16.6 BODE DIAGRAMS

In this section we will discover a quick method of obtaining an *approximate* picture of the amplitude and phase variation of a given transfer function as functions of ω. Accurate curves may, of course, be plotted after calculating values with a programmable calculator or a computer; curves may also be produced directly on the computer.

The Decibel (dB) Scale

The approximate response curve we construct is called an asymptotic plot, or a ***Bode plot,*** or a ***Bode diagram,*** after its developer, Hendrik W. Bode, who was an electrical engineer and mathematician with the Bell Telephone Laboratories. Both the magnitude and phase curves are shown using a logarithmic frequency scale for the abscissa, and the magnitude itself is also shown in logarithmic units called ***decibels*** (dB). We define the value of $|\mathbf{H}(j\omega)|$ in dB as follows:

The decibel is named in honor of Alexander Graham Bell.

$$H_{\text{dB}} = 20 \log |\mathbf{H}(j\omega)|$$

where the common logarithm (base 10) is used. (*A multiplier of 10 instead of 20 is used for power transfer functions,*) The inverse operation is

$$|\mathbf{H}(j\omega)| = 10^{H_{\text{dB}}/20}$$

Before we actually begin a detailed discussion of the technique for drawing Bode diagrams, it will help to gain some feeling for the size of the decibel unit, to learn a few of its important values, and to recall some of the properties of the logarithm. Since $\log 1 = 0$, $\log 2 = 0.30103$, and $\log 10 = 1$, we note the correspondences:

$$|\mathbf{H}(j\omega)| = 1 \Leftrightarrow H_{\text{dB}} = 0$$
$$|\mathbf{H}(j\omega)| = 2 \Leftrightarrow H_{\text{dB}} \approx 6 \text{ dB}$$
$$|\mathbf{H}(j\omega)| = 10 \Leftrightarrow H_{\text{dB}} = 20 \text{ dB}$$

An increase of $|\mathbf{H}(j\omega)|$ by a factor of 10 corresponds to an increase in H_{dB} by 20 dB. Moreover, $\log 10^n = n$, and thus $10^n \Leftrightarrow 20n$ dB, so that 1000 corresponds to 60 dB, while 0.01 is represented as -40 dB. Using only the values already given, we may also note that $20 \log 5 = 20 \log \frac{10}{2} = 20 \log 10 - 20 \log 2 = 20 - 6 = 14$ dB, and thus $5 \Leftrightarrow 14$ dB. Also, $\log \sqrt{x} = \frac{1}{2} \log x$, and therefore $\sqrt{2} \Leftrightarrow 3$ dB and $1/\sqrt{2} \Leftrightarrow -3$ dB.[2]

We will write our transfer functions in terms of $\mathbf{s}$, substituting $\mathbf{s} = j\omega$ when we are ready to find the magnitude or phase angle. If desired, the magnitude may be written in terms of dB at that point.

PRACTICE

16.10 Calculate H_{dB} at $\omega = 146$ rad/s if $\mathbf{H}(\mathbf{s})$ equals (*a*) $20/(\mathbf{s} + 100)$; (*b*) $20(\mathbf{s} + 100)$; (*c*) $20\mathbf{s}$. Calculate $|\mathbf{H}(j\omega)|$ if H_{dB} equals (*d*) 29.2 dB; (*e*) -15.6 dB; (*f*) -0.318 dB.

Ans: -18.94 dB; 71.0 dB; 69.3 dB; 28.8; 0.1660; 0.964.

(2) Note that we are being slightly dishonest here by using $20 \log 2 = 6$ dB rather than 6.02 dB. It is customary, however, to represent $\sqrt{2}$ as 3 dB; since the dB scale is inherently logarithmic, the small inaccuracy is seldom significant.

Determination of Asymptotes

Our next step is to factor $\mathbf{H}(\mathbf{s})$ to display its poles and zeros. We first consider a zero at $\mathbf{s} = -a$, written in a standardized form as

$$\mathbf{H}(\mathbf{s}) = 1 + \frac{\mathbf{s}}{a} \qquad [26]$$

The Bode diagram for this function consists of the two asymptotic curves approached by H_{dB} for very large and very small values of ω. Thus, we begin by finding

$$|\mathbf{H}(j\omega)| = \left|1 + \frac{j\omega}{a}\right| = \sqrt{1 + \frac{\omega^2}{a^2}}$$

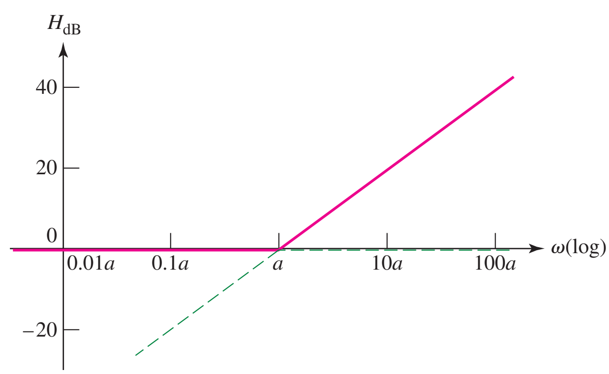
and thus

$$H_{\text{dB}} = 20\log\left|1 + \frac{j\omega}{a}\right| = 20\log\sqrt{1 + \frac{\omega^2}{a^2}}$$

When $\omega \ll a$,

$$H_{\text{dB}} \approx 20\log 1 = 0 \qquad (\omega \ll a)$$

This simple asymptote is shown in Fig. 16.21. It is drawn as a solid line for $\omega < a$, and as a dashed line for $\omega > a$.



