MSO 201A/ESO 209: Probability and Statistics

Assignment – IV

Solutions

Problem No. 1 $\lim_{x \uparrow \infty} F(x, y) = 1 = G(y)$ $(\forall y)$.

G is not a d.f. (marginal d.f.)

$\Rightarrow$ $F(x, y)$ is not a d.f.

Problem No. 2 For rectangle $(\frac{1}{4}, 1] \times (\frac{1}{4}, 1]$

$P(\frac{1}{4} < X \le 1, \frac{1}{4} < Y \le 1) = F(1, 1) - F(\frac{1}{4}, 1) - F(1, \frac{1}{4}) + F(\frac{1}{4}, \frac{1}{4})$

$= -1 < 0$

$\Rightarrow$ F is not a d.f.

Problem No. 3 (a) $\lim_{x \to -\infty} F(x, y) = \lim_{y \to -\infty} F(x, y) = 0$

$\lim_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = 1$

For each $y \in \mathbb{R}$, $F(x, y)$ is right continuous in x and for each $x \in \mathbb{R}$, $F(x, y)$ is right continuous in y.

For rectangle $(a_1, b_1] \times (a_2, b_2]$, $a_1 < b_1$, $a_2 < b_2$, consider

$\Delta = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$

Case I. $a_1 < 0$

$\Delta = F(b_1, b_2) - F(b_1, a_2) \ge 0$ (since for each fixed $b_1 \in \mathbb{R}$, $F(b_1, x) \uparrow$ in x)

Case II $a_2 < 0$

$\Delta = F(b_1, b_2) - F(a_1, b_2) \ge 0$ (since for each fixed $b_2 \in \mathbb{R}$ $F(x, b_2) \uparrow$ in x)

Case III $0 \le a_1 < 1$, $0 \le a_2 < 1$, $0 \le b_1 < 1$, $0 \le b_2 < 1$

$$\Delta = \frac{1 + b_1 b_2}{2} - \frac{1 + b_1 a_2}{2} - \frac{1 + a_1 b_2}{2} + \frac{1 + a_1 a_2}{2}$$

$$= \frac{1}{2}(b_2 - a_2)(b_1 - a_1) \ge 0$$

Case IV $\quad 0 \le a_1 < 1, \quad 0 \le a_2 < 1, \quad 0 \le b_1 < 1 \quad b_2 \geq 1$

$$\Delta = \frac{1+b_1}{2} - \frac{1+a_1}{2} - \frac{1+b_1 a_2}{2} + \frac{1+a_1 a_2}{2}$$

$$= \frac{(b_1 - a_1)(1-a_2)}{2} \geq 0$$

Case V: $\quad 0 \le a_1 < 1, \quad 0 \le a_2 < 1, \quad b_1 \geq 1, \quad 0 \le b_2 < 1$

$$\Delta = \frac{1+b_2}{2} - \frac{1+a_1 b_2}{2} - \frac{1+a_2}{2} + \frac{1+a_1 a_2}{2}$$

$$= \frac{(b_2 - a_2)(1-a_1)}{2} \geq 0$$

Case VI: $\quad 0 \le a_1 < 1 \quad 0 \le a_2 < 1 \quad b_1 \geq 1, \; b_2 \geq 1$

$$\Delta = 1 - \frac{1+a_1}{2} - \frac{1+a_2}{2} + \frac{1+a_1 a_2}{2}$$

$$= \frac{(1-a_1)(1-a_2)}{2} \geq 0$$

Case VII $\quad 0 \le a_1 < 1, \quad a_2 \geq 1, \quad 0 \le b_1 < 1, \quad b_2 \geq 1$

$$\Delta = \frac{1+b_1}{2} - \frac{1+a_1}{2} - \frac{1+b_1}{2} + \frac{1+a_1}{2} = 0$$

Case VIII. $\quad 0 \le a_1 < 1 \quad a_2 \geq 1, \quad b_1 \geq 1, \quad b_2 \geq 1$

$$\Delta = 1 - \frac{1+a_1}{2} - 1 + \frac{1+a_1}{2} = 0$$

Case IX $\quad a_1 \geq 1, \quad 0 \le a_2 < 1, \quad b_1 \geq 1, \quad 0 \le b_2 < 1$

$$\Delta = \frac{1+b_2}{2} - \frac{1+b_2}{2} - \frac{1+a_2}{2} + \frac{1+a_2}{2} = 0$$

Case X $\quad a_1 \geq 1, \quad 0 \le a_2 < 1, \quad b_1 \geq 1, \quad b_2 \geq 1$

$$\Delta = 1 - 1 - \frac{1+a_2}{2} + \frac{1+a_2}{2} = 0$$

Case XI $\quad a_1 \geq 1, \quad a_2 \geq 1, \quad b_1 \geq 1, \quad b_2 \geq 1$

$$\Delta = 1 - 1 - 1 + 1 = 0$$

Thus $\Delta \geq 0$

$\Rightarrow$ F is a d.f.

$\boxed{2/\text{IV}}$

(b) $F(0,0)=\frac{1}{2} \neq \lim_{h\uparrow 0} F(h,h)=0$

$\Rightarrow$ F is not of AC type.

F is discontinuous at all points of the type $(0,y)$, $y<0$ (uncountable # of discontinuities)

$\Rightarrow$ F is not of discrete type.

(c) $F_{X_1}(x)=\lim_{y\to\infty} F(x,y)=\begin{cases} 0, & x<0 \\ \frac{1+x}{2}, & 0\le x<1 \\ 1, & x\ge 1 \end{cases}$

$F_{X_2}(y)=\lim_{x\to\infty} F(x,y)=\begin{cases} 0, & y<0 \\ \frac{1+y}{2}, & 0\le y<1 \\ 1, & y\ge 1 \end{cases}$.

(d) $P(\frac{1}{2}\le X_1\le 1, \frac{1}{4}<X_2<\frac{1}{2})$

$=F(1,\frac{1}{2}-)-F(\frac{1}{2}-,\frac{1}{2}-)-F(1,\frac{1}{4})+F(\frac{1}{2}-,\frac{1}{4})$

$=\frac{3}{4}-\frac{5}{8}-\frac{5}{8}+\frac{9}{16}=\frac{1}{16}$

$P(X_1=1)=F(1,\infty)-F(1-,\infty)=1-1=0$

$P(X_1\ge\frac{3}{2}, X_2<\frac{1}{4})=P(X_2<\frac{1}{4})-P(X_1<\frac{3}{2}, X_2<\frac{1}{4})$

$=F(\infty,\frac{1}{4}-)-F(\frac{3}{2}-,\frac{1}{4}-)$

$=\frac{5}{8}-\frac{5}{8}=0$

(e) Clearly $F(x,y)\neq F_{X_1}(x)F_{X_2}(y)$ $\forall (x,y)\in\mathbb{R}^2$. Thus X_1 and X_2 are not independent.

