Assignment RM5: Extended Vasicek model Monte Carlo simulation for CCR estimation

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1 Case Study 1: Risk Measures under extended Vasicek model for short-rate

1.1 Part 1: Computation of the discount factor curve for every simulation date

We compute the discount factor curve $B(s,\tau)$ using the Hull-White model's affine term-structure form. For each simulation date s, the function `affine_trick` calculates $A(s,\tau)$ and $C(s,\tau)$. The term $C(s,\tau) = \frac{1-e^{-a(\tau-s)}}{a}$ models mean reversion, while $A(s,\tau)$ combines the ratio of bootstrapped discount factors ($\frac{B(t_0,\tau)}{B(t_0,s)}$) with a numerical evaluation of the volatility integral: $\int_{t_0}^{s} \left(\sigma(u,\tau)^2 - \sigma(u,s)^2\right) du$, computed using `scipy.integrate.quad`. This replaces analytical approximations for flexibility in handling non-constant volatility. These terms are then combined to construct $B(s,\tau) = A(s,\tau)e^{-C(s,\tau)x(t)}$, where $x(t)$ is the simulated short rate.

1.2 Derivation of the Discount Factor under the Extended Vasicek Model

We consider the Hull-White (Extended Vasicek) model for the short rate r_t, given by:

$$dr_t = (\theta(t) - ar_t)\,dt + \sigma dW_t,$$

We decompose r_t into a deterministic and a stochastic part:

$$r_t = \phi(t) + x_t,$$

where: x_t satisfies the Ornstein–Uhlenbeck process: $dx_t = -ax_t dt + \sigma dW_t, \quad x_0 = 0.$

1.2.1 To prove

We aim to derive the discount factor:

$$B(s,T) := \mathbb{E}_s\left[\exp\left(-\int_s^T r_u du\right)\right],$$

under the given model.

Substituting $r_u = \phi(u) + x_u$, we obtain:

$$B(s,T) = \mathbb{E}_s\left[\exp\left(-\int_s^T \phi(u)du - \int_s^T x_u du\right)\right].$$

Since $\phi(u)$ is deterministic, so since expectation is a linear operator:

$$B(s,T) = \exp\left(-\int_s^T \phi(u)du\right) \cdot \mathbb{E}_s\left[\exp\left(-\int_s^T x_u du\right)\right].$$

1.2.2 Solution to the Ornstein–Uhlenbeck Process

The solution to the OU process is:

$$x_u = x_s e^{-a(u-s)} + \sigma \int_s^u e^{-a(u-v)} dW_v.$$

Integrating over $u \in [s, T]$, we get:

$$\int_s^T x_u du = x_s C(s,T) + \sigma \int_s^T \int_s^u e^{-a(u-v)} dW_v du.$$

Where:

$$\int_s^T e^{-a(u-s)} du = \frac{1 - e^{-a(T-s)}}{a} := C(s,T).$$

1.2.3 Evaluating the Expectation

Let:

$$Y := \int_s^T x_u du = x_s C(s,T) + \text{Gaussian random variable}.$$

Then:

$$\mathbb{E}_s\left[\exp(-Y)\right] = \exp\left(-x_s C(s,T) + \frac{1}{2}\text{Var}\left[\text{Gaussian random variable}\right]\right).$$

We write

$$\int_s^T x_u\, du = x_s C(s,T) + \sigma \int_s^T C(v,T)\, dW_v,$$

where the second term is Gaussian. Then, by using ito's isometry we get:

$$\mathbb{E}_s\left[\exp\left(-\int_s^T x_u\, du\right)\right] = \exp\left(-x_s C(s,T) + \frac{1}{2}\sigma^2 \int_s^T C(v,T)^2\, dv\right).$$

1.2.4 Final Expression for the Discount Factor

Putting all together:

$$B(s,T) = \exp\left(-\int_s^T \phi(u) du\right) \cdot \exp\left(-x_s C(s,T) + \frac{1}{2}\sigma^2 \int_s^T C(v,T)^2 dv\right).$$

