including the Long Range Arena, neural machine translation, language modeling, and image and speech classification. **(iv) DeiT** [42] is an adaptation of ViT, incorporating a class-token designed for distillation purposes.

Adding positional bias into transformers By design, Vision Transformers are permutation invariant, and thus a lot of work was put into injecting bias into them. Besides the standard positional encoding, the following methods are proposed:

Exponential Moving Average (EMA) The EMA is a common technique for smoothing time-series data and prioritizing recent data points. It is computed using $EMA_t = (1 - \alpha) \cdot EMA_{t-1} + \alpha \cdot u_t$, where EMA_t is the current EMA value, EMA_{t-1} is the previous value and α is the smoothing factor. It is being used in MEGA [35] to incorporate positional awareness bias into the attention.

Other 2-D bias contributions By design, Vision Transformers are permutation invariant, and thus a lot of work was put into injecting 2-D bias into them. A particular research direction emphasizes the introduction of positional bias via various positional encoding methods. For instance, the Swin Transformer [32] employs a learnable bias term referred to as relative positional bias. In contrast, Convit [7] utilizes the relative positional bias while modeling it as a soft inductive bias. This is achieved by initializing the attention function as a convolution and allowing the model to converge toward diverse attention forms. In alternative avenues of research, efforts have been made to modify the attention window through diverse techniques, such as incorporating a two-dimensional local bias by cropping [9] the attention window or integrating convolutional neural networks (CNNs) with attention mechanisms [5], [30]. Recently, an intriguing outcome in image classification was accomplished when the EMA mechanism was applied to image patches, as described in [35].

3 Method

In this section, we present the core of the 2-D SSM layer, which is our main technical contribution. This layer maps a 2-dimensional sequence $u_{i,j}$ to $y_{i,j}$, where $u_{i,j}, y_{i,j} \in \mathbb{R}$ for all $0 \leq i \leq L_1$ and $0 \leq j \leq L_2$. Similarly to previous SSM-based layers, we extend the core of our layer to a multi-directional and multidimensional layer, detailed in Appendix D.

3.1 2-D Recurrent State Space Model as Convolution

Eq. 2 is defined in a recursive manner. However, for reasons of efficiency, it is best to define operators in closed form and in a way that is concurrent with parallel processing. Inspired by previous work [17, 20, 35, 18], we exploit the fact that the SSM is linear and can therefore be expressed as a convolution with a kernel K . To do so, we first unroll the recurrent rule and then describe it in a closed-form formulation.

For simplicity, we assume that the initial states are zeros $\forall j \geq 0 : x^h_{-1,j} = 0$, and $\forall i \geq 0 : x^v_{i,-1} = 0$. The horizontal and vertical states at $i = 0, j = 0$ are:

$$x^h_{0,0} = B_1 u_{0,0}, \quad x^v_{0,0} = B_2 u_{0,0} \tag{4}$$

By applying the recurrent rule once at each axis:

$$x^h_{1,0} = A_1 B_1 u_{0,0} + A_2 B_2 u_{0,0} + B_1 u_{1,0}, \quad x^v_{0,1} = A_3 B_1 u_{0,0} + A_4 B_2 u_{0,0} + B_2 u_{0,1} \tag{5}$$

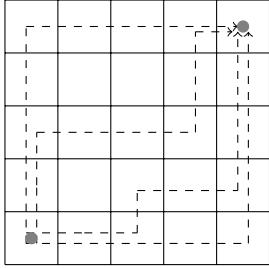


Figure 2: Examples of paths from coordinate $(\hat{i}, \hat{j}) = (0, 0)$ to $(i, j) = (4, 4)$. Each path represents a sequence of recursive calls for Eq. 2.

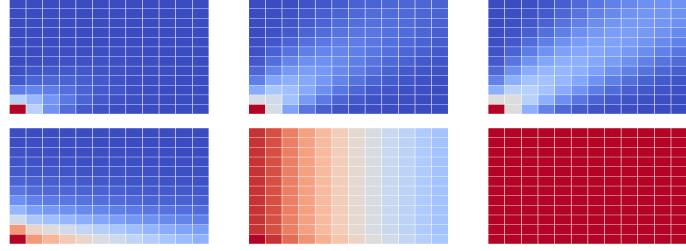


Figure 3: The kernels before and after the modifications of Sec. 3.2. Each column is created by the same $A_1 \dots A_4, B_1, B_2, C_1, C_2 \in \mathbb{R}$ parameters. The first row is the normalized 2-D SSM formulation explained in 2, the second is the outcome of Eq. 12 and performing Eq. 13, which is the kernel formulation we use. The bottom left corner of each heatmap is $K_{0,0}$. The figures demonstrate that before the relaxation, the kernels displayed a diagonal tendency while afterward, they exhibited a more diverse and versatile pattern.

