Random Vectors $(\Omega, \mathcal{F}, P)$

Tue 2 Feb

Let $X_1, X_2 \ldots X_r$ be r discrete random variables on $(\Omega, \mathcal{F}, P)$

Define $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix}$ is an r-dimensional vector

for an $\omega \in \Omega$

$$\underline{X}(\omega) = \begin{bmatrix} X_1(\omega) \\ X_2(\omega) \\ \vdots \\ X_r(\omega) \end{bmatrix} \in R^r$$

Suppose $X_1(\omega) = x_1$, $X_2(\omega) = x_2 \ldots\ldots X_r(\omega) = x_r$

$$\underline{X}(\omega) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix}$$

Definition → Let $X = \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}$ be a vector where X_i is a random variable on $(\Omega, \mathcal{F}, P)$. For every $\underline{x} \in \mathbb{R}^r$, the set $\{\omega \in \Omega \mid \underline{X}(\omega) = \underline{x}\} \in \mathcal{F}$

Then $X : \Omega \to \mathbb{R}^r$ is called as an r-dimensional random vector.

We are interested in $\text{Prob}(\underline{X} = \underline{x})$

Let $\underline{X} = \begin{bmatrix} X_1(\omega) \\ X_2(\omega) \\ \vdots \\ X_r(\omega) \end{bmatrix}$ be a discrete random vector

If $\underline{x}$ denotes the value assumed by the random vector $\underline{X}$, then,

$\{x : P(\underline{X}(\omega) = x) > 0\}$ is finite or countably finite.

<u>Definition</u>: The discrete density / discrete joint p.m.f of the random vector $\underline{X}$ is defined as

$f(x_1, \ldots x_r) = \text{Prob}(X_1 = x_1, X_2 = x_2 \ldots X_r = x_r)$

where $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}$ and $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix}$

In the vector notation

pmf $f(\underline{x}) = P(\underline{X} = \underline{x}) \quad \forall \underline{x} \in R^r$

for a subset $A \subseteq R^r$

$$P(\underline{X} \in A) = \sum_{\underline{x} \in A} f(\underline{x})$$

<u>Definition</u>:- A function f is called as discrete joint p.m.f if

i) $f(\underline{x}) \geq 0 \quad \forall \underline{x} \in R^r$

ii) $\{x : f(\underline{x}) \neq 0\}$ is finite or countably finite

we denote the elements of this set as $x_1, x_2 \ldots$

iii) $\sum_i f(\underline{x}_i) = 1$

ex-1

$X_1 \downarrow$ \ $X_2 \rightarrow$	1	2	3	4
1	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$

$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$\begin{aligned} &P(\underline{X} = \underline{x}) \\ &= P(X_1 = x_1 ; X_2 = x_2) \\ &= f(x_1, x_2) \\ &= f(\underline{x}) \end{aligned}$$

Very easy to verify that this table is a discrete joint pmf

$\text{Prob}(X_1 \geq X_2) = \frac{1}{4} + \frac{1}{16} + \frac{1}{16}$

$$P(X_1 = 1) = \frac{1}{2} = \sum_{x=1}^{4} Prob(X_1 = 1, X_2 = x)$$

$$= \sum_{x_2} f(1, x_2)$$

$$Prob(X_1 = 2) = \sum_{x_2} f(2, x_2) = \frac{1}{2}$$

$$\boxed{f_{X_1}(x_1) = \sum_{x_2} f(x_1, x_2) \quad \forall x_1 \in R_{X_1}}$$

Marginal pmf of X_1

Independent r.v.s :-

Let $X_1, X_2 \ldots X_r$ be r discrete random variable with p.m.fs $f_1, f_2 \ldots f_r$ respectively
- The random variable $X_1 \ldots X_r$ are called as mutually independent if their joint p.m.f is given by $f(x_1, x_2 \ldots x_r) = f_1(x_1) \cdot f_2(x_2) \ldots f_r(x_r)$

Notation :

$f(x_1, x_2 \ldots x_r) = Prob(X_1 = x_1, X_2 = x_2 \ldots X_r = x_r)$

Define $A_1 \subseteq \Omega$ s.t $A_1 = [X_1 = x_1]$

$$A_1 = \{\omega : X_1(\omega) = x_1\}$$

$= Prob(A_1 \cap A_2 \ldots \cap A_r)$ independence of events

$= Prob(A_1) \cdot Prob(A_2) \ldots Prob(A_r)$

$= Prob(X_1 = x_1) \ldots Prob(X_r = x_r)$

$= f_1(x_1) \cdot f_2(x_2) \ldots f_r(x_r)$

The above example $Ex-1$ is not an independent event

Ex Of Independent event

x_1 \ x_2	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

Ex Let X_1 & X_2 be two independent random variable each with geometric distribution with parameter p. Find the distribution of $\min(X_1, X_2)$

$\text{Prob}(\min(X_1, X_2) \geq z)$

$= \text{Prob}(X_1 \geq z \,;\, X_2 \geq z)$

$= \text{Prob}(X_1 \geq z) \cdot \text{Prob}(X_2 \geq z)$ (X_1 X_2 are independent)

