Tutorial 8 Solution

(1)

$$
\begin{array}{c} \text{Branches} \rightarrow \\ A = \end{array}
\begin{array}{c}
\begin{array}{cccccccc} ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0
\end{bmatrix}
\end{array}
$$

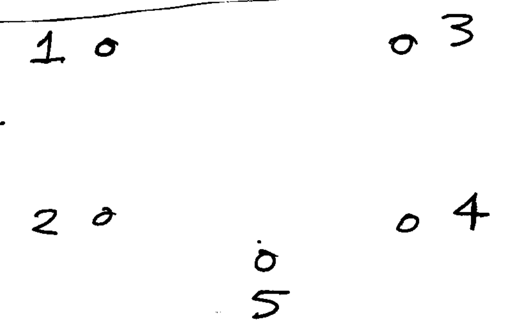
Observe that the sum of all the rows is not zero. (In other words some columns does not have both +1 an -1.) Therefore, the given matrix must be the reduced incidence matrix. We can therefore write the complete incidence matrix as

$$
A = \begin{array}{c}
\begin{array}{cccccccc} ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \end{array} \leftarrow \boxed{\text{Branches}} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{l} \\ -1 \\ -2 \\ -3 \\ -4 \\ -5 \end{array}
$$

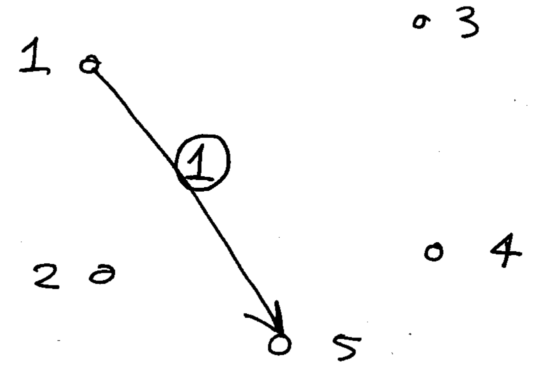
$\uparrow$ $\boxed{\text{nodes}}$

There are 5 nodes and 8 edges

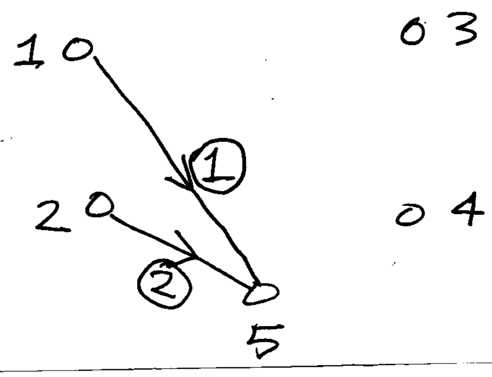
We first draw the 5 nodes arbitrarily



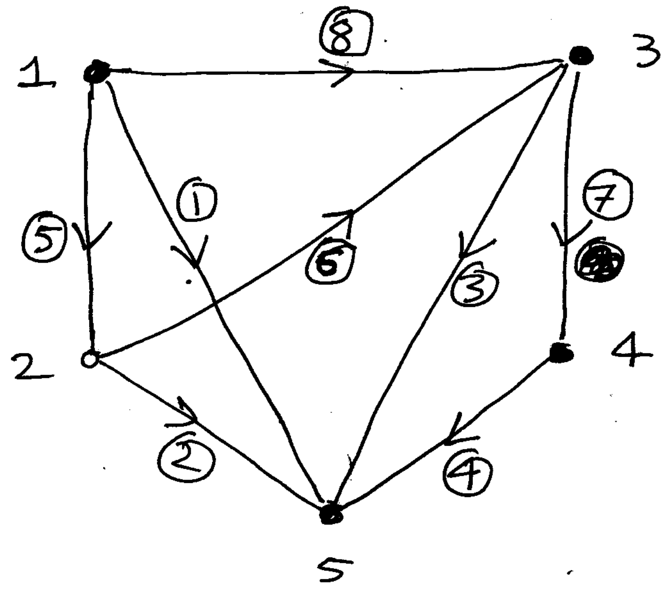
Then observe that branch ① starts at node 1 and stops at node 5 (observe the first column) so we can draw branch ①



Similarly branch ② starts at node 2 and stops at node 5



In this way we can draw all the branches



Do distinguish node numbers from edge numbers, edge numbers are circled.

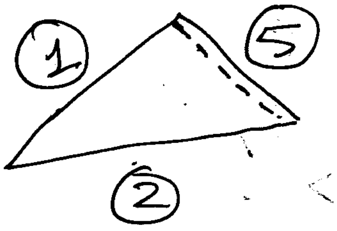
2

$$
\begin{array}{c@{}l}
Q = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{l} -1 \\ -2 \\ -3 \\ -4 \end{array} \text{ cut sets} \\
\text{Branches} \rightarrow \quad \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} &
\end{array}
$$

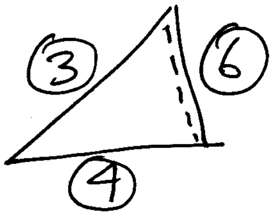
Note that branch 1, 2, 3, 4 are present only in ~~only~~ cutsets 1, 2, 3, 4 respective. Therefore 1, 2, 3, 4 are the twigs of the underlying tree. So 5, 6 & 7 are the links.

Now branch 5 is present in ~~two~~ cut sets 1 & 2 Therefore branches 5, 1 & 2 should form a loop.

