CML, CAPM and APT

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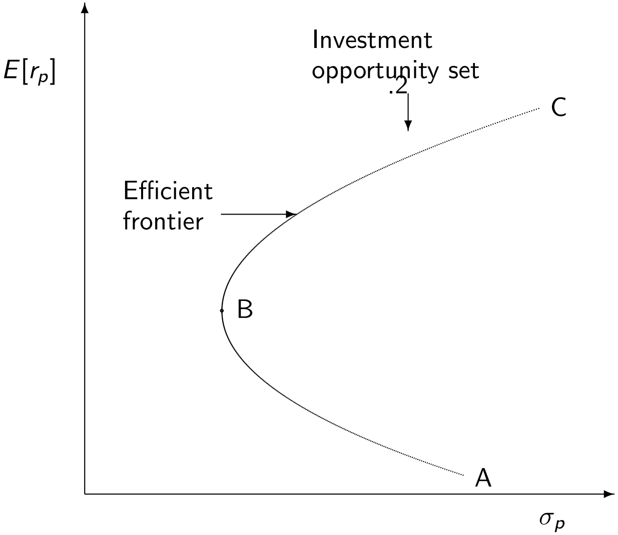

Investment universe and the efficient frontier

Not all opportunities will be chosen by rational investors:

- only those on the *efficient frontier* between
  - *minimum variance portfolio* B and
  - *maximum return portfolio* C

All other opportunities are inefficient:

- they can be replaced by an investment that
  - offers higher return for the same risk
  - or lower risk for the same return

We analyse portfolio selection first without, then with a financial market.

Investors choose portfolios:

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- expressed in their indifference curves
- such that their utility is maximized (i.e. choice is on highest indifference curve)

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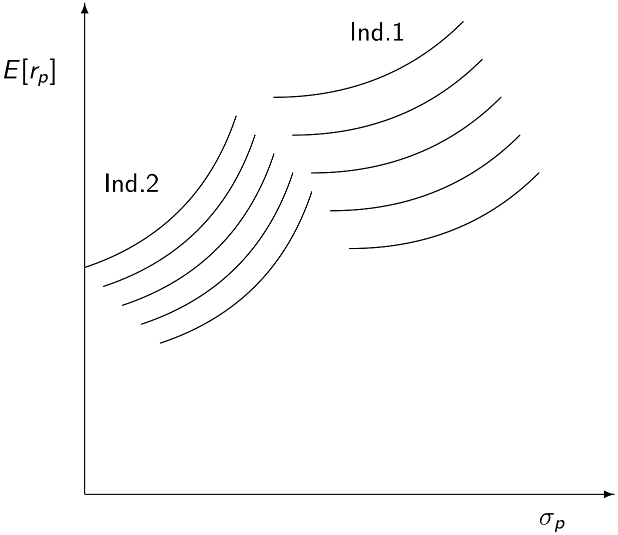
- What do indifference curves look like in a risk-return space?
- Which of the two individuals in the picture is more risk averse?
- In which direction increases utility?



Indifference curves in risk-return space

In this setting, portfolio selection is done in 2 steps:

1. the preferred risk return combination is chosen
    1. as the tangency point of the indifference curve and the efficient frontier
    2. individual preferences have to be known to make that decision!

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1. the preferred risk return combination is chosen
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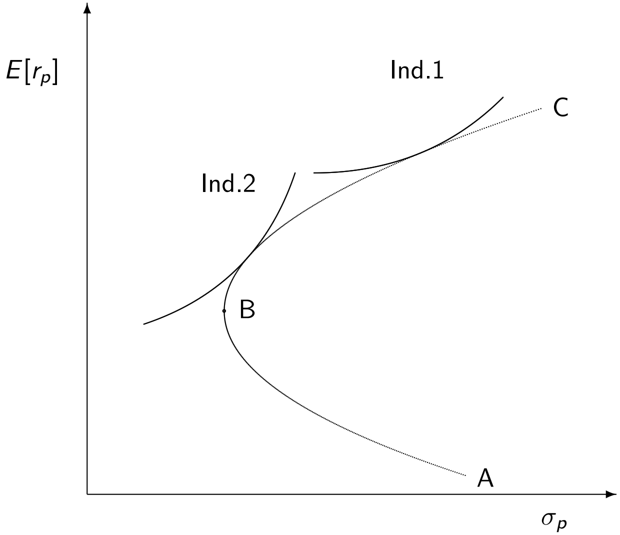
    2. individual preferences have to be known to make that decision!
2. portfolio variance is minimized subject to the restrictions that
    1. the return is not less than the chosen return
    2. the portfolio weights sum to 1
    3. (the portfolio weights are positive, if no short sales are allowed)



Choices along the efficient frontier

Minimization can be done in different ways:

- analytically e.g. with Lagrange multipliers
- numerically

Banks used to provide this as an expensive service
Now you can do it at home with a spreadsheet

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Do you see a practical problem coming up?

- Number of covariances is $I(I-1)/2$, gets very large:
  - $I = 10 \Rightarrow I(I-1)/2 = 45$
  - $I = 100 \Rightarrow I(I-1)/2 = 4950$

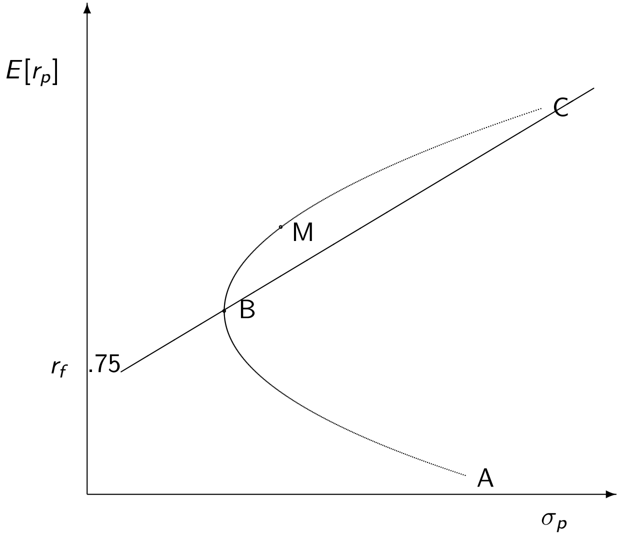
Pricing portfolios in equilibrium

We extend the analysis with a financial market (similar to Fisher's analysis) and market equilibrium

