

With mean μ_y & variance σ_y^2 given by
 $\mu_y = 1216.64$
 $\sigma_y^2 = 99.75$

$$P(x_2 \geq 1080 | x_1 = 1) = P(Y \geq 1080)$$

$$= P\left(\frac{Y - \mu_y}{\sigma_y} \geq \frac{1080 - \mu_y}{\sigma_y}\right)$$

$$= P\left(Z \geq \frac{1080 - 1216.64}{\sqrt{99.75}}\right)$$

$$= P(Z \geq -13.6)$$

$$= 1 - \Phi(-13.6)$$

$$= \Phi(13.6)$$

$$\approx 1$$

25/3/19

Change of variable formula \rightarrow

Let x_1, x_2, \dots, x_n be random variables with joint pdf $f(x_1, \dots, x_n)$

Define $Y_i = \sum_{j=1}^n a_{ij} x_j \quad i = 1, \dots, n$

$$A = [a_{ij}]$$

$$= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & & \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

then clearly

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

if $|A| \neq 0$, $B = A^{-1}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$x_i = \sum_{j=1}^n b_{ij} y_j$$

$$i = 1, \dots, n$$

$$f(y_1, y_2, \dots, y_n) = \frac{f(x_1, x_2, \dots, x_n)}{\det(A)}$$

Non linear transformation case -

$$y_i = g_i(x_1, \dots, x_n) \quad i = 1, \dots, n$$

$$J(x_1, \dots, x_n) = \begin{vmatrix} \frac{dy_1}{dx_1} & \dots & \frac{dy_1}{dx_n} \\ \vdots & & \vdots \\ \frac{dy_n}{dx_1} & \dots & \frac{dy_n}{dx_n} \end{vmatrix}$$

$$f(y_1, \dots, y_n) = \frac{1}{|J(x_1, \dots, x_n)|} f(x_1, \dots, x_n)$$

Ex: let x_1, \dots, x_n be iid $\exp(\lambda)$

$$y_i = x_1 + \dots + x_i \quad i = 1, 2, \dots, n$$

$$y_1 = x_1$$

$$y_2 = x_1 + x_2$$

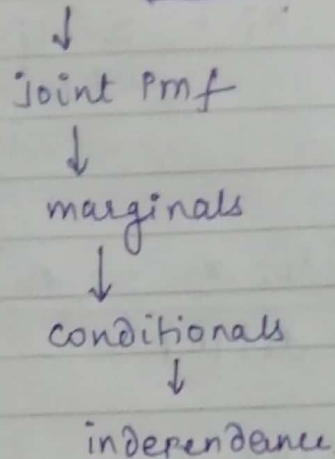
$$y_n = x_1 + x_2 + \dots + x_n$$

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

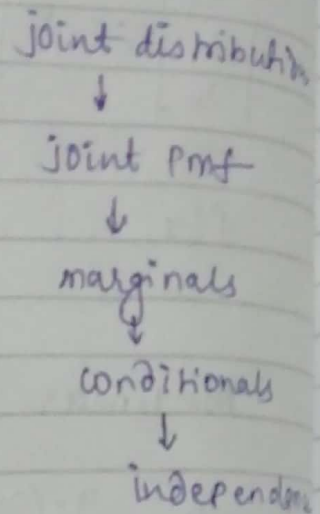
\Rightarrow joint density of y_i 's is same as joint density of x_i 's in this case

Summary

Joint r.v.s discrete etc



Continuous



moments

(mean, variance, median)

mgfs

conditional expectation +
variance.

covariance + correlation

(Bivariate normal)

Transformations

- distribution of sums
convolution / mgf

- distribution of quotients

$\mathbb{R}^n \rightarrow \mathbb{R}$

$$x_1, \dots, x_n \rightarrow x_1 + \dots + x_n$$

$$x_1, x_2 \rightarrow x_1 / x_2$$

$$Y_i = g_i(x_1, \dots, x_n); i = 1, 2, \dots, n \quad \mathbb{R}^n \rightarrow \mathbb{R}^n$$

sampling distribution:-

1) Standard Normal $N(0, 1)$

2) χ^2 distribution with n degrees of freedom.
let (Chi-squared $\tilde{\chi}^2$)

x_1, \dots, x_n be iid $N(0, \sigma^2)$

$$\frac{x_1 + \dots + x_n}{n} \sim N\left(0, \frac{\sigma^2}{n}\right)$$

$$\begin{aligned} \frac{x_1 + \dots + x_n}{\sigma\sqrt{n}} &= \left(\frac{x_1 + \dots + x_n}{n} \right) / \frac{\sigma}{\sqrt{n}} \\ &= \left[\frac{n(\mu, \sigma^2) - \mu}{\sigma} \right] \\ &= N(0, 1) \end{aligned}$$

in Particular $\frac{x_i}{\sigma} \sim N(0, 1)$

$$\Rightarrow \frac{x_i^2}{\sigma^2} \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{for } i = 1, 2, \dots, n$$

From the additivity property of $\Gamma(\alpha_i, \beta)$
we know that

$$\frac{x_1^2}{\sigma^2} + \frac{x_2^2}{\sigma^2} + \dots + \frac{x_n^2}{\sigma^2} \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

$\Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ is χ^2 density with n degrees of freedom.