$= (1-p)^z (1-p)^z = (1-p)^{2z}$

$\min(X_1, X_2) \sim \text{geometric}(1-(1-p)^2)$

Post- Midsem

Sum Of Independent Random Variables

Let X and Y be two independent r.v.s, let $x_1, x_2 \dots x_n$ be the discrete values taken by X

Interested in the event

$\{X+Y=z\}$ $\forall z$

$\parallel$

$\bigcup_i \{X=x_i \,;\, Y=z-x_i\}$ ← Note this union is disjoint

$$P(X+Y=z) = P\left(\bigcup_i \{X=x_i, Y=z-x_i\}\right)$$

$$= \sum_i P(X=x_i, Y=z-x_i)$$

$$= \sum_i P(X=x_i) \cdot P(Y=z-x_i)$$

$$\boxed{f_{X+Y}(z) = \sum_{x} f_X(x) f_Y(z-x)}$$ Convolution for

Expectation

Let $\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix}$ be a given discrete random vector with joint p.m.f. $f_{X_1,\ldots X_r}(x_1, \ldots x_r)$

Let $h(x_1, \ldots x_r)$ be a function of $x_1, x_2 \ldots x_r$

$$E(h(X_1, \ldots X_r)) = \sum_{x_1 \ldots x_r} h(x_1 \ldots x_r) f_{X_1, X_2 \ldots X_r}(x_1, \ldots x_r)$$

- for $r = 2$, $h(x_1, x_2) = x_1 + x_2$

$$E(X_1 + X_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2) f_{X_1, X_2}(x_1, x_2)$$

- for $r = 2$, $h(x_1, x_2) = x_1$

$$E(X_1) = \sum_{x_1} \sum_{x_2} x_1 f_{X_1, X_2}(x_1, x_2)$$

$$= \sum_{x_1} x_1 \sum_{x_2} f_{X_1, X_2}(x_1, x_2)$$

$$= \sum_{x_1} x_1 f_{X_1}(x_1) \longrightarrow \text{Marginal density of } X_1$$

* In general, take $h(x_1, \ldots x_r) = x_i$ for $1 \le i \le r$

$$\boxed{E(X_i) = \sum x_i f_{X_i}(x_i)}$$

Q Let X and Y be independent discrete random variable with joint pmf $f_{X,Y}(x, y)$

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$$E(XY) = \sum_{x,y}\sum xy\, f_{X,Y}(x,y)$$

$$= \sum_x \sum_y xy\, f_X(x), f_Y(y)$$

$$= \sum_x x f_X(x) \cdot \sum_y y f_Y(y)$$

Important

$$\boxed{E(XY) = E(X).E(Y)}$$

Monday to March

$$E(\psi_1(x)\, \psi_2(y)) = \sum_x \sum_y \psi_1(x)\, \psi_2(y)\, f_{X,Y}(x,y)$$

$$= \sum_x \psi_1(x) f_X(x) \cdot \sum_y \psi_2(y) f_Y(y)$$

$$= E(\psi_1(x))\; E(\psi_2(y))$$

X & Y are independent random variable with joint pmf $f_{X,Y}(x,y)$

MGF of X + Y →

$$M_{X+Y}(t) = E(e^{t(x+y)})$$

$$= E(e^{tx} \cdot e^{ty})$$

$$= E(e^{tx}).\, E(e^{ty})$$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Generalising this result,

$X_1, X_2 - - X_n$ are mutually independent discrete r.v.s

$$\boxed{M_{\sum_{i=1}^{n} X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t)}$$

Ex Let $X_i \sim$ Bernoulli (p) for $i = 1, 2 \ldots n$ be i.i.d (independent & identically distributed)

$$M_Y(t) = M_{\sum_{i=1}^{n} X_i}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \prod_{i=1}^{n} (1 - p + pe^t)$$

$$= (1 - p + pe^t)^n$$

$\boxed{\text{Binomial}(n, p) = \text{Sum of } n \text{ independent Bernoulli}(p)}$

Imp Observation

X and Y are independent

$\Rightarrow E(XY) = E(X)E(Y)$ (already prove)

Q Is the converse true ?? $\boxed{\text{No}}$

Ex Let (X, Y) be a discrete r.v with Range $R_{X,Y} = \{(0,1), (0,-1), (1,0), (-1,0)\}$ with each outcome equally likely.

$$p(x,y) = \frac{1}{4} \quad \text{for } (x,y) \in R_{X,Y}$$

$$= 0 \quad \text{otherwise}$$

$E(X) = 0 \qquad E(Y) = 0$

$XY = 0 \quad \Rightarrow \quad E(XY) = 0$

$P(X = 0) = \frac{1}{2}$

$P(Y = 0) = \frac{1}{2}$

$f(X=0, Y=0) = 0 \neq f(X=0) \cdot f(Y=0) = \frac{1}{4}$

Remember

- $E(X+Y) = E(X) + E(Y)$
- $E(aX) = aE(X)$
- $E(X+c) = E(X) + c$

Sum of variance:

Let X & Y be discrete random variables

$$Var(X+Y) = E(X+Y-E(X+Y))^2$$
$$= E[(X-E(X)) + (Y-E(Y))]^2$$
$$= E(X-E(X))^2 + E(Y-E(Y))^2 + \underbrace{2E(X-E(X))(Y-E(Y))}_{= COV(X,Y)}$$

$$\boxed{Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)}$$

Note: Covariance, $Cov(X,Y) = E(X-E(X))(Y-E(Y))$

$$= E(XY - XE(Y) - YE(X) + E(X)E(Y))$$
$$= E(XY) - E(X)E(Y) - \cancel{E(X)E(Y)} + \cancel{E(X)E(Y)}$$

$$\boxed{Cov(X,Y) = E(XY) - E(X)E(Y)}$$

Corollary: If X and Y are independent, then

$Cov(X,Y) = 0$

- Converse is NOT true

$Cov(X,Y) = 0 \not\Rightarrow$ X & Y are independent

If X and Y are independent r.v.s

- $\boxed{Var(X+Y) = Var(X) + Var(Y)}$

- $Var(aX) = E(aX - E(aX))^2$

$= E(aX - aE(X))^2$

$= a^2 Var(X)$

- $Var(c)$ where c is a constant

$= E(c - E(c))^2$

$= E(c-c)^2 = 0$

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* Let $X_1, X_2 \cdots X_n$ be independent r.v with variances $\sigma_1^2, \sigma_2^2 \cdots \sigma_n^2$

$Var(X_1 + X_2 \cdots X_n) = Var(X_1) + Var(X_2) \cdots Var(X_n)$

$= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \cdots \sigma_n^2$

Let $X_1, X_2 \cdots X_n$ be iid with mean μ and variance σ^2

- $E(\bar{X}) = E\left(\frac{X_1 + X_2 \cdots X_n}{n}\right) = E\left(\frac{X_1}{n}\right) + \cdots E\left(\frac{X_n}{n}\right)$

$= \frac{1}{n}\left(E(X_1) + \cdots E(X_n)\right)$

$= \frac{1}{n}(n\mu)$

$\boxed{E(\bar{X}) = \mu}$

- $Var(\bar{X}) = Var\left(\frac{X_1 + X_2 \cdots X_n}{n}\right)$

$= Var\left(\frac{X_1}{n}\right) + \cdots Var\left(\frac{X_n}{n}\right)$

$$Var(\bar{x}) = \frac{1}{n^2} \sum_{i=1}^{n} var(x_i)$$

$$= \frac{1}{n^2} \cdot n\sigma^2$$

$$\boxed{Var(\bar{x}) = \frac{\sigma^2}{n}}$$

Correlation coefficient

Let X, Y be two discrete r.v.s, then correlation coefficient $\rho(X,Y)$ (rho) is defined as

$$\rho_{XY} = \rho = \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

$$= \frac{Cov(X,Y)}{S.D(X)\, SD(Y)}$$

X & Y are independent $\Rightarrow \rho_{XY} = 0$

Theorem: Schwartz inequality

Let X and Y be two random variable with finite second order moments

$$[E(XY)]^2 \leq E(X^2) \cdot E(Y^2)$$

furthermore, equality holds when $P(Y=0)=1$ or $P(X=aY)=1$ for some $a \in R$

Proof

Easy to see that, when $P(Y=0)=1$ or $P(X=aY)=1$ equality holds

In order to prove inequality,

For any $\lambda \in R$

$$0 \le E(X-\lambda Y)^2 = \lambda^2 E(Y^2) - 2\lambda(E(XY)) + E(X^2)$$

Since the above expression is a quadratic in λ and $E(Y^2) > 0$, the min. value of this quadratic expression is achieved at

$$2\lambda E(Y^2) - 2E(XY) = 0$$

$$\Rightarrow \quad \lambda = \frac{E(XY)}{E(Y^2)}$$

Min. value at the point $\frac{E(XY)}{E(Y^2)}$ is given by

$$\left[\frac{E(XY)}{E(Y^2)}\right]^2 E(Y^2) - 2\frac{E(XY)}{E(Y^2)} E(XY) + E(X^2)$$

$$-\frac{[E(XY)]^2}{E(Y^2)} + E(X^2) \ge 0$$

$$\boxed{[E(XY)]^2 \le E(X^2) \cdot E(Y^2)}$$

- Importance of Schwartz inequality

Apply the Schwartz inequality to two r.v

$X - E(X)$ and $Y - E(Y)$

$$[E(X-E(X))(Y-E(Y))]^2 \le E(X-E(X))^2 \cdot E(Y-E(Y))^2$$

$$\Rightarrow \quad [Cov(X,Y)]^2 \le Var(X) \cdot Var(Y)$$

$$\Rightarrow \quad \frac{[Cov(X,Y)]^2}{Var(X) \cdot Var(Y)} \le 1$$

$$\Rightarrow \quad \rho_{XY}^2 = \frac{[\mathrm{cov}(X,Y)]^2}{\mathrm{Var}(X)\cdot \mathrm{Var}(Y)} \le 1$$

$$\Rightarrow \quad \boxed{-1 \le \rho_{XY} \le 1}$$

- $\boxed{\rho_{XY} = \pm 1 \iff P(X = aY) = 1}$, when one random variable is a multiple of another random variable
- $\rho_{XY} = 1$ when $a > 0$ & $\rho_{XY} = -1$ when $a < 0$

