Basics of Probability

- We want to assign probability to an event from the sample space; outcome of an experiment.

- Defn: • Distinct outcomes of an experiment form the sample space Ω.
  • Probability distribution function is a map $P : \Omega \to [0,1]$ s.t.
    i) $\sum_{\omega \in \Omega} P(\omega) = 1$ (i.e. some outcome happens)

  Subsets $S \subseteq \Omega$ are called events.

ii) $P(S) := \sum_{\omega \in S} P(\omega)$ [i.e. event S happens]

↳ $P(\phi) := 0$.

↳ this extends P to a map: $2^{\Omega} \to [0,1]$.

collection of subsets of Ω.

→ We're assuming finite Ω.

- Let us instantiate this in an example.

- Eg. You are in a casino & the dealer has cards numbered 1 to 1000. He asks you to pick a random card: If it's divisible by 2 or 5 then you win Rs. 100; else you lose Rs. 200.

- **Qn:** Is this a good bet for you?

- **Analyse:** • Sample space $\Omega := [1000]$.
  - Events are subsets of Ω, i.e. $\mathcal{E} := 2^{\Omega}$.
  - Favorable event $S := \{n \in \Omega : 2|n \vee 5|n\}$.
  - Card c is picked at random, so $P(\{c\}) := 1/1000$ [by (i)]

$\Rightarrow \forall S \in \mathcal{E},\ P(S) = |S|/1000$ [by (ii)].

$\triangleright\ |S| = \#\{n : 2|n\} + \#\{n : 5|n\} - \#\{n : 10|n\}$
$= 1000/2 + 1000/5 - 1000/10 = \underline{600}$.

$\Rightarrow \triangleright\ P(S) = 600/1000 = 0.6$; $P(S^c) = 0.4$.

$\triangleright$ Odds of win are $0.6 : 0.4 = 1.5 : 1$.

$\Rightarrow$ You should pay <u>less than 150/-</u> to have a favorable game!

— To do probability analysis, you need to define (Ω, P) pair.

- <u>Egs.</u>:
  - Toss a coin 5 times $\Rightarrow |\Omega| = 2^5 = 32$.
  - Throw 3 dices $\Rightarrow |\Omega| = 6^3$
- Put n letters in n envelopes $\Rightarrow |\Omega| = n! \approx n^{n/2}$.

