Black and Scholes Model (BSM)

- Suppose what happens to a call option if the price of a stock price changes frequently.
- We can test the risk-neutral characteristics of such price movements using Binomial pricing model.
- In case of Binomial pricing, things are quite convenient if the price of a call option changes only at the expiry date or at the end of the contract.
- But what happens if the price changes continuously like an American option.

Black and Scholes Model (BSM)

- BSM allows to set-up a portfolio even in this situation. The portfolio will have the payoff identical to that of a call option.

- However, the composition of this portfolio will have to be changed continuously as the time progresses.

- Calculating the value of such portfolio and through that the value of the call option in such a situation appears to be an unwieldy task but Black and Scholes developed a formula that does precisely that.

Black and Scholes Model (BSM)

- The BSM formula starts with

$$C_0 = S_0 N(d_1) - \frac{K}{e^{rT}} N(d_2))$$

C_0 = Equilibrium value of the call option now,

S_0 = Price of the stock (current)

K = Exercise or strike price

e = base of natural logarithm

r = annualized continuously compounded (risk free rate)

T = Length of time in years to the expiration date

$N(\mathrm{d})$ = Value of the cumulative normal density function

Black and Scholes Model (BSM)

- The BSM formula starts with $C_0 = S_0 N(d_1) - \frac{K}{e^{rT}} N(d_2)$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where ln is the natural logarithm

σ is the standard deviation of the continuously compounded annual rate of return of the stock.

Black and Scholes Model (BSM)

- The BSM formula has a great appeal because four of the parameters, namely S_0,K, r, and T are observable.
- Only one of the parameters, namely σ^2 has to be estimated.
- It is noteworthy that the value of a call option is affected by neither risk aversion of the investor nor the expected return on the stock.

Black and Scholes Model (BSM)

- **Example**

  Suppose we have following data for a stock

  $S_0 = \text{Rs.}\,60, \text{E} = \text{K} = \text{Rs.}\,56$, σ of continuously compounded annual returns σ = 0.30.

  Years to maturity = t = 0.5

  Interest rate per annum = 0.14

  Step 1: Calculate d_1 and d_2

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_1 = \frac{\ln\left(\frac{60}{56}\right)+(0.14+((0.30)\hat{}2)/2)0.5}{0.30\sqrt{0.5}} = 0.7614$$

$$d_2 = d_1 - \sigma\sqrt{T}$$ = 0.7614 – 0.2121 = 0.5493

Black and Scholes Model (BSM)

- **Example**

  Step 2: Find $N(d_1)$ and $N(d_2)$

  - $N(d_1)$ and $N(d_2)$ represent the probabilities that a random variable that has a standardized normal distribution will assume values less than d_1 and d_2.
  - The simplest way to find $N(d_1)$ and $N(d_2)$ is to use NORMDIST function of Excel.
  - One can also refer the normal distribution table, $N(0.7614)$ lies between 0.75 and 0.80.
    - As per the table , N (0.75) = 1 – 0.2264 = 0.7736

      N (0.80) = 1 – 0.2119 = 0.7881

  - For a difference of 0.05 (0.80-0.75), the Cumulative Probability increased by 0.0145 (0.7881 – 0.7736).
  - The difference between 0.7614 and 0.75 is 0.0114.
  - This value is indeed a close approximation for the true value 0.7768.

Black and Scholes Model (BSM)

- **Example**

Step 3: Estimate the present value of the exercise price using continuous discounting principle:

  - If it's call option then the $c = Ke^{-rT} = 56e^{-0.14*0.5} = 52.21$

Step 4: We can plug the numbers obtained in the step 3 in the Black-Scholes formula

$$C_0 = S_0 N(d_1) - \frac{K}{e^{rT}} N(d_2))$$

$$C_0 = 60 * 0.7768 - 52.21 * 0.7086$$

$$C_0 = 46.61 - 37 = Rs. 9.61$$

Black and Scholes Model (BSM)

Example:

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

Solution:

$$S_0 = \$\,52, \mathrm{K} = \$\,50, \mathrm{r} = 0.12, \sigma = 0.30, \mathrm{T} = 0.25$$

