Random Numbers

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CS21BTECH11004

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /1/1_1.c
```

Execute the above files using code,

```
gcc 1_1.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution:
Download the following files

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /1/1_2.py
```

Execute the code using command

```
python3 1_2.py
```

Plot 1.1 is obtained,

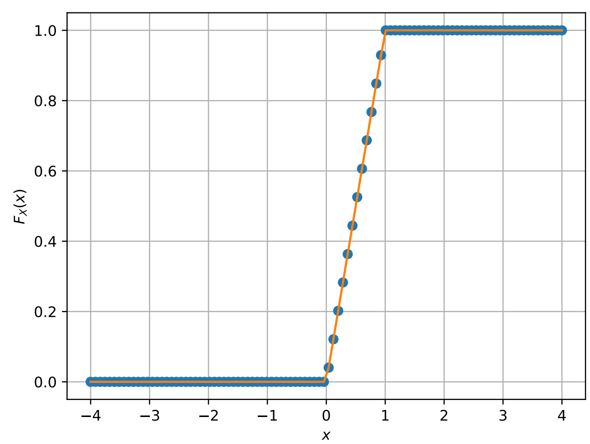

Fig. 1.1

1.3 Find a theoretical expression for $F_U(x)$.
Solution:
Pdf of Uniform distribution between [0,1] is given by,

$$f_U(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)dx \tag{1.3}$$

Case-1: x < 0,

$$F_U(x) = \int_{-\infty}^{x} 0dx \tag{1.4}$$
$$= 0 \tag{1.5}$$

Case-2: x ∈ [0,1],

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{x} 1dx \tag{1.6}$$
$$= x \tag{1.7}$$

Case-3: x > 1,

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{x} 0dx \tag{1.8}$$
$$= 1 \tag{1.9}$$

Hence,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{1.10}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.11}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.12}$$

Write a C program to find the mean and variance of U.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /1/1_4.c
```

Execute the above files using code,

```
gcc 1_4.c −lm
./a.out
```

The following result is obtained

$$E[U] = 0.500007$$
$$\text{var}[U] = 0.083301$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \tag{1.13}$$

Solution:

$F_U(x)$ for uniform distribution,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{1.14}$$

$$E[U] = \int_{-\infty}^{\infty} xdF_U(x) \tag{1.15}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} xdx + \int_{1}^{\infty} 0dx \tag{1.16}$$
$$= \frac{1}{2} \tag{1.17}$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2dF_U(x) \tag{1.18}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} x^2dx + \int_{1}^{\infty} 0dx \tag{1.19}$$
$$= \frac{1}{3} \tag{1.20}$$

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \tag{1.21}$$

From Equation (1.17) and (1.20),

$$E[U - E[U]]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.22}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.23}$$
$$= \frac{1}{12} \approx 0.083 \tag{1.24}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_1.c
```

Execute the above files using code,

```
gcc 2_1.c −lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_2.py
```

Execute the above file using code,

```
python3 2_2.py
```

Plot 2.1 obtained is symmetric about (0,0.5)

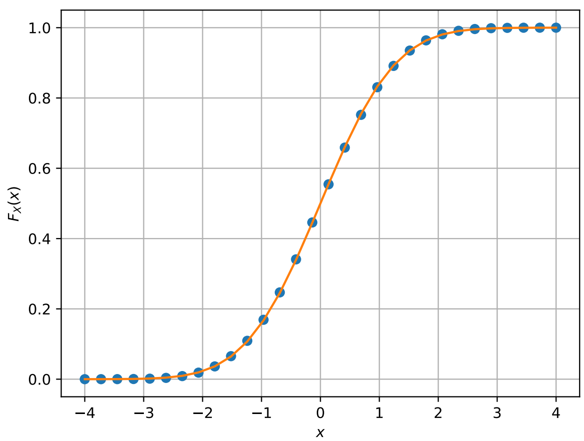

Fig. 2.1

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_3.py
```

Execute the above file using code,

```
python3 2_3.py
```

Plot 2.2 obtained is symmetric about y-axis

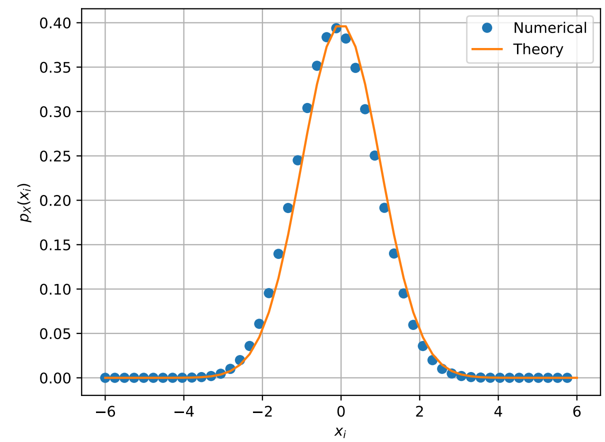

Fig. 2.2

2.4 Find the mean and variance of X by writing a C program.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/soltion/1/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /2/2_4.c
```

Execute the above files using code,