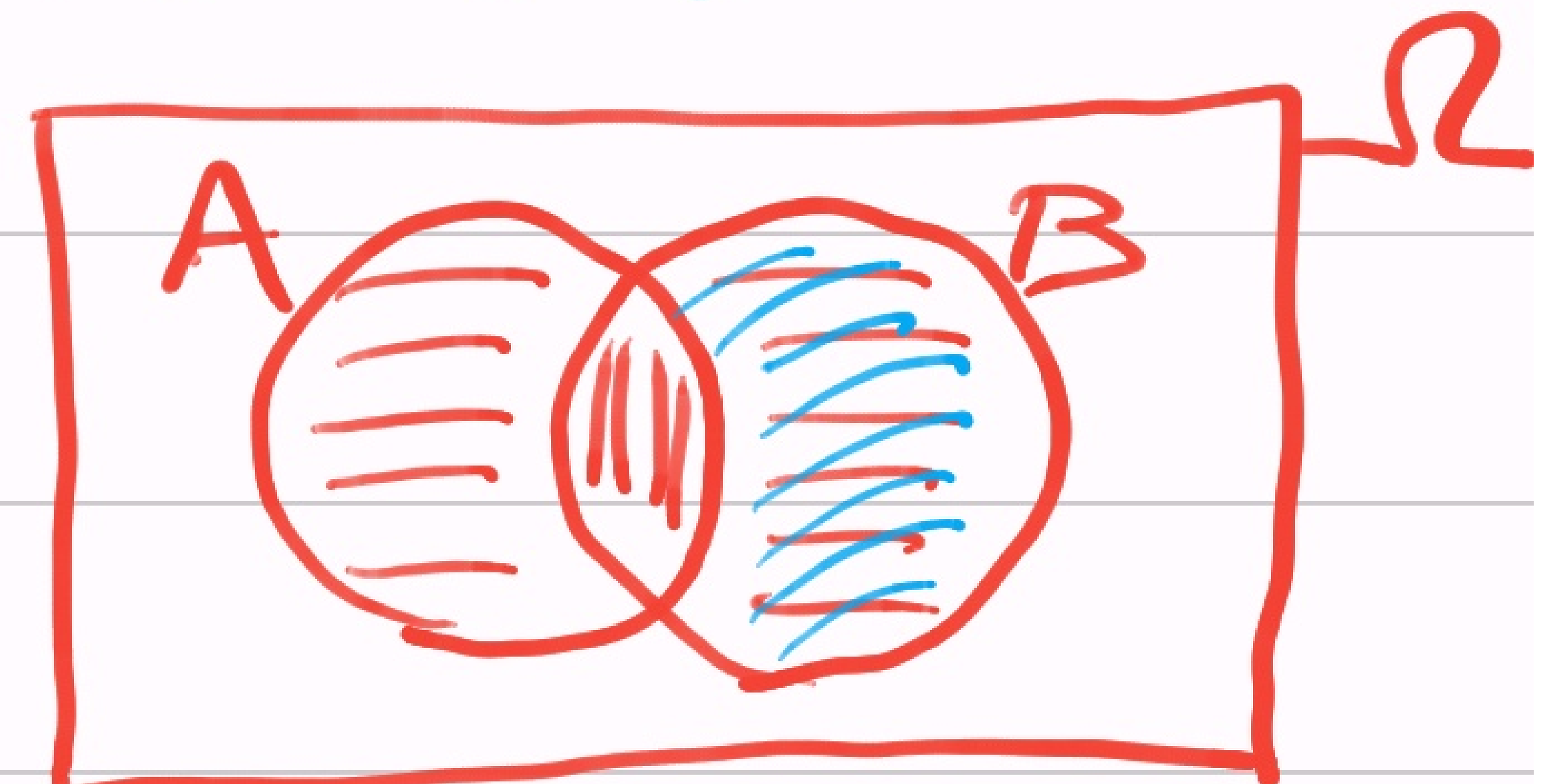
exponential-size spaces

Union, Intersection & Complement

- Since events are subsets, it's natural to look at all the set operations.

- Lemma 1: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: • By defn, $P(S) = \sum_{\omega \in S} P(\omega)$.



$$\Rightarrow P(A \cup B) = \sum_{\omega \in A \cup B} P(\omega)$$

$$= \sum_{\omega \in A} P(\omega) + \sum_{\omega \in B} P(\omega) - \sum_{\omega \in A \cap B} P(\omega)$$

$$= P(A) + P(B) - P(A \cap B).$$

$\square$

- **Lemma 2:** $P(A^c) = 1 - P(A)$.