Step 1: Calculate d_1 and d_2

$$d_1 = \frac{\ln\left(\frac{52}{50}\right) + (0.12 + ((0.30)^\wedge 2)/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.30\sqrt{0.25} = 0.3865$$

Black and Scholes Model (BSM)

Solution:

$S_0 = \$\ 52, \mathrm{K} = \$\ 50, \mathrm{r} = 0.12, \sigma = 0.30, \mathrm{T} = 0.25$

Step 1: Calculate d_1 and d_2

$$d_1 = \frac{\ln\left(\frac{52}{50}\right)+(0.12+((0.30)^\wedge 2)/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$= 52N(0.5365) - 50e^{-0.12\times 0.25}N(0.3865)$$

$$= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504$$

$$= \$5.06$$

Black and Scholes Model (BSM)

- **Example:**
  - What is the price of a European put option on a non-dividend paying stock when the stock is \$69, the strike price is \$70, the risk free interest rate is 5% per annum, the volatility is 35% per annum and the time to maturity is six months?

Solution: $S_0 = 69, \mathrm{K} = 70, \mathrm{r} = 0.05, \sigma = 0.35$ and T = 0.5

Step 1: Calculate d_1 and d_2

$$d_1 = \frac{\ln\left(\frac{69}{70}\right)+(0.05+((0.35)\text{^}2)/2)0.25}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.35\sqrt{0.50} = -\ 0.0809$$

- *The price of the European put is*

$$= 70e^{-0.05\times0.5}N(0.0809) - 69N(-0.1666)$$

$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338$$

$$= 6.40$$

Black and Scholes Model (BSM)

- Assumptions
  - The call option is the European option
  - The stock price is continuous and is distributed lognormally
  - There are no transaction costs and taxes
  - There are no restriction on a penalties for short selling
  - Stock pays no dividend
  - The risk free rate is known and is constant

- **Implied Volatility:**
  - BSM formula required five inputs S_0, K, r, T and σ
  - Out of these, the first four (S_0, K, r, T) can be observed directly.
  - Only σ, the volatility of stock price cannot be observed directly.
  - Practitioners use two approaches to estimate σ
    - The first estimate involves using historical data
    - The second step involves backing out the value of σ using the BSM itself.
    - It is done by using the option price quoted in the market as an input and then solve for the volatility.
    - Such an estimate of stock's volatility is called as ***Implied Volatility (IV)***.
    - IV from one option can be used to value other options on that stock which have the same expiration date.
    - Options with different expiration dates can also be valued if the volatility is not expected to change.

Example:

A call option on a non-dividend-paying stock has a market price of $2.50. The stock price is $15, the strike price is $13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

$$c = 2.5\,, S_0 = 15, K = 13, \mathrm{T} = 0.25, r = 0.05$$

Solution:

- The implied volatility must be calculated using an iterative procedure.
- A volatility of 0.2 (or 20% per annum) gives c = 2.20
- A volatility of 0.3 gives c = 2.32
- A volatility of 0.4 gives c = 2.50
- By interpolation the implied volatility is about 0.396 or 39.6% per annum.

Real options

- Sometimes options are associated with investment opportunities that are not financial instruments.
  - For instance, when operating a factory, a manager may have the option of hiring additional employees or buying new equipment.
  - For instance, if one acquires a piece of land, one has the option to drill for oil, and then later the option of extracting oil if oil is found.
  - In fact, it is possible to view almost any process that allows the control as a process with a series of operational options.
  - These operational options are often termed '***real options***' to emphasize that they involve real activities or real commodities, as opposed to the purely financial commodities.

- Plant Manager's Problem:
  - Some manufacturing plants can be described by a fixed cost per month (for equipment, management and rent) and a variable cost (for material, labor and utilities) that is proportional to the level of production.
  - The total cost of this firms would be $T = F + Vx$, where F is the fixed cost, V is the rate of variable cost and x is the amount of product produced.
  - The profit of the plant in a month in which it operates at level x is

    profit (π) = $px - F - Vx$, where p is the market price of the product.
  - If $p > V$, the firm will operate at x equal to the maximum capacity of the plant.
  - If $p < V$, it will not operate.
  - Hence, the firm has a continuing option to operate, with a strike price equal to the rate of variable cost.