Electrical Technology: Self & Mutual inductances

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Contents

1 Goals of the lesson

After going through this lesson, students are expected to understand about the following.

1. Dot (•) markings on mutually coupled coils and its significance.

2. Self Inductance(L): Flux linkage when the coil carries 1 A current. If the coil caries a current of i, then voltage across it is $L\frac{di}{dt}$ and the terminal through which i is entering is +ve and the other terminal is -ve.

3. Mutual Inductance between two coupled coils (M): Flux linkage with the second coil when 1 A flows through the first coil or flux linkage with the first coil when 1 A flows through the second coil.

4. To conclude that $M_{12} = M_{21} = M$.

5. Let two coils are coupled with self inductances L_1, L_2 and mutual inductance M between them. Also let currents in the two coils be i_1 and i_2.

6. The voltage drop across coil-1 will be sum of two drops, namely $L_1\frac{di_1}{dt}$ and $M\frac{di_2}{dt}$. Similarly the voltage drop across coil-2 will be sum of two drops, namely $L_2\frac{di_2}{dt}$ and $M\frac{di_1}{dt}$.

7. We already know how to ascertain the +ve and -ve sign for voltage drop due to self inductance.

8. Question is how to ascertain +ve and -ve sign of voltages due to mutual inductance and this we can do correctly only if • marking are shown in the coupled coils.

9. If i_1 is entering through the • terminal of the coil-1, then voltage (due to mutual effect) in the second coil $M\frac{di_1}{dt}$, will have +ve sign on the • terminal of the coil-2.

10. Unless we know the instantaneous polarities correctly, of the voltages across the mutually coupled coils, KVL equations in the coils can not be written successfully.

11. Energy stored in a single coil of inductance L and carrying current i is $\frac{1}{2}Li^2$.

12. Energy stored in two coupled coils of self inductances L_1 & L_2 and mutual inductance M is given by $\frac{1}{2}L_1i_1^2 + \frac{1}{2}L_2i_2^2 + Mi_1i_2$ where i_1 and i_2 are the coil currents entering through the • terminals of the coils.

2 Introduction

A stationary coil of turns N may link flux lines and flux linkage with the coil is said to be $\psi = N\phi$. If ϕ happens to be function of time, then there will be induced voltage in the coil given by

$$e = N\frac{d\phi}{dt} = \frac{d(N\phi)}{dt} = \frac{d\psi}{dt}$$

The value of ψ of a particular coils to be calculated carefully, since total flux linkage with the coil may be due to its own current and/or due to a number of other current carrying coils having mutual coupling with the coil being considered.

2.1 Understanding • (Dot)convention

The primary of the transformer shown in figure 1 is energized from a.c source and potential of terminal 1 with respect to terminal 2 is $v_{12} = V_{max}\sin\omega t$. Naturally polarity of 1 is sometimes +ve and some other time it is -ve. The *dot* convention helps us to determine the polarity of the induced voltage in the secondary coil marked with terminals 3 and 4. Suppose at some time t we find that terminal 1 is +ve and it is increasing with respect to terminal 2. At that time what should be the status of the induced voltage polarity in the secondary - whether terminal 3 is +ve or -ve? If possible let us assume terminal 3 is -ve and terminal 4 is positive. If that be current the secondary will try to deliver current to a load such that current comes out from terminal 4 and enters terminal 3. Secondary winding therefore, produces flux in the core in the same direction as that of the flux produced by the primary. So core flux gets strengthened in inducing more voltage. This is contrary to the dictate of Lenz's law which says that the polarity of the induced voltage in a coil should be such that it will try to oppose the cause for which it is due. Hence terminal 3 can not be -ve.