$\boxed{\text{Problem No. 4}}$ Similar to Problem No. 3

$\boxed{\text{Problem No. 5}}$ (a) $\sum_{(x_1, x_2) \in S_{\underline{X}}} f_{\underline{X}}(x_1, x_2) = 1$

$\Rightarrow c[3+5+4+6] = 1 \Rightarrow c = \frac{1}{18}$

(b) $f_{X_1}(x_1) = \begin{cases} c(x_1+2)+c(x_1+4), & x_1 = 1, 2 \\ 0, & o.w. \end{cases} = \begin{cases} 2c(x_1+3), & x_1 = 1, 2 \\ 0, & o.w. \end{cases}$

$f_{X_2}(x_2) = \begin{cases} c(1+2x_2)+c(2+2x_2), & x_2 = 1, 2 \\ 0, & o.w. \end{cases} = \begin{cases} c(3+4x_2), & x_2 = 1, 2 \\ 0, & o.w. \end{cases}$

where $c = \frac{1}{18}$

(c) For $x_1 \in \{1, 2\}$

$f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1, X_2}(x_1, x_2)}{f_{X_1}(x_1)} = \begin{cases} \frac{x_1+2x_2}{2(x_1+3)}, & x_2 = 1, 2 \\ 0, & o.w. \end{cases}$

$E(X_2|X_1=x_1) = \sum_{x_2} x_2 f_{X_2|X_1}(x_2|x_1)$

$= \frac{x_1+2}{2(x_1+3)} + \frac{2(x_1+4)}{2(x_1+3)}$

$= \frac{3x_1+10}{2(x_1+3)}$

$E(X_2^2|X_1=x_1) = \sum_{x_2} x_2^2 f_{X_2|X_1}(x_2|x_1)$

$= \frac{x_1+2}{2(x_1+3)} + \frac{4(x_1+4)}{2(x_1+3)}$

$= \frac{5x_1+18}{2(x_1+3)}$

$Var(X_2|X_1=x_1) = E(X_2^2|X_1=x_1) - (E(X_2|X_1=x_1))^2$

$\boxed{4/10}$

(d) $P\left(X_1 < \frac{X_2}{3}\right) = P\left(X_1 < \frac{1}{3}, X_2 = 1\right) + P\left(X_1 < \frac{2}{3}, X_2 = 2\right) = 0$

$P(X_1 = X_2) = P(X_1 = X_2 = 1) + P(X_1 = X_2 = 2)$

$= 3c + 6c = 9c = \frac{1}{2}$

$P\left(X_1 \geq \frac{X_2}{2}\right) = P\left(X_1 \geq \frac{1}{2}, X_2 = 1\right) + P(X_1 \geq 1, X_2 = 2)$

$= P(X_2 = 1) + P(X_2 = 2) = 1.$

$P(X_1 + X_2 \leq 3) = P(X_1 = 1, X_2 = 1) + P(X_1 = 1, X_2 = 2) + P(X_2 = 2, X_1 = 1)$

$= 1 - P(X_1 = 2, X_2 = 2)$

$= 1 - 6c = \frac{2}{3}$

(e) $E(X_1 X_2) = c\,[1\times 3 + 2\times 5 + 2\times 4 + 4\times 6] = 45c$

$E(X_1) = 2c\,[1\times 4 + 2\times 5] = 28c$

$E(X_1^2) = 2c\,[1\times 4 + 4\times 5] = 48c$

$E(X_2) = c\,[1\times 7 + 2\times 11] = 29c$

$E(X_2^2) = c\,[1\times 7 + 4\times 11] = 51c$

$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = 45c - 29\times 28\,c^2$

$Var(X_1) = E(X_1^2) - (E(X_1))^2 = 48c - (28c)^2$

$Var(X_2) = E(X_2^2) - (E(X_2))^2 = 51c - (29c)^2$

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)\,Var(X_2)}}.$$

$\boxed{\text{Problem No. 6}}$ (a) $\sum_{\underline{x} \in S_{\underline{x}}} f_{\underline{X}}(\underline{x}) = 1$

$\Rightarrow c \sum_{x_2=1}^{2} \sum_{x_1=1}^{x_2} x_1 x_2 = 1$

$\Rightarrow \frac{c}{2} \sum_{x_2=1}^{2} x_2^2(x_2+1) = 1 \Rightarrow c = \frac{1}{7}$

(b) $f_{X_1}(x_1) = \sum_{x_2} f_{X_1 X_2}(x_1, x_2) = \begin{cases} 3c, & x_1 = 1 \\ 4c, & x_1 = 2 \\ 0, & \text{o.w} \end{cases} = \begin{cases} 3/7, & x_1 = 1 \\ 4/7, & x_1 = 2 \\ 0, & \text{o.w} \end{cases}$

$f_{X_2}(x_2) = \sum_{x_1} f_{X_1 X_2}(x_1, x_2) = \begin{cases} c, & x_2 = 1 \\ 6c, & x_2 = 2 \\ 0, & \text{o.w} \end{cases} = \begin{cases} 1/7, & x_2 = 1 \\ 6/7, & x_2 = 2 \\ 0, & \text{o.w.} \end{cases}$

(c) $f_{X_2|X_1}(x_2|1) = \dfrac{f_{X_1 X_2}(1, x_2)}{f_{X_1}(1)} = \begin{cases} \frac{x_2}{3}, & x_2 = 1, 2 \\ 0, & \text{o.w.} \end{cases}$

$E(X_2|X_1=1) = \sum_{x_2} x_2 f_{X_2|X_1}(x_2|1) = = 1 \times \frac{1}{3} + 2 \times \frac{2}{3} = \frac{5}{3}$

$E(X_2^2|X_1=1) = \sum_{x_2} x_2^2 f_{X_2|X_1}(x_2|1) = 1 \times \frac{1}{3} + 4 \times \frac{2}{3} = 3$

$Var(X_2|X_1=1) = E(X_2^2|X_1=1) - (E(X_2|X_1=1))^2 = 3 - \frac{25}{9} = \frac{2}{9}$

(d) $P(X_1 > X_2) = \sum\sum_{x_1 > x_2} f_{\underline{X}}(\underline{x}) = 0$

$P(X_1 = X_2) = \sum\sum_{x_1 = x_2} f_{\underline{X}}(\underline{x}) = c[1\times 1 + 2\times 2] = 5/7$

$P(X_1 < \frac{2}{3} X_2) = P(X_1 < \frac{2}{3}, X_2 = 1) + P(X_1 < \frac{4}{3}, X_2 = 2)$

$= P(X_1 = 1, X_2 = 2) = 2c = 2/7$

$P(X_1 + X_2 \geq 3) = 1 - P(X_1 + X_2 \leq 2)$

$= 1 - P(X_1 = 1, X_2 = 1) = 1 - c = \frac{6}{7}$

(e) $E(X_1 X_2) = \sum\sum_{\underline{x}} x_1 x_2 f_{\underline{X}}(x_1, x_2)$

$= c[1\times 1 + 2\times 2 \cdot 4\times 4] = 21c = 3$

$E(X_1) = 1 \times \frac{3}{7} + 2 \times \frac{4}{7} = \frac{11}{7}$

$E(X_1^2) = 1 \times \frac{3}{7} + 4 \times \frac{4}{7} = \frac{19}{7}$

$E(X_2) = 1 \times \frac{1}{7} + 2 \times \frac{6}{7} = \frac{13}{7}$

$E(X_2^2) = 1 \times \frac{1}{7} + 4 \times \frac{6}{7} = \frac{25}{7}$

$$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$$

$$= 3 - \frac{143}{49} = \frac{4}{49}$$

$$Var(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{19}{7} - \frac{121}{49} = \frac{12}{49}$$

$$Var(X_2) = E(X_2^2) - (E(X_2))^2 = \frac{25}{7} - \frac{169}{49} = \frac{6}{49}$$

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)\,Var(X_2)}} = \frac{4}{\sqrt{72}}.$$

(b) $\rho(X_1, X_2) \neq 0 \Rightarrow X_1$ and X_2 are not independent.

