Spectral Analysis of Time-series

Anshul Thakur Andrew Creagh Prof. David A. Clifton

Department of Engineering Science
University of Oxford

Centre for Doctoral Training in Healthcare Innovation

Table of Contents

Table of Contents

What are Transforms?

- Transform data from one domain to other domain
  - E.g. Time domain to frequency domain
- Why transform data?
  - Analysis of data may be easier in transformed domain
  - Certain operations may only be feasible in transformed domain
- Linear Transformation functions are invertible
- Deep neural networks are non-linear non-invertible transforms!!!

Transforms around us!

- **Discrete Cosine Transform:** Represents a finite signal as a combination of cosine functions of different frequencies
- Allows compression of signals. Used in:
  - digital images (JPEG)
  - digital video (MPEG)
  - digital audio (MP3, Dolby Digital)
  - speech coding
- Made streaming music, games and Netflix possible

Combination of Vectors

- Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \ \ldots \mathbf{v}_n\}$ be a set of N vectors and $\{\alpha_1, \alpha_2, \alpha_3 \ \ldots \alpha_n\}$ be a set of N scalars
- **Linear combination:** $\mathbf{y} = \sum_{i=1}^{N} \alpha_i \mathbf{v}_i$
- **Affine combination:** $\mathbf{y} = \sum_{i=1}^{N} \alpha_i \mathbf{v}_i$ where $\sum_{i=1}^{N} \alpha_i = 1$
- **Conic combination:** $\mathbf{y} = \sum_{i=1}^{N} \alpha_i \mathbf{v}_i$ where $\alpha_i \geq 0$
- **Convex combination:** $\mathbf{y} = \sum_{i=1}^{N} \alpha_i \mathbf{v}_i$ where $\sum_{i=1}^{N} \alpha_i = 1$ and $\alpha_i \geq 0$

Transform your data

- Let $\mathbf{x} \in \mathbb{R}^d$ be an input vector and $\mathbf{W} \in \mathbb{R}^{d \times m}$ be a transformation matrix, a simple linear transform is:

$$\mathbf{y} = \mathbf{W}^T \mathbf{x} \tag{1}$$

- **Random Projections:** Project data to low dimensional space using Random Gaussian matrix

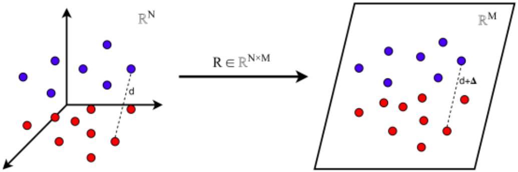


Figure: Random projections for dimensional reduction.

Transform Your Data

- Represent **x** as linear combination of rows of **W**:

$$\mathbf{x} = \sum_{i=1}^{m} y_i \mathbf{w}_i \tag{2}$$

Now, we can represent input vector **x** as coefficients $[y_1, y_2, y_3, \ldots, y_m]$

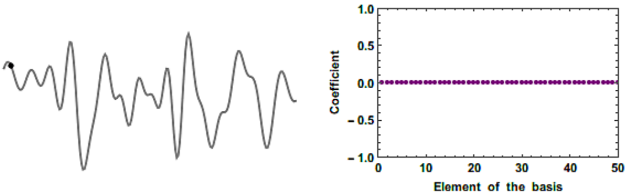


Table of Contents

Types of Transforms

- Transforms can be categorised based on the nature of transformation matrix
  - **Analytical transform: W** is analytically determined
  - **Learned transform: W** is learned from the data itself
- Analytical transforms:
  - Data independent and universal
  - E.g. Fourier and cosine transforms where signal is decomposed onto sinusoidal waves of different frequencies
- Learned transforms:
  - Data dependent
  - **W** contains basis of a subspace or exemplars
  - E.g. PCA as principle components are learned from the data
  - **Dictionary learning:** Learn transformation matrices or dictionaries from data and decompose data onto the atoms of these dictionaries

