with mean μ_y & variance σ_y^2 given by

$\mu_y = 1216.64$

$\sigma_y^2 = 99.75$

$$P(X_2 \geq 1080 \mid X_1 = 1) = P(Y \geq 1080)$$

$$= P\left(\frac{Y - \mu_y}{\sigma_y} \geq \frac{1080 - \mu_y}{\sigma_y}\right)$$

$$= P\left(Z \geq \frac{1080 - 1216.64}{\sqrt{99.73}}\right)$$

$$= P(Z \geq -13.6)$$

$$= 1 - \Phi(-13.6)$$

$$= \Phi(13.6)$$

$$\boxed{\approx 1}$$

25/3/19 Change of variable formula $\rightarrow$

Let $X_1, X_2 \ldots X_n$ be random variables with joint pdf $f(x_1, \ldots x_n)$

Define $Y_i = \sum_{j=1}^{n} a_{ij} X_j \qquad i = 1 \ldots n$

$A = [a_{ij}]$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

then clearly

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = A \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

if $|A| \neq 0$, $B = A^{-1}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad x_i = \sum_{j=1}^{n} b_{ij} y_j$$

$i = 1, \dots n$

$$f(y_1, y_2, \dots y_n) = \frac{f(x_1, x_2 \dots x_n)}{\det |A|}$$

Non linear transformation case -

$y_i = g_i(x_1 \dots x_n) \qquad i = 1 \dots n$

$$J(x_1 \dots x_n) = \begin{vmatrix} \frac{dy_1}{dx_1} & \cdots & \frac{dy_1}{dx_n} \\ \vdots & & \vdots \\ \frac{dy_n}{dx_1} & \cdots & \frac{dy_n}{dx_n} \end{vmatrix}$$

$$f(y_1 \dots y_n) = \frac{1}{|J(x_1 \dots x_n)|} f(x_1 \dots x_n)$$

Ex; let $X_1 \dots X_n$ be iid $\exp(\lambda)$

$Y_i = X_1 + \dots + X_i \qquad i = 1, 2 \dots n$

$Y_1 = X_1$

$Y_2 = X_1 + X_2$

$Y_n = X_1 + X_2 + \dots X_n$

$$A = \begin{bmatrix} 1 & 0 & \cdots & & \\ 1 & 1 & 0 & \cdots & \\ 1 & & & & \\ 1 & 1 & \cdots & & 1 \end{bmatrix}$$

$\Rightarrow$ joint density of Y_i's is same as joint density of X_i's in this case.

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Summary :

joint r.v.s ~~etc~~ discrete

↓

joint pmf

↓

marginals

↓

conditionals

↓

independence

moments —

(mean, variance, median)

mgfs

conditional expectation + variance.

covariance + correlation

Continuous

joint distribution

↓

joint pmf

↓

marginals

↓

conditionals

↓

independence

—

(Bivariate normal

Transformations

- distribution of sums

convolution / mgf

- distribution of quotients

$x_1 \ldots x_n \to x_1 + \ldots + x_n$

$x_1, x_2 \to x_1 / x_2$

$\mathbb{R}^n \to R$

$Y_i = g_i(x_1, \ldots x_n) ; i = 1, 2 \ldots n \mid \mathbb{R}^n - \mathbb{R}^n$

1)

2)

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Sampling distribution:—

1) Standard Normal $N(0,1)$

2) χ^2 distribution with n degrees of freedom.
let (Chi-squared χ^2)
$x_1, \ldots x_n$ be iid $N(0, \sigma^2)$

$$\frac{x_1 + \cdots x_n}{n} \sim N\left(0, \frac{\sigma^2}{n}\right)$$

$$\frac{x_1 + \cdots x_n}{\sigma\sqrt{n}} = \left(\frac{x_1 + \cdots x_n}{n}\right) \Big/ \sigma/\sqrt{n}$$

$$= \left[\frac{n(\mu, \sigma^2) - \mu}{\sigma}\right]$$

$$= N(0,1)$$

in particular $\frac{x_i}{\sigma} \sim N(0,1)$

$$\Rightarrow \frac{x_i^2}{\sigma^2} \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{for } i = 1, 2, \ldots n$$

From the additivity property of $\Gamma(\alpha_i, \beta)$
we know than

$$\frac{x_1^2}{\sigma^2} + \frac{x_2^2}{\sigma^2} + \cdots \frac{x_n^2}{\sigma^2} \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

$\Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ is χ^2 density with n degrees of freedom.

