Network Analysis with Graph Theory

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1 Introduction

A large network having several elements and sources is better to be solved systematically using a computer. Graph theory is adopted to formulate the problem in a systematic way. Graph theory itself is a major topic and several books are available on this topic. In this lecture we shall primarily engage ourselves to the discussion on how graph theory is applied to solve network problems after giving a brief introduction to it. We shall try to answer the following:

1. How to draw a graph of a network?
2. What is a directed graph?
3. What is a node?
4. What is an element in a graph?
5. What is a tree and what is a twig?
6. What is a co-tree and what is a link?
7. What is a cut-set?
8. How to draw directed graph, tree and co-tree for a given network?

9. How to solve for node voltages using [A] matrix?
10. How to solve for fundamental link currents using [B] matrix?
11. How to solve for element voltages of a tree using [Q] matrix?
12. How to establish relations among [A], [B] and [Q] matrices?

2 Network and its Graph

2.1 Network

By now we have a fair idea of a network consisting several sources with various circuit elements. One such network is shown in figure 1(a). Each **box** represent an element which may consist of voltage/current sources and various circuit elements connected in various ways.

There are six number of total branches present. A node is a junction where three or more branch ends meet. In this network there are a total of four nodes present which are marked with • symbols and named as a, b, c and o. We may choose o to be the reference node.

We can choose the directions of the currents (i_{e1} to i_{e6}) quite arbitrarily in various elements as shown with arrows in figure 1(a). Once the current directions are chosen, we can choose the polarity of the voltages across each element (v_{e1} to v_{e6}) such that current flows from +ve polarity to -ve polarity. Initially for the i^{th} element, we shall be concerned with the terminal voltage v_{ei} and current through the the element i_{ei}.

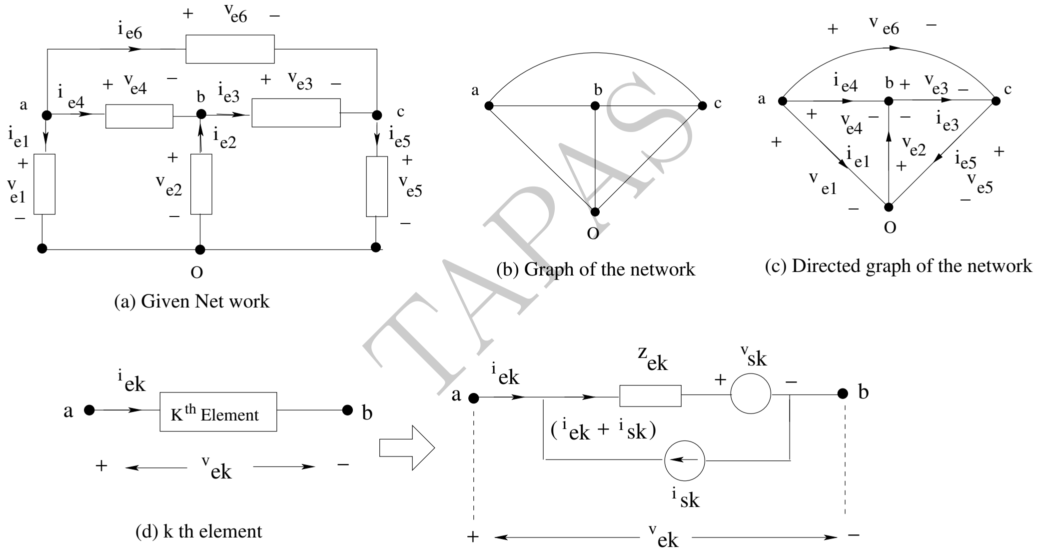


Figure 1:

2.2 Graph & directed graph of the network

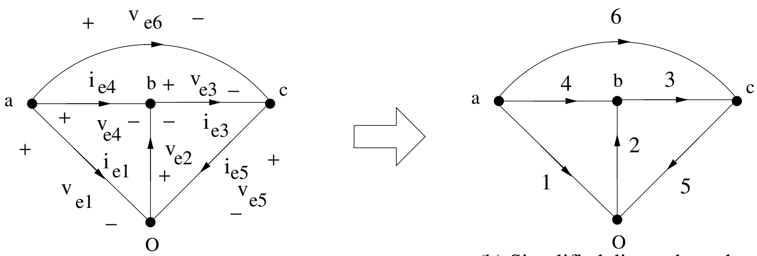
To draw the graph of the network shown in figure 1(a):

1. Draw the nodes with • symbols and write down their names : a, b, c and o.

2. Now join the nodes with continuous firm lines. Don't show the actual circuit elements of the network in the line. Graph of the network of figure 1(a) is shown in figure 1(b). The graph may be called the skeleton of the network. It may be noted that this graph has six elements.

