Assignment

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CS21BTECH11004

1 Uniform Random Numbers

1.1: Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:
Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/1/1_1.c
```

Execute the above files using code,

```
gcc 1_1.c -lm
./a.out
```

1.2: Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = Pr(U \leq x)$$

Solution:
Download the following files

```
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/1/1_2.py
```

Execute the code using command

```
python3 1_2.py
```

Plot 1 is obtained,

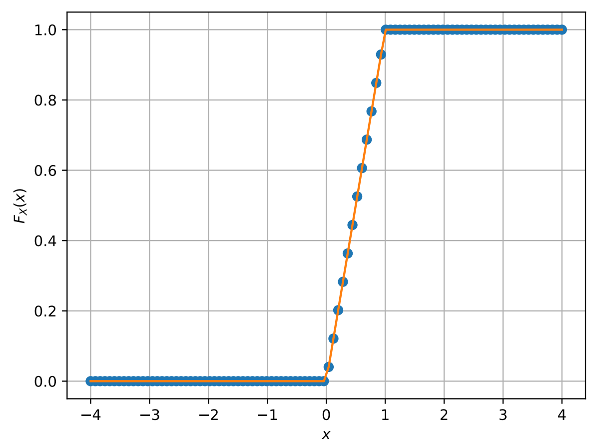

Fig. 1.

1.3: Find a theoretical expression for $F_U(x)$.
Solution:
Pdf of Uniform distribution between [0,1] is given by,

$$f_U(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)dx \tag{2}$$

Case-1: x $<$ 0,

$$F_U(x) = \int_{-\infty}^{x} 0dx \tag{3}$$
$$= 0 \tag{4}$$

Case-2: x $\in$ [0,1],

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{x} 1dx \tag{5}$$
$$= x \tag{6}$$

Case-3: x $>$ 1,

$$F_U(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{x} 0dx \tag{7}$$
$$= 1 \tag{8}$$

Hence,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{9}$$

1.4: The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i$$

and its variance as

$$\text{var}\,[U] = E\,[U - E\,[U]]^2$$

Write a C program to find the mean and variance of U.

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/1/1_4.c
```

Execute the above files using code,

```
gcc 1_4.c -lm
./a.out
```

The following result is obtained

$$E\,[U] = 0.500007$$
$$\text{var}\,[U] = 0.083301$$

1.5: Verify your result theoretically given Z that,

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

Solution:

$F_U(x)$ for uniform distribution,

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases} \tag{10}$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{11}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} xdx + \int_{1}^{\infty} 0dx \tag{12}$$
$$= \frac{1}{2} \tag{13}$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{14}$$
$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} x^2dx + \int_{1}^{\infty} 0dx \tag{15}$$
$$= \frac{1}{3} \tag{16}$$

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \tag{17}$$

From Equation (13) and (16),

$$E[U - E[U]]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{18}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{19}$$
$$= \frac{1}{12} \approx 0.083 \tag{20}$$

2 Central Limit Theorem

2.1: Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{21}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/1/coeffs.h
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_1.c
```

Execute the above files using code,

```
gcc 2_1.c -lm
./a.out
```

2.2: Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:

Download the following files,

```
wget https://github.com/Anshul-Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_2.py
```

Execute the above file using code,

```
python3 2_2.py
```

Plot 2 obtained is symmetric about (0,0.5)

2.3: Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{22}$$

What properties does the PDF have?

Solution:

Download the following files,

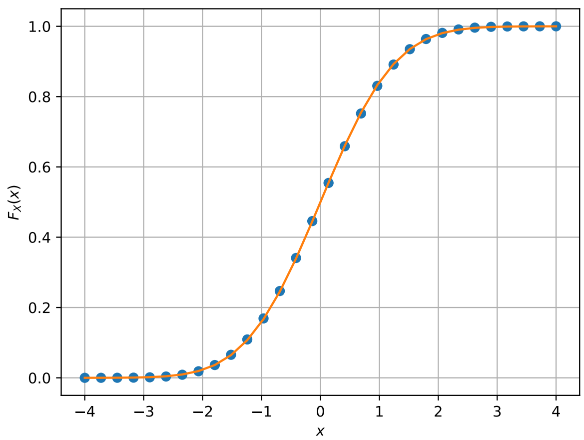

Fig. 2.

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_3.py
```