- Introduction of a financial (money) market
  - adds a new investment opportunity: risk free borrowing and lending
  - is also opportunity to move consumption back and forth in time

Looks trivial, but has profound effects

- changes the shape of the efficient frontier
- all investors want to hold combinations of risk free asset and tangency portfolio M (called *two-fund separation*)



The Capital Market Line

The straight line from r_f through portfolio M is called *Capital Market Line*

- offers higher exp. return than old efficient frontier BC
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Notice that M is tangency point:

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- determined by r_f + returns, var-covar of risky assets
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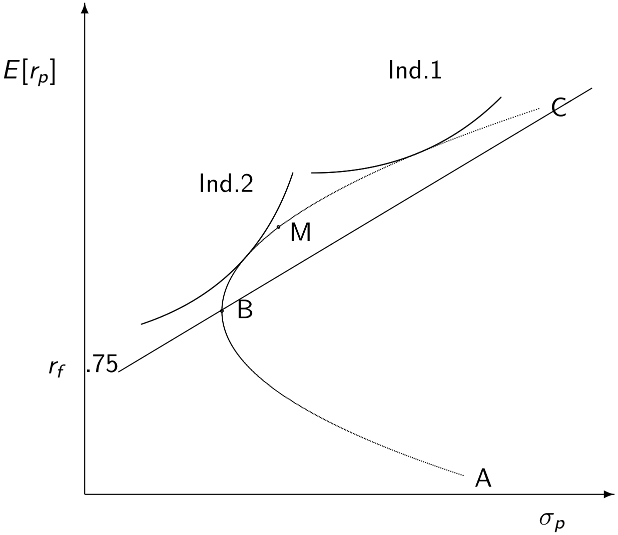
All investors will want to hold M $\Rightarrow$

- individual preferences expressed in proportion risk free investment

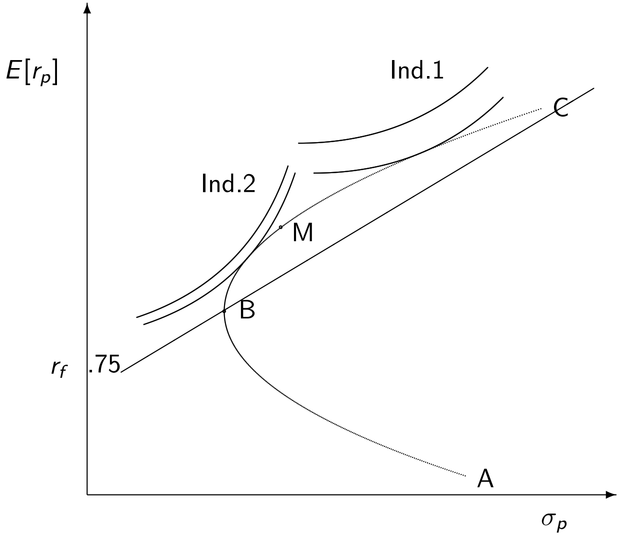
Market equilibrium requires:

- set of market clearing prices
- all assets must be held $\Rightarrow$ prices adjust so that excess demand/supply is zero
- includes risk free asset: risk free rate such that borrowing equals lending
- in tangency portfolio M:
  - all risky assets are held according to their market value weights
  - hence the name *market portfolio*
  - $\Rightarrow$ all investors hold risky assets in same proportions

Result: investors jump to higher indifference curves



Choices along the capital market line



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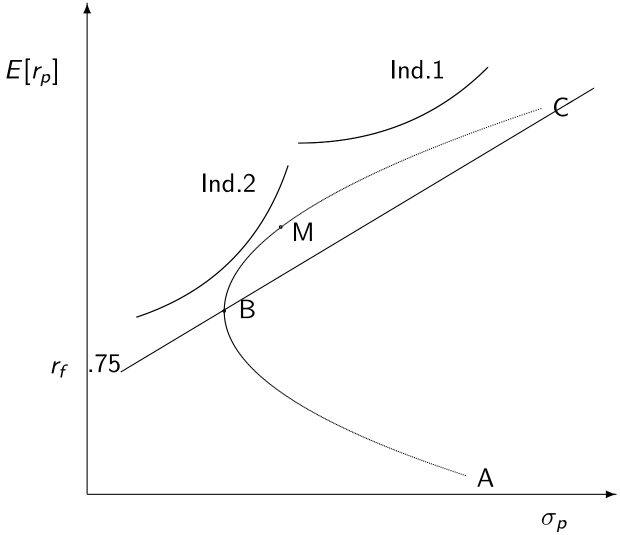

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How does Ind. 1 reach her optimal point on the CML beyond M?

