School of Engineering and Applied Science (SEAS), Ahmedabad University

Probability and Stochastic Processes (MAT277)

Homework Assignment-3

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1. Consider that Alice and Bob have no first girl child then they have k children as mentioned in the question. So, the probability for no girl child is $\left(\dfrac{1}{2^k}\right)$.

   When they have one girl the probability is the complement of no girl child,

   $$\left(1 - \frac{1}{2^k}\right)$$

   The first i^{th} offspring are usually boys, that is to have at least $i \geq 1$ the probability is $\dfrac{1}{2^i}$. Therefore, the predicted number of male children has increased as

   $$\sum_{i=1}^{k}\left(\frac{1}{2^i}\right) = \left(1 - \frac{1}{2^k}\right)$$

   $$\boxed{\therefore \ \text{The expected number of female or male children is same that is } \left(1 - \frac{1}{2^k}\right).}$$

2. (a) **Maximizing the expected amount of money:**
   Given that the commodity sells for $2 per ounce now, we have $1000 to spend.
   Therefore, we can buy
   $$\frac{1,000}{2} = 500 \text{ ounces}$$
   After one week, the price of the commodity can either increase to $4 per ounce or decrease to $1 per ounce. Both possibilities are equally likely.

   i. If the price increases to $4 per ounce:
      - we have 500 ounces.
      - Selling them at $4 per ounce would give us $500 \times 4 = \$2,000$.

   ii. If the price decreases to $1 per ounce:
      - we still have 500 ounces.
      - Selling them at $1 per ounce would give us $500 \times 1 = \$500$.

   Since both the scenarios are equally likely to occur, we calculate the expected value by taking the average of the possible outcomes:

   $$E[\text{Money at the end of the week}] = \frac{2,000 + 500}{2} = \$1,250$$

   $\therefore$ Our expected amount of the commodity at the end of the week will be $\approx$ 1,250 ounces.

   (b) **Maximizing the expected amount of the commodity at the end of the wek:**
   We should wait until the end of the week to buy because the price will either be $1 or $4 per ounce with equal probability.

   Therefore, calculating the expected price at the end of the week by taking average we get, $2.50 per ounce.
   With the given amount, we can buy

   $$\frac{1,000}{2.50} = 400 \text{ ounces}$$

   $\therefore$ Our expected amount of the commodity at the end of the week will be 400 ounces.

   Hence, if our objective is to maximize the expected amount of money, we should buy the commodity now and hold onto it.

   That strategy ensures us obtain the most commodity for your money, taking into account the uncertainty that can take place within the price fluctuations.

3. Let, the variable X denotes the player's winnings. The probability that the player wins their n^{th} game is given by $\left(\dfrac{1}{2}\right)^n$.

Calculating the Expectation:

$$E[X] = \sum_{i=1}^{\infty} XP(X = X)$$

$$= \sum_{i=1}^{\infty} \left[2^n \cdot \left(\frac{1}{2}\right)^n \right]$$

$$= \sum_{i=1}^{\infty} (1)$$

$$= \infty$$

(a) No, I would not be pay $1 million to play this game once.

Considering the best case scenario: $1 million to play once and you win in the first toss, you will take home a big amount $2 million. But on the other hand, if you lose the bet, you will lose the invested amount.(*Not a small amount*)

The fact $E[X] = \infty$ is the expected value in the long run, not for one game.

(b) Yes, the expectation when considering a long run is indeed "something large," so we can expect the profit. Therefore, one can pay $1 million for each game.

