School of Engineering and Applied Science (SEAS), Ahmedabad University

Probability and Stochastic Processes (MAT277)

Homework Assignment-1

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1. **While tossing a biased die, calculate the probability that face 3 has turned up, Given Alex tells either face 3 or face 6 has turned up.**

   (a) *We are given that,*

Face	1	2	3	4	5	6
Probability	0.2	0.22	0.11	0.25	0.15	0.07

Let A be the event that face 3 has turned up and B be the event that face 6 has turned up.

$\therefore \Pr(A) = 0.11 \quad \& \quad \Pr(B) = 0.07$

We know that these two are mutually exclusive events, hence:

$$\begin{aligned}\therefore \Pr(A \cup B) &= \Pr(A) + \Pr(B)\\ &= 0.11 + 0.32\\ &= 0.43\end{aligned}$$

Clearly, here we have to find the conditional probability, $\Pr(A \mid A \cup B)$

$$\Pr(A \mid A \cup B) = \frac{\Pr(A \cap (A \cup B))}{\Pr(A \cup B)}$$

$$= \frac{\Pr(A)}{\Pr(A \cup B)}$$

$$= \frac{0.11}{0.43}$$

$$= 0.2558139535.$$

$$\boxed{\therefore \ \Pr(A \mid A \cup B) \quad \approx \quad 0.2559}$$

Hence, the probability that face 3 has turned up, given either face 3 or face 6 has turned up is approx 25.59%

2. **There exists two events E1 & E2 such that** $\Pr(E1 \mid E2) = 0.45$**,** $\Pr(E2 \mid E1) = 0.5$ **and** $\Pr(E1 \cup E2) = 0.4$

   (a) *Calculate* $\Pr(E1 \cap E2)$*:*

   According to Conditional Probability theorem we can say,

$$\begin{aligned}\Pr(E1 \mid E2)\Pr(E2) &= \Pr(E2 \mid E1)\Pr(E1)\\ 0.45P(E2) &= 0.5P(E1)\end{aligned}$$

   Applying Basic Probaility theorem:

$$\begin{aligned}\Pr(E1 \cup E2) &= \Pr(E1) + \Pr(E2) + \Pr(E1 \cap E2)\\ 0.4 &= 0.9P(E2) + \Pr(E2) + \Pr(E1 \cap E2)\end{aligned}$$

$$\therefore \Pr(E1 \cap E2) = 1.9P(E2) - 0.4 \qquad ......(1)$$

   Now, Substituting these equation in conditional probability equation:

$$\Pr(E1 \mid E2) = \frac{\Pr(E1 \cap E2)}{\Pr(E2)}$$

$$\begin{aligned}0.45 &= \frac{1.9P(E2) - 0.4}{\Pr(E2)}\\ 0.4 &= 1.45P(E2)\end{aligned}$$

$$\therefore \Pr(E2) = 0.275$$

   Substituting value of $\Pr(E2)$ *in equation 1, we get*

$$\therefore \Pr(E1) = 0.2475$$

$$\therefore \Pr(E1 \cap E2) = 0.1225$$

   (b) *Comment on the dependency relation between event E1 and E2:*

   When event E2 occurs, there's a 45% chance that event E1 will occur. On the other hand if event E1 occurs, there's a 50% chance that event E2 will occur. This shows that the events are unsymmetrically related.

   Also upon calculating we can see that, there's some shared occurance happening between two of the given events, as probability of them happening together was found to be 12.25%.

3. **Given Probabilies are:**

    (a) **Let's denote the probability of a Red ball as:** $\Pr(R) = 0.45$

    (b) **Let's denote the probability of a Striped ball as:** $\Pr(S) = 0.3$

    (c) **Let's denote the probability of a Red ball with stripes as:** $\Pr(RS) = \Pr(R \cap S) = 0.2$

    To find the probability that ball is striped given the ball picked is a Red one.

$$\Pr(S \mid R) = \frac{\Pr(R \cap S)}{\Pr(R)}$$

$$= \frac{0.2}{0.45}$$

$$= 0.4444444...$$

$$\boxed{\therefore\ \Pr(S \mid R) \quad \approx \quad 0.4445}$$

Hence, the probability that the ball is striped one given the ball in red ball is approx 44.45%

4. **Let A denote the event where number 8 is obtained while tossing a 8-sided unbiased dice**

$\therefore \Pr(A) = \dfrac{1}{8} =$ p

$\therefore \Pr(A') = 1 - p$

X denotes the number of tosses required to get number 8 as an outcome.