■ FIGURE 16.21 The Bode amplitude plot for $\mathbf{H}(\mathbf{s}) = 1 + \mathbf{s}/a$ consists of the low- and high-frequency asymptotes, shown as dashed lines. They intersect on the abscissa at the corner frequency. The Bode plot represents the response in terms of two asymptotes, both straight lines and both easily drawn.

When $\omega \gg a$,

$$H_{\text{dB}} \approx 20\log\frac{\omega}{a} \qquad (\omega \gg a)$$

At $\omega = a$, $H_{\text{dB}} = 0$; at $\omega = 10a$, $H_{\text{dB}} = 20$ dB; and at $\omega = 100a$, $H_{\text{dB}} = 40$ dB. Thus, the value of H_{dB} increases 20 dB for every 10-fold increase in frequency. The asymptote therefore has a slope of 20 dB/decade. Since H_{dB} increases by 6 dB when ω doubles, an alternate value for the slope is 6 dB/octave. The high-frequency asymptote is also shown in Fig. 16.21, a solid

A ***decade*** refers to a range of frequencies defined by a factor of 10, such as 3 Hz to 30 Hz, or 12.5 MHz to 125 MHz. An ***octave*** refers to a range of frequencies defined by a factor of 2, such as 7 GHz to 14 GHz.

line for $\omega > a$, and a broken line for $\omega < a$. Note that the two asymptotes intersect at $\omega = a$, the frequency of the zero. This frequency is also described as the ***corner, break, 3 dB,*** or ***half-power frequency.***

Smoothing Bode Plots

Let us see how much error is embodied in our asymptotic response curve. At the corner frequency ($\omega = a$),

$$H_{\text{dB}} = 20 \log \sqrt{1 + \frac{a^2}{a^2}} = 3 \text{ dB}$$

Note that we continue to abide by the convention of taking $\sqrt{2}$ as corresponding to 3 dB.

as compared with an asymptotic value of 0 dB. At $\omega = 0.5a$, we have

$$H_{\text{dB}} = 20 \log \sqrt{1.25} \approx 1 \text{ dB}$$

Thus, the exact response is represented by a smooth curve that lies 3 dB above the asymptotic response at $\omega = a$, and 1 dB above it at $\omega = 0.5a$ (and also at $\omega = 2a$). This information can always be used to smooth out the corner if a more exact result is desired.

Multiple Terms

Most transfer functions will consist of more than a simple zero (or simple pole). This, however, is easily handled by the Bode method, since we are in fact working with logarithms. For example, consider a function

$$\mathbf{H}(\mathbf{s}) = K\left(1 + \frac{\mathbf{s}}{s_1}\right)\left(1 + \frac{\mathbf{s}}{s_2}\right)$$

where K = constant, and $-s_1$ and $-s_2$ represent the two zeros of our function $\mathbf{H}(\mathbf{s})$. For this function H_{dB} may be written as

$$\begin{aligned} H_{\text{dB}} &= 20 \log \left| K\left(1 + \frac{j\omega}{s_1}\right)\left(1 + \frac{j\omega}{s_2}\right) \right| \\ &= 20 \log \left[K \sqrt{1 + \left(\frac{\omega}{s_1}\right)^2} \sqrt{1 + \left(\frac{\omega}{s_2}\right)^2} \right] \end{aligned}$$

or

$$H_{\text{dB}} = 20 \log K + 20 \log \sqrt{1 + \left(\frac{\omega}{s_1}\right)^2} + 20 \log \sqrt{1 + \left(\frac{\omega}{s_2}\right)^2}$$

which is simply the sum of a constant (frequency-independent) term $20 \log K$ and two simple zero terms of the form previously considered. In other words, *we may construct a sketch of H_{dB} by simply graphically adding the plots of the separate terms.* We explore this in the following example.

EXAMPLE 16.7

Obtain the Bode plot of the input impedance of the network shown in Fig. 16.22.

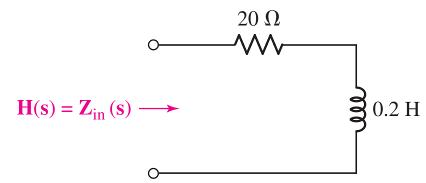


FIGURE 16.22 If $\mathbf{H(s)}$ is selected as $\mathbf{Z}_{\text{in}}(\mathbf{s})$ for this network, then the Bode plot for $\mathbf{H}_{\text{dB}}$ is as shown in Fig. 16.23*b*.

We have the input impedance,

$$\mathbf{Z}_{\text{in}}(\mathbf{s}) = \mathbf{H}(\mathbf{s}) = 20 + 0.2\mathbf{s}$$

Putting this in standard form, we obtain

$$\mathbf{H}(\mathbf{s}) = 20\left(1 + \frac{\mathbf{s}}{100}\right)$$

The two factors constituting $\mathbf{H}(\mathbf{s})$ are a zero at $\mathbf{s} = -100$, leading to a break frequency of $\omega = 100$ rad/s, and a constant equivalent to $20\log 20 = 26$ dB. Each of these is sketched lightly in Fig. 16.23*a*. Since we are working with the logarithm of $|\mathbf{H}(j\omega)|$, we next add together the Bode plots corresponding to the individual factors. The resultant magnitude plot appears as Fig. 16.23*b*. No attempt has been made to smooth out the corner with a +3 dB correction at $\omega = 100$ rad/s; this is left to the reader as a quick exercise.

