Two port network

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1 Introduction

A two port network is shown in figure 1. Let terminal voltage and current at port-1 are respectively V_1 and I_1 while the terminal voltage and current at port-2 are respectively V_2 and I_2. Note convention of the current directions and voltage polarity. Voltage and current of port-1 can be related with Voltage and current of port-2 in many ways in terms of network parameters.

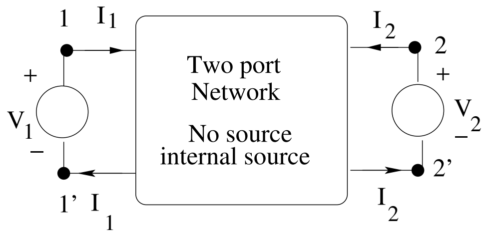


Figure 1:

1.1 Short circuit Admittance parameters

In this case port currents are related with the port voltages as follows.

$$
\begin{aligned}
I_1 &= y_{11}V_1 + y_{12}V_2 \\
I_2 &= y_{21}V_1 + y_{22}V_2
\end{aligned}
$$

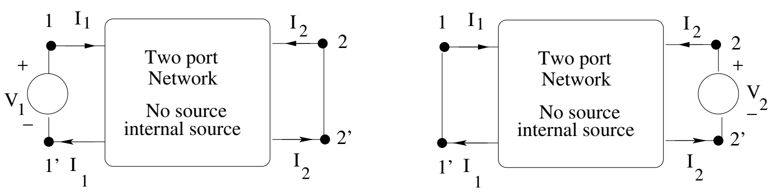
The above equation can be written in a compact form using matrices as follows.

$$
\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}
$$

Y_{11}, Y_{21}, Y_{12} and Y_{22} are called short circuit admittance parameters of the two port network. In the first and second equations, if we make $V_2 = 0$, then

$$
\begin{aligned}
\text{Driving point admittance of port-1, } y_{11} &= \frac{I_1}{V_1}|_{V_2=0} \\
\text{Transfer admittance, } y_{21} &= \frac{I_2}{V_1}|_{V_2=0}
\end{aligned}
$$

This will be achieved, if we exite port-1 with voltage v_1 and keep port-2 shorted as shown in figure 2(a) Similarly, if $V_1 = 0$, y_{12} and y_{22} as under.



(a) For finding Y_{11} and Y_{21} (b) For finding Y_{12} and Y_{22}

Figure 2:

$$\text{Driving point admittance of port-2, } y_{22} = \frac{I_2}{V_2}|_{V_1=0} \quad (1)$$

$$\text{Transfer admittance, } y_{12} = \frac{I_1}{V_2}|_{V_1=0} \quad (2)$$

$$(3)$$

This will be achieved, if we excite port-2 with voltage V_2 and keep port-1 shorted as shown in figure 2(b)

1.2 Open circuit impedance parameters

In this case port voltages are related with the port currents as follows.

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

The above equation can be written in a compact form using matrices as follows.

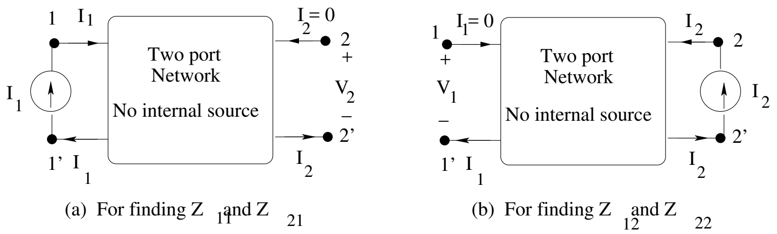
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

z_{11}, z_{21}, z_{12} and z_{22} are called open circuit impedance parameters of the two port network. In the first and second equations, if we make $I_2 = 0$, then

$$\text{Driving point impedance of port-1, } z_{11} = \frac{V_1}{I_1}|_{I_2=0}$$

$$\text{Transfer impedance, } z_{21} = \frac{V_2}{I_1}|_{I_2=0}$$

This will be achieved, if we excite port-1 with current I_1 and keep port-2 opened as shown in figure 3(a) Similarly, with $I_1 = 0$, z_{12} and z_{22} as under.



