

# Random Numbers

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CS21BTECH11004

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## 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

### Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/1/
coeffs.h
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/1/1_1.c
```

Execute the above files using code,

```
gcc 1_1.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

### Solution:

Download the following files

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/1/1_2.py
```

Execute the code using command

```
python3 1_2.py
```

Plot 1.1 is obtained,

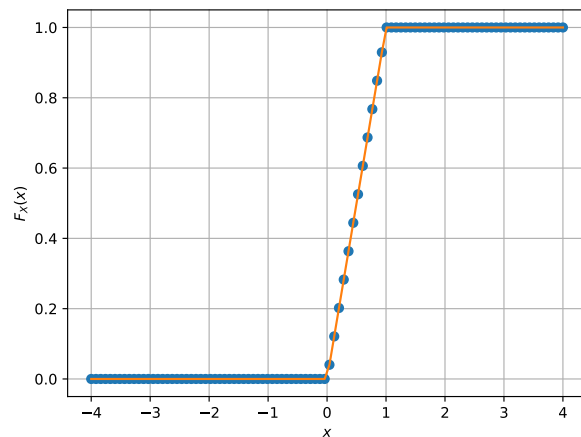


Fig. 1.1

- 1.3 Find a theoretical expression for  $F_U(x)$ .

### Solution:

Pdf of Uniform distribution between  $[0,1]$  is given by,

$$f_U(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases} \quad (1.2)$$

$$F_U(x) = \int_{-\infty}^x f_U(x) dx \quad (1.3)$$

Case-1:  $x < 0$ ,

$$F_U(x) = \int_{-\infty}^x 0 dx \quad (1.4)$$

$$= 0 \quad (1.5)$$

Case-2:  $x \in [0,1]$ ,

$$F_U(x) = \int_{-\infty}^0 0 dx + \int_0^x 1 dx \quad (1.6)$$

$$= x \quad (1.7)$$

Case-3:  $x > 1$ ,

$$F_U(x) = \int_{-\infty}^0 0dx + \int_0^1 1dx + \int_1^x 0dx \quad (1.8)$$

$$= 1 \quad (1.9)$$

Hence,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0, 1] \\ 1, & x > 1 \end{cases} \quad (1.10)$$

1.4 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.11)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.12)$$

Write a C program to find the mean and variance of  $U$ .

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/1/
coeffs.h
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/1/1_4.c
```

Execute the above files using code,

```
gcc 1_4.c -lm
./a.out
```

The following result is obtained

$$E[U] = 0.500007$$

$$\text{var}[U] = 0.083301$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.13)$$

**Solution:**

$F_U(x)$  for uniform distribution,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0, 1] \\ 1, & x > 1 \end{cases} \quad (1.14)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.15)$$

$$= \int_{-\infty}^0 0dx + \int_0^1 xdx + \int_1^{\infty} 0dx \quad (1.16)$$

$$= \frac{1}{2} \quad (1.17)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.18)$$

$$= \int_{-\infty}^0 0dx + \int_0^1 x^2 dx + \int_1^{\infty} 0dx \quad (1.19)$$

$$= \frac{1}{3} \quad (1.20)$$

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \quad (1.21)$$

From Equation (1.17) and (1.20),

$$E[U - E[U]]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.22)$$

$$= \frac{1}{3} - \frac{1}{4} \quad (1.23)$$

$$= \frac{1}{12} \approx 0.083 \quad (1.24)$$

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/1/
coeffs.h
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/2/2_1.c
```

Execute the above files using code,

```
gcc 2_1.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does a CDF have?

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/2/2_2.py
```

Execute the above file using code,

```
python3 2_2.py
```

Plot 2.1 obtained is symmetric about (0,0.5)

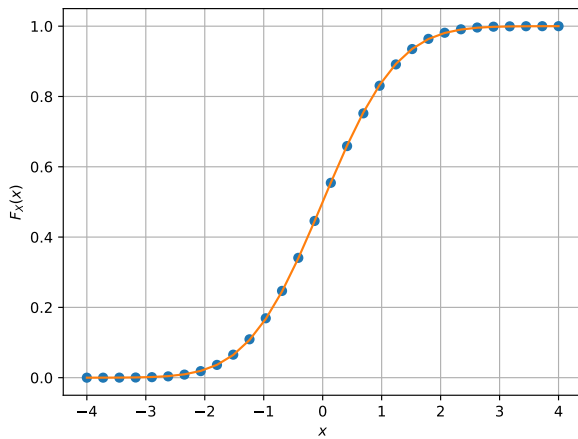


Fig. 2.1

- 2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples in gau.dat. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/2/2_3.py
```

Execute the above file using code,

```
python3 2_3.py
```

Plot 2.2 obtained is symmetric about y-axis

- 2.4 Find the mean and variance of  $X$  by writing a C program.