3. When the directions of the currents (as assumed in the network) are also shown in the graph; we get a directed graph as shown in figure 1(c). We shall always use directed graphs where both direction of currents and polarities of voltages can be shown (figure 1(c)).

So far network analysis is concerned, we shall extensively use directed graph. But the directed graph showing all the voltages and currents with their directions and polarities becomes a clumsy diagram as shown in figure 2(a) Firstly we can easily see that the arrow shown in each element can be also used to indicate the polarity of the element voltage apart from the direction of the



(a) Directed graph showing voltages and currents Representation a bit clumsy

(b) Simplified directed graph Neat Representation

Figure 2:

current. Arrowhead represents the negative polarity of the element voltage v_e and back of the arrow represents the positive polarity of the voltage. Also we can avoid writing i_{ei} and v_{ei} ($i = 1$ to 6) in each element. We shall only write numbers 1 to 6 for each element in the directed graph and for each element we remember that two variable i_{ei} and v_{ei} exist. The simplified and neat directed graph is shown in figure 2(b).

2.3 Tree & Co-Tree of a Graph

For a given graph, a tree and co-tree are obtained as follows:

1. Let us assume the total number of elements in a graph is denoted by e. For the given graph $e = 6$
2. First draw the nodes.
3. Connect the nodes with elements (firm lines), such that no loop is formed and no node is left out alone. Look at figures 3(b).
4. The elements of a tree are called twigs.
5. If the number of nodes is n, then number of twigs, $t = (n - 1)$.
6. For a given graph we can construct many a tree.
7. The elements which are absent in a tree can be shown in a separate diagram (figure 3(c)) by dashed lines after removing the twigs.
8. The elements (with dashed lines), present in a co-tree are called links. Number of links is denoted by L.
9. Number of links, L is obviously equal to $e - t = e - n + 1$
10. Figure 3(d) show tree and co-tree together. Continuous lines are the twigs and the dashed lines are the links.

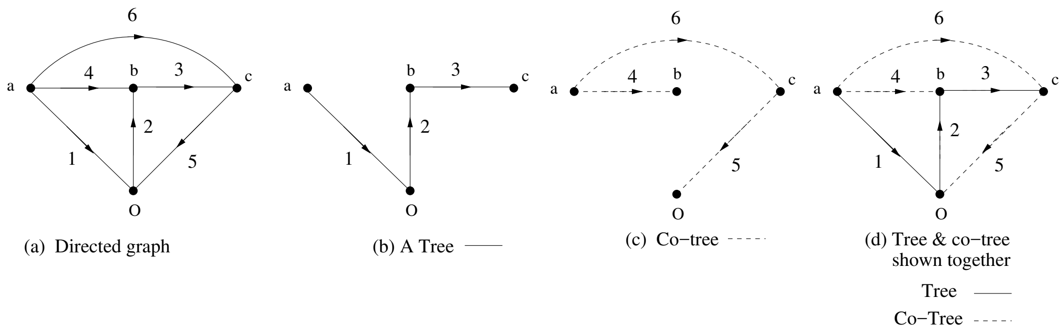


Figure 3:

For a given graph, one can draw several trees and co-trees for a given graph. This has been elaborated in figure 4 where 3 different trees are shown for the same graph.

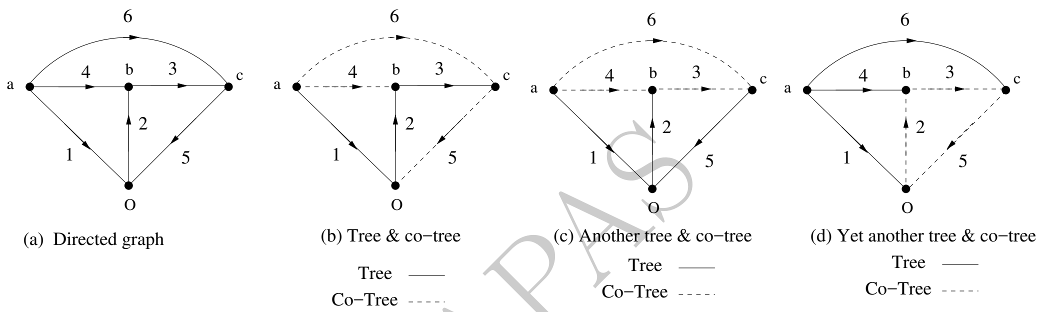


Figure 4:

It is always advantageous to number the twigs first and then number the links. The usefulness of numbering in this fashion will be explained later. We are now ready to use knowledge of graph theory to solve circuit problems in a systematic way.