(a) For finding Z_{11} and Z_{21} (b) For finding Z_{12} and Z_{22}

Figure 3:

$$\text{Driving point impedance of port-2, } z_{22} = \frac{V_2}{I_2}|_{V_1=0} \quad (4)$$

$$\text{Transfer admittance, } z_{12} = \frac{V_1}{I_2}|_{I_1=0} \quad (5)$$

$$(6)$$

This will be achieved, if we excite port-2 with current I_2 and keep port-1 opened as shown in figure 3(b)

1.3 Hybrid parameters

In this case, it is mix bag - v_1 & I_2 are expressed interms of I_1 & V_2. The parameter involved is denoted by h, called hybrid parameters.

port currents are related with the port voltages as follows.

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned}$$

The above equation can be written in a compact form using matrices as follows.

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

h_{11}, h_{21}, h_{12} and h_{22} are called hybrid parameters of the two port network. In the first and second equations, if we make $V_2 = 0$, then

$$\begin{aligned} h_{11} &= \frac{V_1}{I_1} |_{V_2=0} \\ h_{21} &= \frac{I_2}{I_1} |_{V_2=0} \end{aligned}$$

This will be achieved, if we excite port-1 with current I_1 and keep port-2 shorted as shown in figure 4(a) Similarly, if $I_1 = 0$, h_{22} and h_{12} as under.

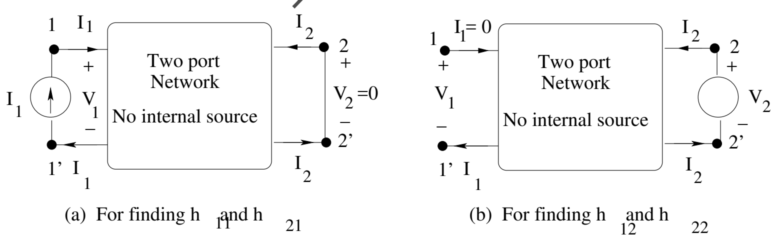


Figure 4:

$$h_{22} = \frac{I_2}{V_2} |_{I_1=0} \tag{7}$$
$$h_{12} = \frac{V_1}{V_2} |_{I_1=0} \tag{8}$$
$$\tag{9}$$

This will be achieved, if we excite port-2 with voltage V_2 and keep port-1 opened as shown in figure 4(b)

It may be noted that h_{12} and h_{21} are dimension less while h_{11} has dimension of impedance and h_{22} has dimension of admittance.

1.4 ABCD parameters

In this case port-1 voltage & current are expressed in terms of port-2 voltage & current. This way of viewing a two port network is particularly suitable for modelling a transmission line : port-1 may be considered to be the sending end of the line and port-2, the receiving end of the line. Port-1 is to be excited from a generator where as, load will be connected at port-2 . People dealing with the transmission line will be interested to know the current delivered to the load from port-2. To be consistent with the convention, we will show I_2 flowing into the network as usual - which means the load current (from left to right) is then $-I_2$. In this case V_1 & I_1 are expressed in terms of V_2 & $-I_2$

The resulting matrix will have A,B,C and D parameters.

Port-1 voltage & current are expressed in terms Port-2 voltage & current as follows.

$$\begin{aligned} V_1 &= AV_2 + B(-I_2) \\ I_1 &= CV_2 + D(-I_2) \end{aligned}$$

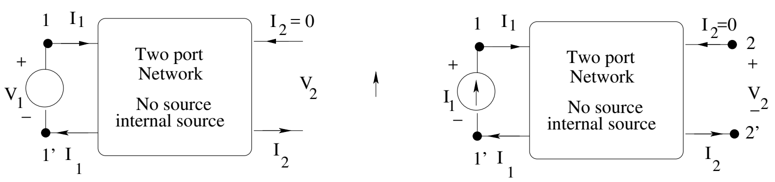
The above equation can be written in a compact form using matrices as follows.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

A, B, C and D are called the transmission line parameters (t parameters) of the two port network. In the first and second equations, if we make $I_2 = 0$, then

$$\begin{aligned} A &= \frac{V_1}{V_2}\Big|_{I_2=0} \\ \text{or, } \frac{1}{A} &= \frac{V_2}{V_1}\Big|_{I_2=0} \\ \text{or, } C &= \frac{I_1}{V_2}\Big|_{I_2=0} \\ \frac{1}{C} &= \frac{V_2}{I_1}\Big|_{I_2=0} \end{aligned}$$

$\frac{1}{A}$, can be found out if we excite port-1 with current V_1 and keep port-2 opened as shown in figure 5(a) and $\frac{1}{C}$ can be found out if we excite port-1 with current I_1 and keep port-2 opened as shown in figure 5(b).