Recall: If X is a non-negative r.v with finite expectation then for $t > 0$

$$P(X \ge t) \le \frac{E(X)}{t}$$

Chebyshev inequality

$$P(|X-\mu| \ge t) \le \frac{\sigma^2}{t^2}$$

- Let $X_1, X_2 \cdots X_n$ be iid r.v.s. Let $\mu = E(X_1)$, $\sigma^2 = \mathrm{Var}(X_1)$

$$E\left(\frac{S_n}{n}\right) = E\left(\frac{X_1 + X_2 \cdots X_n}{n}\right) = \mu$$

$$\mathrm{Var}\left(\frac{S_n}{n}\right) = \mathrm{var}\left(\frac{X_1 + X_2 \cdots X_n}{n}\right) = \frac{\sigma^2}{n}$$

Chebyshev inequality

$$P\left(\left|\frac{S_n}{n} - \mu\right| \ge \delta\right) \le \frac{\mathrm{Var}\left(\frac{S_n}{n}\right)}{\delta^2} = \frac{\sigma^2}{n\delta^2}$$

In particular

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| \ge \delta\right) = 0$$

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \delta\right) = 0$$

WLLN (weak Law of large Numbers)

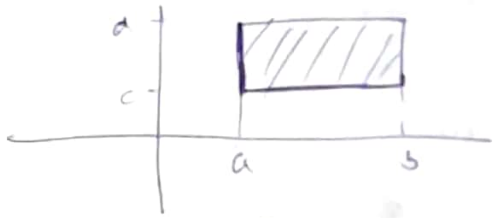
Joint Continuous random Variable

X and Y are continuous r.v.s on the same probability space

$$F(x,y) = \text{Prob}(X \leq x, Y \leq y) \quad -\infty < x, y < \infty$$

Rectangle $R = \{(x,y) \mid a \leq x \leq b \quad c \leq y \leq d\}$

pdf



$$P((X,Y) \in R) = P(a < X \leq b,\ c < Y \leq d)$$

$$= F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

Marginal distributions

$$F_X(x) = P(X \leq x) = F(x, \infty)$$

$$= \lim_{y\to\infty} F(x,y) \quad \text{(Marginal cdf of X)}$$

$$F_Y(y) = P(Y \leq y) = F(\infty, y)$$

$$= \lim_{x\to\infty} F(x,y) \quad \text{(Marginal cdf of Y)}$$

- If there exist a non-negative f^n:– $f(x,y)$ over R^2_y s.t

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

then $f(x,y)$ is called pdf of (X,Y)

$$P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$$

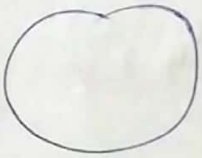
Obviously, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

and $\boxed{\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)}$

Ex let (X,Y) denote the random vector representing the coordinates of randomly chosen point in the circle with center at $(0,0)$ & radius R. If

$$f(x,y) = C \quad \text{for } (x,y) \in \text{circle}$$
$$= 0 \quad \text{o.w}$$

find C



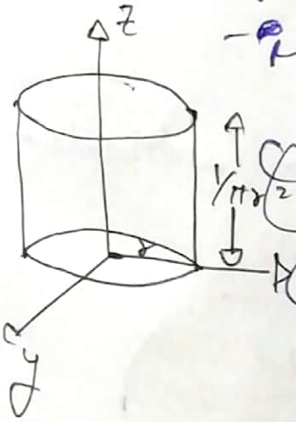
$$\int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-x^2}} C\,dx\,dy = 1$$

$$4\int_0^R \sqrt{R^2-x^2}$$

$$C\,(2R \cdot 2R) = 1$$

$$4\int_0^R \sqrt{R^2-x^2}\,dx$$

$$4C = \frac{1}{4R^2}$$



$\iint$

$$C = \frac{1}{\pi R^2}$$

Ans

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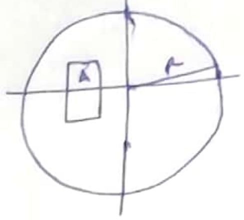

$$f(x,y) = \frac{1}{\pi r^2} \qquad (x,y) \in \text{circle } x^2+y^2 \le r^2$$

$$= 0 \qquad \text{o.w}$$

$$P((x,y) \in A) = \iint_A f(x,y)\,dx\,dy$$

$$= \frac{\text{Area of } A}{\pi R^2}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \qquad \text{Marginal pdf of } X$$

$$= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy$$

$$f_X(x) = \frac{2\sqrt{R^2-x^2}}{\pi R^2} \qquad -R < x < R$$

$$= 0 \qquad \text{otherwise}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{1}{\pi R^2}\,dx$$

$$f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2} \qquad -R < y < R \qquad \text{Marginal pdf of } Y$$

$$= 0 \qquad \text{otherwise}$$

Independence: X and Y are called independent r.v.s if and only if

$$\boxed{f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

(Joint pdf is the product of Marginal pdfs)