3 Graph theory & Nodal method

Look at figures 5(a),(b),(c) where a network and with its graph, tree and co-tree are shown. Note that the twigs are marked from 1 to 3 while the links are numbered as 4 to 6

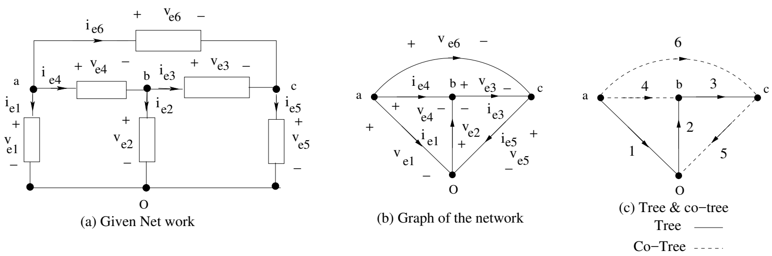


Figure 5:

$$
\begin{aligned}
\text{Total number of nodes } n &= 4 \\
\text{Total number of elements } e &= 6 \\
\text{number of twigs in the tree } t = n - 1 &= 3 \\
\text{number of links in the tree } L = e - t &= 3
\end{aligned}
$$

Now we write the KCL at the nodes a,b, c and o by assigning +ve sign for the current leaving a node.

$$\begin{aligned}
\text{KCL at node } a\text{:} \quad i_{e1} + i_{e4} + i_{e6} &= 0 \\
\text{KCL at node } b\text{:} \quad -i_{e2} + i_{e3} - i_{e4} &= 0 \\
\text{KCL at node } c\text{:} \quad -i_{e3} + i_{e5} - i_{e6} &= 0 \\
\text{KCL at node } o\text{:} \quad -i_{e1} + i_{e2} - i_{e5} &= 0
\end{aligned}$$

It can be easily shown that three of the four KCL equations written above are linearly independent. For example, KCL at node O, can be manufactured from the first three equations. The KCL equations at nodes a, b and c can be written in matrix form as follows.

$$\begin{array}{cc}
\text{Elements} \rightarrow & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} \text{Node } a \\ \text{Node } b \\ \text{Node } c \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix}
\end{array}
\begin{bmatrix} i_{e1} \\ i_{e2} \\ i_{e3} \\ i_{e4} \\ i_{e5} \\ i_{e6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In short hand the KCL equations at the nodes can be written as:

$$[A][i_e] = [0]$$

where,

$$\begin{aligned}
[A] &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix} \\
[i_e] &= \begin{bmatrix} i_{e1} & i_{e2} & i_{e3} & i_{e4} & i_{e5} & i_{e6} \end{bmatrix}^T
\end{aligned}$$

In general size of $[A]$ matrix is $(n-1) \times e$ and size of $[i_e]$ matrix is $e \times 1$

3.1 Expressing element voltages in terms of node voltages

Now if the node voltages, denoted as v_a, v_b and v_c (all voltages measured wrt reference node O) are known then the voltage across the elements can be expressed in terms of node voltages as follows:

$$\begin{aligned}
v_{e1} &= v_a \\
v_{e2} &= -v_b \\
v_{e3} &= v_b - v_c \\
v_{e4} &= v_a - v_b \\
v_{e5} &= v_c \\
v_{e6} &= v_a - v_c
\end{aligned}$$

Writing the above equations in matrix form we get,

$$\begin{array}{cc}
\text{Nodes} \rightarrow & \begin{array}{ccc} a & b & c \end{array} \\
\begin{array}{c} \text{element-1 voltage} \\ \text{element-2 voltage} \\ \text{element-3 voltage} \\ \text{element-4 voltage} \\ \text{element-5 voltage} \\ \text{element-6 voltage} \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}
\end{array}
\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} v_{e1} \\ v_{e2} \\ v_{e3} \\ v_{e4} \\ v_{e5} \\ v_{e6} \end{bmatrix}$$