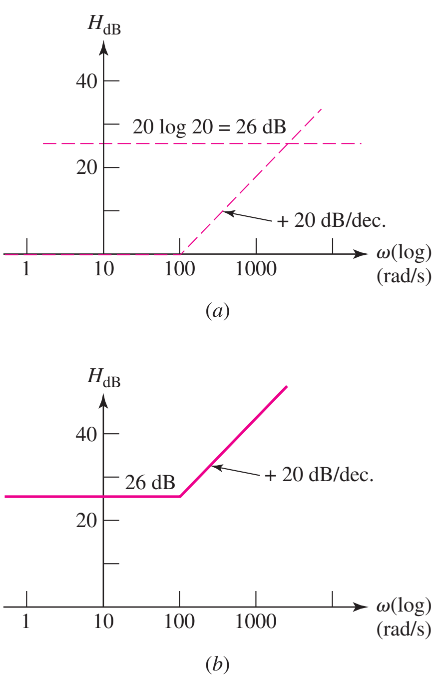


FIGURE 16.23 (*a*) The Bode plots for the factors of $\mathbf{H(s)} = 20(1 + \mathbf{s}/100)$ are sketched individually. (*b*) The composite Bode plot is shown as the sum of the plots of part (*a*).

PRACTICE

16.11 Construct a Bode magnitude plot for $\mathbf{H}(\mathbf{s}) = 50 + \mathbf{s}$.

Ans: 34 dB, $\omega < 50$ rad/s; slope = +20 dB/decade $\omega > 50$ rad/s.

Phase Response

Returning to the transfer function of Eq. [26], we would now like to determine the ***phase response*** for the simple zero,

$$\text{ang}\,\mathbf{H}(j\omega) = \text{ang}\left(1 + \frac{j\omega}{a}\right) = \tan^{-1}\frac{\omega}{a}$$

This expression is also represented by its asymptotes, although three straight-line segments are required. For $\omega \ll a$, ang $\mathbf{H}(j\omega) \approx 0°$, and we use this as our asymptote when $\omega < 0.1a$:

$$\text{ang}\,\mathbf{H}(j\omega) = 0° \qquad (\omega < 0.1a)$$

At the high end, $\omega \gg a$, we have ang $\mathbf{H}(j\omega) \approx 90°$, and we use this above $\omega = 10a$:

$$\text{ang}\,\mathbf{H}(j\omega) = 90° \qquad (\omega > 10a)$$

Since the angle is 45° at $\omega = a$, we now construct the straight-line asymptote extending from 0° at $\omega = 0.1a$, through 45° at $\omega = a$, to 90° at $\omega = 10a$. This straight line has a slope of 45°/decade. It is shown as a solid curve in Fig. 16.24, while the exact angle response is shown as a broken line. The maximum differences between the asymptotic and true responses are $\pm 5.71°$ at $\omega = 0.1a$ and $10a$. Errors of $\mp 5.29°$ occur at $\omega = 0.394a$ and

$2.54a$; the error is zero at $\omega = 0.159a$, a, and $6.31a$. The phase angle plot is typically left as a straight-line approximation, although smooth curves can also be drawn in a manner similar to that depicted in Fig. 16.24 by the dashed line.

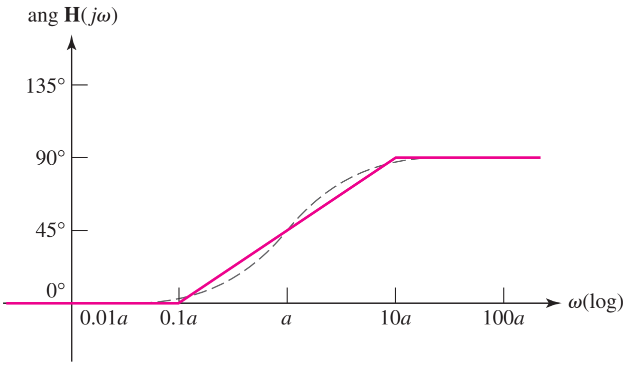


FIGURE 16.24 The asymptotic angle response for $\mathbf{H}(\mathbf{s}) = 1 + \mathbf{s}/a$ is shown as the three straight-line segments in solid color. The endpoints of the ramp are 0° at $0.1a$ and 90° at $10a$. The dashed line represents a more accurate (smoothed) response.

It is worth pausing briefly here to consider what the phase plot is telling us. In the case of a simple zero at $\mathbf{s} = a$, we see that for frequencies much less than the corner frequency, the phase of the response function is 0°. For high frequencies, however ($\omega \gg a$), the phase is 90°. In the vicinity of the corner frequency, the phase of the transfer function varies somewhat rapidly.

PRACTICE

16.12 Draw the Bode phase plot for the transfer function of Example 16.7.

Ans: $0°$, $\omega \leq 10$; $90°$, $\omega \geq 1000$; $45°$, $\omega = 100$; 45°/dec slope, $10 < \omega < 1000$. (ω in rad/s).

