

Computer Science & Engineering Department
I. I. T. Kharagpur

Principles of Programming Languages: CS40032
Elective

Assignment – 2 solutions : λ -Calculus

Marks: 20

Assign Date: January 31, 2020

1. Reduce the following λ -expressions. Show every step of α -, β -, η - and δ - reductions. [2 * 6 = 12]

- (a) $(\lambda z. z) (\lambda y. y y) (\lambda x. x a)$
 (b) $(\lambda z. z) (\lambda z. z z) (\lambda z. z y)$
 (c) $(\lambda x. \lambda y. x y y) (\lambda a. a) b$
 (d) $(\lambda x. \lambda y. x y y) (\lambda y. y) y$
 (e) $(\lambda x. x x) (\lambda y. y x) z$
 (f) $(\lambda x. (\lambda y. (x y)) y) z$

BEGIN SOLUTION

$(\lambda z. z) (\lambda y. y y) (\lambda x. x a)$	// β -reduction = body[sym/replacement]
$(\lambda z. z) (\lambda y. y y) (\lambda x. x a) \rightarrow$	// $z[(\lambda y. y y)]$ replace z with $\lambda y. y$
$(\lambda y. y y) (\lambda x. x a) \rightarrow$	// $y y[(\lambda x. x a)]$ replace y with $\lambda x. x$
$(\lambda x. x a) (\lambda x. x a) \rightarrow$	// $x a[(\lambda x. x a)]$ replace x with $\lambda x. x$
$(\lambda x. x a) a \rightarrow a a$	// $x a[x/a]$ replace x with a
$(\lambda z. z) (\lambda z. z z) (\lambda z. z y)$	
$(\lambda z. z) (\lambda z. z z) (\lambda z. z y) \rightarrow$	// β -reduction: replace z with $\lambda z. z z$
$(\lambda z. z z) (\lambda z. z y) \rightarrow$	// β -reduction: replace z with $\lambda z. z z$
$(\lambda z. z y) (\lambda z. z y) \rightarrow$	// β -reduction: replace z with $\lambda z. z z$
$(\lambda z. z y) y \rightarrow y y$	// β -reduction: replace z with y
$(\lambda x. \lambda y. x y y) (\lambda a. a) b$	
$(\lambda x. \lambda y. x y y) (\lambda a. a) b \rightarrow$	// β -reduction: replace x with $\lambda a. a$
$(\lambda y. (\lambda a. a) y y) b \rightarrow$	// β -reduction: replace y with b
$(\lambda a. a) b b \rightarrow b b$	// β -reduction: replace a with b
$(\lambda x. \lambda y. x y y) (\lambda y. y) y$	
$(\lambda x. \lambda y. x y y) (\lambda y. y) y \rightarrow$	// α -conversion: rename y to a
$(\lambda x. \lambda a. x a a) (\lambda y. y) y \rightarrow$	// β -reduction: replacing x with $\lambda y. y$
$(\lambda a. (\lambda y. y) a a) y \rightarrow$	// β -reduction: replacing a with y
$(\lambda y. y) y y \rightarrow y y$	// β -reduction: replacing y with y
$(\lambda x. x x) (\lambda y. y x) z$	
$(\lambda x. x x) (\lambda y. y x) z \rightarrow$	// β -reduction: replacing x with $\lambda y. y x$
$(\lambda y. y x) (\lambda y. y x) z \rightarrow$	// β -reduction: replacing y with $\lambda y. y x$
$(\lambda y. y x) x z \rightarrow$	// β -reduction: replacing y with x
$x x z$	
$(\lambda x. (\lambda y. (x y)) y) z$	
$(\lambda x. (\lambda y. (x y)) y) z \rightarrow$	// α -conversion: rename y to a
$(\lambda x. (\lambda a. (x a)) y) z \rightarrow$	// β -reduction: replacing x with z
$(\lambda a. (z a)) y \rightarrow$	// β -reduction: replacing a with y
$z y$	

END SOLUTION

2. (a) define $T = \lambda f. \lambda x. \lambda y. \text{if } (= x 0) y \text{ (f (- x 1) (+ y 1))}$
 $Y T = T(Y T)$, is then *add*