**Solution:**

Download the following files,

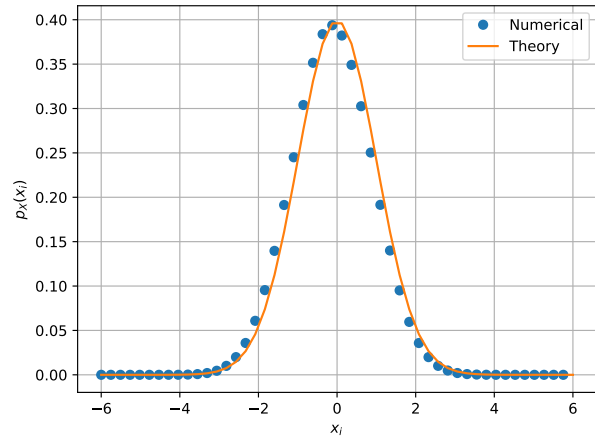


Fig. 2.2

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/1/
coeffs.h
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/2/2_4.c
```

Execute the above files using code,

```
gcc 2_4.c -lm
./a.out
```

The result is,

$$E[U] = 0.000294 \quad (2.3)$$

$$\text{var}[U] = 0.999561 \quad (2.4)$$

- 2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

**Solution:**

$$E[U] = \int_{-\infty}^{\infty} u p_X(u) du \quad (2.6)$$

$$(2.7)$$

$u p_X(u)$  is a odd function.

$$E[U] = 0 \quad (2.8)$$

$$(2.9)$$

$$E[U^2] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \quad (2.10)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \left( u \exp\left(-\frac{u^2}{2}\right) \right) du \quad (2.11)$$

Using integration by parts,

$$E[U^2] = -u \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \quad (2.12)$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \quad (2.13)$$

$$= 1 \quad (2.14)$$

Hence,

$$\text{var}[U] = E[U^2] - [E[U]]^2 \quad (2.15)$$

$$= 1 \quad (2.16)$$

### 3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/
coeffs.h
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/3/3_1.c
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/3/3_1.py
```

Execute the above files using code,

```
gcc 3_1.c -lm
./a.out
python3 3_1.py
```

Plot 3.1 is obtained

3.2 Find a theoretical expression for  $F_V(x)$ .

**Solution:**

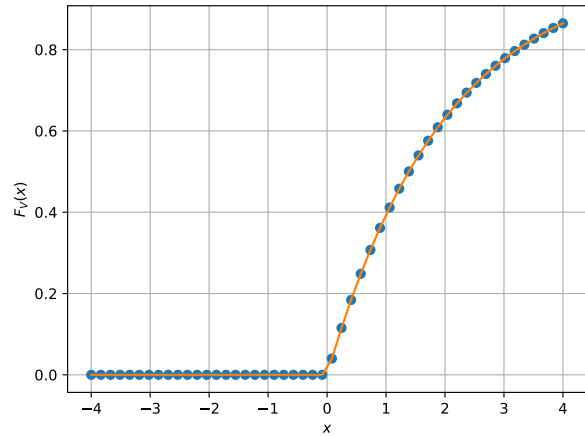


Fig. 3.1

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(U \leq 1 - e^{-\frac{x}{2}}\right) \quad (3.4)$$

$$F_V(x) = F_U\left(1 - e^{-\frac{x}{2}}\right) \quad (3.5)$$

$$= \begin{cases} 0, & 1 - e^{-\frac{x}{2}} < 0 \\ 1 - e^{-\frac{x}{2}}, & 1 - e^{-\frac{x}{2}} \in [0, 1] \\ 1, & 1 - e^{-\frac{x}{2}} > 1 \end{cases} \quad (3.6)$$

Now,

$$1 - e^{-\frac{x}{2}} < 0 \quad (3.7)$$

$$\Rightarrow x < 0 \quad (3.8)$$

$$0 \leq 1 - e^{-\frac{x}{2}} \leq 1 \quad (3.9)$$

$$\Rightarrow x \geq 0 \quad (3.10)$$

$$1 < 1 - e^{-\frac{x}{2}} \quad (3.11)$$

$$\Rightarrow x \in \phi \quad (3.12)$$

Hence,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \geq 0 \end{cases} \quad (3.13)$$

