▨ Correlation coefficient is independent of change of origin and scale.

Let $u = \frac{X-a}{h}$, $V = \frac{Y-b}{K}$ $[\text{as } X = a + hU,\ Y = b + KV]$

To prove $r(X, Y) = r(U, V)$.

$$E(X) = a + h\,E(U) \quad , \quad E(Y) = b + K\,E(V)$$

$$X - E(X) = h\left[U - E(U)\right] \qquad Y - E(Y) = K\left[V - E(V)\right]$$

$$\begin{aligned}
Cov(X, Y) &= E\left\{\{X - E(X)\}\{Y - E(Y)\}\right\} \\
&= E\left\{h\{U - E(U)\} . K\{V - E(V)\}\right\} \\
&= hK\ E\left\{\{U - E(U)\}\{V - E(V)\}\right\} \\
&= hK\ Cov(U, V)
\end{aligned}$$

$$\begin{aligned}
\sigma_X^2 = E\{X - E(X)\}^2 &= E\left\{h^2\{U - E(U)\}^2\right\} \\
&= h^2 \sigma_U^2
\end{aligned}$$

$||^{ly}$ $\sigma_Y^2 = K^2 \sigma_V^2$

$$\Rightarrow \quad r(X, Y)$$

$$= \frac{Cov(X, Y)}{\sigma_X\ \sigma_Y} = \frac{hK\ Cov(U, V)}{hK\,\sigma_U\,\sigma_V}$$

$$= \frac{Cov(U, V)}{\sigma_U\ \sigma_V} = r(U, V).$$

▨ The variables X and Y are connected by the equ^{n} $aX + bY + C = 0$. Show that the correlation coefficient bet^{n} X and Y is -1 if the signs of a and b are alike and $+1$ if they are different.

Given,

$$aX + bY + c = 0$$

$$\Rightarrow a\,E(X) + b\,E(Y) + c = 0$$

$$\Rightarrow a\{X - E(X)\} + b\{Y - E(Y)\} = 0$$

$$Cov(X,Y) = E\{\{X - E(X)\}\{Y - E(Y)\}\}$$

$$= -\frac{b}{a}\{E(Y - E(Y))^2\} = -\frac{b}{a}\sigma_Y^2$$

Again,

$$\sigma_X^2 = E\{X - E(X)\}^2 = \frac{b^2}{a^2}\sigma_Y^2$$

$$\Rightarrow r = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-\frac{b}{a}\sigma_Y^2}{\left|\frac{b}{a}\right|\sigma_Y^2}$$

☒ Calculate correlation coefficient for the following heights (in inches) 28

X: Height of father

Y: Height of son

X: 65 66 67 67 68 69 70 72

Y: 67 68 65 68 72 72 69 71

	X	Y	X^2	Y^2	XY
	65	67	4225	4489	4355
	66	68	4356	4624	4488
	67	65	4489	4225	4355
	67	68	4489	4624	4556
	68	72	4624	5184	4896
	69	72	4761	5184	4968
	70	69	4900	4761	4830
	72	71	5184	5041	5112
Total	544	552	37028	38132	37560

$\bar{X} = \frac{1}{n}\sum X = 68$ $\qquad$ $\bar{Y} = \frac{1}{n}\sum Y = 69$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\frac{1}{n}\sum XY - \bar{X}\bar{Y}}{\sqrt{\frac{1}{n}(\sum X^2 - \bar{X}^2)}\sqrt{\frac{1}{n}(\sum Y^2 - \bar{Y}^2)}}$$

$$= \frac{\frac{1}{8} \times 37560 - 68 \times 69}{\sqrt{4.5 \times 5.5}}$$

$$= \cdot 603.$$

Short-cut Method

X	Y	U = X-68	V = Y-69	U^2	V^2	UV
65	67	-3	-2	9	4	6
66	68	-2	-1	4	1	2
67	65	-1	-4	1	16	4
67	68	-1	-1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
Total		0	0	36	44	24

