Q.1. Consider.
(a) language $L_\epsilon = \{ \langle M \rangle \mid M \text{ accepts } \epsilon \}$

We can use the fact that L_ϵ is undecidable

* $L_\epsilon \leq_m L_1$

We will construct a computable function f that takes input as $\langle M \rangle$ and produces $\langle M_1, M_2 \rangle$ as output such ~~tat~~ that $\epsilon \in L(M) \Leftrightarrow M_1$ takes fewer steps than M_2 on ϵ.

The reduction function f.
<u>Input</u> $\langle M \rangle$

→ Construct a TM M_1 that does the following on every input.
(i) Simulate M on ϵ
(ii) If M accepts ϵ then "accept" and if M rejects ϵ then go into ∞ loop.

→ Construct a TM M_2, that goes into an ∞ loop on every input

<u>Output</u> $\langle M_1, M_2 \rangle$

Note that M_2 always takes ∞ number of steps on ϵ.

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Proof of Correctness

Now,

$\epsilon \in L(M) \Rightarrow M_1$ accepts ϵ in finite number of steps $\Rightarrow M_1$ takes fewer steps ~~tan~~ than M_2 on ϵ.

$\epsilon \notin L(M) \Rightarrow M_1$ goes into an ∞ loop on $\epsilon \Rightarrow M_1$ takes same number of steps as M_2 on ϵ

Therefore, $\epsilon \in L(M) \Leftrightarrow M_1$ takes fewer steps than M_2 on ϵ and hence $L_\epsilon \leq_m L_1$. This proves that L_1 is undecidable.

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Ans(b) Input $\langle M \rangle$

1. On every input of length atmost 2^{340}, run M for at most 2^{340} steps.
2. If M accepts any such input within this time, then "accept" else "reject".

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Note that if an input is accepted within 2^{340} steps then only the first 2^{340} bits of the input are of any relevance to our algorithm. Hence we need to consider only inputs of length at most 2^{340}.

$\Rightarrow$ Decidable.

Ans.(c) Every TM as infinitely many equivalent TMs. Hence every TM is accepted.

* We have to chick ~~weather~~ whether the input correctly encodes a TM or not.

$\Rightarrow$ Decidable

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A.1.(d) Consider.

$\overline{FIN} = \{ <M> \mid L(M) \text{ is infinite} \}$

We have : $\overline{FIN}$ is undecidable.

$\rightarrow \overline{FIN} \leq_m L_4$

We will construct a computable function f that takes as input $<M>$ and produces an output $<M_1, M_2>$ such that ~~L(M) is infinite ⟺~~

$L(M)$ is $\infty \iff L(M_1) \cap L(M_2)$ is ∞

Reduction function f.

$\boxed{\text{Input}}$ $<M>$

1. Set $M_1 := M$
2. Construct a TM M_2 that accepts all inputs ($L(M_2) = \Sigma^*$)

$\boxed{\text{Output}}$ $<M_1, M_2>$

Proof of Correctness.

Now,

$L(M)$ is infinite $\iff L(M_1) \cap \Sigma^*$ is $\infty \iff L(M_1) \cap L(M_2)$ is ∞

$\Rightarrow L(M)$ is $\infty \iff L(M_1) \cap L(M_2)$ is ∞ and hence $\overline{FIN} \leq_m L_4$.

Hence L_4 is undecidable

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$L' = \{0w \mid w \in L\} \cup \{1w \mid w \notin L\}$

Given - L is TR and $\bar{L}$ is non-TR

A.2: We prove that L' is non-TR

Claim $\bar{L} \leq_m L'$

We will construct a computable function f that takes an input w and gives ~~output~~ ~~ot~~ out an output w' such that.

$$w \in \bar{L} \Leftrightarrow w' \in L'$$

Reduction : f

Input : w, Add the character '1' at the left end of the input w and output the resultant string i.e. $w' = 1w$.

Output : w'

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Proof of Correctness

Now, $w \in \bar{L} \Rightarrow w \notin L \Rightarrow w' \in L'$ from the definition of L'.

$w \notin \bar{L} \Rightarrow w \in L \Rightarrow w' \notin L'$ since if $w' \in L'$ then $w \notin L$ (as w' starts with 1) which would be a contradiction.

Therefore, $w \in \bar{L} \Leftrightarrow w' \in L'$ & hence $\bar{L} \leq_m L'$

This proves that L' is non-TR.

What about $\bar{L'}$?

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A 3. (a) $f : \langle M, w \rangle \rightarrow \langle N \rangle$

$$\begin{array}{ccc} \langle M, w \rangle & \rightarrow & \langle N \rangle \\ \downarrow & & \downarrow \\ A_{TM} & & INFINITE_{TM} \end{array}$$

Input : $\langle M, w \rangle$
Output : $\langle N \rangle$

Construct TM N that on input x does the following.

→ Simulate M on W.
→ If M rejects W then "reject"
→ If M accepts W then check if x is of the form $0^n 1^n$.
→ If it is of the form $0^n 1^n$ then accept otherwise reject

Proof:

$$\langle M, W \rangle \in A_{TM} \Rightarrow M \text{ accepts } W.$$
$$\Rightarrow W(N) = \{0^n 1^n \mid n \geq 0\}$$
$$\Rightarrow \langle N \rangle \in INFINITE_{TM}$$

$$\langle M, W \rangle \notin A_{TM} \Rightarrow M \text{ doesn't accept } W$$
$$\Rightarrow W(N) = \phi$$
$$\Rightarrow \langle N \rangle \notin INFINITE_{TM}$$

Hence $A_{TM} \leq_M INFINITE_{TM}$

∴ It is undecidable

A.3(b) $f : \langle M, w \rangle \rightarrow \langle N \rangle$

(For $A_{TM} \leq_M ALL_{TM}$)

Input : $\langle M, W \rangle$

Output : $\langle N \rangle$

Construct TM N that on input x does the following

→ Simulate M on W

→ If M reject W then "reject"

→ If M accepts W then "accept".

<u>Proof</u>:

$\langle M, W \rangle \in A_{TM} \Rightarrow M$ accepts W

$\Rightarrow L(N) = \Sigma^*$

$\Rightarrow N \in ALL_{TM}$

$\langle M, W \rangle \notin A_{TM} \Rightarrow M$ dosen't accept W

$\Rightarrow L(N) = \phi$

$\Rightarrow \langle N \rangle \notin ALL_{TM}$

~~∴ A~~ $\therefore A_{TM} \leq_M ALL_{TM}$

$\Rightarrow ALL_{TM}$ is undecidable.

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