$$x^v_{1,0} = B_2 u_{1,0}, \quad x^h_{0,1} = B_1 u_{0,1} \quad (6)$$

Next, we compute $x^h_{1,1}$ given $x^h_{0,1}$ and $x^v_{0,1}$.

$$x^h_{1,1} = A_1 A_3 B_1 u_{0,0} + A_1 A_4 B_2 u_{0,0} + A_1 B_2 u_{0,1} + A_2 B_1 u_{0,1} + B_1 u_{1,1}, \quad (7)$$

$$= k_{1,1}^h u_{0,0} + k_{1,0}^h u_{0,1} + k_{0,0}^h u_{1,1} \quad (8)$$

and in general

$$x^h_{i,j} = \sum_{0 \leq \hat{i} \leq i} \sum_{0 \leq \hat{j} \leq j} k_{\hat{i}, \hat{j}}^h u_{\hat{i}, \hat{j}}, \quad x^v_{i,j} = \sum_{0 \leq \hat{i} \leq i} \sum_{0 \leq \hat{j} \leq j} k_{\hat{i}, \hat{j}}^v u_{\hat{i}, \hat{j}}, \quad (9)$$

where as explained in the notation, each element $k_{\hat{i}, \hat{j}}^h$, $k_{\hat{i}, \hat{j}}^v$ is an aggregation of $A_1, A_2, A_3, A_4, B_1, B_2$ multiplications (e.g Eq. 7) and is associated with a single path from coordinate $(0, 0)$ to (i, j) , as presented in Fig 2.

By plugging Eq. 9 in Eq. 2 one obtains:

$$y_{i,j} = C_1 x^h_{i,j} + C_2 x^v_{i,j} = \sum_{0 \leq \hat{i} \leq i} \sum_{0 \leq \hat{j} \leq j} (C_1 k_{\hat{i}, \hat{j}}^h + C_2 k_{\hat{i}, \hat{j}}^v) u_{\hat{i}, \hat{j}} \quad (10)$$

and the convolutional kernel K is

$$\forall i, j : K_{i,j} = C_1 k_{\hat{i}, \hat{j}}^h + C_2 k_{\hat{i}, \hat{j}}^v \quad (11)$$

3.2 Efficient and Stable Parameterization

Parameter diagonalization Computing K is difficult for two reasons. Firstly, the number of elements in each $k_{i,j}$ is exponential in i and j , since it is equivalent to the number of paths from $(0, 0)$ to (i, j) . Secondly, calculating the powers of non-diagonal matrices A_1, A_2, A_3, A_4 becomes costly when L_1, L_2 are large. To overcome these challenges, we parameterized the system matrices A_1, A_2, A_3, A_4 as diagonal. This allows for efficient summation of at most $2L_{max}$ elements in $k_{i,j}^h$. Although diagonalization limits the expressiveness of our SSM, previous works have shown its effectiveness in one-dimensional cases [18, 20, 36, 21].

Limiting the parameters $A_i \in \mathbb{R}^{N \times N}$ is now a diagonal matrix. We'll denote the eigenvalues of A_i by $(\alpha_1, \alpha_2 \dots \alpha_N) := \text{diag}(A_i)$. Each α_i value behaves separately until the computation of K (Eq. 11). Thus, for $\alpha_i > 1$, $\lim_{z \rightarrow \infty} \alpha^z = \infty$. Therefore, we limit $\alpha_i \in [0, 1]$ by parameterizing it by $\alpha_i := \text{sigmoid}(\dot{\alpha}_i)$ and optimizing $\dot{\alpha}_i$ instead.

Normalization There is a scaling problem that arises from Eq. 7. Since $k_{1,1}^h = A_1A_3B_1 + A_1A_4B_2$, even if $A_1A_3B_1, A_1A_4B_2 \leq 1$, $k_{1,1}^h$ can be greater than 1. The same behavior makes the kernel explode.

We would like to keep $\forall i, j : 0 \leq k_{i,j}^h \leq 1$ and thus we employ a straightforward normalization mechanism, in which we divide by two each time we compute Eq. 2. This Eq. is thus replaced by:

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = 0.5 \begin{bmatrix} A_1A_2 \\ A_3A_4 \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j} \quad (12)$$

Relaxation of the kernel When imposing a "divide by two" normalization for every $k_{i,j}$, we impose a kernel formulation that is much more biased towards modeling diagonal kernels, i.e., if $|i - j|$ is small, $k_{i,j}^h$ is much larger than when $|i - j|$ is large.