Pf: • Lemma 1 $\Rightarrow$ $P(A \cup A^c) = P(A) + P(A^c) - P(A \cap A^c)$

$\Rightarrow$ $P(\Omega) = P(A) + P(A^c) - P(\phi)$

$\Rightarrow$ $1 = P(A) + P(A^c)$. $\square$

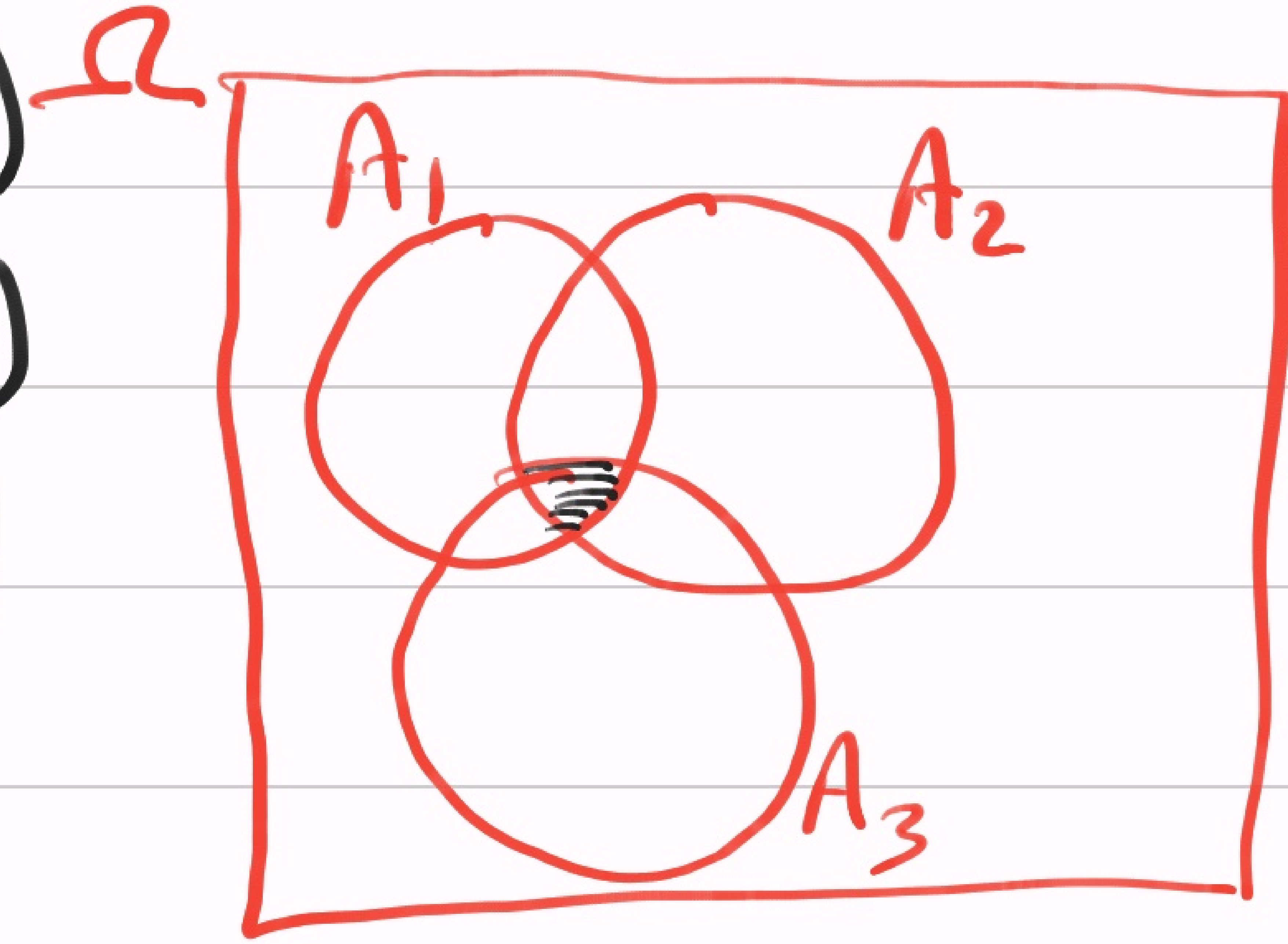
▷ $P(A \cup B) = P(A) + P(B - A) = P(B) + P(A - B)$.

▷ $P(A - B) = P(A) - P(B) \underbrace{+ P(B - A)}$.

• Recall $A \cap B = (A^c \cup B^c)^c$. Use this on $P(\cdot)$.

Qn: What about Union of 3 events?

$\triangleright$ $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$
$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3)$
$+ P(A_1 \cap A_2 \cap A_3)$. [Why?]



$\rightarrow$ The inclusion-exclusion for the probability fn. P is:

Lemma 3: $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq S \subseteq [n]} (-1)^{|S|-1} \cdot P\left(\bigcap_{i \in S} A_i\right)$.

Pf:

- Say, $\omega \in \bigcup_{i=1}^{n} A_i$ appears in exactly k of the A_i's. Wlog, $A_1, \ldots, A_k$.

• In RHS events, element w is counted exactly $\sum_{\phi \neq S \subseteq [k]} (-1)^{|S|-1} \cdot 1$ times. Qn: Is this $= 1$?

• Break the sum into $|S| = 1, 2, \ldots, k$:

$$= (-1)^{1-1} \cdot \binom{k}{1} + (-1)^{2-1} \cdot \binom{k}{2} + \cdots + (-1)^{k-1} \cdot \binom{k}{k}$$

$$= 1 - (1-1)^k = 1.$$

$\Rightarrow$ LHS = RHS. $\square$

Classic Eg.: You randomly assign n letters to n distinctly addressed envelopes. What is the chance that all letters get wrongly delivered?

[Such permutations are called <u>derangements</u>.]

<u>Analyse:</u>

- Sample space $\Omega :=$ permutations on n letters.

  (e.g. $l_1 l_3 l_2$ means l_2, l_3 are in wrong envelopes.)

- Favorable event $S := \{\pi \in \Omega \mid \forall i,\ \pi(i) \neq i\}$
- Prob. distribution fn. $P(\{\pi\}) = 1/n!$.

  $\Rightarrow P(S) = |S|/n!$. $\rightarrow$ Let's flip the problem:

- Let $A_i := \{\pi \in \Omega \mid \pi(i) = i\}$, i.e. ith letter correct.