In short notation, the above equations are written as: In short hand the KCL equations at the nodes can be written as:

$$[A]^T\,[v_n] = [v_e]$$

where,

$$\begin{aligned}
[A]^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \\
[v_n] &= \begin{bmatrix} v_a & v_b & v_c \end{bmatrix}^T = \text{node voltages} \\
[v_e] &= \begin{bmatrix} v_{e1} & v_{e2} & v_{e3} & v_{e4} & v_{e5} & v_{e6} \end{bmatrix}^T = \text{element voltages}
\end{aligned}$$

In nodal method of solving a circuit, we write KCL at the nodes and then solve for the node voltages. To summarize the results obtained we get two equations:

$$\begin{aligned}
[A]\,[i_e] &= [0] \\
[A]^T\,[v_n] &= [v_e]
\end{aligned}$$

4 Graph theory & loop method

In any circuit if you choose an arbitrary loop, KVL (sum of the voltages in the loop = 0) will be satisfied. In graph theory the selection of loops are done following a particular rule. These loops are called the *fundamental loops*.

What is a fundamental loop?

In the tree of a graph, if one and only one link is inserted then a loop will be formed and this is called a fundamental loop. Obviously the number of fundamental loops will be equal to the number of links. For the graph we are considering here, there will be three fundamental loops as the number of links is three. Links have already be numbered as 3, 4 and 5. We shall name the loops as L_4, L_5 and L_6. Referring to figure **??**(c), loop L_4 has elements 4, 2 & 1; loop L_5 has elements 5, 2 & 3 and loop L_6 has elements 6, 3, 2, & 1; Now we write the KVL in the fundamental loops L_4, L_5 and L_6.

$$\begin{aligned}
\text{KVL in loop } L_4\text{:}\quad -v_{e1} - v_{e2} + v_{e4} &= 0 \\
\text{KVL in loop } L_5\text{:}\quad v_{e2} + v_{e3} + v_{e5} &= 0 \\
\text{KVL in loop } L_6\text{:}\quad -v_{e1} - v_{e2} - v_{e3} + v_{e6} &= 0
\end{aligned}$$

In using matrices, the above equations can be written as:

$$\begin{array}{c} \text{Elements} \rightarrow \\ \hline \text{KVL in } L_4 \\ \text{KVL in } L_5 \\ \text{KVL in } L_6 \end{array}
\left[\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right]
\begin{bmatrix} v_{e1} \\ v_{e2} \\ v_{e3} \\ v_{e4} \\ v_{e5} \\ v_{e6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In short hand the KVL equations in the fundamental loops can be written as:

$$[B][v_e] = [0]$$

where,

$$[B] = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

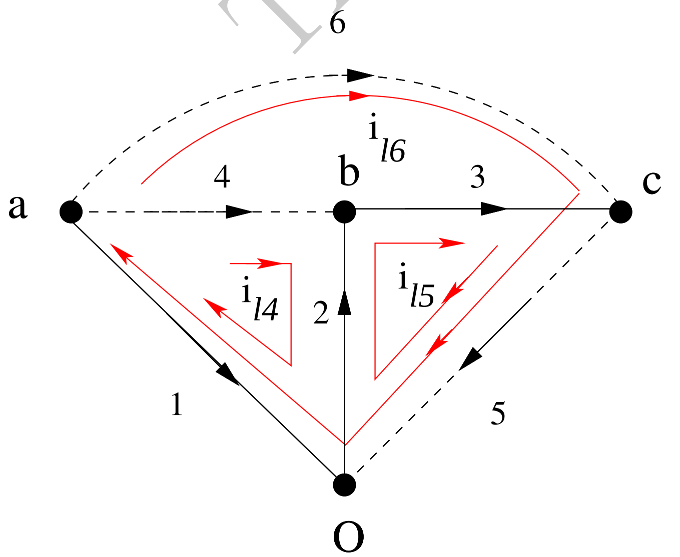
$$[v_e] = \begin{bmatrix} v_{e1} & v_{e2} & v_{e3} & v_{e4} & v_{e5} & v_{e6} \end{bmatrix}^T$$

In general size of $[B]$ matrix is $l \times e$ and size of $[v_e]$ matrix is $e \times 1$

4.1 Expressing element currents in terms of fundamental loop currents

Now if the link (loop) currents , denoted by i_{l4}, i_{l5} and i_{l6} are known then the current through the elements can be expressed in terms of fundamental loop currents. For getting these relations we refer to figure 6

$$\begin{aligned}
i_{e1} &= -i_{l4} - i_{l6} \\
i_{e2} &= -i_{l4} + i_{l5} - i_{l6} \\
‘i_{e3} &= i_{l5} - i_{l6} \\
i_{e4} &= i_{l4} \\
i_{e5} &= i_{l5} \\
i_{e6} &= i_{l6}
\end{aligned}$$



