Random Numbers

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CS21BTECH11004

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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /1/1_1.c
```

Execute the above files using code,

```
gcc 1_1.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution:
Download the following files

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /1/1_2.py
```

Execute the code using command

```
python3 1_2.py
```

Plot 1.1 is obtained,

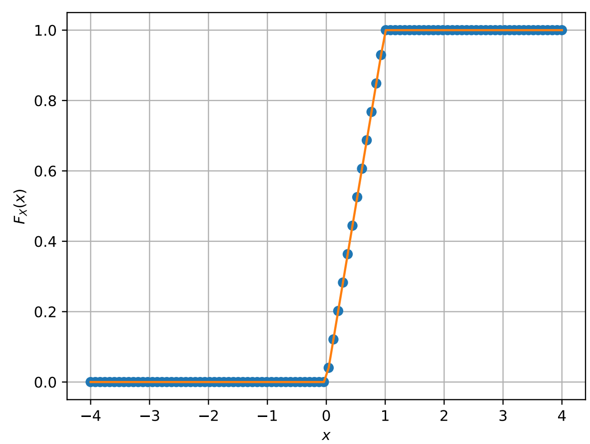

Fig. 1.1

1.3 Find a theoretical expression for $F_U(x)$.
Solution:
Pdf of Uniform distribution between [0,1] is given by,

$$f_U(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)dx \tag{1.3}$$

Case-1: x < 0,

$$F_U(x) = \int_{-\infty}^{x} 0dx \tag{1.4}$$

$$= 0 \tag{1.5}$$

Case-2: x ∈ [0,1],

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{x} 1dx \tag{1.6}$$
$$= x \tag{1.7}$$

Case-3: x > 1,

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{x} 0dx \tag{1.8}$$
$$= 1 \tag{1.9}$$

Hence,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{1.10}$$

1.4 The mean of U is defined as

$$E\left[U\right] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.11}$$

and its variance as

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \tag{1.12}$$

Write a C program to find the mean and variance of U.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /1/1_4.c
```

Execute the above files using code,

```
gcc 1_4.c −lm
./a.out
```

The following result is obtained

$$E\left[U\right] = 0.500007$$
$$\text{var}\left[U\right] = 0.083301$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \tag{1.13}$$

Solution:

$F_U(x)$ for uniform distribution,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{1.14}$$

$$E[U] = \int_{-\infty}^{\infty} xdF_U(x) \tag{1.15}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} xdx + \int_{1}^{\infty} 0dx \tag{1.16}$$
$$= \frac{1}{2} \tag{1.17}$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.18}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} x^2dx + \int_{1}^{\infty} 0dx \tag{1.19}$$
$$= \frac{1}{3} \tag{1.20}$$

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \tag{1.21}$$

From Equation (1.17) and (1.20),

$$E[U - E[U]]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.22}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.23}$$
$$= \frac{1}{12} \approx 0.083 \tag{1.24}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_1.c
```

Execute the above files using code,

```
gcc 2_1.c −lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_2.py
```

Execute the above file using code,

```
python3 2_2.py
```

Plot 2.1 obtained is symmetric about (0,0.5)

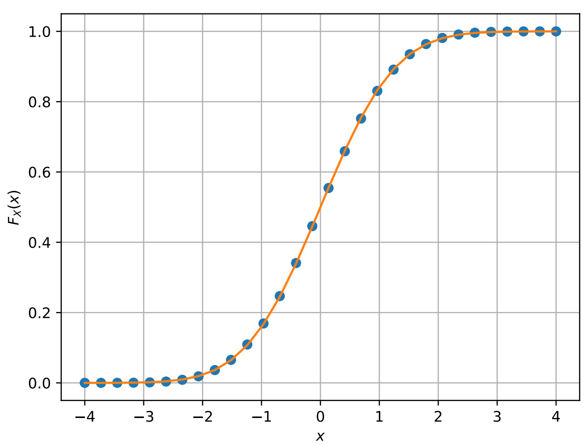

Fig. 2.1

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_3.py
```

