3.a. Consider the following density

$$f(x) = ce^{-x^2/18} \text{ for } x > 0.$$

(a) Find c.
(b) Let X be a random variable with pdf f. Find $M_X(t)$ for $-\infty < t < \infty$.
(c) Use $M_X(t)$ to compute $Var(X)$. [1+2+2]

3.b. Let $Y_1, \ldots, Y_n$ be random variables and $b_1, \ldots, b_n$ be positive numbers. Prove that

$$\textstyle\sum_{i=1}^n b_i \sqrt{Var(Y_i)} \leq \sqrt{\sum_{i=1}^n b_i} \sqrt{\sum_{i=1}^n b_i Var(Y_i)}.$$ [2]

Solution 3.a.

3(a) : Given, $f_X(x) = c\, e^{-x^2/18}$, $x > 0$

$$\int_0^\infty f_X(x)\, dx = 1$$

$$\Rightarrow c \int_0^\infty e^{-x^2/18}\, dx = 1$$

$$\Rightarrow \quad c = \frac{1}{3}\sqrt{\frac{2}{\pi}} \quad \text{———} \quad \boxed{1 \text{ mark}}$$

3(b) :- $M_X(t) = \frac{1}{3}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{tx - \frac{x^2}{18}}\, dx$

$$= \frac{1}{3}\sqrt{\frac{2}{\pi}}\, e^{\frac{9t^2}{2}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{x^2}{9} - 2\cdot\frac{x}{3}\cdot 3t + 9t^2\right)} dx$$

$$= \frac{1}{3}\sqrt{\frac{2}{\pi}}\, e^{9t^2/2} \int_0^\infty e^{-\frac{1}{2}\left(\frac{x}{3} - 3t\right)^2} dx$$

$$= \sqrt{\frac{2}{\pi}}\, e^{9t^2/2} \int_{-3t}^\infty e^{-z^2/2}\, dz$$

Put, $\frac{x}{3} - 3t = z$

$dx = 3\, dz$

$$= 2e^{9t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-3t}^\infty e^{-z^2/2}\, dz$$

$$= 2e^{9t^2/2}\left[1 - \Phi(-3t)\right]$$

$$= 2e^{9t^2/2}\, \Phi(3t), \quad t \in \mathbb{R}, \quad \left[\because 1 - \Phi(-z) = \Phi(z)\right]$$

$$\text{———} \quad \boxed{2 \text{ marks}}$$

3(c): $\psi_X(t) = \log M_X(t) = \log 2 + \frac{9t^2}{2} + \log \Phi(3t)$

$$\frac{\partial}{\partial t}\psi_X(t) = 9t + 3\,\frac{\phi(3t)}{\Phi(3t)}$$

$$\frac{\partial^2}{\partial t^2}\psi_X(t) = 9 + 3\left[\frac{\Phi(3t).\,\phi'(3t).3 - 3.(\phi(3t))^2}{(\Phi(3t))^2}\right]$$

$$\Rightarrow Var(X) = \left.\frac{\partial^2}{\partial t^2}\psi_X(t)\right|_{t=0}$$

$$= 9 + 9\left[\frac{\frac{1}{2}.0 - \frac{1}{2\pi}}{(\frac{1}{2})^2}\right] \qquad \left[\because \phi'(t) = -t\,\phi(t) \Rightarrow \phi'(0) = 0\right]$$

$$= 9 + 36\left(-\frac{1}{2\pi}\right)$$

$$= 9\left(1 - \frac{2}{\pi}\right) \quad \text{———} \quad \boxed{\text{2 marks}}$$

Solution 3.b.

Let $X = y_i$ w.p. p_i for $1 \le i \le n$.

$\sum_{i=1}^{n} p_i = 1$, $p_i > 0$, and $y_i > 0$.

Using Jensen's inequality with $f(x) = \sqrt{x}$,

$$E(\sqrt{X}) \le \sqrt{E(X)}$$

$$\Rightarrow \sum_{i=1}^{n} \sqrt{y_i}\,p_i \le \sqrt{\sum_{i=1}^{n} y_i p_i} \quad \text{———} \quad \boxed{\text{1 mark}}$$

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Take $y_i = Var(Y_i)$ and $p_i = \frac{b_i}{\sum_{i=1}^{n} b_i}$

$$\Rightarrow \frac{\sum_{i=1}^{n} b_i \sqrt{Var(Y_i)}}{\sum_{i=1}^{n} b_i} \leq \frac{\sqrt{\sum_{i=1}^{n} b_i Var(Y_i)}}{\sqrt{\sum_{i=1}^{n} b_i}}$$

$$\Rightarrow \sum_{i=1}^{n} b_i \sqrt{Var(Y_i)} \leq \sqrt{\sum_{i=1}^{n} b_i} \sqrt{\sum_{i=1}^{n} b_i Var(Y_i)}$$ — $\boxed{\text{1 mark}}$