4. From the properties of expectation,

$$E[a] = a, \quad \text{here } a \text{ is a constant.}$$

$$E[a - X] = a - E[X], \quad \text{here } X \text{ is a random variable.}$$

Consider any constant function y as function of random variable X as below.

$$y = (X - E[X])^k \quad \ldots (1)$$

Expand R.H.S of eq.(1) using Binomial's expansion,

$$(X - E[X])^k = \binom{k}{0} X^k (-E[X])^0 + \binom{k}{1} X^{k-1} (-E[X])^1 + \ldots + \binom{k}{k} X^0 (-E[X])^k$$

$$= X^k + (-E[X])^k - \binom{k}{1} X^{k-1} (E[X]) + \ldots + (-E[X])^k$$

substituting these values in eq.(1),

$$y - X^k = (E[X])^k - \binom{k}{1} X^{k-1} (E[X]) + \ldots + \binom{k}{k-1} X(-E[X])^{k-1} \qquad \ldots (2)$$

Apply expectation on both sides of eq.(2),

$$E[y - X^k] = E\left[(E[X])^k - \binom{k}{1} X^{k-1} (E[X]) + \ldots + \binom{k}{k-1} X(-E[X])^{k-1} \right] \qquad \ldots (3)$$

On applying properties of expectation from the eq.(3),

$$E[X^k] = (E[X])^k - \binom{k}{1} E[X^{k-1}(E[X])] + \ldots + \binom{k}{k-1} E[X(-E[X])^{k-1}] \qquad \ldots (4)$$

From the above eq.(4) it is interpreted that,

$$(E[X^k]) > (E[X])^k$$

Hence, proved.

5. (a) It is given that the length of the streaks is

$$k = \frac{\log_2 n + 1}{2} \text{ where } n \text{ is some power of 2.}$$

Consider the length of streaks be denoted by k and Y_i be that term of sequence of flips which represents 1 whenever the streak of length k starts from the i-th flip.

The length of the streak will be k from the i-th flip if the flips $Y_i's$ are all same for $i = 1, \ldots, n-k+1$. therefore

$$\sum_{i=1}^{n-k+1} Y_i$$

will be the total number of streaks of length k.
linearity of expectations states that

$$E\left[\sum_{i=1}^{n-k+1} Y_i\right] = \sum_{i=1}^{n-k+1} E[Y_i] = \sum_{i=1}^{n-k+1} Pr(Y_i = 1).$$

The Y_i will have value 1 if $k-1$ flips produce the same result as X_i. Since the dealt coin is fair and flips are independent, the probability that Y_i is 1 will be

$$Pr(Y_i = 1) = \left(\frac{1}{2}\right)^{k-1}.$$

Hence, the value is,

$$E\left[\sum_{i=1}^{n-k+1} Y_i\right] = \sum_{i=1}^{n-k+1} \left(\frac{1}{2}\right)^{k-1}.$$

substituting, $\log_2 n + 1$ for k

$$\begin{aligned} E\left[\sum_{i=1}^{n-k+1} Y_i\right] &= \sum_{i=1}^{n-k+1} \left(\frac{1}{2}\right)^{\log_2 n} \\ &= \sum_{i=1}^{n-k+1} \left(\frac{1}{2^{\log_2 n}}\right) \\ &= \sum_{i=1}^{n-k+1} \frac{1}{n} \\ &= \frac{n-k+1}{n} \\ &= \frac{n - \log_2 n - 1 + 1}{n} \\ &= 1 - \frac{\log_2 n}{n} \end{aligned}$$

Since $\frac{\log_2 n}{n}$ is of order 1, the expected number of streaks of order k is $1 - o(1)$ with respect to n.

Hence, proved.

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(b) Let n be the number of terms in the sequence of flips and the length of streaks be denoted by k.

If they are grouped into distinct blocks of k consecutive flips, then there will be $\dfrac{n}{k}$ such blocks.

Let E_1 be the event that the sequence of n terms does not have any streaks of length k and E_2 be the event that none of the blocks have a streak of length k.

It is easy to observe that E_1 is possible if E_2 holds, that is if none of the blocks have the streaks of length k then sequence will also not have any such streak.

Therefore, $Pr(E_1) \leq Pr(E_2)$.

Since the coin is fair and flips are independent, the probability that a block of k consecutive flips has a streak of length k will be $\left(\dfrac{1}{2}\right)^{k-1}$. Therefore, the probability that a block does not have any streak of length k is $\left[1 - \left(\dfrac{1}{2^{k-1}}\right)\right]$.