Noting $\tilde{\chi}^2$ is sum of squares of ' n ' independent standard normal r.v.s.
(n ; dof (degree of freedom))

Theorem; let $Y_1, \dots, Y_2, \dots, Y_m$ be independent χ^2 random variables with d.o.f k_1, \dots, k_m respectively. Then

$$Y_1 + Y_2 + \dots + Y_m \sim \chi^2_{k_1 + \dots + k_m}$$

Ratio of χ^2 densities

density of Y_1/Y_2 is denoted as

$$\text{let } Y_1 \sim \chi^2_{k_1}$$

$$F(k_1, k_2)$$

$$Y_2 \sim \chi^2_{k_2}$$

& Y_1 & Y_2 are independent.

Then the random variable defined by the ratio Y_1/Y_2 is called an F distributed random variable with Y_1/Y_2 (k_1, k_2) d.o.f.

→ Quotient of 2 gamma densities.

$$\text{Particular case; } k_1 = 1 = F(1, k_2)$$

$$F(1, k_2) = \frac{\text{square of std. normal}}{\text{sum of squares of std. normal}}$$

Q/W t-distribution:

(CCT)

let X be std. normal r.v. & Y be a χ^2 r.v. with n dof & X & Y are independent of each other.

$$X \sim N(0, 1) \quad \& \quad X \& Y \text{ are indep.}$$

$$Y \sim \chi^2_n$$

then $\frac{X}{\sqrt{Y/n}}$ is said to follow t distribution with n dof.

$$\frac{X}{\sqrt{Y/n}} \sim t_n \quad \text{clearly } t_n^2 = F(1, n)$$

H/W → derive pdf of t_n .

Central limit Theorem;

let x_1, x_2, \dots be iid with μ & variance σ^2 .

To study the distribution of $\sum_{i=1}^n x_i = S_n$

Notice $\rightarrow E(S_n) = n\mu$

$\text{Var}(S_n) = n\sigma^2$

Suppose x_1 has density f then $f_{S_n}(x) = \sum_y f_{S_{n-1}}(y) f(x-y)$

if x_i 's are discrete

$$f_{S_n}(x) = \int_{-\infty}^{\infty} f_{S_{n-1}}(y) f(x-y) dy \text{ if}$$

x_i 's are continuous

$$\text{Note: } S_n^* = \frac{S_n - E S_n}{\sqrt{\text{Var}(S_n)}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\sim N(0, 1)$$

(CLT) central limit Theorem;

let x_1, x_2, \dots be iid with mean μ & variance σ^2 .

let $S_n = x_1 + \dots + x_n$ then

$$\lim_{n \rightarrow \infty} P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right] = \Phi(x)$$

$-\infty < x < \infty$

where $\Phi(x) = \text{CDF of } N(0, 1)$

$$\text{Note that } P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right] = F_{S_n^*}(x) = \text{CDF of } S_n^*$$

CLT states $\lim_{n \rightarrow \infty} F_{S_n^*}(x) = \Phi(x) \quad -\infty < x < \infty$

Observe:

$$* P(S_n \leq x) \approx \Phi\left(\frac{x - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{x - E S_n}{\sqrt{\text{Var}(S_n^*)}}\right)$$

$$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq \frac{x - n\mu}{\sigma\sqrt{n}}\right)$$

→ Generally for $n \geq 25$ these approximation are really very good.

Ex. let $x_i \sim \exp(1)$ for $i = 1, 2, \dots, n$

$$\lambda e^{-\lambda} = e^{-1}$$

$$S_n = x_1 + \dots + x_n$$

using
CLT

$$\text{Prob}(S_n \leq x) = \Phi\left(\frac{x - n}{\sqrt{n}}\right) \quad -\infty < x < \infty$$

ex: Suppose life of a bulb after it is installed follow exponential distribution with mean = 10 days as soon as the bulb runs out, another bulb with same characteristic is installed what is the probability that 50 bulbs are req. in 1 year.

$$X_i = \text{life of } i^{\text{th}} \text{ bulb after it is installed } \exp\left(\frac{1}{10}\right)$$

$$S_{50} = x_1 + \dots + x_{50}$$

The probability of interest $P(S_{50} \leq 365)$

$$\approx \Phi\left(\frac{365 - 500}{\sqrt{5000}}\right) \rightarrow E(S_n)$$

\swarrow
 \searrow
 $\sqrt{\text{Var } S_n}$

$$= \Phi(-1.91)$$

$$= 0.028$$