(b)

add 2 5

$$\begin{aligned} &= (Y\ T)\ 2\ 5 \\ &= T\ (Y\ T)\ 2\ 5 \\ &= (\lambda \textcolor{red}{f}.\lambda x.\lambda y.\text{if}\ (= x\ 0)\ y\ (\textcolor{red}{f}\ (-\ x\ 1)\ (+\ y\ 1)))\ (Y\ T)\ 2\ 5 \quad ,\ \beta\text{-reduction: } f \rightarrow (Y\ T) \\ &= (\lambda \textcolor{red}{x}.\lambda y.\text{if}\ (= \textcolor{red}{x}\ 0)\ y\ ((Y\ T)\ (-\ \textcolor{red}{x}\ 1)\ (+\ y\ 1)))\ 2\ 5 \quad ,\ \beta\text{-reduction: } x \rightarrow 2 \\ &= (\lambda \textcolor{red}{y}.\text{if}\ (= 2\ 0)\ \textcolor{red}{y}\ ((Y\ T)\ (-\ 2\ 1)\ (+\ \textcolor{red}{y}\ 1)))\ 5 \quad ,\ \beta\text{-reduction: } y \rightarrow 5 \\ &= \text{if}\ (= 2\ 0)\ 5\ ((Y\ T)\ (-\ 2\ 1)\ (+\ 5\ 1)) \quad ,\ \delta\text{-reduction} \\ &= (Y\ T)\ (-\ 2\ 1)\ (+\ 5\ 1) \\ &= (Y\ T)\ 1\ 6 \\ &= T\ (Y\ T)\ 1\ 6 \\ &= (\lambda \textcolor{red}{f}.\lambda x.\lambda y.\text{if}\ (= x\ 0)\ y\ (\textcolor{red}{f}\ (-\ x\ 1)\ (+\ y\ 1)))\ (Y\ T)\ 1\ 6 \quad ,\ \beta\text{-reduction: } f \rightarrow (Y\ T) \\ &= (\lambda \textcolor{red}{x}.\lambda y.\text{if}\ (= \textcolor{red}{x}\ 0)\ y\ ((Y\ T)\ (-\ \textcolor{red}{x}\ 1)\ (+\ y\ 1)))\ 1\ 6 \quad ,\ \beta\text{-reduction: } x \rightarrow 1 \\ &= (\lambda \textcolor{red}{y}.\text{if}\ (= 1\ 0)\ \textcolor{red}{y}\ ((Y\ T)\ (-\ 1\ 1)\ (+\ \textcolor{red}{y}\ 1)))\ 6 \quad ,\ \beta\text{-reduction: } y \rightarrow 6 \\ &= \text{if}\ (= 1\ 0)\ 6\ ((Y\ T)\ (-\ 1\ 1)\ (+\ 6\ 1)) \quad ,\ \delta\text{-reduction} \\ &= (Y\ T)\ (-\ 1\ 1)\ (+\ 6\ 1) \\ &= (Y\ T)\ 0\ 7 \\ &= T\ (Y\ T)\ 0\ 7 \\ &= (\lambda \textcolor{red}{f}.\lambda x.\lambda y.\text{if}\ (= x\ 0)\ y\ (\textcolor{red}{f}\ (-\ x\ 1)\ (+\ y\ 1)))\ (Y\ T)\ 0\ 7 \quad ,\ \beta\text{-reduction: } f \rightarrow (Y\ T) \\ &= (\lambda \textcolor{red}{x}.\lambda y.\text{if}\ (= \textcolor{red}{x}\ 0)\ y\ ((Y\ T)\ (-\ \textcolor{red}{x}\ 1)\ (+\ y\ 1)))\ 0\ 7 \quad ,\ \beta\text{-reduction: } x \rightarrow 0\ T \\ &= (\lambda \textcolor{red}{y}.\text{if}\ (= 0\ 0)\ \textcolor{red}{y}\ ((Y\ T)\ (-\ 0\ 1)\ (+\ \textcolor{red}{y}\ 1)))\ 7 \quad ,\ \beta\text{-reduction: } y \rightarrow 7 \\ &= \text{if}\ (= 0\ 0)\ 7\ ((Y\ T)\ (-\ 0\ 1)\ (+\ 7\ 1)) \quad ,\ \delta\text{-reduction} \\ &= 7 \end{aligned}$$