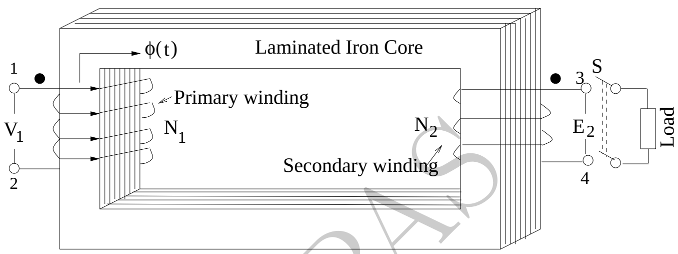
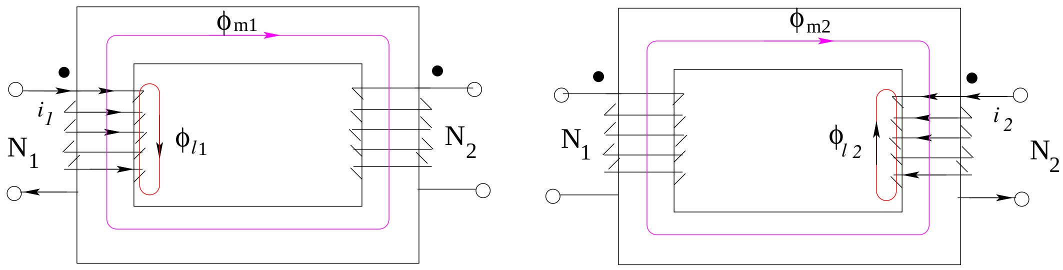


Figure 1: Understanding • convention.

If terminal 3 is +ve then we find that secondary will drive current through the load leaving from terminal 3 and entering through terminal 4. Therefore flux produced by the secondary clearly opposes the primary flux fulfilling the condition set by Lenz's law. Thus when terminal 1 is +ve terminal 3 of the secondary too has to be positive. In mutually coupled coils dots are put at the appropriate terminals of the primary and secondary merely to indicative the status of polarities of the voltages. Dot terminals will have at any point of time identical polarities. In the transformer of figure 1 it is appropriate to put dot markings on terminal 1 of primary and terminal 3 of secondary. It is to be noted that that if the *sense* of the windings are known (as in figure 1), then one can ascertain with confidence where to place the dot markings without doing any testing whatsoever. In practice however, only a pair of primary terminals and a pair of secondary terminals are available to the user and the sense of the winding can not be ascertained at all. In such cases the dots can be found out by doing some simple tests such as *polarity test* or *d.c kick test.*
In circuit problems involving coupled coils, coils are shown with • markings. At any instant polarities of the dot terminals, will be alike.

3 Self and Mutual inductances

Consider a transformer having two coils as shown in figure 2. These two coils have a very strong magnetic coupling through the iron core. Let us imagine that coil-1 alone is energized with current i_1. Then most of the total flux (ϕ_1) created will be confined to the iron and this flux is called the mutual flux ϕ_{m1} for obvious reason as ϕ_{m1} is also going to link the second coil. A little flux called leakage flux (ϕ_{l1}) will also be created and it will close its path through air as shown in figure 2(a). Leakage flux will be always less by many order, as reluctance of air is much much higher compared to iron.



(a) Only coil−1 carries current

(b) Only coil−2 carries current

Figure 2: Coupled coils

$$
\begin{aligned}
\text{Let, } \phi_1 &= \text{Total flux created by coil-1} \\
\phi_{l1} &= \text{Leakage flux created by coil-1} \\
\phi_{m1} &= \text{Mutual flux created by coil-1} \\
\text{Then, } \phi_1 &= \phi_{m1} + \phi_{l1} \\
\text{Also let, } L_1 &= \text{Self inductance of coil-1} \\
L_{l1} &= \text{Leakage inductance of coil-1} \\
M_{21} \,\&\, M_{12} &= \text{Mutual inductance between coil-1 \& coil-2} \\
\text{Now by definition,, } L_1 &= \frac{N_1\phi_1}{i_1} \\
\text{or, } L_1 &= \frac{N_1\phi_{m1}}{i_1} + \frac{N_1\phi_{l1}}{i_1} \\
\text{or, } L_1 &= \frac{N_1}{N_2}\left(\frac{N_2\phi_{m1}}{i_1}\right) + \frac{N_1\phi_{l1}}{i_1} \\
\text{or, } L_1 &= aM_{21} + L_{l1} \text{ where, } a = \frac{N_1}{N_2}
\end{aligned}
$$