* In the above case, X & Y are NOT Independent

Easy way to generate Example:

let X & Y be two independent cont r.v.s with pdfs $f_1(x)$ and $f_2(y)$ resp.

$$f_{XY}(x,y) = f_1(x) f_2(y)$$

$$f_{XY}(x,y) \geq 0 \quad \forall x,y$$

$$\iint_{x\,y} f_{XY}(x,y)\,dx\,dy = \iint f_1(x) f_2(y)\,dx\,dy$$

$$= \int f_1(x)\,dx \cdot \int f_2(y)\,dy$$

Ex

$X \sim N(0,1)$

$Y \sim N(0,1)$

$$\phi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

$$\phi_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \qquad -\infty < y < \infty$$

$$f_{XY}(x,y) = \phi_X(x) \cdot \phi_Y(y)$$

$$= \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} \qquad -\infty < x,y < \infty$$

Ex let X & Y have joint density

$$f(x,y) = c\,e^{-\frac{(x^2-xy+y^2)}{2}} \qquad -\infty < x,y < \infty$$

What is the value of c?

Marginal of X

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= c\int_{-\infty}^{\infty} e^{-\frac{(x^2-xy+y^2)}{2}}\,dy$$

$$= c\int_{-\infty}^{\infty} e^{-\left[\left(y-\frac{x}{2}\right)^2+\frac{3x^2}{4}\right]/2}\,dy$$

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$$= c\, e^{-\frac{3x^2}{8}} \int_{-\infty}^{\infty} e^{-(y-\frac{x}{2})^2/2}\, dy$$

$$= c\, e^{-\frac{3x^2}{8}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\, du \qquad \left(y - \tfrac{x}{2} = u\right)$$

$$f_X(x) = c\, e^{-\frac{3x^2}{8}} \sqrt{2\pi}$$

$1 = \int_{-\infty}^{\infty} c\, e^{-\frac{3x^2}{8}} \sqrt{2\pi}\, dx \quad x = \frac{\sqrt{3}u}{2} \Rightarrow 1 = c \times \frac{2}{\sqrt{3}} 2\pi \Rightarrow$ $\boxed{c = \frac{\sqrt{3}}{4\pi}}$ Ans

Monday 18-Mar

Distributions of Sums & quotients

Let X and Y be ~~[illegible]~~ cont. r.v.s with joint pdf $f(x,y)$

Define $Z = \varphi(X, Y)$

For a fixed $z \in \mathbb{R}$, we are interested in the event $\{Z \le z\}$

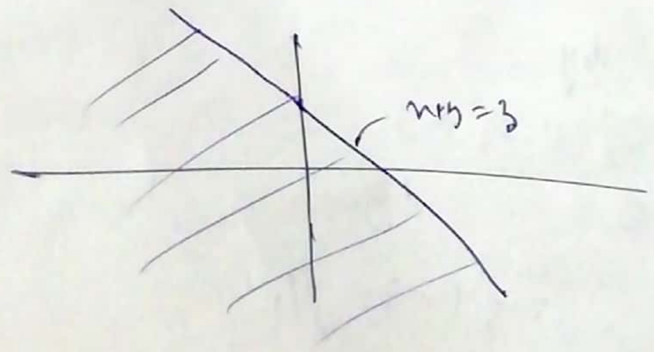
By $A_z \subseteq \mathbb{R}^2$, define

$$A_z = \{(x,y) \mid \varphi(x,y) \le z\}$$

$$F_Z(z) = P(Z \le z)$$
$$= P((X,Y) \in A_z)$$
$$= \iint_{A_z} f(x,y)\, dx\, dy$$

Distribution of Sums $\quad (\varphi(X,Y) = X + Y)$

$$A_z = \{(x,y) \mid x + y \le z\}$$



$$F_Z(z) = \iint_{A_z} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-x} f(x,y)\, dy \right] dx$$

Substitute $y = v - x$ in the inner integral

$$F_Z(z) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z} f(x, v-x)\, dv \right] dx$$

$$F_Z(z) = \int_{-\infty}^{z} \left(\int_{-\infty}^{\infty} f(x, v-x)\, dx \right) dv$$

Thus the pdf of $X+Y$ is given by

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x)\, dx \qquad -\infty < z < \infty$$