Execute the above file using code,

```
python3 2_3.py
```

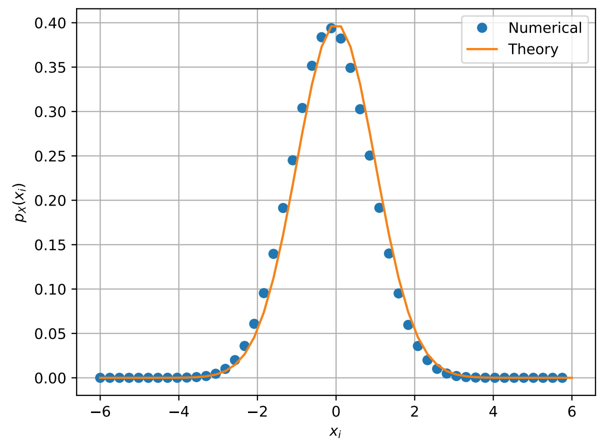

Fig. 2.2

Plot 2.2 obtained is symmetric about y-axis

2.4 Find the mean and variance of X by writing a C program.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/soltion/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_4.c
```

Execute the above files using code,

```
gcc 2_4.c −lm
./a.out
```

The result is,

$$E[U] = 0.000294 \tag{2.3}$$
$$\text{var}[U] = 0.999561 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution:

$$E[U] = \int_{-\infty}^{\infty} u p_X(u) du \tag{2.6}$$
$$\tag{2.7}$$

$up_X(u)$ is a odd function.

$$E[U] = 0 \tag{2.8}$$

$$\tag{2.9}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \tag{2.10}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\left(u \exp\left(-\frac{u^2}{2}\right)\right) du \tag{2.11}$$

Using integration by parts,

$$E\left[U^2\right] = -u\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \tag{2.12}$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \tag{2.13}$$

$$= 1 \tag{2.14}$$

Hence,

$$\text{var}[U] = E\left[U^2\right] - [E[U]]^2 \tag{2.15}$$

$$= 1 \tag{2.16}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.

Solution:

Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /3/3_1.c
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /3/3_1.py
```

Execute the above files using code,

```
gcc 3_1.c −lm
./a.out
python3 3_1.py
```

Plot 3.1 is obtained

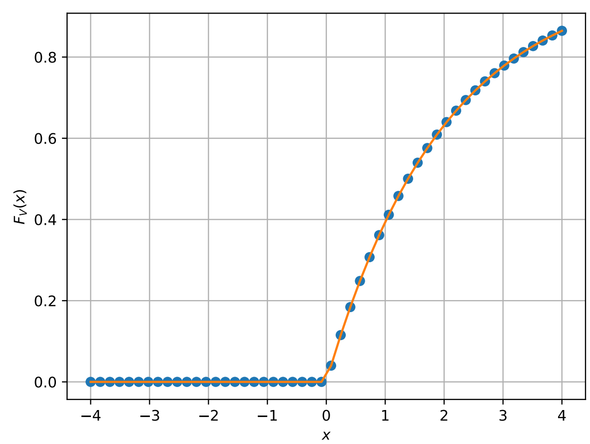

Fig. 3.1

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = Pr(V \le x) \tag{3.2}$$

$$= Pr(-2\ln(1-U) \le x) \tag{3.3}$$

$$= Pr\left(U \le 1 - e^{-\frac{x}{2}}\right) \tag{3.4}$$

$$F_V(x) = F_U\left(1 - e^{-\frac{x}{2}}\right) \tag{3.5}$$

$$= \begin{cases} 0, & 1 - e^{-\frac{x}{2}} < 0 \\ 1 - e^{-\frac{x}{2}}, & 1 - e^{-\frac{x}{2}} \in [0,1] \\ 1, & 1 - e^{-\frac{x}{2}} > 1 \end{cases} \tag{3.6}$$

