

Assignment

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CS21BTECH11004

1 UNIFORM RANDOM NUMBERS

1.1: Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/1/1_1.c
```

Execute the above files using code,

```
gcc 1_1.c -lm
./a.out
```

1.2: Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = Pr(U \leq x)$$

Solution:

Download the following files

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/1/1_2.py
```

Execute the code using command

```
python3 1_2.py
```

Plot 1 is obtained,

1.3: Find a theoretical expression for $F_U(x)$.

Solution:

Pdf of Uniform distribution between $[0,1]$ is given by,

$$f_U(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$F_U(x) = \int_{-\infty}^x f_U(x) dx \quad (2)$$

Case-1: $x < 0$,

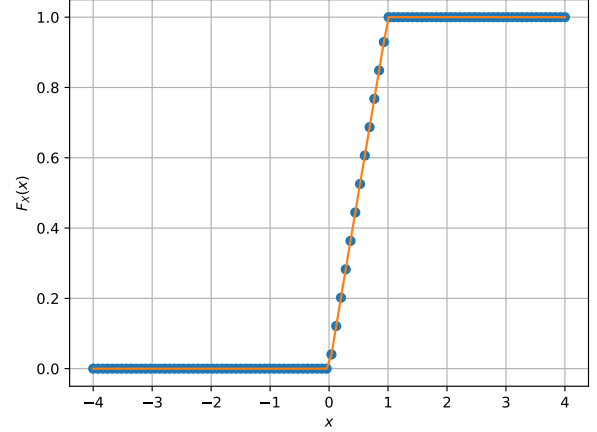


Fig. 1.

$$F_U(x) = \int_{-\infty}^x 0 dx \quad (3)$$

$$= 0 \quad (4)$$

Case-2: $x \in [0,1]$,

$$F_U(x) = \int_{-\infty}^0 0 dx + \int_0^x 1 dx \quad (5)$$

$$= x \quad (6)$$

Case-3: $x > 1$,

$$F_U(x) = \int_{-\infty}^0 0 dx + \int_0^1 1 dx + \int_1^x 0 dx \quad (7)$$

$$= 1 \quad (8)$$

Hence,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0, 1] \\ 1, & x > 1 \end{cases} \quad (9)$$

1.4: The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2$$

Write a C program to find the mean and variance of U .

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/1/1_4.c
```

Execute the above files using code,

```
gcc 1_4.c -lm
./a.out
```

The following result is obtained

$$E[U] = 0.500007$$

$$\text{var}[U] = 0.083301$$

1.5: Verify your result theoretically given Z that,

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

Solution:

$F_U(x)$ for uniform distribution,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0, 1] \\ 1, & x > 1 \end{cases} \quad (10)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (11)$$

$$= \int_{-\infty}^0 0 dx + \int_0^1 x dx + \int_1^{\infty} 0 dx \quad (12)$$

$$= \frac{1}{2} \quad (13)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (14)$$

$$= \int_{-\infty}^0 0 dx + \int_0^1 x^2 dx + \int_1^{\infty} 0 dx \quad (15)$$

$$= \frac{1}{3} \quad (16)$$

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \quad (17)$$

From Equation (13) and (16),

$$E[U - E[U]]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (18)$$

$$= \frac{1}{3} - \frac{1}{4} \quad (19)$$

$$= \frac{1}{12} \approx 0.083 \quad (20)$$

2 CENTRAL LIMIT THEOREM

2.1: Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (21)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/2/2_1.c
```

Execute the above files using code,

```
gcc 2_1.c -lm
./a.out
```

2.2: Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/2/2_2.py
```

Execute the above file using code,