If X and Y are independent

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

If X & Y are non-negative independent r.v s

$$f_{X+Y}(z) = \int_{0}^{z} f_X(x)\, f_Y(z-x)\, dx \qquad 0 < z < \infty$$

$$= 0 \qquad \text{elsewhere}$$

Ex $X, Y \sim \exp(\lambda)$ are independent

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$
$$= 0 \qquad \text{otherwise}$$

$$f_Y(y) = \lambda e^{-\lambda y} \qquad y \geq 0$$
$$= 0 \qquad \text{otherwise}$$

$$f_{X+Y}(z) = \int_{0}^{z} f_X(x) \cdot f_Y(z-x)\, dx \qquad z \geq 0$$

$$= 0 \qquad \text{otherwise}$$

for $z \geq 0$

$$f_{X+Y}(z) = \int_0^z \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)} \, dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$\boxed{f_{X+Y}(z) = \lambda^2 z e^{-\lambda z}} \quad z \geq 0$$

$$= 0 \quad \text{otherwise}$$

Thus $X+Y \sim \text{gamma}(2, \lambda)$

Pdf of $\text{Gamma}(\alpha, \lambda) = \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \quad x \geq 0$

Ep X, Y are iid $U(0,1)$ $\quad f_{X+Y} = ?$

$$f_{X+Y} = \int_0^z f_X(x) \cdot f_Y(z-x) \, dx$$

$f_X(x) \cdot f_Y(z-x)$ takes values only 0 and 1

$f_X(x) \, f_Y(z-x)$ takes value 1 when

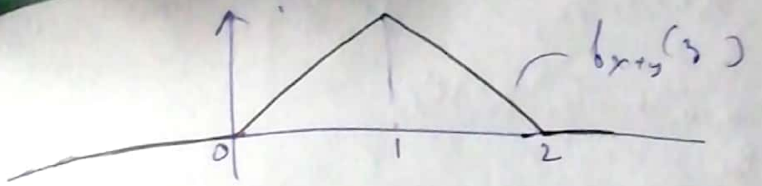
$0 \leq x \leq 1$ and $0 \leq z-x \leq 1$

If $0 \leq z \leq 1$, then the integrand has value 1 on the set $0 \leq x \leq z$

$$\boxed{f_{X+Y}(z) = z} \quad 0 \leq z \leq 1$$

If $1 < z \leq 2$, then the integrand has value 1 on the set $z-1 \leq x \leq 1$

$$\boxed{f_{X+Y}(z) = 2-z} \quad 1 \leq z \leq 2$$



Ex: Let $X \sim \Gamma(\alpha_1, \lambda)$ & $Y \sim \Gamma(\alpha_2, \lambda)$ be independent r.v.s. Then $X+Y \sim \Gamma(\alpha_1+\alpha_2, \lambda)$

$$f_X(x) = \frac{\lambda^{\alpha_1} x^{\alpha_1 - 1} e^{-\lambda x}}{\Gamma(\alpha_1)} \qquad x > 0$$

$$f_Y(y) = \frac{\lambda^{\alpha_2} y^{\alpha_2 - 1} e^{-\lambda y}}{\Gamma(\alpha_2)} \qquad y > 0$$

for $z > 0$

$$f_{X+Y}(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^z e^{-\lambda z} x^{\alpha_1 - 1} (z-x)^{\alpha_2 - 1} \, dx$$

Homework

Prove ~~$f_{X+Y}(z)$~~ $X+Y \sim \Gamma(\alpha_1+\alpha_2, \lambda)$ (try it with MGF approach)

Ex: Let X, Y be independent r.v.s with $X \sim N(\mu_1, \sigma_1^2)$ $\quad Y \sim N(\mu_2, \sigma_2^2)$

Then what is the pdf of $X+Y$??

(Another Approach is MGF)

$$M_{X+Y}(t) = E(e^{t(X+Y)})$$

$$= E(e^{tX} \cdot e^{tY})$$

$$= E(e^{tX}) \cdot E(e^{tY}) \qquad (X \,\&\, Y \text{ are independent})$$

$$\boxed{M_{X+Y}(t) = M_X(t) \cdot M_Y t}$$ for independent X & Y

$$M_X(t) = e^{\mu_1 t + \sigma_1^2 t^2/2} \qquad M_Y(t) = e^{\mu_2 t + \sigma_2^2 t^2/2}$$

$$M_{X+Y} = e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2}$$

$$\Rightarrow \quad X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

- If X & Y are cont. r.v.s with joint pdf $f_{X,Y}(x,y)$

$$E(\theta(x,y)) = \iint \theta(x,y) \cdot f_{X,Y}(x,y)\, dx\, dy$$

- If X & Y are independent

$$E(\theta(x,y)) = \iint \theta(x,y) \cdot f_X(x) \cdot f_Y(y)\, dy\, dx$$

- If X & Y are indep. & $\theta(x,y) = \theta_1(x) \cdot \theta_2(y)$

$$E(\theta(x,y)) = \int \theta_1(x) f_X(x)\, dx \cdot \int \theta_2(y) f_Y(y)\, dy$$

$$= E(\theta_1(x)) \cdot E(\theta_2(y))$$

[This is justification of MGF result we have used in the previous example]

Generalize the result

1. $X_i \sim \exp(\lambda)$ are iid for $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} X_i \sim \text{gamma}(n, \lambda)$$