### 4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/
coeffs.h
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/4/4_1.c
```

Execute the above files using code,

```
gcc 4_1.c -lm
./a.out
```

#### 4.2 Find the CDF of $T$ .

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/4/4_2.py
```

Execute the above files using code,

```
python3 4_2.py
```

Plot 4.1 is obtained

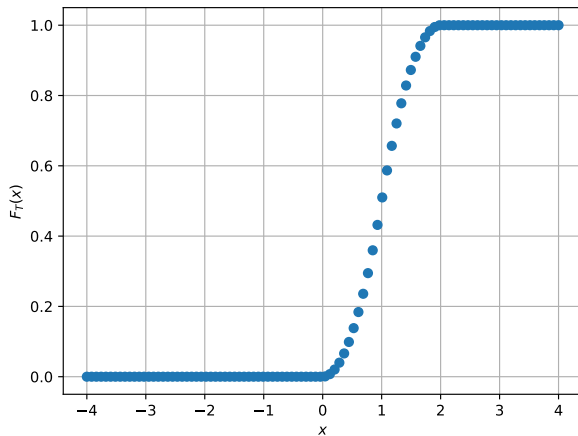


Fig. 4.1

#### 4.3 Find the PDF of $T$ .

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/4/4_3.py
```

Execute the above files using code,

```
python3 4_3.py
```

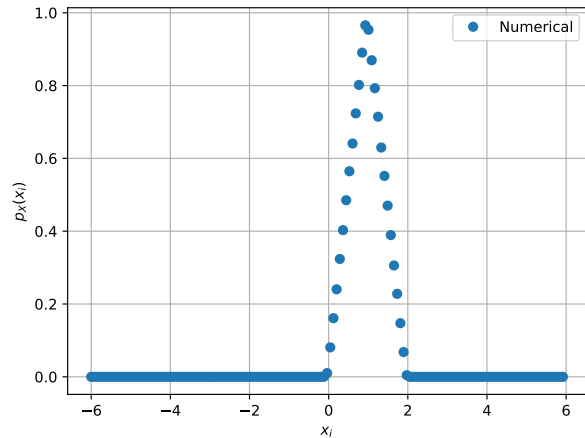


Fig. 4.2

Plot 4.2 is obtained

#### 4.4 Find the theoretical PDF and CDF of $T$ .

**Solution:**

The CDF of  $T$  is given by

$$F_T(t) = \Pr(T \leq t) = \Pr(U_1 + U_2 \leq t) \quad (4.2)$$

Since  $U_1, U_2 \in [0, 1] \implies U_1 + U_2 \in [0, 2]$   
Therefore, if  $t \geq 2$ , then  $U_1 + U_2 \leq t$  is always true and if  $t < 0$ , then  $U_1 + U_2 \leq t$  is always false.

Now, fix the value of  $U_1$  to be some  $x$

$$x + U_2 \leq t \implies U_2 \leq t - x \quad (4.3)$$

If  $0 \leq t \leq 1$ , then  $x$  can take all values in  $[0, t]$

$$F_T(t) = \int_0^t \Pr(U_2 \leq t - x) p_{U_1}(x) dx \quad (4.4)$$

$$= \int_0^t F_{U_2}(t - x) p_{U_1}(x) dx \quad (4.5)$$

$$0 \leq x \leq t \implies 0 \leq t - x \leq t \leq 1 \quad (4.6)$$

$$\implies F_{U_2}(t - x) = t - x \quad (4.7)$$

$$F_T(t) = \int_0^t (t - x) \cdot 1 \cdot dx \quad (4.8)$$

$$= tx - \frac{x^2}{2} \Big|_0^t \quad (4.9)$$

$$= \frac{t^2}{2} \quad (4.10)$$

If  $1 < t < 2$ ,  $x$  can only take values in  $[0, 1]$  as

$$U_1 \leq 1$$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot dx \quad (4.11)$$

$$0 \leq x \leq t-1 \implies 1 \leq t-x \leq t \quad (4.12)$$

$$t-1 \leq x \leq 1 \implies 0 < t-1 \leq t-x \leq 1 \quad (4.13)$$

$$F_T(t) = \int_0^{t-1} 1 dx + \int_{t-1}^1 (t-x) dx \quad (4.14)$$

$$= t-1 + t(1-(t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \quad (4.15)$$

$$= t-1 + 2t - t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \quad (4.16)$$

$$= -\frac{t^2}{2} + 2t - 1 \quad (4.17)$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \quad (4.18)$$

The PDF of  $T$  is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \quad (4.19)$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \quad (4.20)$$

4.5 Verify your results through a plot.

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/4/4_5_1.py
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/4/4_5_2.py
```

