paper by Shannon [4], in a book by Keister, Ritchie, and Washburn [2], and in a report by the staff of the Harvard University Computation Laboratory [5].

[1] Boole, G.: *An Investigation of the Laws of Thought*, Dover, New York, 1854.
[2] Keister, W., S. A. Ritchie, and S. Washburn: *The Design of Switching Circuits*, Van Nostrand, New York, 1951.
[3] Shannon, C. E.: "A symbolic analysis of relay and switching circuits," *Trans. AIEE*, vol. 57, pp. 713.723, 1938.
[4] Shannon, C. E.: "The synthesis of two-terminal switching circuits," *Bell System Tech. J.*, vol. 28, pp. 59–98, 1949.
[5] Staff of the Computation Laboratory: "Synthesis of electronic computing and control circuits," *Annals*, vol. 27, Harvard University Press, Cambridge MA, 1951.

Problems

Problem 3.1. Prove the properties in Eqs. (3.3) through (3.12).

Problem 3.2. Using mathematical induction, prove De Morgan's theorem for n variables,

$$[f(x_1, x_2, \ldots, x_n, 0, 1, +, \cdot)]' = f(x_1', x_2', \ldots, x_n', 1, 0, \cdot, +).$$

Problem 3.3. Simplify the following algebraic expressions:
(a) $x' + y' + xyz'$
(b) $(x' + xyz') + (x' + xyz')(x + x'y'z)$
(c) $xy + wxyz' + x'y$
(d) $a + a'b + a'b'c + a'b'c'd + \cdots$
(e) $xy + y'z' + wxz'$
(f) $w'x' + x'y' + w'z' + yz$

Problem 3.4. Find, by inspection, the complement of each of the following expressions and then simplify it.
(a) $x'(y' + z')(x + y + z')$
(b) $(x + y'z')(y + x'z')(z + x'y')$
(c) $w' + (x' + y + y'z')(x + y'z)$

Problem 3.5. Demonstrate, without using perfect induction, whether each of the following equations is valid.
(a) $(x + y)(x' + y)(x + y')(x' + y') = 0$
(b) $xy + x'y' + x'yz = xyz' + x'y' + yz$
(c) $xyz + wy'z' + wxz = xyz + wy'z' + wxy'$
(d) $xy + x'y' + xy'z = xz + x'y' + x'yz$

Problem 3.6. Given $AB' + A'B = C$, show that $AC' + A'C = B$.

Problem 3.7. Find the values of two-valued variables A, B, C, and D by solving the following set of simultaneous equations:

$$\begin{aligned} A' + AB &= 0, \\ AB &= AC, \\ AB + AC' + CD &= C'D. \end{aligned}$$

Problem 3.8. Prove that if $w'x + yz' = 0$, then

$$wx + y'(w' + z') = wx + xz + x'z' + w'y'z.$$

Problem 3.9. Define a connective operator * for two-valued variables A, B, and C as follows:

$$A * B = AB + A'B'.$$

Let $C = A * B$. Determine which of the following is valid:
(a) $A = B * C$
(b) $B = A * C$
(c) $A * B * C = 1$

Problem 3.10. Determine the canonical sum-of-products representation of the following functions:
(a) $f(x, y, z) = z + (x' + y)(x + y')$
(b) $f(x, y, z) = x + (x'y' + x'z)'$

Problem 3.11. Show the truth table for each of the following functions and find its simplest product-of-sums form (i.e., the form with the minimum number of literals).
(a) $f(x, y, z) = xy + xz$
(b) $f(x, y, z) = x' + yz'$

Problem 3.12. By adding redundant factors or terms to the expression $uvw + uwxy + uvxz + xyz$, it may be simplified as follows:

$$\begin{aligned} uvw + uwxy + uvxz + xyz &= uw(v + xy) + xz(uv + y) \\ &= uw(uv + xy) + xz(uv + xy) \\ &= (uw + xz)(uv + xy). \end{aligned}$$

Factor each of the following expressions into a product of two factors such that the resulting expression has the least number of literals:
(a) $wxyz + w'x'y'z' + w'xy'z + wx'yz'$
(b) $vwx + vwyz + wxy + vxyz$

Problem 3.13. The dual f_{d} of a function $f(x_1, x_2, \ldots, x_n)$ is obtained by interchanging the operations of logical addition and multiplication and by interchanging constants 0 and 1 within any expression for that function.
(a) Show that $f_{\mathrm{d}} = f'(x_1', x_2', \ldots, x_n')$.
(b) Find a three-variable function that is its own dual. Such a function is called *self-dual*.
(c) Prove that for any function f and any two-valued variable A, which may or may not be a variable in f, the function

$$g = Af + A'f_{\mathrm{d}}$$

is self-dual.