As there are $\dfrac{n}{k}$ blocks,

$$Pr(E_2) = \left(1 - \frac{1}{2^{k-1}}\right)^{\frac{n}{k}}.$$

Using the facts that $k = 1 \leq \log_2 n - 2\log_2 \log_2 n$, $\dfrac{n}{k} \leq \dfrac{n}{\log_2 n}$, and $1 - x \leq e^{-x}$ and substituting them in the equation of the probability of E_2.

$$Pr(E_2) \leq \left(1 - \frac{1}{2^{(\log_2 n - 2\log_2 \log_2 n)}}\right)^{\frac{n}{\log_2 n}}$$

$$= \left(\frac{\log_2^2 n}{n}\right)^{\frac{n}{\log_2 n} - 1}$$

$$\leq \exp\left[-\frac{n}{\log_2 n}\left(\frac{\log_2^2 n}{n}\right)\right]$$

$$= \exp\left[-\log_2 n\left(\frac{\log_2 n}{n}(\log_2 n - 1)\right)\right]$$

$$Pr(E_2) = \exp\left[-\log_2 n\left(1 - \frac{\log_2 n}{n}\right)\right]$$

It is obtained that $Pr(E_2) = \exp\left[-\log_2 n\left(1 - \dfrac{\log_2 n}{n}\right)\right]$.

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Simplifing further using the properties of logarithmic function, we get:

$$Pr(E_2) = \exp\left[-\log_2 n \left(1 - \frac{\log_2 n}{n}\right)\right]$$

$$\leq \exp\left(-\log_2 n \left(\frac{1}{\log_2 e}\right)\right)$$

$$= \exp\left(\frac{\log_2 n}{\log_2 e}\right)$$

$$= \exp(-\ln n)$$

It follows that

$$Pr(E_2) \leq \frac{1}{n}.$$

Hence, it is proved that

$$\mathbf{Pr(no\ streak\ of\ length\ at\ least}\ [\log_2 n - 2\log_2 \log_2 n]) < \frac{1}{n}.$$

6. (a) Let f be a convex function on the interval I. If $x_1, x_2, \ldots, x_n \in I$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non negative real numbers such that $\sum_{i=1}^{n} \lambda_i = 1$, then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \le \sum_{i=1}^{n} \lambda_i f(x_i).$$

The case for $n = 2$ is true by the definition of convex. Assume the relation holds for n, then we have

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = f\left(\sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1} x_{n+1}\right) = f\left(\lambda_{n+1} \cdot x_{n+1} + (1 - \lambda_{n+1})\frac{1}{1 - \lambda_{n+1}}\sum_{i=1}^{n} \lambda_i x_i\right)$$

$$\le \lambda_{n+1} \cdot f(x_{n+1}) + (1 - \lambda_{n+1}) \cdot f\left(\frac{1}{1 - \lambda_{n+1}}\sum_{i=1}^{n} \lambda_i x_i\right)$$

$$= \lambda_{n+1} \cdot f(x_{n+1}) + (1 - \lambda_{n+1}) \cdot f\left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} \cdot x_i\right)$$

$$\le \lambda_{n+1} \cdot f(x_{n+1}) + (1 - \lambda_{n+1}) \cdot \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} \cdot f(x_i)$$

$$= \sum_{i=1}^{n} \lambda_i \cdot f(x_i) + \lambda_{n+1} \cdot f(x_{n+1})$$

$$= \sum_{i=1}^{n+1} \lambda_i f(x_i).$$

Hence, by induction, the theorem is proven.

(b) We all know that expectation is denoted by:

$$E[X] = \sum_{i=1}^{n} X_i P(X_i)$$

Since, $P(X_i) \ge 0$, let $\lambda_i = P(X_i) \ge 0$ & $\sum_{i=1}^{n} P(X_i) = 1$

$$\therefore f(E[X]) = f\left(\sum_{i}^{n} x_i P(X_i)\right)$$

$$= f\left(\sum_{i}^{n} X_i \lambda_i\right) \le \sum_{i}^{n} \lambda_i f(X_i)$$

$$= \sum_{i}^{n} p(X_i) f(X_i) \le E[f(X)]$$

Hence, Proved

7. (a)   i. To find $\mathbf{E}[(4X-1)^2]$, we can use the properties of expected value.

