

Probability and Stochastic Processes (MAT277)

Homework Assignment-4

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1. To find the probability density function (PDF) of the random variable  $Z = aX^2$ , where  $X$  is a normal random variable with mean 0 and variance  $\sigma^2$ , and  $a > 0$ , we'll use the method of transformation of variables.

Let's denote the PDF of  $X$  as  $f_X(x)$ , and the PDF of  $Z$  as  $f_Z(z)$ . We'll first find the cumulative distribution function (CDF) of  $Z$  and then differentiate it to obtain the PDF. The CDF of  $Z$ ,

$$F_Z(z) = P(Z \leq z) = P(aX^2 \leq z)$$

Since  $a > 0$ , we can rewrite this as:

$$F_Z(z) = P(|X| \leq \frac{z}{a})$$
$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

So, differentiating to find the PDF:

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= 2 \cdot f_X\left(\sqrt{z/a}\right) \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{z/a}{2\sigma^2}\right) \\ &= \frac{2}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{z}{2a\sigma^2}\right) \end{aligned}$$

So, the probability density function (PDF) of the random variable  $Z = aX^2$  is:

$$f_Z(z) = \frac{2}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{z}{2a\sigma^2}\right)$$

2. A random variable  $X$  is uniformly distributed over the interval  $(0, 1)$  and related to  $Y$  by,

$$\tan\left(\frac{\pi Y}{2}\right) = e^X \implies Y = \frac{2}{\pi} \arctan(e^X)$$

$$\therefore \frac{dY}{dX} = \frac{2}{\pi} \cdot \frac{1}{1 + e^{2X}}$$

Applying the transformation rule, we get:

$$f_Y(y) = f_X(x) \left| \frac{dY}{dX} \right| = 1 \times \frac{2}{\pi} \cdot \frac{1}{1 + e^{2X}}$$

Since  $X$  is expressed in terms of  $Y$  through the initial transformation,  $e^X = \tan\left(\frac{\pi Y}{2}\right)$ , the *PDF* can be expressed in terms of  $Y$  as follows:

$$f_Y(y) = \left(\frac{2}{\pi}\right) \left( \frac{1}{1 + \tan^2\left(\frac{\pi y}{2}\right)} \right)$$

Using the identity  $1 + \tan^2(z) = \sec^2(z)$ , we get:

$$f_Y(y) = \left(\frac{2}{\pi}\right) \left( \frac{1}{\sec^2\left(\frac{\pi y}{2}\right)} \right) = \frac{2}{\pi} \cos^2\left(\frac{\pi y}{2}\right)$$

By solving for  $Y$ , computing the derivative with respect to  $X$ , and applying the transformation rule, the resulting *PDF* for  $Y$  is  $f_Y(y) = \frac{2}{\pi} \cos^2\left(\frac{\pi y}{2}\right)$ , valid for  $y$  in the interval  $(0, 1)$ .

3. A line passing through  $(0, l)$  can be described by its slope  $m$ , with the x-intercept occurring when  $y = 0$ . This leads to the relation  $X = -\frac{l}{m}$ , where  $m$  is the slope of the line.

To express the PDF of  $X$ , we consider the angle  $\theta$  the line makes with the x-axis, with  $\theta$  uniformly distributed between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$ . The slope  $m$  is related to  $\theta$  by  $m = \tan(\theta)$ .

when doing the transformation from  $\theta$  to  $X$ :

$$X(\theta) = -\frac{l}{\tan(\theta)}$$

Differentiating  $X$  with respect to  $\theta$ :

$$\frac{dX}{d\theta} = \frac{l}{\sin^2(\theta)}$$

Given the uniform distribution of  $\theta$ , the PDF of  $\theta$ ,  $f_\theta(\theta)$ , is  $\frac{1}{\pi}$  for  $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . Applying the transformation rule, the PDF of  $X$  in terms of  $\theta$  is:

$$f_X(\theta) = \frac{|l|}{\pi \sin^2(\theta)}$$

4. To find the probability density function (PDF) of the random variable  $Y$  given different transformations of the random variable  $X$ , we will use the method of transformations.