```
python3 2_2.py
```

Plot 2 obtained is symmetric about (0,0.5)

2.3: Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (22)$$

What properties does the PDF have?

Solution:

Download the following files,

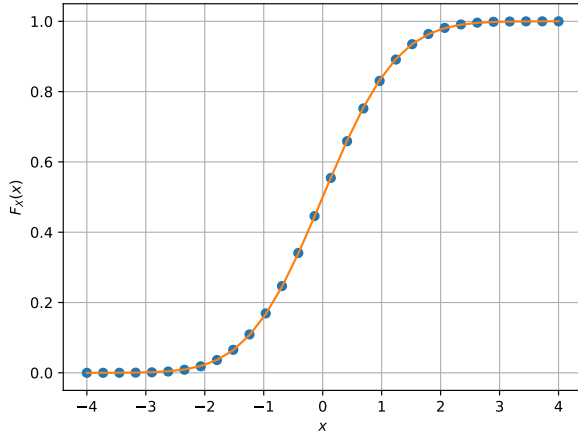


Fig. 2.

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/2/2_3.py
```

Execute the above file using code,

```
python3 2_3.py
```

Plot 3 obtained is symmetric about y-axis

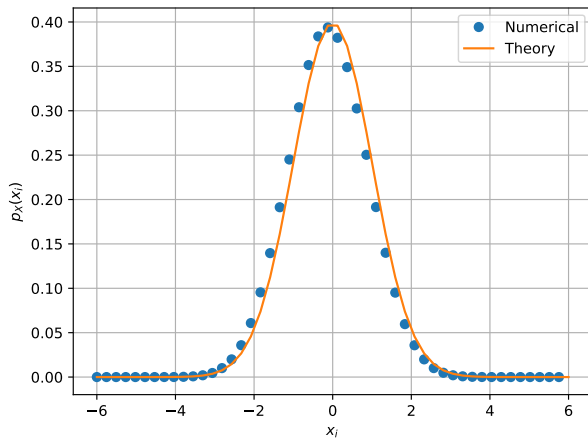


Fig. 3.

2.4: Find the mean and variance of X by writing a C program.

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/2/2_4.c
```

Execute the above files using code,

```
gcc 2_4.c -lm
./a.out
```

The result is,

$$E[U] = 0.000294 \quad (23)$$

$$\text{var}[U] = 0.999561 \quad (24)$$

2.5: Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad -\infty < x < \infty, \quad (25)$$

repeat the above exercise theoretically.

Solution:

$$E[U] = \int_{-\infty}^{\infty} up_X(u) du \quad (26)$$

$$(27)$$

$up_X(u)$ is a odd function.

$$E[U] = 0 \quad (28)$$

$$(29)$$

$$E[U^2] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \quad (30)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \left(u \exp\left(-\frac{u^2}{2}\right) \right) du \quad (31)$$

Using integration by parts,

$$E[U^2] = -u \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \quad (32)$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \quad (33)$$

$$= 1 \quad (34)$$

Hence,

$$\text{var}[U] = E[U^2] - [E[U]]^2 \quad (35)$$

$$= 1 \quad (36)$$

3 FROM UNIFORM TO OTHER

3.1: Generate samples of

$$V = -2 \ln(1 - U) \quad (37)$$

and plot its CDF.

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/3/3_1.c
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/3/3_1.py
```

Execute the above files using code,

```
gcc 3_1.c -lm
./a.out
python3 3_1.py
```

Plot 4 is obtained

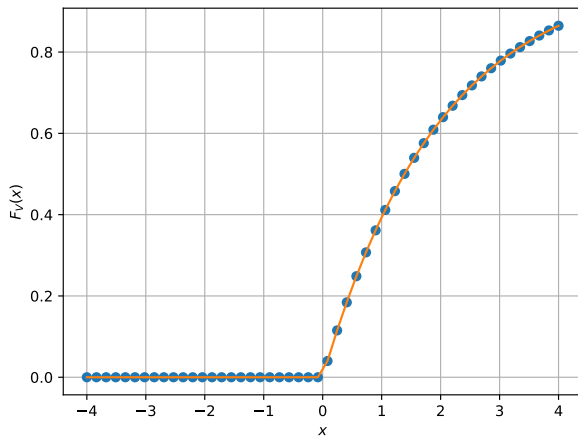


Fig. 4.