(a) To determine 1/A

(b) To determine 1/C

Figure 5:

Similarly, with $V_2 = 0$, B and D as under.

$$B = \frac{V_1}{-I_2}\Big|_{V_2=0} \tag{10}$$

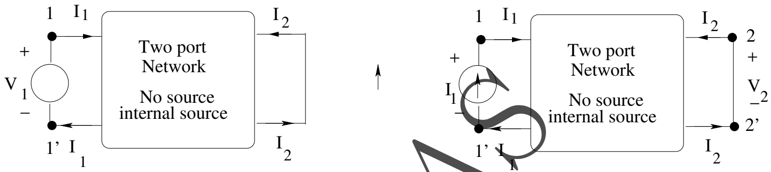
$$\text{or, } \frac{1}{B} = \frac{-I_2}{V_1}\Big|_{V_2=0} \tag{11}$$

$$D = \frac{I_1}{-I_2}\Big|_{V_2=0} \tag{12}$$

$$\text{or, } \frac{1}{D} = \frac{-I_2}{I_1}\Big|_{V_2=0} \tag{13}$$

$$\tag{14}$$

$\frac{1}{B}$, can be found out if we excite port-1 with voltage V_1 and keep port-2 shorted as shown in figure 6(a) and $\frac{1}{D}$ can be found out if we excite port-1 with current I_1 and keep port-2 shorted as shown in figure 6(b).



(a) To determine 1/B (b) To determine 1/D

Figure 6:

2 Relationship among y, z, h and t parameters

We have seen that a given two port network can be modelled either in terms of y or z or h or t parameters. These models are written in the form of matrix equations as follows.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

2.1 Relation between y and Z parameters

The relationship between y and z parameters is straight forward.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Multiplying both sides by inverse of y matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Multiplying both sides by inverse of y matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{(y_{11}y_{12} - y_{21}y_{12})} \begin{bmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Thus by comparing we get,

$$\begin{aligned} z_{11} &= \frac{y_{22}}{(y_{11}y_{12} - y_{21}y_{12})} \\ z_{12} &= \frac{-y_{12}}{(y_{11}y_{12} - y_{21}y_{12})} \\ z_{21} &= \frac{-y_{21}}{(y_{11}y_{12} - y_{21}y_{12})} \\ z_{22} &= \frac{y_{11}}{(y_{11}y_{12} - y_{21}y_{12})} \end{aligned}$$

In the same way, y matrix can be expressed in terms of z matrix.

2.2 Relation between h and y parameters

Basic equations involving y and h parameters are rewritten below.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Here obviously matrix inversion does not help to express y parameters in terms of h parameters or vice versa. In this case, the two equations involving y parameters are to be algebrically manipulated, to cast them in hybrid format. The first equation of y parameters, we get

$$\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ \text{Manipulating this, we rewrite} \\ V_1 &= \frac{1}{y_{11}}I_1 - \frac{y_{12}}{y_{11}}V_2 \\ \therefore\ h_{11} &= \frac{1}{y_{11}} \\ \text{and } h_{12} &= -\frac{y_{12}}{y_{11}} \end{aligned}$$

Now using the second equation of y parameters, we get

$$\begin{aligned} I_2 &= y_{21}V_1 + y_{22}V_2 \\ \text{But we already have, } V_1 &= \frac{1}{y_{11}}I_1 - \frac{y_{12}}{y_{11}}V_2 \\ \text{Putting this } V_1\text{, in the first equation we get,} \\ I_2 &= y_{21}\left(\frac{1}{y_{11}}I_1 - \frac{y_{12}}{y_{11}}V_2\right) + y_{22}V_2 \end{aligned}$$

$$\begin{aligned}
\text{or, } I_2 &= \frac{y_{21}}{y_{11}} I_1 + \left(y_{22} - \frac{y_{12}y_{21}}{y_{11}} \right) V_2 \\
\therefore\ h_{21} &= \frac{y_{21}}{y_{11}} \\
\text{and } h_{22} &= \left(y_{22} - \frac{y_{12}y_{21}}{y_{11}} \right)
\end{aligned}$$

In the same way the reverse relations i,e., y in terms of h can be found out.

