School of Engineering and Applied Science (SEAS), Ahmedabad University

Probability and Stochastic Processes (MAT277)

Homework Assignment-4

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1. The mean function $\mu_X(t)$ of a random process is defined as:

   $\mu_X(t) = \quad E[X(t)]$

   Given the member functions, we'll calculate the mean function using linearity of expectation:

   $\mu_X(t) = E[X(t)] = E[\sum_{i=1} R_X(t) \ \ \text{Rxi}(t)]$

   $\mu_X(t) = R\sum_{i=1}^{5} E[x_i(t)]$

   We'll calculate each term individually:

   (a) $E\left[x_1(t)\right] = E\left[\int_{\frac{\pi}{2}}^{\pi} \text{Rt}\sin(t)\text{dt}\right] = [\mathbf{R}\ \ (\text{sint}\,(\mathbf{t})\ \mathbf{-tcos}\,(\mathbf{t})]_{\frac{\pi}{2}}^{\pi} = \mathbf{E}[\,(\pi - \mathbf{1})\,\mathbf{R}]$

   (b) $E\left[x_2(t)\right] = E\left[\int_0^{\pi} -Rt\cos(t)\text{dt}\right] = \mathbf{e}[\mathbf{2R}]$

   (c) $E[x_3(t)] = E[\pi\left(1 + R\int_0^{\pi}(cos(t) - sin(t))\text{dt}\right)] = \mathbf{E}[\pi\,(\mathbf{1} - \mathbf{2R})\,]$

   (d) $E\left[x_4(t)\right] = E\left[1 - R\int_0^{\pi}(\cos(t) + \sin(t))\,\text{dt}\right] = \mathbf{E}[\mathbf{1} - \mathbf{2R}]$

   (e) $E[x_5(t)] = E[1 + R\int_{\frac{\pi}{2}}^{\pi}(sin(t) + cos(t))\text{dt}] = \ \ E[1]$

   (a) $R_{X,X}(t_1, t_2) = E[X(t_1)X(t_2)]$

      i. $E[x_1\,(t_1)\,x_1(t_2)] = E[\,\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{2}}^{\pi} R^2 t_1 t_2 \sin(t_1)\sin(t_2)\,dt_1 dt_2]$

      ii. $E[x_2\,(t_1)\,x_2(t_2)] = E[\,\int_0^{\pi}\int_0^{\pi} -R^2 t_1 t_2 \cos(t_1)\cos(t_2)\,dt_1 dt_2]$

      iii. $E[x_3\,(t_1)\,x_3(t_2)] = E[\pi^2(1 + R^2\int_0^{\pi}\int_0^{\pi}(\cos(t_1) - \sin(t_1))\,(\cos(t_2) - \sin(t_2))\,dt_1 dt_2)$

      iv. $E[x_4\,(t_1)\,x_4(t_2)] = E[\,(1 - Rt_1)\,(1 - Rt_2)\int_0^{\pi}\int_0^{\pi}(\cos(t_1) + \sin(t_1))\,(\cos(t_2) + \sin(t_2))\,dt_1 dt_2$

      v. $E[x_5\,(t_1)\,x_5(t_2)] = E[\,(1 + Rt_1)\,(1 + Rt_2)\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{2}}^{\pi}(\sin(t_1) + \cos(t_1))\,(\sin(t_2) + \cos(t_2))\,dt_1 dt_2]$

   WSS or SSS?

   To determine if the process is Wide Sense Stationary (WSS) or Strict Sense Stationary (SSS), we need to check if its mean function is constant and if its autocorrelation function depends only on the time difference. If both conditions are met, the process is WSS. If the autocorrelation function is independent of both time variables, it is SSS.

   Here, it is SSS.

2. For t = 0, Z (0) is equally likely to be 0 or 1, so:

   $\mu Z(0) = \frac{1}{2}{\cdot}0{+}\frac{1}{2}{\cdot}1 = \frac{1}{2}$

   For any other time, t > 0, given that a transition from 0 to 1 or from 1 to 0 occurs randomly, the process doesn't have a drift term, and thus the mean function remains the same:

   $\mu Z(t) = \frac{1}{2}$

   (a) The autocorrelation function $R_{Z,Z}\,(t_1,t_2)\,R_{Z,Z}(t_1,t_2)$ represents the correlation between the values of the process Z at times t1t1 and t2t2. In this case, we need to find $R_{Z,Z}(t{+}\tau,t)R_{Z,Z}(t{+}\tau,t)$.

