Circuit Analysis with Laplace transform

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1 Introduction

To solve a circuit using Laplace transform is always advantageous because

1. The differential equations are transformed to algebraic equations after taking Laplace transform.

2. Finally to get the time domain expression of the variable, one has to take Laplace inverse.

3. Both steady state and transient solutions are obtained in one stroke.

1.1 R, L & C in s-domain

The voltage-current relationship of a resistance R is given by

$$\begin{aligned}\text{In time domain } v(t) &= Ri(t)\\ \text{Taking LT of both sides } V(s) &= RI(s)\\ \text{or, } R &= \frac{V(s)}{I(s)}\end{aligned}$$

The voltage-current relationship of an inductor L **having no initial current** is given by

$$\begin{aligned}\text{In time domain } v(t) &= L\frac{di(t)}{dt}\\ \text{Taking LT of both sides } V(s) &= sLI(s)\\ \text{or, } sL &= \frac{V(s)}{I(s)}\end{aligned}$$

Thus an initially relaxed inductor has an impedance of sL in s-domain.

The voltage-current relationship of a capacitor C **having no initial voltage** is given by

$$\begin{aligned}\text{In time domain } i(t) &= C\frac{dv(t)}{dt}\\ \text{Taking LT of both sides } I(s) &= sCV(s)\\ \text{or, } \frac{1}{sC} &= \frac{V(s)}{I(s)}\end{aligned}$$

Thus an initially relaxed capacitor may be considered to have an impedance of $\frac{1}{sC}$ in s-domain.

Therefore if a circuit has R, L & C, in the transformed domain s one can redraw the circuit replacing L & C respectively sL and $\frac{1}{sC}$ provided the inductor had no initial current and capacitor had no initial voltage.

1.2 Initially charged inductor and capacitor in s-domain

Recall that an inductor L, with initial current $i(0)$ can be represented as parallel combination of an uncharged inductor L with a constant current source $i(0)$ in time domain. Also recall that a capacitor C, with initial voltage $v(0)$ can be represented as series combination of an uncharged capacitor c with a constant voltage source $v(0)$ in time domain. Note LT of $i(0)$ is $\frac{i(0)}{s}$ in case of inductor and LT of $v(0)$ is $\frac{v(0)}{s}$.

The above discussion is summarized in figure 1.

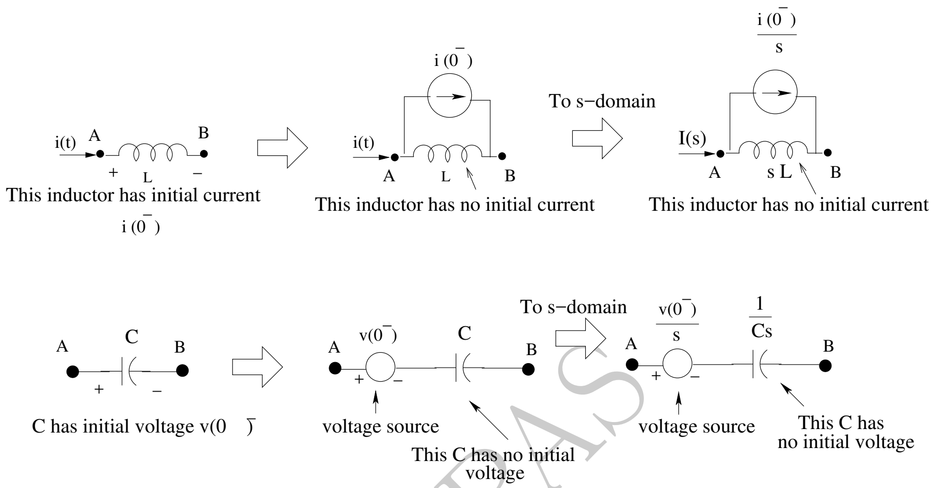


Figure 1:

The advantages we get by redrawing the circuit in s-domain are as follows:

1. Now **it is not necessary to write the differential equation in time domain** then take Laplace transform.

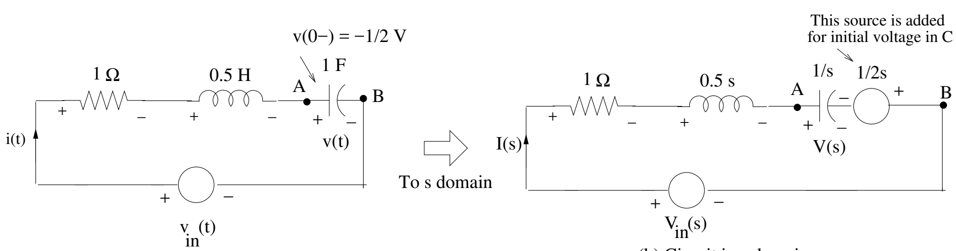
2. All voltages now will be of the form $V(s)$ and all currents will be $I(s)$.