In the same way, if coil-2 alone is energized as shown in figure 2(b), self, mutual and leakage inductance of the second coil can be worked out as:

$$
\begin{aligned}
\text{Let, } \phi_2 &= \text{Total flux created by coil-2} \\
\phi_{l2} &= \text{Leakage flux created by coil-2} \\
\phi_{m2} &= \text{Mutual flux created by coil-2} \\
\text{Then, } \phi_2 &= \phi_{m2} + \phi_{l2} \\
\text{Also let, } L_2 &= \text{Self inductance of coil-2} \\
L_{l2} &= \text{Leakage inductance of coil-2} \\
M_{12} &= \text{Mutual inductance between coil-1 \& coil-2} \\
\text{Now by definition,, } L_2 &= \frac{N_2\phi_2}{i_2} \\
\text{or, } L_2 &= \frac{N_2}{N_1}\left(\frac{N_1\phi_{m2}}{i_2}\right) + \frac{N_2\phi_{l2}}{i_2} \\
\text{or, } L_2 &= \frac{M_{12}}{a} + L_{l2} \text{ where, } a = \frac{N_1}{N_2}
\end{aligned}
$$

Let the reluctance of the core of the magnetic circuit shown in figure 2 be $\mathcal{R}$.Then,

$$\begin{aligned}
M_{21} &= \frac{N_2\phi_{m1}}{i_1} \\
\text{or, } M_{21} &= \frac{N_2\phi_{m1}}{i_1} \\
\text{but, } \phi_{m1} &= \frac{N_1 i_1}{\mathcal{R}} \\
\text{so, } M_{21} &= \frac{N_2}{i_1}\frac{N_1 i_1}{\mathcal{R}} = \frac{N_1 N_2}{\mathcal{R}} \\
\text{In the same way,} & \\
M_{12} &= \frac{N_1\phi_{m2}}{i_2} \\
\text{or, } M_{12} &= \frac{N_1\phi_{m2}}{i_2} \\
\text{but, } \phi_{m2} &= \frac{N_2 i_2}{\mathcal{R}} \\
\text{so, } M_{12} &= \frac{N_1}{i_2}\frac{N_2 i_2}{\mathcal{R}} = \frac{N_1 N_2}{\mathcal{R}} \\
\text{Therefore we conclude that, } M_{21} &= M_{12} = M
\end{aligned}$$

We are now ready to handle circuits which has mutually coupled coils with various known inductances. To begin with, it is explained how to write down the KVL equations in mutually coupled coils in terms of flux linkages and inductances and how to get expression for energy stored in such a system.

3.1 Polarities of the voltages in coupled coils

Let two coils are coupled with self inductances L_1, L_2 and mutual inductance M between them. Also let currents in the two coils be i_1 and i_2. The voltage drop across coil-1 will be sum of two drops, namely $L_1\frac{di_1}{dt}$ and $M\frac{di_2}{dt}$. Similarly the voltage drop across coil-2 will be sum of two drops, namely $L_2\frac{di_2}{dt}$ and $M\frac{di_1}{dt}$. We already know how to ascertain the +ve and -ve sign for voltage drop due to self inductance. Question is how to ascertain +ve and -ve sign of voltages due to mutual inductance and this we can do correctly only if • marking are shown in the coupled coils. For the beginners my suggestion is to mark the terminals (which are not marked with •), with □ sign as shown in figure 3

The rules for the appropriate signs of the self and mutual voltages are determined by applying the following the rules.

1. Voltage drop due to Self Inductance(L):**If the coil caries a current of i, then voltage across it is $L\frac{di}{dt}$ and the terminal through which i is entering is +ve and the other terminal is -ve**.

2. Voltage drop in coil-2 due to Mutual Inductance(M):**If i_1 is entering through the • terminal of the coil-1, then voltage (due to mutual effect) in the second coil $M\frac{di_1}{dt}$, will have +ve sign on the • terminal of the coil-2**.

   OR

   Voltage drop in coil-2 due to Mutual Inductance(M):**If i_1 is entering through the □ terminal of the coil-1, then voltage (due to mutual effect) in the second coil $M\frac{di_1}{dt}$, will have +ve sign on the □ terminal of the coil-2**.

3. Voltage drop in coil-1 due to Mutual Inductance(M):**Similarly if, i_2 is entering through the • terminal of the coil-2, then voltage (due to mutual effect) in the second coil $M\frac{di_2}{dt}$, will have +ve sign on the • terminal of the coil-1**.

OR

Voltage drop in coil-1 due to Mutual Inductance(M):**Similarly if, i_2 is entering through the □ terminal of the coil-2, then voltage (due to mutual effect) in the second coil $M\frac{di_2}{dt}$, will have +ve sign on the □ terminal of the coil-1**.