$\boxed{\text{Problem No. 7}}$ (a) For $x \in (0, 1)$

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)\,dy = 1 \Rightarrow c(x)\int_x^1 y\,dy = 1 \Rightarrow c(x) = \frac{2}{1-x^2}.$$

(b)
$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\,f_X(x) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{o.w} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} 4y^3, & 0<y<1 \\ 0, & \text{o.w.} \end{cases}$$

(c) For $y \in (0, 1)$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{y^2}, & 0<x<y \\ 0, & \text{o.w.} \end{cases}$$

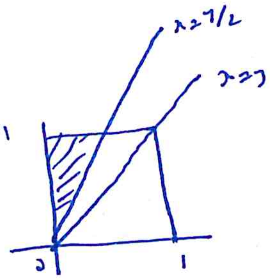
$$E(X|Y=y) = \frac{2}{y^2}\int_0^y x^2\,dx = \frac{2}{3}y$$

$$E(X^2|Y=y) = \frac{2}{y^2}\int_0^y x^3\,dx = \frac{y^2}{2}$$

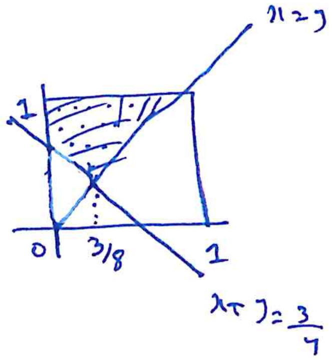
$$Var(X|Y=y) = E(X^2|Y=y) - (E(X|Y=y))^2 = \frac{y^2}{2} - \frac{4}{9}y^2 = \frac{y^2}{18}$$

(d)
$$P(X < \tfrac{Y}{2}) = \iint_{x<\frac{y}{2}} f_{X,Y}(x,y)\,dx\,dy =$$

$$= \int_0^1 \int_0^{y/2} 8xy\,dx\,dy = \frac{1}{4}$$



$$P\left(X+Y \geq \frac{3}{4}\right) = \int_0^{3/8} \int_{\frac{3}{4}-x}^{1} 8xy \, dy \, dx + \int_{3/8}^{1} \int_{x}^{1} 8xy \, dy \, dx$$



$$P(X=2Y) = \iint_{x=2y} f_{\underline{X}Y}(\underline{x}, y) \, dx \, dy = 0$$

(e) $E(XY) = \int_0^1 \int_0^y 8x^2y^2 \, dx \, dy = \frac{4}{9}$

$E(X) = \int_0^1 \int_0^y 8x^2y \, dx \, dy = \frac{8}{15}$

$E(X^2) = \int_0^1 \int_0^y 8x^3y \, dx \, dy = \frac{1}{3}$

$E(Y) = \int_0^1 \int_0^y 8xy^2 \, dx \, dy = \frac{4}{5}$

$E(Y^2) = \int_0^1 \int_0^y 8xy^3 \, dx \, dy = \frac{2}{3}$

$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{9} - \frac{32}{75} = \frac{4}{225}$

$\mathrm{Var}(X) = \frac{1}{3} - \frac{64}{225} = \frac{11}{225}$; $\mathrm{Var}(Y) = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

(f) $\rho(X,Y) \neq 0 \Rightarrow X$ and Y are not independent.

$\boxed{\text{Problem No. 8}}$ (a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\underline{X}}(\underline{x}) \, d\underline{x} = 1$

$$\Rightarrow c \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{1}{x_1 x_2} \, dx_3 \, dx_2 \, dx_1 = 1 \Rightarrow c = 1$$

(b) $f_{X_2}(x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, x_2, x_3) \, dx_1 \, dx_3$

$$= \begin{cases} \int_{x_2}^{1} \int_0^{x_2} \frac{1}{x_1 x_2} \, dx_3 \, dx_1, & 0 < x_2 < 1 \\ 0, & \text{o.w} \end{cases} = \begin{cases} -\ln x_2, & 0 < x_2 < \\ 0, & \text{o.w} \end{cases}$$

(c) For $0 < y < x < 1$

$$f_{X_2|X_1,X_3}(x_2|x,y) = \frac{f_{X_1,X_2,X_3}(x,x_2,y)}{f_{X_1,X_3}(x,y)} = \begin{cases} \frac{c(x,y)}{x_2}, & y < x_2 < x \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} \frac{1}{\ln\left(\frac{x}{y}\right) x_2}, & y < x_2 < x \\ 0, & \text{o.w.} \end{cases}$$

$$E(X_2 | (X_1,X_3) = (x,y)) = \frac{1}{\ln\left(\frac{x}{y}\right)} \int_y^x \frac{x_2}{x_2}\, dx_2 = \frac{x-y}{\ln \frac{x}{y}}$$

$$E(X_2^2 | (X_1,X_3) = (x,y)) = \frac{1}{\ln\left(\frac{x}{y}\right)} \int_y^x \frac{x_2^2}{x_2}\, dx_2 = \frac{x^2-y^2}{2 \ln \frac{x}{y}}$$

$$Var(X_2 | (X_1,X_3) = (x,y)) = E(X_2^2 | (X_1,X_3)=(x,y)) - \left(E(X_2 | (X_1,X_3)=(x,y))\right)^2$$

(d) $$P(X_2 < \tfrac{X_1}{2}) = \int_0^1 \int_0^{x_1/2} \int_0^{x_2} \frac{1}{x_1 x_2}\, dx_3\, dx_2\, dx_1 = \frac{1}{2}$$

$$P(X_3 = 2X_2 > \tfrac{X_1}{2}) = 0$$

(e) $$E(X_1 X_2) = \int_0^1 \int_0^{x_1} \int_0^{x_2} dx_3\, dx_2\, dx_1 = \frac{1}{6}$$

$$E(X_1) = \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{1}{x_2}\, dx_3\, dx_2\, dx_1 = \frac{1}{2}$$

$$E(X_1^2) = \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{x_1}{x_2}\, dx_3\, dx_2\, dx_1 = \frac{1}{3}$$

$$E(X_2) = \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{1}{x_1}\, dx_3\, dx_2\, dx_1 = \frac{1}{4}$$

$$E(X_2^2) = \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{x_2}{x_1}\, dx_3\, dx_2\, dx_1 = \frac{1}{9}$$

$$Var(X_1) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}; \quad Var(X_2) = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

$$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2) = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)\, Var(X_2)}} = \frac{1}{24} \times \frac{\sqrt{12} \times 12}{\sqrt{7}}$$

(f) $\rho(X_1, X_2) \neq 0 \Rightarrow X_1$ and X_2 are not independent
$\Rightarrow X_1, X_2, X_3$ are not independent.

$\boxed{9/10}$

Problem No. 9

(a)-(b) clearly $P((X_1, X_2) = (0,0)) = P((X_1, X_2) = (1,0)) = P((X_1, X_2) = (0,1))$

$= P((X_1, X_2) = (1,1)) = \frac{1}{4}$

Also $(X_1, X_2) \stackrel{d}{=} (X_2, X_3) \stackrel{d}{=} (X_1, X_3)$

$P(X_i = 0) = P(X_i = 1) = \frac{1}{2}, \quad i = 1, 2, 3$

Thus X_1, X_2 and X_3 are pair-wise independent.

$P(X_1 = 0, X_2 = 0, X_3 = 0) = \frac{1}{4} \neq P(X_1 = 0) P(X_2 = 0) P(X_3 = 0) = \frac{1}{8}$

$\Rightarrow X_1, X_2$ and X_3 are not independent.

(c) $P(X_1 + X_2 = 0, X_3 = 1) = P(X_1 + X_2 = 2, X_3 = 1) = \frac{1}{4}$

$P(X_1 + X_2 = 1, X_3 = 0) = \frac{1}{2} \quad P(X_1 + X_2 = 0) = \frac{1}{4}$

$P(X_1 + X_2 = 1) = \frac{1}{2}, \quad P(X_1 + X_2 = 2) = \frac{1}{4}$.

clearly

$P(X_1 + X_2 = 0, X_3 = 1) = \frac{1}{4} \neq P(X_1 + X_2 = 0) P(X_3 = 1) = \frac{1}{8}$

$\Rightarrow X_1 + X_2$ and X_3 are not independent.

Problem No. 10

(a)-(b) $f_{X_1, X_2}(x_1, x_2) = \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(x_1, x_2, x_3)\, dx_3$

$= \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}}, \quad -\infty < x_i < \infty \quad i = 1, 2$

$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1, X_2}(x_1, x_2)\, dx_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}}, \quad -\infty < x_1 < \infty$

By symmetry

$f_{X_i, X_j}(x_i, x_j) = \frac{1}{2\pi} e^{-\frac{x_i^2 + x_j^2}{2}}, \quad -\infty < x_i, x_j < \infty \quad i \neq j$

$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}, \quad -\infty < x_i < \infty \quad i = 1, 2, 3$

(since $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$ and $\int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}}\, dx = 0$).

Thus X_1, X_2, X_3 are pair-wise independent but not independent.