(a) The probability that X = 6:
we use the equation:

$$\Pr(X) = \Pr(A')^{(X-1)}.\Pr(A)$$

$$\begin{aligned}\Pr(X_6) &= (1-p)^5.p \\ &= (\frac{7}{8})^5.\frac{1}{8} \\ &= (0.875)^5 * (0.125) \\ &= 0.0641136169\end{aligned}$$

$$\boxed{\therefore\ \Pr(X_6) \quad \approx \quad 0.06412}$$

(b) Conditional Probability that $X \leq 6$ given $X < 9$:

$$\begin{aligned}\Pr(X \leq 6 \mid X < 9) &= \frac{\Pr(X \leq 6 \cap X < 9)}{\Pr(X < 9)} \\ &= \frac{\Pr(X \leq 6)}{\Pr(X < 9)} \\ &= \frac{\sum_{i=1}^{6}((1-p)^{(i-1)} * p)}{\sum_{j=1}^{8}((1-p)^{(j-1)} * p)} \\ &= \frac{\sum_{i=1}^{6}(\frac{7}{8}^{(i-1)}\frac{1}{8})}{\sum_{j=1}^{8}(\frac{7}{8}^{(j-1)}\frac{1}{8})}\end{aligned}$$

Using the formula of Sum of Geometric series

$$a * (\frac{1-r^n}{1-r}) \ where\ a = \frac{1}{8},\ r = \frac{7}{8} \tag{1}$$

upon solving, we get

$$\textstyle\sum_{i=1}^{6}(\frac{7}{8}^{(i-1)}) \text{ and } \sum_{j=1}^{8}(\frac{7}{8}^{(j-1)})$$

$$\frac{\sum_{i=1}^{6}(\frac{7}{8}^{(i-1)} * \frac{1}{8})}{\sum_{j=1}^{8}(\frac{7}{8}^{(j-1)} * \frac{1}{8})} = \frac{0.551204}{0.656391}$$

$$\boxed{\therefore\ \Pr(X \leq 6 \mid X < 9) \quad = \ 0.8397}$$

Hence, the probability that $X \leq 6$ *given* $X < 9$ *is* 0.8397 .

5. **Given Probabilies are:**

   (a) **probability that an employee arrives late:** $\Pr(A_l) = 0.15$

   (b) **probability that an employee leaves early:** $\Pr(L_e) = 0.25$

   (c) **probability that an employee arrives late and leaves early:** $\Pr(A_l \cap L_e) = 0.08$

We need to find the probability of the employee arriving early given that he leaves late:

$$\Pr(A_l'|L_e') = \frac{\Pr(A_l' \cap L_e')}{\Pr(L_e')}$$

$$= \frac{\Pr(A_l \cup L_e)'}{\Pr(L_e')}$$

$$= \frac{(\Pr(A_l) + \Pr(L_e) - \Pr(A_l \cup L_e))'}{\Pr(L_e')}$$

$$= \frac{1 - (\Pr(A_l) + \Pr(L_e) - \Pr(A_l \cup L_e))}{1 - \Pr(L_e)}$$

$$= \frac{1 - (0.15 + 0.25 - 0.08)}{0.75}$$

$$= 0.9066666667$$

$$\boxed{\therefore \ \Pr(A_l'|L_e') \quad = \quad 0.9067}$$

Hence, the probability of the employee arriving early given that he leaves late is 80%.

6. Given S = { 1, 2, ..., n}, and X is a subset of S where if coins lands a heads then that particular element is added to X, and otherwise not.

   (a) For each coin toss there are two possible outcomes: either it is included in X or not.
   Given that a fair coin is tossed and all tosses are independent, the probability of it being in X is $(\frac{1}{2})$ and that of not being in X is $(\frac{1}{2})$.

   As there are total n element/coins in the set S, and each element has 2 possible outcomes,
   Total number of possible outcomes will be: 2^n.

   Since each outcome is equally likely to be include in set X or not, i.e $(\frac{1}{2})$, and there are total of 2^n outcomes, and each of them having equal probability of occuring, i.e $(\frac{1}{2})^n$.

