School of Engineering and Applied Science (SEAS), Ahmedabad University

Probability and Stochastic Processes (MAT277)

Homework Assignment-2

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1. **Let event A be the probability that Omega is significant:** $\Pr(A) = 0.75$
   Let there be an event that the Algorithm Delta produces a positive result, given that Omega is significant: $\Pr(B|A) = 1 - 0.15 = 0.85$
   Let there be an event that the Algorithm Delta produces a positive result, given that Omega is insignificant: $\Pr(B|A') = 0.15$
   The probability that Omega is insignificant: $\Pr(A') = 0.25$

   The Probability that the message is significant given Algorithm Delta is positive is,

$$Pr(A|B) = \frac{Pr(B|A).Pr(A)}{Pr(B|A).Pr(A) + Pr(B|A').Pr(A')}$$

$$= \frac{(0.85).(0.75)}{(0.85).(0.75) + (0.15).(0.25)}$$

$$= \frac{0.6375}{0.675}$$

$$= 0.9444444$$

$$\boxed{\therefore \ \Pr(A|B) \quad \approx \quad 0.945}$$

   Hence, as the probability that message contains critical information is less than 95%, we should opt for further analysis.

2. **An undirected graph G with n nodes is given.**
The min-cut algorithm works by iteratively contracting edges until only two nodes remain, representing the two disjoint sets of the minimum cut. The contraction process merges nodes, and the algorithm repeats until only two super-nodes are left.

(a) The total number of edges in the graph is given by the binomial coefficient $\binom{n}{2}$, which represents all possible ways to choose 2 nodes out of n. This is equal to $\dfrac{n(n-1)}{2}$.

During the contraction process, each contraction operation results in a unique cut. Since there are $\dfrac{n(n-1)}{2}$ distinct edges in the graph, there can be at most $\dfrac{n(n-1)}{2}$ distinct cuts formed during the contraction process.

$\therefore$ we can argue that there can be at most $\dfrac{n(n-1)}{2}$ distinct min-cut sets in a graph G with n nodes using the analysis of the min-cut algorithm.

(b)

3. **Let A denote the event where number 6 is obtained while tossing a 6-sided unbiased dice.**

(a) The probability of rolling a 6 on a single trial is: $\Pr(A) = \frac{1}{6}$

The probability of not rolling a 6 on a single trial is: $\Pr(A') = \frac{5}{6}$

Now, let us consider the event that a six occurs on the k^{th} roll. This implies that there were $k-1$ consecutive non-six outcomes followed by a six. The probability of this sequence is given by:

$$\Pr(X = k) = \Pr(A')^{(X-1)}.\Pr(A)$$

$$\Pr(X = k) = \left(\frac{5}{6}\right)^{k-1} \cdot \left(\frac{1}{6}\right)$$

(b) To Check the legitimacy of the PMF, we need to satisfy two conditions:

i. Each probability of P(X = k) is non-negative for all possible values of k.

$$\Pr(X = k) \geq 0$$

$$\left(\frac{5}{6}\right)^{k-1} \cdot \left(\frac{1}{6}\right) \geq 0$$

This is ture because both the terms, $\left(\frac{5}{6}\right)^{k-1}$ and $\left(\frac{1}{6}\right)$ are non-negative.

ii. The sum of all probabilities over all possible values of k is equal to 1.

$$\sum_{k=1}^{n} P(X = k) = \sum_{k=1}^{n} \left(\frac{5}{6}\right)^{k-1} \cdot \left(\frac{1}{6}\right)$$

It's a geometric series where the sum can be represented as:

$$\sum_{k=1}^{n} \left(\frac{5}{6}\right)^{k-1} \cdot \left(\frac{1}{6}\right) = \left(\frac{1}{6}\right) \cdot \frac{1-\left(\frac{5}{6}\right)^n}{1-\frac{5}{6}}$$
$$= 1 - \left(\frac{5}{6}\right)^n$$

This term approaches 1 as n tends to infinity, the sum of probabilities converges to 1.

$\therefore$ This PMF is legitimate.

4. **The PMF of random variable X is given by:**

$$p(i) = \frac{c.\lambda^i}{i!}, \qquad i = 0, 1, 2, ...$$

Given λ is a positive value.

(a) P{X=0} can be obtained by simply Substituting $i = 0$ in the above equation.

$$P\{X = 0\} = p(0) = \frac{c.\lambda^0}{0!}$$

$$= c$$

$$\boxed{\therefore P\{X = 0\} = c}$$

(b) P{X > 2} can also be written in other form as: 1 - P{X ≤ 2}

$$P\{X \leq 2\} = p(0) + p(1) + p(2)$$

$$= c + \frac{c.\lambda^1}{1!} + \frac{c.\lambda^2}{2!}$$

$$P\{X > 2\} = 1 - \left(c + c.\lambda + \frac{c.\lambda^2}{2}\right)$$

$$\boxed{\therefore P\{X > 2\} = 1 - \left(\frac{c.\lambda^2 + 2c.\lambda + 2c}{2}\right)}$$