Now,

$$1 - e^{-\frac{x}{2}} < 0 \tag{3.7}$$

$$\implies x < 0 \tag{3.8}$$

$$0 \le 1 - e^{-\frac{x}{2}} \le 1 \tag{3.9}$$

$$\implies x \ge 0 \tag{3.10}$$

$$1 < 1 - e^{-\frac{x}{2}} \tag{3.11}$$

$$\implies x \in \phi \tag{3.12}$$

Hence,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \ge 0 \end{cases} \tag{3.13}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110/blob/main/Assignment/solution/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110/blob/main/Assignment/solution/4/4_1.c
```

Execute the above files using code,

```
gcc 4_1.c −lm
./a.out
```

4.2 Find the CDF of T.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110/blob/main/Assignment/solution/4/4_2.py
```

Execute the above files using code,

```
python3 4_2.py
```

Plot 4.1 is obtained

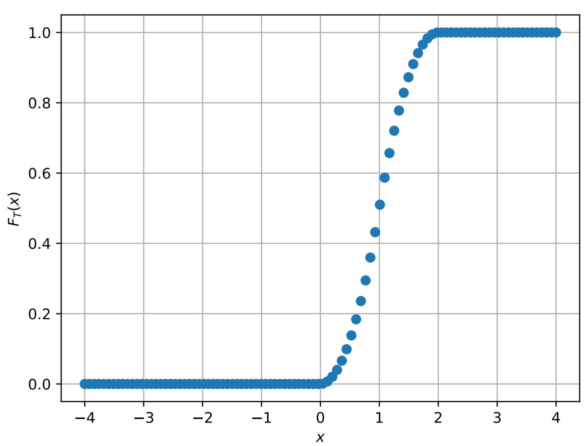

Fig. 4.1

4.3 Find the PDF of T.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110/blob/main/Assignment/solution/4/4_3.py
```

Execute the above files using code,

```
python3 4_3.py
```

Plot 4.2 is obtained

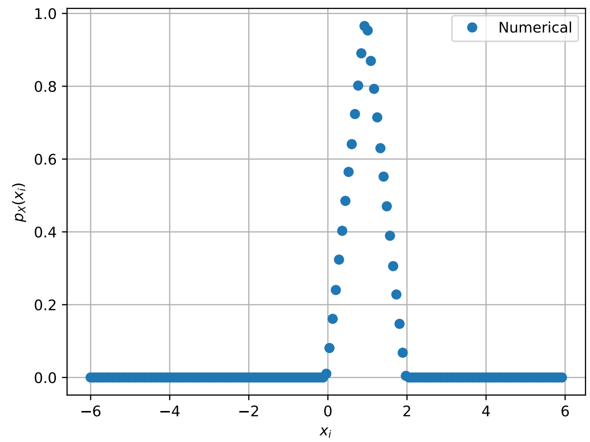

Fig. 4.2

4.4 Find the theoretical PDF and CDF of T.
Solution:
The CDF of T is given by

$$F_T(t) = \Pr(T \le t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$

Since $U_1, U_2 \in [0,1] \implies U_1 + U_2 \in [0,2]$
Therefore, if $t \ge 2$, then $U_1 + U_2 \le t$ is always true and if $t < 0$, then $U_1 + U_2 \le t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \le t \implies U_2 \le t - x \tag{4.3}$$

If $0 \le t \le 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr(U_2 \le t - x)\, p_{U_1}(x) \mathrm{d}x \tag{4.4}$$

$$= \int_0^t F_{U_2}(t-x) p_{U_1}(x) \mathrm{d}x \tag{4.5}$$

$$0 \le x \le t \implies 0 \le t - x \le t \le 1 \tag{4.6}$$

$$\implies F_{U_2}(t-x) = t - x \tag{4.7}$$

$$F_T(t) = \int_0^t (t-x) \cdot 1 \cdot \mathrm{d}x \tag{4.8}$$

$$= tx - \frac{x^2}{2}\Big|_0^t \tag{4.9}$$

$$= \frac{t^2}{2} \tag{4.10}$$

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot \mathrm{d}x \tag{4.11}$$

$$0 \leq x \leq t-1 \implies 1 \leq t-x \leq t \tag{4.12}$$

$$t-1 \leq x \leq 1 \implies 0 < t-1 \leq t-x \leq 1 \tag{4.13}$$

$$F_T(t) = \int_0^{t-1} 1\mathrm{d}x + \int_{t-1}^{1} (t-x)\mathrm{d}x \tag{4.14}$$

$$= t-1+t(1-(t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \tag{4.15}$$

$$= t-1+2t-t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \tag{4.16}$$

$$= -\frac{t^2}{2} + 2t - 1 \tag{4.17}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.18}$$

The PDF of T is given by

$$p_T(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_T(t) \tag{4.19}$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.20}$$

4.5 Verify your results through a plot.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /4/4_5_1.py
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /4/4_5_2.py
```