   $\therefore$ Set X is equally likely to be any one of the 2^n possible subsets.

   (b) X and Y are two sets choosen independently and uniformly at random from 2^n subsets of set S.
   Note: here X and Y are not representing the outcomes of individual coin flips, they represent two subsets randomly chosen from set S.

   i. $\Pr(X \subseteq Y)$:
   For each element in the set S, there are two possibilities, it is included in X or it is not. Similarly, there are two possibilities for each element regarding Y, it is included in Y or it is not.
   Probability that a specific element is in X is $(\frac{1}{2})$, and that it is in Y is also $(\frac{1}{2})$.
   Now for X to be subset of Y, every element in X must also be ther in Y, Hence Probaility that a element is in both is: $(\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$.

$$\boxed{\therefore\ \Pr(X \subseteq Y) = (\frac{1}{4})^n}$$

   (as there as n elements)

   ii. $\Pr(X \cup Y = \{1, 2, ..., n\})$:
   Probability that a specific element is in either of X or Y, is the sum of probability of a element being in X and probability of a element being in Y.

$$
\begin{aligned}
\Pr(X \cup Y = \{1,2,...,n\}) &= 1 - \Pr(X^{'} \cup\ Y^{'}) \\
&= 1 - \Pr(X^{'})\Pr(Y^{'}) \\
&= 1 - (\frac{1}{2})^n(\frac{1}{2})^n \\
&= \left(\frac{3}{4}\right)^n
\end{aligned}
$$

$$\boxed{\therefore \Pr(X \cup Y = \{1,2,...,n\}) = \left(\frac{3}{4}\right)^n}$$

Hence, it is certain that union of X and Y is indeed set S.

7. We know that there are several different min-cut sets in the graph.

   Let us consider $C_1, ..., C_k$ be the distinct minimum cuts of the graph.
   Let $\mathcal{E}_i$ be the event that C_i is output using the analysis of the randomized min-cut algorithm.
   Since the event $\mathcal{E}_i$ is disjoint, it makes all these randomized events disjoint as follows:

   $$\textstyle\sum_{i,j} \ \Pr[\mathcal{E}_i] \leq 1.$$

   By the analysis of the randomized min-cut algorithm, is showed that:

   $$\Pr[\mathcal{E}_i] = \tfrac{n(n-1)}{2}$$

   for every i, which then implies that

   $$k \leq \tfrac{n(n-1)}{2}.$$

   This holds true as the n-cycle has exactly $\binom{n}{2}$ minimum cuts.
   Hence, from the above explanation, it is concluded that there can be at most $\frac{n(n-1)}{2}$ distinct min cut-sets in a graph.

8.

(a)

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{\substack{i,j=1 \\ i<j}}^{n} P(E_i \cap E_j) + \ldots + (-1)^{l+1} \sum_{\substack{i_1,\ldots,i_l=1 \\ i_1<\cdots<i_l}}^{n} P\left(\bigcap_{r=1}^{l} E_{i_r}\right) + \ldots$$

It is given that $E_1, \ldots, E_n$ be the n events.

Let l is odd, then for events $E_1, \ldots, E_n$, the relation is as follows:

$$P\left(\bigcup_{i=1}^{l} E_i\right) \leq \sum_{i=1}^{l} P(E_i) - \sum_{\substack{i,j=1 \\ i<j}}^{l} P(E_i \cap E_j) + \ldots + (-1)^{l+1} \sum_{\substack{i_1,\ldots,i_l=1 \\ i_1<\cdots<i_l}}^{l} P\left(\bigcap_{r=1}^{l} E_{i_r}\right) \ldots \quad (1)$$

Now, let l is even, then for events $E_1, \ldots, E_n$, the relation is as follows:

$$P\left(\bigcup_{i=1}^{l} E_i\right) \geq \sum_{i=1}^{l} P(E_i) - \sum_{\substack{i,j=1 \\ i<j}}^{l} P(E_i \cap E_j) + \ldots + (-1)^{l+1} \sum_{\substack{i_1,\ldots,i_l=1 \\ i_1<\cdots<i_l}}^{l} P\left(\bigcap_{r=1}^{l} E_{i_r}\right) \ldots \quad (2)$$

From equation (1) and (2), the relation is as follows:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{\substack{i,j=1 \\ i<j}}^{n} P(E_i \cap E_j) + \ldots + (-1)^{l+1} \sum_{\substack{i_1,\ldots,i_l=1 \\ i_1<\cdots<i_l}}^{n} P\left(\bigcap_{r=1}^{l} E_{i_r}\right) + \ldots$$

Hence, inclusion-exclusion principle is proved.

