School of Engineering and Applied Science (SEAS), Ahmedabad University

Probability and Stochastic Processes (MAT277)

Homework Assignment-4

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1. To find the probability density function (PDF) of the random variable $Z = aX^2$, where X is a normal random variable with mean 0 and variance σ^2, and $a > 0$, we'll use the method of transformation of variables.
   Let's denote the PDF of X as $f_X(x)$, and the PDF of Z as $f_Z(z)$. We'll first find the cumulative distribution function (CDF) of Z and then differentiate it to obtain the PDF. The CDF of Z,

$$F_Z(z) = P(Z \leq z) = P(aX^2 \leq z)$$

   Since $a > 0$, we can rewrite this as:

$$F_Z(z) = P(|X| \leq \frac{z}{a})$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot exp^{-\dfrac{x^2}{2\sigma^2}}$$

   So, differentiating to find the PDF:

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= 2.f_X\left(\sqrt{z/a}\right) \\ &= 2.\frac{1}{\sqrt{2\pi}\sigma} \cdot exp\left(-\frac{z/a}{2\sigma^2}\right) \\ &= \frac{2}{\sqrt{2\pi}\sigma} \cdot exp\left(-\frac{z}{2a\sigma^2}\right) \end{aligned}$$

   So, the probability density function (PDF) of the random variable $Z = aX^2$ is:

$$f_Z(z) = \frac{2}{\sqrt{2\pi}\sigma} \cdot exp\left(-\frac{z}{2a\sigma^2}\right)$$

2. A random variable X is uniformly distributed over the interval (0, 1) and related to Y by,

$$\tan\left(\frac{\pi Y}{2}\right) = e^X \implies Y = \frac{2}{\pi}\arctan(e^X)$$

$$\therefore \quad \frac{dY}{dX} = \frac{2}{\pi} \cdot \frac{1}{1+e^{2X}}$$

Applying the the transformation rule, we get:

$$f_Y(y) = f_X(x)\left|\frac{dY}{dX}\right| = 1 \times \frac{2}{\pi} \cdot \frac{1}{1+e^{2X}}$$

Since X is expressed in terms of Y through the initial transformation, $e^X = \tan\left(\frac{\pi Y}{2}\right)$, the PDF can be expressed in terms of Y as follows:

$$f_Y(y) = \left(\frac{2}{\pi}\right)\left(\frac{1}{1+\tan^2\left(\frac{\pi y}{2}\right)}\right)$$

Using the identity $1 + \tan^2(z) = \sec^2(z)$, we get:

$$f_Y(y) = \left(\frac{2}{\pi}\right)\left(\frac{1}{sec^2\left(\frac{\pi y}{2}\right)}\right) = \frac{2}{\pi}\cos^2\left(\frac{\pi y}{2}\right)$$

By solving for Y, computing the derivative with respect to X, and applying the transformation rule, the resulting PDF for Y is $f_Y(y) = \frac{2}{\pi}\cos^2\left(\frac{\pi y}{2}\right)$, valid for y in the interval $(0, 1)$.

3. A line passing through $(0, l)$ can be described by its slope m, with the x-intercept occurring when $y = 0$. This leads to the relation $X = -\frac{l}{m}$, where m is the slope of the line.

   To express the PDF of X, we consider the angle θ the line makes with the x-axis, with θ uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The slope m is related to θ by $m = \tan(\theta)$.

   when doing the transformation from θ to X:

$$X(\theta) = -\frac{l}{\tan(\theta)}$$

   Diffrentiating X with respect to θ:

$$\frac{dX}{d\theta} = \frac{l}{\sin^2(\theta)}$$

   Given the uniform distribution of θ, the PDF of θ, $f_\theta(\theta)$, is $\frac{1}{\pi}$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Applying the transformation rule, the PDF of X in terms of θ is:

$$f_X(\theta) = \frac{|l|}{\pi \sin^2(\theta)}$$

4. To find the probability density function (PDF) of the random variable Y given different transformations of the random variable X, we will use the method of transformations.