$$r(U,V) = \frac{Cov(U,V)}{\sigma_U \sigma_V} = \frac{3}{\sqrt{4.5 \times 5.5}} = .603$$

$$\bar{U} = \frac{1}{n} \Sigma U = 0$$

$$\bar{V} = \frac{1}{n} \Sigma V = 0$$

$$Cov(U,V) = \frac{1}{n} \Sigma UV - \bar{U}\bar{V}$$

$$= \frac{1}{8} \times 24 = 3$$

$$\sigma_U^2 = \left\{\frac{1}{n} \Sigma (U^2) - (\bar{U})^2\right\} = \frac{1}{8} \times 36 = 4.5$$

$$\sigma_V^2 = \frac{1}{n} \Sigma V^2 - (\bar{V})^2 = \frac{1}{8} \times 44 = 5.5$$

▨ The independent variables X and Y are defined by:

$$f(x) = \begin{cases} 4ax & 0 < x < r \\ 0 & \text{otherwise} \end{cases} \qquad f(y) = \begin{cases} 4by & 0 \leq y < s \\ 0 & \text{otherwise} \end{cases}$$

Show that $\text{Corr.}(U, V) = \frac{b-a}{b+a}$,

where $U = X + Y, \quad V = X - Y.$

$$\text{Cov}(U, V) = \text{Cov.}(X + Y, X - Y)$$

$$= \underline{\text{Cov.}(X, X) - \text{Cov.}(X, Y) + \text{Cov.}(Y, X) - \text{Cov}(Y, Y)}$$

$$= \sigma_X^2 - \sigma_Y^2$$

$$\text{Var}(U) = \sigma_X^2 + \sigma_Y^2 \qquad \text{Var}(V) = \sigma_X^2 + \sigma_Y^2$$

$$E(X) = \frac{2r}{3}, \quad E(X^2) = \frac{r^2}{2} \qquad V(X) = \frac{1}{36a}$$

$$\left[\int_0^r f(x)\,dx = 1 \Rightarrow a = \frac{1}{2r^2} \right]$$

$$V(Y) = \frac{1}{36b} \Rightarrow \frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} = \frac{b-a}{b+a}$$

▨ Joint probability distⁿ is given by

Y↓ / X→	−1	+1
0	$\frac{1}{8}$	$\frac{3}{8}$
1	$\frac{2}{8}$	$\frac{2}{8}$

Find correlation coefficient between X and Y.

$$E(X) = (-1)\,\frac{3}{8} + 1\left(\frac{5}{8}\right) = \frac{1}{4} \qquad E(X^2) = 1.\frac{3}{8} + 1.\frac{5}{8} = 1$$

$$\text{Var}(X) = \frac{15}{16}, \quad \text{Var}(Y) = \frac{1}{4}$$

$$E(XY) = 0.(-1).\frac{1}{8} + 0.1.\frac{3}{8} + (-1).1.\frac{2}{8} + 1.1.\frac{2}{8} = 0$$

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y) = -\frac{1}{8}.$$

$$\rho = -.2582$$

▨ Given $f(x_1, x_2) = 6x_1 \quad 0 < x_1 < x_2 < 1$

$\qquad\qquad = 0 \quad$ otherwise

Find the correlation coefficient ρ.

Marginal distn of X_1 is

$$f_1(x_1) = 6x_1(1 - x_1) \quad 0 < x_1 < 1$$

Marginal distn of X_2 is

$$f_2(x_2) = 3x_2^2 \qquad 0 < x_2 < 1$$

$E(X_1) = \int_0^1 6x_1^2(1 - x_1)\,dx = \frac{1}{2} \qquad E(X_2) = \frac{3}{4}.$

