2.a. Let X_i follow the Exponential(2) distribution for $i = 1, 2, 3$, and they are independent. Define $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_1 + X_2$. Find the joint distribution of (Y_1, Y_2). Are Y_1 and Y_2 independent? Give clear arguments. [2+2]

2.b. A random variate from the standard Cauchy distribution is given to you. Explain how you would use this random variate to generate a random variate from the following pdf:

$$f(x) = \frac{e^x}{(1 + e^x)^2} \text{ for } -\infty < x < \infty.$$

[3]

S2

2.a.

We have $X_i \sim$ Exponential(2) ; $i = 1, 2, 3$.

The joint pdf of $\underline{X} = (X_1, X_2, X_3)$ is

$$f_{\underline{X}}(\underline{x}) = 8 e^{-2(x_1+x_2+x_3)} \quad ; \quad x_1 > 0,\ x_2 > 0,\ x_3 > 0$$

$Y_1 = X_1 + X_2 + X_3$

$Y_2 = X_1 + X_2$

define $Y_3 = X_1$

$X_1 = Y_3$

$X_2 = Y_2 - Y_3$

$X_3 = Y_1 - Y_2$

Jacobian

$$|J| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = 1$$

(1 mark)

$\therefore$ the joint pdf of $\underline{Y} = (Y_1, Y_2, Y_3)$ is

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}(y_3,\ y_2 - y_3,\ y_1 - y_2)\,|J| = 8 \cdot e^{-2y_1} \quad ; \quad 0 < y_3 < y_2 < y_1 < \infty$$

Now, $$f_{Y_1,Y_2}(y_1, y_2) = \int_0^{y_2} f_{\underline{Y}}(\underline{y})\, dy_3 \quad , \quad 0 < y_2 < y_1 < \infty$$

$$= \int_0^{y_2} 8\, e^{-2y_1}\, dy_3 \quad , \quad 0 < y_2 < y_1 < \infty$$

(1 mark)

$$= 8\, y_2\, e^{-2y_1} \quad , \quad 0 < y_2 < y_1 < \infty$$

$$f_{Y_1}(y_1) = \int_0^{y_1} f_{Y_1,Y_2}(y_1, y_2)\, dy_2 = \int_0^{y_1} 8\, y_2\, e^{-2y_1}\, dy_2$$

$$= 4\, y_1^2\, e^{-2y_1} \quad ; \quad y_1 > 0$$

$\therefore$ $Y_1 \sim$ Gamma$(3, 2)$

S2

$$f_{Y_2}(y_2) = \int_{y_2}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_1 = \int_{y_2}^{\infty} 8\,y_2\,e^{-2y_1}\,dy_1$$

$$= 4\,y_2\,e^{-2y_2}, \quad y_2 > 0$$ (1 mark)

$Y_2 \sim$ Gamma $(2, 2)$

Since, $f_{Y_1,Y_2}(y_1,y_2) \neq f_{Y_1}(y_1)\,f_{Y_2}(y_2)$, ~~for~~ $0 < y_2 < y_1 < \infty$

Hence Y_1 and Y_2 are not independent (1 mark)

$[\Rightarrow \quad Y_1 \not\perp\!\!\!\perp Y_2.]$

2.b. $f_1(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$

$$F_1(x) = \frac{1}{2} + \frac{1}{\pi} \cdot \tan^{-1}(x) \quad , \quad X_1 \sim f_1$$

$$U_1 = F_1(X_1) = \frac{1}{2} + \frac{1}{\pi} \cdot \tan^{-1}(X_1) \sim U(0,1)$$

Given $f_X(x) = \frac{e^x}{(1+e^x)^2} \quad , -\infty < x < \infty$

$$F(X) = \frac{1}{1+e^{-X}} \stackrel{D}{=} U_1$$

$$\Rightarrow \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(X_1) = \frac{1}{1+e^{-X}}$$

$$\Rightarrow 1+e^{-X} = \frac{1}{\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(X_1)}$$

$$\Rightarrow e^{-X} = \frac{1 - \frac{1}{2} - \frac{1}{\pi} \tan^{-1}(X_1)}{\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(X_1)} = \frac{\frac{1}{2} - \frac{1}{\pi} \tan^{-1}(X_1)}{\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(X_1)}$$

$$\Rightarrow X = \log_e \left[\frac{\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(X_1)}{\frac{1}{2} - \frac{1}{\pi} \tan^{-1}(X_1)} \right]$$