Convolution and circuit response

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1 Convolution in LTI Continuous Time System

For a given LTI system, if we know the response of the system $h(t)$ when the excitation or input is an unit impulse $\delta(t)$, then we can find out the response or output of the system $y(t)$ for any arbitrary input signal $x(t)$. Look at the following steps (LHS being the input and RHS is the corresponding output).

$$\begin{aligned}
\text{It is given that, } \delta(\tau) &\rightarrow h(\tau) \\
\therefore\ \delta(t-\tau) &\rightarrow h(t-\tau) \ \because \text{ time invariant system} \\
\therefore\ x(\tau)\,d\tau\ \delta(t-\tau) &\rightarrow x(\tau)\,d\tau h(t-\tau) \ \because \text{ linear system} \\
\text{or, } x(\tau)\,\delta(t-\tau)\,d\tau &\rightarrow x(\tau)\,h(t-\tau)\,d\tau \text{ applying superposition} \\
\text{or, } \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau &\rightarrow \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \because \text{ Linear system} \\
\text{but, } \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau &= x(t) \\
x(t) &\rightarrow \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau
\end{aligned}$$

To summaries: The output $y(t)$ of a linear time invariant continuous time varying system for any arbitrary input $x(t)$ can be obtained by evaluating the following integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \textbf{ where } h(t) \textbf{ is impulse response}$$

This integral is defined as the *convolution* of the two signal $x(t)$ and $h(t)$ and this operation is written in short form mathematically as $y(t) = x(t) \circledast h(t)$. Convolving two signals $x(t)$ and $h(t)$ means we are doing $x(t) \circledast h(t)$. Thus,

$$y(t) = x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \text{ where } h(t) \text{ is impulse response}$$

It may be noted that the convolution operation is commutative i.e., $x(t) \circledast h(t) = h(t) \circledast x(t)$, which means:

$$y(t) = x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

1.1 Steps to obtain convolution $x(t) \circledast h(t)$

1. Sketch the given signals $x(t)$ and $h(t)$ against time.
2. Recall we have to calculate $y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$ where τ is the running variable.
3. Draw $x(\tau)$ and $h(\tau)$ against running variable τ
4. Sketch $h(-\tau)$ and then $h(t-\tau)$.
5. Find out the intervals where overlap occurs.
6. Decide about the limits of integration in the overlap zones.
7. Carry out the integration as τ varies from $-\infty$ to $+\infty$
8. Final result $y(t)$ will be function of time t. Sketch $y(t)$.

Example-1

For the two rectangular pulses $x(t)$ and $h(t)$ shown in figure 1(a), find out $y(t) = x(t) \circledast h(t)$.

Solution

- In figure 1(a), $x(t)$ and $h(t)$ are shown as functions of time t.
- In figure 1(b), $x(\tau)$ and $h(-\tau)$ are shown as functions of time τ.
- In figure 1(c), $h(t-\tau)$ is shown for a general value of t.
- Now we have to find out the overlap regions as $h(t-\tau)$ is bodily moved from left to right.

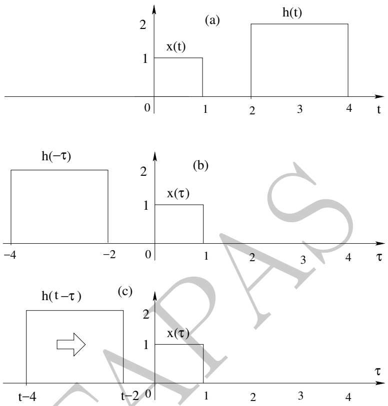

Figure 1

- As can be seen, so long $t-2 < 0$ or $t < 2$,, no overlap occurs. Thus, no overlap so long $-\infty < t < 2$. Therefore $y(t) = \int_{-\infty}^{2} x(\tau)\, h(t-\tau)\, d\tau = 0$.
- First overlap starts when $t-2 > 0$ or $t > 2$ which is pictorially depicted in figure 2(b).How long does the first overlap continue? Since the duration of the signal $h(t)$ is more than that of the signal $x(t)$ so long $t-2 < 1$ or $t < 3$. Combining these two, we conclude first overlap continues during $2 < t < 3$ and the integration limit for τ will be from 0 to $t-2$. Hence,

$$
\begin{aligned}
y(t) &= \int_{\tau=0}^{(t-2)} x(\tau)\, h(t-\tau)\, d\tau \\
&= \int_{\tau=0}^{(t-2)} 1 \times 2\, d\tau = 2t - 4
\end{aligned}
$$