$$E[(4X-1)^2] = E[16X^2 - 8X + 1]$$

By using linearity of expectation:

$$E[16X^2 - 8X + 1] = 16E[X^2] - 8E[X] + E[1]$$

We know that,

$$Var(X) = E[X^2] - (E[X])^2$$
$$\therefore E[X^2] = Var(X) + (E[X])^2$$

substituting this into the expression,we get:

$$16(Var(X) + (E[X])^2) - 8E[X] + 1 = 16(1 + 3^2) - (8 \times 3) + 1 = 16(10) - 24 + 1$$
$$= 160 - 24 + 1 = 137$$

$$\boxed{\therefore E\left[(4X-1)^2\right] = 137}$$

ii. Now, let's find $\mathbf{Var(5-2X)}$:

$$E[5 - 2X] = 5 - 2E[X]$$
$$5 - (2 \times 3) = 5 - 6 = -1$$

$$Var(5 - 2X) = E[(5 - 2X)^2] - [E(5 - 2X)]^2$$
$$= E[25 - 20X + 4X^2] - (-1)^2$$

Using the properties of expectation:

$$25 - 20E[X] + 4E[X^2] - 1 = 25 - (20 \times 3) + (4 \times 10) - 1$$
$$= 25 - 60 + 40 - 1 = 4$$

$$\boxed{\therefore Var(5 - 2X) = 4}$$

(b) The total number of selecting 2 cards from a deck of 52 cards is given by: $n(T) = \binom{52}{2} = 1326$.

Now, considering both the selected cards are of the same suit: $n(ss) = \binom{4}{1}.\binom{13}{2} = 312$.

Similarly, if both the selected cards are of the same color: $n(sc) = \binom{2}{1}.\binom{26}{2} = 650$.

$$\therefore Pr(ss) = \frac{312}{1326} \approx 0.2353$$

$$\therefore Pr(sc) = \frac{650}{1326} \approx 0.4902$$

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∴ It implies that $P(X = 1) = Pr(sc)$ ...[the player won \$1]
∴ It implies that $P(X = 2) = Pr(ss)$ ...[the player won \$2]

For other cases, it's given that a player will lose \$0.50

$$\therefore P(X = -0.50) = 1 - P(X = 1) - P(X = 2)$$
$$= 1 - [0.2353 + 0.4902]$$
$$= 0.2745$$

We know that,

$$Var(X) = E[X^2] - (E[X])^2 \quad \ldots(i)$$
$$E[X^2] = \sum x^2 . P(X = x) \quad \& \quad E[X] = \sum x . P(X = x)$$

$$\therefore E[X^2] = (2^2 \times 0.2353) + (1^2 \times 0.4902) + (-0.5^2 \times 0.2745)$$
$$= 1.5$$

$$\therefore E[X] = (2 \times 0.2353) + (1 \times 0.4902) + (-0.5 \times 0.2745)$$
$$= 0.8235$$

$$Var(X) = E[X^2] - (E[X])^2$$
$$= 1.5 - (0.8235^2)$$
$$= 0.8218$$

∴ The variance of the amount won by the player is, \$0.8218.

(c) Let's say we have two teams A & B.
If Team A has a winning Probability of p, then winning probability of Team B will be $(1 - p)$.
Let N be total number of games played when $i = 2$.
Hence, N can take only two values $i.e$, 2 OR 3 (Since for atleast one of them to win, atmost 3 games are required).
$P(N = 2) = P(N = 2, A\ wins) + P(N = 2, B\ wins)$
$= p^2 + (1 - p)^2$

$P(N = 3) = P(N = 3, A\ wins) + P(N = 3, B\ wins)$
• For A to win when $N = 3$, A has to win the 3rd game and any of the first two games.