Execute the above files using code,

```
python3 4_5_1.py
python3 4_5_2.py
```

Plot 4.3 is obtained is pdf Plot 4.4 is obtained

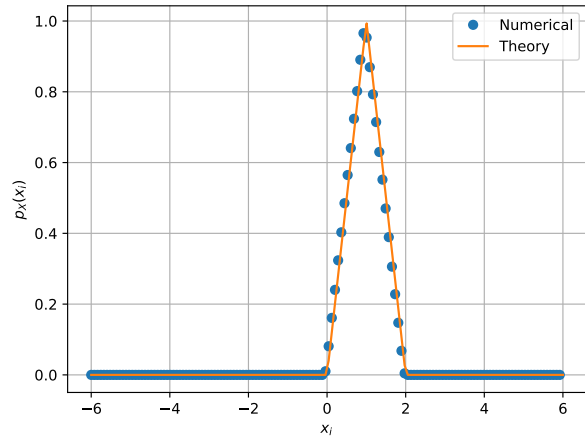


Fig. 4.3

is cdf

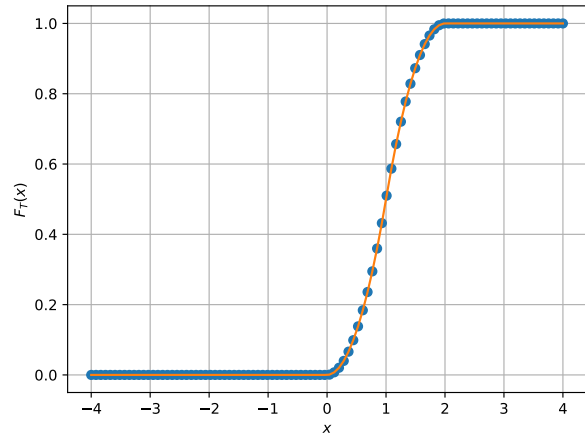


Fig. 4.4

## 5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ .

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/5/5.1.c
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/
coeffs.h
```

Execute the above files using code,

```
gcc 5.1.c -lm
./a.out
```

### 5.2 Generate

$$Y = AX + N \quad (5.1)$$

where  $A = 5$  dB,  $X \in \{1, -1\}$  is Bernoulli and  $N \sim \mathcal{N}(0, 1)$ .

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/5/5.2.c
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution/
coeffs.h
```

Execute the above files using code,

```
gcc 5.2.c -lm
./a.out
```

### 5.3 Plot Y using a scatter plot.

**Solution:**

Download the following files,

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/5/5.3.py
```

Execute the above files using code,

```
python3 5.3.py
```

Plot 5.1 is obtained

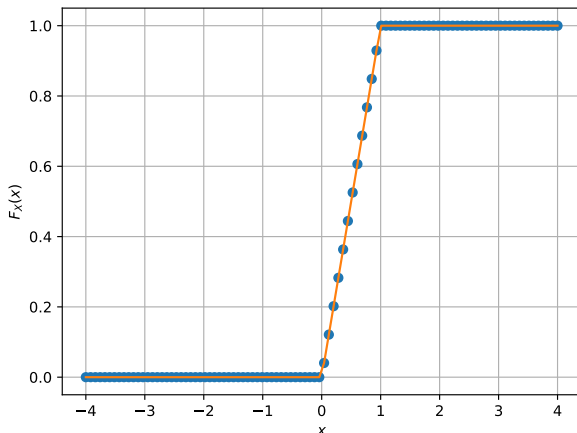


Fig. 5.1

### 5.4 Guess how to estimate $X$ from $Y$ .

**Solution:**

From the plot of  $Y$ , we see that the estimate model can be written as

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ 0 & Y < 0 \end{cases} \quad (5.2)$$

### 5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.3)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.4)$$

**Solution:**

Letting  $X = 1$  and  $X = -1$  respectively, we see the number of mismatched data points to compute the error probabilities. The simulation is coded in

```
$ wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/5/5.5.py
```

and can be run by typing