- Similarly during $3 < t < 4$, second overlap lasts (figure 2(c)) and the integration limit will be 0 to 1. Hence,

$$
\begin{aligned}
y(t) &= \int_{\tau=0}^{1} x(\tau)\, h(t-\tau)\, d\tau \\
&= \int_{\tau=0}^{1} 1 \times 2\, d\tau = 2
\end{aligned}
$$

- Third overlap lasts (figure 2(d)) during $4 < t < 5$ and the integration limit will be $t-4$ to 1. Hence,

$$\begin{aligned} y(t) &= \int_{\tau=(t-4)}^{1} x(\tau)\, h(t-\tau)\, d\tau \\ &= \int_{\tau=(t-4)}^{1} 1 \times 2\, d\tau = 10 - 2t \end{aligned}$$

- Finally when $t > 5$ (figure 2(e)), no overlap takes place and $y(t) = 0$

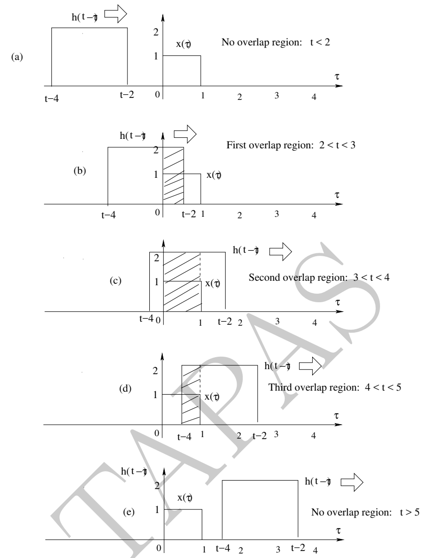

Figure 2

We can now sketch $y(t)$ for $-\infty < t < +\infty$ as shown in figure 3 along with $x(t)$ and $h(t)$.

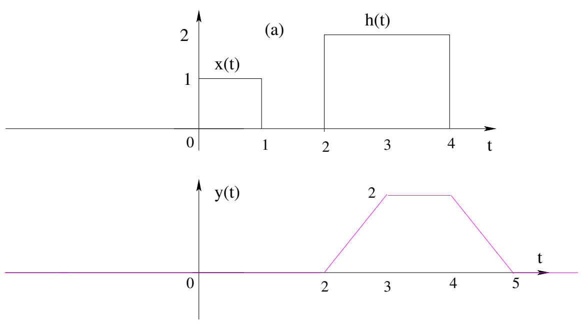

Figure 3: Output $y(t)$

Example-2

Find out the current response $i(t)$ of an $R-L$ series circuit when the excitation is $u(t)$ using the principle of convolution. Impulse response of the circuit is known to be $h(t) = i(t) = \frac{1}{L}e^{-\frac{R}{L}t}$.

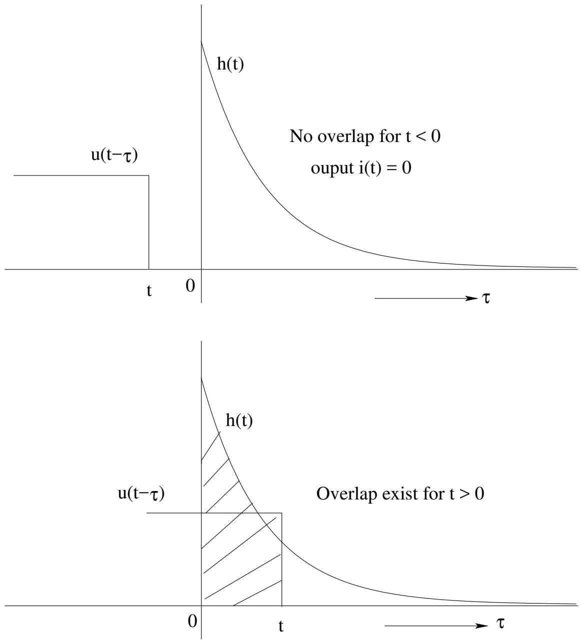

Figure 4

1.1.1 Solution

In figure we have sketched $h(\tau)$ and $u(t-\tau)$. Here the situation is much simpler. We can immediately conclude that there is no overlap for $t < 0$ and for $t > 0$ overlap continues. So output (current $i(t)$) is zero for $t < 0$.