Types of Transforms: Deep Learning

- Deep neural networks can be seen as cascades of learned transforms:

$$\mathbf{y} = \mathbf{W}_3\mathbf{W}_2\mathbf{W}_1\mathbf{x} \tag{3}$$

But this is linear! (Deep dictionary learning)

$$\mathbf{y} = \text{ReLU}(\mathbf{W}_3\text{ReLU}(\mathbf{W}_2\text{ReLU}(\mathbf{W}_1\mathbf{x}))) \tag{4}$$

- DNNs learn these transforms simultaneously using gradient descent
- Purpose is to learn embedding or $\mathbf{y}$-space where classes are separable

Table of Contents

Why Fourier Analysis?

- In healthcare, we often deal with $1d$ signals or time-series data:
  - EEG, ECG and SPO$_2$, Blood pressure
  - Speech signals
  - Breathing sound recordings
- Limited features can be extracted from signals in time-domain:
  - Zero crossing rate, autoregression coefficients and slope sign changes
  - How far can we really go with time-domain analysis?
  - Deep neural networks armed with Conv-1D filters can extract meaningful features
- Fourier analysis helps us to represent a signal as linear combination of sinusoidal components of different frequencies
  - Provides the frequency profile of a signal

Why Fourier Analysis?

- time-domain analysis shows how a signal changes over time
- frequency-domain analysis shows how the signal's energy is distributed over a range of frequencies
- frequency-domain representation also includes information on the phase shift
  - Phase tells you how all the frequency components align in time

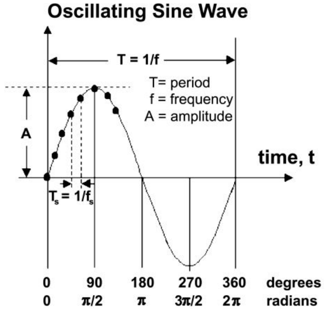
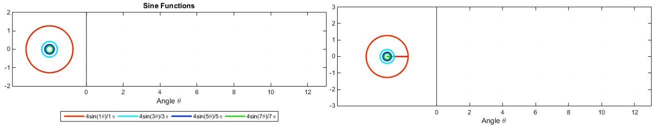


Figure: Phase and amplitude of a sine wave.

Fourier Series

- Any periodic signal can be represented as combination of sines and cosines:

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t) \tag{5}$$



Fourier Series

- Coefficients can be worked out as:

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt, \quad n = 0, 1, 2, 3 \ldots \tag{6}$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt, \quad n = 1, 2, 3 \ldots \tag{7}$$

- We are analysing $f(t)$ as mixtures of sines and cosines with fundamental frequency ω and harmonics $2\omega, 3\omega$ etc.
- $\omega = 2\pi/T$, where T is period of the signal
- $T = 1/f$, f is frequency of the signal and $\omega = 2\pi f$

Fourier Series: New Basis

- We have used sines and cosines of different frequencies as basis
- An alternation formulation based on different set of basis functions i.e. exponentials

$$e^{ix} = \cos x + i \sin x \tag{8}$$

- A representation based on this family of functions is called complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \tag{9}$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{10}$$

- C_n are normally complex

Fourier Transform

- Fourier transform and its associated inverse:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{11}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \tag{12}$$

- $F(\omega)$ is frequency spectrum of $f(t)$
  - No longer representing signal as sum of fundamental frequency and harmonics
- Can make analysis and certain operations easier:
  - Transform to frequency domain, remove higher frequency bands and reconstruct

Fourier Transform: Complex Frequency Spectrum

- Sinusoids are complex, so $F(\omega)$ will be complex
- Complex numbers: $z = x + iy$, x being real and y is imaginary
  - Magnitude of z=$\sqrt{x^2 + y^2}$, phase of $z = \tan^{-1}(y/x)$
- Similarly, we can compute magnitude spectrum and phase of $F(\omega)$
- Most of the spectral analysis is limited to magnitude spectrum
- But phase has more information about signal!

