1.a. Let $X_1, X_2, \ldots, X_n$ be i.i.d. observations from $N(0, 3\tau^2)$ with $\tau > 0$. Find the asymptotic distribution of $T_n := \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Further, find a function g and constant $\beta > 0$ (independent of τ) such that

$$\sqrt{n}[g(T_n) - g(k\tau^2)] \xrightarrow{D} N(0, \beta^2) \text{ as } n \to \infty.$$

[1+2]

1.b. A random sample $X_1, X_2, \ldots, X_n$ is obtained from the probability density function:

$$f(x|\theta) = \begin{cases} \dfrac{x^{30}}{\theta^2} \exp\left(- \dfrac{x^{31}}{31\theta^2}\right) & x > 0, \\ 0 & \text{else.} \end{cases}$$

Find the maximum likelihood estimate and the method of moments estimate of θ. [2+1]

2.a. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables with $\mathbb{E}(X_i) = 0$ for all $i \in \mathbb{N}$. Suppose that $Var(X_i) \leq 1$ for all $i \in \mathbb{N}$ and $\left|Cov(X_i, X_j)\right| = 0.26^{|i-j|}$ for $i \neq j$.

(i) Find an upper bound of $\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j)$, which is linear in n.

(ii) Examine whether $\frac{S_n}{n} := \frac{1}{n}\sum_{i=1}^{n} X_i$ converges in probability to a limit, or not. Give clear arguments.

[2+2]

2.b. A randomly selected product from a factory can belong to any one of the three distinct categories I, II and III with probabilities $\theta^2, 2\theta(1-\theta)$ and $(1-\theta)^2$, respectively, where $0 < \theta < 1$. Suppose in a random sample of 100 products, we obtain 10, 60 and 30 products belonging to categories I, II and III, respectively.
Find the maximum likelihood estimator of θ. [2]