Given the probability density function (PDF) of X as:

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

(a) $\mathbf{Y = 1 - X^3}$

We start by finding the cumulative distribution function (CDF) of Y and then differentiate it to get the PDF of Y.

i. Finding the CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(1 - X^3 \leq y)$$

Solve for X:

$$X \leq (1-y)^{1/3}$$

$$F_Y(y) = P(X \leq (1-y)^{1/3})$$

$$F_Y(y) = \int_{-\infty}^{(1-y)^{1/3}} \frac{1}{\pi(1+x^2)} dx$$

Let $u = 1 + x^2$, then $du = 2xdx$, and $dx = \frac{du}{2x}$. The integral becomes:

$$F_Y(y) = \frac{1}{2\pi} \int_2^{\frac{1}{(1-y)^{2/3}}} \frac{1}{u} du$$

$$= \frac{1}{2\pi} \ln|u| \Big|_2^{\frac{1}{(1-y)^{2/3}}}$$

$$= \frac{1}{2\pi} \ln\left(\frac{1}{(1-y)^{2/3}}\right) - \frac{1}{2\pi}\ln(2)$$

$$= -\frac{1}{2\pi}\ln(1-y) - \frac{1}{3\pi}\ln(2)$$

ii. Finding the PDF of Y:

differentiating the CDF $F_Y(y)$ with respect to y, we get the PDF $f_Y(y)$:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= -\frac{1}{2\pi}\left(-\frac{1}{1-y}\right)$$

$$= \frac{1}{2\pi(1-y)}$$

(b) $\mathbf{Y = arctan(X)}$

Similar to the previous transformation, we find the CDF and then differentiate to get the PDF.

i. Finding the CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(\arctan(X) \leq y)$$

$$= P(X \leq \tan(y))$$

$$F_Y(y) = \int_{-\infty}^{\tan(y)} \frac{1}{\pi(1+x^2)} dx$$

This integral can be recognized as the inverse tangent function:

$$F_Y(y) = \frac{1}{\pi}[\arctan(\tan(y)) - \arctan(-\infty)]$$

$$F_Y(y) = \frac{1}{\pi}\left[y - \left(-\frac{\pi}{2}\right)\right]$$

$$F_Y(y) = \frac{1}{\pi}\left(y + \frac{\pi}{2}\right)$$

ii. Finding the PDF of Y:

differentiating the CDF $F_Y(y)$ with respect to y to get the PDF $f_Y(y)$:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{1}{\pi}$$

Hence, for $Y = \arctan(X)$, the PDF of Y is a constant function with value $\frac{1}{\pi}$ within the interval where $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Outside of this interval, the PDF is zero.

5. Given X is a random variable on $(0, \infty)$ with pdf

$$f(x) = e^{-x}, \quad x \in (0, \infty)$$

Now given Y is a random variable on $(0, \infty)$ such that

$$Y = X^2$$

$$X = \sqrt{Y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

and

$$f(y) = e^{-\sqrt{y}}, \quad y \in (0, \infty)$$

So PDF for Y,

$$f_Y(y) = f(y) \left| \frac{dx}{dy} \right|$$

$$f_Y(y) = e^{-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{2} \cdot \frac{e^{-\sqrt{y}}}{\sqrt{y}}, \quad y \in (0, \infty)$$

6. To evaluate the probability density function (pdf) of the random variable X given by the expression:

$$x = \frac{1}{2}\left[1 + \frac{2}{\sqrt{2\pi}}\int_0^{Y-\ \theta} \exp\left(-\frac{t^2}{2}\right) dt\right]$$

$$g(t) = e^{-\frac{t^2}{2}}$$

To find the $PDF\ f(x)$, we differentiate the given expression with respect to Y (as X is dependent on Y) using the chain rule:

$$f(Y) = \frac{dX}{dY}$$

$$f(Y) = \frac{1}{2} \times \frac{1}{\sqrt{2\pi}} \times e^{-\frac{\left(\frac{Y-\theta}{\sigma_y}\right)^2}{2}} \times \frac{1}{\sigma_y}$$

$$f(Y) = \frac{1}{\sigma_y\sqrt{2\pi}} \times e^{-\frac{\left(\frac{Y-\theta}{\sigma_y}\right)^2}{2}}$$

This is the Probability Density Function (PDF) of the random variable Y.

7. The properties of normal distributions play a crucial role in determining the distribution of a linear combination of two independent normal random variables. Specifically, for $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, the distribution of $Z = X - Y$ can be found as follows:

   The expected value (mean) of Z is given by the difference of the means of X and Y:

   $$\mu_Z = E[Z] = E[X - Y] = E[X] - E[Y] = \mu_X - \mu_Y.$$

   The variance of Z is given by the sum of the variances of X and Y, since variance is additive for independent variables, and subtraction of variables is equivalent to adding a negative variable:

   $$\sigma_Z^2 = Var[Z] = Var[X - Y] = Var[X] + Var[-Y] = \sigma_X^2 + \sigma_Y^2.$$

   This is because $Var[-Y] = Var[Y]$ for any random variable Y.