Importance of Phase

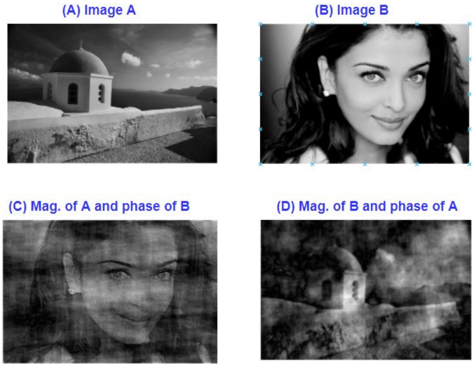


Figure: Impact of phase on reconstruction of an image.

Discarding Negative Frequencies

- $F(\omega)$ contains infinite frequencies and ω can be negative
  - $cos(-\omega t) = cos(\omega t)$ and $sin(-\omega t) = -sin(\omega t)$

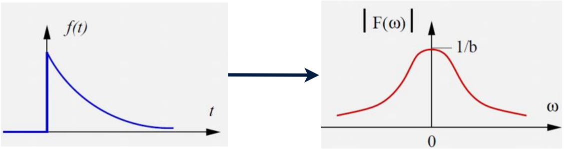


Figure: Frequency spectrum for positive frequencies is mirror of spectrum for negative frequencies.

- Just discard the negative frequency components from frequency spectrum

Frequency spectrum of EEG signal

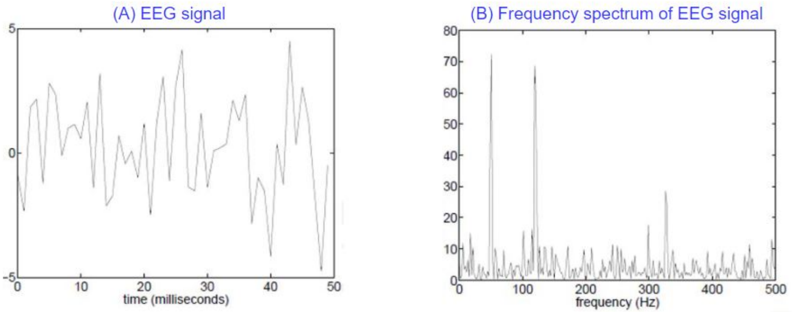
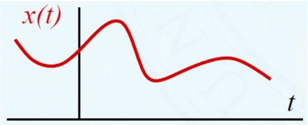


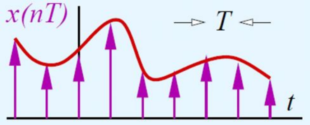
Figure: Frequency spectrum of EEG signal.

Table of Contents

Discrete Signals



(A) Continuous Signal



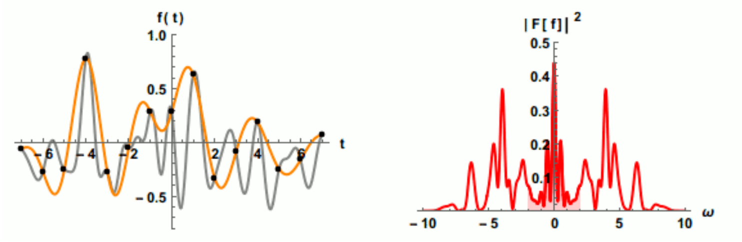
(B) Discrete Signal

Figure: Discrete vs continuous signal.

- We work on sampled signals
  - **Sampling:** Record the values at fixed interval of time
  - **Sampling frequency (f_s):** Number of samples obtained in 1 second. E.g. 20 kHz implies 20,000 samples are obtained in 1 second

Nyquist-Shannon Sampling Theorem

- If $x(t)$ contains no frequency higher than B hertz, perfect reconstruction is guaranteed if $B < f_S/2$
- **Nyquist Frequency:** $f_s/2$, Highest frequency represented at sample rate f_S