Now you must look at figures 3 (a) to (d), for understanding clearly how the above rules have been applied to get the polarities of self and mutual voltages which are shown by circles (◯) with correct polarities. The terminals which are not marked with • (dot), are shown with □ square symbol. When you are used to it, □ square symbol is really not necessary.

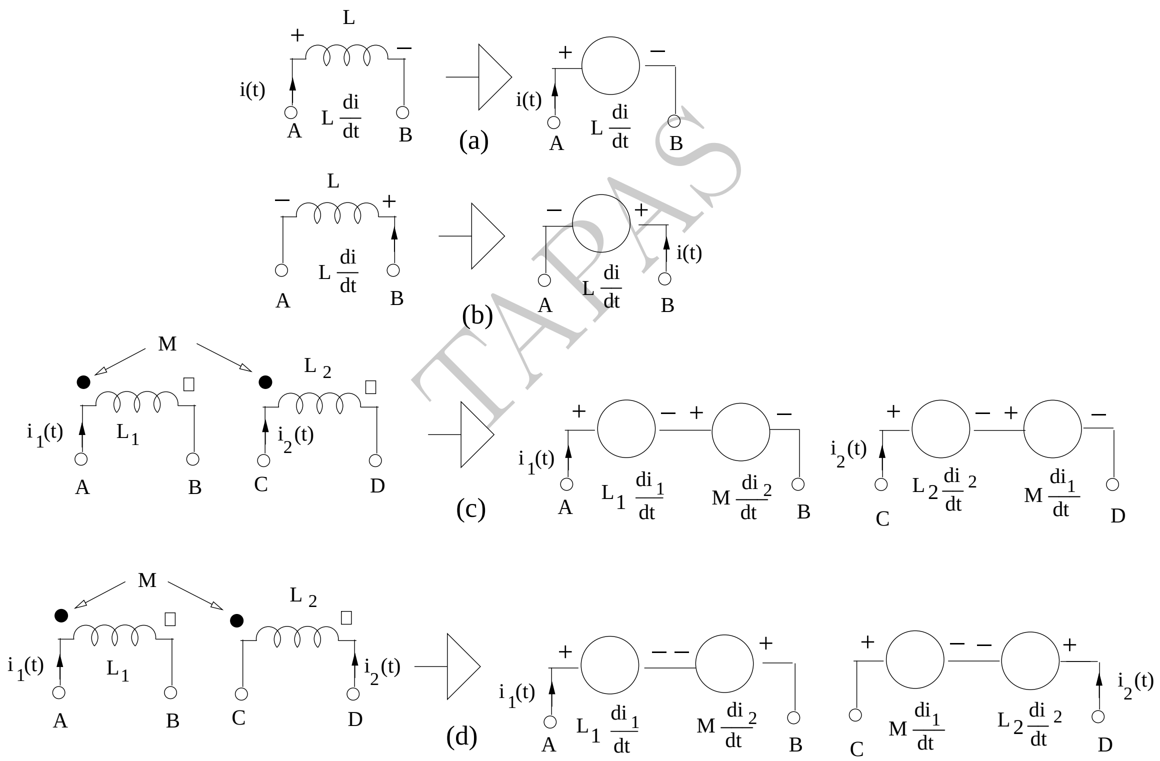


Figure 3: Understanding voltages in mutually coupled coils.

3.2 Energy stored in the coupled coils

Consider the circuit shown in figure 4, where the coupled circuits are supplied with voltages v_1 and v_2 and currents drawn are i_1 and i_2 respectively.

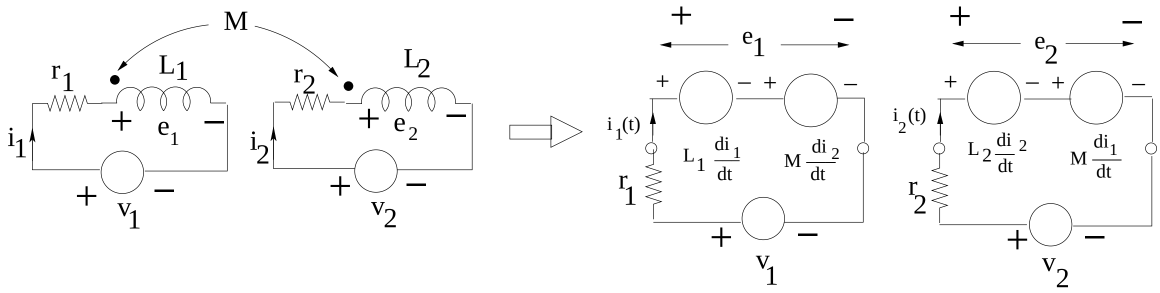


Figure 4: Energy stored in mutual coils.

