

Solving for Coefficients of Minimum Jerk Trajectories

Minimum Jerk Trajectory

$$x^*(t) = \operatorname{argmin}_{x(t)} \int_0^T \mathcal{L}(\ddot{x}, \dot{x}, x, t) dt = \operatorname{argmin}_{x(t)} \int_0^T \dot{x}^2 dt$$

We can solve the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{x}} \right) - \frac{d^3}{dt^3} \left(\frac{\partial \mathcal{L}}{\partial x^{(3)}} \right) = 0$$

to get the condition:

$$x^{(6)} = 0$$

Thus, we want a trajectory of the form:

$$x(t) = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	a	0	0
$t = T$	b	0	0

Position constraints: $x(t) = c_5t^5 + c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$

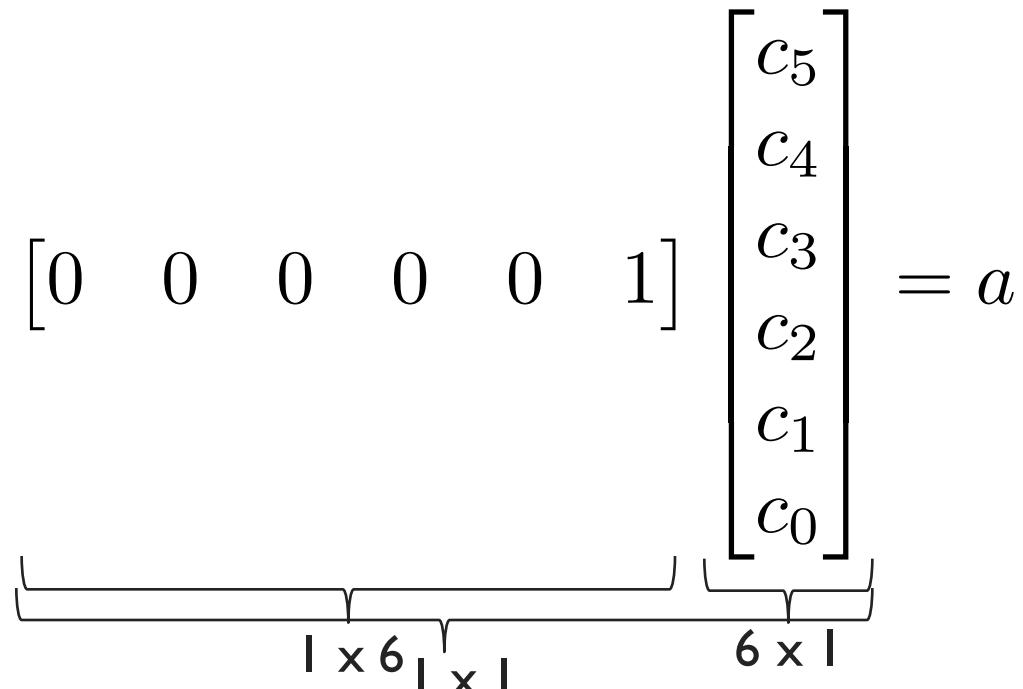
$$x(0) = c_0 = a$$

$$x(T) = c_5(T)^5 + c_4(T)^4 + c_3(T)^3 + c_2(T)^2 + c_1(T) + c_0 = b$$

Solving for Coefficients

Position constraints in matrix form:

$$x(0) = c_0 = a$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = a$$


The diagram illustrates the dimensions of the matrix equation. The row vector $[0 \ 0 \ 0 \ 0 \ 0 \ 1]$ is labeled 1×6 . The column vector $\begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}$ is labeled 6×1 . The resulting product is a 1×1 scalar equal to a .

