School of Engineering and Applied Science (SEAS), Ahmedabad University

Probability and Stochastic Processes (MAT277)

Homework Assignment-4

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1. To find the probability density function (PDF) of the random variable $Z = aX^2$, where X is a normal random variable with mean 0 and variance σ^2, and $a > 0$, we'll use the method of transformation of variables.
   Let's denote the PDF of X as $f_X(x)$, and the PDF of Z as $f_Z(z)$. We'll first find the cumulative distribution function (CDF) of Z and then differentiate it to obtain the PDF.

   (a) The CDF of Z,

   $$F_Z(z) = P(Z \leq z) = P(aX^2 \leq z)$$

   Since $a > 0$, we can rewrite this as:

   $$F_Z(z) = P(X^2 \leq \frac{z}{a})$$

   CDF of X^2:

   $$F_{X^2}(t) = P(X^2 \leq t)$$

   Since X is a standard normal random variable, X^2 follows a chi-square distribution with one degree of freedom ($\chi^2(1)$).

   $$F_{X^2}(t) = P(X^2 \leq t) = P(|X| \leq \sqrt{t})$$

   CDF of the standard normal distribution, is given by:

   $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

   So,

   $$F_{X^2}(t) = 2 \cdot \Phi(\sqrt{t}) - 1$$

   (a) differentiating to find the PDF:

   $$f_Z(z) = \frac{d}{dz} F_Z(z)$$

   $$= \frac{d}{dz} \left(2 \cdot \Phi \left(\sqrt{\frac{z}{a}} \right) - 1 \right)$$

   $$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{a}} \cdot e^{-\frac{z}{2a}} \cdot \frac{1}{\sqrt{z}}$$

   $$= \frac{1}{2\sqrt{\pi a}} \cdot \frac{1}{\sqrt{z}} \cdot e^{-\frac{z}{2a}}$$

So, the probability density function (PDF) of the random variable $Z = aX^2$ is:

$$f_Z(z) = \frac{1}{2\sqrt{\pi a}} \cdot \frac{1}{\sqrt{z}} \cdot e^{-\dfrac{z}{2a}}$$

The probability density function (PDF) of the random variable $Z = aX^2$, where X is a normal random variable with mean 0 and variance σ^2, and $a > 0$, is given by:

$$f_Z(z) = \frac{1}{2\sqrt{\pi a}} \cdot \frac{1}{\sqrt{z}} \cdot e^{-\dfrac{z}{2a}}$$

2. A random variable X is uniformly distributed over the interval (0, 1) and related to Y by,

$$\tan\left(\frac{\pi Y}{2}\right) = e^X \implies Y = \frac{2}{\pi}\arctan(e^X)$$

$$\therefore \quad \frac{dY}{dX} = \frac{2}{\pi} \cdot \frac{1}{1+e^{2X}}$$

Applying the the transformation rule, we get:

$$f_Y(y) = f_X(x)\left|\frac{dY}{dX}\right| = 1 \times \frac{2}{\pi} \cdot \frac{1}{1+e^{2X}}$$

Since X is expressed in terms of Y through the initial transformation, $e^X = \tan\left(\frac{\pi Y}{2}\right)$, the PDF can be expressed in terms of Y as follows:

$$f_Y(y) = \left(\frac{2}{\pi}\right)\left(\frac{1}{1+\tan^2\left(\frac{\pi y}{2}\right)}\right)$$

Using the identity $1 + \tan^2(z) = \sec^2(z)$, we get:

$$f_Y(y) = \left(\frac{2}{\pi}\right)\left(\frac{1}{sec^2\left(\frac{\pi y}{2}\right)}\right) = \frac{2}{\pi}\cos^2\left(\frac{\pi y}{2}\right)$$

By solving for Y, computing the derivative with respect to X, and applying the transformation rule, the resulting PDF for Y is $f_Y(y) = \frac{2}{\pi}\cos^2\left(\frac{\pi y}{2}\right)$, valid for y in the interval $(0, 1)$.

3. Any straight line passing through the point $(0, l)$ can be represented by the equation $y = mx + l$, where m is the slope of the line.

   From the line equation, we get $x = -\frac{l}{m}$.

   Since m can take any real value, the x-intercept can take any real value as well, except $x = 0$ (as the line cannot intersect the x-axis at the origin).

   As we're drawing the line randomly, we can assume that the probability of the line having any particular slope m is uniformly distributed between negative and positive infinity.

   Therefore, the Probability Density Function (PDF) $f(x)$ is:

$$f(x) = \begin{cases} k, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

   Where k is a constant representing the uniform probability density over the entire real line except $x = 0$.