Additional Considerations in Creating Bode Plots

We next consider a simple pole,

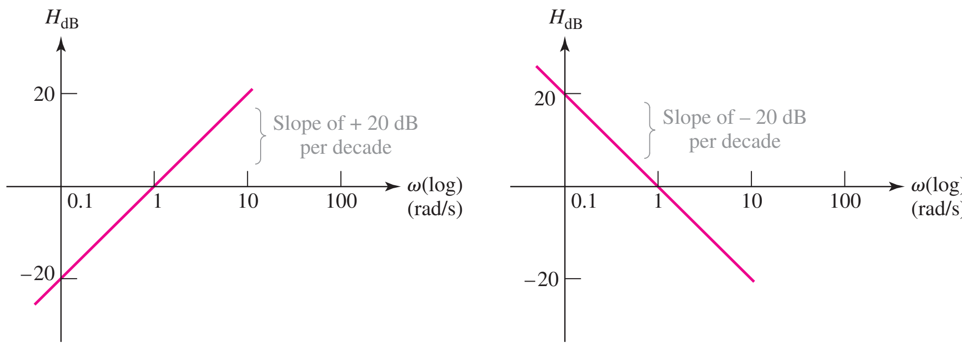
$$\mathbf{H}(\mathbf{s}) = \frac{1}{1 + \mathbf{s}/a} \qquad [27]$$

Since this is the reciprocal of a zero, the logarithmic operation leads to a Bode plot which is the *negative* of that obtained previously. The amplitude is 0 dB up to $\omega = a$, and then the slope is -20 dB/decade for $\omega > a$. The angle plot is 0° for $\omega < 0.1a$, $-90°$ for $\omega > 10a$, and $-45°$ at $\omega = a$, and it has a slope of $-45°$/decade when $0.1a < \omega < 10a$. The reader is encouraged to generate the Bode plot for this function by working directly with Eq. [27].

Another term that can appear in $\mathbf{H}(\mathbf{s})$ is a factor of $\mathbf{s}$ in the numerator or denominator. If $\mathbf{H}(\mathbf{s}) = \mathbf{s}$, then

$$H_{\mathrm{dB}} = 20 \log |\omega|$$

Thus, we have an infinite straight line passing through 0 dB at $\omega = 1$ and having a slope everywhere of 20 dB/decade. This is shown in Fig. 16.25*a*. If the $\mathbf{s}$ factor occurs in the denominator, a straight line is obtained having a slope of -20 dB/decade and passing through 0 dB at $\omega = 1$, as shown in Fig. 16.25*b*.



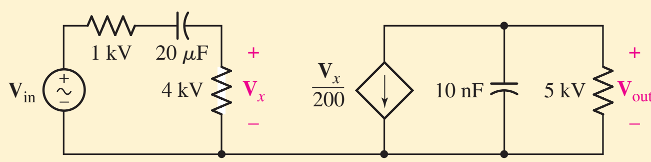
(*a*) Draw the phase plots youselves (*b*)

■ FIGURE 16.25 The asymptotic diagrams are shown for (*a*) $\mathbf{H}(\mathbf{s}) = \mathbf{s}$ and (*b*) $\mathbf{H}(\mathbf{s}) = 1/\mathbf{s}$. Both are infinitely long straight lines passing through 0 dB at $\omega = 1$ and having slopes of ± 20 dB/decade.

Another simple term found in $\mathbf{H}(\mathbf{s})$ is the multiplying constant K. This yields a Bode plot which is a horizontal straight line lying $20 \log |K|$ dB above the abscissa. It will actually be below the abscissa if $|K| < 1$.

EXAMPLE 16.8

Obtain the Bode plot for the gain of the circuit shown in Fig. 16.26.



■ FIGURE 16.26 If $\mathbf{H}(\mathbf{s}) = \mathbf{V}_{\mathrm{out}}/\mathbf{V}_{\mathrm{in}}$, this amplifier is found to have the Bode amplitude plot shown in Fig. 16.27*b*, and the phase plot shown in Fig. 16.28.

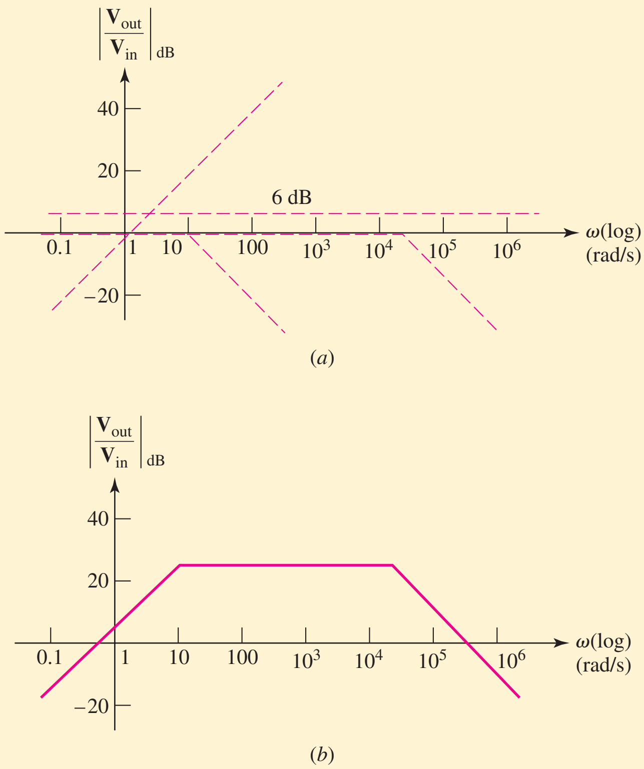
We work from left to right through the circuit and write the expression for the voltage gain,

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_{\mathrm{out}}}{\mathbf{V}_{\mathrm{in}}} = \frac{4000}{5000 + 10^6/20\mathbf{s}} \left(-\frac{1}{200}\right) \frac{5000(10^8/\mathbf{s})}{5000 + 10^8/\mathbf{s}}$$

which simplifies (mercifully) to

$$\mathbf{H}(\mathbf{s}) = \frac{-2\mathbf{s}}{(1 + \mathbf{s}/10)(1 + \mathbf{s}/20{,}000)} \qquad [28]$$

We see a constant, $20 \log |-2| = 6$ dB, break points at $\omega = 10$ rad/s and $\omega = 20{,}000$ rad/s, and a linear factor $\mathbf{s}$. Each of these is sketched in Fig. 16.27*a*, and the four sketches are added to give the Bode magnitude plot in Fig. 16.27*b*.