3.2: Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = \Pr(V \leq x) \quad (38)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (39)$$

$$= \Pr(U \leq 1 - e^{-\frac{x}{2}}) \quad (40)$$

$$F_V(x) = F_U(1 - e^{-\frac{x}{2}}) \quad (41)$$

$$= \begin{cases} 0, & 1 - e^{-\frac{x}{2}} < 0 \\ 1 - e^{-\frac{x}{2}}, & 1 - e^{-\frac{x}{2}} \in [0, 1] \\ 1, & 1 - e^{-\frac{x}{2}} > 1 \end{cases} \quad (42)$$

Now,

$$1 - e^{-\frac{x}{2}} < 0 \quad (43)$$

$$\implies x < 0 \quad (44)$$

$$0 \leq 1 - e^{-\frac{x}{2}} \leq 1 \quad (45)$$

$$\implies x \geq 0 \quad (46)$$

$$1 < 1 - e^{-\frac{x}{2}} \quad (47)$$

$$\implies x \in \phi \quad (48)$$

Hence,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \geq 0 \end{cases} \quad (49)$$

4 TRIANGULAR DISTRIBUTION

4.1: Generate

$$T = U_1 + U_2 \quad (50)$$

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/4/4_1.c
```

Execute the above files using code,

```
gcc 4_1.c -lm
./a.out
```

4.2: Find the CDF of T .

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/4/4_2.py
```

Execute the above files using code,

```
python3 4_2.py
```

Plot 5 is obtained

4.3: Find the PDF of T .

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/4/4_3.py
```

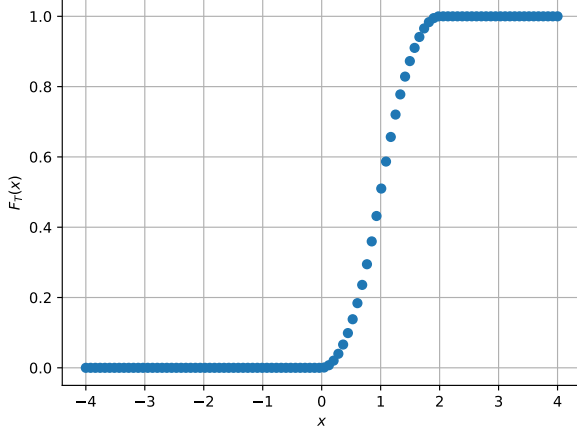


Fig. 5.

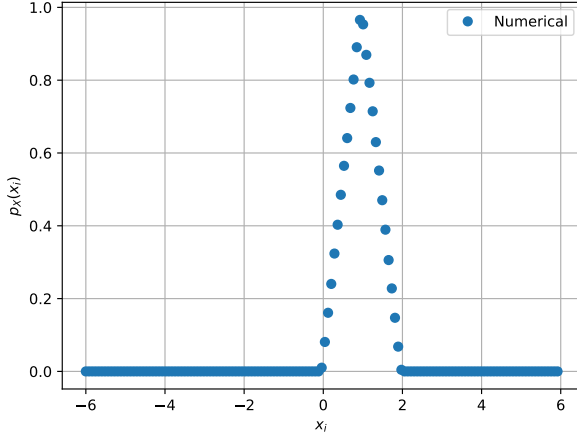


Fig. 6.

Execute the above files using code,