At any time t, the power input to the inductances (figure 4) is given by:

$$\begin{aligned}
\text{voltage drop in coil-1 } e_1 &= L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \\
\text{voltage drop in coil-1 } e_2 &= L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \\
\text{Power absorbed by coils } p &= e_1 i_1 + e_2 i_2 \\
p &= \left(L_1\frac{di_1}{dt} + M\frac{di_2}{dt}\right) i_1 + \left(L_2\frac{di_2}{dt} + M\frac{di_1}{dt}\right) i_2 \\
\therefore \text{ Energy supplied to coils in } dt,\ dw &= p\,dt \\
dW &= \left(L_1\frac{di_1}{dt} + M\frac{di_2}{dt}\right) i_1\,dt + \left(L_2\frac{di_2}{dt} + M\frac{di_1}{dt}\right) i_2\,dt \\
&= L_1\,di_1\,i_1 + M\,di_2\,i_1 + L_2\,di_2\,i_2 + M\,di_1\,i_2 \\
&= L_1\,i_1\,di_1 + L_2\,i_2\,di_2 + M(di_2\,i_1 + di_1\,i_2) \\
dW &= L_1\,i_1\,di_1 + L_2\,i_2\,di_2 + M\,d(i_1 i_2)
\end{aligned}$$

If the currents in the coil are brought to the levels of I_1 and I_2 from zero in time t, then the energy stored in the magnetic field is obtained by integrating the above equation and energy stored at time t is given by:

$$\begin{aligned}
W &= L_1\int_0^{I_1} i_1\,di_1 + L_2\int_0^{I_2} i_2\,di_2 + M\int_0^{I_1 I_2} d(i_1 i_2) \\
\text{or, } W &= \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2
\end{aligned}$$

Example

Equivalent inductance of two coupled coils when one coil is shorted

Two mutually coupled coils having self inductances L_1 and L_2 and mutual inductance M are shown in figure (5) with the second coil terminals shorted. The equivalent inductance looking into first coil is to be determined. The KVL in the first and the second coils are given by:

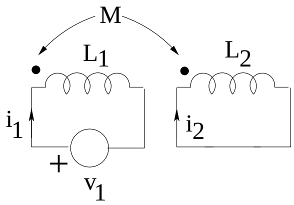


Figure 5: Coil-2 is shorted.

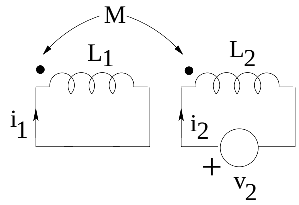


Figure 6: Coil-1 is shorted.

$$
\begin{aligned}
v_1 &= L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \\
0 &= L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \\
\text{or, } \frac{di_2}{dt} &= -\frac{M}{L_2}\frac{di_1}{dt}
\end{aligned}
$$

Now putting the value of $\frac{di_2}{dt}$ in the first equation we get,

$$
\begin{aligned}
v_1 &= L_1\frac{di_1}{dt} - \frac{M^2}{L_2}\frac{di_i}{dt} \\
\text{or, } v_1 &= (L_1 - \frac{M^2}{L_2})\frac{di_i}{dt} \\
\text{Thus the equivalent inductance is, } L_{eq} &= L_1 - \frac{M^2}{L_2}
\end{aligned}
$$

In the same way, if coil-1 is shorted as shown in figure (6) the equivalent inductance of the configuration looking from coil-2 is given by the following.