$V(X_1) = E(X_1^2) - \{E(X_1)\}^2 \qquad V(X_2) = E(X_2^2) - \{E(X_2)\}^2$

$\qquad = \frac{1}{20} \qquad\qquad\qquad\qquad = \cdot 39$

$$E(X_1X_2) = \int_0^1 \int_0^{x_2} x_1x_2\, f(x_1, x_2)\, dx_1\, dx_2$$

$$= \frac{2}{5}.$$

$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)\,E(X_2) = \frac{1}{40}.$

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_{X_1}\,\sigma_{X_2}} = \cdot 179.$$

▨ Let $f(x, y) = \begin{cases} 4xy\, e^{-(x^2+y^2)} & x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{Otherwise} \end{cases}$

Prove that density f^n of $U = \sqrt{X^2+Y^2}$ is

$$h(u) = \begin{cases} 2u^3 e^{-u^2} & 0 \leqslant u < \infty \\ 0 & \text{Otherwise} \end{cases}$$

→ Assume $U = \sqrt{X^2+Y^2}$, $V = X$

$$\frac{1}{J} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & 1 \\ \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} = -\frac{y}{\sqrt{x^2+y^2}}$$

The joint pdf of U and V is given by

$$g(u, v) = f(x, y)\,|J| = 4xy\, e^{-(x^2+y^2)} \cdot \frac{\sqrt{y^2+x^2}}{y}$$

$$= 4x\sqrt{x^2+y^2}\, e^{-(x^2+y^2)}$$

$$= 4uv\, e^{-u^2} \qquad \begin{matrix} u \geqslant 0 \\ 0 < v \leqslant u \end{matrix}$$

Marginal density f^n. of $U = \sqrt{x^2+y^2}$ is

$$h(u) = \int_0^u g(u, v)\, dv = 4u e^{-u^2} \int_0^u v\, dv = \begin{cases} 2u^3 e^{-u^2} & u \geqslant 0 \\ 0 & \text{Otherwise} \end{cases}$$

Bivariate Normal Distribution

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

for $-\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$

$-\infty < \mu_1, \mu_2 < \infty \quad \sigma_1, \sigma_2 > 0, \ -1 \leq \rho \leq 1$

Joint probability

$P(a_1 \leq X_1 \leq b_1, a_2 \leq X_2 \leq b_2)$ is defined as

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2)\, dx_1\, dx_2$$

Marginal density

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \Rightarrow X_1 \sim N(\mu_1, \sigma_1^2)$$

$\parallel^{ly}$

$$f_2(x_2) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}. \quad \Rightarrow X_2 \sim N(\mu_2, \sigma_2^2)$$

Another expression for $f(x_1, x_2)$ is

$$f(x_1, x_2) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2} \times$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_1(1-\rho^2)^{1/2}} e^{-\frac{1}{2\sigma_1(1-\rho^2)}\left[x_1 - \left\{\mu_1 + \rho\sigma_1\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right\}\right]^2}$$

Thus,

$$f_{X_1/X_2=x_2}(x_1/x_2) = \frac{f(x_1, x_2)}{f(x_2)}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(1-\rho^2)^{1/2}} \cdot e^{-\frac{1}{2\sigma_1(1-\rho^2)}\left[x_1 - \left\{\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2)\right\}\right]^2}$$

$$\Rightarrow$$

$$\boxed{X_1/_{X_2=x_2} \sim N\left(\mu_1 + \rho\sigma_1\left(\frac{x_2-\mu_2}{\sigma_2}\right),\ \sigma_1^2(1-\rho^2)\right)}$$

Similarly,

$$\boxed{X_2/_{X_1=x_1} \sim N\left(\mu_2 + \rho\sigma_2\left(\frac{x_1-\mu_1}{\sigma_1}\right),\ \sigma_2^2(1-\rho^2)\right)}$$

Theorem If $(X_1, X_2) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then the marginal and conditional distribution of X_1, X_2 conditional distn $X_1/_{X_2 = x_2}$, $X_2/_{X_1 = x_1}$ are all univariate normal.

Conversely, if marginal and conditional distributions are univariate normal then the joint distribution will be bivariate normal.