   Since Z(t) only takes values 0 and 1, its autocorrelation function is:

   $R_{Z,Z}(t_1,t_2) = E[Z(t_1)Z(t_2)]{-}E[Z(t_1)]E[Z(t_2)]$

   Given the transition probabilities, we need to find the joint probability distribution of Z at times t and $t+\tau$.

   The joint probability distribution $P(Z(t{+}\tau) = 1, Z(t) = 0)P(Z(t{+}\tau) = 1, Z(t) = 0)$ is the probability that A occurs, and B doesn't occur, given by:

   $P(A \cap B') = pA(1 - pB)P(A \cap B') = pA(1 - pB)$

   Similarly, $P(Z(t + \tau) = 0, Z(t) = 1)P(Z(t + \tau) = 0, Z(t) = 1)$ is the probability that B occurs, and A doesn't occur:

   $P(A' \cap B) = (1 - pA)pBP(A' \cap B) = (1 - pA)pB$

   And $P(Z(t + \tau) = Z(t))P(Z(t + \tau) = Z(t))$ is the probability that neither A nor B occurs:

   $P(A' \cap B') = (1 - pA)(1 - pB)P(A' \cap B') = (1 - pA)(1 - pB)$

   Given the transition probability formula, $pA = \alpha\tau 1 + \alpha\tau pA = 1 + \alpha\tau\alpha\tau$ and $pB = \alpha\tau 1 + \alpha\tau pB = 1 + \alpha\tau\alpha\tau$, the autocorrelation function becomes:

   $RZ, Z(t + \tau, t) = \frac{\alpha\tau}{1+\alpha\tau} - \left(\frac{1}{2}\right)^2$

   Stationarity:

   To determine whether Z(t) is stationary, we need to check if its mean function and autocorrelation function are time-invariant.

   Since the mean function $\mu Z(t)$ is constant and the autocorrelation function $RZ, Z(t{+}\tau, t)$depends only on the time difference τ, Z(t) is stationary.

3. • To determine which of the given functions can be the autocorrelation function of a random process, we need to check if they satisfy the properties of a valid autocorrelation function:

   (a) Non-negativity: The autocorrelation function must be non-negative for all values of τ .

   (b) Even Function: The autocorrelation function must be an even function, meaning it is symmetric about $\tau = 0$.

   (c) Bounded: The autocorrelation function must be bounded.

      i. $f(\tau) = sin(2\pi f_0\tau) f(\tau) = \sin(2\pi f 0\tau)$: This function is not necessarily non-negative for all values of τ. Also, it is not an even function. Therefore, it does not satisfy the properties of a valid autocorrelation function.

      ii. $f(\tau) = \tau^2$: This function is non-negative for all real values of τ, but it is not necessarily an even function. Hence, it doesn't satisfy the second property of a valid autocorrelation function.

      iii. $f(\tau) = 1-\tau$ for $\mid \tau \mid \leq 1$

$1 + \tau$ for $\mid \tau \mid > 1$

This function satisfies the non-negativity property and is symmetric about $\tau = 0$, making it an even function. It is also bounded. Therefore, this function can be the autocorrelation function of a random process.

So, the correct answer is c.

4. Given that $X(t) = A+Bt^3X(t) = A+Bt^3$, where A and B are independent random variables uniformly distributed on [0, 2].

   (a) $\mu_X(t) = E[X(t)]$

      Since A and B are uniformly distributed on [0, 2], the mean of A is $E[A] = \frac{1}{2}(2-0) = 1$ and the mean of B is $E[B] = \frac{1}{2}\ (2-0) = 1$

   $\mu X(t) = E[A + Bt^3] = E[A] + t^3E[B] = 1 + t^3$

      So, the mean function $\mu_X(t) = 1 + t^3\mu X(t) = 1 + t^3$.

   (a) The autocorrelation function $R_{X,X}(t_1, t_2)$ represents the correlation between the values of the process X at times $t1$ and $t2$.