Execute the above files using code,

```
python3 4_5_1.py
python3 4_5_2.py
```

Plot 4.3 is obtained is pdf Plot 4.4 is obtained

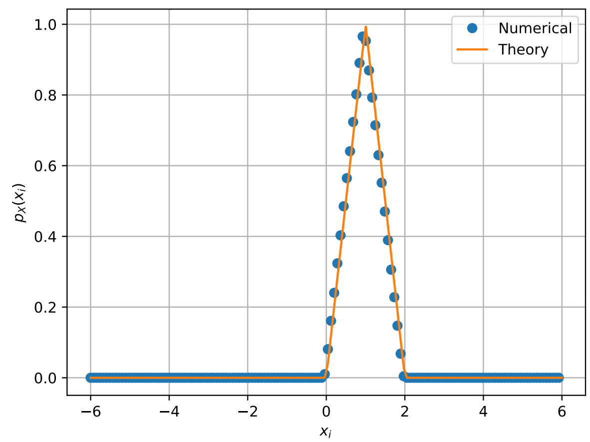

Fig. 4.3

is cdf

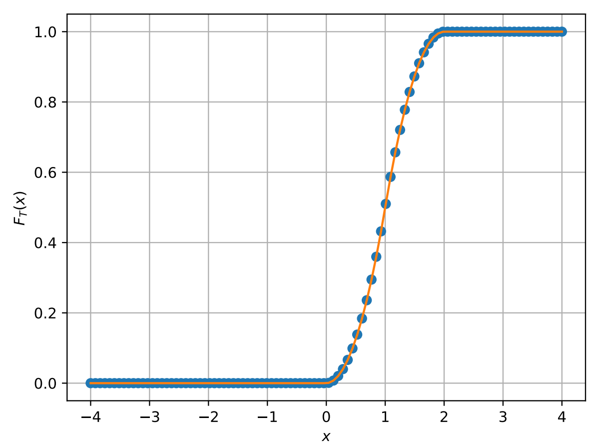

Fig. 4.4

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /5/5.1.c
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/
    coeffs.h
```

Execute the above files using code,

```
gcc 5.1.c −lm
./a.out
```

5.2 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB, $X \in \{1, -1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /5/5.2.c
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/
    coeffs.h
```

Execute the above files using code,

```
gcc 5.2.c −lm
./a.out
```

5.3 Plot Y using a scatter plot.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /5/5.3.py
```

Execute the above files using code,