- By borrowing some amount risk free and investing *more than her money* in the market portfolio.
  - M is expected to earn more than r_f
  - if expectation is realized, difference $r_m - r_f$ is added to return, which will be $> r_m$
  - but if realized $r_m < r_f$, her return may be $< r_f$, risk is increased

Capital market line:

- equilibrium risk-return relation for *efficient* portfolios
- only valid when all risk comes from share of market portfolio M in any portfolio p

Expression for CML can easily be derived:

- invest x in M and $(1-x)$ risk free
- this portfolio has expected return of:

$$E(r_p) = (1-x)r_f + xE(r_m)$$

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$\sigma_p = x\sigma_m$ which means: $x = \frac{\sigma_p}{\sigma_m}$

Substituting this back in return relation eliminates x:

$$E(r_p) = (1 - \frac{\sigma_p}{\sigma_m}) r_f + \frac{\sigma_p}{\sigma_m} E(r_m)$$

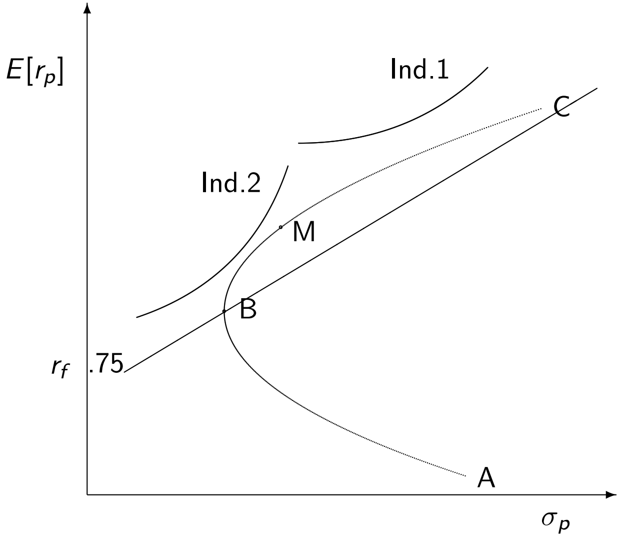
$$E(r_p) = r_f - \frac{\sigma_p}{\sigma_m} r_f + \frac{\sigma_p}{\sigma_m} E(r_m)$$

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m} \sigma_p$$

- r_f = time value of money
- $\frac{E(r_m) - r_f}{\sigma_m}$ = price per unit of risk, the *market price of risk*
- σ_p = volume of risk

Capital market line is linear

- Intuition: in Markowitz' mean-variance model
  - return is function of a quadratic (σ_p^2)
  - marginal return (1^{st} derivative) will be linear
  - marginal risk-return trade-off is constant
- If marginal risk-return trade-off is constant
  - it is the same for all market participants
  - regardless of their attitudes to risk
    (shape of their indifference curves)
- By consequence, managers can use market price of risk
  - don't have to know preferences, risk attitude of shareholders
  - allows separation of ownership and management



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Capital Asset Pricing Model CAPM

Capital Market Line only valid for efficient portfolios

- combinations of risk free asset and market portfolio M
- all risk comes from market portfolio

What about inefficient portfolios or individual stocks?

- don't lie on the CML, cannot be priced with it
- need a different model for that

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What needs to be changed in the model:
the market price of risk $((E(r_m) - r_f)/\sigma_m)$,
or the measure of risk σ_p?

CAPM is more general model, developed by Sharpe

Consider a two asset portfolio:

- one asset is market portfolio M, weight $(1 - x)$
- other asset is individual stock i, weight x

Note that this is an inefficient portfolio

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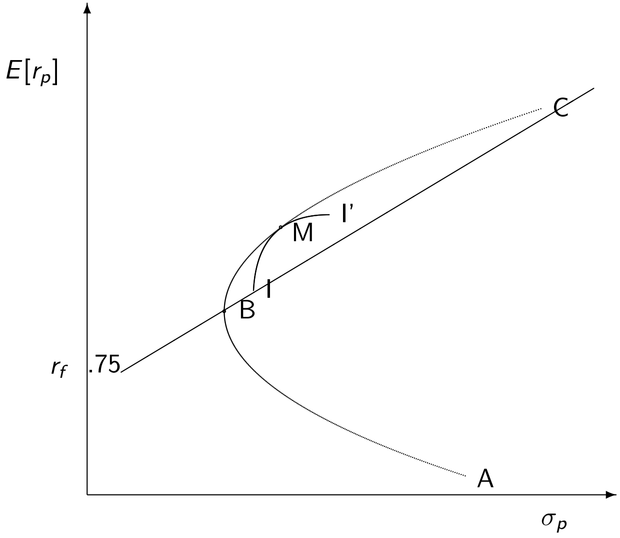
Analyse what happens if we vary proportion x invested in i

- begin in point I, 100% in i, x=1
- in point M, x=0, but asset i is included in M with its market value weight
- to point I', x<0 to eliminate market value weight of i



Portfolios of asset i and market portfolio M

Risk-return characteristics of this 2-asset portfolio:

$$E(r_p) = xE(r_i) + (1 - x)E(r_m)$$

$$\sigma_p = \sqrt{[x^2\sigma_i^2 + (1 - x)^2\sigma_m^2 + 2x(1 - x)\sigma_{i,m}]}$$

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$$\frac{\partial \sigma_p}{\partial x} = \frac{1}{2}\left[x^2\sigma_i^2 + (1 - x)^2\sigma_m^2 + 2x(1 - x)\sigma_{i,m}\right]^{-\frac{1}{2}} \times \left[2x\sigma_i^2 - 2\sigma_m^2 + 2x\sigma_m^2 + 2\sigma_{i,m} - 4x\sigma_{i,m}\right]$$

First term of $\partial\sigma_p/\partial x$ is $\frac{1}{2\sigma_p}$, so:

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Isolating x gives:

$$\frac{\partial\sigma_p}{\partial x} = \frac{x(\sigma_i^2 + \sigma_m^2 - 2\sigma_{i,m}) + \sigma_{i,m} - \sigma_m^2}{\sigma_p}$$

At point M all funds are invested in M so that:

- $x = 0$ and $\sigma_p = \sigma_m$

Note also that:

- i is already included in M with its market value weight
- economically x represents *excess* demand for i
- in equilibrium M excess demand is zero