Notice; χ^2 is sum of squares of 'n' independent standard normal r.v.s.
(n; dof (degree of freedom))

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Theorem; let $Y_1, Y_2 \dots Y_m$ be independent χ^2 random variables with dofs $k_1 \dots k_m$ respectively

then

$$Y_1 + Y_2 + \dots + Y_m \sim \chi^2_{k_1 + \dots + k_m}$$

Ratio of χ^2 densities let $Y_1 \sim \chi^2_{k_1}$ $Y_2 \sim \chi^2_{k_2}$ & Y_1 & Y_2 are independent	density of Y_1/Y_2 is denoted as $F(k_1, k_2)$

then the random variable defined by the ratio Y_1/k_1 is called an F distributed random variable with Y_2/k_2 (k_1, k_2) dofs.

→ Quotient of 2 gamma densities.

Particular case; $k_1 = 1$ $=$ $F(1, k_2)$

$$F(1, k_2) = \frac{\text{square of std. normal}}{\text{sum of squares of std. normal}}$$

H/W **t-distribution;** (CLT)

let X be std. normal r.v. & Y be a χ^2 r.v. with n dof & X & Y are independent of each other.

$X \sim N(0,1)$ & X & Y are indep.

$Y \sim \chi^2_n$

then $\dfrac{X}{\sqrt{Y/n}}$ is said to follow t distribution with n dof.

$$\frac{X}{\sqrt{Y/n}} \sim t_n \qquad \text{clearly } t_n^2 = F(1, n)$$

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H/W → derive pdf of t_n:

Central limit Theorem;

let $x_1, x_2 \ldots$ be iid with μ & variance σ^2.

To study the distribution of $\sum_{i=1}^{n} x_i = S_n$

Notice → $E(S_n) = n\mu$

$var(S_n) = n\sigma^2$

Suppose X_1 has density f then $f_{S_n}(x) = \sum_y f_{S_{n-1}}(y) f(x-y)$

if X_i's are discrete

$$f_{S_n}(x) = \int_{-\infty}^{\infty} f_{S_{n-1}}(y) f(x-y)\, dy$$ if

X_i's are continuous

Note:- $$S_n^* = \frac{S_n - ES_n}{\sqrt{var(S_n)}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$\sim N(0,1)$

(CLT) central limit theorem;

let $x_1, x_2 \ldots$ be iid with mean μ & variance σ^2.

let $S_n = x_1 + \ldots x_n$ then

$$\lim_{n\to\infty} P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right] = \Phi(x)$$

$-\infty < x < \infty$

where $\Phi(x)$ = CDF of $n(0,1)$

Note that $P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right] = F_{S_n^*}(x)$

= CDF of S_n^*

CLT states $\lim_{n\to\infty} F_{S_n^*}(x) = \Phi(x) \quad -\infty < x < \infty$

Observe:

* $P(S_n \le x) \approx \Phi\left(\frac{x - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{x - ES_n}{\sqrt{var(S_n^*)}}\right)$

$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le \frac{x - n\mu}{\sigma\sqrt{n}}\right)$

→ Generally for $n \geq 25$ these approximation are really very good.

Ex. let $x_i \sim exp(1)$ for $i = 1, 2 \ldots n$

$\lambda e^{-\lambda} = e^{-1}$

$S_n = x_1 + \cdots x_n$

using CLT $Prob(S_n \le x) = \Phi\left(\frac{x - n}{\sqrt{n}}\right) \quad -\infty < x < \infty$

ex: Suppose life of a bulb after it is installed follow exponential distribution with mean = 10 days as soon as the bulb runs out, another bulb with same characteristic is installed what is the probability that 50 bulbs are req. in 1 year.

X_i = life of i^{th} bulb after it is installed $exp\left(\frac{1}{10}\right)$

$S_{50} = x_1 + \cdots x_{50}$

The probability of interest $P(S_{50} \le 365)$

$\approx \Phi\left(\frac{365 - \overset{E(S_n)}{500}}{\underset{\sqrt{var\, S_n}}{\sqrt{5000}}}\right)$

$= \Phi(-1.91)$

$= 0.028$