   $R_{\text{XX}}(t_1, t_2) = E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)]$

   $R_{\text{XX}}(t_1, t_2) = E[(A + Bt_1^3)(A + Bt_2^3)] - \left(1 + t_1^3\right)\left(1 + t_2^3\right) R_{\text{XX}}(t_1, t_2) = E[(A + Bt_1^3)(A + Bt_2^3)] - (1 + t_1^3)(1 + t_2^3)$

      Since A and B are independent, we have:

   $E[A2] = Var[A] + (E[A])^2 = \frac{1}{3}(2-0)^2 + 1 = \frac{4}{3}$

   $E[B2] = Var[B] + (E[B])2 = \frac{1}{3}(2-0)^2 + 1 = \frac{4}{3}$

   $E[AB] = E[A]E[B] = 1$

   $R_{\text{XX}}(t_1, t_2) = E[A_2] + t_1^3E[B_2] + t_2^3E[B_2] + t_1^3t_2^3E[AB] - (1 + t_1^3)(1 + t_2^3)$

   $R_{\text{XX}}\,(t_1, t_2) = \frac{1}{3} + \frac{1}{3}\left(t_1^3 + t_2^3\right)$

2. (a) $X_t = W_3$

Stationarity:

This process is stationary because it's a function of a single random variable W_3, which is independent and identically distributed with mean 0 and variance 1. Since W_3 has a constant mean and variance, X_t does not depend on time.

Mean Function:

The mean function μX_t is simply the mean of W_3, which is 0.

Autocovariance Function:

Since Xt is a constant, its autocovariance function $Cov(Xt, Xt + \tau)$ is also constant.

$Cov(Xt, Xt + \tau) = Cov(W3, W3 + \tau) = Var(W3) = 1$

So, the mean function is $\mu X_t = 0$and the autocovariance function is $Cov(Xt, Xt + \tau) = 1$.

(a) $X_t = t + W^3$

Stationarity:

This process is not stationary because it has a linear trend t, which introduces a dependency on time. The mean of X_t changes with time due to the linear term.

Mean Function:

The mean function μX_t changes with time due to the linear term tt.

Autocovariance Function:

The autocovariance function also changes with time due to the linear term tt, making it non-stationary.

(a) $Xt = W_2$

Stationarity:

This process is stationary because it's a function of a single random variable $W2$ which is i.i.d with mean 0 and variance 1. Since W_2 has a constant mean and variance, Xt does not depend on time.

Mean Function:

The mean function μX_t is simply the mean of W_2, which is 0.

Autocovariance Function:

Since Xt is a constant, its autocovariance function $Cov(Xt, Xt + \tau)$ is also constant.

$Cov(Xt, Xt + \tau) = Cov(W2, W2 + \tau) = Var(W2) = 1$

So, the mean function is $\mu Xt = 0$ and the autocovariance function is $Cov(Xt, Xt + \tau) = 1$.

3. $X_t = W_t\,(1 - W_{t\ -1})\,Z_t$

Solution 7:

Given the autocorrelation function definition:

$R_{f,f}(\tau) = \int_{-\infty}^{\infty} f(t)f(t+\tau)\,dt$

We can rewrite $R_{f,f}(0)$ as:

$R_{f,f}(0) = \int_{-\infty}^{\infty} f(t)f(t)\,dt = \int -\infty\infty[f(t)]2\,dt$

Let $g(\tau) = R_{f,f}(0){-}R_{f,f}(\tau)$

We want to show that g(τ) $\geq$0 for all τ. Let's calculate the derivative of g(τ) with respect to τ:

$\frac{\mathrm{dg}}{\mathrm{d}\tau} = \frac{d}{\mathrm{d}\tau}(R_{f,f}(0) - R_{f,f}(\tau))$

$= \ -\frac{d}{d\tau}(\int_{-\infty}^{\infty} f(t)f(t+\tau)\mathrm{dt})$

Now, let's evaluate this expression at τ=0:

$\frac{\mathrm{Dg}}{\mathrm{d}\tau}\quad |\,\tau = 0\ \ = -\int -\infty^{\infty} f(t)f'(t)\,\mathrm{dt}$

$\frac{\mathrm{dg}}{\mathrm{d}\tau}\ \ |\tau = 0 = -\int -\infty^{\infty} f(t)f'(t)dt$

$= -\frac{1}{2}\,[f(t)]^2\ |_{-\infty}^{\infty}= 0$

This result implies that g(τ) is maximized at τ=0, as its derivative with respect to τ is zero at τ=0. Therefore, $Rf,f(0) \geq Rf,f(\tau)$ for all τ, which proves that the autocorrelation function of any real function f(t)f(t) has a maximum value at τ=0.