Problem 3.14
(a) Show that $f(A, B, C) = A'BC + AB' + B'C'$ is a universal operation.
(b) Assuming that a constant value 1 is available, show that $f(A, B) = A'B$ (together with the constant) is a universal operation.

Problem 3.15. For each of the following, prove or show a counter-example.

(a) If $A \oplus B = 0$ then $A = B$.
(b) If $A \oplus C = B \oplus C$ then $A = B$.
(c) $A \oplus B = A' \oplus B'$.
(d) $(A \oplus B)' = A' \oplus B = A \oplus B'$.
(e) $A \oplus (B + C) = (A \oplus B) + (A \oplus C)$.
(f) If $A \oplus B \oplus C = D$ then $A \oplus B = C \oplus D$ and $A = B \oplus C \oplus D$.

Problem 3.16. Any function of two variables can be represented, with proper choice of truth values for the a's, as

$$f(x, y) = a_0x'y' + a_1x'y + a_2xy' + a_3xy.$$

(a) Prove that each representation below can also be used to specify any function of two variables. Show how to obtain the b's and c's from the a's.

$$f(x, y) = b_0 \oplus b_1y \oplus b_2x \oplus b_3xy,$$
$$f(x, y) = c_0x'y' \oplus c_1x'y \oplus c_2xy' \oplus c_3xy.$$

Hint: Compare coefficients by choosing appropriate values for x and y.

(b) Prove that if a function $f(x_1, x_2, \ldots, x_n)$ is represented in a canonical sum-of-products form then all OR operations may be replaced by EXCLUSIVE-OR operations.

Problem 3.17. Prove that any function $f(x_1, x_2, \ldots, x_n)$ can be expressed in a complement-free form as follows:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = d_0 &\oplus d_1x_1 \oplus d_2x_2 \oplus \cdots \oplus d_nx_n \\ &\oplus d_{n+1}x_1x_2 \oplus d_{n+2}x_1x_3 \oplus \cdots \oplus d_{n(n+1)/2}x_{n-1}x_n \\ &\oplus d_{[n(n+1)/2]+1}x_1x_2x_3 \oplus \cdots d_{2^n-1}x_1x_2\cdots x_n, \end{aligned}$$

where $d_0, d_1, \ldots, d_{2^n-1}$ are two-valued variables.

Problem 3.18. Prove that the expansion of any switching function of n variables $f(y_1, y_2, \ldots, y_s, z_1, z_2, \ldots, z_{n-s})$ with respect to the variables $z_1, z_2, \ldots, z_{n-s}$ is given by

$$\begin{aligned} &f(y_1, y_2, \ldots, y_s, z_1, z_2, \ldots, z_{n-s}) \\ &\quad = \sum_{i=1}^{2^{n-s}-1} f_i(y_1, y_2, \ldots, y_s)g_i(z_1, z_2, \ldots, z_{n-s}), \end{aligned}$$

where

$$\begin{aligned} f_0(y_1, y_2, \ldots, y_s) &= f(y_1, y_2, \ldots, y_s, 0, 0, \ldots, 0), \\ f_1(y_1, y_2, \ldots, y_s) &= f(y_1, y_2, \ldots, y_s, 0, 0, \ldots, 1), \\ &\vdots \\ f_{2^{n-s}-1}(y_1, y_2, \ldots, y_s) &= f(y_1, y_2, \ldots, y_s, 1, 1, \ldots, 1) \end{aligned}$$

and where $g_i(z_1, z_2, \ldots, z_{n-s})$ is the product term whose decimal representation is i, e.g., $g_0 = z_1'z_2' \ldots z_{n-s}'$. Note that the distinction between the y's and the z's is only for convenience and has no other significance.

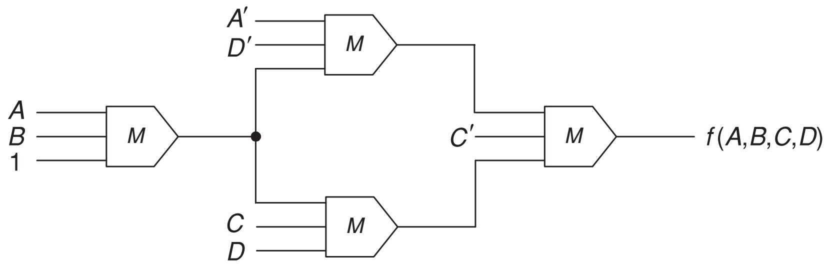
Hint: Use Shannon's expansion theorem as given in Eq. (3.25) and finite induction on s.