This simplifies marginal risk to:

$$\left.\frac{\partial \sigma_p}{\partial x}\right|_{x=0} = \frac{\sigma_{i,m} - \sigma_m^2}{\sigma_p} = \frac{\sigma_{i,m} - \sigma_m^2}{\sigma_m}$$

So the slope of the risk-return trade-off at equilibrium point M is:

$$\left.\frac{\partial E(r_p)/\partial x}{\partial \sigma_p/\partial x}\right|_{x=0} = \frac{E(r_i) - E(r_m)}{\left(\sigma_{i,m} - \sigma_m^2\right)/\sigma_m}$$

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Solving for $E(r_i)$ gives:

$$\begin{aligned} E(r_i) &= r_f + (E(r_m) - r_f)\frac{\sigma_{i,m}}{\sigma_m^2} \\ &= r_f + (E(r_m) - r_f)\beta_i \end{aligned}$$

$$E(r_i) = r_f + (E(r_m) - r_f)\beta_i$$

This is the *Capital Asset Pricing Model*

- Sharpe was awarded the Nobel prize for this result
- Its graphical representation is known as the
  - *Security Market Line*
- Pricing relation for entire investment universe
  - including inefficient portfolios
  - including individual assets
- clear price of risk: $E(r_m) - r_f$
- clear measure of risk: β

CAPM formalizes risk-return relationship:

- well-diversified investors value assets according to their contribution to portfolio risk
  - if asset i increases portf. risk $E(r_i) > E(r_p)$
  - if asset i decreases portf. risk $E(r_i) < E(r_p)$
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Offers other insights as well. Look at 4 of them:

1. Systematic and unsystematic risk
2. Risk adjusted discount rates
3. Certainty equivalents
4. Performance measures

1. Systematic & unsystematic risk

- The CML is pricing relation for *efficient* portfolios:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p$$

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The SML valid for all investments, incl. inefficient portfolios and individual stocks:

$$E(r_p) = r_f + (E(r_m) - r_f)\beta_p$$

we can write β as:

$$\beta_p = \frac{cov_{p,m}}{\sigma_m^2} = \frac{\sigma_p \sigma_m \rho_{p,m}}{\sigma_m^2} = \frac{\sigma_p \rho_{p,m}}{\sigma_m}$$

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Compare with CML:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p$$

The difference between CML and SML is in volume part:

- SML only prices the *systematic risk*
  - is therefore valid for all investment objects.
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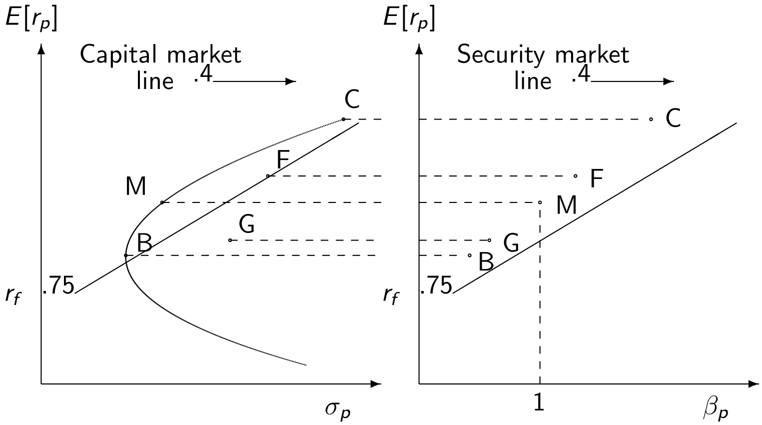
  - otherwise, CML uses 'wrong' risk measure
- difference is correlation term, that is ignored in CML
  - efficient portfolios only differ in proportion M in it
  - so all efficient portfolios are perfectly positively correlated: $\rho_{M,(1-x)M} = 1$
  - if $\rho_{p,m} = 1 \Rightarrow \sigma_p \rho_{p,m} = \sigma_p$ and $CML = SML$



Systematic and unsystematic risk

2. CAPM and discount rates

Recall general valuation formula for investments:

$$Value = \sum^{t} \frac{Exp\left[Cash\ flows_t\right]}{\left(1 + discount\ rate_t\right)^t}$$

Uncertainty can be accounted for in 3 different ways:

1. Adjust discount rate to *risk adjusted discount rate*
2. Adjust cash flows to *certainty equivalent cash flows*
3. Adjust probabilities (expectations operator) from normal to *risk neutral* or *equivalent martingale probabilities*

Use of CAPM as risk adjusted discount rate is easy
CAPM gives expected (=required) return on portfolio P as:

$$E(r_p) = r_f + (E(r_m) - r_f)\beta_p$$

But return is also:

$$E(r_p) = \frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}}$$

Discount rate:

- links expected end-of-period value, $E(V_{p,T})$, to value now, $V_{p,0}$
- found by equating the two expressions:

$$\frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}} = r_f + (E(r_m) - r_f)\beta_p$$

solving for $V_{p,0}$ gives:

$$V_{p,0} = \frac{E(V_{p,T})}{1 + r_f + (E(r_m) - r_f)\beta_p}$$

- r_f is the time value of money
- $(E(r_m) - r_f)\beta_p$ is the adjustment for risk
- together they form the risk adjusted discount rate

3. Certainty equivalent formulation

The second way to account for risk:

- adjust uncertain cash flow to a *certainty equivalent*
- can (and should) be discounted with risk free rate

Requires some calculations, omitted here

$$\frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}} = r_f + (E(r_m) - r_f)\beta_p$$

can be written as:

$$V_{p,0} = \frac{E(V_{p,T}) - \lambda cov\left(V_{p,T}, r_m\right)}{1 + r_f}$$

This is the *certainty equivalent formulation* of the CAPM:

- uncertain end-of-period value is adjusted by
  - the market price of risk, λ:

$$\lambda = \frac{E(r_m) - r_f}{\sigma_m^2}$$

  - $\times$ the volume of risk, i.e. cov.(end-of-period value, return on market portfolio)
- The resulting certainty equivalent value is discounted at the risk free rate to find the present value.