```
python3 5.3.py
```

Plot 5.1 is obtained

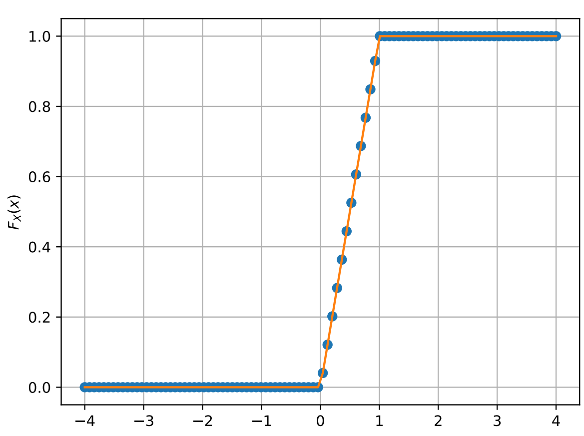

Fig. 5.1

5.4 Guess how to estimate X from Y.
Solution:
From the plot of Y, we see that the estimate model can be written as

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ 0 & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution:
Letting $X = 1$ and $X = -1$ respectively, we see the number of mismatched data points to compute the error probabilities. The simulation is coded in

```
$ wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /5/5.5.py
```

and can be run by typing

```
$ python3 5.5.py
```

The results are

$$P_{e|0} = 5.02 \times 10^{-4} \tag{5.5}$$
$$P_{e|1} = 5.24 \times 10^{-4} \tag{5.6}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution:
Here, $\Pr(X = 1) = \Pr(X = -1) = 0.5$. Thus,

$$P_e = \Pr(X = 1) P_{e|1} + \Pr(X = -1) P_{e|0} \tag{5.7}$$
$$= \frac{1}{2}\left(P_{e|0} + P_{e|1}\right) = 5.13 \times 10^{-4} \tag{5.8}$$

5.7 Verify by plotting the theoretical P_e wrt A from 0 dB to 10 dB.
Solution:

$$P_{e|0} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.9}$$
$$= \Pr(Y > 0 | X = -1) \tag{5.10}$$
$$= \Pr(AX + N > 0 | X = -1) \tag{5.11}$$
$$= \Pr(N > A) = Q(A) \tag{5.12}$$

since X and N are independent. Writing a similar expression for $P_{e|1}$ and noting that

$$\Pr(N < -A) = \Pr(N > A) = Q(A) \quad (5.13)$$

it follows that $P_e = Q(A)$. This is the idea used to plot the theoretical P_e. The plot is coded both in the rectangular axes and the semilog-y axes. Download the relevant codes using

```
wget https://github.com/Anshul-Sangrame/
     AI1110/blob/main/Assignment/solution
     /5/5.7.py
wget https://github.com/Anshul-Sangrame/
     AI1110/blob/main/Assignment/solution
     /5/5.7_semilog.py
```

Execute the above files using code,