Problem 3.19. The *majority function* $M(x, y, z)$ is equal to 1 when two or three of its arguments equal 1, that is,

$$M(x, y, z) = xy + xz + yz = (x + y)(x + z)(y + z)$$

(a) Show that $M(a, b, M(c, d, e)) = M(M(a, b, c), d, M(a, b, e))$.
(b) Show that $M(x, y, z)$, the complementation operation, and the constant 0 form a functionally complete set of operations.
(c) Find the simplest switching expression $f(A, B, C, D)$ corresponding to the network of Fig. P3.19.

Fig. P3.19



Problem 3.20. A safe has five locks, v, w, x, y, and z, all of which must be unlocked for the safe to open. The keys to the locks are distributed among five executives in the following manner:

A has keys for locks v and x;
B has keys for locks v and y;
C has keys for locks w and y;
D has keys for locks x and z;
E has keys for locks v and z.

(a) Determine the minimum number of executives required to open the safe.
(b) Find all the combinations of executives that can open the safe. Write an expression $f(A, B, C, D, E)$ which specifies when the safe can be opened as a function of which executives are present.
(c) Who is the "essential executive" without whom the safe cannot be opened?

Problem 3.21. You are presented with a set of requirements under which an insurance policy will be issued. The applicant must be

1. a married female 25 years old or over, or
2. a female under 25, or
3. a married male under 25 who has not been involved in a car accident, or
4. a married male who has been involved in a car accident, or
5. a married male 25 years or over who has not been involved in a car accident.

Variables w, x, y, and z assume truth value 1 in the following cases:

$w = 1$ if the applicant has been involved in a car accident;
$x = 1$ if the applicant is married;
$y = 1$ if the applicant is a male;
$z = 1$ if the applicant is under 25.

(a) Find an algebraic expression that assumes the value 1 whenever the policy should be issued.
(b) Simplify algebraically the above expression and suggest a simpler set of requirements.

Problem 3.22. Five soldiers, A, B, C, D, and E, volunteer to perform an important military task if the following conditions are satisfied.

1. Either A or B or both must go.
2. Either C or E, but not both, must go.
3. Either both A and C go or neither goes.
4. If D goes then E must also go.
5. If B goes then A and D must also go.

Define variables A, B, C, D, E such that an unprimed variable will mean that the corresponding soldier has been selected to go. Determine the expression that specifies the combinations of volunteers that can get the assignment.

Problem 3.23
(a) Show a series–parallel network that realizes the transmission function $T = A(B + C'D') + A'B'$.
(b) Show an AND, OR, NOT gate network that realizes the function $T = A'B + AB'C + B'C'$, assuming that only unprimed inputs are available.

Problem 3.24. Prove that a Boolean algebra of three elements $B = \{0, 1, a\}$ cannot exist.

Problem 3.25. Prove that for every Boolean algebra:
(a) $a + a'b = a + b$;
(b) if $a + b = a + c$ and $a' + b = a' + c$ then $b = c$;
(c) if $a + b = a + c$ and $ab = ac$ then $b = c$.

Problem 3.26. Prove that the partial ordering of all positive integers dividing number 30 is a Boolean algebra of eight elements, $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$.
(a) Draw the corresponding Hasse diagram.
(b) Define the binary operations by their operations on the integers.
(c) For each element a in B, specify its complement a'.

Problem 3.27. An alternative definition of Boolean algebra is by means of the *Huntington postulates*, which are given as follows:

Definition A Boolean algebra is a set B of elements $a, b, c, \ldots$ with the following properties.

1. B has two binary operations $+$ and $\cdot$, which satisfy the idempotent laws $a + a = a$ and $a \cdot a = a$, the commutative laws $a + b = b + a$ and $a \cdot b = b \cdot a$, the associative laws $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and the absorption laws $a + (a \cdot b) = a$ and $a \cdot (a + b) = a$.
2. The operations are mutually distributive:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad a + (b \cdot c) = (a + b) \cdot (a + c).$$

3. There exist in B two universal bounds 0 and 1, which satisfy

$$0 + a = a, \quad 0 \cdot a = 0, \quad 1 + a = 1, \quad 1 \cdot a = a.$$

4. The Boolean algebra B has a unary operation of complementation, which assigns to every element a in B an element a' in B such that

$$a \cdot a' = 0, \qquad a + a' = 1.$$

Derive the following properties of Boolean algebras directly from the above Huntington postulates.

(a) For each a in B, there exists a *unique* a' in B.
(b) For every a in B, $(a')' = a$.
(c) For every Boolean algebra, $0' = 1$ and $1' = 0$.
(d) In any Boolean algebra,

$$(a + b)' = a' \cdot b' \quad \text{and} \quad (a \cdot b)' = a' + b'.$$