▨ The amount of rainfall recorded at weather station in January is RV X and the amount in February at the same station is RV Y. Suppose $(X, Y) \sim BVN(6, 4, 1, \cdot 25, \cdot 1)$. Find $P(X \leq 5)$ and $P(Y \leq 5 / X = 5)$.

$$P(X \leq 5) = P\left(Z \leq \frac{5-6}{1}\right) = \Phi(-1) = 0.1587.$$

$$Y/_{X=5} \sim N\left(4 + \cdot 1 \times \frac{\cdot 5}{1}(5-6)\ ,\ 0.25\,(1 - \cdot 01)\right)$$

$$= N(3.975\ ,\ 0.2475)$$

$$P(Y \leq 5 / X = 5) = P\left(Z \leq \frac{5 - 3.975}{\cdot 4975}\right) = \Phi(2.06)$$

$$= \cdot 9803.$$

▨ A deck of n numbered cards is thoroughly snffled and the cards are inserted into n numbered cells one by one. If the card number 'i' falls in the cell 'i', we count it as a match, otherwise not. Find the mean and variance of total number of such matches.

→ Total number of matches 's' is given by where, $S = X_1 + X_2 + \cdots + X_n$

$$X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ card falls in } `i' \text{ th cell} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_1 + X_2 + \cdots + X_n) =$ ~~0.P(X_1)+1.~~

$$= \sum_{i=1}^{n} 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = n \cdot \frac{1}{n} = 1.$$

$$Var(X_1 + X_2 + \cdots + X_n)$$

$$= Var(X_1) + \cdots + Var(X_n) + 2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} Cov(X_i, X_j)$$

$$Var(X_i) = E(X_i^2) - (E(X_i))^2$$

$$\text{Now, } E(X_i^2) = 1^2 \cdot P(X_i = 1) + 0^2 \cdot P(X_i = 0)$$

$$= \frac{1}{n}$$

$$\Rightarrow var(X_i) = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2},$$

$$Cov(x_i, x_j) = E(x_i x_j) - E(x_i)\, E(x_j)$$

$$E(x_i x_j) = 1 \cdot P(x_i x_j = 1) + 0 \cdot P(x_i x_j = 0)$$

$x_i x_j = 1 \Rightarrow$ i^{th} & j^{th} cards are in their matching position

so, $$P(x_i x_j = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

Thus,

$$Cov(x_i x_j) = \frac{(n-2)!}{n!} \overset{-\frac{1}{n^2}}{\wedge} = \frac{1}{n(n-1)} - \frac{1}{n^2}$$

$$= \frac{n - n + 1}{n^2(n-1)}$$

$$= \frac{1}{n^2(n-1)}$$

Therefore,

$Var(s)$

$$= n \cdot \left(\frac{n-1}{n^2}\right) + 2\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} \frac{1}{n^2(n-1)}$$

$$= \frac{n-1}{n} + 2 \cdot {}^{n}C_{2} \cdot \frac{1}{n^2(n-1)} = \frac{n-1}{n} + \frac{1}{n} = 1.$$

The life a tube (x_1) and the filament diameter (x_2) are distributed as BVN (2000, 0.1, 2500, ·01, ·87). If the filament diameter is ·098, What is the probability that the tube will last 1950 hours?

Solⁿ:
$$X_1 / X_2 = \cdot 098 \sim N\left(2000 + \cdot 87 \cdot \frac{50}{\cdot 1}(\cdot 098 - \cdot 1),\right.$$
$$\left. 2500\left(1 - (0.87)^2\right)\right)$$

$$\equiv N\left(\overset{1999\cdot\cancel{9}3}{2000\cdot 87}, \; 607\cdot 25\right)$$

$$P\left(X_1 > 1950 / X_2 = \cdot 098\right)$$

$$= P\left(Z > \frac{1950 - \underline{2000\cdot 87}}{24\cdot 6526}\right)$$

$$= P\left(Z > -\underline{2\cdot 06}\right)$$

$$= \underline{0\cdot 9803.}$$

Development of Poisson Process

Assumptions

→ Number of arrivals during non-overlapping time intervals are independent RV.