4. Performance measures

CML and SML relate expected return to risk

- can be reformulated as *ex post performance measures*
- relate realized returns to observed risk

Sharpe uses slope of CML for this:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p \Rightarrow$$
$$\frac{E(r_p) - r_f}{\sigma_p} = \frac{E(r_m) - r_f}{\sigma_m}$$

Left hand side is *return-to-variability ratio* or *Sharpe ratio*

In ex post formulation:

$$\text{Sharpe ratio:} \quad SR_p = \frac{\bar{r}_p - \bar{r}_f}{\widehat{\sigma}_p}$$

- SR_p is Sharpe ratio of portfolio p
- $\bar{r}_p$ is portfolio's historical average return $\bar{r}_p = \sum_t r_{pt}/T$
- $\bar{r}_f$ is historical average risk free interest rate
- $\widehat{\sigma}_p$ is stand. dev.portf. returns: $\widehat{\sigma}_p = \sqrt{\sum_t (r_{pt} - \bar{r}_p)^2/T}$
- T is number of observations (periods)

Sharpe ratios widely used to:

- rank portfolios, funds or managers
- identify poorly diversified portfolios (too high $\widehat{\sigma}_p$)
- identify funds that charged too high fees ($\overline{r}_p$ too low)

Sharpe ratio can be adapted:

- measure the risk premium over other benchmark than r_f
  - also known as the *information ratio*
- measure risk as semi-deviation (downward risk)
  - known as *Sortino ratio*

Treynor ratio uses security market line, β as risk measure:

$$\text{Treynor ratio:} \quad TR_p = \frac{\overline{r}_p - \overline{r}_f}{\widehat{B}_p}$$

$\widehat{B}_p$ is estimated from historical returns

- Treynor ratio usually compared with risk premium market portfolio
- is TR for portfolio with β of 1

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What does the CAPM predict about the TR of different assets and portfolios?

Treynor ratio uses security market line, β as risk measure:

$$\text{Treynor ratio:} \quad TR_p = \frac{\overline{r}_p - \overline{r}_f}{\widehat{B}_p}$$

$\widehat{B}_p$ is estimated from historical returns

- Treynor ratio usually compared with risk premium market portfolio
- is TR for portfolio with β of 1

What does the CAPM predict about the TR of different assets and portfolios?
All assets lie on SML $\Rightarrow$ all have same TR

Jensen's alpha also based on CAPM

- measures portfolio return in excess of CAPM
- found by regressing portfolio risk-premium on market portfolio's risk-premium:

$$r_{pt} - r_{ft} = \widehat{\alpha}_p + \widehat{B}_p(r_{mt} - r_{ft}) + \widehat{\varepsilon}_{pt}$$

- taking averages and re-writing gives Jensen's alpha:

$$\text{Jensen's alpha :} \quad \widehat{\alpha}_p = \overline{r}_p - (\overline{r}_f + \widehat{B}_p(\overline{r}_m - \overline{r}_f))$$

We will meet these performance measures again in market efficiency tests

Assumptions CAPM is based on:

- Financial markets are perfect and competitive:
  - no taxes or transaction costs, all assets are marketable and perfectly divisible, no limitations on short selling and risk free borrowing and lending
  - large numbers of buyers and sellers, none large enough to individually influence prices, all information simultaneously and costlessly available to all investors
- Investors
  - maximize expected utility of their end wealth by choosing investments based on their mean-variance characteristics over a single holding period
  - have homogeneous expectations w.r.t. returns (i.e. they observe same efficient frontier)

Assumptions have different backgrounds and importance

- Some make modelling easy, model doesn't break down if we include phenomena now 'assumed away':
    - no taxes or transaction costs, all assets are marketable and divisible

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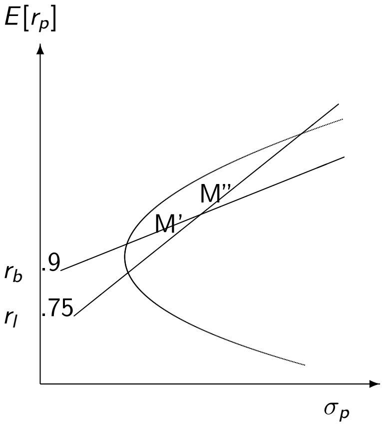
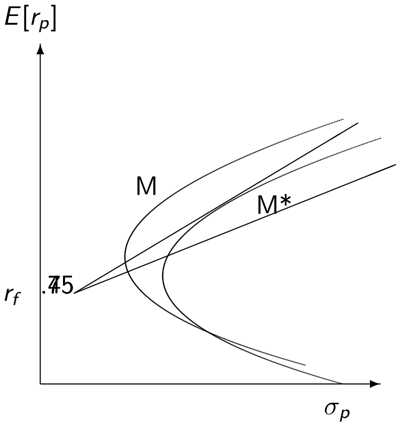
- Another points at unresolved shortcoming of the model:
  - single holding period clearly unrealistic, real multi-period model not available
- Still others have important consequences:
  - different borrowing and lending rates invalidate same risk-return trade-off for all (see picture)
  - if investors see different frontiers, effect comparable to restriction, e.g. ethical and unethical investments (see picture)



CML with different borrowing and lending rates



CML with heterogeneous expectations

Key assumption is:

Investors maximize expected utility of their end wealth by choosing investments based on their mean-variance characteristics

- Is the 'behavioural assumption' (assertion):
- the behaviour (force) that drives the model into equilibrium
- Mean variance optimization *must* take place for the model to work

We did not explicitly say anything about mean-variance in utility theory. Is that special for Markowitz' analysis? Not quite