■ FIGURE 16.27 (*a*) Individual Bode magnitude sketches are made for the factors (-2), $(\mathbf{s})$, $(1 + \mathbf{s}/10)^{-1}$, and $(1 + \mathbf{s}/20{,}000)^{-1}$. (*b*) The four separate plots of part (*a*) are added to give the Bode magnitude plots for the amplifier of Fig. 16.26.

PRACTICE

16.13 Construct a Bode magnitude plot for $\mathbf{H}(\mathbf{s})$ equal to (*a*) $50/(\mathbf{s} + 100)$; (*b*) $(\mathbf{s} + 10)/(\mathbf{s} + 100)$; (*c*) $(\mathbf{s} + 10)/\mathbf{s}$.

Ans: (*a*) −6 dB, $\omega < 100$; −20 dB/decade, $\omega > 100$; (*b*) −20 dB, $\omega < 10$; +20 dB/decade, $10 < \omega < 100$; 0 dB, $\omega > 100$; (*c*) 0 dB, $\omega > 10$; −20 dB/decade, $\omega < 10$.

Before we construct the phase plot for the amplifier of Fig. 16.26, let us take a few moments to investigate several of the details of the magnitude plot.

value of the combined magnitude plot may be found easily at selected points by considering the asymptotic value of each factor of $\mathbf{H(s)}$ at the point in question. For example, in the flat region of Fig. 16.27*a* between $\omega = 10$ and $\omega = 20{,}000$, we are below the corner at $\omega = 20{,}000$, and so we represent $(1 + \mathbf{s}/20{,}000)$ by 1; but we are above $\omega = 10$, so $(1 + \mathbf{s}/10)$ is represented as $\omega/10$. Hence,

$$\begin{aligned} H_{\text{dB}} &= 20 \log \left| \frac{-2\omega}{(\omega/10)(1)} \right| \\ &= 20 \log 20 = 26 \text{ dB} \qquad (10 < \omega < 20{,}000) \end{aligned}$$

We might also wish to know the frequency at which the asymptotic response crosses the abscissa at the high end. The two factors are expressed here as $\omega/10$ and $\omega/20{,}000$; thus

$$H_{\text{dB}} = 20 \log \left| \frac{-2\omega}{(\omega/10)(\omega/20{,}000)} \right| = 20 \log \left| \frac{400{,}000}{\omega} \right|$$

Since $H_{\text{dB}} = 0$ at the abscissa crossing, $400{,}000/\omega = 1$, and therefore $\omega = 400{,}000$ rad/s.

Many times we do not need an accurate Bode plot drawn on printed semilog paper. Instead we construct a rough logarithmic frequency axis on simple lined paper.

EXAMPLE 16.9

Draw the phase plot for the transfer function given by Eq. [28], H(s) = −2s/[(1 + s/10)(1 + s/20,000)].

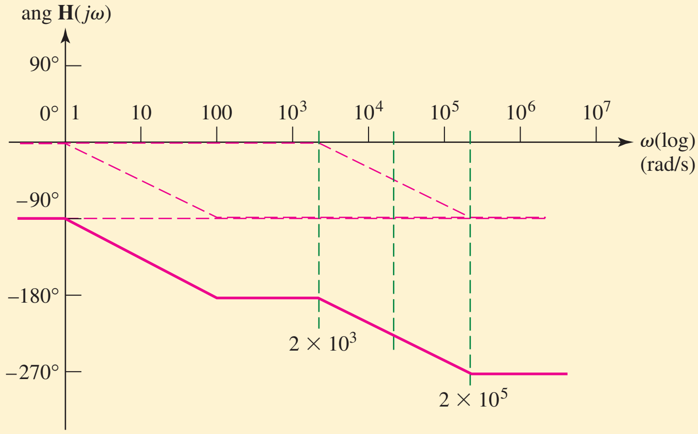
We begin by inspecting $\mathbf{H}(j\omega)$:

$$\mathbf{H}(j\omega) = \frac{-j2\omega}{(1 + j\omega/10)(1 + j\omega/20{,}000)} \qquad [29]$$

The angle of the numerator is a constant, -90°.

The remaining factors are represented as the sum of the angles contributed by break points at $\omega = 10$ and $\omega = 20{,}000$. These three terms appear as broken-line asymptotic curves in Fig. 16.28, and their sum is shown as the solid curve. An equivalent representation is obtained if the curve is shifted upward by 360°.