Closed path for Fundamental Loop currents

i_{l4} , i_{l5} and i_{l6}

Figure 6:

Writing the above equations in matrix form we get,

$$\begin{array}{r|ccc} \text{Fundamental loops} \rightarrow & 4 & 5 & 6 \\ \hline \text{element-1 current} & -1 & 0 & -1 \\ \text{element-2 current} & -1 & 1 & -1 \\ \text{element-3 current} & 0 & 1 & -1 \\ \text{element-4 current} & 1 & 0 & 0 \\ \text{element-5 current} & 0 & 1 & 0 \\ \text{element-6 current} & 0 & 0 & 1 \end{array} \begin{bmatrix} i_{l4} \\ i_{l5} \\ i_{l6} \end{bmatrix} = \begin{bmatrix} i_{e1} \\ i_{e2} \\ i_{e3} \\ i_{e4} \\ i_{e5} \\ i_{e6} \end{bmatrix}$$

It is interesting to note that the matrix involved in transforming the link currents to element currents is $[B]^T$ In short notation, the above equations are written as:

$$[B]^T\,[i_l] = [i_e]$$

In short notation, the above equations are written as:

where,

$$\begin{aligned} [B]^T &= \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ [i_l] &= \begin{bmatrix} i_{l4} & i_{l5} & i_{l6} \end{bmatrix}^T = \text{Fundamental loop currents} \\ [i_e] &= \begin{bmatrix} i_{e1} & i_{e2} & i_{e3} & i_{e4} & i_{e5} & i_{e6} \end{bmatrix}^T = \text{element voltages} \end{aligned}$$

In loop method of solving a circuit, we write KVL in the fundamental loops and then solve for the look currents. To summarize the results obtained we get two equations:

$$\begin{aligned} [B]\,[v_e] &= [0] \\ [B]^T\,[i_l] &= [i_e] \end{aligned}$$

4.2 Comments on [A] and [B] matrices

1. From the directed graph [A] matrix can be constructed very easily by inspection. Row wise entry corresponds to a particular node.
2. Go to a particular node assign +1 for element current leaving the node or -1 if current is converging the node.
3. Matrix [A] can be divided (partitioned) vertically into two parts as

   $$[A] = [\,[A_t]|[A_l]\,]$$

   where, $[A_t]$ is of size $(n-1)\times t$ corresponding to twigs and $[A_l]$ is of size $(n-1)\times l$ corresponding to links.
4. Matrix $[B]$ too can be constructed from the graph of the network.
5. We have to write down the KVL equations in the fundamental loops to get the elements of $[B]$ matrix which will be + or -1.

6. Matrix [B] can be divided (partitioned) vertically into two parts as

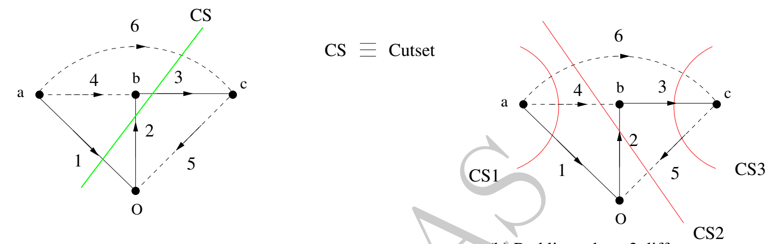
$$[B] = [\,[B_t]|[B_l]\,] = [[B_t]|[U_l]]$$

where, $[B_t]$ is of size $l \times t$ corresponding to twigs and $[B_l]$ is of size $l \times l$ corresponding to links. It is may be noted that $[B_l]$ is a square matrix of size $l \times l$ and it is an Identity matrix. Thus $[B_l] = [U_l]$.

7. Matrices $[A]$ and $[B]$ are orthogonal which means that $[A][B]^T = [0]$

5 Graph theory & Cutset method

A network or graph, you imagine to be cut in two halves such that in both the halves at least one node is present. The green line in figure 7(a), is a cutset denoted by CS. While drawing a cut set



(a) Green line is a general cutset

(b) Red lines show 3 different Fundamental CUT SET

Figure 7:

it may cut several twigs and or links. As can be seen, the green cutset has cut twigs 1,2 & 3 and a link 6. A cut set which cuts **only and only one twig** is called a **fundamental cutset**. So the green cutset is not a fundamental cutset as it cuts three twigs. In figure 7(b), three different cutsets have been shown with red lines. If you look carefully each one of them cuts a single twig. Hence these three cutsets are fundamental cutsets. The name of a cut set is given depending on which twig it cuts. CS1, CS2 and CS3 therefore are the names of the three cutsets.