Given the probability density function (PDF) of  $X$  as:

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

(a)  $Y = 1 - X^3$

We start by finding the cumulative distribution function (CDF) of  $Y$  and then differentiate it to get the PDF of  $Y$ .

i. Finding the CDF of  $Y$ :

$$F_Y(y) = P(Y \leq y) = P(1 - X^3 \leq y)$$

Solve for  $X$ :

$$\begin{aligned} X &\leq (1 - y)^{1/3} \\ F_Y(y) &= P(X \leq (1 - y)^{1/3}) \\ F_Y(y) &= \int_{-\infty}^{(1-y)^{1/3}} \frac{1}{\pi(1+x^2)} dx \end{aligned}$$

Let  $u = 1 + x^2$ , then  $du = 2x dx$ , and  $dx = \frac{du}{2x}$ . The integral becomes:

$$\begin{aligned} F_Y(y) &= \frac{1}{2\pi} \int_2^{(1-y)^{2/3}+1} \frac{1}{u} du \\ &= \frac{1}{2\pi} \ln |u| \Big|_2^{(1-y)^{2/3}+1} \\ &= \frac{1}{2\pi} \ln \left( \frac{1}{(1-y)^{2/3}} \right) - \frac{1}{2\pi} \ln(2) \\ &= -\frac{1}{2\pi} \ln(1-y) - \frac{1}{3\pi} \ln(2) \end{aligned}$$

ii. Finding the PDF of  $Y$ :

differentiating the CDF  $F_Y(y)$  with respect to  $y$ , we get the PDF  $f_Y(y)$ :

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= -\frac{1}{2\pi} \left( -\frac{1}{1-y} \right) \\ &= \frac{1}{2\pi(1-y)} \end{aligned}$$

(b)  $Y = \arctan(X)$

Similar to the previous transformation, we find the CDF and then differentiate to get the PDF.

i. Finding the CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(\arctan(X) \leq y)$$

$$= P(X \leq \tan(y))$$

$$F_Y(y) = \int_{-\infty}^{\tan(y)} \frac{1}{\pi(1+x^2)} dx$$

This integral can be recognized as the inverse tangent function:

$$F_Y(y) = \frac{1}{\pi} [\arctan(\tan(y)) - \arctan(-\infty)]$$

$$F_Y(y) = \frac{1}{\pi} \left[ y - \left( -\frac{\pi}{2} \right) \right]$$

$$F_Y(y) = \frac{1}{\pi} \left( y + \frac{\pi}{2} \right)$$

ii. Finding the PDF of Y:

differentiating the CDF  $F_Y(y)$  with respect to  $y$  to get the PDF  $f_Y(y)$ :

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{1}{\pi}$$

Hence, for  $Y = \arctan(X)$ , the PDF of Y is a constant function with value  $\frac{1}{\pi}$  within the interval

where  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ . Outside of this interval, the PDF is zero.

5. Given  $X$  is a random variable on  $(0, \infty)$  with pdf

$$f(x) = e^{-x}, \quad x \in (0, \infty)$$

Now given  $Y$  is a random variable on  $(0, \infty)$  such that

$$Y = X^2$$

$$X = \sqrt{Y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

and

$$f(y) = e^{-\sqrt{y}}, \quad y \in (0, \infty)$$

So *PDF* for  $Y$ ,

$$f_Y(y) = f(x) \left| \frac{dx}{dy} \right|$$

$$f_Y(y) = e^{-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{2} \cdot \frac{e^{-\sqrt{y}}}{\sqrt{y}}, \quad y \in (0, \infty)$$

6. To evaluate the probability density function (pdf) of the random variable X given by the expression:

$$x = \frac{1}{2} \left[ 1 + \frac{2}{\sqrt{2\pi}} \int_0^{Y-\theta} \exp\left(-\frac{t^2}{2}\right) dt \right]$$

$$g(t) = e^{-\frac{t^2}{2}}$$

To find the *PDF*  $f(x)$ , we differentiate the given expression with respect to Y (as X is dependent on Y) using the chain rule:

$$\begin{aligned} f(Y) &= \frac{dX}{dY} \\ f(Y) &= \frac{1}{2} \times \frac{1}{\sqrt{2\pi}} \times e^{-\frac{\left(\frac{Y-\theta}{\sigma_y}\right)^2}{2}} \times \frac{1}{\sigma_y} \\ f(Y) &= \frac{1}{\sigma_y \sqrt{2\pi}} \times e^{-\frac{\left(\frac{Y-\theta}{\sigma_y}\right)^2}{2}} \end{aligned}$$

This is the Probability Density Function (PDF) of the random variable Y.

7. The properties of normal distributions play a crucial role in determining the distribution of a linear combination of two independent normal random variables. Specifically, for  $X \sim N(\mu_X, \sigma_X^2)$  and  $Y \sim N(\mu_Y, \sigma_Y^2)$ , the distribution of  $Z = X - Y$  can be found as follows:

The expected value (mean) of  $Z$  is given by the difference of the means of  $X$  and  $Y$ :

$$\mu_Z = E[Z] = E[X - Y] = E[X] - E[Y] = \mu_X - \mu_Y.$$

The variance of  $Z$  is given by the sum of the variances of  $X$  and  $Y$ , since variance is additive for independent variables, and subtraction of variables is equivalent to adding a negative variable:

$$\sigma_Z^2 = \text{Var}[Z] = \text{Var}[X - Y] = \text{Var}[X] + \text{Var}[-Y] = \sigma_X^2 + \sigma_Y^2.$$

This is because  $\text{Var}[-Y] = \text{Var}[Y]$  for any random variable  $Y$ .