Now using the convolution theorem, the response due to unit step voltage is:

$$\begin{aligned} i(t) &= \int_{-\infty}^{t} u(t-\tau)h(\tau)\,d\tau \\ \text{or, } i(t) &= \int_{0}^{t} h(\tau)\,d\tau \\ \text{or, } i(t) &= \int_{0}^{t} \frac{1}{L}\,e^{-\frac{R}{L}\tau}\,d\tau \\ \text{Finally } i(t) &= \frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right)u(t) \end{aligned}$$

The result obtained correct; same as obtained by other means.

1.2 Example-3

Find out the time response $i(t)$ when a series $R-L$ circuit is excited with a sinusoidal voltage i.e., $f(t) = V_{max}\sin\omega t\,u(t)$.

Answer

We know that the unit impulse response of a series $R-L$ circuit is:

$$h(t) = i_h(t) = \frac{1}{L}\,e^{-\frac{R}{L}t}$$

Look at figure 5, where $h(t-\tau)$ and $h(\tau)$ have been shown. Here also it is straight forward to conclude that $i(t) = 0$ for $t < 0$ as no overlap exist during this time. Overlap of course exists for

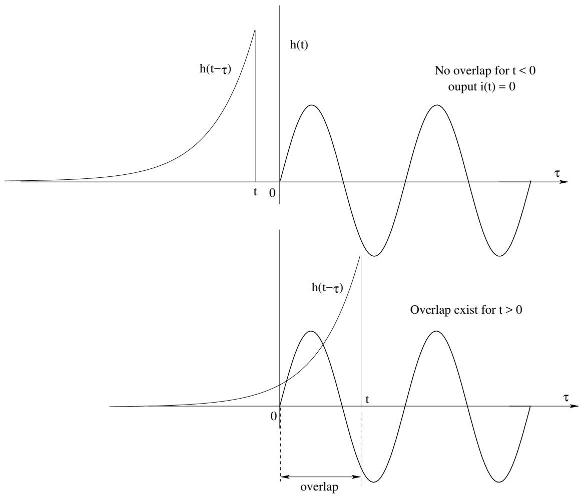

Figure 5

$t > 0$. Now using the convolution theorem, the response due to $f(t) = V_{max} \sin \omega t \, u(t)$ is:

$$
\begin{aligned}
i(t) &= \int_{-\infty}^{t} V_{max} \sin \omega\tau \; u(\tau)\, h(t-\tau)\, d\tau \\
\text{or, } i(t) &= \int_{0}^{t} V_{max} \sin \omega\tau \, h(t-\tau)\, d\tau \\
\text{or, } i(t) &= \int_{0}^{t} \frac{1}{L} e^{-\frac{R}{L}(t-\tau)}\, V_{max} \sin \omega\tau \, d\tau \\
\text{or, } i(t) &= \frac{V_{max}}{L} e^{-\frac{R}{L}t} \int_{0}^{t} e^{\frac{R}{L}\tau} \sin \omega\tau \; d\tau \\
\text{but we know, } \int e^{ax} \sin bx \, dx &= \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) \\
\text{identifying } a = \tfrac{R}{L} \ \&\ b = \omega & \\
\text{or, } i(t) &= \frac{V_{max}}{L} e^{-\frac{R}{L}t} \frac{e^{\frac{R}{L}\tau}}{\sqrt{\frac{R^2}{L^2}+\omega^2}} \sin\left(\omega\tau - \tan^{-1}\frac{\omega L}{R}\right)\Bigg|_{\tau=0}^{t} \\
\text{Simplifying, we get, } i(t) &= \frac{V_{max}}{(\sqrt{R^2+\omega^2 L^2})} \sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right) u(t) \\
&\quad - \frac{V_{max}}{(\sqrt{R^2+\omega^2 L^2})} e^{-\frac{R}{L}t} \sin\left(-\tan^{-1}\frac{\omega L}{R}\right) u(t)
\end{aligned}
$$

The result obtained correct; same as obtained by other means.

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