KCL at the cutsets

At a particular cutset (no matter whether it is a fundamental cutset or not), current going into the cut set is equal to the current going out of the cutset. In other word algebraic sum of the currents at a cutset is zero. For example, at the green cutset of figure 7(a),

$$i_{e1} + i_{e3} + i_{e6} = i_{e2} \text{ or, } i_{e1} + i_{e3} + i_{e6} - i_{e2} = 0$$

Our focus will be in the fundamental cutsets since the relevant KCL equations will be linearly independent.

5.1 KCL equations at the fundamental cutsets & $[Q]$ matrix

Now we refer to figure 7(b) where three cutsets in red color) CS1, CS2 and CS3 are shown. we want to write KCL at each cutset. Recall that each cutset is associated with a twig. The direction of the twig current is assumed to be positive while writing the equations. Now we write the KCL at the nodes a,b, c and o by assigning +ve sign for the current leaving a node.

$$\begin{aligned}
\text{KCL at node CS1: } \; i_{e1} + i_{e4} + i_{e6} &= 0 \\
\text{KCL at node CS2: } \; -i_{e2} + i_{e4} - i_{e5} + i_{e6} &= 0 \\
\text{KCL at node CS3: } \; -i_{e3} + i_{e5} - i_{e6} &= 0
\end{aligned}$$

It can be easily shown that three

$$\begin{array}{c} \text{Elements} \rightarrow \\ \hline \text{CS1} \\ \text{CS2} \\ \text{CS3} \end{array} \left[\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{bmatrix} i_{e1} \\ i_{e2} \\ i_{e3} \\ i_{e4} \\ i_{e5} \\ i_{e6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In short hand the KCL equations at the cut sets can be written as:

$$[Q][i_e] = [0]$$

where,

$$\begin{aligned} [Q] &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \\ [i_e] &= \begin{bmatrix} i_{e1} & i_{e2} & i_{e3} & i_{e4} & i_{e5} & i_{e6} \end{bmatrix}^T \end{aligned}$$

In general size of $[Q]$ matrix is $t \times e$ and size of $[i_e]$ matrix is $e \times 1$

5.1.1 Expressing element voltages in terms of element voltages of the twigs

Now if the element voltages of the twigs denoted as $v_{et1}, v_{et2} v_{et3}$ are known then the voltage across the elements can be expressed in terms $v_{et1}, v_{et2} v_{et3}$ as follows:

$$\begin{aligned} v_{e1} &= v_{et1} \\ v_{e2} &= v_{et2} \\ v_{e3} &= v_{et3} \\ v_{e4} &= v_{et1} + v_{et2} \\ v_{e5} &= -v_{et2} - v_{et3} \\ v_{e6} &= v_{et1} + v_{et2} + v_{et3} \end{aligned}$$

Writing the above equations in matrix form we get,

$$\begin{array}{c} \text{Twigs} \rightarrow \\ \hline \text{element-1 voltage} \\ \text{element-2 voltage} \\ \text{element-3 voltage} \\ \text{element-4 voltage} \\ \text{element-5 voltage} \\ \text{element-6 voltage} \end{array} \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{array}\right] \begin{bmatrix} v_{et1} \\ v_{et2} \\ v_{et3} \end{bmatrix} = \begin{bmatrix} v_{e1} \\ v_{e2} \\ v_{e3} \\ v_{e4} \\ v_{e5} \\ v_{e6} \end{bmatrix}$$

In short notation, the above equations are written as: In short hand the KCL equations at the nodes can be written as:

$$[Q]^T [v_{et}] = [v_e]$$

where,

$$
\begin{aligned}
[Q]^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\
[v_{et}] &= \left[\begin{array}{ccc} v_{et1} & v_{et2} & v_{et3} \end{array} \right]^T = \text{Twig voltages} \\
[v_e] &= \left[\begin{array}{cccccc} v_{e1} & v_{e2} & v_{e3} & v_{e4} & v_{e5} & v_{e6} \end{array} \right]^T = \text{element voltages}
\end{aligned}
$$

In cutset method of solving a circuit, we write KCL at the fundamental cutsets and then solve for the twig voltages. To summarize the results obtained we get two equations:

$$
\begin{aligned}
[Q]\,[i_e] &= [0] \\
[Q]^T\,[v_{et}] &= [v_e]
\end{aligned}
$$

Properties of $[Q]$ matrix

1. Matrix [Q] can be divided (partitioned) vertically into two parts as

$$[Q] = [\,[Q_t]|[Q_l]\,] = [\,[U_t]|[Q_l]]$$

   where, $[Q_t]$ is of size $t \times t$ corresponding to twigs and $[Q_l]$ is of size $t \times l$ corresponding to links.