Combining these properties, we deduce that  $Z$  follows a normal distribution with mean  $\mu_Z$  and variance  $\sigma_Z^2$ :

$$Z \sim N(\mu_Z, \sigma_Z^2) = N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

Thus, the random variable  $Z = X - Y$  is normally distributed with parameters derived from those of  $X$  and  $Y$ , stating the fundamental property that linear combinations of independent normal random variables also follow a normal distribution.



8. Could not Solve

9. **(a): Probability density  $f(x, y)$  of the system of random variables  $(X, Y)$  is given:**

Given the joint probability density function  $f(x, y)$  for  $(X, Y)$ , we need to find the probability density function (PDF) of  $Z = \frac{X}{Y}$ . The PDF of  $Z$  is given by:

$$f_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |J| \cdot f(x, y) dx dy$$

where  $J$  is the Jacobian determinant.

Since the joint probability density function  $f(x, y)$  is given, let's denote it as  $f_{X,Y}(x, y)$ . Then, we have:

$$f_{X,Y}(x, y) = f(x, y)$$

The Jacobian determinant  $J$  is calculated as:

$$J = \left| \frac{\partial(x, y)}{\partial(z)} \right|$$

For the transformation  $Z = \frac{X}{Y}$ , we have:

$$\begin{aligned} Z &= \frac{X}{Y} \\ X &= ZY \end{aligned}$$

Taking partial derivatives with respect to  $X$  and  $Y$ , we get:

$$\begin{aligned} \frac{\partial X}{\partial Z} &= Y \\ \frac{\partial X}{\partial Y} &= Z \end{aligned}$$

Therefore, the Jacobian determinant  $J$  is:

$$J = \left| \frac{\partial(x, y)}{\partial(z)} \right| = \left| \frac{\partial(X, Y)}{\partial(Z)} \right| = |YZ| = |ZY| = |Z|$$

The PDF of  $Z$  is given by:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Z| \cdot f_{X,Y}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |z| \cdot f(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |z| \cdot f(x, y) dx dy \end{aligned}$$

**(b): X and Y are independent random variables obeying Rayleigh's distribution law:**

Given that  $X$  and  $Y$  are independent random variables obeying Rayleigh's distribution, we have the probability density functions:

$$f_X(x) = \begin{cases} \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2}\right) & \text{for } x \geq 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y}{a^2} \exp\left(-\frac{y^2}{2a^2}\right) & \text{for } y \geq 0 \\ 0 & \text{for } y \leq 0 \end{cases}$$

Since  $X$  and  $Y$  are independent, their joint PDF is the product of their individual PDFs:

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) \cdot f_Y(y) \\ &= \begin{cases} \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2}\right) \cdot \frac{y}{a^2} \exp\left(-\frac{y^2}{2a^2}\right) & \text{for } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{xy}{a^4} \exp\left(-\frac{x^2+y^2}{2a^2}\right) & \text{for } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{xy}{a^4} \exp\left(-\frac{x^2+y^2}{2a^2}\right) \\ f_Z(z) &= \int_0^\infty \int_0^\infty |z| \cdot \frac{xy}{a^4} \exp\left(-\frac{x^2+y^2}{2a^2}\right) dx dy \end{aligned}$$

First, let's integrate with respect to  $x$ :

$$\int_0^\infty \frac{xy}{a^4} \exp\left(-\frac{x^2+y^2}{2a^2}\right) dx$$

Let's substitute  $u = x^2 + y^2$ , then  $du = 2x dx$ .

$$\begin{aligned} &\frac{1}{2} \int_0^\infty \frac{1}{a^4} e^{-u/(2a^2)} du \\ &= -\frac{1}{2} \left[ e^{-u/(2a^2)} \right]_0^\infty = -\frac{1}{2} (0 - 1) = \frac{1}{2} \end{aligned}$$

Now, let's integrate with respect to  $y$  from 0 to  $\infty$ :

$$\begin{aligned} f_Z(z) &= |z| \cdot \frac{1}{2} \cdot \int_0^\infty dy \\ &= \frac{|z|}{2} \cdot [y]_0^\infty = \frac{|z|}{2} \cdot (\infty - 0) = \infty \end{aligned}$$

Therefore, we can state that the resulting PDF  $f_Z(z)$  is not properly normalized. It appears that the integral diverges, indicating that the PDF  $f_Z(z)$  does not exist.

10. **(a) Probability density  $f(x, y)$  for the system of random variables  $(X, Y)$  is given:**

If the joint probability density function  $f(x, y)$  is given, we can directly compute the PDF of  $R$  using the transformation method.