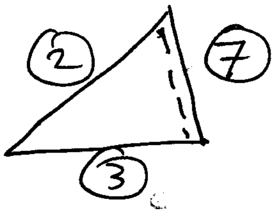
Symbols used
- - - - - : links
———— twigs



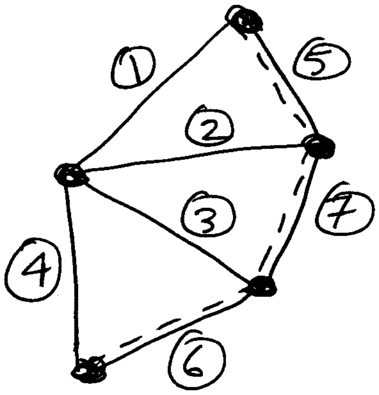
Similarly, branch 6 is present in cutsets 3 and 4. Therefore branches 6, 3 & 4 should form a loop



similarly branches 7, 2 and 3 will form a loop.

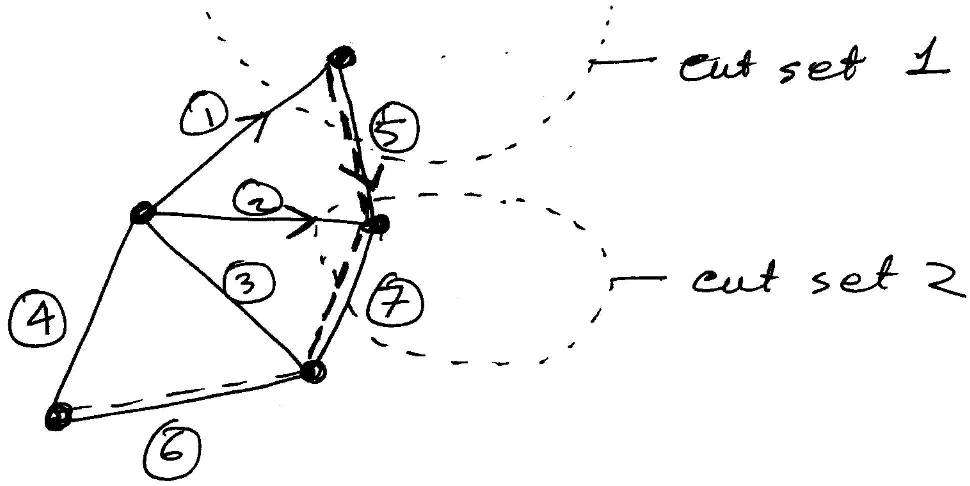


Now if we cleverly combine the above three figures we will get



Now we have to put the arrows consistently with the Q matrix

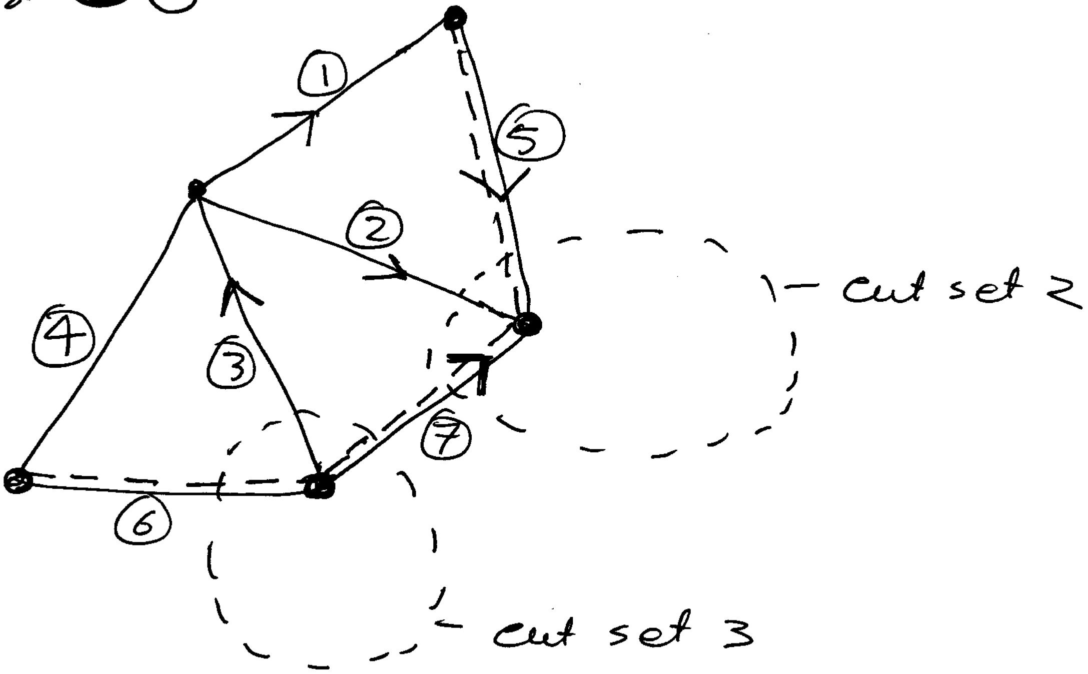
From column 5 of Q, we can say with respect to cutset 1 5 and 1 are in opposite direction, but 5 and 2 are in same direction with respect to cut set 2