(c) Joint pdf of (X_i, X_j) is (see above)

$$f_{X_i, X_j}(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, \quad -\infty < x, y < \infty.$$

$\boxed{\text{Problem No. 11}}$ (a) Let $W = X+Y$. The joint p.m.f. of (W, Z) is

(w, z)	$(2, 0)$	$(3, 1)$	$(4, 0)$	$(5, 1)$
$f_{W,Z}(w, z)$	$\frac{2}{15}$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{4}{15}$

The p.m.f. of W is

$$f_W(w) = \begin{cases} \frac{2}{15}, & w = 2 \\ \frac{1}{3}, & w = 3 \\ \frac{4}{15}, & w = 4, 5 \\ 0, & \text{o.w.} \end{cases}$$

The p.m.f. of Z is

$$f_Z(z) = \begin{cases} \frac{2}{5}, & z = 0 \\ \frac{3}{5}, & z = 1 \\ 0, & \text{o.w.} \end{cases}$$

Clearly $f_{W,Z}(w, z) \neq f_W(w) f_Z(z)$, $\forall (w, z)$

$\Rightarrow W = X+Y$ and Z are not independent

(b) $E(WZ) = 0 \times \frac{2}{15} + 3 \times \frac{1}{3} + 0 \times \frac{4}{15} + 5 \times \frac{4}{15} = \frac{7}{3}$

$E(W) = 2 \times \frac{2}{15} + 3 \times \frac{1}{3} + 4 \times \frac{4}{15} + 5 \times \frac{4}{15} = \frac{11}{3}$

$E(W^2) = 4 \times \frac{2}{15} + 9 \times \frac{1}{3} + 16 \times \frac{4}{15} + 25 \times \frac{4}{15} = \frac{217}{15}$

$E(Z) = 0 \times \frac{2}{15} + 1 \times \frac{1}{3} + 0 \times \frac{4}{15} + 1 \times \frac{4}{15} = \frac{3}{5}$

$E(Z^2) = 0 \times \frac{2}{15} + 1 \times \frac{1}{3} + 0 \times \frac{4}{15} + 1 \times \frac{4}{15} = \frac{3}{5}$

$$Var(W) = E(W^2) - (E(W))^2 = \frac{217}{15} - \frac{121}{9} = \frac{46}{45}$$

$$Var(Z) = E(Z^2) - (E(Z))^2 = \frac{3}{5}\left(1-\frac{3}{5}\right) = \frac{6}{25}$$

$$Cov(W, Z) = E(WZ) - E(W)E(Z) = \frac{7}{3} - \frac{33}{15} = \frac{2}{15}$$

$$\rho(W, Z) = \frac{Cov(W, Z)}{\sqrt{Var(W)\,Var(Z)}} = \frac{2}{15}\,\frac{\sqrt{25 \times 45}}{\sqrt{6 \times 46}}.$$

$\boxed{\text{Problem No. 12}}$ (a) Through observation we have

$$f_{X_1 X_2 X_3}(x_1, x_2, x_3) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3), \quad \underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

where

$$f_{X_1}(x_1) = \begin{cases} 1, & 0 < x_1 < 1 \\ 0, & \text{o.w} \end{cases}; \quad f_{X_2}(x_2) = \begin{cases} e^{-x_2}, & x_2 > 0 \\ 0, & \text{o.w} \end{cases}; \quad f_{X_3}(x_3) = \begin{cases} 2e^{-2x_3}, & x_3 > 0 \\ 0 & \text{o.w} \end{cases}.$$

$\Rightarrow$ X_1, X_2, X_3 are independent.

(b) X_1, X_2, X_3 are independent

$\Rightarrow$ (X_1, X_2) and X_3 are independent

$\Rightarrow$ $X_1 + X_2$ and X_3 are independent

(c) See (a)

(d) Since X_1 and X_2 are independent

$$f_{X_1|X_2}(x_1|2) = f_{X_1}(x_1) = \begin{cases} 1 & 0 < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$\boxed{\text{Problem No. 13}}$ $Cov(X_i, X_j) = \sigma_i \sigma_j \rho_{ij}, \quad i \neq j$

$\Rightarrow E(X_i X_j) = \mu_i \mu_j + \rho_{ij}\sigma_i\sigma_j, \quad i \neq j$

$$Cov(Y, Z) = E((Y - E(Y))(Z - E(Z))) = E\left[\left(\sum_{i=1}^n a_i (X_i - \mu_i)\right)\left(\sum_{j=1}^n b_j (X_j - \mu_j)\right)\right]$$

$$= E\left[\sum_{i=1}^n \sum_{j=1}^n a_i b_j (X_i - \mu_i)(X_j - \mu_j)\right] = \sum_{i=1}^n \sum_{j=1}^n a_i b_j\, Cov(X_i, X_j)$$

$$= \sum_{i=1}^n a_i b_i \sigma_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n a_i b_j\, Cov(X_i, X_j).$$

Problem No. 14 $Var(X) = E(X^2) - \{E(X)\}^2 = 2 = Var(Y)$

$$Cov\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) = \frac{2}{9} Var(X) + \frac{1}{9} Cov(X, Y) + \frac{4}{9} Cov(X, Y) + \frac{2}{9} Var(Y)$$

$$= \frac{8}{9} + \frac{5}{9} Cov(X, Y) = \frac{8}{9} + \frac{5}{9} \times \frac{1}{3} \times \sqrt{2} \times \sqrt{2}$$

$$= \frac{34}{27}$$

$$Var\left(\frac{X}{3} + \frac{2Y}{3}\right) = Var\left(\frac{2X}{3} + \frac{Y}{3}\right) = \frac{Var(X)}{9} + \frac{4}{9} Var(Y) + \frac{4}{9} Cov(X, Y)$$

$$= \frac{10}{9} + \frac{8}{27} = \frac{38}{27}$$

$$\Rightarrow \rho\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) = \frac{34}{38}.$$

Problem No. 15 (a) $Var\left(\sum_{i=1}^{n} p_i X_i\right) = \sum_{i=1}^{n} p_i^2 Var(X_i) + 2 \sum\sum_{1 \le i < j \le n} p_i p_j Cov(X_i, X_j)$

$$\le \sum_{i=1}^{n} \left[p_i \sqrt{Var(X_i)}\right]^2 + 2 \sum\sum_{1 \le i < j \le n} p_i p_j \sqrt{Var(X_i) Var(X_j)}$$

$$= \left(\sum_{i=1}^{n} p_i \sqrt{Var(X_i)}\right)^2$$

$$\Rightarrow \sqrt{Var\left(\sum_{i=1}^{n} p_i X_i\right)} \le \sum_{i=1}^{n} p_i \sqrt{Var(X_i)}$$

For proving the other inequality consider a r.v. Y s.t. $P(Y = a_i) = p_i$, $i = 1, \ldots, n$, for some positive real constants $a_1, \ldots, a_n$. Then

$$E(Y) \ge \left(E(\sqrt{Y})\right)^2 \quad \text{(Jensen's inequality)}$$

$$\Rightarrow \sum_{i=1}^{n} a_i p_i \ge \left(\sum_{i=1}^{n} \sqrt{a_i}\, p_i\right)^2$$

Now taking $a_i = Var(X_i)$, $i = 1, \ldots, n$, we get the result.

(b) Take $p_i = \frac{1}{n}$, $i = 1, \ldots, n$ in (a).