Thus, we relax the normalization in the first row and column as follows. When calculating $k_{i,0}^h, k_{0,j}^h$ we use Eq. 2. Additionally, for $K_{i,0}, K_{0,j}$, we use $\hat{C}_1 = 2C_1, \hat{C}_2 = 2C_2$ in the following manner:

$$K_{0,j} = \hat{C}_1 k_{0,j}^h + \hat{C}_2 k_{0,j}^v \quad (13)$$

Figure 3 illustrates examples of different kernels before and after the relaxation.

We note that these modifications are straightforward and probably not optimal. The development of optimal normalization according to the kernel formulation is an interesting direction for future work.

3.3 Computation and Complexity

Training Complexity The training process has two parts. First, we calculate the kernel K . The calculation itself is explained thoroughly in Appendix 8 and has a time complexity of $O(L_{tot}L_{max}N)$, which is not dependent on B , and it is much faster than naive computation thanks to several design choices, such as parameters diagonalization, pre-processing, and a sophisticated caching procedure.

Next, we apply a convolution between K and U , using the classical procedure of FFT, element-wise multiplication, and inverse FFT. The complexity of this step is $O(BL_{tot} \log(L_{tot}))$ where B is the batch size. Hence, the total complexity is:

$$O(L_{max}NL + B \log(L_{tot})L_{tot}) \quad (14)$$

Inference Complexity During inference, the 2-D SSM can pre-compute the convolution kernels, resulting in an additional computation cost of only the 2-dimensional convolution of the layer input and the signal, which takes $L_{tot} \log(L_{tot})$ operations. Therefore, the additional computation overhead relative to the vanilla ViT is minimal, as the quadratic complexity dominates the overall complexity.

3.4 Complex and Real SSMs

While most SSM-based deep learning models, e.g S4 [17] or DLR [21], use a complex SSM, MEGA used EMA, which can be interpreted as a restriction of the diagonal SSM to real numbers. Our 2-D SSM layer can be built over real or complex parametrization. The complex-SSM based model that we examined is described in detail in Appendix E.1.

4 Model Analysis

In this section, we study the expressiveness of the 2-D SSM layer, as well as its unique spatial inductive bias.

4.1 Expressiveness

We compare the expressiveness of our 2-D SSM layer with S4ND [37], a very recent layer that is also based on multidimensional multi-axes SSM. We first introduce the key differences between our layer and S4ND, and then demonstrate the expressiveness gap.

The relationship between 2-D SSM and S4ND The main difference between S4ND and 2-D SSM is that S4ND runs a standard 1-D SSM over each axis independently, and those functions are combined to form a global kernel. In contrast, our model learns multi-dimensional functions over multi-axes data directly. This difference arises from the fact that the 2-D SSM has additional system matrices, A_2, A_3 , which aggregate and process information from different axes.

2D-SSM is a generalization of S4ND When restricted to 2-dimensional space, given a 2-D SSM model, S4ND is obtained by restricting A_2, A_3 to be zeros, and setting A_1, A_4 as the system matrices of S4ND. Additionally, to replicate S4ND in 2D-SSM, one should initialize the states of the 2D-SSM by the kernels factorized from S4ND.

Tensor rank as a criterion for expressiveness Over the years, several criteria were proposed for measuring the expressiveness of neural and classical models, such as VC-dimension [44], norms, and Rademacher complexity [1]. Inspired by [3], we employ tensor rank as our measure, and prove the followings theorems:

Theorem 4.1. *The 8 parameters of the 2-D SSM can express full-rank kernels*

Theorem 4.2. *S4ND can express kernels of rank 1 only.*

Assumptions For simplicity, we assume that both the 2-D SSM and S4ND layers contain one channel and one hidden dimension. In this case, the SSM matrices are scalars. When the assumption about the number of channels is omitted, the rank of kernels that S4ND can express increases from 1 to N . However, this comes with the cost of MNr additional parameters, where M is the number of axes and r is the required rank for S4ND. It should be noted that the authors of S4ND did not evaluate the performance of models with $r > 1$.

Under these assumptions, the proof of Theorem 4.1 is specified in Appendix C.1. The proof of 4.2 is trivial, and derives from the fact that to compute a global multi-axis kernel K , S4ND takes the outer product operation on the per-axis kernels $k_m \in \mathbf{C}^{L_m \times 1}$ for all $m \in [M]$. Since each kernel is a vector, it is clear that:

$$\text{rank}(K) = \text{rank}(k_1 \otimes k_2 \otimes \dots k_M) = 1 \quad (15)$$

4.2 Image-Specific Inductive Bias

Two-dimensional Position-Awareness Our layer is grounded by a two-dimensional linear recurrent (Eq. 2 ,Fig. 1, left), as a result, positional information is taken into account by design when the kernel K is computed from the parameters A, B, C , and D . This is a unique property without a counterpart in other modern layers. For instance, transformers lack positional bias and rely on additional positional encoding, while convolutional layers do not inherently encode explicit positional information; however, they learn to extract features based on local spatial dependencies.