Exact values can also be obtained for the asymptotic phase response. For example, at $\omega = 10^4$ rad/s, the angle in Fig. 16.28 is obtained from the numerator and denominator terms in Eq. [29]. The numerator angle is -90°. The angle for the pole at $\omega = 10$ is -90°, since ω is greater than 10 times the corner frequency. Between 0.1 and 10 times the corner frequency, we recall that the slope is -45° per decade for a simple pole. For the break point at 20,000 rad/s, we therefore calculate the angle, $-45^\circ \log(\omega/0.1a) = -45^\circ \log[10{,}000/(0.1 \times 20{,}000)] = -31.5^\circ$.



■ FIGURE 16.28 The solid curve displays the asymptotic phase response of the amplifier shown in Fig. 16.26.

The algebraic sum of these three contributions is $-90^\circ - 90^\circ - 31.5^\circ = -211.5^\circ$, a value that appears to be moderately near the asymptotic phase curve of Fig. 16.28.

PRACTICE

16.14 Draw the Bode phase plot for $\mathbf{H}(\mathbf{s})$ equal to (*a*) $50/(\mathbf{s}+100)$; (*b*) $(\mathbf{s}+10)/(\mathbf{s}+100)$; (*c*) $(\mathbf{s}+10)/\mathbf{s}$.

Ans: (*a*) 0°, $\omega < 10$; -45°/decade, $10 < \omega < 1000$; -90°, $\omega > 1000$; (*b*) 0°, $\omega < 1$; $+45^\circ$/decade, $1 < \omega < 10$; 45°, $10 < \omega < 100$; -45°/decade, $100 < \omega < 1000$; 0°, $\omega > 1000$; (*c*) -90°, $\omega < 1$; $+45^\circ$/decade, $1 < \omega < 100$; 0°, $\omega > 100$.

Higher-Order Terms

The zeros and poles that we have been considering are all first-order terms, such as $\mathbf{s}^{\pm 1}$, $(1+0.2\mathbf{s})^{\pm 1}$, and so forth. We may extend our analysis to higher-order poles and zeros very easily, however. A term $\mathbf{s}^{\pm n}$ yields a magnitude response that passes through $\omega = 1$ with a slope of $\pm 20n$ dB/decade; the phase response is a constant angle of $\pm 90n^\circ$. Also, a multiple zero, $(1+\mathbf{s}/a)^n$, must represent the sum of n of the magnitude-response curves, or n of the phase-response curves of the simple zero. We therefore obtain an asymptotic magnitude plot that is 0 dB for $\omega < a$ and has a slope of $20n$ dB/decade when $\omega > a$; the error is $-3n$ dB at $\omega = a$, and $-n$ dB at $\omega = 0.5a$ and $2a$. The phase plot is 0° for $\omega < 0.1a$, $90n^\circ$ for $\omega > 10a$, $45n^\circ$ at $\omega = a$, and a straight line with a slope of $45n^\circ$/decade for $0.1a < \omega < 10a$, and it has errors as large as $\pm 5.71n^\circ$ at two frequencies.

The asymptotic magnitude and phase curves associated with a factor such as $(1+\mathbf{s}/20)^{-3}$ may be drawn quickly, but the relatively large errors associated with the higher powers should be kept in mind.

Complex Conjugate Pairs

The last type of factor we should consider represents a conjugate complex pair of poles or zeros. We adopt the following as the standard form for a pair of zeros:

$$\mathbf{H}(\mathbf{s}) = 1 + 2\zeta\left(\frac{\mathbf{s}}{\omega_0}\right) + \left(\frac{\mathbf{s}}{\omega_0}\right)^2$$

The quantity ζ is the damping factor introduced in Sec. 16.1, and we will see shortly that ω_0 is the corner frequency of the asymptotic response.

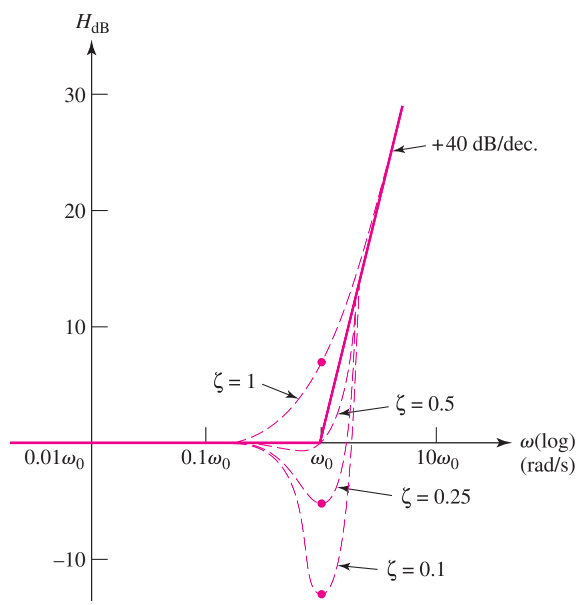
If $\zeta = 1$, we see that $\mathbf{H}(\mathbf{s}) = 1 + 2(\mathbf{s}/\omega_0) + (\mathbf{s}/\omega_0)^2 = (1 + \mathbf{s}/\omega_0)^2$, a second-order zero such as we have just considered. If $\zeta > 1$, then $\mathbf{H}(\mathbf{s})$ may be factored to show two simple zeros. Thus, if $\zeta = 1.25$, then $\mathbf{H}(\mathbf{s}) = 1 + 2.5(\mathbf{s}/\omega_0) + (\mathbf{s}/\omega_0)^2 = (1 + \mathbf{s}/2\omega_0)(1 + \mathbf{s}/0.5\omega_0)$, and we again have a familiar situation.