```
python3 4_3.py
```

Plot 6 is obtained

4.4: Find the theoretical expressions for the PDF and CDF of T .

Solution:

The CDF of T is given by

$$F_T(t) = \Pr(T \leq t) = \Pr(U_1 + U_2 \leq t) \quad (51)$$

Since $U_1, U_2 \in [0, 1] \implies U_1 + U_2 \in [0, 2]$
Therefore, if $t \geq 2$, then $U_1 + U_2 \leq t$ is always true and if $t < 0$, then $U_1 + U_2 \leq t$ is always false.

Now, fix the value of U_1 to be some x

$$x + U_2 \leq t \implies U_2 \leq t - x \quad (52)$$

If $0 \leq t \leq 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr(U_2 \leq t - x) p_{U_1}(x) dx \quad (53)$$

$$= \int_0^t F_{U_2}(t - x) p_{U_1}(x) dx \quad (54)$$

$$0 \leq x \leq t \implies 0 \leq t - x \leq t \leq 1 \quad (55)$$

$$\implies F_{U_2}(t - x) = t - x \quad (56)$$

$$F_T(t) = \int_0^t (t - x) \cdot 1 \cdot dx \quad (57)$$

$$= tx - \frac{x^2}{2} \Big|_0^t \quad (58)$$

$$= \frac{t^2}{2} \quad (59)$$

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t - x) \cdot 1 \cdot dx \quad (60)$$

$$0 \leq x \leq t - 1 \implies 1 \leq t - x \leq t \quad (61)$$

$$t - 1 \leq x \leq 1 \implies 0 < t - 1 \leq t - x \leq 1 \quad (62)$$

$$F_T(t) = \int_0^{t-1} 1 dx + \int_{t-1}^1 (t - x) dx \quad (63)$$

$$= t - 1 + t(1 - (t - 1)) - \frac{1}{2} + \frac{(t - 1)^2}{2} \quad (64)$$

$$= t - 1 + 2t - t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \quad (65)$$

$$= -\frac{t^2}{2} + 2t - 1 \quad (66)$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \quad (67)$$

The PDF of T is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \quad (68)$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \quad (69)$$

4.5: Verify your results through a plot

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/4/4_5_1.py
wget https://github.com/Anshul-Sangrame/AI1110
/blob/main/Assignment/solution/4/4_5_2.py
```

Execute the above files using code,

```
python3 4_5_1.py
python3 4_5_2.py
```

Plot 7 is obtained is pdf

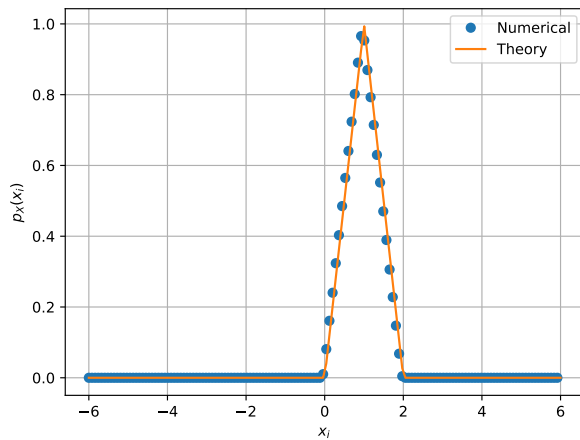


Fig. 7.

Plot 8 is obtained is cdf

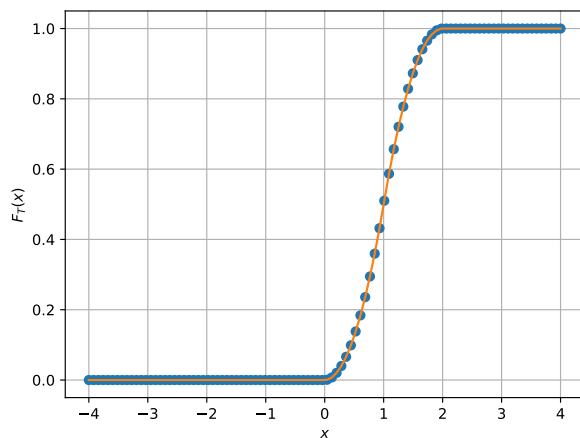


Fig. 8.