Mean variance optimization fits in with general economic theory under 2 possible scenario's (assumptions):

1. Asset returns are jointly normally distributed
    1. means, variances and covariances completely describe return distributions (higher moments zero)
    2. no other information required for investment decisions

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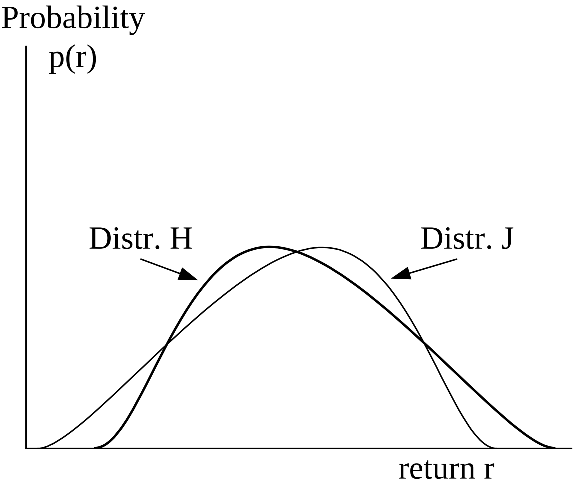
1. Asset returns are jointly normally distributed
    1. means, variances and covariances completely describe return distributions (higher moments zero)
    2. no other information required for investment decisions
2. Investors have quadratic utility functions
    1. If $U(W) = \alpha + \beta W - \gamma W^2$; choosing a portfolio to maximize U only depends on $E[W]$ and $E[W^2]$, i.e. expected returns and their (co-)variances
    2. means investors only care about first 2 moments

Do investors ignore higher moments? Which would you chose?



2 mirrored distributions with identical mean and stand.dev.

Empirical tests of the CAPM

Require approximations and assumptions:

- model formulated in expectations
- has to be tested with historical data
- gives returns a function of β, not directly observable

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Tested with a two pass regression procedure:

1. time series regression of individual assets
2. cross section regression of assets' βs on returns

First pass, time series regression estimates βs:

$$r_{it} - r_{ft} = \widehat{\alpha}_i + \widehat{\beta}_i(r_{mt} - r_{ft}) + \widehat{\varepsilon}_{it}$$

- regresses asset risk premia on market risk premia
- for each asset separately
- market approximated by some index
- usually short observation periods (weeks, months)
- result is called *characteristic line*
- slope coefficient is estimated beta of asset i, $\widehat{\beta}_i$

Second pass, cross section regression estimates risk premia:

$$\overline{rp}_i = \gamma_0 + \gamma_1 \widehat{\beta}_i + \gamma_{2n}(testvar_n) + \widehat{u}_i$$

- regresses average risk premia on $\widehat{\beta}$
- rp averaged over observation period $\overline{rp}_i = \sum_t (r_{it} - r_{ft})/T$
- β can also be estimated over prior period

Some more details:

- usually done with portfolios, not individual assets
- over longer periods (years)
- with rolling time window (drop oldest year, add new year)
- often includes other variables (testvars)

What does the CAPM predict about the coefficients of 2^{nd} pass regression?

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1. $\gamma_0 = 0$
2. $\gamma_1 = \overline{rp}_m$
3. $\gamma_2 = 0$
4. and relation should be linear in β
   e.g. β^2 as testvar should not be significant
5. R^2 should be reasonably high

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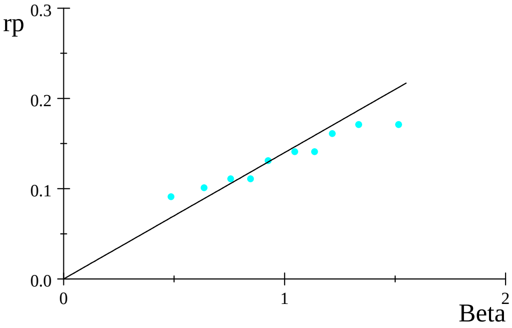
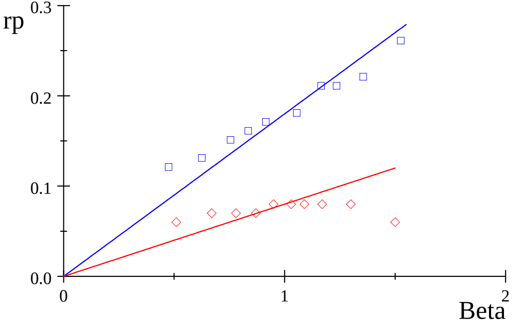
For 10 portfolios, β plotted against risk premium:



Black, 1931-1991, line is $\overline{rp}_m \times \beta$



Black, 1931-1965 (blue) and 1966-1991 (red), lines are $\overline{rp}_m \times \beta$

Black's results are typical for many other studies:

1. $\gamma_0 > 0$ (i.e. too high)
2. $\gamma_1 < \overline{rp}_m$ but $\gamma_1 > 0$ (i.e. too low)
    1. in recent data, γ_1 is lower than before
    2. even close to zero ('Beta is dead')
3. linearity generally not rejected
4. other variables are significantly $\neq 0$, so other factors play a role:
    1. small firm effect
    2. book-to-market effect
    3. P/E ratio effect
5. R^2 ?

Roll's critique: can CAPM be tested at all?

