CS201

Mathematics For Computer Science

Indian Institute of Technology, Kanpur

Assignment 3

Due by: Nov 21, 2020

Instructions.

- Solutions should be mandatorily LaTeXed using the template shared and submitted through GradeScope before time. Mention Group Numbers and member names in solutions (refer template instructions).
- Clearly express solutions avoiding unnecessary details. Everything discussed in class is not required to be proved again. And anything non-trivial must be proved.
- Write the solutions on your own. Acknowledge the source wherever required. Keep in my mind department's Anti-Cheating Policy.

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1. (a) Find the generating function for the following recurrence relation.

$$f(n+1) = \begin{cases} 1 & \text{if } n+1=0 \\ \sum_{i=0}^{n} f(i)f(n-i) & \text{if } n \geq 0 \end{cases}$$

   (b) Using the generating function and generalised binomial theorem for $\sqrt{1+y}$, find a closed form for $f(n)$.

2. Define n-varaiate polynomials P_d and Q_d as:

$$\begin{aligned} P_d(x_1, x_2, \ldots, x_n) &= \sum_{\substack{J \subseteq [1,n] \\ |J|=d}} \prod_{r \in J} x_r \\ Q_d(x_1, x_2, \ldots, x_n) &= \sum_{\substack{0 \leq i_1, i_2, \ldots, i_n \leq d \\ i_1+i_2+\cdots+i_n=d}} \prod_{r=1}^{n} x_r^{i_r}, \end{aligned}$$

and $P_0(x_1, x_2, \ldots, x_n) = 1 = Q_0(x_1, x_2, \ldots, x_n)$. Show that for any $d > 0$:

$$\sum_{m=0}^{d} (-1)^m P_m(x_1, x_2, \ldots, x_n) Q_{d-m}(x_1, x_2, \ldots, x_n) = 0.$$

3. (a) Let $\alpha \in$ and N be a natural number. Using pigeon-hole principle, show that there exists integers p and q such that $1 \leq q \leq N$ and

$$|q\alpha - p| \leq \frac{1}{N}$$

   (b) Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ and N be a natural number. Using pigeon-hole principle, show that there exists integers $p_1, p_2, \ldots, p_n, q$ such that $1 \leq q \leq N^n$ and for all $i \in \{1, \ldots, n\}$

$$|\alpha_i - \frac{p_i}{q}| \leq \frac{1}{q^{1+1/n}}$$

4. Give a proof for Ramsey's theorem for general case.

5. Consider the set $S_n = \{f \mid f : [n] \to [n] \text{ and } f \text{ is a bijection}\}$ which contains all bijective mapping from $[n]$ to $[n]$ where $[n] = \{1, 2, 3, \ldots, n\}$. In other words, any $f \in S_n$ simply permutes the elements in $[n]$.

   (a) A mapping $f \in S_n$ is called a **transposition** if there exists (i, j) such that $0 \leq i \neq j \leq n$ and

$$f(k) = \begin{cases} j & \text{if } k = i \\ i & \text{if } k = j \\ k & \text{otherwise} \end{cases}$$

   Show that any $g \in S_n$ can be written as a finite product $f_1 \circ f_2 \circ \cdots \circ f_m$ where each f_i is a transposition in S_n.

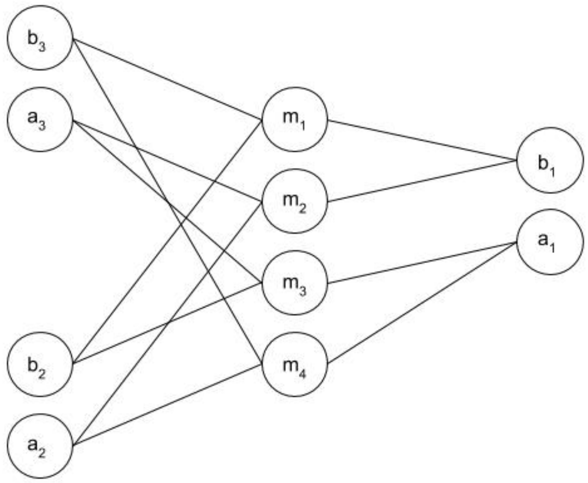
   (b) The **parity** of a function f in S_n denoted by $N(f)$ is defined as the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $f(i) > f(j)$. Show that

$$N(f) \equiv m \ (\mathit{mod}\ 2)$$

   where $f = g_1 \circ g_2 \circ \cdots \circ g_m$ and each g_i is a transposition in S_n.

6. Let $G = (V, E)$ be a graph where V is the vertex set and E is the edge set. A

bijective mapping $f : V \to V$ is an **automorphism** if it has the property that $(u, v) \in E \iff (f(u), f(v)) \in E$. Consider the following graph.



Let $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2, b_3\}, M = \{m_1, m_2, m_3, m_4\}$. Then, the vertex set of the above graph is $V = A \cup B \cup M$. Consider a bijective mapping $g : A \cup B \to A \cup B$ such that $g(a_i) \in \{a_i, b_i\}$ and $g(b_i) \in \{a_i, b_i\}$ for all $i \in \{1, 2, 3\}$, i.e., g maps the ordered pair $[a_i, b_i]$ to either $[a_i, b_i]$ (no swap) or $[b_i, a_i]$ (swap).

Show that g can be extended to an automorphism f for the above graph if and only if the number of swaps performed by g is even.