$$
L_{eq} = L_2 - \frac{M^2}{L_1}
$$

4 Solving circuit problems with couple coils

Write down the KVL equations in the two meshes for the circuit shown in figure 7

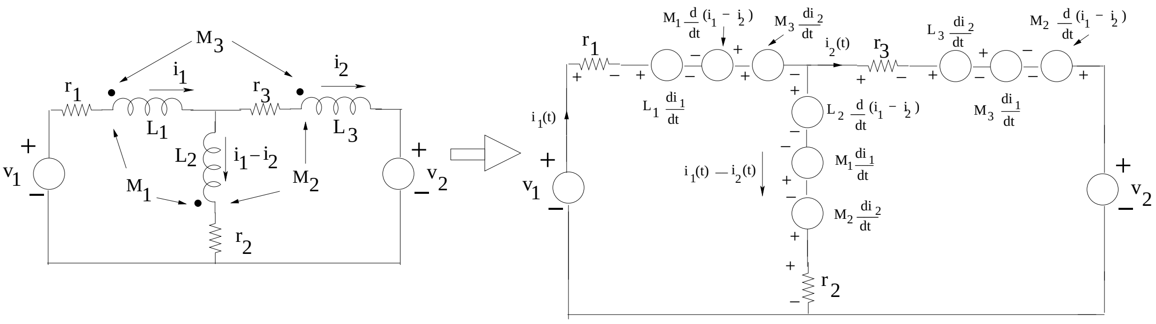


Figure 7: Write KVL equations in the meshes.

We first assume the currents (quite arbitrarily) in various branches as shown. KVL equation in mesh-1.

$$
v_1 - r_1 i_1 - L_1\frac{di_1}{dt} + M_1\frac{d(i_1-i_2)}{dt} - M_3\frac{di_2}{dt} - L_2\frac{d(i_1-i_2)}{dt} + M_1\frac{di_1}{dt} + M_2\frac{di_2}{dt} - (i_1-i_2)r_2 = 0
$$

and KVL equation in mesh-2

$$v_2 - M_2\frac{d(i_1 - i_2)}{dt} + M_3\frac{di_1}{dt} + L_3\frac{di_2}{dt} + r_3 i_2 - L_2\frac{d(i_1 - i_2)}{dt} + M_1\frac{di_1}{dt} + M_2\frac{di_2}{dt} - (i_1 - i_2)r_2 = 0$$

The above two equations are general for any time varying $v_1(t)$ & $v_2(t)$ can now be solved for the currents. **However, if $v_1(t)$ & $v_2(t)$ happen to be sinusoidally varying with angular frequency ω, then the KVL equations, in phasor form can be obtained by replacing $\frac{d}{dt}$ with $j\omega$ as follows:**

$$\bar{V}_1 - r_1\bar{I}_1 - j\omega L_1\bar{I}_1 + j\omega M_1(\bar{I}_1 - \bar{I}_2) - j\omega M_3\bar{I}_2 - j\omega L_2(\bar{I}_1 - \bar{I}_2) + j\omega M_1\bar{I}_1 + j\omega M_2\bar{I}_2 - (\bar{I}_1 - \bar{I}_2)r_2 = 0$$

$$\bar{V}_2 - j\omega M_2(\bar{I}_1 - \bar{I}_2) + j\omega M_3\bar{I}_1 + j\omega L_3\bar{I}_2 + r_3\bar{I}_2 - j\omega L_2(\bar{I}_1 - \bar{I}_2) + j\omega M_1\bar{I}_1 + j\omega M_2\bar{I}_2 - (\bar{I}_1 - \bar{I}_2)r_2 = 0$$

With a little understanding and practice, the KVL equations can be directly written in phasor form without first writing the equations in time domain and then getting equations in phasor form. Solve more problems on this topic.

Problem-1

Two coils having self inductances L_1 & L_2 with mutual inductance between them to be M are connected in parallel as shown in figures 8(a) and (b). Calculate looking in equivalent inductance L_{AB} and L_{CD} as shown in figures 8(a) and (b).

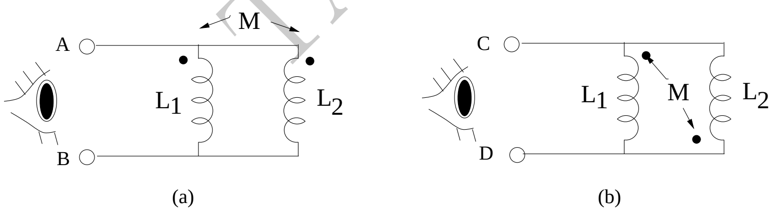


Figure 8: Equivalent L_{AB} =? L_{CD} = ?