Problem No. 16 Let (X, Y) be a r.v. s.t.

$$P((X,Y)=(x_i, y_i)) = \frac{1}{n}, \quad i=1,\dots,n.$$

Then

$$E(XY) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i, \quad E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i = 0 = \frac{1}{n}\sum_{i=1}^{n} y_i = E(Y)$$

$$E(X^2) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = Var(X), \quad E(Y^2) = \frac{1}{n}\sum_{i=1}^{n} y_i^2 = Var(Y)$$

$$\rho^2(X,Y) \le 1 \Rightarrow Cov^2(X,Y) \le Var(X)\,Var(Y)$$

$$\Rightarrow \left(\frac{1}{n}\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)$$

$$\Rightarrow \left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

Problem No. 17 $E(XY) = \frac{49}{15}$, $E(X) = \frac{7}{5}$, $E(X^2) = \frac{11}{5}$, $Var(X) = \frac{6}{25}$

$E(Y) = \frac{34}{15}$, $E(Y^2) = \frac{86}{15}$, $Var(Y) = \frac{134}{225}$, $Cov(X,Y) = \frac{7}{45}$

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{7}{\sqrt{804}}$$

Problem No. 18 (a) $\psi_{Y,Z}(t_1,t_2) = \ln M_{Y,Z}(t_1,t_2) = \frac{t_1^2}{1-2t_2} - \ln(1-2t_2)$, $t_1 \in \mathbb{R}$, $t_2 < \frac{1}{2}$

$$\frac{\partial}{\partial t_1}\psi_{Y,Z}(t_1,t_2) = \frac{2t_1}{1-2t_2}, \quad \frac{\partial^2}{\partial t_1^2}\psi_{Y,Z}(t_1,t_2) = \frac{2}{1-2t_2}$$

$$\frac{\partial^2}{\partial t_2 \partial t_1}\psi_{Y,Z}(t_1,t_2) = \frac{4t_1}{(1-2t_2)^2}$$

$$Cov(Y,Z) = \left[\frac{\partial^2}{\partial t_2 \partial t_1}\psi_{Y,Z}(t_1,t_2)\right]_{t=0} = 0$$

$$\Rightarrow Cov(Y,Z) = 0$$

(b) $M_Y(t_1) = M_{Y,Z}(t_1, 0) = e^{t_1^2}$, $t_1 \in \mathbb{R}$

$M_Z(t_2) = M_{Y,Z}(0, t_2) = \frac{1}{1-2t_2}$, $t_2 < \frac{1}{2}$

$M_{Y,Z}(t_1,t_2) \ne M_Y(t_1)\,M_Z(t_2)$, $\forall (t_1,t_2) \in (-a,a)\times(-a,\frac{1}{2})$

$\Rightarrow$ Y and Z are not independent (although $Cov(Y, Z) = 0$).

(c) $M_{Y+Z}(t) = E(e^{t(Y+Z)})$

$= M_{Y,Z}(t, t), \quad t < \frac{1}{2}$

$= \dfrac{e^{\frac{t^2}{1-2t}}}{1-2t}, \quad t < \frac{1}{2}.$

$\boxed{\text{Problem No. 19}}$ (a) $\psi_{Y,Z}(\underline{t}) = \ln M_{Y,Z}(\underline{t}) = \dfrac{t_1^2 + t_2^2 + t_1 t_2}{2}, \quad \underline{t} \in \mathbb{R}^2$

$\dfrac{\partial}{\partial t_1} \psi_{Y,Z}(\underline{t}) = \dfrac{2t_1 + t_2}{2}, \quad \dfrac{\partial^2}{\partial t_1^2} \psi_{Y,Z}(\underline{t}) = 1, \quad \dfrac{\partial^2}{\partial t_2 \partial t_1} \psi_{Y,Z}(\underline{t}) = \dfrac{1}{2}$

$\dfrac{\partial}{\partial t_2} \psi_{Y,Z}(\underline{t}) = \dfrac{2t_2 + t_1}{2}, \quad \dfrac{\partial^2}{\partial t_2^2} \psi_{Y,Z}(\underline{t}) = 1$

$\Rightarrow \quad Cov(Y, Z) = \left[\dfrac{\partial^2}{\partial t_2 \partial t_1} \psi_{Y,Z}(\underline{t})\right]_{\underline{t}=\underline{0}} = \dfrac{1}{2}$

$Var(Y) = \left(\dfrac{\partial^2}{\partial t_1^2} \psi_{Y,Z}(\underline{t})\right]_{\underline{t}=\underline{0}} = 1 = Var(Z)$

$\Rightarrow \quad \rho(Y, Z) = \dfrac{1}{2}$

(b) $\rho(Y, Z) \neq 0 \Rightarrow Y$ and Z are not independent.

(c) $M_{Y-Z}(t) = E(e^{t(Y-Z)}) = M_{Y,Z}(t, -t) = e^{t^2/2}, \quad t \in \mathbb{R}.$

$\boxed{\text{Problem No. 20}}$ (a) $S_{\underline{Y}} = \{(0,0), (-1,1), (1,1), (0,2)\}$

$f_{Y_1,Y_2}(y_1, y_2) = P(X_1 - X_2 = y_1, \; X_1 + X_2 = y_2)$

$= P\left(X_1 = \dfrac{y_1 + y_2}{2}, \; X_2 = \dfrac{y_2 - y_1}{2}\right)$

$$= \begin{cases} \left(\frac{2}{3}\right)^{y_2} \left(\frac{1}{3}\right)^{2-y_2}, & \underline{y} \in S_{\underline{Y}} \\ 0, & \text{o.w.} \end{cases}$$

(b) $f_{Y_1}(y_1) = \sum_{y_2} f_{Y_1,Y_2}(y_1,y_2)$

$$= \begin{cases} \frac{2}{9}, & y_1 = -1 \\ \frac{5}{9}, & y_1 = 0 \\ \frac{2}{9}, & y_1 = 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_{Y_2}(y_2) = \sum_{y_1} f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{1}{9}, & y_2 = 0 \\ \frac{4}{9}, & y_2 = 1 \\ \frac{4}{9}, & y_2 = 2 \\ 0, & \text{o.w.} \end{cases}$$

(c) $E(Y_2) = 0 \times \frac{1}{9} + 1 \times \frac{4}{9} + 2 \times \frac{4}{9} = \frac{4}{3}$

$E(Y_2^2) = 0 \times \frac{1}{9} + 1 \times \frac{4}{9} + 4 \times \frac{4}{9} = \frac{20}{9}$

$Var(Y_2) = E(Y_2^2) - (E(Y_2))^2 = \frac{4}{9}$

$$E(Y_1 Y_2) = 0 \times \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^2 + (-1)\left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} + 1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1}$$
$$+ 0 \times \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{2-2} = 0$$

$E(Y_1) = -\frac{2}{9} + \frac{2}{9} = 0$

$\Rightarrow Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 0$

(d) $P(Y_1 = 0, Y_2 = 0) = \frac{1}{9} \neq P(Y_1 = 0)\, P(Y_2 = 0) = \frac{5}{9} \times \frac{1}{9}$

$\Rightarrow$ Y_1 and Y_2 are not independent (although $Cov(Y_1, Y_2) = 0$).

$\boxed{\text{Problem No. 21}}$ (a)

$X_1, \dots, X_n$ is a random sample and $X_1 - \mu \overset{d}{=} \mu - X_1$

$\Rightarrow (X_1 - \mu, \dots, X_n - \mu) \overset{d}{=} (\mu - X_1, \dots, \mu - X_n)$

$\Rightarrow$ r-th smallest of $\{X_1 - \mu, \dots, X_n - \mu\}$ $\overset{d}{=}$ r-th smallest of $\{\mu - X_1, \dots, \mu - X_n\}$

$\Rightarrow \quad X_{r:n} - \mu \overset{d}{=} \mu - X_{n-r+1:n}, \quad r=1,\dots,n$

(b) By (a)

$$E(X_{r:n} - \mu) = E(\mu - X_{n-r+1:n})$$

$$\Rightarrow E(X_{r:n} + X_{n-r+1:n}) = 2\mu, \quad r=1,\dots,n.$$

(c) Taking $r = \frac{n+1}{2}$ in (b) we get

$$E\left(X_{\frac{n+1}{2}:n}\right) = \mu$$

(d) Using (a) for $r = \frac{n+1}{2}$ we get

$$X_{\frac{n+1}{2}:n} - \mu \overset{d}{=} \mu - X_{\frac{n+1}{2}:n}$$

$$\Rightarrow P\left(X_{\frac{n+1}{2}:n} - \mu > 0\right) = P\left(\mu - X_{\frac{n+1}{2}:n} > 0\right)$$

$$\Rightarrow P\left(X_{\frac{n+1}{2}:n} > \mu\right) = P\left(X_{\frac{n+1}{2}:n} < \mu\right)$$

$$\Rightarrow P\left(X_{\frac{n+1}{2}:n} > \mu\right) = \frac{1}{2} \quad \left(\text{as } P\left(X_{\frac{n+1}{2}:n} = \mu\right) = 0 \text{ since } X\text{'s are AC}\right)$$