Roll argues: CAPM produces only 1 testable hypothesis:

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Roll argues: CAPM produces only 1 testable hypothesis:

the market portfolio is mean-variance efficient

Argument based on following elements:

- There is only 1 ex ante efficient market portfolio using the whole investment universe
- includes investments in human capital, venture idea's, collectors' items as wine, old masters' paintings etc.
- is unobservable
- tested with ex post sample of market portfolio, e.g. S&P 500 index, MSCI, Oslo Børs Benchmark Index

Gives rise to benchmark problem:

- sample may be mean-variance efficient, while the market portfolio is not
- or the other way around

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But if sample is ex post mean-variance efficient:

- mathematics dictate that $\beta's$ calculated relative to sample portfolio will satisfy the CAPM
- means: all securities will plot on the SML

Only test is whether portfolio we use is really the market portfolio $\Rightarrow$ untestable

A simple practical application of what we have learned so far

Suppose you are very risk averse, what would you choose:

1. A very risky share of 250 in a company you expect to perform badly in the near future
2. A risk free bond of 235

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What would you chose:

1. 250 today
2. 235 today

What do we learn from this?

- Financial markets provide information needed to value alternatives
  - nature of the bond and stock already reflected in price
  - nobody needs stocks or bonds to allocate consumption over time
  - everybody prefers more to less

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- Financial markets provide information needed to value alternatives
  - nature of the bond and stock already reflected in price
  - nobody needs stocks or bonds to allocate consumption over time
  - everybody prefers more to less
- Financial decisions can be made rationally by maximizing value regardless of risk preferences or expectations
  - risky share and risk free bond have the same value for risk averse student and rich businessman
  - doesn't matter where the money comes from
  - simply choose highest PV, reallocate later

Financial markets give the opportunity to:

- expose to risk / eliminate risk
- move consumption back and forth in time

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On well functioning financial markets:

- prices are 'fair', i.e. arbitrage free
- arbitrage brings about the 'Law of one price':
    - same assets have same price
    - asset value comes from its cash flow pattern over time/scenario's
    - if same pattern can be constructed with different combination of assets, price must the same
    - if not, buying what is cheap and selling what is expensive will drive prices to same level

Arbitrage

Arbitrage is strategy to profit from mispricing in markets

Formally, an arbitrage strategy:

- either requires
  - investment ≤ 0 today, while
  - all future pay-offs ≥ 0 and
  - at least one payoff > 0
- or requires
  - investment < 0 today ($=$profit) and
  - all future pay-offs ≥ 0

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Less formally:

- either costs nothing today + payoff later
- or payoff today without obligations later

Arbitrage Pricing Theory

- Introduced by Ross (1976)
- Does not assume that investors maximize utility based on stocks' mean-variance characteristics
- Instead, assumes stock returns are generated by a multi-index, or multi-factor, process
- More general than CAPM, gives room for more than 1 risk factor
- Widely used, e.g. Fama-French 3 factor model

Introduce with detour over *single index model*

Single index model

So far, we used whole variance-covariance matrix

- With I stocks, calls for $\frac{1}{2}$I(I-1) covariances
- Gives practical problems for large I
- plus: non marked related part of covariance low/erratic

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Single index model is practical way around this:

- Assumes there is *only 1* reason why stocks covary: they all respond to changes in market as a whole
- Stocks respond in different degrees (measured by β)
- But stocks do not respond to unsystematic (not marked related) changes in other stocks' values

Can be formalized by writing return on stock i as:

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

$r_i, r_m =$ return stock i, market
$\alpha =$ expected value non marked related return
$\varepsilon =$ random element of non marked related return, with $E(\varepsilon) = 0$ and variance $= \sigma_\varepsilon^2$
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1. $cov(r_m, \varepsilon_i) = 0$: random element of non marked related return not correlated with market return
2. $cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$: random elements of non marked related returns are uncorrelated

Means that variance, covariance of stocks is:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2 \qquad \sigma_{i,j} = \beta_i \beta_j \sigma_m^2$$

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Simplifies analysis of large portfolios drastically:

- have to calculate each stock's α, β and σ_ε^2
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- plus r_m and σ_m^2, i.e. 3I $+ 2 <$ I $+ \frac{1}{2}$I(I-1)
- for 100 stock portfolio
  - full var-covar has $100 \times 99/2$ =4950 covar's + 100 var's
  - index model uses $3 \times 100 + 2 = 302$

The single index model

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

can also be looked upon as a *return generating process* :
The returns on any investment consist of:

- α_i expected return not related to the return on the market
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Return generating process easily extended to more indices (or factors):

- 'split' market index in several industry indices (industrials, shipping, financial,....)
- general economic factors (interest rate, oil price,...)

Expression for stock returns then becomes:

$$r_i = \alpha_i + b_{1i}F_1 + b_{2i}F_2 + ... + b_{Ki}F_K + \varepsilon_i$$

b_{1i} = sensitivity of stock i for changes in factor F_1
F_1 = return on factor 1, etc.

The multi-factor (-index) model assumes that:

- factors are uncorrelated: $cov(F_m, F_k) = 0$ for all $m \neq k$
- residuals uncorrelated with factors $cov(F_k, \varepsilon_i) = 0$
- residuals of different stocks uncorrelated $cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$

Arbitrage Pricing Theory

- Arbitrage pricing theory builds on such a multi-factor return generating process
- Distinguishes between
  - *expected* part of stock returns
  - *unexpected* part
- Unexpected part (risk) consists of
  - systematic (or market) risk
  - and unsystematic (or idiosyncratic) risk
- Market risk not expressed as covar with market but as sensitivity to (any) number of risk factors

To derive pricing relation, start with return generating process:

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which can be re-written as:

$$r_i = E(r_i) + \sum_{k=1}^{K} b_{ik}(F_k - E(F_k)) + \varepsilon_i$$

- $E(r_i) =$ is expected return of stock i
- $b_{ik} =$ is sensitivity of stock i to factor k
- $F_k =$ return of factor k, with $E(F_k - E(F_k)) = 0$ ($\Rightarrow$ fair game: expectations accurate in long run)
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Terms after $E(r_i)$ are 'error' part of process:

- describe deviation from expected return
- b_{ik} is sensitivity for *unexpected* factor changes
- expected part included in $E(r_i)$

