

CS201

MATHEMATICS FOR COMPUTER SCIENCE I

LECTURE 3

OUTLINE

1 SETS, ORDERS, AND PROOFS

COMPARING CARDINALITIES

- In the last lecture, we showed that $|\mathbb{Q}| = |\mathbb{Z}| = |2\mathbb{Z}| = \aleph_0$.
- Is $|\mathbb{R}| = |\mathbb{Z}|$?
 - ▶ No.
 - ▶ Proof by diagonalization.
- It will be useful to first show that $|\mathbb{N}| = \aleph_0$.
 - ▶ Map

$$f(n) = \begin{cases} 2n+1 & \text{if } n \geq 0 \\ -2n & \text{if } n < 0 \end{cases}$$

is a bijection between \mathbb{Z} to \mathbb{N} .

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CARDINALITY OF \mathbb{R}

THEOREM

$$|\mathbb{R}| \neq \aleph_0.$$

PROOF.

- Consider a mapping $f : \mathbb{N} \mapsto \mathbb{R}$.
- Define a real number r as follows:
 - ▶ The number is between 0 and 1.
 - ▶ Check if the n th digit after decimal of $f(n)$ is 0. If yes, n th digit after decimal of r is set to 1.
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- Number r is well-defined and so $r \in \mathbb{R}$.
- Suppose $f(m) = r$.
- What is the value of m th digit of r ?
 - ▶ If it is 0, then by definition of r , m th digit would be 1.
 - ▶ If it is non-zero, then by definition of r , m th digit would be 0.
- In either case, we have an impossibility and so our assumption that $f(m) = r$ is incorrect.
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- Previous theorem shows that \mathbb{R} has a **higher** level of infinity than \mathbb{Z} or \mathbb{Q} .
- Let $|\mathbb{R}| = \aleph_1$.
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DEFINITION

We say that $|A| \leq |B|$ if there exists a one-to-one map from A to B .

- A one-to-one mapping from A to B embeds A into B without losing any information.
- Hence the definition is logical.
- It holds true for all sets, whether finite or infinite.
- The definition also fits with the Cantor-Bernstein-Schroeder Theorem which, in this notation, states that if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

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MORE INFINITIES

- Are there infinities beyond \aleph_1 ?
- Yes, infinitely many!

DEFINITION (POWER SET)

For any set A , its power set, denoted by $\mathcal{P}(A)$ is defined as:

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}.$$

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THEOREM

For any A , $|A| < |\mathcal{P}(A)|$.

PROOF.

- If A is finite, then $|\mathcal{P}(A)| = 2^{|A|}$, hence $|A| < |\mathcal{P}(A)|$.
- Suppose A is infinite.
- Then mapping $f(a) = \{a\}$ is a one-to-one map from A to $\mathcal{P}(A)$ showing $|A| \leq |\mathcal{P}(A)|$.

$|A| < |B|$ means $|A| \leq |B|$ and $|A| \neq |B|$.

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- Let h be a map from A to $\mathcal{P}(A)$.
- Define a subset B of A as:

$$B = \{x \mid x \in A \ \& \ x \notin h(x)\}.$$

- Suppose there exists $y \in A$ such that $h(y) = B$.
- Is $y \in B$?
 - ▶ If $y \in B = h(y)$ then by definition of B , $y \notin B$.
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MORE INFINITIES

- Therefore, we can construct infinitely many levels of infinities as:

$$\aleph_1 = |\mathbb{R}| < \mathcal{P}(\mathbb{R}) < \mathcal{P}(\mathcal{P}(\mathbb{R})) < \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R}))) < \dots$$

- Following shows that $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Z})|$:

- ▶ Define map $f : \mathbb{R} \mapsto \mathcal{P}(\mathbb{Z})$ as:

$$f(n.d_1d_2d_3\cdots) = \{n, 10^{|n|+1}d_1, 10^{|n|+2}d_2, 10^{|n|+3}d_3, \dots\}.$$

- ▶ The key property of above set is that $n < 10^{|n|+1}d_1 < 10^{|n|+2}d_2 < 10^{|n|+3}d_3 < \dots$.
- ▶ f is one-to-one since two real numbers would differ in either their integral part or a digit after decimal.
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- Following shows that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$:

► Define map $g : \mathcal{P}(\mathbb{N}) \mapsto \mathbb{R}$ as:

$$g(A) = 0.b_1b_2b_3\cdots,$$

where $b_i = 1$ if $i \in A$ else $b_i = 0$.

- g is one-to-one since for two distinct subsets of \mathbb{N} there exists an i that belongs to one but not other. And so the output of g on the two subsets will differ on i th digit after decimal.
- It is easy to give a one-to-one map from $\mathcal{P}(\mathbb{Z})$ to $\mathcal{P}(\mathbb{N})$, showing that

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where $\mathcal{P}^i(\mathbb{Z})$ stands for taking power-set i times and $\mathcal{P}^0(\mathbb{Z}) = \mathbb{Z}$.

- We get an infinite sequence of infinities of cardinalities

$$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \dots$$

- Are there infinities even higher than all of these?
- Yes! One can show that construction of infinities can be continued indefinitely.

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- Are there infinities in between \aleph_i and \aleph_{i+1} for any i ?
- **Continuum Hypothesis** states that there exist no infinities in between.
- It has been shown that Continuum Hypothesis can neither be proved nor disproved in **Zermelo-Fraenkel** set theory.
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