From column 7 of Q we can say

7 and 2 are in same direction with respect to cutset 2

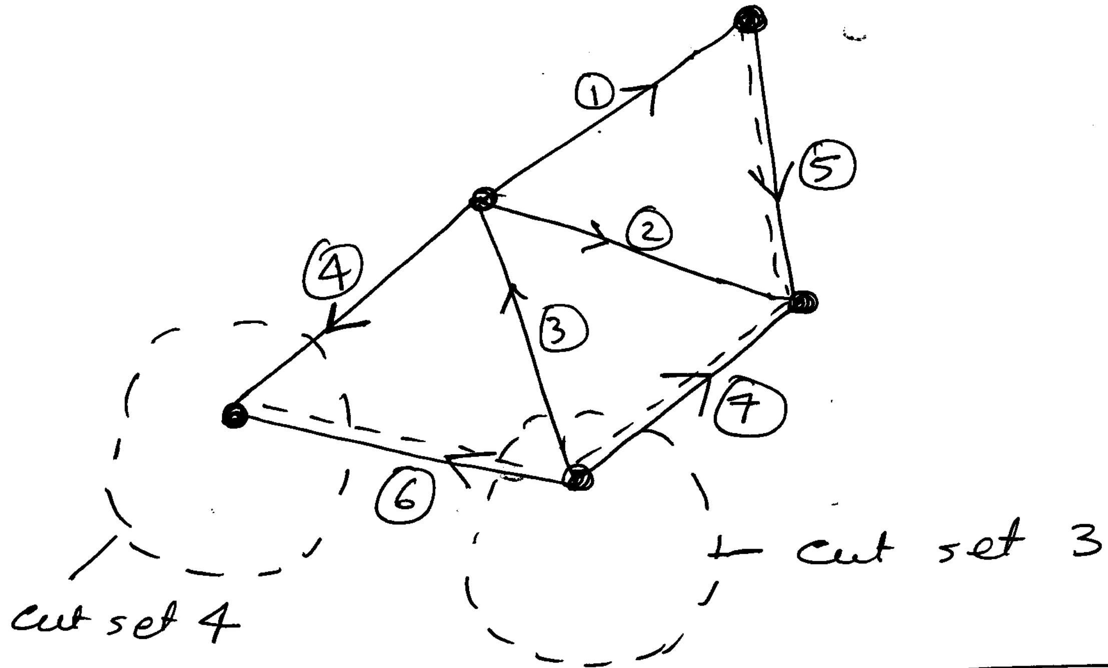
7 and 3 are in same direction with respect to cutset 3

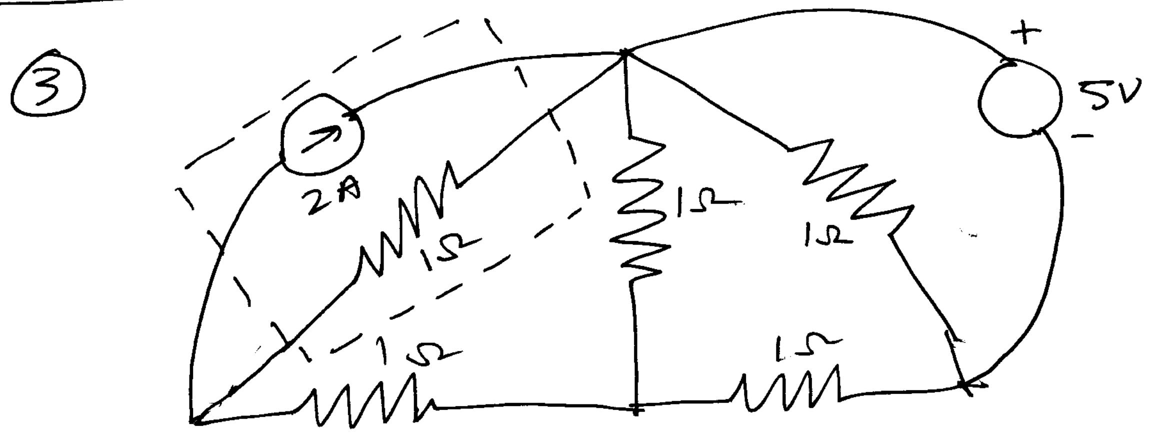


Finally from column 6 of Q we see that

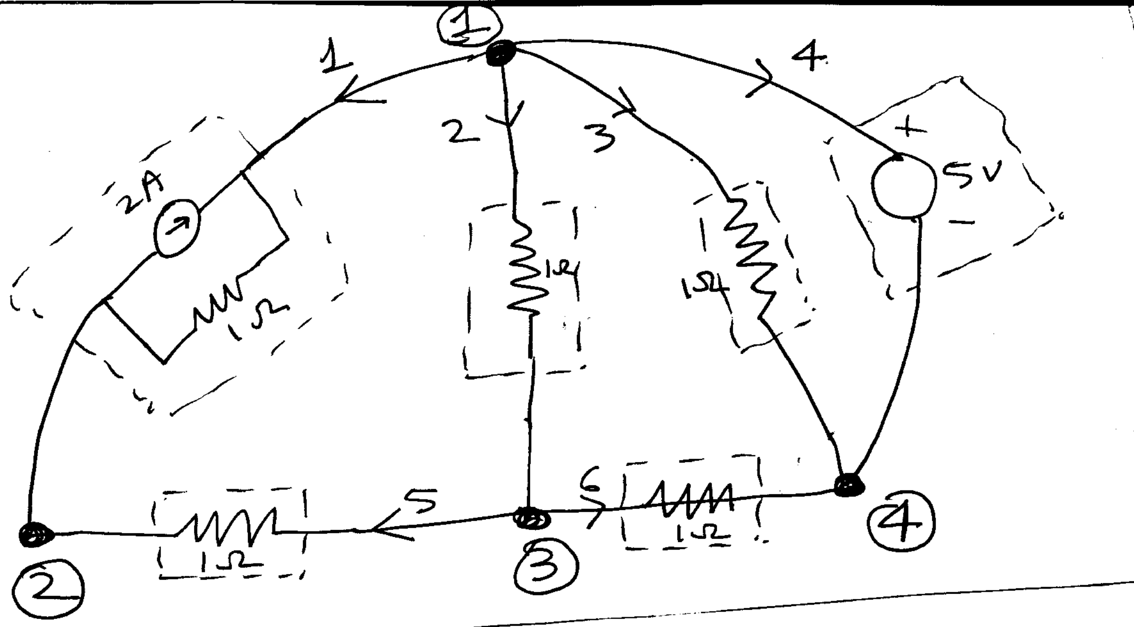
6 & 3 are in same direction with respect to cut set 3

