Independent events

- eg. 1 : What is the probability of obtaining a Head again, given that the first-toss was H ?
- eg. 2 : Student S misses a day in school with $P = 1/10$. What is the probability of S missing Tuesday given that she missed Monday?

- In eg. 1, the two tosses are "independent".
- In eg. 2, the two misses are "correlated". As, S may have illness or taking a long weekend off.

- **Defn:** Events $A, B \in \mathcal{E}$ are independent if
$$P(A|B) = P(A).$$

▷ A, B independent $\iff$ $P(B|A) = P(B) \iff$
$$P(A \cap B) = P(A) \cdot P(B).$$

▷ $A \cap B = \phi \implies A, B$ dependent

Pf: $0 = P(A \cap B) \neq P(A) \cdot P(B) \neq 0$.

$\implies A, B$ cannot be independent. $\square$

▷ Converse is false!

- Eg. A bin has 2 R & 2 B balls. You pick two balls sequentially and randomly.

R_1 : event that 1st ball picked is R.
B_2 : " " 2nd " " " B.

▷ R_1, B_2 are dependent.

▷ If you pick with replacement, then R_1, B_2 are independent.

- Historically, conditional probability is tied to a very influential theorem: —

Bayes Theorem (1700s)

- **Hypothesis Testing:** Suppose a theory predicts that it will rain today, with probability p. If it rained today (A), then what is the probability that the theory is correct (B)?

- Qn: How does $P(B|A)$ relate to $p = P(A|B)$?

Theorem (Bayes): $P(B|A) = P(B) \cdot P(A|B) \,/\, P(A)$

$$= \frac{P(B) \cdot P(A|B)}{P(B) \cdot P(A|B) + P(B^c) \cdot P(A|B^c)}.$$

Pf: Simply by $P(A \cap B) = \cdots$. $\square$

- Medical diagnosis is a fertile ground for Bayes.
- Eg.1. Suppose an RT-PCR Covid19 test has 80% correctness. Assume that only 20% of the population is covid19 infected.

  I took the test & it was positive.

Qn: What is the chance that I'm infected?

Analyze: events $\begin{cases} A: \text{I'm infected;} \\ B: \text{Test outputs positive (on me).} \end{cases}$

$$\triangleright\ P(A|B) = \frac{P(A)\cdot P(B|A)}{\text{"} \quad + \ P(A^c)\cdot P(B|A^c)} = \frac{0.2\times 0.8}{0.2\times 0.8 + 0.8\times 0.2}$$

$= 1/2 \; . = 50\%$!

$\Rightarrow$ This test is of no use ; it's as good as tossing a coin !

↳ Base Rate Fallacy

- This already gives a "feeling" of Bayes theorem.

- Eg.2. Monty Hall Fallacy : From Monty Hall's TV show called "Let's make a deal" (1960s).
  - There are 3 closed doors in the show. One hides a car & the other two have goats.

- You are asked to pick a door (without opening).
- The host Monty opens one of the <u>other</u> two doors to reveal <u>a</u> goat.

Qn: Should you <u>switch</u> your door choice (with the third one) ?

↳ Since Monty hasn't given any "new info", you shouldn't change your decision ??

<u>Analyse:</u> • Let the doors be $[3]$; with door-1 being your pick & door-2 being Monty's pick.

• Events $\begin{cases} D_i : & \text{the car is behind door-}i. \\ M_2 : & \text{Monty opens door-2.} \end{cases}$

[These are already conditioned on door-1 being picked.]

▷ $P(D_i) = 1/3$, $\forall i$.

▷ $P(M_2 | D_1) = 1/2$. But $P(M_2) = ?$

– P(door-1 is good | Monty picked door-2)

$$= P(D_1 | M_2) = P(D_1) \cdot P(M_2 | D_1) / P(M_2)$$

$$= \frac{1/3 \cdot 1/2}{\text{"} + P(D_2) \cdot P(M_2 | D_2) + P(D_3) \cdot P(M_2 | D_3)}$$

$$= \frac{1/6}{1/6 + 1/3 \cdot 0 + 1/3 \cdot 1} = \frac{1/6}{1/2} = \frac{1}{3} < 50\%.$$