$\boxed{\text{Problem 10.22}}$ (a)

$$(X_1, X_2, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n) \overset{d}{=} (X_i, X_2, \dots, X_{i-1}, X_1, X_{i+1}, \dots, X_n), \quad i=1,\dots,n$$

$$\Rightarrow E\left(\frac{X_i}{X_1 + X_2 + \dots + X_{i-1} + X_i + X_{i+1} + \dots + X_n}\right) = E\left(\frac{X_1}{X_i + X_2 + \dots + X_{i-1} + X_1 + X_{i+1} + \dots + X_n}\right), \quad i=1,\dots,n$$

$$\Rightarrow E\left(\frac{X_i}{\sum_{j=1}^n X_j}\right) = E\left(\frac{X_1}{\sum_{j=1}^n X_j}\right) = c$$

$$\Rightarrow \sum_{i=1}^n E\left(\frac{X_i}{\sum_{j=1}^n X_j}\right) = nc \Rightarrow E\Bigg(\underbrace{\sum_{i=1}^n \frac{X_i}{\sum_{j=1}^n X_j}}_{=1}\Bigg) = nc$$

$\Rightarrow \quad c = \frac{1}{n}$

$$\Rightarrow \quad E\left(\frac{X_i}{\sum_{j=1}^{n} X_j} \right) = \frac{1}{n}, \quad i=1,\dots,n$$

$$\Rightarrow \quad E\left(\frac{X_1 + \dots + X_k}{X_1 + \dots + X_n} \right) = \sum_{i=1}^{k} E\left(\frac{X_i}{\sum_{j=1}^{n} X_j} \right) = \frac{k}{n}.$$

(b) As in (a)

$(X_1, X_2, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n) \overset{d}{=} (X_i, X_2, \dots, X_{i-1}, X_1, X_{i+1}, \dots, X_n)$
$i=1,\dots,n$

$$\Rightarrow E(X_1 \mid X_1 + X_2 + \dots + X_{i-1} + X_i + X_{i+1} + \dots + X_n = t)$$

$$= E(X_i \mid X_i + X_2 + \dots + X_{i-1} + X_1 + X_{i+1} + \dots + X_n = t)$$

$$\Rightarrow E\left(X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = E\left(X_1 \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = c(t), \text{ (say)}$$

$$\Rightarrow \sum_{i=1}^{n} E\left(X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = n\,c(t)$$

$$\Rightarrow E\left(\sum_{i=1}^{n} X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = n\,c(t)$$

$$\Rightarrow \quad t = n\,c(t) \Rightarrow c(t) = \frac{t}{n}$$

$$\Rightarrow E\left(X_1 \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = \frac{t}{n}$$

(c) $X_1, X_2, \dots, X_n$ is a random sample

$\Rightarrow X_1, \dots, X_r$ is a random sample

$\Rightarrow (X_1, \dots, X_r) \overset{d}{=} (X_{\alpha_1}, \dots, X_{\alpha_r})$, $\forall$ permutation $\underset{\sim}{\alpha} = (\alpha_1, \dots, \alpha_r)$ of $(1, \dots, r)$

$\Rightarrow P(X_1 < \dots < X_r) = P(X_{\alpha_1} < \dots < X_{\alpha_r})$, $\forall$ permutation $\underset{\sim}{\alpha}$ of $(1, \dots, r)$

Since $\underset{\sim}{X}$ is of AC type

$$\sum_{\underset{\sim}{\alpha}} P(X_{\alpha_1} < \dots < X_{\alpha_r}) = 1 \Rightarrow P(X_1 < \dots < X_r) = \frac{1}{\lfloor r}.$$

$\boxed{\text{Problem No. 23}}$ The jt p.m.f. of $\underset{\sim}{X} = (X_1, X_2)$ is

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = f(x_1)f(x_2) = \begin{cases} \theta^2 (1-\theta)^{x_1+x_2-2}, & (x_1, x_2) \in \mathbb{N}\times\mathbb{N} \\ 0, & \text{o.w.} \end{cases}$$

$$S_{\underset{\sim}{X}} = \{1, 2, \dots\} \times \{1, 2, \dots\} = \mathbb{N}\times\mathbb{N}$$

(a)
$$f_{Y_1}(y) = \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ \max\{x_1, x_2\}=y}} f_{\underset{\sim}{X}}(\underset{\sim}{x})$$

$$= \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ x_1=x_2=y}} f_{\underset{\sim}{X}}(\underset{\sim}{x}) + \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ x_1<x_2,\ x_2=y}} f_{\underset{\sim}{X}}(\underset{\sim}{x}) + \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ x_1>x_2,\ x_2=y}} f_{\underset{\sim}{X}}(\underset{\sim}{x})$$

$$= \theta^2(1-\theta)^{2y-2} + \sum_{x=y+1}^{\infty} \theta^2(1-\theta)^{x+y-2} + \sum_{x=y+1}^{\infty} \theta^2(1-\theta)^{x+y-2}$$

$$= \theta(2-\theta)(1-\theta)^{2y-2}, \quad y\in\{1, 2, \dots\}$$

$$\Rightarrow f_{Y_1}(y) = \begin{cases} \theta(2-\theta)(1-\theta)^{2y-2}, & y\in\{1, 2, \dots\} \\ 0, & \text{o.w.} \end{cases}$$

(b)
$$f_{Y_2}(y) = \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ \max\{x_1, x_2\}-\min\{x_1, x_2\}=y}} f_{\underset{\sim}{X}}(\underset{\sim}{x})$$

Case I $\quad y=0$

$$f_{Y_2}(y) = \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ x_1=x_2}} f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \sum_{x=1}^{\infty} \theta^2(1-\theta)^{2x-2} = \frac{\theta}{2-\theta}$$

Case II $\quad y\in\{1, 2, \dots\}$

$$f_{Y_2}(y) = \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ x_2-x_1=y \\ x_1<x_2}} f_{\underset{\sim}{X}}(\underset{\sim}{x}) + \sum_{\substack{\underset{\sim}{x}\in S_{\underset{\sim}{X}} \\ x_1-x_2=y \\ x_1>x_2}} f_{\underset{\sim}{X}}(\underset{\sim}{x})$$

$$= \sum_{x=1}^{\infty} \theta^2 (1-\theta)^{2x+y-2} + \sum_{x=1}^{\infty} \theta^2 (1-\theta)^{2x+y-2}$$

$$= \frac{2\theta(1-\theta)^y}{2-\theta}.$$

Thus

$$f_{Y_2}(y) = \begin{cases} \frac{\theta}{2-\theta}, & y=0 \\ \frac{2\theta(1-\theta)^y}{2-\theta}, & y \in \{1, 2, \dots\} \\ 0, & \text{o.w.} \end{cases}$$

(c) $f_{\underline{Y}}(y_1, y_2) = P(\min\{X_1, X_2\} = y_1, \max\{X_1, X_2\} - \min\{X_1, X_2\} = y_2)$

$$= P(\min\{X_1, X_2\} = y_1, \max\{X_1, X_2\} = y_1 + y_2)$$

<u>Case I</u>: $y_1 \in \{1, 2, \dots\}, \quad y_2 = 0$

$$f_{\underline{Y}}(y_1, y_2) = P(\max\{X_1, X_2\} = \min\{X_1, X_2\} = y_1)$$

$$= P(X_1 = X_2 = y_1) = \theta^2 (1-\theta)^{2y_1 - 2}$$

<u>Case II</u> $y_1 \in \{1, 2, \dots\}, \quad y_2 \in \{1, 2, \dots\}$

$$f_{\underline{Y}}(y_1, y_2) = P(\min\{X_1, X_2\} = y_1, \max\{X_1, X_2\} = y_1 + y_2)$$

$$= P(X_1 = y_1, X_2 = y_1 + y_2) + P(X_1 = y_1 + y_2, X_2 = y_1)$$

$$= 2\theta^2 (1-\theta)^{2y_1 + y_2 - 2}$$

Thus

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} \theta^2 (1-\theta)^{2y_1 - 2} & \underline{y} \in \mathbb{N} \times \{0\} \\ 2\theta^2 (1-\theta)^{2y_1 + y_2 - 2}, & \underline{y} \in \mathbb{N} \times \mathbb{N} \\ 0, & \text{o.w} \end{cases}$$

(d) clearly $f_{\underline{Y}}(\underline{y}) = f_{Y_1}(y_1) f_{Y_2}(y_2), \quad \forall \underline{y} \in \mathbb{R}^2$

$\Rightarrow Y_1$ and Y_2 are independent.