6 & 4 are in same direction with respect to cut set 4

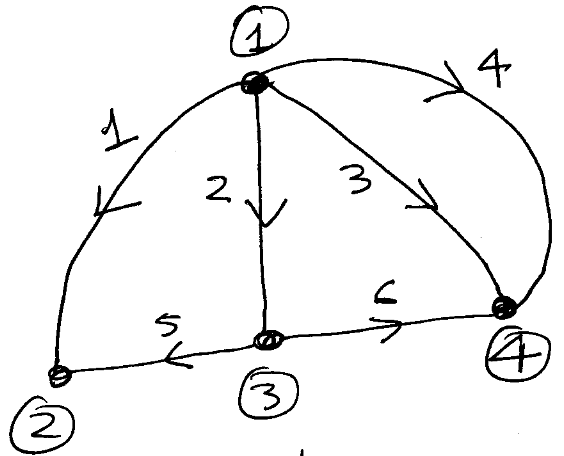




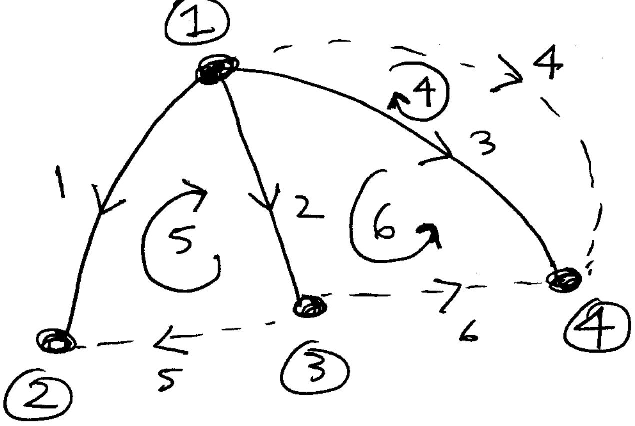
Since we have to use mesh analysis if we treat the current source alone as an independent branch then the ~~volt~~ ~~admi~~ impedence of that branch will be infinite (α). We will not be able to solve the problem. So we will ~~treat~~ treat the current source together with its (parallel) ~~branch~~ adjacent 1Ω resistance as a single branch.



Node numbers are in circle



graph.



Tree and loops

Symbols

—— twigs

- - - links

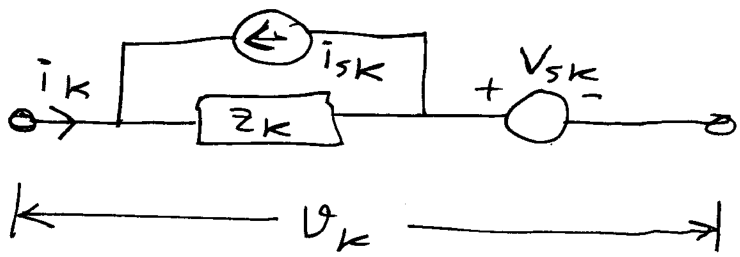
↻ loops

Note that twigs are numbered first & then the links

$B =$

loops \ Edges →	1	2	3	4	5	6
4	0	0	−1	+1	0	0
5	−1	+1	0	0	+1	0
6	0	+1	−1	0	0	+1

Now the general structure of a branch can be chosen as



Therefore, $v_k = (i_k + i_{sk}) z_k + V_{sk}$

considering all edges we can write

$$\begin{bmatrix} \vdots \\ v_k \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & z_k & \\ & & \ddots \end{bmatrix} \left(\begin{bmatrix} \\ i_k \\ \\ \end{bmatrix} + \begin{bmatrix} i_{sk} \\ \\ \\ \end{bmatrix} \right) + \begin{bmatrix} \\ V_{sk} \\ \\ \end{bmatrix}$$

For the given graph

$$z_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} i_{sk} \end{bmatrix} = \begin{bmatrix} +2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_{sk} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ +5 \\ 0 \\ 0 \end{bmatrix}$$

[carefully consider the polarities of sources & corresponding ± sign]

Now $\left[V_k\right] = \left[Z_k\right]\left(\left[i_k\right] + \left[i_{sk}\right]\right) + \left[V_{sk}\right]$

Multiplying both side with B

$$\Rightarrow B\left[V_k\right] = B\left[Z_k\right]\left(\left[i_k\right] + \left[i_{sk}\right]\right) + B\left[V_{sk}\right]$$

$$= 0$$

$$\left[\because B\left[V_k\right] = 0 \text{ (KVL)}\right]$$

$$\therefore B\left[Z_k\right]\left[i_k\right] = -B\left[V_{sk}\right] - B\left[Z_k\right]\left[i_{sk}\right]$$

$$\Rightarrow B\left[Z_k\right]B^T\left[i_L\right] = -B\left[V_{sk}\right] - B\left[Z_k\right]\left[i_{sk}\right]$$

where $\left[i_L\right] = \begin{bmatrix} i_{L4} \\ i_{L5} \\ i_{L6} \end{bmatrix}$ = loop currents

Now let us put the values of B, $\left[Z_k\right]$, $\left[V_{sk}\right]$, $\left[I_{sk}\right]$ etc.