Solving for Coefficients

Position constraints in matrix form:

$$x(0) = c_0 = a$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = a$$

Solving for Coefficients

Position constraints in matrix form:

$$x(T) = c_5(T)^5 + c_4(T)^4 + c_3(T)^3 + c_2(T)^2 + c_1(T) + c_0 = b$$

$$\begin{bmatrix} T^5 & T^4 & T^3 & T^2 & T & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = b$$

Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	a	0	0
$t = T$	b	0	0

Velocity constraints: $\dot{x}(t) = 5c_5t^4 + 4c_4t^3 + 3c_3t^2 + 2c_2t + c_1$

$$\dot{x}(0) = c_1 = 0$$

$$\dot{x}(T) = 5c_5(T)^4 + 4c_4(T)^3 + 3c_3(T)^2 + 2c_2(T) + c_1 = 0$$

Solving for Coefficients

Velocity constraints in matrix form:

$$\dot{x}(0) = c_1 = 0$$

$$\dot{x}(T) = 5c_5(T)^4 + 4c_4(T)^3 + 3c_3(T)^2 + 2c_2(T) + c_1 = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	a	0	0
$t = T$	b	0	0

Acceleration constraints: $\ddot{x}(t) = 20c_5t^3 + 12c_4t^2 + 6c_3t^2 + 2c_2$

$$\ddot{x}(0) = 2c_2 = 0$$

$$\ddot{x}(T) = 20c_5(T)^3 + 12c_4(T)^2 + 6c_3(T)^2 + 2c_2 = 0$$

Solving for Coefficients

Acceleration constraints in matrix form:

$$\ddot{x}(0) = 2c_2 = 0$$

$$\ddot{x}(T) = 20c_5(T)^3 + 12c_4(T)^2 + 6c_3(T)^2 + 2c_2 = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	a	0	0
$t = T$	b	0	0

Combine constraints into one matrix expression:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example I: Find the Minimum Jerk Trajectory

Find the minimum jerk trajectory with boundary conditions:

Position	Velocity	Acceleration
$t = 0$	$a = 0$	0
$t = T = 1$	$b = 5$	0

$$Ax = b$$

$$x = A^{-1}b$$

$A \longrightarrow$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20 & 12 & 6 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \leftarrow b$$

$\uparrow x$

Example I: Find the Minimum Jerk Trajectory

$$x = \begin{bmatrix} 30 \\ -75 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \longleftrightarrow \quad \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}$$

$$x(t) = 30t^5 - 75t^4 + 50t^3$$

Example I: Find the Minimum Jerk Trajectory

We can verify that this trajectory does in fact satisfy all boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	$a = 0$ ✓	0	0
$t = T = 1$	$b = 5$ ✓	0	0

$$x(t) = 30t^5 - 75t^4 + 50t^3$$

$$x(0) = 0$$

$$x(1) = 30(1)^5 - 75(1)^4 + 50(1)^3 = 5$$

Example I: Find the Minimum Jerk Trajectory

We can verify that this trajectory does in fact satisfy all boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	$a = 0$ ✓	0 ✓	0
$t = T = 1$	$b = 5$ ✓	0 ✓	0

$$\dot{x}(t) = 150t^4 - 300t^3 + 150t^2$$

$$\dot{x}(0) = 0$$

$$\dot{x}(1) = 150 - 300 + 150 = 0$$

Example I: Find the Minimum Jerk Trajectory

We can verify that this trajectory does in fact satisfy all boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	$a = 0$ ✓	0 ✓	0 ✓
$t = T = 1$	$b = 5$ ✓	0 ✓	0 ✓

$$\ddot{x}(t) = 600t^3 - 900t^2 + 300t$$

$$\ddot{x}(0) = 0$$

$$\ddot{x}(1) = 600 - 900 + 300 = 0$$

Find the Minimum Jerk Trajectory

The order of coefficients in the matrix of unknowns matters.

$$\begin{bmatrix} T^5 & T^4 & T^3 & T^2 & T & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = b$$

Find the Minimum Jerk Trajectory

The order of coefficients in the matrix of unknowns matters.

$$\begin{bmatrix} 1 & T & T^2 & T^3 & T^4 & T^5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = b$$

Find the Minimum Jerk Trajectory

The order of coefficients in the matrix of unknowns matters.

$$\begin{bmatrix} T^4 & T & T^2 & T^5 & T^3 & 1 \end{bmatrix} \begin{bmatrix} c_4 \\ c_1 \\ c_2 \\ c_5 \\ c_3 \\ c_0 \end{bmatrix} = b$$