Furthermore, as can be seen in Sec. 5, our method is highly effective in inserting positional bias into the transformer, even outperforming positional encoding in some cases.

Controllable Unrestricted Spatial Context A significant limitation of both CNNs and transformers is the need to choose an appropriate patch size, which can result in a loss of global or local context. In contrast, our layer implements a controllable global convolution, which benefits from a global context, and can control the effective receptive field and efficiently capture spatial local dependencies. Moreover, our layer is not confined to patch-like dependencies and can effectively capture diverse local and global features in horizontal, vertical, or diagonal directions (See 1, middle).

Modeling Symmetries Similar to other CNNs, our layer exhibits translation equivariance as the kernels slide across the input during computation. Additionally, as detailed in Appendix D, our layer’s core extends over multiple directions, which allows it to accommodate rotations and reflections naturally.

Parameter Efficiency In contrast to CNNs, which parameterize filters of size $H \times W$ with at least HW parameters, our layer has a fixed and small number of parameters (9 parameters per channel, $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, D \in \mathbb{R}$), however, those parameters can be expanded into unbounded two-dimensional kernels. Furthermore, we use parameter sharing for the SSM parameterization across channels, similarly to CNNs or state-space layers and denote n_{ssm} as the number of non-shared channels.

5 Experiments

We assess our 2-D SSM layer as an inductive bias within various ViT-based backbones. We demonstrate the universality of our method by incorporating it into various backbones, such as ViT, DeiT, Swin, and report improved results over the baselines on ImageNet-1k, Celeb-A, Tiny-ImageNet and CIFAR-100, with negligible additional parameters and without hyperparameter tuning, except for stochastic depth. For a comprehensive overview of the experimental setup, please see Appendix F

Swin, DeiT and ViT We adopted a straightforward approach to incorporate our 2-D SSM layer into the aforementioned backbone structures. Prior to the Transformer Block, we apply the 2-D SSM to the input signal $u \in R^{L_1 \times L_2 \times D}$, as illustrated in Fig. 1. We highlight that in the case of Swin Transformer, the 2-D SSM operates on the patch level and not the window level and therefore injects additional 2-D bias between windows.

We tested **Swin and ViT** over the small datasets Tiny-ImageNet and CIFAR-100, using the results reported by [29] as baseline. As shown in Tab. 1, with the ViT backbone, we improve 0.8% on CIFAR-100 and 0.5% on Tiny-ImageNet, and with the Swin backbone, we improve 3.25% on CIFAR-100 and 4.25% on Tiny ImageNet.

The **DeiT and Swin** backbones were tested on the large-scale Celeb-A dataset [34]. This dataset involves a 40-way multi-label attribute classification. We report aggregate accuracy across all 40 tasks. As can be seen in Tab. 2, the complex version outperforms the real in all experiments and achieves 1.41%, 0.72%, 0.6%, and 0.3% improvements over the baseline for DeiT sizes Tiny, Small, and Base, and Swin-Tiny respectively.

Mega In Mega, we replace the EMA mechanism with 2-D SSM, which means that we only perform our layer on Q, K . We compare original Mega (with EMA) vs Mega-ablate (without EMA) and Mega 2-D SSM. We examined our model on CIFAR-10 Grayscale in an isotropic manner (without decreasing the image size along the architecture, and without patches), which is part of the Long Range Arena benchmark [41]. As shown in Tab. 3, we improved the result by almost 1% over MEGA, obtaining state-of-the-art results. We also conduct an experiment on the ImageNet-1K dataset [8], and as shown in Tab. 4, we improve over MEGA’s ViT-T results by $\sim 0.4\%$ in both Top 1 accuracy and Top 5 accuracy. Finally, we check other small datasets (Tab. 1) and find superior results for combining Mega with 2-D SSM over baseline or other methods.

Comparisons against S4ND layer S4ND is the only N-Dimensional SSM-based layer known to us; we compare our results against it by substituting SSM in ViT, Mega, and Swin. We conducted experiments on CIFAR100 and Tiny Imagenet. As indicated in Tab. 1, S4ND performs very poorly when integrated into the original ViT backbone on CIFAR-100 (lower than the baseline by 1.21%)

Table 1: Results using ViT, MEGA and Swin backbones on the Tiny ImageNet (T-IN) and CIFAR-100 (C100) datasets. No hyperparameter tuning except for stochastic depth.