$$RHS = -B\left[V_{sk}\right] - B\left[Z_k\right]\left[i_{sk}\right]$$

$$= -B\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= -B\left(\begin{bmatrix}0\\0\\0\\5\\0\\0\end{bmatrix} - \begin{bmatrix}2\\0\\0\\0\\0\\0\end{bmatrix}\right) = -B\begin{bmatrix}-2\\0\\0\\5\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}-5\\+2\\0\end{bmatrix}$$ (putting the value of B)

Now $$B\left[Z_k\right]B^T = B\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&0&&\\&&&&1&\\&&&&&1\end{bmatrix}\begin{bmatrix}0&-1&0\\0&1&1\\-1&0&-1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}0&0&-1&1&0&0\\-1&1&0&0&1&0\\0&1&-1&0&0&1\end{bmatrix} \times \begin{bmatrix}0&-1&0\\0&1&1\\-1&0&-1\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}+1&0&1\\0&3&1\\+1&1&3\end{bmatrix}$$

$$\therefore \begin{bmatrix}+1&0&1\\0&3&1\\+1&1&3\end{bmatrix}\left[i_L\right] = \begin{bmatrix}-5\\+2\\0\end{bmatrix}$$

You may now use a matrix inversion or solve the equations in any other manner

$$\boxed{\begin{aligned}&\cancel{+i_{L4} + i_{L6} = -5}\\&\cancel{3i_{L5} + i_{L6} = 2}\\&-i_{L1} + i_{L2} = 0 \Rightarrow i_{L1} = i_{L2}\end{aligned}}$$

$$\Rightarrow i_{L4} + i_{L6} = -5 \quad \text{--- (i)}$$

$$3 i_{L5} + i_{L6} = 2 \quad \text{--- (ii)}$$

$$i_{L4} + i_{L5} + 3 i_{L6} = 0 \quad \text{--- (iii)}$$

$$\text{(iii)} - \text{(i)} - 2\,\text{(ii)} \Rightarrow -5 i_{L5} = 1 \Rightarrow i_{L5} = -\frac{1}{5}$$

$$\therefore i_{L6} = 2 + \frac{3}{5} = \frac{13}{5}$$

and $i_{L4} = \frac{1}{5} - \frac{39}{5} = -\frac{38}{5}$

Now the branch voltages and currents can be found easily

---

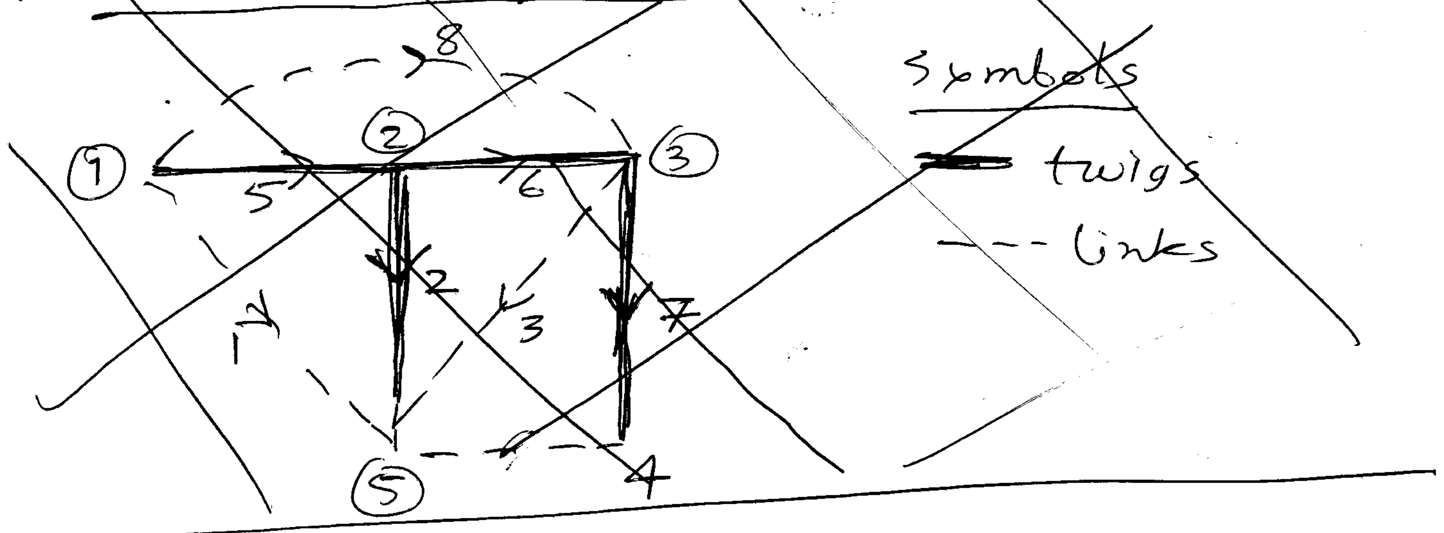
(6)

$$A = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & & & & 1 & & & 1 \\ & 1 & & & -1 & 1 & & \\ & & 1 & & & -1 & 1 & -1 \\ & & & 1 & & & -1 & \\ -1 & -1 & -1 & -1 & & & & \end{bmatrix} \end{array}$$