2. It is may be noted that $[B_t]$ is a square matrix of size $t \times t$ and it is an Identity matrix. Thus $[B_t] = [U_t]$.

3. Also matrices $[B]$ and $[Q]$ are orthogonal which means that $[B][Q]^T = [0]$

6 Relationship among $[A]$, $[B]$ and $[Q]$ matrices

Before we start establishing relationships, let us summarize the results obtained in previous sections below.

Equations involving matrix $[A]$

$$
\begin{aligned}
[A]\,[i_e] &= [0] \\
[A]^T\,[v_n] &= [v_e] \\
[A] &= [\,[A_t]|[A_l]] \\
\therefore\ [A]^T &= [\,[A_t]|[A_l]]^T = \begin{bmatrix} [A_t]^T \\ [A_l]^T \end{bmatrix}
\end{aligned}
$$

Equations involving matrix $[B]$

$$
\begin{aligned}
[B]\,[v_e] &= [0] \\
[B]^T\,[i_l] &= [i_e] \\
[B] &= [\,[B_t]|[U_l]] \\
\text{where, } [U_l] &= \text{Identity matrix of size } l \times l
\end{aligned}
$$

$$\therefore\ [B]^T = [\,[B_t]|[U_l]]^T = \begin{bmatrix} [B_t]^T \\ \hline [U_l] \end{bmatrix}$$

Equations involving matrix $[Q]$

$$[Q]\ [i_e] = [0]$$
$$[Q]^T\ [v_{et}] = [v_e]$$
$$[Q] = [\,[U_t]|[Q_l]]$$
$$\text{where, } [U_t] = \text{Identity matrix of size } t \times t$$
$$\therefore\ [Q]^T = [\,[U_t]|[Q_l]]^T = \begin{bmatrix} [U_t] \\ \hline [Q_l]^T \end{bmatrix}$$

Relationships are found out based on the facts

- $[A]$ and $[B]$ are orthogonal.
- Which means $[A][B]^T = [0]$
- $[B]$ and $[Q]$ are orthogonal.
- Which means $[B][Q]^T = [0]$

6.1 Relating $[A]$ and $[B]$

$$\begin{aligned} [A]\,[B]^T &= [0] \\ \text{or, } [\,[A_t]|[A_l]] \begin{bmatrix} [B_t]^T \\ \hline [U_l] \end{bmatrix} &= [0] \\ \text{or, } [A_t]\,[B_t]^T + [A_l]\,[U_l] &= [0] \\ \text{or, } [A_t]\,[B_t]^T &= -\,[A_l] \\ \text{or, } [B_t]^T &= -\,[A_t]^{-1}\,[A_l] \end{aligned}$$

It may be noted that $[B_l] = [U_l]$ is already known to be an unitary matrix. $[B_t]$ is now obtained from the above equation. Matrix $[B]$ can be thus obtained from the $[A]$ matrix. However, to calculate $[B_t]$, one has to take inverse of $[A_t]$ which may be tedious.

6.2 Relating $[Q]$ and $[B]$

$$\begin{aligned} [Q]\,[B]^T &= [0] \\ \text{or, } [\,[U_t]|[Q_l]] \begin{bmatrix} [B_t]^T \\ \hline [U_l] \end{bmatrix} &= [0] \\ \text{or, } [B_t]^T + [Q_l] &= [0] \\ \text{or, } [B_t]^T &= -\,[Q_l] \end{aligned}$$

$[B_t]$ is now obtained from the above equation in terms of $[Q_l]$. Matrix $[B]$ can thus be obtained from the $[Q]$ matrix. In this case, $[B_t]$ is simply equal to $[Q_l]$ and no inversion of matrix is necessary. Matrix $[Q]$ can be developed from the cutsets.