→ $\exists \lambda$, such that for small Δt
Prob (exactly one arrival in Δt) $= \lambda \Delta t$

→ Prob (exactly zero arrival) $= 1 - \lambda \Delta t$

→ Prob (2 or more arrivals) in $0 \Delta t$
s.t. $\frac{0 \Delta t}{\Delta t} \to 0$ as $\Delta t \to 0$

where λ is mean arrival rate / occurence rate.

Then if $P(x) = P(X_t = x) = p_x(t)$
prob. that x no. of arrivals in t.
$x = 0, 1, 2 \ldots$

$$P_x(t + \Delta t) = \underbrace{p_x(t)}_{x \text{ arrivals in } t} \underbrace{(1 - \lambda \Delta t)}_{\substack{\text{No arrivals} \\ \text{in } [t, t+\Delta t]}} + \underbrace{P_{x-1}(t) \cdot \lambda \Delta t}_{\substack{(x-1) \text{ arrivals} \\ \text{in } t \text{ and} \\ 1 \text{ arrival in } \Delta t.}}$$

$$\frac{p_x(t+\Delta t) - p_x(t)}{\Delta t} = -\lambda p_x(t) + \lambda p_{x-1}(t)$$

For $\Delta t \to 0$

$$p_x'(t) = \lambda p_{x-1}(t) - \lambda p_x(t)$$

$$p_0'(t) = \lambda p_{-1}(t) - \lambda P_0(t) = -\lambda p_0(t)$$

$$\Rightarrow p_0(t) = e^{-\lambda t}$$

$$p_1'(t) = \lambda p_0(t) - \lambda p_1(t)$$

$$= -\lambda p_1(t) + \lambda e^{-\lambda t}$$

$$\Rightarrow p_1(t) = e^{-\lambda t}(\lambda t)$$

$$\Rightarrow p_2(t) = \frac{e^{-\lambda t}(\lambda t)^2}{2!}$$

$$\vdots$$

$$\Rightarrow p_x(t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

Suppose that average number of telephone calls arriving at the switchboard of an operator is 30 calls per hour.

(i) What is The prob that no calls arrive in 3 minute periods?

(ii) What is The prob that more than 5 calls in 5 mins period?

Here, $\lambda = 30$, $t = 1$ hr. $\Rightarrow \lambda t = \frac{1}{2}$ per minute.

$$P_0(3) = \frac{e^{-\frac{1}{2}\times 3}\left(\frac{3}{2}\right)^0}{0!} \simeq .22$$

$$P\left(x(5) > 5\right) = \sum_{i=6}^{\infty} \frac{e^{-\frac{1}{2}\times 5}\left(\frac{5}{2}\right)^i}{i!} \cong .42 .$$

▨ Two RVs X and Y have The following joint pdf

$$f(x,y) = \begin{cases} 2-x-y & 0\leq x\leq 1,\ 0\leq y\leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

Find (a) Marginal pdf of X and Y (b) Cond. density fn. (c) Var(X), Var(Y) (d) Covariance of X and Y.

Marginal

$$f_X(x) = \int_0^1 (2-x-y)\,dy = \frac{3}{2} - x$$

$$f_Y(y) = \int_0^1 (2-x-y)\,dx = \frac{3}{2} - y$$

Conditional

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{2-x-y}{3/2-y} \qquad 0<x,y<1$$

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{2-x-y}{3/2-x} \qquad 0<x,y<1$$

E(X), E(X²), Var(X) & Same for Y

$$E(X) = \int_0^1 x f_X(x)\,dx = \frac{5}{12} \qquad E(Y) = \frac{5}{12}$$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \frac{1}{4} \qquad E(Y^2) = \frac{1}{4}$$

$$V(X) = \frac{1}{4} - \frac{25}{144} = \frac{11}{144} \qquad V(Y) = \frac{11}{144}$$

E(XY), Cov(X, Y)

$$E(XY) = \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy = \frac{1}{6}.$$

$$\text{Cov}(X,Y) = E(XY) - E(X)\,E(Y) = -\frac{1}{144}.$$

▨ $$f(x,y) = \begin{cases} \alpha^{-2} e^{-(x+y)/\alpha} & x, y > 0, \ \alpha > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the distribution of $\frac{1}{2}(X-Y)$.