<u>Branch voltages interms of node voltages</u>

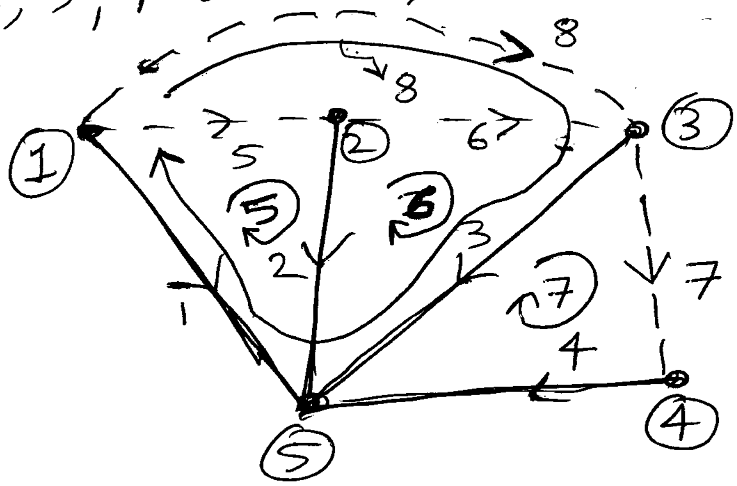
$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \\ V_{e4} \\ V_{e5} \\ V_{e6} \\ V_{e7} \\ V_{e8} \end{bmatrix} = A^T \begin{bmatrix} V_{n1} \\ V_{n2} \\ V_{n3} \\ V_{n4} \\ V_{n5} \end{bmatrix}$$

~~Chosen tree and loops~~



Tree and loops

You may choose any tree of your choice for this problem. But note that edges 1, 2, 3, 4 (i.e. the first 4 edges form a tree). So I am choosing 1, 2, 3, 4 as my tree.

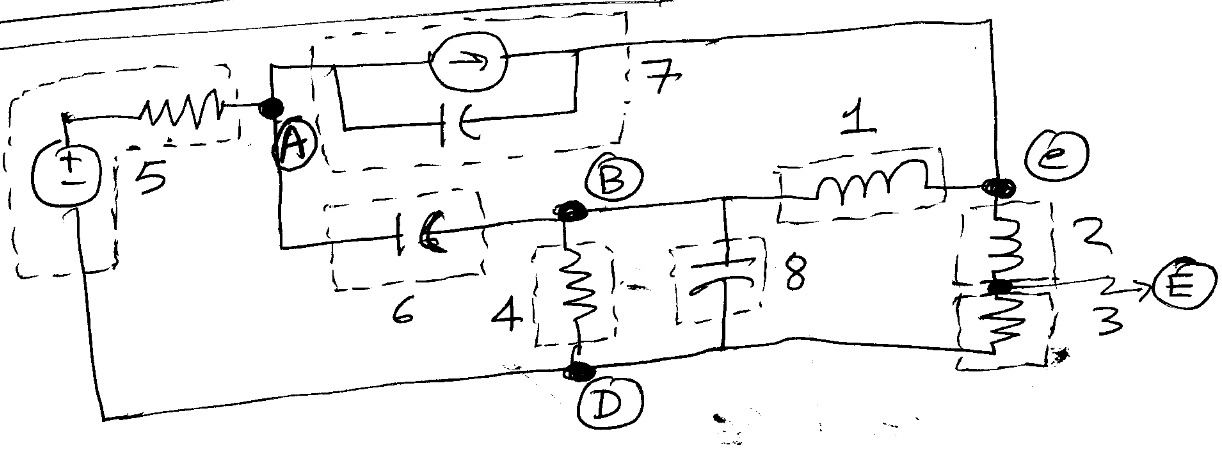


$$\therefore\ B = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \text{loops} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} -1 & +1 & & & +1 & & & \\ & -1 & +1 & & & +1 & & \\ & & -1 & +1 & & & +1 & \\ -1 & & +1 & & & & & +1 \end{bmatrix} \end{array}$$

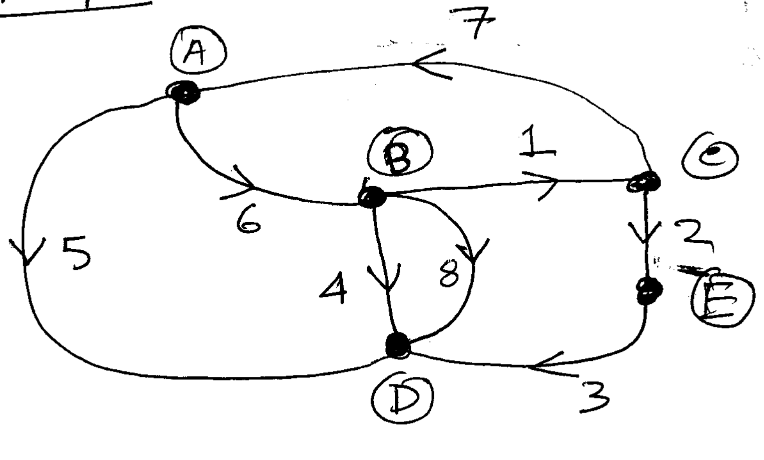
Branch currents in terms of loop currents

$$\begin{bmatrix} i_{e1} \\ i_{e2} \\ i_{e3} \\ i_{e4} \\ i_{e5} \\ i_{e6} \\ i_{e7} \\ i_{e8} \end{bmatrix} = B^T \begin{bmatrix} i_{L5} \\ i_{L6} \\ i_{L7} \\ i_{L8} \end{bmatrix}$$

(5)

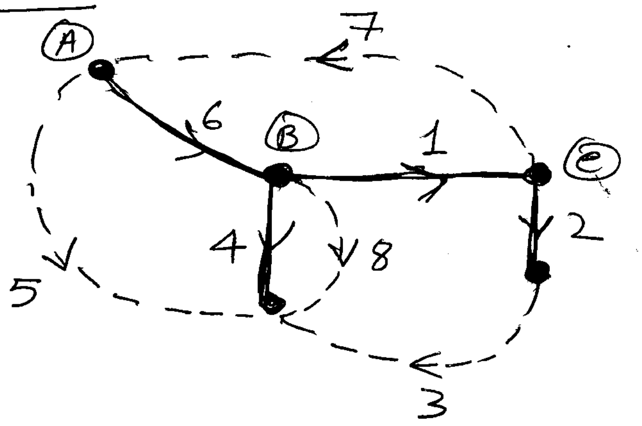


:- Graph



Note that Node E is unnecessary, but since edge 2 and 3 are given as 2 different edges in the question, we are following it.