Discrete Fourier Transform

- Discrete Fourier Transform and its inverse:

$$X[k] = \sum_{n=0}^{N-1} x[n] \;\; exp\left(\frac{-j2\pi kn}{N}\right) \tag{13}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \;\; exp\left(\frac{j2\pi kn}{N}\right) \tag{14}$$

- Fast Fourier Transform (FFT) is an efficient algorithm for computing DFT
- **FFT:** Transforms a series of 2^n time-series data into frequency spectrum of 2^n components
- 2^n frequency components cover frequency range $[0, f_s]$

Discrete Fourier Transform

Practicalities of using FFT

- Suppose there is a signal sampled at 1000 Hz and it has $512\,(2^9)$ samples
- 512 frequency components between $[0, fs]$
- Each component covers $1000/512 = 1.95$ Hz

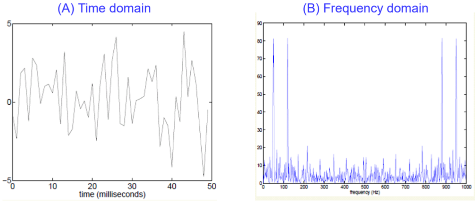


Figure: Time domain to frequency domain.

Table of Contents

Short-term Fourier Transform

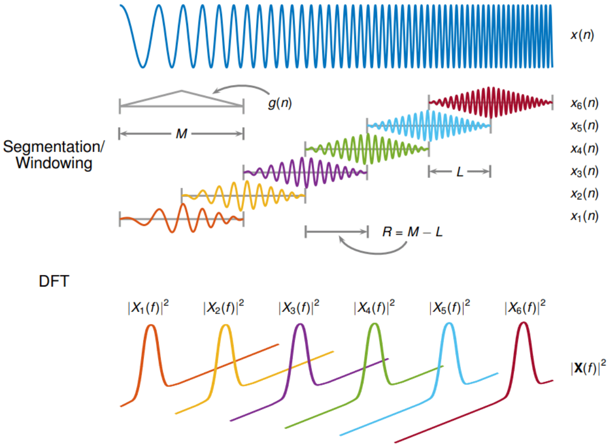


Figure: Short-term Fourier Transform

Short-term Fourier Transform

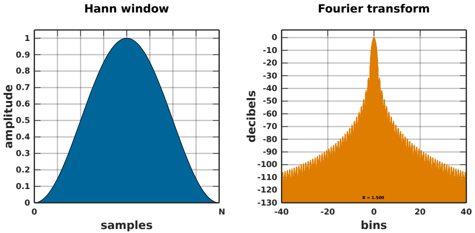


Figure: Windowing function: Hann window

- DFT can't capture time-frequency modulations in non-stationary signals
- **Short-term Fourier transform (STFT):**
  - Divide input signal into overlapping segments of M samples
  - Analysis window is applied to each segment to avoid spectral ringing
  - DFT is applied to each segment and magnitude spectrum is computed
  - $\mathbf{X}(f) = [\mathbf{x}_1(f)\ \mathbf{x}_2(f)\ \mathbf{x}_3(f) \ldots \mathbf{x}_k(f)]$

Windowing

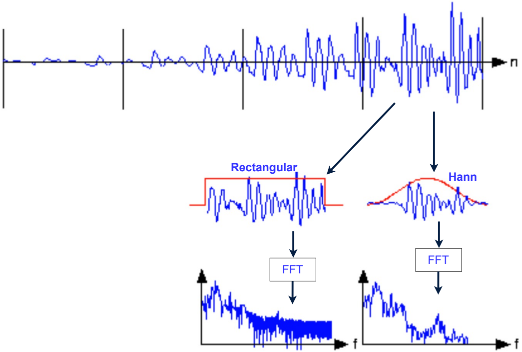


Figure: Difference in frequency profiles due to rectangular and Hann window.