Let $U = \frac{1}{2}(X-Y)$, $v = y$

$$\Rightarrow x = 2u + v, \quad y = v$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 2.$$

$$g(u,v) = f(x,y)\,|J| = \frac{2}{\alpha^2} e^{-\frac{2}{\alpha}(u+v)}$$

where $-\infty < u < \infty$

$v > 0$ if $u \geq 0$

$v > -2u$ if $u < 0$.

Marginal dist: ~~form~~ u

$$g_u(u) = \int_{-2u}^{\infty} \frac{2}{\alpha^2} e^{-\frac{2}{\alpha}(u+v)} dv = \frac{1}{\alpha} e^{\frac{2u}{\alpha}}, \quad v > 0.$$

▨ Let $(X, Y) \sim N(5, 10, 1, 25, \rho)$

(i) If $\rho > 0$, find ρ

when $P(4 < Y < 16 / X = 5) = \cdot 954$.

Conditional dis^{n}.

$$f_{Y/x}(y/x) \sim N\left((\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X),\ \sigma_Y^2(1-\rho^2)\right)$$

$$\Rightarrow P(Y/X = 5) \sim N\left(10 + \rho \frac{5}{1}(5-5),\ 25(1-\rho^2)\right)$$

$$\Rightarrow \text{Given } P(4 < Y < 16 / X = 5) = \cdot 954$$

$$\Rightarrow P\left(\frac{4-10}{5(1-\rho^2)^{1/2}} < \frac{Y-10}{5(1-\rho^2)^{1/2}} < \frac{16-10}{5(1-\rho^2)^{1/2}} \Big/ X = 5\right) = \cdot 954$$

$$\Rightarrow P\left(\frac{-6}{k} < Z < \frac{6}{k}\right) = \cdot 954$$

or, $\frac{6}{k} = 2$ or $k = 3 \Rightarrow \boxed{\rho = \frac{4}{5} = \cdot 8.}$

(ii) If $\rho = 0$ find $P(X + Y) \leq 16$.

$$X + Y \sim N(\mu_X + \mu_Y,\ \sigma_X^2 + \sigma_Y^2)$$

i.e $\sim N(15, 26)$

$$P\left(Z \leq \frac{16-15}{\sqrt{26}}\right) = \cdot 5793.$$

Additive property of poisson distribution.

Let $x_1, x_2 \dots x_n$ be independent Poisson (distribution) RVs with $x_i \sim P(\lambda)$, $i = 1, 2 \dots n$.

Let $S_n = x_1 + x_2 + \dots + x_n$

$$\Rightarrow M_{S_n}(t) = \prod_{i=1}^{n} M_{x_i}(t) = \prod_{i=1}^{n} e^{\lambda_i (e^t - 1)}$$

$$= e^{\sum \lambda_i (e^t - 1)}$$

$$\Rightarrow S_n \sim P(\Sigma \lambda_i)$$

Additive property of Geometric distribution.

Let $x_1, x_2 \dots x_n$ be iid Geo(p). Then if $S_n = \sum x_i$

$$M_{S_i}(t) = \prod_{i=1}^{n} M_{x_i}(t) = \left(\frac{pe^t}{1 - qe^t}\right)^n \quad qe^t < 1.$$

$$\Rightarrow S_i \sim NB(n, p)$$

Additive property of Negative binomial dist^n.

Let $x_1, x_2 \dots x_n$ be iid $NB(r_i, p)$, $i = 1, 2 \dots n$

$$S_n = \sum x_i$$

We get $S_n \sim NB(\Sigma r_i, p)$

Additive property of Gamma distn.