Tree



Since the edge numbers are already given in the question we are not able to name the twigs first and the the links.

Cutset matrix

$$
Q = \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\left[\begin{array}{cccccccc}
+1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & +1 & +1 & +1 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & +1 & +1 & -1 & 0
\end{array}\right]
\end{array}
$$

Relation between twig and branch voltages

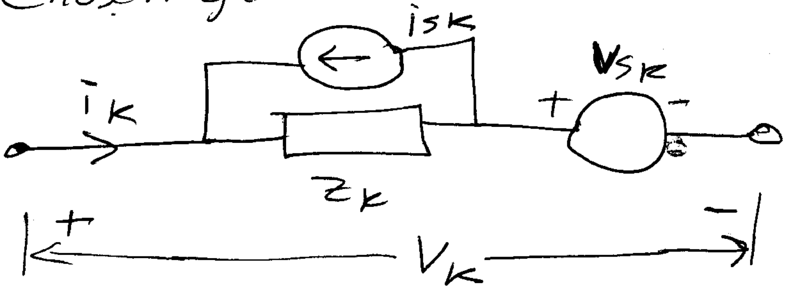
$$
\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \\ V_{e4} \\ V_{e5} \\ V_{e6} \\ V_{e7} \\ V_{e8} \end{bmatrix} = Q^T \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e4} \\ V_{e6} \end{bmatrix}
$$

KCL

$$
Q \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \\ \vdots \\ V_{e8} \end{bmatrix} = 0
$$

Circuit equations

Chosen general structure of any branch



$$\therefore\ i_k = (V_k - V_{sk})/Z_k - i_{sk}$$

$$= Y_k (V_k - V_{sk}) - i_{sk}$$

$$\left[\text{where } Y_k = \frac{1}{Z_k}\right]$$

For all branches

$$[i_k] = [Y_k]\left([V_k] - [V_{sk}]\right) - [i_{sk}]$$

$$\Rightarrow Q[i_k] = 0 = Q[Y_k]\left([V_k] - [V_{sk}]\right) - Q[i_{sk}]$$

$$\Rightarrow Q[Y_k][V_k] = Q[Y_k][V_{sk}] + Q[i_{sk}]$$

$$\Rightarrow Q[Y_k]Q^T[\text{twig voltages}] = Q[Y_k][V_{sk}] + Q[i_{sk}]$$

$$= Q\left([Y_k][V_{sk}] + [i_{sk}]\right)$$

For the given graph

$$[V_{sk}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad [i_{sk}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i \\ 0 \end{bmatrix}$$

[be careful about the $\pm$ sign]

Q is derived before

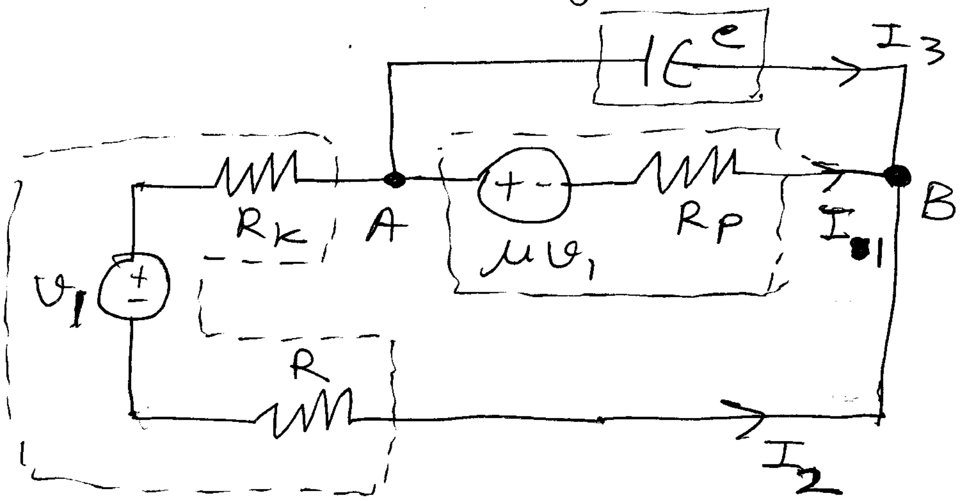
and $[y_k] = \begin{bmatrix} 1/L_1 s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/L_2 s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/R_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/R_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/R_5 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_6 s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_7 s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_8 s \end{bmatrix}$

and $[\text{twig voltages}] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e4} \\ V_{e6} \end{bmatrix}$