   Combining these properties, we deduce that Z follows a normal distribution with mean μ_Z and variance σ_Z^2:

   $$Z \sim N(\mu_Z, \sigma_Z^2) = N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

   Thus, the random variable $Z = X - Y$ is normally distributed with parameters derived from those of X and Y, stating the fundamental property that linear combinations of independent normal random variables also follow a normal distribution.

8. **Could not Solve**

9. **(a): Probability density $f(x, y)$ of the system of random variables (X, Y) is given:**

Given the joint probability density function $f(x, y)$ for (X, Y), we need to find the probability density function (PDF) of $Z = \frac{X}{Y}$. The PDF of Z is given by:

$$f_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |J| \cdot f(x, y)\, dx\, dy$$

where J is the Jacobian determinant.

Since the joint probability density function $f(x, y)$ is given, let's denote it as $f_{X,Y}(x, y)$. Then, we have:

$$f_{X,Y}(x, y) = f(x, y)$$

The Jacobian determinant J is calculated as:

$$J = \left| \frac{\partial(x, y)}{\partial(z)} \right|$$

For the transformation $Z = \frac{X}{Y}$, we have:

$$Z = \frac{X}{Y}$$
$$X = ZY$$

Taking partial derivatives with respect to X and Y, we get:

$$\frac{\partial X}{\partial Z} = Y$$

$$\frac{\partial X}{\partial Y} = Z$$

Therefore, the Jacobian determinant J is:

$$J = \left| \frac{\partial(x, y)}{\partial(z)} \right| = \left| \frac{\partial(X, Y)}{\partial(Z)} \right| = |YZ| = |ZY| = |Z|$$

The PDF of Z is given by:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Z| \cdot f_{X,Y}(x, y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |z| \cdot f(x, y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |z| \cdot f(x, y)\, dx\, dy \end{aligned}$$

(b): X and Y are independent random variables obeying Rayleigh's distribution law:

Given that X and Y are independent random variables obeying Rayleigh's distribution, we have the probability density functions:

$$f_X(x) = \begin{cases} \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2}\right) & \text{for } x \geq 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y}{a^2} \exp\left(-\frac{y^2}{2a^2}\right) & \text{for } y \geq 0 \\ 0 & \text{for } y \leq 0 \end{cases}$$

Since X and Y are independent, their joint PDF is the product of their individual PDFs:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$$

$$= \begin{cases} \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2}\right) \cdot \frac{y}{a^2} \exp\left(-\frac{y^2}{2a^2}\right) & \text{for } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{xy}{a^4} \exp\left(-\frac{x^2 + y^2}{2a^2}\right) & \text{for } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X,Y}(x, y) = \frac{xy}{a^4} \exp\left(-\frac{x^2 + y^2}{2a^2}\right)$$

$$f_Z(z) = \int_0^\infty \int_0^\infty |z| \cdot \frac{xy}{a^4} \exp\left(-\frac{x^2 + y^2}{2a^2}\right) dx\, dy$$

First, let's integrate with respect to x:

$$\int_0^\infty \frac{xy}{a^4} \exp\left(-\frac{x^2 + y^2}{2a^2}\right) dx$$

Let's substitute $u = x^2 + y^2$, then $du = 2x\, dx$.

$$\frac{1}{2} \int_0^\infty \frac{1}{a^4} e^{-u/(2a^2)}\, du$$

$$= -\frac{1}{2} \left[e^{-u/(2a^2)}\right]_0^\infty = -\frac{1}{2}(0 - 1) = \frac{1}{2}$$

Now, let's integrate with respect to y from 0 to ∞:

$$f_Z(z) = |z| \cdot \frac{1}{2} \cdot \int_0^\infty dy$$

$$= \frac{|z|}{2} \cdot [y]_0^\infty = \frac{|z|}{2} \cdot (\infty - 0) = \infty$$

Therefore, we can state that the resulting PDF $f_Z(z)$ is not properly normalized. It appears that the integral diverges, indicating that the PDF $f_Z(z)$ does not exist.

10. **(a) Probability density $f(x,y)$ for the system of random variables (X,Y) is given:**

If the joint probability density function $f(x,y)$ is given, we can directly compute the PDF of R using the transformation method.