A new case arises when $0 \le \zeta \le 1$. There is no need to find values for the conjugate complex pair of roots. Instead, we determine the low- and high-frequency asymptotic values for both the magnitude and phase response, and then apply a correction that depends on the value of ζ.

For the magnitude response, we have

$$H_{\text{dB}} = 20\log|\mathbf{H}(j\omega)| = 20\log\left|1 + j2\zeta\left(\frac{\omega}{\omega_0}\right) - \left(\frac{\omega}{\omega_0}\right)^2\right| \qquad [30]$$

When $\omega \ll \omega_0$, $\mathbf{H}_{\text{dB}} = 20\log|1| = 0$ dB. This is the low-frequency asymptote. Next, if $\omega \gg \omega_0$, only the squared term is important, and $\mathbf{H}_{\text{dB}} = 20\log|-(\omega/\omega_0)^2| = 40\log(\omega/\omega_0)$. We have a slope of +40 dB/decade. This is the high-frequency asymptote, and the two asymptotes intersect at 0 dB, $\omega = \omega_0$. The solid curve in Fig. 16.29 shows this asymptotic representation of the magnitude response. However, a correction must be applied



■ FIGURE 16.29 Bode amplitude plots are shown for $\mathbf{H}(\mathbf{s}) = 1 + 2\zeta(\mathbf{s}/\omega_0) + (\mathbf{s}/\omega_0)^2$ for several values of the damping factor ζ.

in the neighborhood of the corner frequency. We let $\omega = \omega_0$ in Eq. [30] and have

$$H_{\text{dB}} = 20 \log \left| j2\zeta \left(\frac{\omega}{\omega_0} \right) \right| = 20 \log(2\zeta) \qquad [31]$$

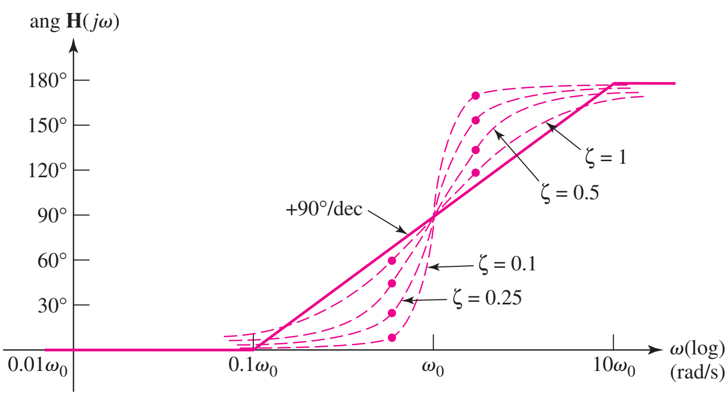
If $\zeta = 1$, a limiting case, the correction is +6 dB; for $\zeta = 0.5$, no correction is required; and if $\zeta = 0.1$, the correction is -14 dB. Knowing this one correction value is often sufficient to draw a satisfactory asymptotic magnitude response. Figure 16.29 shows more accurate curves for $\zeta = 1$, 0.5, 0.25, and 0.1, as calculated from Eq. [30]. For example, if $\zeta = 0.25$, then the exact value of H_{dB} at $\omega = 0.5\omega_0$ is

$$H_{\text{dB}} = 20 \log |1 + j0.25 - 0.25| = 20 \log \sqrt{0.75^2 + 0.25^2} = -2.0 \text{ dB}$$

The negative peaks do not show a minimum value exactly at $\omega = \omega_0$, as we can see by the curve for $\zeta = 0.5$. The valley is always found at a slightly lower frequency.

If $\zeta = 0$, then $\mathbf{H}(j\omega_0) = 0$ and $H_{\text{dB}} = -\infty$. Bode plots are not usually drawn for this situation.

Our last task is to draw the asymptotic phase response for $\mathbf{H}(j\omega) = 1 + j2\zeta(\omega/\omega_0) - (\omega/\omega_0)^2$. Below $\omega = 0.1\omega_0$, we let ang $\mathbf{H}(j\omega) = 0°$; above $\omega = 10\omega_0$, we have ang $\mathbf{H}(j\omega) = \text{ang}\,[-(\omega/\omega_0)^2] = 180°$. At the corner frequency, ang $\mathbf{H}(j\omega_0) = \text{ang}\,(j2\zeta) = 90°$. In the interval $0.1\omega_0 < \omega < 10\omega_0$, we begin with the straight line shown as a solid curve in Fig. 16.30. It extends from $(0.1\omega_0, 0°)$, through $(\omega_0, 90°)$, and terminates at $(10\omega_0, 180°)$; it has a slope of 90°/decade.



■ FIGURE 16.30 The straight-line approximation to the phase characteristic for $\mathbf{H}(j\omega) = 1 + j2\zeta(\omega/\omega_0) - (\omega/\omega_0)^2$ is shown as a solid curve, and the true phase response is shown for $\zeta = 1$, 0.5, 0.25, and 0.1 as broken lines.

We must now provide some correction to this basic curve for various values of ζ. From Eq. [30], we have

$$\text{ang}\,\mathbf{H}(j\omega) = \tan^{-1} \frac{2\zeta(\omega/\omega_0)}{1 - (\omega/\omega_0)^2}$$

One accurate value above and one below $\omega = \omega_0$ may be sufficient to give an approximate shape to the curve. If we take $\omega = 0.5\omega_0$, we find ang $\mathbf{H}(j0.5\omega_0) = \tan^{-1}(4\zeta/3)$, while the angle is $180° - \tan^{-1}(4\zeta/3)$ at

$\omega = 2\omega_0$. Phase curves are shown as broken lines in Fig. 16.30 for $\zeta = 1, 0.5, 0.25$, and 0.1; heavy dots identify accurate values at $\omega = 0.5\omega_0$ and $\omega = 2\omega_0$.