```
gcc 2_4.c −lm
./a.out
```

The result is,

$$E[U] = 0.000294 \tag{2.3}$$
$$\text{var}[U] = 0.999561 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution:

$$E[U] = \int_{-\infty}^{\infty} u p_X(u) du \tag{2.6}$$
$$\tag{2.7}$$

$up_X(u)$ is a odd function.

$$E[U] = 0 \tag{2.8}$$
$$\tag{2.9}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \tag{2.10}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \left(u \exp\left(-\frac{u^2}{2}\right)\right) du \tag{2.11}$$

Using integration by parts,

$$E\left[U^2\right] = -u\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \tag{2.12}$$
$$= 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \tag{2.13}$$
$$= 1 \tag{2.14}$$

Hence,

$$\text{var}\left[U\right] = E\left[U^2\right] - \left[E\left[U\right]\right]^2 \tag{2.15}$$
$$= 1 \tag{2.16}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{3.1}$$

and plot its CDF.

Solution:

Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /3/3_1.c
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /3/3_1.py
```

Execute the above files using code,

```
gcc 3_1.c −lm
./a.out
python3 3_1.py
```

Plot 3.1 is obtained

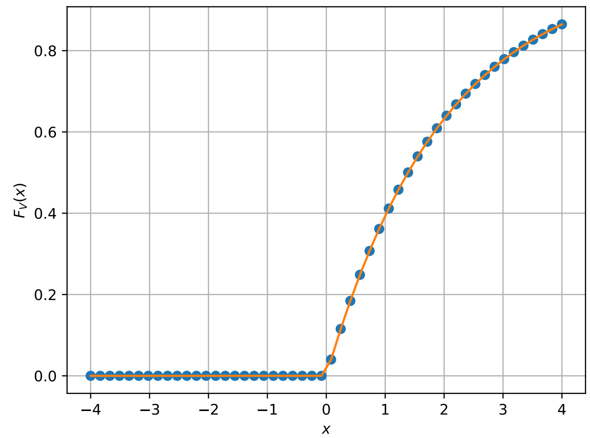

Fig. 3.1

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = Pr\left(V \le x\right) \tag{3.2}$$
$$= Pr\left(-2\ln\left(1 - U\right) \le x\right) \tag{3.3}$$
$$= Pr\left(U \le 1 - e^{-\frac{x}{2}}\right) \tag{3.4}$$
$$F_V(x) = F_U\left(1 - e^{-\frac{x}{2}}\right) \tag{3.5}$$
$$= \begin{cases} 0, & 1 - e^{-\frac{x}{2}} < 0 \\ 1 - e^{-\frac{x}{2}}, & 1 - e^{-\frac{x}{2}} \in [0, 1] \\ 1, & 1 - e^{-\frac{x}{2}} > 1 \end{cases} \tag{3.6}$$

Now,

$$1 - e^{-\frac{x}{2}} < 0 \tag{3.7}$$
$$\implies x < 0 \tag{3.8}$$

$$0 \le 1 - e^{-\frac{x}{2}} \le 1 \tag{3.9}$$
$$\implies x \ge 0 \tag{3.10}$$

$$1 < 1 - e^{-\frac{x}{2}} \tag{3.11}$$
$$\implies x \in \phi \tag{3.12}$$

Hence,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \ge 0 \end{cases} \tag{3.13}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:

Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution/
    coeffs.h
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /4/4_1.c
```

Execute the above files using code,

```
gcc 4_1.c −lm
./a.out
```

4.2 Find the CDF of T.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /4/4_2.py
```

Execute the above files using code,

```
python3 4_2.py
```

Plot 4.1 is obtained

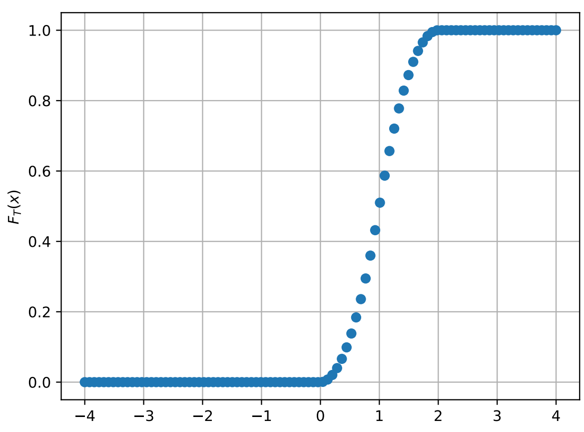

Fig. 4.1

4.3 Find the PDF of T.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /4/4_3.py
```

Execute the above files using code,