3. Ratio of $V(s)$ and $I(s)$ of an element gives the impedance of the element $Z(s)$.

4. In s-domain also, you can adopt any method (mesh analysis, nodal method etc.) to solve for $I(s)$ in any branch.

1.2.1 Example-1

In the circuit shown in figure 2(a),it is given that capacitor voltage $v(0_{-}) = -\frac{1}{2}$ Volt. (i) Redraw the circuit in s-domain showing the initial condition. (ii) Calculate $i(t)$ for $t \geq 0$ when the input voltage $v_{in} = t$ for $t \geq 0$.



(a) Circuit in time domain (b) Circuit in s domain

Figure 2:

1.2.2 Solution

We first redraw the circuit in s-domain as shown in figure 2(b). Since capacitor had initial voltage, its representation between A and B will be an uncharged capacitor and the initial voltage 1/2 V in series - note carefully the polarity of the voltage.

$$\begin{aligned}
I(s) &= \frac{V_{in}(s) + \frac{1}{2s}}{1 + 0.5s + \frac{1}{s}} \\
\text{Now,}\ \ v_{in} &= t\,u(t) \\
\therefore\ V_{in}(s) &= \frac{1}{s^2} \\
\text{So,}\ I(s) &= \frac{\frac{1}{s^2} + \frac{1}{2s}}{1 + 0.5s + \frac{1}{s}} \\
&= \frac{s+2}{s^2+2s+2} \\
\therefore\ I(s) &= \frac{s+2}{s(s^2+2s+2)} = \frac{K}{s} + \frac{As+B}{s^2+2s+2} \\
K &= \left.\frac{s+2}{(s^2+2s+2)}\right|_{s=0} = 1 \\
\therefore\ I(s) &= \frac{s+2}{s(s^2+2s+2)} = \frac{1}{s} + \frac{As+B}{s^2+2s+2}
\end{aligned}$$

To know A and B equate the numerators of both the sides.

$$\begin{aligned}
(s^2+2s+2) + s(As+B) &= s+2 \\
\text{so,}\ A+1 &= 0 \text{ or, } A = -1 \\
B+2 &= 1 \text{ or, } B = -1 \\
\therefore\ I(s) &= \frac{1}{s} - \frac{s+1}{s^2+2s+2} \\
\text{or,} I(s) &= \frac{1}{s} - \frac{s+1}{(s+1)^2+1} \\
\text{Taking Laplace inverse: } i(t) &= \left(1 - e^{-t}\cos\,t\right)u(t)
\end{aligned}$$

Example-2

Once we have drawn the circuit in s-domain showing the initial conditions for inductor current and capacitor voltage, we can adopt any known method of solving the circuit. For example, consider a circuit given in s-domain with some initial current in the inductor as shown in figure 3(a). The capacitor had no initial voltage.

Nodal method in s-domain

Let us apply nodal method to calculate the the node voltages V_{AO} and V_{BO} in s-domain. So **by inspection** we get the following two equations.

$$\begin{aligned}
\left(\frac{1}{R_1} + \frac{1}{sL}\right)V_{AO}(s) - \frac{1}{sL}V_{BO}(s) &= \frac{V(s)}{R_1} - \frac{i(0)}{s} \\
-\frac{1}{sL}V_{AO}(s) + \left(\frac{1}{R_3} + \frac{1}{sL} + \frac{1}{R_2 + 1/sC}\right)V_{BO}(s) &= \frac{i(0)}{s}
\end{aligned}$$

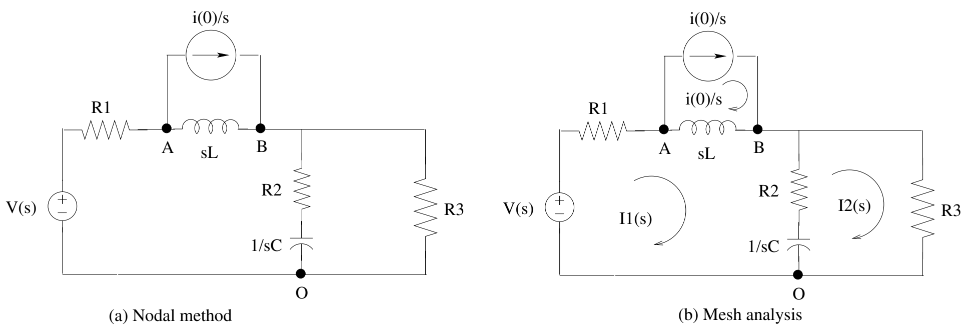


Figure 3:

Solving these two algebraic equations $V_{AO}(s)$ and $V_{BO}(s)$ can be obtained as function of s. Hence $v_{AO}(t)$ and $v_{BO}(t)$ by taking Laplace inverse. If we are also interested to find current say in the branch BO, then we first try to get $I_{BO}(s)$ and take its inverse as follows.