$\triangleright \quad S = \left(\bigcup_{i \in [n]} A_i\right)^c$.

$\Rightarrow$ Suffices to find $\quad P\left(\bigcup_{i \in [n]} A_i\right) = ?$

→ Apply inclusion-exclusion:

$$P\Big(\bigcup_i A_i\Big) = \sum_{\emptyset \neq S \subseteq [n]} (-1)^{|S|+1} \cdot P\Big(\bigcap_{i \in S} A_i\Big)$$ ← letters in S correctly placed.

$$= \sum_S (-1)^{|S|+1} \cdot \frac{(n-|S|)!}{n!}$$ ← rest of the places in π are "free".

→ How to simplify it? Rewrite wrt $|S| =: k$.

$$= \sum_{k \in [n]} (-1)^{k+1} \cdot \frac{(n-k)!}{n!} \times \binom{n}{k} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$$

▷ $$P(S) = 1 - P\Big(\bigcup_i A_i\Big) = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots -$$

$$= e^{-1} \text{ as } n \to \infty$$

$\approx 0.3678\ldots$ quite a high probability!

Exercise: What's the prob. that k letters are placed correctly, & the rest not?

Probability for infinite Ω

- Here, we may not be able to go one element at a time. e.g. $\sum_{w \in S} P(w)$ may be undefined!

Instead we work with subsets/events directly.

Defn: • Collection Σ of subsets of Ω is called sigma-algebra if:

(i) $\Omega \in \Sigma$

(ii) Σ closed under complement $[A \in \Sigma \Rightarrow A^c \in \Sigma]$

(iii) " " " countable union.

[A <u>countable</u> infinite index set I is in bijection with $\mathbb{N}$, $\mathbb{Z}$ & $\mathbb{Q}$.]

[... but <u>not</u> with $\mathbb{R}$!]

So, $\bigcup_{i \in \mathbb{N}} A_i$ is allowed inside $\mathcal{E}$.

- <u>Probability distribution</u> function $P : \mathcal{E} \longrightarrow [0,1]$ that satisfies: (i) $P(\Omega) = 1$.

  (ii) $\forall$ <u>disjoint</u> $A, B \in \mathcal{E}$: $P(A \cup B) = P(A) + P(B)$.

<u>Exercise</u>: • Above defines $P(\bigcup_{i \in I} A_i)$, for <u>countable</u> I.

• " " $P(\bigcap_{i \in I} A)$, " " ".

• $\forall A, B \in \mathcal{E}$: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- Egs. of sigma-algebra: • $\mathcal{F} = \{\phi, \Omega\}$

• $\mathcal{F} = \{\phi, \Omega, A, A^c\}$, for any $A \subseteq \Omega$.

P(A) may not be 1/2 !

• $\mathcal{F} = 2^{\Omega} := \{A \mid A \subseteq \Omega\}$

$\left(:= \{ f : \Omega \to \{0,1\} \} \right)$

- Eg. with $|\Omega| = \infty$: Pick "random" $x \in [0,1]$. What's the chance that $x \leq 0.5$?