$\therefore P(N = 3, A\ wins) = p.(1 - p).p + (1 - p).p.p$
$= 2.p^2.(1 - p)$

• Similarly for B to win when $N = 3$, we get:

$\therefore P(N = 3, B\ wins) = p.(1 - p).(1 - p) + (1 - p).p.(1 - p)$
$= 2.p.(1 - p)^2$

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We get the Expectation:

$$
\begin{aligned}
E[N] &= 2.P(N=2) + 3.P(N=3) \\
&= \left[2. \left(p^2 + (1-p)^2\right)\right] + \left[3. \left(2p^2(1-p) + 2p(1-p)^2\right)\right] \\
&= -2p^2 + 2p + 2
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E[N^2] &= 2^2.P(N=2) + 3^2.P(N=3) \\
&= \left[4. \left(p^2 + (1-p)^2\right)\right] + \left[9. \left(2p^2(1-p) + 2p(1-p)^2\right)\right] \\
&= -10p^2 + 10p + 4
\end{aligned}
$$

The variance can we calculated by,

$$
\begin{aligned}
Var(N) &= E[N^2] - (E[N])^2 \\
&= \left[-10p^2 + 10p + 4\right] - \left[-2p^2 + 2p + 2\right]^2 \\
&= -4p^4 + 8p^3 - 6p^2 + 2p
\end{aligned}
$$

• **The 2nd part can't be proved as the value of probability can't be more than 1 (we are given 12).**

8. (a) Determine the value of

$$Pr\left(\frac{1}{2} \le X \le \frac{3}{4}\right) \text{ and } Pr\left(X \le \frac{1}{2} \,\middle|\, \frac{1}{4} \le X \le \frac{3}{4}\right)$$

We know that X be a uniform random variable on $[0,1]$. Thus, The density function of X is given as follows:

$$f(x) = 1, \quad 0 < x < 1$$

Conditional probability for continuous random variable is defined as

$$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$$

Substitute $E = X \le \frac{1}{2}$ and $F = \frac{1}{4} \le X \le \frac{3}{4}$ in

$$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$$

as follows:

$$Pr\left(X \le \frac{1}{2} \,\middle|\, \frac{1}{4} \le X \le \frac{3}{4}\right) = \frac{Pr\left(\left(X \le \frac{1}{2}\right) \cap \left(\frac{1}{4} \le X \le \frac{3}{4}\right)\right)}{Pr\left(\frac{1}{4} \le X \le \frac{3}{4}\right)}$$

$$= \frac{Pr\left(\frac{1}{4} \le X \le \frac{1}{2}\right)}{Pr\left(\frac{1}{4} \le X \le \frac{3}{4}\right)}$$

It is known that the probability of the interval $[a, b]$ is given by the integral

$$Pr(a \le X \le b) = \int_a^b f(x) \, dx$$

applying this to find the probability of the interval [a,b]:

$$Pr\left(X \le \frac{1}{2} \,\middle|\, \frac{1}{4} \le X \le \frac{3}{4}\right) = \frac{\int_{1/4}^{1/2} 1 \, dx}{\int_{1/4}^{3/4} 1 \, dx}$$

$$= \frac{[x]_{1/4}^{1/2}}{[x]_{1/4}^{3/4}}$$

$$= \frac{1}{2}$$

substituting $E = X \le \frac{1}{4}$ and $F = \frac{1}{3} \le X \le \frac{2}{3}$ in

$$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$$

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$$Pr\left(X \le \frac{1}{4} \,\middle|\, X \le \frac{1}{3}\right) = \frac{Pr\left(\left(X \le \frac{1}{4}\right) \cap \left(\left(X \le \frac{1}{3}\right) \cup \left(X \ge \frac{2}{3}\right)\right)\right)}{Pr\left(\left(X \le \frac{1}{3}\right) \cup \left(X \ge \frac{2}{3}\right)\right)}$$

$$= \frac{Pr\left(X \le \frac{1}{4}\right)}{Pr\left(\left(X \le \frac{1}{3}\right) \cup \left(X \ge \frac{2}{3}\right)\right)}$$