   To find k, we can integrate $f(x)$ over its entire range (excluding $x = 0$) and set the result equal to 1, since the total probability density over all possible values must equal 1.

$$\int_{-\infty}^{-\epsilon} k\,dx + \int_{\epsilon}^{\infty} k\,dx = 1$$

   where ϵ is a small positive value approaching zero.

$$\begin{aligned} 2k \int_{\epsilon}^{\infty} dx &= 1 \\ 2k\,[x]_{\epsilon}^{\infty} &= 1 \\ 2k(\infty - \epsilon) &= 1 \\ 2k \cdot \infty &= 1 \\ 2k \cdot \infty &\approx 1 \\ k \cdot \infty &\approx \frac{1}{2} \\ k &\approx 0 \end{aligned}$$

   The constant k represents the uniform probability density over the real line except at $x = 0$. Integrating the probability density function $f(x)$ over its entire range (excluding $x = 0$) and setting it equal to 1 yields $k \approx 0$, indicating that $f(x)$ is effectively zero at $x = 0$, consistent with the notion that the line cannot intersect the x-axis at the origin.

4. To find the probability density function (PDF) of the random variable Y given different transformations of the random variable X, we will use the method of transformations.

   Given the probability density function (PDF) of X as:

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

   (a) $\mathbf{Y = 1 - X^3}$

   We start by finding the cumulative distribution function (CDF) of Y and then differentiate it to get the PDF of Y.

   i. Finding the CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(1 - X^3 \leq y)$$

   Solve for X:

$$X \leq (1-y)^{1/3}$$

$$F_Y(y) = P(X \leq (1-y)^{1/3})$$

$$F_Y(y) = \int_{-\infty}^{(1-y)^{1/3}} \frac{1}{\pi(1+x^2)} dx$$

   Let $u = 1 + x^2$, then $du = 2xdx$, and $dx = \frac{du}{2x}$. The integral becomes:

$$F_Y(y) = \frac{1}{2\pi} \int_2^{\frac{1}{(1-y)^{2/3}}} \frac{1}{u} du$$

$$= \frac{1}{2\pi} \ln|u| \Bigg|_2^{\frac{1}{(1-y)^{2/3}}}$$

$$= \frac{1}{2\pi} \ln\left(\frac{1}{(1-y)^{2/3}}\right) - \frac{1}{2\pi}\ln(2)$$

$$= -\frac{1}{2\pi}\ln(1-y) - \frac{1}{3\pi}\ln(2)$$

   ii. Finding the PDF of Y:

   differentiating the CDF $F_Y(y)$ with respect to y, we get the PDF $f_Y(y)$:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= -\frac{1}{2\pi}\left(-\frac{1}{1-y}\right)$$

$$= \frac{1}{2\pi(1-y)}$$

   (b) $\mathbf{Y = arctan(X)}$

   Similar to the previous transformation, we find the CDF and then differentiate to get the PDF.

i. Finding the CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(\arctan(X) \leq y)$$

$$= P(X \leq \tan(y))$$

$$F_Y(y) = \int_{-\infty}^{\tan(y)} \frac{1}{\pi(1+x^2)} dx$$

This integral can be recognized as the inverse tangent function:

$$F_Y(y) = \frac{1}{\pi}[\arctan(\tan(y)) - \arctan(-\infty)]$$

$$F_Y(y) = \frac{1}{\pi}\left[y - \left(-\frac{\pi}{2}\right)\right]$$

$$F_Y(y) = \frac{1}{\pi}\left(y + \frac{\pi}{2}\right)$$

ii. Finding the PDF of Y:

differentiating the CDF $F_Y(y)$ with respect to y to get the PDF $f_Y(y)$:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{1}{\pi}$$

Hence, for $Y = \arctan(X)$, the PDF of Y is a constant function with value $\frac{1}{\pi}$ within the interval where $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Outside of this interval, the PDF is zero.

5. Given X is a random variable on $(0, \infty)$ with pdf

$$f(x) = e^{-x}, \quad x \in (0, \infty)$$

Now given Y is a random variable on $(0, \infty)$ such that

$$Y = X^2$$

$$X = \sqrt{Y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

and

$$f(y) = e^{-\sqrt{y}}, \quad y \in (0, \infty)$$

So PDF for Y,

$$f_Y(y) = f(y)\left|\frac{dx}{dy}\right|$$

$$f_Y(y) = e^{-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{2} \cdot \frac{e^{-\sqrt{y}}}{\sqrt{y}}, \quad y \in (0, \infty)$$

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