Given  $R = \sqrt{X^2 + Y^2}$ , the Jacobian determinant of the transformation is  $\frac{\partial(x, y)}{\partial(r)} = \frac{r}{\sqrt{x^2 + y^2}}$ .

So, the PDF of  $R$  is:

$$f_R(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{r} f(x, y) dx dy$$

**(b) Random variables  $X$  and  $Y$  are independent and obey the same normal distribution  $N(0, \sigma)$ :**

Given independent normal distributions for  $X$  and  $Y$ , we can exploit the fact that the sum of squares of independent standard normal variables follows a chi-squared distribution.

Since  $X$  and  $Y$  are independent,  $X^2$  and  $Y^2$  are also independent. Therefore,  $R^2 = X^2 + Y^2$  follows a chi-squared distribution with 2 degrees of freedom, which is equivalent to an exponential distribution with parameter  $\frac{1}{2\sigma^2}$ .

$$f_R(r) = \frac{r}{\sigma^2} \cdot e^{-\left(\frac{r^2}{2\sigma^2}\right)}$$

**(c) Random variables  $X$  and  $Y$  are independent normal random variables with probability density  $f(x, y)$ :**

Given that the joint probability density function  $f(x, y)$  for the system of random variables  $(X, Y)$  is:

$$f(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-h)^2 + y^2}{2\sigma^2}}$$

We want to find the probability density function (PDF) for the modulus of the radius vector  $R = \sqrt{X^2 + Y^2}$ .

We'll use the transformation method. The transformation is  $R = \sqrt{X^2 + Y^2}$ . To find the PDF of  $R$ , we need to calculate:

$$f_R(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{r} f(x, y) dx dy$$

Substituting the given expression for  $f(x, y)$ , we have:

$$\begin{aligned} f_R(r) &= \frac{1}{2\pi\sigma^2 r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(x-h)^2 + y^2}{2\sigma^2}} dx dy \\ &= \frac{1}{2\pi\sigma^2 r} \int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2\sigma^2}} \left( \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy \right) dx \end{aligned}$$

The inner integral  $\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$  is simply the integral of a standard normal distribution, and it equals  $\sqrt{2\pi\sigma^2}$ .

$$= \frac{1}{2\pi\sigma^2 r} \int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2\sigma^2}} \cdot \sqrt{2\pi\sigma^2} dx$$

$$= \frac{1}{r} \int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2\sigma^2}} dx$$

Now, we have an integral of a Gaussian function, which integrates to  $\sqrt{2\pi\sigma^2}$ .

$$f_R(r) = \frac{1}{r} \cdot \sqrt{2\pi\sigma^2}$$

$$f_R(r) = \frac{\sqrt{2\pi\sigma^2}}{r}$$

**(d) Random variables  $X$  and  $Y$  are independent normal random variables with mean  $\mu_x = \mu_y = 0$  and variances  $\sigma_x^2$  and  $\sigma_y^2$ , respectively:**

Since  $X$  and  $Y$  are independent, their squares  $X^2$  and  $Y^2$  are also independent. Therefore,  $R^2 = X^2 + Y^2$  follows a chi-squared distribution with 2 degrees of freedom.

The PDF of  $R$  will be similar to case (b):

$$f_R(r) = \frac{r}{\sigma_x \sigma_y} \cdot e^{-\left(\frac{r^2}{2(\sigma_x^2 + \sigma_y^2)}\right)}$$

11. **We have the quadratic equation:**

$$x^2 + \alpha x + \beta = 0,$$

**whose both roots take all values from -1 to +1 with equal probabilities.**

Now, let's denote the roots of the quadratic equation as  $r_1$  and  $r_2$ .

From the quadratic formula, we have:

$$r_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \quad (1)$$

$$r_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \quad (2)$$

We know that both roots can take all values from -1 to +1 with equal probabilities. Since the roots are symmetric to the coefficient  $\alpha$ , we can assume that  $r_1$  takes values from -1 to +1 with equal probabilities, and so does  $r_2$ . This means that the distribution of the sum and the product of the roots are uniform.

$$r_1 + r_2 = -\alpha \quad (3)$$

$$r_1 r_2 = \beta \quad (4)$$

Therefore, the probability density function of  $\alpha$  is given by the distribution of the sum of two independent uniform random variables between -1 and +1, which is a triangular distribution with the density.

$$f(\alpha) = \begin{cases} 1 - |\alpha|, & \text{if } -1 \leq \alpha \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the probability density function of  $\beta$  is given by the distribution of the product of two Independent uniform random variables between -1 and +1 which is a distribution that peaks at zero and has a maximum density of 3/4 at  $\beta = 0$ .

$$g(\beta) = \begin{cases} \frac{3}{4}(1 - \beta^2), & \text{if } -1 \leq \beta \leq 1 \\ 0, & \text{otherwise} \end{cases}$$