$\boxed{20/ \text{N}}$

(e) $f_{Y_1}(y_1) = \sum_{y_2=0}^{\infty} f_Y(y_1, y_2)$

$$= \theta^2(1-\theta)^{2y_1-2} + \sum_{y_2=1}^{\infty} 2\theta^2(1-\theta)^{2y_1+y_2-2}$$

$$= \theta(2-\theta)(1-\theta)^{2y_1-2}, \quad y_1 \in \{1, 2, \dots\}$$

$$\Rightarrow f_{Y_1}(y_1) = \begin{cases} \theta(2-\theta)(1-\theta)^{2y_1-2}, & y_1 \in \{1, 2, \dots\} \\ 0, & \text{o.w.} \end{cases}$$

Similarly

$$f_{Y_2}(y_2) = \sum_{y_1=1}^{\infty} f_{Y_1,Y_2}(y_1, y_2)$$

$$= \sum_{y_1=1}^{\infty} \theta^2(1-\theta)^{2y_1-2}, \quad \text{if } y_2 = 0$$

$$= \frac{\theta}{2-\theta}, \quad \text{if } y_2 = 0$$

For $y_2 \in \{1, 2, \dots\}$

$$f_{Y_2}(y_2) = \sum_{y_1=1}^{\infty} 2\theta^2(1-\theta)^{2y_1+y_2-2} = \frac{2\theta(1-\theta)^{y_2}}{2-\theta}$$

$$\Rightarrow f_{Y_2}(y_2) = \begin{cases} \frac{\theta}{2-\theta}, & y_2 = 0 \\ \frac{2\theta(1-\theta)^{y_2}}{2-\theta}, & y_2 \in \{1, 2, \dots\} \\ 0, & \text{o.w.} \end{cases}$$

$\boxed{\text{Problem No. 24}}$ (a)

$$P(Y_1 = y) = P(X_1 + X_2 = y) = \begin{cases} \frac{4}{9}, & y = 1 \\ \frac{5}{9}, & y = 2 \\ 0, & \text{o.w.} \end{cases}$$

(b) $$P(Y_2 = y) = P(X_2 + X_3 = y) = \begin{cases} \frac{4}{9}, & y = 1 \\ \frac{5}{9}, & y = 2 \\ 0, & \text{o.w.} \end{cases}$$

(c) $f_{\underline{Y}}(y_1, y_2) = P(X_1+X_2=y_1,\ X_2+X_3=y_2)$

$$= \begin{cases} \frac{2}{9}, & \text{if } (y_1,y_2) = (1,1),\ (1,2),\ (2,1) \\ \frac{1}{3}, & \text{if } (y_1,y_2) = (2,2) \\ 0, & \text{o.w.} \end{cases}$$

(d) $P(Y_1=1, Y_2=1) = \frac{2}{9} \neq P(Y_1=1)\,P(Y_2=1)$

$\Rightarrow$ Y_1 and Y_2 are not independent

(e) $$f_{Y_1}(y_1) = \sum_{y_2} f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{4}{9}, & y_1=1 \\ \frac{5}{9}, & y_1=2 \\ 0, & \text{o.w.} \end{cases}$$

By symmetry

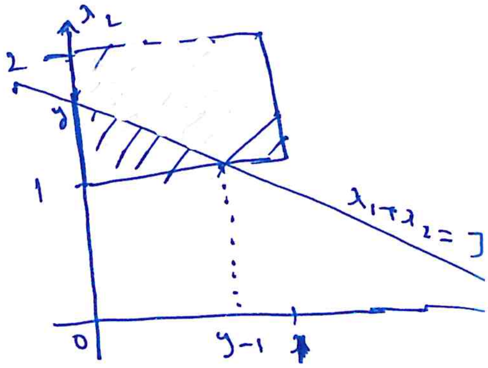
$$f_{Y_2}(y_2) = \begin{cases} \frac{4}{9}, & y_2=1 \\ \frac{5}{9}, & y_2=2 \\ 0, & \text{o.w.} \end{cases}$$

$\boxed{\text{Problem No. 25}}$ The joint p.d.f. of $\underline{X}=(X_1, X_2)$ is

$$f_{\underline{X}}(x_1,x_2) = f_1(x_1)\,f_2(x_2) = \begin{cases} 1, & 0<x_1<1,\ 1<x_2<2 \\ 0, & \text{o.w.} \end{cases}$$

(a) $S_Y^0 = (1,3)$. Clearly for $y<1$ $F_Y(y)=0$ and for $y\geq 3$ $F_Y(y)=1$. For $1\leq y<3$

$F_Y(y) = P(X_1+X_2 \leq y)$

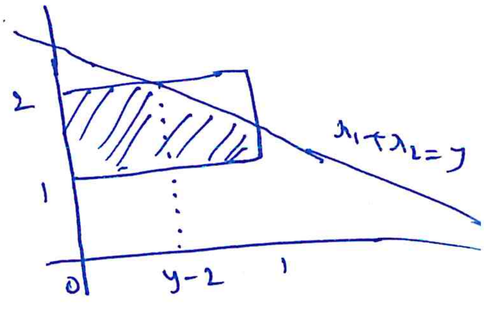


Case I $1\leq y<2$

$$F_Y(y) = \int_1^{y}\int_0^{y-x_2} dx_2\,dx_1 = \frac{(y-1)^2}{2}$$

Case II $2\leq y<3$

$$F_Y(y) = 1 - \int_{y-2}^{1}\int_{y-x_1}^{1} dx_2\,dx_1 = 1-\frac{(3-y)^2}{2}$$



$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 1 \\ \frac{(y-1)^2}{2}, & 1 \le y < 2 \\ 1 - \frac{(3-y)^2}{2}, & 2 \le y < 3 \\ 1, & y \ge 3 \end{cases}$$

Thus the p.d.f. of Y is

$$f_Y(y) = \begin{cases} y-1, & 1 < y < 2 \\ 3-y, & 2 < y < 3 \\ 0, & \text{o.w.} \end{cases}$$

(b) $S^o_{\underset{\sim}{X}} = (0,1) \times (1,2)$, $Y = h_1(x_1, x_2) = x_1 + x_2$, $Z = h_2(x_1, x_2) = x_1 - x_2$. The transformation $\underset{\sim}{h} = (h_1, h_2) : S_{\underset{\sim}{X}} \to \mathbb{R}^2$ is 1-1 with inverse transformation

$$x_1 = h_1^{-1}(y, z) = \frac{y+z}{2}, \quad x_2 = h_2^{-1}(y, z) = \frac{y-z}{2}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$0 < x_1 < 1$, $1 < x_2 < 2 \Leftrightarrow 0 < y+z < 2$, $2 < y-z < 4$

$\Rightarrow S^o_{Y,Z} = \{(y, z) \in \mathbb{R}^2 : 0 < y+z < 2,\ 2 < y-z < 4\}$

The joint p.d.f. of (Y, Z) is

$$f_{Y,Z}(y, z) = f_{\underset{\sim}{X}}\left(\frac{y+z}{2}, \frac{y-z}{2}\right) \left|-\frac{1}{2}\right| I_{S_{Y,Z}}(y, z)$$

$$= \begin{cases} \frac{1}{2}, & 0 < y+z < 2 < y-z < 4 \\ 0, & \text{o.w.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y, z)\, dz = \begin{cases} \int_{\max\{-y,\, y-4\}}^{\min\{2-y,\, y-2\}} \frac{1}{2}\, dz, & \text{if } 1 < y < 3 \\ 0, & \text{o.w.} \end{cases}$$