Execute the above file using code,

```
python3 2_3.py
```

Plot 3 obtained is symmetric about y-axis

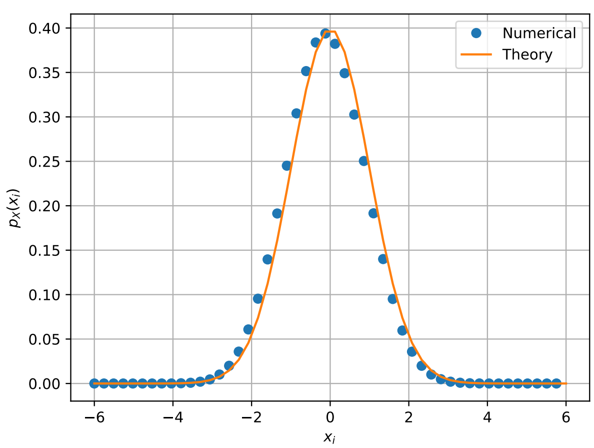

Fig. 3.

2.4: Find the mean and variance of X by writing a C program.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/soltion/1/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/2/2_4.c
```

Execute the above files using code,

```
gcc 2_4.c −lm
./a.out
```

The result is,

$$E[U] = 0.000294 \tag{23}$$
$$\text{var}[U] = 0.999561 \tag{24}$$

2.5: Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{25}$$

repeat the above exercise theoretically.
Solution:

$$E[U] = \int_{-\infty}^{\infty} u p_X(u) du \tag{26}$$
$$\tag{27}$$

$u p_X(u)$ is a odd function.

$$E[U] = 0 \tag{28}$$
$$\tag{29}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \tag{30}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \left(u \exp\left(-\frac{u^2}{2}\right)\right) du \tag{31}$$

Using integration by parts,

$$E\left[U^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du \tag{32}$$
$$E\left[U^2\right] = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = 1 \tag{33}$$

Hence,

$$\text{var}[U] = E\left[U^2\right] - [E[U]]^2 \tag{34}$$
$$= 1 \tag{35}$$

3 From Uniform to Other

3.1: Generate samples of

$$V = -2\ln(1-U) \tag{36}$$

and plot its CDF.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/3/3_1.c
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/3/3_1.py
```

Execute the above files using code,

```
gcc 3_1.c −lm
./a.out
python3 3_1.py
```

Plot 4 is obtained

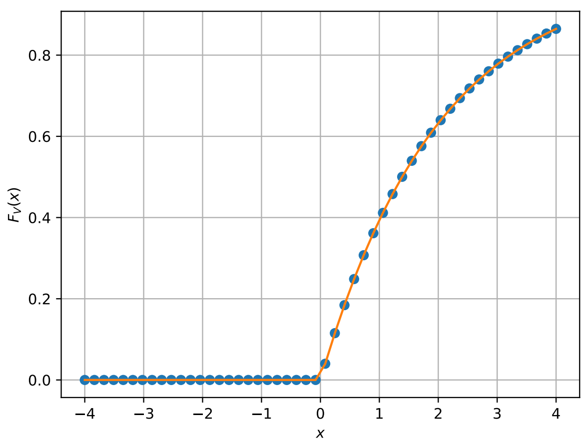

Fig. 4.

3.2: Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = Pr\left(V \leq x\right) \tag{37}$$
$$= Pr\left(-2\ln\left(1-U\right) \leq x\right) \tag{38}$$
$$= Pr\left(U \leq 1 - e^{-\frac{x}{2}}\right) \tag{39}$$
$$F_V(x) = F_U\left(1 - e^{-\frac{x}{2}}\right) \tag{40}$$
$$= \begin{cases} 0, & 1 - e^{-\frac{x}{2}} < 0 \\ 1 - e^{-\frac{x}{2}}, & 1 - e^{-\frac{x}{2}} \in [0,1] \\ 1, & 1 - e^{-\frac{x}{2}} > 1 \end{cases} \tag{41}$$

Now,

$$1 - e^{-\frac{x}{2}} < 0 \tag{42}$$
$$\implies x < 0 \tag{43}$$

$$0 \leq 1 - e^{-\frac{x}{2}} \leq 1 \tag{44}$$
$$\implies x \geq 0 \tag{45}$$

$$1 < 1 - e^{-\frac{x}{2}} \tag{46}$$
$$\implies x \in \phi \tag{47}$$

Hence,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \geq 0 \end{cases} \tag{48}$$

4 Triangular Distribution

4.1: Generate

$$T = U_1 + U_2 \tag{49}$$

Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/coeffs.h
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/4/4_1.c
```

Execute the above files using code,