Model	C100	T-IN	Train Time	# Of Parameters
ViT	73.81	57.07	1x	2.71M (1x)
ViT w/ S4ND.	72.60	56.10	2.22x	2.72M (1.003x)
ViT w/ SSM-r.	74.07	57.56	2.66x	2.72M (1.003)
ViT w/ SSM-c.	73.91	57.66	2.89x	2.73M (1.01x)
Mega-ablate	74.82	56.43	1.1x	2.75M (1x)
Mega	72.27	54.49	1.28x	2.98M (1.08xx)
Mega w/ S4ND	74.9	56.65	1.46x	2.80M (1.02x)
Mega 2-D SSM-r	76.02	57.95	2.03x	2.80M (1.02x)
Mega 2-D SSM-c	75.09	56.51	2.03x	2.84M (1.03x)
Swin	76.87	60.87	1x	7.15M (1x)
Swin-reprod.	77.98	61.29	1x	7.15M (1x)
Swin w/ EMA	77.01	60.13	1.39x	7.52M (1.05x)
Swin w/ S4ND	79.26	64.6	1.29x	7.18M (1.004x)
Swin w/ SSM-r	80.12	65.77	2.16x	7.25M (1.01x)
Swin w/ SSM-c	3.28	12.76	2.18x	7.26M (1.02x)

Table 2: Results for DeiT and Swin of different sizes on the Celeb-A dataset.

Model	Top 1	#Param
DeiT-T	88.43	5.532M
DeiT-T w. SSM-r	89.76	5.537M
DeiT-T w. SSM-c	89.84	5.541M
DeiT-S	89.66	21.681M
DeiT-S w. SSM-r	90.24	21.688M
DeiT-S w. SSM-c	90.38	21.691M
DeiT-B	90.13	85.829M
DeiT-B w. SSM-r	90.45	85.841M
DeiT-B w. SSM-c	90.73	85.845M
Swin-T	91.48	27.550M
Swin-T w. SSM-r	91.68	27.556M
Swin-T w. SSM-c	91.78	27.558M

Table 3: Accuracy on the CIFAR-10 grayscale classification task, which is part of the Long Range Arena.

Models	Image - LRA
Transformer [45]	42.94
S4-v1	87.26
S4-v2	88.65
CCNN-1D [26]	88.90
CCNN-2D [26]	91.12
S4ND [37]	89.90
Hyena [38]	91.20
Mega Ablate	81.00
Mega	90.44
MEGA 2-D SSM	91.31

Table 4: ImageNet-1K accuracy of MEGA variants.

Model	Top 1	Top 5	# of Parameters
MEGA-ablate	66.97	88.17	5.91M
EMA	69.73	89.76	6.21M
2D-SSM-R	70.11	90.19	5.96M

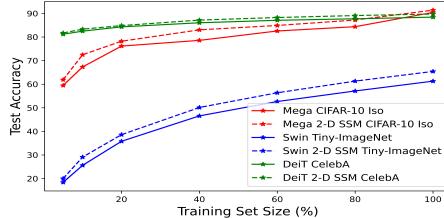


Figure 3: The effect of the training set size.

and sub-optimally when integrated into Swin and MEGA (achieves 1% or more lower accuracy on CIFAR-100 and Tiny-ImageNet for both backbones).

Sample Complexity We examine the behavior of our model with different backbones on different datasets over the baseline. As can be seen in Fig. 4, 2-D SSM maintains improved results over the baseline for all backbones, which shows the data-efficient quality of our model.

Removing the positional encoding (PE) We compare the empirical results obtained with real vs. complex kernels, with and without PE in Tab. 5. Evidently, complex-SSM can be superior or inferior to real-based SSM, depending on the scenario. Additionally, our findings indicate that complex-SSM has a tendency to exhibit instability during training, which can result in poor performance. Stabilizing these models is an important direction for future research.

Running ViT backbones without PE decreases performance dramatically. In contrast, when our 2D-SSM layer is inserted into these backbones, they benefit from PE, and even without PE they outperform the original backbones with PE. These findings support a innovative approach to introducing positional bias in ViT: rather than encoding positional information directly into the representation, it can be integrated into the computation by incorporating positional-dependent operators.

Table 5: Ablations. For each model and dataset, we examine the effect of using original positional encoding and complex (C) vs. real (R) SSM. The column δ represents the average difference for models with and without PE. As can be seen, our models are much more resistant to PE removal.

Dataset:	Tiny-INet (Swin)		CIFAR100 (ViT)		CelebA (DeiT-T)		CIFAR10 (Mega-ISO)		Avg.
	with PE	w/o PE	with PE	w/o PE	with PE	w/o PE	with PE	w/o PE	
Baseline	61.29	58.97	73.26	64.09	88.43	87.99	90.44	75.21	-6.79
+Ours (R)	65.77	65.44	74.07	74.89	89.76	89.63	91.31	90.68	-0.07
+Ours (C)	3.28	2.16	73.91	74.67	89.84	89.83	90.46	90.79	-0.01

6 Limitations

Despite the promising results presented in this paper, there are several limitations that should be considered. First, the current implementation of our proposed layer has relatively slow training times, as shown by the wall-clock measurements presented in the Experiments section 5. This slow training time may be even more pronounced when applying our method to a longer two-dimensional sequence of patches, which could limit its applicability to tasks that require handling multi-dimensional long-range dependencies. One possible approach to mitigating this challenge is to use multi-dimensional parallel scanners, which could potentially reduce the training time of our layer. The main idea is to extend the work of S5 [40], which leverages 1-D parallel scanners to apply SSM on 1-D sequences to multi-dimensional parallel scanners and multi-dimensional sequences.