We know that $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

Here

$$Pr\left(\left(X \le \frac{1}{3}\right) \cap \left(X \ge \frac{2}{3}\right)\right) = 0$$

so,

$$Pr\left(\left(X \le \frac{1}{3}\right) \cup \left(X \ge \frac{2}{3}\right)\right) = Pr\left(X \le \frac{1}{3}\right) + Pr\left(X \ge \frac{2}{3}\right)$$

$$Pr\left(X \le \frac{1}{4} \,\middle|\, \left(X \le \frac{1}{3}\right) \cup \left(X \ge \frac{2}{3}\right)\right) = \frac{Pr\left(X \le \frac{1}{4}\right)}{Pr\left(X \le \frac{1}{3}\right) + Pr\left(X \ge \frac{2}{3}\right)}$$

It is known that the probability of the interval $[a, b]$ is given by the integral

$$Pr(a \le X \le b) = \int_a^b f(x) \; dx$$

$$Pr\left(X \le \frac{1}{4}\middle|\left(X \le \frac{1}{3}\right) \cup \left(X \ge \frac{2}{3}\right)\right) = \frac{\int_0^{1/4} 1 \; dx}{\int_0^{1/3} 1 \; dx + \int_{2/3}^1 1 \; dx}$$

$$= \frac{[x]_0^{1/4}}{[x]_0^{1/3} + [x]_{2/3}^1}$$

$$= \frac{\dfrac{1}{4} - 0}{\left(\dfrac{1}{3} - 0\right) + \left(1 - \dfrac{2}{3}\right)}$$

$$= \frac{\dfrac{1}{4}}{\dfrac{1}{3} + \dfrac{1}{3}}$$

$$= \frac{\left(\dfrac{1}{4}\right)}{\left(\dfrac{2}{3}\right)}$$

$$= \frac{3}{8}$$

Hence, the values are

$$\boxed{Pr\left(X \le \frac{1}{4}\middle|X \le \frac{1}{3}\right) = \frac{1}{2} \quad \text{and} \quad Pr\left(X \le \frac{1}{4}\middle|\left(X \le \frac{1}{3}\right) \cup \left(X \ge \frac{2}{3}\right)\right) = \frac{3}{8}.}$$

(b) Find the expected value of the kth largest of the n random variables.
The objective is to find $E[\max\{x_1, x_2, \ldots, x_n\}]$
Therefore,

$$\begin{aligned}
F_{\max\{x_1,x_2,\ldots,x_n\}}(a) &= P(x_1 \le a, x_2 \le a, \ldots, x_n \le a) \\
&= F_x(a)F_x(a)\ldots F_x(a) \\
&= (1 - e^{\lambda_1 a})(1 - e^{\lambda_2 a})\ldots(1 - e^{\lambda_n a})
\end{aligned}$$

Now, since

$$\begin{aligned}
E\left[F_{\max\{x_1,x_2,\ldots,x_n\}}\right] &= \int_0^\infty 1 - F_{\max\{x_1,x_2,\ldots,x_n\}}(a) \; da \\
&= \int_0^\infty (1 - e^{\lambda_1 a})(1 - e^{\lambda_2 a})\ldots(1 - e^{\lambda_n a}) \; da \\
&= \int_0^\infty 1 - (1 - e^a)(1 - e^a)\ldots(1 - e^a) \; da
\end{aligned}$$

Since, parameter λ is 1, that is,
$$\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$$

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This implies;

$$\int_0^\infty (1-(1-e^a)^n)\,da = \int_0^\infty 1-\left(1-ne^a+\frac{n(n-1)e^{2a}}{2!}-\ldots-\frac{n(n-1)(n-r+1)(-e^a)^r}{r!}+\ldots\infty\right)$$

expanding the binomial expression,

$$(1-x)^n = 1+(-nx)+\frac{n(n-1)}{2}x^2+\ldots+\frac{(-1)^r(n(n-1)\ldots(n-r+1))}{r!}x^r+\ldots\infty$$

Therefore,

$$E[\max\{x_1,x_2,\ldots,x_n\}] = \int_0^\infty ne^a-\frac{n(n-1)e^{2a}}{2!}+\ldots+\frac{(-1)^r(n(n-1))}{r!}-(n-r+1)e^{ar}\,da$$

$$= \left[ne^a-\frac{n(n-1)e^{2a}}{4}+\ldots+(-1)^r\frac{n(n-1)-(n-r+1)e^{ar}}{(rr!)}\right]_0^\infty$$

$$= \infty$$

(c) It is known that one person must wait exactly for 15 minutes for the other person. So, the inequality when two people meet can be written as: $|X-Y| \le 15$.