- Signal may change very abruptly at the frame's edges
- Transform of such a segment reveals a curious oscillation
- Due to proper windowing, signal decays gracefully as it nears the edges

Spectrogram

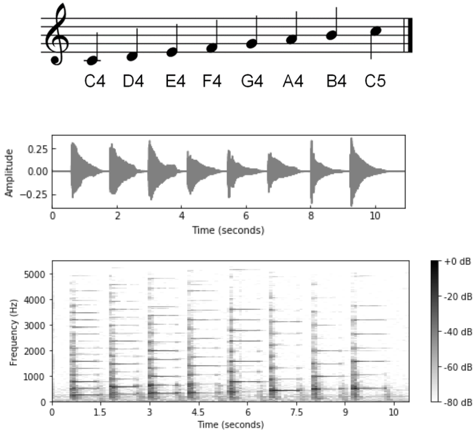


Figure: Spectrogram of an audio file containing C-major scale on piano.

Play

Why Spectrograms?

- Provides a way to analyse complex signals such as speech and soundscapes
- Spectrograms and features derived from it have provide state-of-the-art performance in many acoustic classification tasks
- Spectrograms allow us to use classic CNN architectures for $1d$ signal classification

Uncertainty in STFT: Time-frequency Resolution

- Smaller frame size or analysis window results in better time resolution but poor frequency resolution and vice-versa

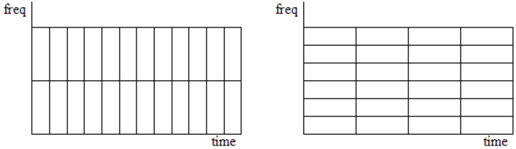


Figure: Conflict between time and frequency resolution.

- N samples in a window produces $N/2$ unique coefficients that represent frequencies between $[0, f_s/2]$
- Two coefficients are f_s/N Hz apart
- Increasing N will improve frequency resolution but decrease time-resolution

Why Phase is Ignored?

- Phase Wrapping problem: ARCTANGENT function can output values in $[-\pi, \pi]$. However, the magnitude of phase doesn't fit this range
  - So, ARCTANGENT wraps the phase in $[-\pi, \pi]$

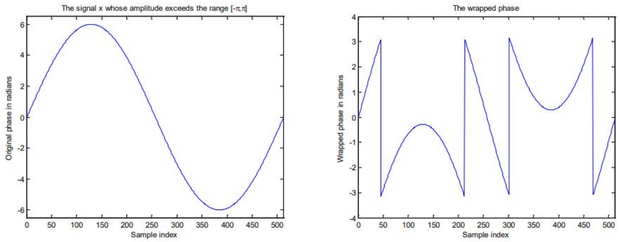


Figure: Phase wrapping problem.

- Difficult to recover phase
- Methods exists and phase has shown some amazing performance in speech and speaker recognition tasks

Why Phase is Ignored?

- Phase unwrapping methods such as all-pole group delay are susceptible to noise
- Limiting their use in many real-world scenarios
- So, should we just accept that phase has to be ignored?
- Not really: This loss of important phase information provides motivation for using raw waveforms as input to neural networks (Conv-1d networks)

Table of Contents

Further Reading: Discrete Cosine Transforms

Figure: DCT and Compression.

Further Reading: Multi-resolution Spectrograms

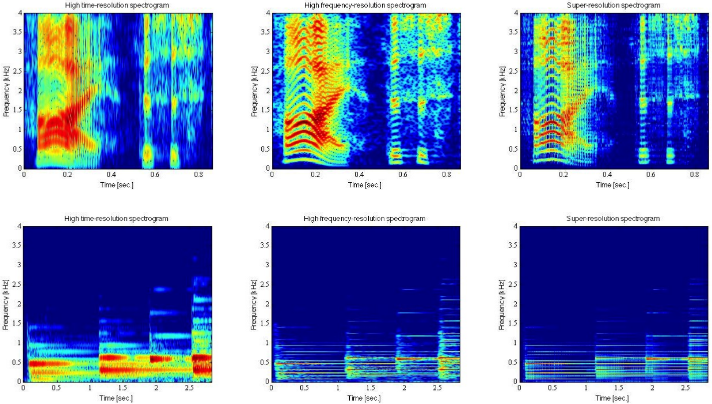


Figure: Multi-resolution spectrograms: Better time and frequency resolution.