7 Conclusions

We present a novel spatial SSM-based layer that is more general than existing ones, encodes positional information by design, and is able to model spatial relations more expressively than other SSMs, including S4ND. When added to various ViT backbones, it is able to improve classification results on the various benchmarks without optimizing any aspect or other parts of the architecture. In future work, we would like to study the behavior of the layer in the context of spatial vision tasks, such as video processing, image segmentation, phrase grounding, and image inpainting. In the last task, the recursive view of the layer could be applied directly to impute missing pixels efficiently.

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A Computing the kernel

We discuss $x_{i,j}^h, k_{i,j}^h$. The same calculations hold for $x_{i,j}^v, k_{i,j}^v$.

$k_{i,j}^h$ can be written as:

$$\forall i, j : k_{i,j}^h = \sum_z^{2*L_{max}} c_z A_1^{z_1} A_2^{z_2} A_3^{z_3} A_4^{z_4} B_{z_5} \quad (16)$$

For brevity and since it is not material for the method, we limit our exposition of the different power combinations of the system matrices A_1, A_2, A_3, A_4 and the input matrices B_1, B_2 . As noted above, for each $k_{i,j}^h$ there are at most $2L_{max}$ elements.

Pre-Processing The problem of finding c_z for each element in the summation is a generalization of Pascal's triangle. In order to calculate the kernel, we calculate all the coefficients and the powers of $A_1 \dots A_4$ up to the size of L_1, L_2 and cache them before the training process.

During training, we employ a matrix multiplication process to compute $k_{i,j}^h$ with the learned parameters $A_1, \dots, A_4, B_1, B_2$.

Thus, for each cell there are at most $2L_{max}$ elements, and for each element, we save a constant number of χ values (the values of z and c_z). As a result, the size of the cached matrix is bounded by $\mathcal{O}(L_{tot}L_{max})$.

It should be noted that currently in our method we cache the coefficients as One Hot Encoding and not the coefficient itself, and thus in our specific implementation we need to multiply the time complexity and memory complexity by L_{max} .

B Time and Memory Complexity of Our Method

To understand the complexity of our method, we will outline each step of the computation, starting from calculating the cache, creating the kernel during training, and computing the output Y afterward.

As before, for brevity, we will refer only to the $x_{i,j}^h, k_{i,j}^h$ matrices caching, but the same holds for the vertical matrices. For simplicity, we assume $H = 1$ (number of channels).

Caching As noted in Section 3.3, for each cell $k_{i,j}^h$ in the horizontal kernel there are at most $2L_{max}$ elements. Also as noted in Section 3.3, $z_1, z_2, z_3, z_4 \leq 2L_{max}$. Thus, for each element in Eq. 16 we save z_1, z_2, z_3, z_4, z_5 and α_z values. In total, for each cell, we save $\chi_1 L_{max}$ values, where χ_1 is a small constant. We have L_{tot} cells and thus the total coefficient cached tensor for calculating K_h is sized

$$\chi_1 * L_{max} L_{tot} \quad (17)$$

From now on we will denote the tensors of horizontal coefficients as $CACHE \in \mathbb{R}^{\chi_1 \times L_{tot} \times L_{max}}$. Notice that there is a $CACHE$ tensor for each parameter, meaning $CACHE_{A_1}^h, CACHE_{A_2}^h, CACHE_{A_3}^h, CACHE_{A_4}^h, CACHE_B^h$.

Creating the Kernel For brevity, we use real-valued diagonal $A_i \in [0, 1]^N$ (after the sigmoid). First, we calculate the Vandermonde Matrix for each A_1, A_2, A_3, A_4 eigenvalues up to the highest power that exists in the kernel, which is $2L_{max}$, and denote $VAN_i = Vandermonde(A_i) \in \mathbb{R}^{2L_{max} \times N}$.

