Assignment

Anshul Sangrame
CS21BTECH11004

1 Uniform Random Numbers

1.1: Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution:

Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/1/1_1.c
```

Execute the above files using code,

```
gcc 1_1.c −lm
./a.out
```

1.2: Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = Pr(U \leq x)$$

Solution:

Download the following files

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/1/1_2.py
```

Execute the code using command

```
python3 1_2.py
```

Plot 1 is obtained,

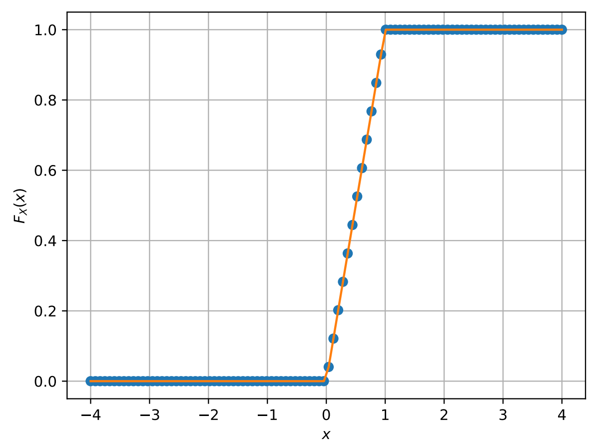

Fig. 1.

1.3: Find a theoretical expression for $F_U(x)$.

Solution:

Pdf of Uniform distribution between [0,1] is given by,

$$f_U(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)dx \tag{2}$$

Case-1: x $<$ 0,

$$F_U(x) = \int_{-\infty}^{x} 0dx \tag{3}$$
$$= 0 \tag{4}$$

Case-2: x $\in$ [0,1],

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{x} 1dx \tag{5}$$
$$= x \tag{6}$$

Case-3: x $>$ 1,

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{x} 0dx \tag{7}$$
$$= 1 \tag{8}$$

Hence,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{9}$$

1.4: The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i$$

and its variance as

$$\text{var}\,[U] = E\left[U - E\left[U\right]\right]^2$$

Write a C program to find the mean and variance of U.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/1/1_4.c
```

Execute the above files using code,

```
gcc 1_4.c −lm
./a.out
```

The following result is obtained

$$E\left[U\right] = 0.500007$$
$$\text{var}\left[U\right] = 0.083301$$

1.5: Verify your result theoretically given Z that,

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

Solution:
$F_U(x)$ for uniform distribution,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{10}$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{11}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} x dx + \int_{1}^{\infty} 0dx \tag{12}$$
$$= \frac{1}{2} \tag{13}$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{14}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} x^2 dx + \int_{1}^{\infty} 0dx \tag{15}$$
$$= \frac{1}{3} \tag{16}$$

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \tag{17}$$

From Equation (13) and (16),

$$E[U - E[U]]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{18}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{19}$$
$$= \frac{1}{12} \approx 0.083 \tag{20}$$

2 Central Limit Theorem

2.1: Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{21}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_1.c
```

Execute the above files using code,

```
gcc 2_1.c −lm
./a.out
```

2.2: Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_2.py
```

Execute the above file using code,

```
python3 2_2.py
```

Plot 2 obtained is symmetric about (0,0.5)

2.3: Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{22}$$

What properties does the PDF have?
Solution:
Download the following files,

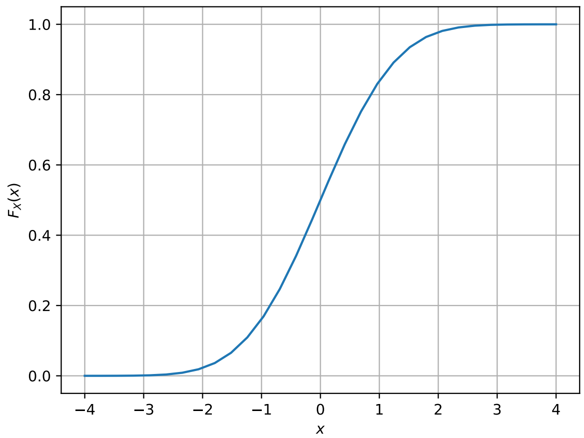

Fig. 2.

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_3.py
```