Let $X_1, X_2 \dots X_n$ iid $\text{Gamma}(r_i, \lambda)$, $i = 1, 2 \dots n$

$$\Rightarrow S_n = \sum X_i \sim \text{Gamma}\left(\sum r_i, \lambda\right)$$

Linearity property of Normal distribution.

Let $X_1, X_2 \dots X_n$ be iid normal Rvs and $X_i \sim N(\mu_i, \sigma_i^2). \forall i$

then $$Y = \sum_{i=1}^{n} (a_i X_i + b_i)$$

$$\sim N\left(\sum_{i=1}^{n}(a_i\mu_i + b_i),\ \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

If X_1 and X_2 are not independent

$$\begin{aligned}\text{Var}(X_1 + X_2) &= E(X_1 + X_2)^2 - (EX_1 + EX_2)^2 \\ &= EX_1^2 + EX_2^2 + 2EX_1X_2 - (EX_1)^2 - (E(X_2))^2 \\ &\qquad - 2(EX_1)(EX_2) \\ &= \text{Var}(X_1) + \text{Var}(X_2) + 2\,\text{Cov}(X_1, X_2)\end{aligned}$$

Further properties...

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, X_j)$$

$$\text{Cov}\left(\sum_{1}^{n} X_i, \sum_{1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j)$$

Additive property of Exponential distn.

Let $X_1, X_2 \cdots X_n$ be iid Exp (λ).

$$\Rightarrow S_n = \sum_i X_i \sim \underline{\text{Gamma}\,(n, \lambda)}.$$

▨ The life of an electric system is $Y = X_1 + X_2 + X_3 + X_4$ where the system lifes X_1, X_2, X_3, X_4 are idependent each having exponential distn with mean 4 hrs. What is the probability that the system will operate atleast 24 hrs?

Here, $X_i \sim \text{Exp}\,(1/4)$

$$\Rightarrow Y = \sum_{i=1}^{4} X_i \sim \text{Gamma}\,(4, 1/4)$$

$$P(Y \geqslant 24) = \int_{24}^{\infty} \frac{1}{4^4 \lceil 4} e^{-x/4} x^3 \, dx .$$

$$= \cdot 1512 .$$

▨ If $X_1, X_2 \sim \text{Bin}\,(n, p)$, prove that $X_1 + X_2 \sim \text{Bin}\,(2n, p)$

Find the distn of $X_1 - X_2$. ?

▨ Let X and Y Bin (n, p). Find the joint dist^{n}. of U and V where $U = \frac{X}{Y+1}$, $V = Y+1$.

X varies from $0, 1, 2 \dots n$

Y ...... $0, 1, 2 \dots n$

V ...... $1, 2, 3 \dots n+1$

U ...... $0, 1, \frac{1}{2}, \frac{1}{3} \dots \frac{1}{n+1}$

$2, \frac{2}{2}, \frac{2}{3}, \frac{2}{4} \dots \frac{2}{n+1}$

....

$n, \frac{n}{2}, \frac{n}{3}, \frac{n}{4} \dots \frac{n}{n+1}$

$$P(U = u, V = v)$$

$$= P(X = uv, Y = v - 1)$$

$$= P(X = uv)\ P(Y = v - 1)$$

$$= \left({}^{n}C_{uv}\ p^{uv}\ q^{n-uv}\right) \cdot \left({}^{n}C_{v-1}\ p^{v-1}\ q^{n-v+1}\right)$$

▨

Y \ X	−1	0	1
−2	1/6	1/12	1/6
1	1/6	1/12	1/6
2	1/12	0	1/12

If $U = |X|$ and $V = Y^2$

Find joint pmf of U, V

V \ U	0	1
1	1/12	1/3
4	1/12	1/12

▨ Let (x, y) have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1+xy}{4} & |x|<1 \,,\ |y|<1 \\ 0 & \text{otherwise} \end{cases}$$

If $U = X^2$ and $V = Y^2$, then find joint pdf of (U, V).