$=$

$$= \begin{cases} \int_{-y}^{y-2} \frac{1}{2}\, dz & 1 < y < 2 \\ \int_{y-4}^{2-y} \frac{dz}{2}, & 2 < y < 3 \\ 0, & \text{o.w} \end{cases}$$

$$= \begin{cases} y-1, & 1 < y < 2 \\ 3-y, & 2 < y < 3 \\ 0, & \text{o.w} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\, dy = \begin{cases} \int_{\max\{-z,\, z+2\}}^{\min\{2-z,\, z+4\}} \frac{dy}{2}, & -2 < z < 0 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \int_{-z}^{z+4} \frac{dy}{2} & -2 < z < -1 \\ \int_{z+2}^{2-z} \frac{dy}{2} & -1 < z < 0 \\ 0, & \text{o.w} \end{cases}$$

$$= \begin{cases} 2+z, & -2 < z < -1 \\ -z, & -1 < z < 0 \\ 0, & \text{o.w} \end{cases}$$

(c) Clearly $f_{Y,Z}(y,z) \neq f_Y(y)\, f_Z(z)$, $\forall (y,z)$

$\Rightarrow$ Y and Z are not independent.

$\boxed{\text{Problem No. 26}}$ The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is

$$f_{\underline{X}}(x_1, x_2) = f(x_1)\, f(x_2) = \begin{cases} \frac{1}{4}, & -1 < x_1, x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$S^{o}_{\underline{X}} = (-1, 1) \times (-1, 1)$.

(a) $S_Y^{\circ} = (-1, 2)$. Thus for $y < -1$ $F_Y(y) = 0$ and for $y \geq 2$, $F_Y(y) = 1$. For $-1 \leq y < 2$

$$F_Y(y) = P(|X_1| + X_2 \leq y)$$

$$= P(X_2 - X_1 \leq y,\ X_1 < 0) + P(X_1 + X_2 \leq y,\ X_1 > 0)$$

$$= P(X_2 - y \leq X_1 < 0) + P(0 < X_1 \leq y - X_2)$$

$$= P(X_2 \leq y,\ X_2 - y \leq X_1 < 0) + P(X_2 \leq y,\ 0 < X_1 \leq y - X_2)$$

Case I $\quad -1 \leq y < 0$ $\qquad$ $-1 \quad y \quad 0 \quad 1$

$$F_Y(y) = \int_{-1}^{y} \int_{\max\{-1,\, x_2 - y\}}^{0} \frac{1}{4}\, dx_1\, dx_2 + \int_{-1}^{y} \int_{0}^{\min\{1,\, y - x_2\}} \frac{1}{4}\, dx_1\, dx_2 .$$

Note that

$$x_2 \geq y - 1 \iff y - x_2 \leq 1 \text{ or } x_2 - y \geq -1$$

$y-1 \quad -1 \quad y \quad 0$

$$F_Y(y) = \int_{-1}^{y} \int_{x_2 - y}^{0} \frac{1}{4}\, dx_1\, dx_2 + \int_{-1}^{y} \int_{0}^{y - x_2} \frac{1}{4}\, dx_1\, dx_2$$

$$= \frac{(y+1)^2}{4}$$

Case II $\quad 0 \leq y < 1$ $\qquad$ $-1 \quad y-1 \quad 0 \quad y \quad 1$

$$F_Y(y) = \int_{-1}^{y} \int_{\max\{-1,\, x_2 - y\}}^{0} \frac{1}{4}\, dx_1\, dx_2 + \int_{-1}^{y} \int_{0}^{\min\{1,\, y - x_2\}} \frac{1}{4}\, dx_1\, dx_2$$

$$= \int_{-1}^{y-1} \int_{-1}^{0} \frac{1}{4}\, dx_1\, dx_2 + \int_{y-1}^{y} \int_{x_2 - y}^{0} \frac{1}{4}\, dx_1\, dx_2$$

$$+ \int_{-1}^{y-1} \int_{0}^{1} \frac{1}{4}\, dx_1\, dx_2 + \int_{y-1}^{y} \int_{0}^{y - x_2} \frac{1}{4}\, dx_1\, dx_2$$

$$= \frac{2y+1}{4}$$

Case III $\quad 1 \le y < 2$

$$F_Y(y) = \int_{-1}^{1}\int_{\max\{-1,\, x_2 - y\}}^{0} \frac{1}{4}\, dx_1\, dx_2 + \int_{-1}^{1}\int_{0}^{\min\{1,\, y-x_2\}} \frac{1}{4}\, dx_2\, dx_1$$

$$= \int_{-1}^{y-1}\int_{-1}^{0} \frac{1}{4}\, dx_1\, dx_2 + \int_{y-1}^{1}\int_{x_2-y}^{0} \frac{1}{4}\, dx_1\, dx_2$$

$$+ \int_{-1}^{y-1}\int_{0}^{1} \frac{1}{4}\, dx_1\, dx_2 + \int_{y-1}^{1}\int_{0}^{y-x_2} \frac{1}{4}\, dx_1\, dx_2$$

$$= \frac{4y - y^2}{4}$$

Thus

$$F_Y(y) = \begin{cases} 0, & y < -1 \\ \frac{(y+1)^2}{4}, & -1 \le y < 0 \\ \frac{2y+1}{4}, & 0 \le y < 1 \\ \frac{4y-y^2}{4}, & 1 \le y < 2 \\ 1, & y \ge 2 \end{cases}$$

Clearly the p.d.f. of Y is

$$f_Y(y) = \begin{cases} \frac{y+1}{2}, & -1 < y < 0 \\ \frac{1}{2}, & 0 < y < 1 \\ \frac{2-y}{2}, & 1 < y < 2 \\ 0, & \text{o.w.} \end{cases}$$

(b) Let $S_1^0 = (-1, 0) \times (-1, 1)$ and $S_2^0 = (0, 1) \times (-1, 1)$

We can take $\underset{\sim}{S_X} = S_1^0 \cup S_2^0$. Let $\underset{\sim}{h} = (h_1, h_2)$,

$h_1(x_1, x_2) = |x_1| + x_2$, $h_2(x_1, x_2) = x_2$.

On S_1, $\underset{\sim}{h}$ is 1-1 with inverse image

$h_{1,1}^{-1}(y, z) = z - y$ $\quad h_{1,2}^{-1}(y, z) = z$

Jacobian $\quad J_1 = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

$\underset{\sim}{h}(S_1) = \{(y, z) \in \mathbb{R}^2 : -1 \leq z - y < 0, -1 \leq z \leq 1\}$

On S_2, $\underset{\sim}{h}$ is 1-1 with inverse image

$h_{2,1}^{-1}(y, z) = y - z$, $\quad h_{2,2}^{-1}(y, z) = z$

Jacobian $\quad J_2 = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$\underset{\sim}{h}(S_2^0) = \{(y, z) \in \mathbb{R}^2 : 0 < y - z < 1, -1 < z < 1\}$

Thus the jt. p.d.f of (Y, Z) is

$$f_{Y,Z}(y, z) = f_{\underset{\sim}{X}}(z - y, z)\, |-1|\, I_{\underset{\sim}{h}(S_1)}(y, z) + f_{\underset{\sim}{X}}(y - z, z)\, I_{\underset{\sim}{h}(S_2)}(y, z)$$

$$= \begin{cases} \frac{1}{2}, & 0 < y - z < 1, \quad -1 < z < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y, z)\, dz = \begin{cases} \displaystyle\int_{\max\{-1, y-1\}}^{\min\{1, y\}} \frac{1}{2}\, dz, & -1 < y < 2 \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} \frac{y+1}{2}, & -1 < y < 0 \\ \frac{1}{2}, & 0 < y < 1 \\ \frac{2-y}{2}, & 1 < y < 2 \\ 0, & \text{o.w} \end{cases}$$