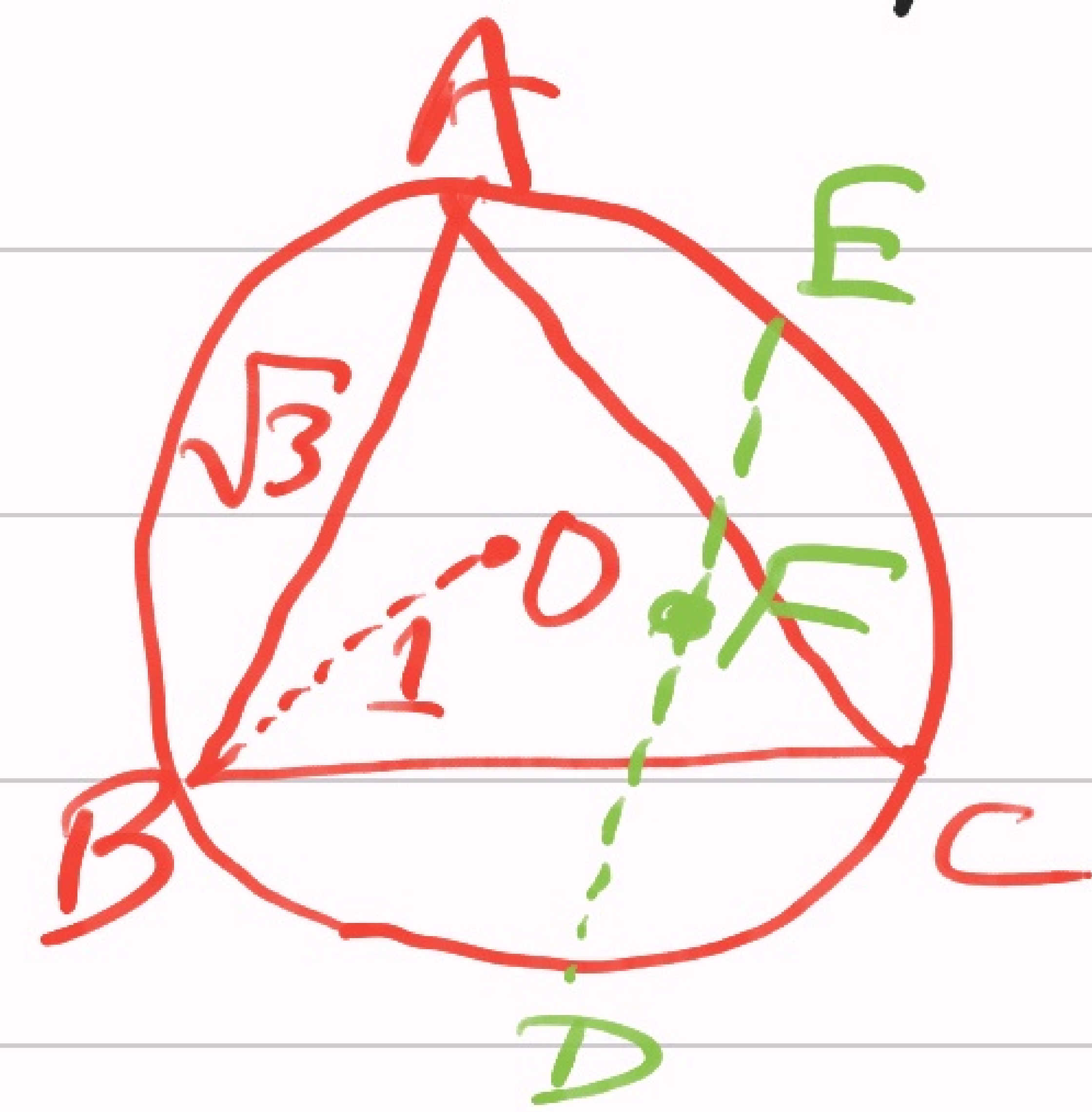
• $\underline{\mathcal{F}} := \{ \phi, \Omega := [0,1], \underbrace{[0, 0.5]}_{=:A}, \underbrace{[0.5, 1]}_{= A^c} \}$

• Intuitive $P(\cdot)$ is: $P(A) = 1/2$ ← uniform distribution

• Consistent with:

$\mathcal{F} := \{[a,b] \mid 0 \leq a \leq b \leq 1\} \cup \{\phi\}$, with $P([a,b]) = (b-a)$.

- The prior example of random chord DE in a unit circle: $P(|DE| \geq \sqrt{3}) = ?$



• $\Omega_1 := \{(D=A, E) \in \overline{Circ}^2\}$ ← boundary of Circ.

$\mathcal{F}_1 := \{\phi, \Omega_1, \underbrace{\{(A,E) \mid |AE| \geq \sqrt{3}\}}_{=:A_1}, A_1^c\}$.

$P_1(\cdot)$ uniform on Ω_1.

• $\Omega_2 := \{F \in Circ\}$ ← area of Circ.

$\mathcal{F}_2 := \{\phi, \Omega_2, \underbrace{\{F \mid |DE| \geq \sqrt{3}\}}_{=:A_2}, A_2^c\}$.

$P_2(\cdot)$ uniform on Ω_2.

- $\Omega_3 := \{ OF \mid F \in \text{Circ} \}$

  $\mathcal{F}_3 := \{ \phi, \Omega_3, \underbrace{\{ OF \mid |DE| \geq \sqrt{3} \}}_{=:A_3}, A_3^c \}$.

  P_3 <u>uniform</u> on Ω_3 wrt $|OF|$.

$\Rightarrow$ Thus the probabilities $\frac{1}{3}, \frac{1}{4}, \frac{1}{2}$ (resp.) are all correct!

$\triangleright$ Defn. of (Ω, P) is critical!