Minimum Velocity Trajectories from the Euler-Lagrange Equations

Minimum Velocity Trajectory

Find the function x(t) such that:

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dot{x}, x, t\right) dt$$

$$= \underset{x(t)}{\operatorname{argmin}} \int_0^T \dot{x}^2 dt$$

Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

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Minimum Velocity Trajectory

Euler-Lagrange equation:

$$\boxed{\frac{d}{dt}}\boxed{\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)} - \boxed{\frac{\partial \mathcal{L}}{\partial x}} = 0$$

Cost function: $\mathcal{L}\left(\dot{x}, x, t\right) = \left(\dot{x}\right)^2$

Euler-Lagrange terms:

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right) = 0$$

No x appears in $\mathcal{L}$

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 2\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}(2\dot{x}) = 2\ddot{x}$$

Minimum Velocity Trajectory

Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

Euler-Lagrange terms: $2\ddot{x} - 0 = 0 \longrightarrow 2\ddot{x} = 0 \longrightarrow \ddot{x} = 0$

Integrate to get the velocity: $\dot{x} = c_1$

Integrate to get position: $x(t) = c_1 t + c_0$

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