5. The event that the contestant knows the Answer to Question-1: E_1
The event that the contestant knows the Answer to Question-2: E_2
The random variable representing winnings for answering Question-1 correctly: X_1
The random variable representing winnings for answering Question-2 correctly: X_2
The amount won for answering Question-1 correctly: V_1
The amount won for answering Question-2 correctly: V_2
The probability of knowing the answer to Question-1: P_1
The probability of knowing the answer to Question-2: P_2

Case-i If they attempt Question-1 first:
• Get Question 1 right and Question 2 right
• Get Question 1 right and Question 2 wrong
• Get Question 1 wrong
So the expected winnings for attempting Question-1 first is given by:

$$P_1.P_2.(V_1 + V_2) + P_1.(1 - P_2).(V_1) + (1 - P_1).0$$

Case-ii If they attempt Question-2 first:
• Get Question 2 right and Question 1 right
• Get Question 2 right and Question 1 wrong
• Get Question 2 wrong
So the expected winnings for attempting Question-2 first is given by:

$$P_2.P_1.(V_1 + V_2) + P_2.(1 - P_1).(V_2) + (1 - P_2).0$$

Hence, To maximize expected winnings, the contestant should attempt the question first that yields the higher expected value from these above two equations.

6. **A PMF is given by**

$$P(X = i) = log_{10}\left(\frac{i+1}{i}\right), \quad i = 1, 2, 3, ..., 9$$

(a) to prove,

$$\sum_{i=1}^{9} log_{10}\left(\frac{i+1}{i}\right) = 1$$

$\therefore$ Solving **L.H.S**:

$$\begin{aligned}
\sum_{i=1}^{9} log_{10}\left(\frac{i+1}{i}\right) &= log_{10}\left(\frac{2}{1}\right) + log_{10}\left(\frac{3}{2}\right) + ... \; log_{10}\left(\frac{10}{9}\right) \\
&= log_{10}\left[\left(\frac{2}{1}\right) \cdot \left(\frac{3}{2}\right) \cdot ... \cdot \left(\frac{10}{9}\right)\right] \\
&= log_{10}\,(10) \\
&= 1 \\
&= \textbf{R.H.S}
\end{aligned}$$

Hence, Proved.

(b) $P(X \leq j)$

$$\begin{aligned}
\sum_{i=1}^{j} log_{10}\left(\frac{i+1}{i}\right) &= log_{10}\left(\frac{2}{1}\right) + log_{10}\left(\frac{3}{2}\right) + ... \; log_{10}\left(\frac{j+1}{j}\right) \\
&= log_{10}\left[\left(\frac{2}{1}\right) \cdot \left(\frac{3}{2}\right) \cdot ... \cdot \left(\frac{j+1}{j}\right)\right] \\
&= log_{10}\,(j+1)
\end{aligned}$$

$$\boxed{\therefore P\{X \leq j\} = log_{10}\,(j+1)}$$

7. Let us denote G as the number of correct guesses,
Let $P(G = k)$ denote the probability of having exactly k correct guesses.
This signifies that the guesses are continuously correct till k^{th} guess, which is incorrect.
The probability of guessing correct card for the 1^{st} time: $\left(\frac{1}{n}\right)$,
The probability of guessing correct card for the 2^{nd} time: $\left(\frac{1}{n-1}\right)$,
.
.
.
The probability of guessing correct card for the k^{th} time: $\left(\frac{1}{n-k+1}\right)$,

$$P(G = k) = \left(\frac{1}{n}\right) \cdot \left(\frac{1}{n-1}\right) \cdot \left(\frac{1}{n-2}\right) \cdot ... \cdot \left(\frac{1}{n-k+2}\right) \cdot \left(\frac{1}{n-k+1}\right)$$

We can rewrite the product of denominator's as:

$$n \cdot (n-1) \cdot (n-2) \cdot ... \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

$\therefore$

$$P(G = k) = \left(\frac{1}{n \cdot (n-1) \cdot (n-2) \cdot ... \cdot (n-k+1)}\right)$$

$$= \left(\frac{1}{\frac{n!}{(n-k)!}}\right)$$

$$= \frac{(n-k)!}{n!}$$

$$\boxed{\therefore P(G = k) = \frac{(n-k)!}{n!}}$$

8. From the distribution function of the random variable X:

For $1 \le x < 3$:

$$P(X = 1) = F(1) - F(-\infty) = \frac{1}{4} - 0 = \frac{1}{4}$$

For $3 \le x < 4$:

$$P(X = 3) = F(3) - F(1) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}$$

For $4 \le x < 6$:

$$P(X = 4) = F(4) - F(3) = \frac{3}{4} - \frac{5}{8} = \frac{1}{8}$$

For $6 \le x < 7$:

$$P(X = 6) = F(6) - F(4) = \frac{7}{8} - \frac{3}{4} = \frac{1}{8}$$

For $x \ge 7$:

$$P(X = 7) = 1 - F(6) = 1 - \frac{7}{8} = \frac{1}{8}$$

Therefore, the probability mass function (PMF) of the random variable X is:

$$P(X = x) = \begin{cases} \frac{1}{4} & \text{if } 1 \le x < 3 \\ \\ \frac{3}{8} & \text{if } 3 \le x < 4 \\ \\ \frac{1}{8} & \text{if } 4 \le x < 7, x \ge 7 \\ \\ 0 & \text{otherwise} \end{cases}$$