6.3 Relating $[Q]$ and $[A]$

We have already obtained,

$$\begin{aligned} [B_t]^T &= -[A_t]^{-1}[A_l] \\ \text{and } [B_t]^T &= -[Q_l] \\ \therefore\ [A_t]^{-1}[A_l] &= -[Q_l] \end{aligned}$$

7 Solving circuit with graph theory

In the earlier sections we have dealt with variables $[v_e]$ and $[i_e]$ which are nothing but the terminal voltages & and the currents of elements. When we say that we want to solve a circuit with known sources and known circuit elements, we must know what exist in the boxes representing elements. The situation is depicted in figure 8 where a general k^th element connected between nodes a and b has been shown. A general element will consist of a voltage source v_{sk} in series with an impedance z_{ek} and across this series combination a current source i_{sk} is present.

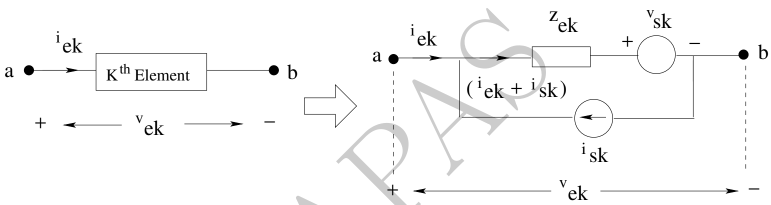


Figure 8:

7.1 Finding fundamental loop currents

Referring to the k^{th} element shown in figure 8, v_{ek} can be written as,

$$v_{ek} = z_{ek}i_{ek} + v_{sk} + z_{ek}i_{sk}$$

The above equation can be written for all the elements by putting element number k. In consolidated form it appears as follows.

$$\begin{aligned} [v_e] &= [z_e][i_e] + [v_s] + [z_e][i_s] \\ \text{Pre-multiplying} &\quad \text{both sides by } [B] \\ [B][v_e] &= [B][z_e][i_e] + [B][v_s] + [B][z_e][i_s] \\ \text{but, } [B][v_e] &= [0] \\ \therefore\ [B][z_e][i_e] + [B][v_s] + [B][z_e][i_s] &= [0] \\ \text{or, } [B][z_e][i_e] &= -[B][v_s] - [B][z_e][i_s] \\ \text{but, } [i_e] &= [B]^T[i_l] \\ \\ \text{or, } [B][z_e][B]^T[i_l] &= -[B][v_s] - [B][z_e][i_s] \end{aligned}$$

7.2 Finding fundamental node voltages

Writing KCL at node a in figure 8 gives

$$i_{ek} + i_{sk} = \frac{(v_{ek} - v_{sk})}{z_{ek}} \tag{1}$$

$$\text{or, } i_{ek} = \frac{v_{ek}}{z_{ek}} - \frac{v_{sk}}{z_{ek}} - i_{sk} \tag{2}$$

$$\text{or, } i_{ek} = y_{ek}v_{ek} - y_{ek}v_{sk} - i_{sk} \tag{3}$$

$$\tag{4}$$

The above equation can be written for all the elements by putting element number k. In consolidated form and it appears as follows.

$$\begin{aligned}
[i_e] &= [Y_e]\,[v_e] - [i_s] - [Y_e]\,[v_s] \\
\text{Pre-multiplying} &\quad \text{both sides by } [A] \\
[A]\,[i_e] &= [A]\,[Y_e]\,[v_e] - [A]\,[i_s] - [A]\,[Y_e]\,[v_s] \\
\text{but, } [A]\,[i_e] &= [0] \\
\therefore\ [A]\,[Y_e]\,[v_e] &= [A]\,[i_s] + [A]\,[Y_e]\,[v_s] \\
\therefore\ [A]\,[Y_e]\,[A]^T\,[v_n] &= [A]\,[i_s] + [A]\,[Y_e]\,[v_s]
\end{aligned}$$

7.3 Finding twig voltages (v_{et})

We have already got,

$$\begin{aligned}
[i_e] &= [Y_e]\,[v_e] - [i_s] - [Y_e]\,[v_s] \\
\text{Pre-multiplying} &\quad \text{both sides by } [Q] \\
[Q]\,[i_e] &= [Q]\,[Y_e]\,[v_e] - [Q]\,[i_s] - [Q]\,[Y_e]\,[v_s] \\
\text{but, } [Q]\,[i_e] &= [0] \\
\therefore\ [Q]\,[Y_e]\,[v_e] &= [Q]\,[i_s] + [Q]\,[Y_e]\,[v_s] \\
\therefore\ [Q]\,[Y_e]\,[Q]^T\,[v_{et}] &= [Q]\,[i_s] + [Q]\,[Y_e]\,[v_s]
\end{aligned}$$