Again, we have L_{tot} cells in the horizontal kernel, each cell having $2 * L_{max}$ elements. We take $CACHE_{A_i}$ which holds for each element its A_i power, z_i and creates a matrix that holds for each element its corresponding $A_i^{z_i}$ value.

$$O_{A_i}^h = VAN_i[CACHE_{A_i}^h] \in \mathbb{R}^{L_{tot} \times 2L_{max} \times N} \quad (18)$$

Now we multiply the matrices element-wise to obtain the final value of each element:

$$O_{pre-addition}^h = O_{A_1}^h \odot O_{A_2}^h \odot O_{A_3}^h \odot O_{A_4}^h \odot O_B^h \odot O_\alpha^h \in \mathbb{R}^{L_{tot} \times L_{2L_{max}} \times N} \quad (19)$$

where \odot denotes element-wise multiplication. Now for each cell, we sum all the elements in the summation $k_{i,j}$, meaning summing over the second dimension:

$$O_{post-addition}^h = \text{sum}(O_{pre-addition}^h, d=1) \in \mathbb{R}^{L_{tot} \times N} \quad (20)$$

Again, all the above steps are employed for the vertical axis as well, thus we are finally able to compute the kernel by using $C_1, C_2 \in \mathbb{R}^{N \times 1}$:

$$K = O_{post-addition}^h C_1 + O_{post-addition}^v C_2 \in \mathbb{R}^{L_1 \times L_2} \quad (21)$$

Remembering that we actually used n_{ssm} channels, the kernel size is $K \in \mathbb{R}^{L_1 \times L_2 \times n_{ssm}}$. It should be noted here that the calculation of the kernel is not dependent on the batch size B .

Forward Pass Let B denote the batch size. We would like to convert input signal $U \in \mathbb{R}^{B \times L_1 \times L_2 \times H}$ to output signal $Y \in \mathbb{R}^{B \times L_1 \times L_2 \times H}$. After calculating the kernel, we convert U, K to the frequency domain through FFT:

$$U_f = FFT(U), K_f = FFT(K) \quad (22)$$

$$Y = IFFT(U_f \odot K_f) \quad (23)$$

This whole process costs us $O(BHL_{tot} \log(L_{tot}))$

Thus, our total forward pass time complexity is:

$$\mathcal{O}(\chi L_{TOT} L_{max} n_{ssm} N + BHL_{TOT} \log L_{TOT}) \quad (24)$$

Implementation detail We implemented the caching and matrix multiplication process with One Hot Encoding Vector of the powers and not by using floats representing the powers themselves. Thus, the size of each $COEFF_i^h$ in our implementation is multiplied by L_{max} , as is the time complexity of Eq. 18 and 19.

C Expressiveness

Theorem C.1. *One channel of 2-D SSM can express full-rank kernels*

Proof. We start by restricting the system, output and input matrices:

$$A_1 = A_2 = A_3 = 1, \quad A_4 = 0, \quad C_1 = 1, C_2 = 0, \quad B_1 = 1, B_2 = 0 \quad (25)$$

For simplicity we assume that the values of the initial states are 0:

$$\forall i : i = -1 \rightarrow x_{i,j} = 0, \quad \forall j : j = -1 \rightarrow x_{i,j} = 0 \quad (26)$$

It suffices to show that (i) K is a triangular matrix, and (ii) the diagonal of K contains non-zero elements. First, by plugging 25,26 into the recurrent rule 2, it can be simplified:

$$y_{i,j} = x^h_{i,j}, \quad x^h_{i,j} = x^h_{i,j-1} + x^v_{i,j-1} + u_{i,j}, \quad x^v_{i,j} = x^h_{i-1,j} \quad (27)$$

Given this simplification 27, both (i) and (ii) can be easily proven by induction on the diagonals of K .

To provide more insight into the proof, Eq. 28 illustrates the values of K .

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ \vdots & \ddots & 1 & 3 & 6 \\ \vdots & \ddots & \ddots & 1 & 4 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \quad (28)$$

And in general, since its clear from 27 that

$$y_{i,j} = x^h_{i,j} = y_{i,j-1} + y_{i-1,j-1} + u_{i,j}, \quad \forall j \rightarrow k_{0,j} = 1 \quad (29)$$

It easy to understand that the upper triangular of K can obtained from a rotation of Pascal's triangle.

□

D Model Extension

D.1 Bidirectional SSM

Using the State Space model, when calculating $x_{i,j}^h$ one only considers $x_{\hat{i},\hat{j}}$ where $\hat{i} \leq i, \hat{j} \leq j$. To benefit from bidirectionality, we employ a version where we transpose the kernel in two or four directions (shown in the ablation experiments) and then sum the results. A similar mechanism for the 1-D case is used in S4, MEGA, and elsewhere.