```
python3 5.7.py
python3 5.7_semilog.py
```

Plot 5.2 obtained in rectangular axis

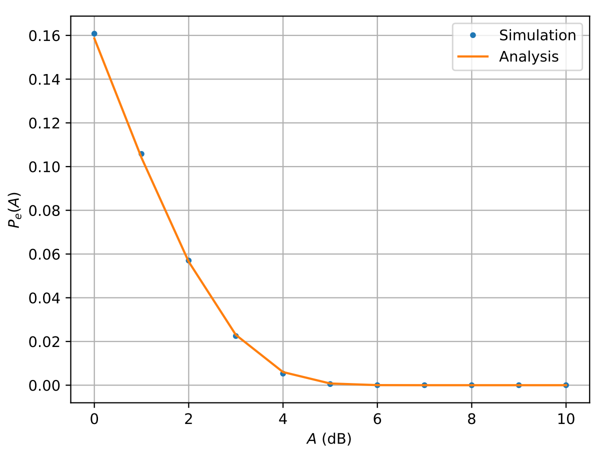

Fig. 5.2

Plot 5.3 obtained in semilog-y axis

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution:
To estimate X from Y, we now consider the following:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \quad (5.14)$$

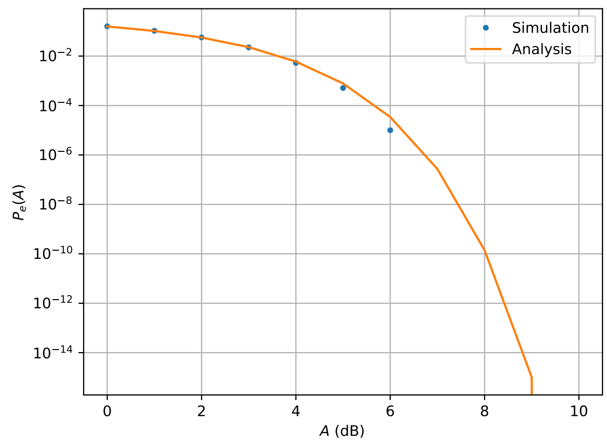

Fig. 5.3

Therefore,

$$\begin{aligned} P_e &= \Pr(X = -1)\, Q(A + \delta) \\ &\quad + \Pr(X = 1)\, Q(A - \delta) \quad (5.15) \\ &= \frac{1}{2}\left(Q(A + \delta) + Q(A - \delta)\right) \quad (5.16) \end{aligned}$$

To minimise P_e, we differentiate the above equation wrt δ:

$$0 = \frac{d}{d\delta}\left(\frac{1}{2}(Q_N(A - \delta) + Q_N(A + \delta))\right) \quad (5.17)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}\right) \quad (5.18)$$

Therefore,

$$(\delta - A)^2 = (\delta + A)^2 \quad (5.19)$$

$$\implies \delta = 0 \quad (5.20)$$

5.9 Repeat the above exercise when

$$p_x(0) = p$$

Solution:

$$P_e = P_{e|0}\, p + P_{e|1}(1 - p) \quad (5.21)$$

$$= pQ_N(A - \delta) + (1 - p)Q_N(A + \delta) \quad (5.22)$$

Differentiating as before, we get:

$$0 = p\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - (1-p)\frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}} \tag{5.23}$$

Taking ln on both sides we have:

$$\ln p - \frac{(\delta-A)^2}{2} = \ln 1 - p - \frac{(\delta+A)^2}{2} \tag{5.24}$$

$$\implies 2\delta A = \ln\frac{1-p}{p} \tag{5.25}$$

$$\implies \delta = \frac{1}{2A}\ln\frac{1-p}{p} \tag{5.26}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution:

Using theorem of conditional probabilities,

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \tag{5.27}$$

$$p_{X|Y}(x|y) = \frac{p_X(x)Pr(Ax+N=y)}{p_Y(y)} \tag{5.28}$$

$$p_{X|Y}(x|y) = \frac{p_X(x)Pr(N=y-Ax)}{p_Y(y)} \tag{5.29}$$

$$p_{X|Y}(x|y) = \frac{p_X(x)p_N(y-Ax)}{p_Y(y)} \tag{5.30}$$

$$p_{X|Y}(x|y) = \frac{\frac{1}{\sqrt{2\pi}}p_X(x)e^{-\frac{(y-Ax)^2}{2}}}{p_Y(y)} \tag{5.31}$$

$$\tag{5.32}$$

When $X = 1$, we have:

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)} \tag{5.33}$$

$$= \frac{(1-p)\,\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p)\,\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}} \tag{5.34}$$

$$= \frac{(1-p)\,e^{2yA}}{p+(1-p)\,e^{2yA}} \tag{5.35}$$

Similarly when $X = -1$,

$$p_{X|Y}(-1|y) = \frac{p}{p+(1-p)e^{2yA}} \tag{5.36}$$

Therefore, when $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$, we have:

$$\frac{(1-p)\,e^{2yA}}{p+(1-p)\,e^{2yA}} > \frac{p}{p+(1-p)\,e^{2yA}} \tag{5.37}$$

$$e^{2yA} > \frac{p}{(1-p)} \tag{5.38}$$

$$y > \frac{1}{2A}\ln\frac{p}{(1-p)} \tag{5.39}$$

Therefore, when Eq. (5.39), we can assert that $X = 1$, and $X = -1$ otherwise.

6 Gaussian to Other

6.1 gaytam singh