(b)

$$P\left(\bigcup_{i=1}^{l} E_i\right) \leq \sum_{i=1}^{l} P(E_i) - \sum_{1\leq i<j\leq l} P(E_i \cap E_j) + \ldots + (-1)^{l+1} \sum_{1\leq i_1<i_2<\cdots<i_l\leq l} P\left(E_{i_1} \cap \ldots \cap E_{i_k}\right)$$

It is given that l is odd.
According to the Bonferroni inequality which states that for odd k in $1, 2, \ldots, n$,

$$P\left(\bigcup_{i=1}^{l} E_i\right) \leq \textstyle\sum_{j=1}^{k} (-1)^{j+1} S_j,$$

, and for even k in $1, 2, \ldots, n$,

$$P\left(\bigcup_{i=1}^{l} E_i\right) \geq \textstyle\sum_{j=1}^{k} (-1)^{j+1} S_j,$$

where,

$$S_1 = \sum_{i=1}^{l} \Pr(E_i), \quad S_2 = \sum_{1\leq i<j\leq l} \Pr(E_i \cap E_j), \quad \text{and} \quad S_k = \sum_{1\leq i_1<i_2<\cdots<i_k\leq l} \Pr(E_{i_1} \cap \ldots \cap E_{i_k}) \text{ for } k = 3, 4, \ldots, n.$$

Expand the right-hand side of the inequality

$$P\left(\bigcup_{i=1}^{l} E_i\right) \leq \sum_{j=1}^{k} (-1)^{j+1} S_j,$$

as follows:

$$(-1)^1 S_1 + (-1)^2 S_2 + (-1)^3 S_3 + \ldots + (-1)^k S_k = S_1 - S_2 + S_3 - \ldots + (-1)^k S_k.$$

Solve the above expression further as follows:

$$P\left(\bigcup_{i=1}^{l} E_i\right) \leq S_1 - S_2 + S_3 - \ldots + (-1)^k S_k.$$

Substitute S_j in the above expression. It is found that

$$P\left(\bigcup_{i=1}^{l} E_i\right) \leq \sum_{i=1}^{l} P(E_i) - \sum_{1\leq i<j\leq l} P(E_i \cap E_j) + \ldots + (-1)^{l+1} \sum_{1\leq i_1<i_2<\cdots<i_l\leq l} P\left(E_{i_1} \cap \ldots \cap E_{i_l}\right).$$

Hence, the inequality for l odd is proved.

c. Recall the Bonferroni inequality which states that for odd k in $1, 2, \ldots, n$,

$$P\left(\bigcup_{i=1}^{l} E_i\right) \leq \sum_{j=1}^{k} (-1)^{j+1} S_j,$$

and for even k in $1, 2, \ldots, n$,

$$P\left(\bigcup_{i=1}^{l} E_i\right) \geq \sum_{j=1}^{k} (-1)^{j+1} S_j,$$

where,

$$S_1 = \sum_{i=1}^{l} P(E_i), \quad S_2 = \sum_{1\leq i<j\leq l} P(E_i \cap E_j), \quad S_k = \sum_{1\leq i_1<i_2<\cdots<i_k\leq l} P\left(E_{i_1} \cap \ldots \cap E_{i_k}\right) \text{ for } k = 3, 4, \ldots, n.$$

Expand the right-hand side of the inequality

$$P\left(\bigcup_{i=1}^{l} E_i\right) \geq \sum_{j=1}^{k} (-1)^{j+1} S_j,$$

as follows:

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Solve the above expression further as follows:

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Substitute S_j in the above expression. It is found that

$$P\left(\bigcup_{i=1}^{l} E_i\right) \geq \sum_{i=1}^{l} P(E_i) - \sum_{1\leq i<j\leq l} P(E_i \cap E_j) + \ldots + (-1)^{k} \sum_{1\leq i_1<i_2<\cdots<i_k\leq l} P\left(E_{i_1} \cap \ldots \cap E_{i_l}\right).$$

Hence, the Bonferroni inequality is proved.