D.2 Multi-Axis Multidimensional SSM

To enrich the dependencies that can be captured by our kernels, we use a weighted linear combination of kernels. Under the assumption that the system matrices are diagonal, the N coordinates of the states are not dependent on each other, and can thus be calculated independently. Specifically, Eq. 9 can be re-written separately per coordinate : $\forall g \in [N]$

$$x_{i,j}^h[g] = \sum_{0 \leq \hat{i} \leq i} \sum_{0 \leq \hat{j} \leq j} k_{\hat{i},\hat{j}}^h[g] u_{\hat{i},\hat{j}} \quad (30)$$

Therefore, increasing N will increase the number of kernels that make up K . By using this structure, the kernel can capture a variety of dependencies, where each coordinate focuses on a different type of dependency. Furthermore, this extension adds relatively negligible runtime when working with large batches, since the batch dimension does not affect the complexity when the kernel is computed. Therefore, increasing N will increase the number of kernels that compose K .

E Justify Design Choices

E.1 Our Complex 2D-SSM

As explained in Sec. 3.3, our models employ real rather than complex SSMs. For reproducibility, here we provide a detailed description of our complex SSM variant: The complex variant of our 2-D SSM model, still assumes $\forall t, u_{i,j} \in \mathbb{R}^1, \forall t, y_{i,j} \in \mathbb{R}^1$ and employs:

$$A_1, A_2, A_3, A_4 \in \mathbb{C}^{N \times N}, B_1, B_2 \in \mathbb{C}^{N \times 1}, C_1, C_2 \in \mathbb{C}^{1 \times N} \quad (31)$$

(diagonal matrices as above), and therefore

$$x_{i,j}^h \in \mathbb{C}^N, x_{i,j}^v \in \mathbb{C}^N \quad (32)$$

The output remains a real number $y_{i,j} \in \mathbb{R}^1$, and thus the real part of Eq. 2 is used, namely $y_{i,j}^{out} = \text{Re}(y_{i,j})$

For complex SSM, we save $\hat{A}_{i_{angle}}, \hat{A}_{i_{radius}} \in \mathbb{R}^{N \times N}$, and we calculate $A_i \in \mathbb{C}^{N \times N}$ in the following manner:

$$A_{i_{angle}} = 2\pi \text{sigmoid}(\hat{A}_{i_{angle}}), \quad A_{i_{radius}} = \text{sigmoid}(\hat{A}_{i_{radius}}) \quad (33)$$

$$A_i = A_{i_{radius}} * (\cos(A_{i_{angle}}) + i * \sin(A_{i_{angle}})) \in \mathbb{C}^{N \times N} \quad (34)$$

The same goes for B_1, B_2 . As for C_1, C_2 , we perform the same operation without limiting the radius size (not applying a sigmoid to $\hat{C}_{i_{radius}}$).

E.2 No Weight Decay on the SSM core

While vision transformers and MEGA regularize the model via weight decay, SSM-based layers typically do not apply this [20, 17], since it drastically lowers the models' ability to learn, especially in the context of long-range dependencies. In general, higher values of the A_i, B, C parameters do not seem to correspond with overfitting. Therefore, our method does not employ weight decay on those parameters.

F Experimental setup

We use PyTorch for all experiments. As a deliberate decision we choose to not perform hyper-parameter tuning of the backbone and training procedure, apart from stochastic depth. All experiment results were averaged over seeds = [0, 1, 2]. For all datasets and backbones, we set $n_{ssm} = 8$, $N = 16$ for all SSM-Based variants (SSM-2D real & complex and S4ND).

Cifar-100 and Tiny imagenet For both datasets, we use as a baseline the experiments performed by [29]. This means we follow DeiT’s [42] application of a long list of data augmentation and regularization methods, including Cutmix [46], Mixup [47], stochastic depth [24], repeated augmentation [23], Rand-Augment [4], and random erasing [48]. AdamW was used as the optimizer. Weight decay was set to 0.05 (apart from SSM layer where it was set to 0), batch size to 128, and warm-up to 10. All models were trained for 100 epochs, and cosine learning rate decay was used. The initial learning rate was set to 0.003. In certain scenarios, we noticed that our models converge faster compared to the baseline approach. We discovered that a slight modification, specifically doubling the stochastic depth, proved to be instrumental in maximizing the model’s performance.

When comparing S4ND, we used the same parameter scheme being used in 2-D SSM to perform a valid comparison, by making $C \in \mathbb{C}^{n_{ssm}, N}$ instead of $C \in \mathbb{C}^{H, N}$ as in the original paper.

CelebA For Celeb-A, the original image size is 178x218, it is resized to 224x224 to match DeiT [42] backbone and patch size. The dataset includes a 40-way multi-label attribute classification. We are reporting an average accuracy of all 40 tasks. We use the same data augmentation, and hyperparameters as DeiT, and train the models for 20 epochs, similar to the training procedure of S4ND [37] on this datasets.

Imagenet and CIFAR-10 Grayscale We use the exact same training procedure including hyper-parameters, data augmentation and training environment as used in the git repository of the baseline [35] for those datasets.