Readers are requested to find out relation among parameters of z & h, z & t, y & t etc.

3 Reciprocal two port network

A network is said to be reciprocal if the ratio of response current in a short circuited port and the excitation voltage applied to the other port remains same if the excitation and response are interchanged between the ports.

Alternatively a network is said to be reciprocal if the ratio of response voltage in a open circuited port and the excitation current applied to the other port remains same if the excitation and response are interchanged between the ports.

We shall find out propoerties of the parameters if the two port network is reciprocal.

3.1 Reciprocal two port network : y paramers

Recall the basic equation involving y parameters:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Consider port-1 is excited with a voltage V_1 and port-2 is short circuited $(V_2 = 0)$ with a response current I_2, then from second equation we get,

$$\frac{I_2}{V_1} = y_{21}$$

Now interchange the source and response positions i.e., port-2 is excited with a voltage V_2 and port-1 is short circuited $(V_1 = 0)$ with a response current I_1, then from the first equation we get,

$$\frac{I_1}{V_2} = y_{12}$$

If the two port network is to be reciprocal then,

$$\frac{I_2}{V_1} = \frac{I_1}{V_2}$$

which means that,

$$y_{21} = y_{12}$$

3.2 Reciprocal two port network : z paramers

Recall the basic equation involving y parameters:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Now consider port-2 is excited with a current source I_2 with port-1 open circuited ($I_1 = 0$). Then from first equation we get,

$$\frac{V_1}{I_2} = z_{12}$$

Now consider port-1 is excited with a current source I_1 with port-2 open circuited ($I_2 = 0$). Then from second equation we get,

$$\frac{V_2}{I_1} = z_{21}$$

If the two port network is to be reciprocal then,

$$\frac{V_1}{I_2} = \frac{V_2}{I_1}$$

which means that,

$$z_{12} = z_{21}$$

3.3 Reciprocal two port network : h paramers

Recall the basic equation involving h parameters:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Consider port-1 is excited with a voltage source V_1 with port-2 short circuited ($V_2 = 0$). Then,

$$\frac{V_1}{I_1} = h_{11} \tag{15}$$

$$\frac{I_2}{I_1} = h_{21} \tag{16}$$

$$\therefore \text{ Dividing these two, we get } \frac{I_2}{V_1} = \frac{h_{21}}{h_{11}} \tag{17}$$

Now consider port-2 is excited with a voltage source V_2 with port-1 short circuited ($V_1 = 0$). Then from first equation we get,

$$\begin{aligned} 0 &= h_{11}I_1 + h_{12}V_2 \\ \text{or, } \frac{I_1}{V_2} &= -\frac{h_{12}}{h_{11}} \end{aligned}$$

If the two port network is to be reciprocal then,

$$\frac{I_2}{V_1} = \frac{I_1}{V_2}$$

which means that,

$$\frac{h_{21}}{h_{11}} = -\frac{h_{12}}{h_{11}} \quad \therefore \quad h_{21} = -h_{12}$$

3.4 Reciprocal two port network : t paramers

Recall the basic equation involving t parameters:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Consider port-1 is excited with a voltage source V_1 with port-2 short circuited ($V_2 = 0$). Then,

$$V_1 = -BI_2 \text{ then } \frac{I_2}{V_1} = -\frac{1}{B}$$

Now consider port-2 is excited with a voltage source V_2 with port-1 short circuited ($V_1 = 0$). Then we get,

$$\begin{aligned} 0 &= AV_2 - BI_2 \text{ or, } I_2 = \frac{AV_2}{B} \\ \text{and } I_1 &= CV_2 - DI_2 = CV_2 - D\frac{AV_2}{B} \\ \text{or, } \frac{I_1}{V_2} &= C - D\frac{A}{B} = \frac{(BC - AD)}{B} \end{aligned}$$

If the two port network is to be reciprocal then,

$$\frac{I_2}{V_1} = \frac{I_1}{V_2} \text{ or, } -\frac{1}{B} = \frac{(BC - AD)}{B}$$

which finally means that,

$$\text{Determinant of T matrix is } AD - BC = 1$$

It may be noted that if a two port network is reciprocal then three parameters are independent. To summarise the results for a two port network to be reciprocal, we conclude that:

1. If the network is modelled with y parameters: $y_{12} = y_{21}$
2. If the network is modelled with z parameters: $z_{12} = z_{21}$
3. If the network is modelled with h parameters: $h_{12} = -h_{21}$
4. If the network is modelled with t parameters: $AD - BC = 1$

4 Symmetrical two port network

In a symmetrical two port network input and output port can be interchanged without changing the voltage and current in the port. In other words there is no distinction can be made between (i) when port-1 is made an input port and port-2 is made an output port (ii) when port-2 is made an input port and port-1 is made an output port. This will be possible if the driving point impedance/admittance looking from port-1 and port-2 are equal. If a network is symmetric, it can be shown that :

1. If the network is modelled with y parameters: $y_{11} = y_{22}$
2. If the network is modelled with z parameters: $z_{11} = z_{22}$
3. If the network is modelled with h parameters: $h_{11}h_{22} - h_{12}h_{21} = 1$
4. If the network is modelled with t parameters: $A = D$

4.1 Symmetrical two port network : y parameters

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Excite port-1 with I_1 with port-2 shorted and we get:

$$\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 = y_{11}V_1 \\ \therefore \frac{I_1}{V_1} &= y_{11} \end{aligned}$$

Now, excite port-2 with I_2 with port-1 shorted and we get:

$$\begin{aligned} I_2 &= y_{21}V_1 + y_{22}V_2 = y_{22}V_2 \\ \therefore \frac{I_2}{V_2} &= y_{22} \end{aligned}$$

For the network to be symmetric:

$$\frac{I_1}{V_1} = \frac{I_2}{V_2} \text{ or, } y_{11} = y_{22}$$

4.2 Symmetrical two port network : z parameters

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Excite port-1 with V_1 with port-2 opened and we get:

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 = z_{11}I_1 \\ \therefore \frac{V_1}{I_1} &= z_{11} \end{aligned}$$

Now, excite port-2 with V_2 with port-1 opened and we get:

$$\begin{aligned} V_2 &= z_{21}I_1 + z_{22}I_2 = z_{22}I_2 \\ \therefore \frac{V_2}{I_2} &= z_{22} \end{aligned}$$

For the network to be symmetric:

$$\frac{V_1}{I_1} = \frac{V_2}{I_2} \text{ or, } z_{11} = z_{22}$$

4.3 Symmetrical two port network : h parameters

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Excite port-1 with voltage with port-2 open circuited.

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ 0 &= h_{21}I_1 + h_{22}V_2 \\ \text{or, } V_2 &= -\frac{h_{21}}{h_{22}}I_1 \\ \text{Put this } V_2 \text{ in first equation} & \\ V_1 &= h_{11}I_1 + h_{12}\left(-\frac{h_{21}}{h_{22}}I_1\right) \\ \therefore \frac{V_1}{I_1} &= \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} \end{aligned}$$

Now excite port-2 with a voltage with port-1 open circuited.

$$\begin{aligned} V_1 &= h_{12}V_2 \\ I_2 &= h_{22}V_2 \\ \therefore \frac{V_2}{I_2} &= \frac{1}{h_{22}} \end{aligned}$$

Now for network to be symmetric:

$$\frac{V_1}{I_1} = \frac{V_2}{I_2} \quad \text{or,} \quad \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} = \frac{1}{h_{22}}$$

Which finally means:

$$h_{11}h_{22} - h_{12}h_{21} = 1 \text{ i.e., Determinant of } h \text{ matrix} = 1$$

4.4 Symmetrical two port network : t parameters

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Excite port-1 with voltage with port-2 open circuited.

$$\begin{aligned} V_1 &= AV_2 \\ I_1 &= CV_2 \\ \therefore \frac{V_1}{I_1} &= \frac{A}{C} \end{aligned}$$

Now excite port-2 with a voltage with port-1 open circuited.

$$\begin{aligned} V_1 &= AV_2 - BI_2 \\ 0 &= CV_2 - DI_2 \\ \therefore \frac{V_2}{I_2} &= \frac{D}{C} \end{aligned}$$

Now for network to be symmetric:

$$\frac{V_1}{I_1} = \frac{V_2}{I_2} \quad \text{or,} \quad \frac{A}{C} = \frac{D}{C} \text{which means,} \;\; A = D$$