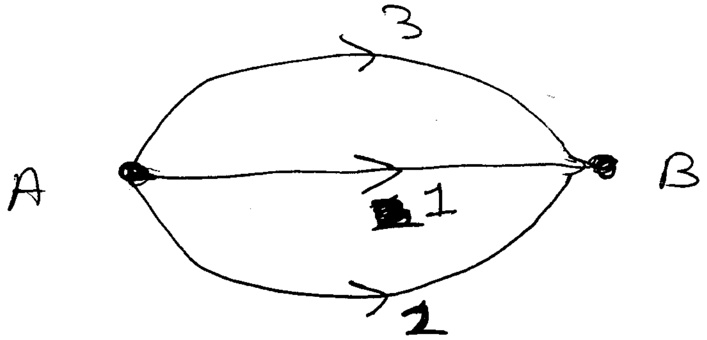
And the nodal equation is

$$\boxed{Q[Y_k]Q^T[\text{twig voltages}] = Q([Y_k][V_{sk}] + [i_{sk}])}$$

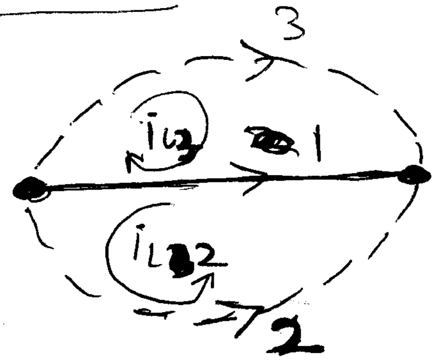
④ To make the solution simpler let us change the branch and node numberings from what is given in the question.



Graph



Tree and loops



Loop matrix

$$B = \begin{array}{c} \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{c|c|c} -1 & +1 & 0 \\ \hline -1 & 0 & +1 \end{array}\right] \end{array}$$

Branch impedence matrix

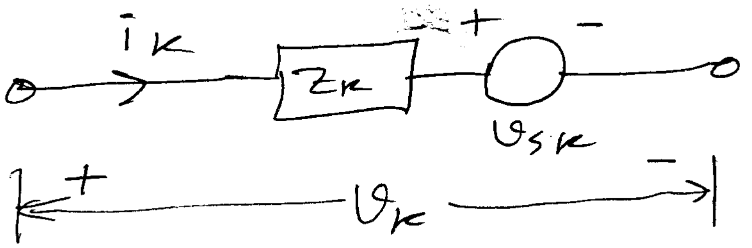
$$[Z_k] = \left[\begin{array}{c|c|c} R_p & 0 & 0 \\ \hline 0 & R+R_k & 0 \\ \hline 0 & 0 & \frac{1}{Cs} \end{array}\right]$$

Source voltage matrix

$$[V_{sk}] = \begin{bmatrix} \mu v_1 \\ v_1 \\ 0 \end{bmatrix}$$

Loop equation

chosen general structure of branches



[we need not use a current source, since there is no current source in this problem]

$$v_k = Z_k i_k + v_{sk}$$

$$\Rightarrow [v_k] = [Z_k][i_k] + [v_{sk}]$$

$$\Rightarrow B[v_k] = 0 = B[Z_k][i_k] + B[v_{sk}]$$

$$\Rightarrow B[Z_k][i_k] = -B[v_{sk}]$$

$$\Rightarrow B[Z_k]B^T[\text{loop currents}] = -B[v_{sk}]$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_p & & \\ & R+R_k & \\ & & \frac{1}{Cs} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{L_2} \\ i_{L_3} \end{bmatrix}$$

$$= -\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu v_1 \\ v_1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -R_p & -R_p \\ R+R_k & 0 \\ 0 & 1/Cs \end{bmatrix} \begin{bmatrix} i_{L_2} \\ i_{L_3} \end{bmatrix} = -\begin{bmatrix} 1-\mu \\ -\mu \end{bmatrix} v_1$$

$$\Rightarrow \begin{bmatrix} R_p+R+R_k & R_p \\ R_p & R_p+\frac{1}{Cs} \end{bmatrix} \begin{bmatrix} i_{L_2} \\ i_{L_3} \end{bmatrix} = -\begin{bmatrix} 1-\mu \\ -\mu \end{bmatrix} v_1 \quad \text{--- (i)}$$

Solution of equation

Matrix inversion may now be difficult so we may do other manipulations.

The goal is to find source current only that means we need only $(-i_{L_2})$

From equation (i) we have

$$(R_p+R+R_k)\, i_{L_2} + R_p\, i_{L_3} = -(1-\mu) v_1 \quad \text{--- (ii)}$$

$$R_p\, i_{L_2} + \left(R_p+\frac{1}{Cs}\right) i_{L_3} = +\mu v_1 \quad \text{--- (iii)}$$

$$\text{(ii)} \times R_p - \text{(iii)} \times (R+R_p+R_k) \Rightarrow$$

$$\left(R_p^2 - (R+R_p+R_k)\left(R_p+\frac{1}{Cs}\right)\right) i_{L_2} = \left(-R_p(1-\mu) + (R+R_p+R_k)\mu\right) \times v_1$$

from (ii) $\times \left(R_p + \frac{1}{cs}\right)$ − (iii) $\times R_p \Rightarrow$

$$\left(\left(R+R_p+R_k\right)\left(R_p+\frac{1}{cs}\right)-R_p^2\right) i_{L2} = \left((\mu-1)\left(R_p+\frac{1}{cs}\right)-\mu R_p\right) v_1$$

$$\therefore i_{L2} = \frac{(\mu-1)\left(R_p+\frac{1}{cs}\right)-\mu R_p}{\left(R+R_p+R_k\right)\left(R_p+\frac{1}{cs}\right)-R_p^2} v_1$$

$$\therefore \text{Source current} = -i_{L2} = \frac{R_p+\frac{1}{cs}-\mu\left(\frac{1}{cs}\right)}{\left(R_p+\frac{1}{cs}\right)\left(R+R_p+R_k\right)-R_p^2}$$

[KINDLY CHECK MY CALCULATION, I AM NOT VERY SURE]

$$\therefore \text{Driving point admittance} = \frac{R_p+\frac{1}{cs}-\mu\left(\frac{1}{cs}\right)}{\left(R_p+\frac{1}{cs}\right)\left(R+R_p+R_k\right)-R_p^2}$$