Grouping the deterministic terms into a function $A(s,T)$, define:

$$A(s,T) := \exp\left(-\int_s^T \phi(u) du + \frac{1}{2}\sigma^2 \int_s^T C(v,T)^2 dv\right),$$

and

$$C(s,T) := \frac{1 - e^{-a(T-s)}}{a}.$$

Then the final form is: $B(s,T) = A(s,T) \cdot \exp\left(-x_s C(s,T)\right).$ ■

1.3 Part 2: Expected Exposure (EE) Profile

The EE profile is derived by simulating 250,000 paths of the short rate $x(t)$ under the Hull-White dynamics. At each future date, the swap's mark-to-market (MtM) is calculated using the precomputed $A(s,\tau)$ and $C(s,\tau)$, with cash flows discounted along each simulated path. Exposure is defined as $\max(\text{MtM}, 0)$, and the EE at each date is the average of these exposures across all paths. The code includes a normalization step where the EE is divided by the probability of positive MtM $\mathbb{P}(\text{MtM} > 0)$ to condition the exposure on scenarios where the swap is in-the-money. This ensures the EE reflects the expected loss given default. Lastly, this is a normalised conditional EE (the average given positive MtM), and not the standard EE used in the CCR metrics.

1.3.1 Remark:

The resulting vector can be viewed in the corresponding attached jupyter notebook.

1.4 Part 3: 95% Potential Future Exposure (PFE) Profile

For each of the 32 future dates, the 95% PFE is computed by sorting the 250,000 simulated exposures $\max(\text{MtM}, 0)$ in ascending order and extracting the 95th percentile value using `np.percentile(positive_MtM, 95)`. This represents the worst-case exposure at a 95% confidence level, i.e., the level not exceeded by 95% of the simulated paths. The PFE profile highlights the time-varying tail risk of the swap, with higher PFE values indicating periods of heightened counterparty risk.

1.4.1 Remark:

The resulting vector can be viewed in the corresponding attached jupyter notebook.

1.5 Part 4: Expected Positive Exposure (EPE)

The EPE is calculated as the time-average of the EE profile across all 32 dates. This provides a single metric representing the average exposure over the swap's lifetime. The ratio of EPE to the swap's notional is derived to serve as a simplified parameter for approximating EPE without full simulations. For a receiver IRS, this ratio is not directly applicable because the exposure profile flips: the payer's EPE corresponds to the receiver's Expected Negative Exposure (ENE). The receiver's exposure would involve $\max(-\mathrm{MtM}, 0)$, requiring a separate parameter to capture its risk profile.

1.5.1 Formulation Used:

EPE is the time-averaged EE over 32 dates:

$$\mathrm{EPE} = \frac{1}{32} \sum_{i=1}^{32} \mathrm{EE}(t_i).$$

1.5.2 Results:

- **EPE(Collateral):** 0.005095 (reduction: 58%)
- **EPE(Without Collateral):** 0.012144

1.6 Part 5: Peak-PFE using PFE profile

The Peak-PFE is identified as the maximum value across all 32 PFE values in the PFE profile. This is computed using `np.max(potential_future_exposure)`, which scans the 95th percentile exposures at each date and selects the highest value. The Peak-PFE represents the most severe exposure the bank faces over the swap's lifetime at the 95% confidence level, serving as a critical metric for capital allocation and risk limits.

1.6.1 Result:

- **Peak-PFE(Collateral):** 0.109501 (reduction: 85.9%)
- **Peak-PFE(Without Collateral):** 0.776836

1.7 Part 6:

We assume collateral is posted annually, resetting exposure to zero on posting dates. At these dates, exposure is artificially set to zero (via a conditional check), reducing the EE and PFE metrics. However, this implementation simplifies collateral mechanics: it does not dynamically track collateral balances, thresholds or remuneration at the risk-free rate. The current logic highlights the directional impact of collateral (lower exposures) but lacks the granularity to capture real-world collateral agreements.