9. We want to compute the conditional probability that the gambler wins i (where $i = 1, 2, 3$), given that he wins a positive amount. The conditional probability is given by:

$$P(X = i|X > 0) = \frac{P(X = i \ \cap \ X > 0)}{P(X > 0)}$$

First, we calculate the probability of winning a positive amount:

$$P(X > 0) = \frac{13}{55} + \frac{1}{11} + \frac{1}{165}$$

Finally, we substitute these value into the conditional probability formula to obtain:

$$P(X = 1|X > 0) = \frac{\frac{13}{55}}{\frac{13}{55} + \frac{1}{11} + \frac{1}{165}} = 0.7\overline{09}$$

$$\boxed{\therefore P(X = 1|X > 0) = 0.7\overline{09}}$$

$$P(X = 2|X > 0) = \frac{\frac{1}{11}}{\frac{13}{55} + \frac{1}{11} + \frac{1}{165}} = 0.\overline{27}$$

$$\boxed{\therefore P(X = 2|X > 0) = 0.\overline{27}}$$

$$P(X = 3|X > 0) = \frac{\frac{1}{165}}{\frac{13}{55} + \frac{1}{11} + \frac{1}{165}} = 0.0\overline{18}$$

$$\boxed{\therefore P(X = 3|X > 0) = 0.0\overline{18}}$$

10. (a) A fair coin is tossed n times, Let X represent the difference between the number of Heads and the number of Tails obtained.

Each toss of the coin can result in either a Head or a Tail. Therefore, the possible outcomes for X depend on the number of Heads and Tails obtained.

Considering the extreme cases:
1. If there are n heads and 0 tails, $X = n$ (all heads).
2. If there are 0 heads and n tails, $X = -n$ (all tails).

Hence, the possible values of X range from $-n$ to n, inclusive. Therefore, the set of possible values for X is $\{-n, -(n-2), -(n-4), \ldots, (n-4), (n-2), n\}$.

(b) The probability mass function (PMF) describes the probability of each possible outcome,
For $n = 3$:

i. $P(X = -3)$: The probability of having 3 tails and 0 heads.

$$P(X = -3) = \left(\frac{1}{2^3}\right) = \frac{1}{8}$$

ii. $P(X = -1)$: The probability of having 2 tails and 1 head or 1 tail and 2 heads.

$$P(X = -1) = \left(\frac{3}{2^3}\right) = \frac{3}{8}$$

iii. $P(X = 1)$: The probability of having 2 heads and 1 tail or 1 head and 2 tails.

$$P(X = 1) = \left(\frac{3}{2^3}\right) = \frac{3}{8}$$

iv. $P(X = 3)$: The probability of having 3 heads and 0 tails.

$$P(X = 3) = \left(\frac{1}{2^3}\right) = \frac{1}{8}$$

11. **We've been tasked with comparing poisson ratio and binomial Probability for some cases were n is large but p is very small, but their product *i.e* ($\lambda = n \cdot p$) is a moderate number.** Here are the various cases to compare:

   (a) $P(X = 2)$ when $n = 8, p = 0.1$:
   Binomial Probability:

$$P(X = 2) = \binom{8}{2}(0.1)^2(0.9)^6$$
$$= 0.1488$$

   Poisson Approximation:

$$P(X = 2) = \frac{e^{-0.8} \cdot (0.8)^2}{2!}$$
$$= 0.1438$$

   (b) $P(X = 9)$ when $n = 10, p = 0.95$:
   Binomial Probability:

$$P(X = 9) = \binom{10}{9}(0.95)^9(0.05)^1$$
$$= 0.3151$$

   Poisson Approximation:

$$P(X = 9) = \frac{e^{-9.5} \cdot (9.5)^9}{9!}$$
$$= 0.13$$

   (c) $P(X = 0)$ when $n = 10, p = 0.1$:
   Binomial Probability:

$$P(X = 0) = \binom{10}{0}(0.1)^0(0.9)^{10}$$
$$= 0.3487$$

   Poisson Approximation:

$$P(X = 0) = \frac{e^{-1} \cdot (1)^0}{0!}$$
$$= 0.3679$$

   (d) $P(X = 4)$ when $n = 9, p = 0.2$:
   Binomial Probability:

$$P(X = 4) = \binom{9}{4}(0.2)^4(0.8)^5$$
$$= 0.066$$

   Poisson Approximation:

$$P(X = 4) = \frac{e^{-1.8} \cdot (1.8)^4}{4!}$$
$$= 0.0723$$