Execute the above file using code,

```
python3 2_3.py
```

Plot 3 obtained is symmetric about y-axis

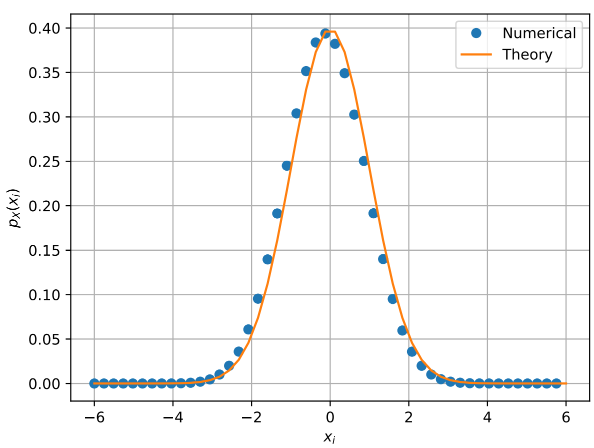

Fig. 3.

2.4: Find the mean and variance of X by writing a C program.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/soltion/1/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_4.c
```

Execute the above files using code,

```
gcc 2_4.c −lm
./a.out
```

The result is,

$$E\left[U\right] = 0.000294 \tag{23}$$
$$\text{var}\left[U\right] = 0.999561 \tag{24}$$

2.5: Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{25}$$

repeat the above exercise theoretically.
Solution:

$$E\left[U\right] = \int_{-\infty}^{\infty} u p_X(u) du \tag{26}$$
$$\tag{27}$$

$up_X(u)$ is a odd function.

$$E\left[U\right] = 0 \tag{28}$$
$$\tag{29}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \tag{30}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \left(u \exp\left(-\frac{u^2}{2}\right)\right) du \tag{31}$$

Using integration by parts,

$$E\left[U^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du \tag{32}$$
$$E\left[U^2\right] = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = 1 \tag{33}$$

Hence,

$$\text{var}\left[U\right] = E\left[U^2\right] - \left[E\left[U\right]\right]^2 \tag{34}$$
$$= 1 \tag{35}$$

3 From Uniform to Other

3.1: Generate samples of

$$V = -2\ln\left(1-U\right) \tag{36}$$

and plot its CDF.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/3/3_1.c
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/3/3_1.py
```

Execute the above files using code,

```
gcc 3_1.c −lm
./a.out
python3 3_1.py
```

Plot 4 is obtained

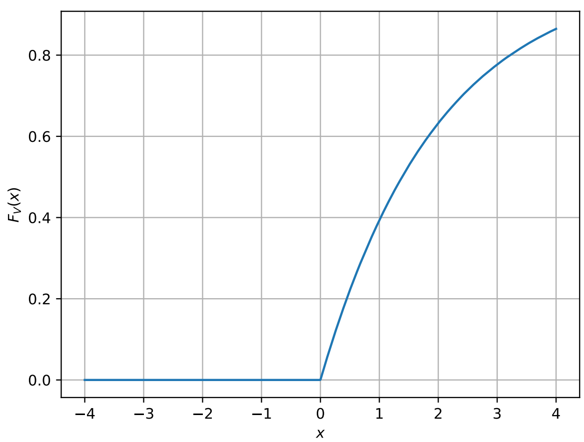

Fig. 4.

3.2: Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = Pr\left(V \leq x\right) \tag{37}$$
$$= Pr\left(-2\ln\left(1-U\right) \leq x\right) \tag{38}$$
$$= Pr\left(U \leq 1-e^{-\frac{x}{2}}\right) \tag{39}$$
$$F_V(x) = F_U\left(1-e^{-\frac{x}{2}}\right) \tag{40}$$
$$= \begin{cases} 0, & 1-e^{-\frac{x}{2}} < 0 \\ 1-e^{-\frac{x}{2}}, & 1-e^{-\frac{x}{2}} \in [0,1] \\ 1, & 1-e^{-\frac{x}{2}} > 1 \end{cases} \tag{41}$$

Now,

$$1-e^{-\frac{x}{2}} < 0 \tag{42}$$
$$\implies x < 0 \tag{43}$$

$$0 \leq 1-e^{-\frac{x}{2}} \leq 1 \tag{44}$$
$$\implies x \geq 0 \tag{45}$$

$$1 < 1-e^{-\frac{x}{2}} \tag{46}$$
$$\implies x \in \phi \tag{47}$$

Hence,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1-e^{-\frac{x}{2}}, & x \geq 0 \end{cases} \tag{48}$$