Credit: *SUPER-RESOLUTION SPECTROGRAM USING COUPLED PLCA, Nam et. al*

Further Reading: Wavelet Transforms

- **Wavelet transform:** Good time resolution for high-frequency events and good frequency resolution for low-frequency events
- **Wavelet:** A rapidly decaying wave like oscillation with zero mean

Figure: Wavelet.

Wavelets

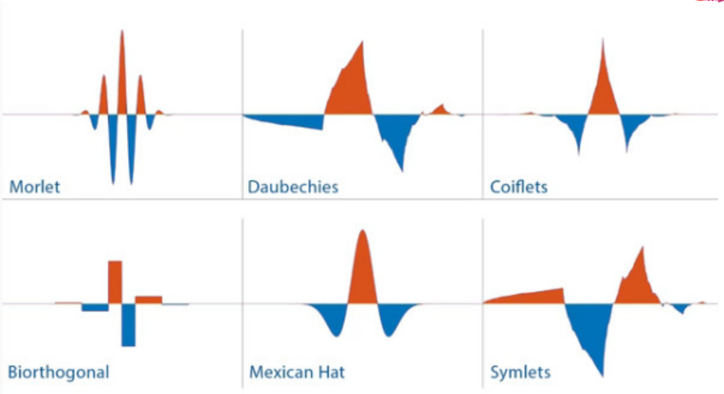


Figure: Some common wavelets.

Wavelets: Scaling and Shifting

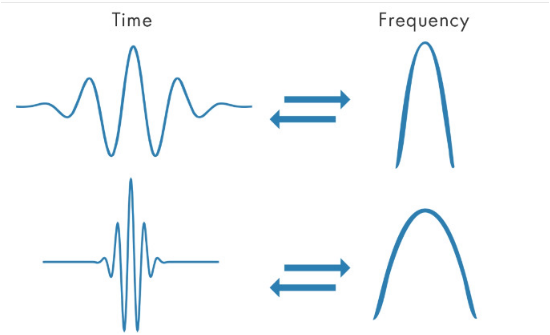


Figure: Scaling of wavelets.

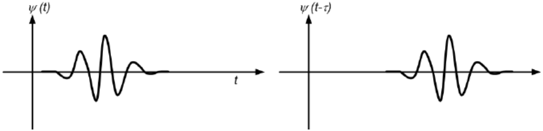


Figure: Shifting of wavelets.

Continuous Wavelet Transform

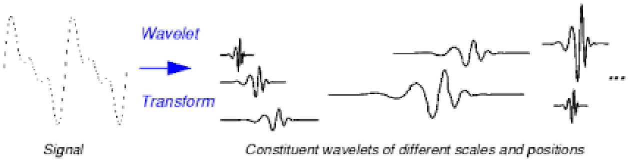


Figure: Representing a signal as combination of scaled and shifted wavelets.

STFT vs Continuous Wavelet Transform

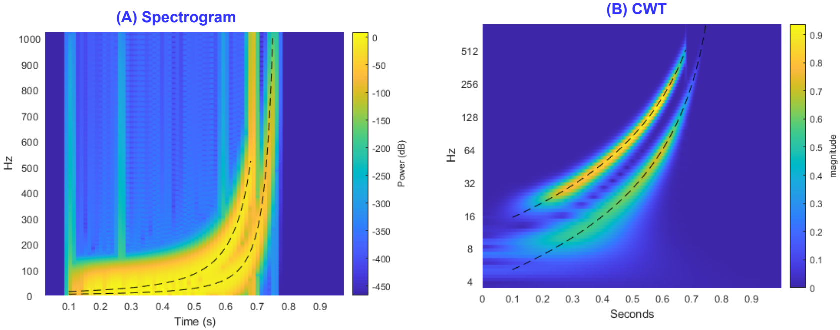


Figure: Difference in time-frequency resolution.

Thank you