Next, construct portfolio, I assets, weights x_i, then portfolio return is:

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In well diversified portfolios, idiosyncratic risk (last term) disappears

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   the expected return must be zero:
   - $\sum_i x_i E(r_i) = 0$

These three no-arbitrage conditions can be interpreted as orthogonality conditions from linear algebra:

1. $\sum_i x_i = 0$ means:
   - vector of weights is orthogonal to a vector of 1's
2. $\sum_i x_i b_{ik} = 0$ means:
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This means that the last vector, $E(r_i)$, must be a linear combination of the other 2:

$$E(r_i) = \lambda_0 + \lambda_1 b_{1i} + \lambda_2 b_{2i} + \ldots + \lambda_k b_{ki}$$

To give lambda's economic meaning:

- construct risk free portfolio:
  - earns risk free rate
  - has zero value for all b_{ij}
  - $r_f = \lambda_0 + \lambda_1 0 + .. + \lambda_k 0 \Rightarrow \lambda_0 = r_f$

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- construct portfolio only sensitive to factor 1:
  - sensitivity 1 for factor 1 and zero value for all other b_{ij} :
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- repeat for all factors

Gives usual form of APT as equilibrium relation:

$$E(r_i) = r_f + \sum_{k=1}^{K} b_{ik}(E(F_k) - r_f)$$

Example

Illustrates APT with 3 well diversified portfolios and their sensitivities to 2 factors, priced to give these returns:

	P_1	P_2	P_3
r_p	.18	.15	.12
b_1	1.5	0.5	0.6
b_2	0.5	1.5	0.3

Portfolio returns are functions of

- risk free rate and 2 factor returns (risk premia)
- portfolios' sensitivities

Example (cont.'ed)

Factor returns and r_f found by solving 3 APT equations:

$$.18 = \lambda_0 + \lambda_1 \times 1.5 + \lambda_2 \times .5$$
$$.15 = \lambda_0 + \lambda_1 \times .5 + \lambda_2 \times 1.5$$
$$.12 = \lambda_0 + \lambda_1 \times .6 + \lambda_2 \times .3$$

which gives $\lambda_0 = 0.075\,, \lambda_1 = 0.06$ and $\lambda_2 = 0.03$

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Equilibrium relation $E(r_i) = .075 + .06b_{1i} + .03b_{2i}$

- defines return plane in 2 risk dimensions
- all investments must lie on this plane
- otherwise arbitrage opportunities exist

Example (cont.'ed)

Suppose you make a portfolio:

- with b_1=.75 and b_2=.7
- you figure it is somewhere between P_1 and P_2
- price it to offer a .16 return, also between P_1 and P_2

What happens?

Example (cont.'ed)

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What happens?

You go bankrupt quickly! You offer this arbitrage opportunity:

- construct arbitrage portfolio of $.2P_1 + .3P_2 + .5P_3$, has:
- $b_1 = .2 \times 1.5 + .3 \times .5 + .5 \times .6 = .75$
- $b_2 = .2 \times .5 + .3 \times 1.5 + .5 \times .3 = .7$
- return of $.2 \times .18 + .3 \times .15 + .5 \times .12 = .141$

Example (cont.'ed)

Arbitrage strategy:

- buy what is cheap (your portfolio)
- sell what is expensive (arbitrage portfolio)

	Cfl_{now}	Cfl_{later}	b_1	b_2
buy your portfolio	-1	1.160	.75	.7
sell arbitrage.portfolio	1	-1,141	-.75	-.7
2-5net result	0	.019	0	0

Profit of .019 is risk free, zero sensitivity to both factors

Empirical tests of APT

- require same assumptions & approximations as CAPM
- done with similar two pass regression procedure:
  - time series regression to estimate sensitivities
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Example: split total market in 2 industry indices:

- manufacturing (F_{man})
- trade (F_{trad})

1. First pass regression: estimate sensitivities

$$r_{it} - r_{f\ t} = \widehat{\alpha}_i + \widehat{\beta}_{man,i}(F_{man,t} - r_{ft}) + \widehat{\beta}_{trad,i}(F_{trad,t} - r_{ft}) + \widehat{\varepsilon}_{it}$$

for all individual assets

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2. Then calculate average risk premia ($\overline{rp}_i$) etc. over same/subsequent period and estimate risk factor premia in second pass regression:

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3. APT predictions tested by:
   - γ_0 should be zero
   - γ_1 should be $\overline{F_{man} - r_f}$
   - γ_2 should be $\overline{F_{trad} - r_f}$

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More difficult if we use:

- business characteristics
  - size, book-to-market value, price-earnings ratio, etc.
- general economic variables
  - interest rate, oil price, exchange rates, etc.

No observed risk premia, difficult to be 'complete'
$\Rightarrow$ omitted variable bias

Example: Fama-French three factor model

- estimated on monthly data 1963-1991
- all stocks on all US exchanges (NYSE, ASE, NASDAQ)
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  - book-to-market: high (top 30%), middle, low (bottom 30%)
    - each month portfolio returns calculated
    - difference: HML, high minus low
    - approximates premium book-to-market related risk factor

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Fama-French three factor model formulated as:

$$E(r_i) - r_f = \widehat{a}_i + \widehat{b}_i[E(r_m) - r_f] + \widehat{s}_i E(SMB) + \widehat{h}_i E(HML)$$

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But: more recent research shows that the model's relevance has diminished over time.

Summarizing, Arbitrage Pricing Theory:

- Rests on different assumptions than CAPM
- Is more general than CAPM
  - makes less restrictive assumptions
  - allows more factors, more realistic
- Is less precise than CAPM
  - does not give a volume of risk (what or even how many factors to use)
  - does not give a price of risk (no expression for factor risk premia, have to be estimated empirically)
- has interesting applications in risk management, default prediction, etc.