$$F_{U,V}(u, v) = P\left(U \leqslant u \,,\ V \leqslant v\right)$$

$$= P\left(-\sqrt{u} \leqslant X \leqslant \sqrt{u} \,,\ -\sqrt{v} \leqslant Y \leqslant \sqrt{v}\right)$$

$$= \int_{-\sqrt{v}}^{\sqrt{v}} \int_{-\sqrt{u}}^{\sqrt{u}} \left(\frac{1+xy}{4}\right) dx\, dy$$

$$= \sqrt{u}\,\sqrt{v}$$

▨ Let $X, Y \overset{iid}{\sim} U(0, 1)$, $U = X + Y$, $V = X - Y$

$$x = \frac{u+v}{2} \,,\ y = \frac{u-v}{2} \qquad J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

Joint pdf of X, Y is

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} 1 & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

So the joint pdf of U and V is

$$f_{U,V}(u, v) = \begin{cases} 1/2 & 0 \leqslant u+v \leqslant 2 \quad 0 < u - v < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow 0 < u < 2$$

$$-1 < v < \cancel{2}\ 1$$

Theorem: Let $X = (X_1, X_2 \ldots X_n)$ be an n-dimensional continuous random vector with joint pdf $f_{\underline{X}}(\underline{x})$, $\underline{x} = (x_1, x_2 \ldots x_n)$. Let (a) $u_i = g_i(\underline{x})$, $i = 1, 2 \ldots n$ be a one-to-one transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e. inverse transformation $x_1 = h_1(\underline{u})$, $x_2 = h_2(\underline{u}) \ldots x_n = h_n(\underline{u})$, $\underline{u} = (u_1, u_2 \ldots u_n)$ defined over the range of transformation

(b) Assume that the mapping and inverse are both continuous,

(c) Assume that partial derivatives $\frac{\partial x_i}{\partial u_j}$, $i, j = 1, 2 \ldots n$ exist and are continuous

(d) Assume that the Jacobian J of transformations does not vanish in the range of transformation

where,
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

Then the random vector $\underline{U} = (U_1, U_2 \ldots U_n)$ is continuous and has joint pdf given by

$$f_{\underline{U}}(\underline{u}) = f_{\underline{X}}(h_1(\underline{u}), h_2(\underline{u}) \ldots h_n(\underline{u}))\,|J|.$$

▨ Let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Exp}(1)$

$$Y_1 = X_1 + X_2 + X_3 \,, \quad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3} \,, \quad Y_3 = \frac{X_1}{X_1 + X_2}$$

$$\Rightarrow Y_1 \sim \overset{\text{Gamma}}{\cancel{\text{Geo}}}\,(3, 1)$$

Again

$$x_1 = y_1\, y_2\, y_3$$

$$x_2 = y_1\, y_2 (1 - y_3)$$

$$x_3 = y_1 (1 - y_2)$$

$$J = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ y_2(1-y_3) & y_1(1-y_2) & -(y_1 y_2) \\ 1-y_2 & -y_1 & 0 \end{vmatrix} = -y_1^2 y_2$$

Now, $f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \prod_{i=1}^{3} f_{X_i}(x_i) = \begin{cases} e^{-\sum x_i} & x_i > 0 \\ & i = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$

Then joint pdf of $(Y_1, Y_2, Y_3) = \underline{Y}$ (say) is

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} e^{-y_1} \cdot y_1^2 y_2 & y_1 > 0, \quad y_2, y_3 \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Marginal dist^{n}s are $f_{Y_1}(y_1) = e^{-y_1} \cdot \frac{y_1^2}{2} \quad y_1 > 0$

$$f_{Y_2}(y_2) = 2y_2 \quad 0 < y_2 < 1$$

$$f_{Y_3}(y_3) = 1 \quad 0 < y_3 < 1$$

Here $f_{\underline{Y}}(y) = \prod_{i=1}^{3} f_{Y_i}(y_i) \Rightarrow$ independent.