Master Theorem

Recurrence equations

Substitution method
recursive tree method

Already seen them with examples.

These are quite general methods to solve method to solve recurrence equations.

However, when applying these methods, we need to start from scratch in each case. Tedious..

We look at a theorem, which whenever applicable, yields the asymptotic bounds immediately.

It is applicable in many cases of analysis of divide and conquer algorithms.

Thm Suppose we have recurrence equation

$$T(n) = b\,T(n/a) + f(n)$$

where $a > 1$, $b \geq 1$ are constants and n/a is either $\left\lfloor \frac{n}{a} \right\rfloor$ or $\left\lceil \frac{n}{a} \right\rceil$

Then

(i) $f(n)$ is $O(n^{\log_a b - \varepsilon})$ for some $\varepsilon > 0$ then

$T(n)$ is $\theta(n^{\log_a b})$

(ii) $f(n)$ is $\theta(n^{\log_a b})$ then $T(n)$ is $\theta(n^{\log_a b} \log n)$

(iii) $f(n)$ is $\Omega(n^{\log_a b + \varepsilon})$ for some $\varepsilon > 0$ and

$b f(n/a) \leq c f(n)$ for some $c < 1$ and all sufficiently large n

then $T(n)$ is $\theta(f(n))$

Remarks 1. Recurrence equation has the form that a problem of size n is divided into b subproblems each of size n/a. Effort to divide into subproblems as well as to combine the solution of these subproblems is $f(n)$.

2. $\log_a b$ is pivotal.

3. All three cases are exclusive, depend on growth of $f(n)$ relative to $n^{\log_a b}$

Example:

(a) $T(n) = 2T(n/4) + 1$

$b = 2, \quad a = 4 \qquad \log_a b = \log_4 2 = \frac{1}{2}$

1 is $O(n^{1/2 - \varepsilon})$ for any $0 < \varepsilon \leq \frac{1}{2}$

so the first case of the theorem is applicable.

$T(n)$ is $\Theta(n^{1/2})$ or $\Theta(\sqrt{n})$

(b) $T(n) = 2T(n/4) + \sqrt{n}$

$n^{\log_a b} = \sqrt{n}$ $\quad f(n)$ is $\Theta(n^{\log_a b})$

Case (ii) applies and

$T(n)$ is $\Theta(n^{1/2} \cdot \log n) = \Theta(\sqrt{n} \log n)$

(c) $T(n) = 2T(n/4) + n$

$f(n) = n$ is $\Omega(n^{1/2+\varepsilon})$ for any $\frac{1}{2} \leq \varepsilon \leq 1$

$b f(n/a) = 2\left(\frac{n}{4}\right) = \frac{n}{2} = \frac{1}{2} \cdot f(n)$ $\quad c = \frac{1}{2}$, holds for all n

Case (iii) applies $T(n)$ ~~is~~ $\Theta(n)$.

(d) $T(n) = 2T(n/4) + n^2$

again n^2 in $\Omega(n^{1/2+\varepsilon})$ $\quad 0 \le \varepsilon \le \frac{3}{2}$

$b f(n/a) = 2\left(\frac{n}{4}\right)^2 = 2 \cdot \frac{n^2}{16} = \frac{1}{8} n^2 = c \cdot f(n)$ $\quad c = \frac{1}{8} < 1$

holds for all n.

$T(n)$ in $\theta(n^2)$.

Example

Recurrence equation

[Strassen's algo]

$$T(n) = 7T\left(\frac{n}{2}\right) + \theta(n^2)$$

$b = 7, \quad a = 2$

$\log_a b = \log_2 7 = 2.81$

n^2 is $O(n^{2.81 - \varepsilon})$ for any ε, $0 \leq \varepsilon \leq 0.81$

Case (i) applies

$$T(n) \text{ is } \theta(n^{\log_2 7}) = \theta(n^{2.81})$$

Example $T(n) = 4T(n/2) + n^2 \log n$

$b = 4, \quad a = 2 \qquad \log_2 4 = 2$

$n^2 \log n$ is not $\Theta(n^2)$ or $O(n^{2-\varepsilon})$

Cases (i) & (ii) do not apply.

Does case (iii) apply?

Is there some $\varepsilon > 0$ s.t. $n^2 \log n$ is $\Omega(n^{2+\varepsilon})$?

$$n^2 \log n \geq c n^{2+\varepsilon} \text{ for all } n \geq n_0$$

$\Rightarrow \quad \log n \geq c\, n^{\varepsilon}$ for all $n \geq n_0$

This does not hold. n^{ε} grows faster than $\log n$ for any $\varepsilon > 0$.

Case (iii) does not apply.

Master theorem is not applicable.

Exercise Give an upper bound on $T(n)$.

[Hint: Use recursion tree method, assume n is a power of 2]

Proof of master theorem is not in our syllabus.

Not covering the proof but will give some idea.

First consider $n = a^m$ (n is a power of a)

$$T(n) = \begin{cases} b\,T(n/a) + f(n) & n > 1 \\ d_0 & n = 1 \end{cases}$$

Exercise 1

Using recursion tree method

show that $T(n) = \sum_{k=0}^{m-1} b^k f\left(\frac{n}{a^k}\right) + d_0 b^m$

Exercise 2 Consider various cases of 'f' as in master theorem to prove the master theorem (when n is a power of a).

When n is not a power of a. floor, or ceiling fn needs to assumed in the recurrence equation.

In the recursion tree nodes at each level will correspond to

$\lfloor n/a \rfloor$ or $\lceil n/a \rceil$, $\underbrace{\lfloor \lfloor n/a \rfloor / a \rfloor}$ . . .

can be estimated $\frac{n}{a} - 1 < \lfloor n/a \rfloor \leq \frac{n}{a}$

$(\frac{n}{a} - 1)/a < \lfloor \lfloor n/a \rfloor / a \rfloor \leq \frac{n}{a^2}$

$\frac{n}{a^2} - \frac{1}{a}$

Analysis can be done in each case of master theorem

For Details, refer to the book.

Good proof, but not needed for the course.