```
python3 4_3.py
```

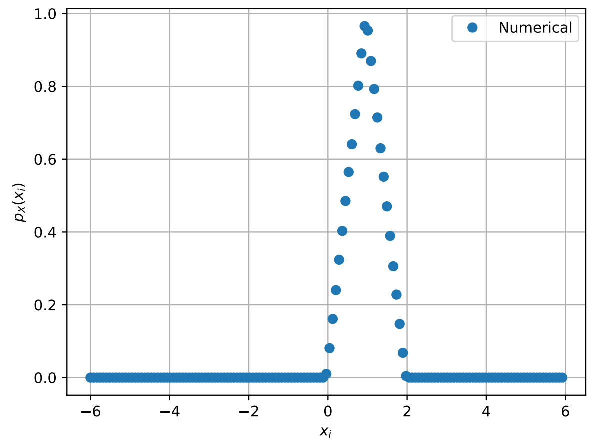

Fig. 4.2

Plot 4.2 is obtained

4.4 Find the theoretical PDF and CDF of T.
Solution:
The CDF of T is given by

$$F_T(t) = \Pr(T \leq t) = \Pr(U_1 + U_2 \leq t) \quad (4.2)$$

Since $U_1, U_2 \in [0,1] \implies U_1 + U_2 \in [0,2]$
Therefore, if $t \geq 2$, then $U_1 + U_2 \leq t$ is always true and if $t < 0$, then $U_1 + U_2 \leq t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \leq t \implies U_2 \leq t - x \quad (4.3)$$

If $0 \leq t \leq 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr(U_2 \leq t - x)\, p_{U_1}(x)\mathrm{d}x \quad (4.4)$$
$$= \int_0^t F_{U_2}(t - x) p_{U_1}(x)\mathrm{d}x \quad (4.5)$$

$$0 \leq x \leq t \implies 0 \leq t - x \leq t \leq 1 \quad (4.6)$$
$$\implies F_{U_2}(t - x) = t - x \quad (4.7)$$

$$F_T(t) = \int_0^t (t - x) \cdot 1 \cdot \mathrm{d}x \quad (4.8)$$
$$= tx - \frac{x^2}{2}\bigg|_0^t \quad (4.9)$$
$$= \frac{t^2}{2} \quad (4.10)$$

If $1 < t < 2$, x can only take values in $[0, 1]$ as

$U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot \mathrm{d}x \tag{4.11}$$

$$0 \leq x \leq t-1 \implies 1 \leq t-x \leq t \tag{4.12}$$

$$t-1 \leq x \leq 1 \implies 0 < t-1 \leq t-x \leq 1 \tag{4.13}$$

$$F_T(t) = \int_0^{t-1} 1\mathrm{d}x + \int_{t-1}^{1} (t-x)\mathrm{d}x \tag{4.14}$$

$$= t - 1 + t(1-(t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \tag{4.15}$$

$$= t - 1 + 2t - t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \tag{4.16}$$

$$= -\frac{t^2}{2} + 2t - 1 \tag{4.17}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.18}$$

The PDF of T is given by

$$p_T(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_T(t) \tag{4.19}$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.20}$$

4.5 Verify your results through a plot.

Solution:

Download the following files,

```
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /4/4_5_1.py
wget https://github.com/Anshul−Sangrame/
    AI1110/blob/main/Assignment/solution
    /4/4_5_2.py
```

Execute the above files using code,

```
python3 4_5_1.py
python3 4_5_2.py
```

Plot 4.3 is obtained is pdf Plot 4.4 is obtained

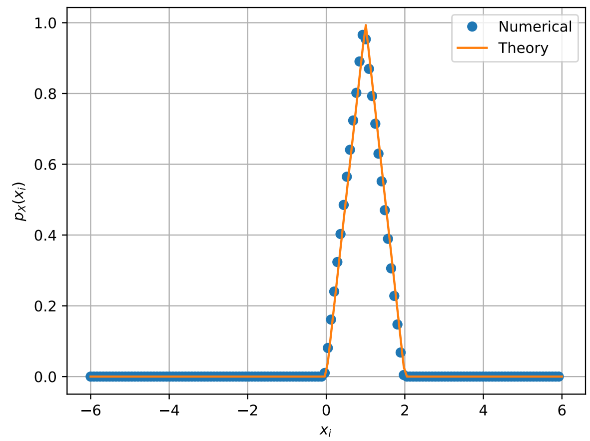

Fig. 4.3

is cdf

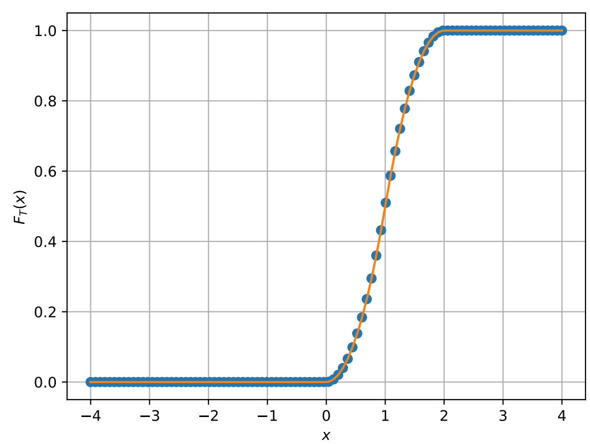

Fig. 4.4

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.