Consider that X and Y are random variables, which are independent and uniformly distributed on the interval $[0, 60]$.

The inequality is divided as follows:

For the first 15 minutes, the probability is $\left(y+\frac{1}{4}\right)$. For the next half an hour, the probability is $\left(\frac{1}{2}\right)$ and for the next 15 minutes, the probability is $\left(\frac{5}{4}-y\right)$.

Therefore we calculate the required probability $P(|X-Y| \le 15)$:

$$P(|X-Y| \le 15) = \int_{-\infty}^\infty \int_{y-15}^{y+15} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^{y+15} \int_{y-15}^{y+15} f_X(x)f_Y(y)\,dx\,dy$$

$$= \int_0^{60} \int_{y-15}^{y+15} f_X(x)\,dx\,dy$$

It is obtained from the problem that

$$\int_{y-15}^{y+15} f_X(x)\,dx = \begin{cases} y+\frac{1}{4}, & 0 \le y < 15 \\ \frac{1}{2}, & 15 \le y < 45 \\ \frac{5}{4}-y, & 45 \le y \le 60 \end{cases}$$

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$$P(|X - Y| \leq 15) = \int_0^{15} \left(y + \frac{1}{4} \right) dy + \int_{15}^{45} \left(\frac{1}{2} \right) dy + \int_{45}^{60} \left(\frac{5}{4} - y \right) dy$$

$$= \frac{1}{2} \left(\frac{1}{4} \right)^2 + \left(\frac{1}{4} \right)^2 + \left(\frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{4} \right)^2 + \left(\frac{1}{4} \right)^2$$

$$= \frac{7}{16}$$

9.

```python
def Given_func(X):
    return 1/X
```
[82] ✓ 0.0s

```python
p = 1/4
n = 20
```
[83] ✓ 0.0s

```python
# Define a function to test Jensen's inequality
def jensens_inequality_test(func, X):
    """
    Tests Jensen's inequality for a given function and random variable X.

    Args:
    - func (function): The function for which Jensen's inequality is to be tested.
    - X (list of tuples): Probability mass function (PMF) in the format (value, prob

    Returns:
    - lhs (float): Left-hand side of Jensen's inequality.
    - rhs (float): Right-hand side of Jensen's inequality.
    """
    # Compute the expectation of the function
    E_fx = expectation([(func(value), probability) for value, probability in X])

    # Compute the function applied to the expectation of X
    fx_E_X = func(expectation(X))

    return E_fx, fx_E_X
```
[84] ✓ 0.0s

Figure 1: Function

```
X_arrays = [
    [(1, 0.25), (2, 0.25), (3, 0.25), (4, 0.25)],
    [(5, 0.25), (6, 0.25), (7, 0.25), (8, 0.25)],
    [(9, 0.25), (10, 0.25), (11, 0.25), (12, 0.25)],
    [(13, 0.25), (14, 0.25), (15, 0.25), (16, 0.25)],
    [(17, 0.25), (18, 0.25), (19, 0.25), (20, 0.25)]
]
functions = [Given_func]
✓ 0.0s
```

Figure 2: PMF

```
Current X: [(9, 0.25), (10, 0.25), (11, 0.25), (12, 0.25)]
Testing function: Given_func

Expectation of f(X): 0.09633838383838383
f(Expectation of X): 0.09523809523809523

Jensen's inequality holds.

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Current X: [(13, 0.25), (14, 0.25), (15, 0.25), (16, 0.25)]
Testing function: Given_func

Expectation of f(X): 0.06937957875457876
f(Expectation of X): 0.06896551724137931

Jensen's inequality holds.

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Current X: [(17, 0.25), (18, 0.25), (19, 0.25), (20, 0.25)]
Testing function: Given_func

Expectation of f(X): 0.05425266597867217
f(Expectation of X): 0.05405405405405406

Jensen's inequality holds.
```

Figure 3: Output