```
$ python3 5.5.py
```

The results are

$$P_{e|0} = 5.02 \times 10^{-4} \quad (5.5)$$

$$P_{e|1} = 5.24 \times 10^{-4} \quad (5.6)$$

### 5.6 Find $P_e$ assuming that $X$ has equiprobable symbols.

**Solution:**

Here,  $\Pr(X = 1) = \Pr(X = -1) = 0.5$ . Thus,

$$P_e = \Pr(X = 1) P_{e|1} + \Pr(X = -1) P_{e|0} \quad (5.7)$$

$$= \frac{1}{2} (P_{e|0} + P_{e|1}) = 5.13 \times 10^{-4} \quad (5.8)$$

### 5.7 Verify by plotting the theoretical $P_e$ wrt $A$ from 0 dB to 10 dB.

**Solution:**

$$P_{e|0} = \Pr(\hat{X} = 1|X = -1) \quad (5.9)$$

$$= \Pr(Y > 0|X = -1) \quad (5.10)$$

$$= \Pr(AX + N > 0|X = -1) \quad (5.11)$$

$$= \Pr(N > A) = Q(A) \quad (5.12)$$

since  $X$  and  $N$  are independent. Writing a

similar expression for  $P_{e|1}$  and noting that

$$\Pr(N < -A) = \Pr(N > A) = Q(A) \quad (5.13)$$

it follows that  $P_e = Q(A)$ . This is the idea used to plot the theoretical  $P_e$ . The plot is coded both in the rectangular axes and the semilog-y axes. Download the relevant codes using

```
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/5/5.7.py
wget https://github.com/Anshul-Sangrame/
AI1110/blob/main/Assignment/solution
/5/5.7_semilog.py
```

Execute the above files using code,

```
python3 5.7.py
python3 5.7_semilog.py
```

Plot 5.2 obtained in rectangular axis

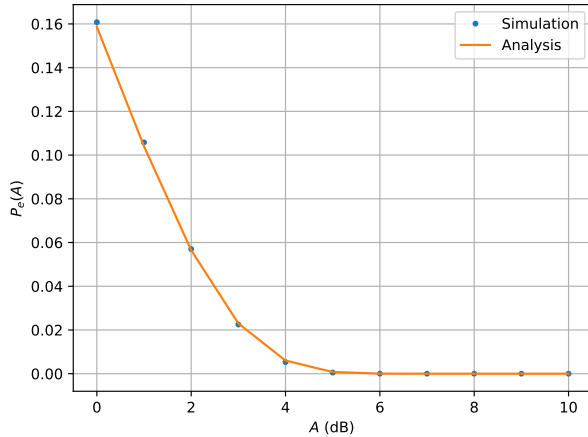


Fig. 5.2

Plot 5.3 obtained in semilog-y axis

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that maximizes the theoretical  $P_e$ .

**Solution:**

To estimate  $X$  from  $Y$ , we now consider the following:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \quad (5.14)$$

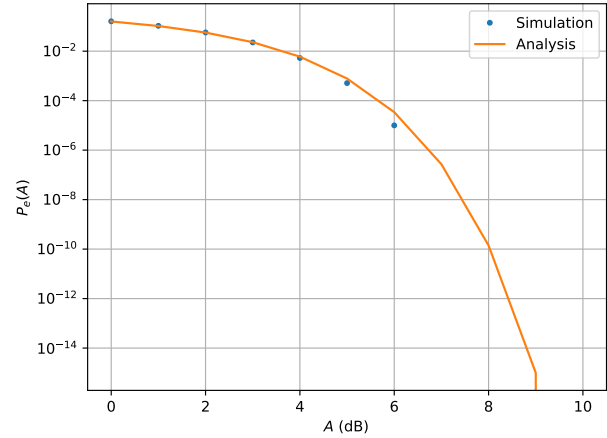


Fig. 5.3

Therefore,

$$P_e = \Pr(X = -1) Q(A + \delta) + \Pr(X = 1) Q(A - \delta) \quad (5.15)$$

$$= \frac{1}{2} (Q(A + \delta) + Q(A - \delta)) \quad (5.16)$$

To minimise  $P_e$ , we differentiate the above equation wrt  $\delta$ :

$$0 = \frac{d}{d\delta} \left( \frac{1}{2} (Q_N(A - \delta) + Q_N(A + \delta)) \right) \quad (5.17)$$

$$= \frac{1}{2} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \right) \quad (5.18)$$

Therefore,

$$(\delta - A)^2 = (A + \delta)^2 \quad (5.19)$$

$$\Rightarrow \delta = 0 \quad (5.20)$$