$$\begin{aligned} I_{BO}(s) &= \frac{V_{BO}(s)}{R_2 + 1/Cs} \\ i_{BO}(t) &= \text{Laplace inverse of } I_{BO}(s) \end{aligned}$$

Mesh analysis in s-domain

Let us now apply Mesh analysis to solve the above circuit. For mesh analysis refer to figure 3(b) where there are three meshes of which the current in the top mesh is known as $\frac{i(0)}{s}$. Assuming the other mesh currents as $I_1(s)$ and $I_2(s)$, KVL in the two meshes can be written **by inspection** as follows.

$$\begin{aligned} (R_1 + R_2 + sL + 1/sC)\, I_1(s) - (R_2 + 1/sC)\, I_2(s) - sL\frac{i(0)}{s} &= V(s) \\ -(R_2 + 1/sC)\, I_1(s) + (R_2 + R_3 + 1/sC)\, I_2(s) &= 0 \end{aligned}$$

Solving these two algebraic equations $I_1(s)$ and $I_2(s)$ can be obtained as function of s. Hence $i_1(t)$ and $i_2(t)$ by taking Laplace inverse. If we are also interested to find current say in the branch BO, then we first try to get $I_{BO}(s)$ and take its inverse as follows.

$$\begin{aligned} I_{BO}(s) &= I_1(s) - I_2(s) \\ i_{BO}(t) &= \text{Laplace inverse of } I_{BO}(s) \end{aligned}$$

Thevenin theorem in s-domain

For the same circuit figure 3(a), we want to find out the current through the resistance R_3. by applying Thevenin theorem. So to calculate Z_{th} refer to circuit shown in figure 4(a), where voltage source is shorted and current source is shown open circuited. Obviously:

$$Z_{th} = \frac{(R_1 + sL)(R_2 + 1/sC)}{R_1 + sL + R_2 + 1/sC}$$

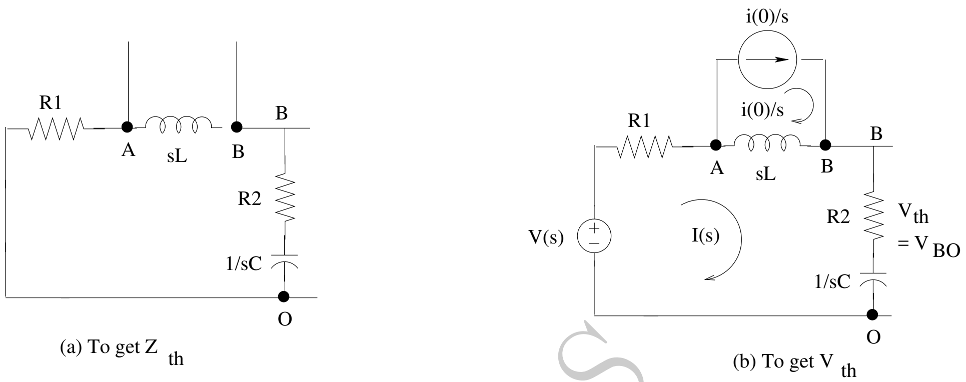


Figure 4:

To obtain Thevenin voltage we have to keep all the sources in the circuit as shown in figure 4(b) and calculate $V_{BO}(s)$ as follows.

$$\begin{aligned} V_{th}(s) = V_{BO}(s) &= I(s)(R_2 + 1/sC) \\ I(s) \text{ can be obtained from:} & \\ (R_1 + sL + R_2 + 1/sC)\, I(s) - sL\frac{i(0)}{s} &= V(s) \\ \therefore \text{ current through } R_3 &= \frac{V_{th}}{Z_{th}(s) + R_3} \end{aligned}$$

2 Conclusions

1. It is suggested that *better avoid writing differential equations in time domain first and take Laplace transform.*

2. By doing so, you can save lot of time.

3. You redraw the given time domain circuit, in s-domain replacing L by sL, C by $1/sC$. Note R remains R - unchanged.

4. Connect current source $\frac{i(0)}{s}$ across sL (with correct direction) if inductor had initial current.

5. Connect voltage source $\frac{v(0)}{s}$ in series with $1/sC$ (with correct polarity)if capacitor had initial voltage.

6. Now be in s-domain circuit, to find current in any branch $I_k(s)$ or voltage $V_k(s)$ across any element by any method you like.

7. Finally, take Laplace inverse of $I_k(s)$ or $V_k(s)$ to get $i_k(t)$ or $v_k(t)$.

8. Laplace transform (and corresponding inverse transform) of some standard and useful function such $u(t)$, $\sin \omega t$, $\cos \omega t$ etc. and frequently used properties of LT must be at your finger tips to solve a circuit problem efficiently and with pleasure.

9. Also revisit your knowledge in partial fraction.