2. $X_i \sim \text{gamma}(\alpha_i, \lambda)$ are independent for $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} X_i \sim \text{gamma}\left(\sum_{i=1}^{n} \alpha_i, \lambda\right)$$

3. $X_i \sim N(\mu_i, \sigma_i^2)$ are independent for $i = 1, \dots n$

$$\sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Distribution of Quotients

Let X and Y be two cont. r.v.s with joint pdf $f_{XY}(x,y)$ or $f(x,y)$

What is the density of for $z = Y/X$

$$A_z = \{(x,y) \mid y/x \le z\}$$

If $x < 0$, then $y/x \le z \Leftrightarrow y \ge xz$

$$A_z = \{(x,y) \mid x < 0 \ \& \ y \ge xz\} \cup \{(x,y) \mid x > 0 \ \& \ y \le xz\}$$

$$F_{Y/X}(z) = \iint_{A_z} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{0} \left[\int_{xz}^{\infty} f(x,y)\,dy\right] dx + \int_{0}^{\infty} \left[\int_{-\infty}^{xz} f(x,y)\,dy\right] dx$$

Substitute $y = xv$ $(dy = x\,dv)$ in inner integral

$$F_{Y/X}(z) = \int_{-\infty}^{0} \left[\int_{z}^{-\infty} x\, f(x, xv)\,dv\right] dx + \int_{0}^{\infty} \left[\int_{-\infty}^{z} x\, f(x, xv)\,dv\right] dx$$

$$\boxed{F_{Y/X}(z) = \int_{-\infty}^{z} \left[\int_{-\infty}^{\infty} |x|\, f(x, xv)\,dx\right] dv}$$ cdf

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} |x|\, f(x, xz)\,dx$$ pdf

For X, Y non-negative & independent

$$f_{Y/X}(z) = \int_0^\infty x f_X(x) \cdot f_Y(xz)\, dx \qquad 0 < z < \infty$$

Ex. let X & Y be independent r.v with densities $\Gamma(\alpha_1, \lambda)$ & $\Gamma(\alpha_2, \lambda)$ resp.

Prove:
$$\boxed{f_{Y/X}(z) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2)} \frac{z^{\alpha_2-1}}{(z+1)^{\alpha_1+\alpha_2}}} \qquad z \geq 0$$

$$= 0 \qquad \text{o.w}$$

Recall, $f_X(x) = \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} e^{-\lambda x} x^{\alpha_1-1} \quad x > 0$

$$f_Y(y) = \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} e^{-\lambda y} y^{\alpha_2-1} \quad y > 0$$

$$f_{Y/X}(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2)} z^{\alpha_2-1} \int_0^\infty x^{\alpha_1+\alpha_2-1} e^{-x\lambda(z+1)}\, dx$$

$$= \frac{\lambda^{\alpha_1+\alpha_2} z^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{\Gamma(\alpha_1+\alpha_2)}{[\lambda(z+1)]^{\alpha_1+\alpha_2}}$$

Proved

Ex. let X & Y be independent $N(0,\sigma^2)$ r.v.s
find the density of Y^2/X^2

$X^2, Y^2 \sim \Gamma\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$

$$f_{Y^2/X^2}(z) = \frac{1}{\pi(z+1)\sqrt{z}} \qquad z > 0$$

$$= 0 \qquad \text{otherwise}$$

Homework: X & Y are independent $N(0,\sigma^2)$ r.v.s find the density of Y/X^2

Conditional densities

let (X,Y) be a discrete random vector

Conditional probability

$$P(Y=y/X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$= \frac{f(x,y)}{f_X(x)}$$

where $f(x,y)$ is joint pmf of (X,Y) & $f_X(x)$ is the marginal pmf of X

Definition : let X & Y be continuous r.v.s with joint pdf $f(x,y)$. Then the conditional density of Y given X, denoted as $f_{Y|X}(y/x)$ is defined as

$$f_{Y|X}(y/x) = \frac{f(x,y)}{f_X(x)} \quad 0 < f_X(x) < \infty$$

$$= 0 \quad \text{otherwise}$$

$$P(a \le Y \le b / X=x) = \int_a^b f_{Y/X}(y/x)\, dy$$

Also, observe

$$\boxed{f(x,y) = f_X(x)\, f_{Y/X}(y/x)} \quad -①$$

If X & Y are independent.

$$\boxed{f_{Y/X}(y/x) = f_Y(y)} \quad (\text{from } ①)$$

Ex $f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-(x^2-xy+y^2)/2}$ $\quad -\infty < x, y < \infty$

$X \sim N(0, \frac{4}{3})$

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{\sqrt{3}}{4\pi} e^{-(x^2-xy+y^2)/2}}{\frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-\frac{3x^2}{8}}}$$

$$f_{Y/x}(y/x) = \frac{1}{\sqrt{2\pi}} e^{-(y-\frac{x}{2})^2/2}$$

Thus, $f_{Y/x}(y/x)$ is $N(\frac{x}{2}, 1)$

Prob $(0 \le Y \le 2 / X = 0) = \phi(2) - \phi(0)$

where $\phi(x)$ is CDF of $N(0,1)$

Prob $(0 \le Y \le 2 / X = 2) = 2\phi(1) - 1$

Ex Let $X \sim U[0,1]$ and the r.v $Y \sim U[0,X]$

find joint densities of X, Y & marginal density of Y

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_{Y/X}(y/x) = \begin{cases} \frac{1}{x} & 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(x,y) = f_X(x) \cdot f_{Y/x}(y/x)$$

$$= \begin{cases} \frac{1}{x} & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_y^1 \frac{1}{x}\, dx$$