Given $R = \sqrt{X^2+Y^2}$, the Jacobian determinant of the transformation is $\dfrac{\partial(x,y)}{\partial(r)} = \dfrac{r}{\sqrt{x^2+y^2}}$.

So, the PDF of R is:

$$f_R(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{r} f(x,y)\,dx\,dy$$

(b) Random variables X and Y are independent and obey the same normal distribution $N(0,\sigma)$:

Given independent normal distributions for X and Y, we can exploit the fact that the sum of squares of independent standard normal variables follows a chi-squared distribution.

Since X and Y are independent, X^2 and Y^2 are also independent. Therefore, $R^2 = X^2+Y^2$ follows a chi-squared distribution with 2 degrees of freedom, which is equivalent to an exponential distribution with parameter $\dfrac{1}{2\sigma^2}$.

$$f_R(r) = \frac{r}{\sigma^2}\cdot e^{-\left(\frac{r^2}{2\sigma^2}\right)}$$

(c) Random variables X and Y are independent normal random variables with probability density $f(x,y)$:

Given that the joint probability density function $f(x,y)$ for the system of random variables (X,Y) is:

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-h)^2+y^2}{2\sigma^2}}$$

We want to find the probability density function (PDF) for the modulus of the radius vector $R = \sqrt{X^2+Y^2}$.

We'll use the transformation method. The transformation is $R = \sqrt{X^2+Y^2}$. To find the PDF of R, we need to calculate:

$$f_R(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{r} f(x,y)\,dx\,dy$$

Substituting the given expression for $f(x,y)$, we have:

$$f_R(r) = \frac{1}{2\pi\sigma^2 r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x-h)^2+y^2}{2\sigma^2}}\,dx\,dy$$

$$= \frac{1}{2\pi\sigma^2 r}\int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2\sigma^2}} \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy\right) dx$$

The inner integral $\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy$ is simply the integral of a standard normal distribution, and it equals $\sqrt{2\pi\sigma^2}$.

$$= \frac{1}{2\pi\sigma^2 r}\int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2\sigma^2}}\cdot\sqrt{2\pi\sigma^2}\,dx$$

$$= \frac{1}{r}\int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2\sigma^2}}\,dx$$

Now, we have an integral of a Gaussian function, which integrates to $\sqrt{2\pi\sigma^2}$.

$$f_R(r) = \frac{1}{r}\cdot\sqrt{2\pi\sigma^2}$$

$$f_R(r) = \frac{\sqrt{2\pi\sigma^2}}{r}$$

(d) Random variables X and Y are independent normal random variables with mean $\mu_x = \mu_y = 0$ and variances σ_x^2 and σ_y^2, respectively:

Since X and Y are independent, their squares X^2 and Y^2 are also independent. Therefore, $R^2 = X^2 + Y^2$ follows a chi-squared distribution with 2 degrees of freedom.

The PDF of R will be similar to case (b):

$$f_R(r) = \frac{r}{\sigma_x\sigma_y}\cdot e^{-\left(\frac{r^2}{2(\sigma_x^2+\sigma_y^2)}\right)}$$

11. **We have the quadratic equation:**

$$x^2 + \alpha x + \beta = 0,$$

whose both roots take all values from -1 to +1 with equal probabilities.
Now, let's denote the roots of the quadratic equation as r_1 and r_2.
From the quadratic formula, we have:

$$r_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \tag{1}$$

$$r_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \tag{2}$$

We know that both roots can take all values from -1 to +1 with equal probabilities. Since the roots are symmetric to the coefficient α, we can assume that r_1 takes values from -1 to +1 with equal probabilities, and so does r_2. This means that the distribution of the sum and the product of the roots are uniform.

$$r_1 + r_2 = -\alpha \tag{3}$$

$$r_1^{\cdot} r_2 = \beta \tag{4}$$

Therefore, the probability density function of α is given by the distribution of the sum of two independent uniform random variables between - 1 and +1, which is a triangular distribution with the density.

$$f(\alpha) = \begin{cases} 1 - |\alpha|, & \text{if } -1 \leq \alpha \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the probability density function of β is given by the distribution of the product of two Independent uniform random variables between -1 and +1 which is a distribution that peaks at zero and has a maximum density of 3/4 at $\beta = 0$.

$$g(\beta) = \begin{cases} \frac{3}{4}(1 - \beta^2), & \text{if } -1 \leq \beta \leq 1 \\ 0, & \text{otherwise} \end{cases}$$