If the quadratic factor appears in the denominator, both the magnitude and phase curves are the *negatives* of those just discussed. We conclude with an example that contains both linear and quadratic factors.

EXAMPLE 16.10

Construct the Bode plot for the transfer function $\mathbf{H(s)} = 100{,}000\mathbf{s}/[(\mathbf{s}+1)(10{,}000+20\mathbf{s}+\mathbf{s}^2)]$.

Let's consider the quadratic factor first and arrange it in a form such that we can see the value of ζ. We begin by dividing the second-order factor by its constant term, 10,000:

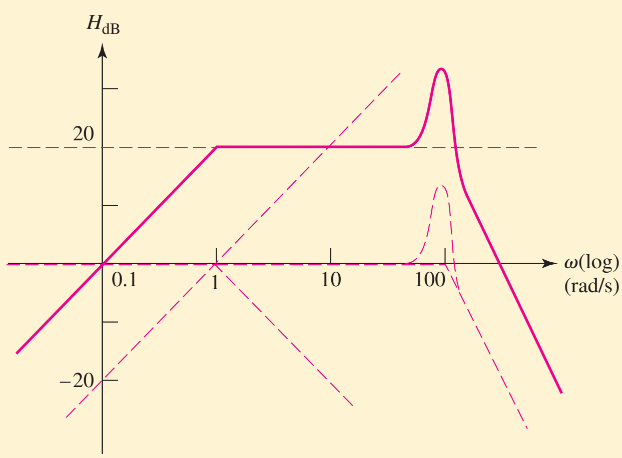
$$\mathbf{H}(\mathbf{s}) = \frac{10\mathbf{s}}{(1+\mathbf{s})(1+0.002\mathbf{s}+0.0001\mathbf{s}^2)}$$

An inspection of the $\mathbf{s}^2$ term next shows that $\omega_0 = \sqrt{1/0.0001} = 100$. Then the linear term of the quadratic is written to display the factor 2, the factor $(\mathbf{s}/\omega_0)$, and finally the factor ζ:

$$\mathbf{H}(\mathbf{s}) = \frac{10\mathbf{s}}{(1+\mathbf{s})[1+(2)(0.1)(\mathbf{s}/100)+(\mathbf{s}/100)^2]}$$

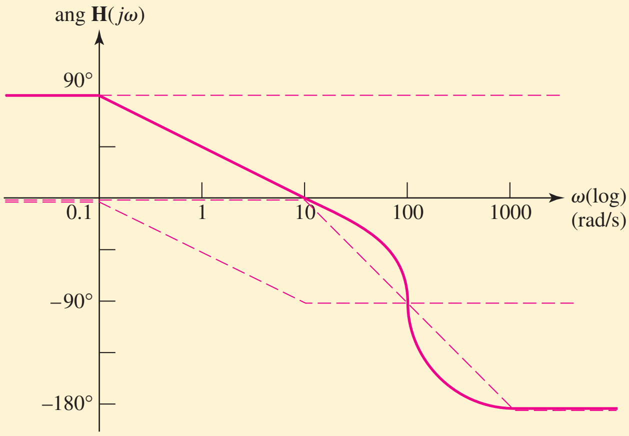
We see that $\zeta = 0.1$.

The asymptotes of the magnitude-response curve are sketched in lightly in Fig. 16.31: 20 dB for the factor of 10, an infinite straight line through $\omega = 1$ with a +20 dB/decade slope for the $\mathbf{s}$ factor, a corner at $\omega = 1$ for the simple pole, and a corner at $\omega = 100$ with a slope of −40 dB/decade for the second-order term in the denominator. Adding these four curves and supplying a correction of +14 dB for the quadratic factor lead to the heavy curve of Fig. 16.31.



■ FIGURE 16.31 The Bode magnitude plot of the transfer function $\mathbf{H(s)} = \dfrac{100{,}000\mathbf{s}}{(\mathbf{s}+1)(10{,}000+20\mathbf{s}+\mathbf{s}^2)}$.

The phase response contains three components: +90° for the factor $\mathbf{s}$; 0° for $\omega < 0.1$, −90° for $\omega > 10$, and −45°/decade for the simple pole; and 0° for $\omega < 10$, −180° for $\omega > 1000$, and −90° per decade for the quadratic factor. The addition of these three asymptotes plus some improvement for $\zeta = 0.1$ is shown as the solid curve in Fig. 16.32.



■ FIGURE 16.32 The Bode phase plot of the transfer function $\mathbf{H}(\mathbf{s}) = \dfrac{100,000\mathbf{s}}{(\mathbf{s}+1)(10,000+20\mathbf{s}+\mathbf{s}^2)}$.

PRACTICE

16.15 If $\mathbf{H}(\mathbf{s}) = 1000\mathbf{s}^2/(\mathbf{s}^2 + 5\mathbf{s} + 100)$, sketch the Bode amplitude plot and calculate a value for (*a*) ω when $H_{\text{dB}} = 0$; (*b*) H_{dB} at $\omega = 1$; (*c*) H_{dB} as $\omega \to \infty$.

Ans: 0.316 rad/s; 20 dB; 60 dB.