```
gcc 4_1.c −lm
./a.out
```

4.2: Find the CDF of T.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/4/4_2.py
```

Execute the above files using code,

```
python3 4_2.py
```

Plot 5 is obtained
4.3: Find the PDF of T.
Solution:
Download the following files,

```
wget https://github.com/Anshul−Sangrame/AI1110
    /blob/main/Assignment/solution/4/4_3.py
```

Execute the above files using code,

```
python3 4_3.py
```

Plot 6 is obtained
4.4: Find the theoretical expressions for the PDF and CDF of T.
Solution:

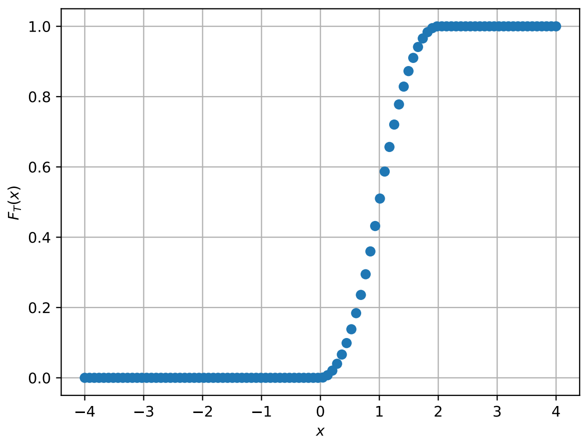

Fig. 5.

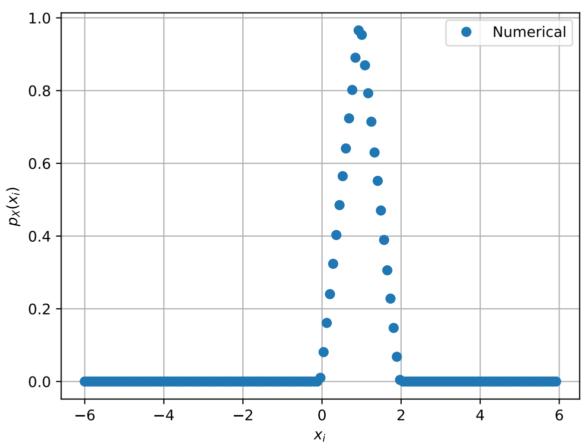

Fig. 6.

The CDF of T is given by

$$F_T(t) = \Pr\left(T \leq t\right) = \Pr\left(U_1 + U_2 \leq t\right) \tag{50}$$

Since $U_1, U_2 \in [0,1] \implies U_1 + U_2 \in [0,2]$ Therefore, if $t \geq 2$, then $U_1 + U_2 \leq t$ is always true and if $t < 0$, then $U_1 + U_2 \leq t$ is always false.

Now, fix the value of U_1 to be some x

$$x + U_2 \leq t \implies U_2 \leq t - x \tag{51}$$

If $0 \leq t \leq 1$, then x can take all values in $[0,t]$

$$F_T(t) = \int_0^t \Pr\left(U_2 \leq t - x\right) p_{U_1}(x) \mathrm{d}x \tag{52}$$

$$= \int_0^t F_{U_2}(t-x) p_{U_1}(x) \mathrm{d}x \tag{53}$$

$$0 \leq x \leq t \implies 0 \leq t - x \leq t \leq 1 \tag{54}$$

$$\implies F_{U_2}(t-x) = t - x \tag{55}$$

$$F_T(t) = \int_0^t (t-x) \cdot 1 \cdot \mathrm{d}x \tag{56}$$

$$= tx - \frac{x^2}{2}\bigg|_0^t \tag{57}$$

$$= \frac{t^2}{2} \tag{58}$$

If $1 < t < 2$, x can only take values in $[0,1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot \mathrm{d}x \tag{59}$$

$$0 \leq x \leq t - 1 \implies 1 \leq t - x \leq t \tag{60}$$

$$t - 1 \leq x \leq 1 \implies 0 < t - 1 \leq t - x \leq 1 \tag{61}$$

$$F_T(t) = \int_0^{t-1} 1 \mathrm{d}x + \int_{t-1}^1 (t-x) \mathrm{d}x \tag{62}$$

$$= t - 1 + t(1 - (t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \tag{63}$$

$$= t - 1 + 2t - t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \tag{64}$$

$$= -\frac{t^2}{2} + 2t - 1 \tag{65}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \dfrac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \tag{66}$$

The PDF of T is given by

$$p_T(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_T(t) \tag{67}$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \tag{68}$$