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$$f_Y(y) = \begin{cases} -\log y & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Baye's Rule

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)\, f_{Y/X}(y/x)}{\int_{-\infty}^{\infty} f_X(x)\, f_{Y/X}(y/x)\, dx}$$

Tue - 19 - Mar

Bivariate Normal Density

Random vector (X_1, X_2) is said to follow bivariate normal density if its joint pdf is given by

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$-\infty < x_1 < \infty$

$-\infty < x_2 < \infty$

Probability Computation

$$P(a_1 \le X_1 \le b_1, a_2 \le X_2 \le b_2)$$

$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)\, dx_1\, dx_2$$

The range of parameters

$-\infty < \mu_1 < \infty$

$-\infty < \mu_2 < \infty$

$\sigma_1 > 0$, $\sigma_2 > 0$

$-1 < \rho < 1$

The marginal densities

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\, dx_2$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma}\right)^2} \quad -\infty < x_1 < \infty$$

Marginal density of X_1 is $N(\mu_1, \sigma_1^2)$

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\, dx_1$$

$$f_{X_2}(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2} \qquad -\infty < x_2 < \infty$$

Marginal density of X_2 is $N(\mu_2, \sigma_2^2)$

$E(X_1) = \mu_1$, $Var(X_1) = \sigma_1^2$

$E(X_2) = \mu_2$, $var(X_2) = \sigma_2^2$

$$Cov(X_1, X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1-\mu_1)(x_2-\mu_2) f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

* ρ can not attain values -1 and $+1$, because in that case $X = aY$, means we are looking at the same Random variable, not two different Random variable (No sense of talking about joint PDF)

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1 \cdot \sigma_2} \qquad E(XY) - E(X)E(Y)$$

⇒ In general, independence ⇒ uncorrelatedness and converse is <u>NOT</u> true

⇒ In case of bivariate normal density, if X_1 & X_2 are uncorrelated ($\rho = 0$) ⇒ X_1, X_2 are independent

(joint = Product of marginal)

The conditional densities

$$f_{X_1/X_2} = \frac{f(x_1,x_2)}{f_{X_2}(x_2)}$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2\sigma_1^2(1-\rho^2)}\left\{x_1 - \left[\mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(x_2-\mu_2)\right]\right\}^2\right]$$

$$f_{X_2/X_1}(x_2/x_1) = \frac{f(x_1, x_2)}{f_{X_1}(x_1)}$$

$$= \frac{1}{\sigma_2 \sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2\sigma_2^2(1-\rho^2)} \left\{ x_2 - \left[\mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1-\mu_1)\right]\right\}^2 \right\}$$

Thus, Conditional densities X_2/X_1 and X_1/X_2 are both Normal densities

$$X_2/(X_1 = x_1) \sim N\left(\mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1-\mu_1), \sigma_2^2(1-\rho^2)\right)$$

$$X_1/(X_2 = x_2) \sim N\left(\mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(x_2-\mu_2), \sigma_1^2(1-\rho^2)\right)$$

Ex (X_1, X_2) is a bivariate normal r.v with parameter $\mu_1 = 0.2$, $\mu_2 = 1100$, $\sigma_1^2 = 0.02$, $\sigma_2^2 = 525$, $\rho = 0.9$

Compute $E(X_2/x_1) = \mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1 - \mu_1)$

$$= 1100 + 0.9\left(\frac{\sqrt{525}}{\sqrt{0.02}}\right)(x_1 - 0.2)$$

$$= 145.8x_1 + 1070.84$$

$E(X_2/x_1=1) = 145.8 \times 1 + 1070.84$ Ans
$= 1216.64$

$P(X_2 \geq 1080/x_1=1) = ?$

$Var(X_2/x_1=1) = \sigma_2^2(1-\rho^2) = 99.75$

- Note that $Y = X_2/x_1=1$ is a Normal density with mean μ_y and variance σ_y^2 given by

$\mu_y = 1216.64$

$\sigma_y^2 = 99.75$

$P(X_2 \geq 1080/x_1=1) = P(Y \geq 1080)$

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$$= P\left(\frac{Y-\mu_Y}{\sigma_Y} \geq \frac{1080-\mu_Y}{\sigma_Y}\right)$$

$$= P\left(Z \geq \frac{1080-1216.64}{\sqrt{99.75}}\right)$$

Z = Standard Normal Random variable

$$= P(Z \geq -13.6)$$

$$= 1 - \phi(-13.6)$$

$$= \phi(13.6) \approx 1$$

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