1. Three distinct numbers are selected from $\{1, 2, 3, \ldots, 20, 21\}$ randomly without replacement. What is the probability that the sum of these numbers is divisible by 3? [5 marks]

Answer: 0.3368

There are 2 ways in which one can select three numbers from $\{1, 2, \cdots, 21\}$ whose sum is divisible by 3.

<u>case (i)</u> All the numbers selected leave different remainder when divided by 3.

The possibilities for remainder are 0, 1, 2.

$\#$ divisible by 3 (remainder 0) $= \# \{3, 6, \cdots, 21\} = 7$

$\#$ ~~divisible by~~

$\#$ (remainder 1) $= \# \{1, 4, \cdots, 19\} = 7$

$\#$ (remainder 2) $= \# \{2, 5, \cdots, 20\} = 7$

Thus, three numbers can be chosen from these sets in $7 \cdot 7 \cdot 7 = \underline{343}$ ways.

<u>case (ii)</u> All numbers selected leave same remainder

These numbers come from $\{3, 6, \cdots, 21\}$ or $\{1, 4, \cdots, 19\}$ or $\{2, 5, 8 \cdots, 20\}$ in $3\binom{7}{3}$ ways

Thus required probability $= \dfrac{3\binom{7}{3} + 343}{\binom{21}{3}}$

$= \dfrac{32}{95} = \boxed{0.3368}$

2. A batch of 580 students taking the course Probability and Statistics is divided into three sections A,B,C. Section A has 180 students while Sections B and C have 200 students each. After the examination, it is observed that 11 students from Section A scored EX grade while 7 students failed the course. In Section B, 9 students scored EX while 12 students failed the course. In Section C, 13 students scored EX while 12 students failed the course.

(a) If a randomly selected student from the batch has scored EX grade, what is the probability that the student is from Section A? [1 mark]

Answer: $\frac{1}{3} = 0.3333$

(b) If a randomly selected student from the batch has failed the course, what is the probability that the student belongs to either Section B or Section C? [2 marks]

Answer: $\frac{24}{31} = 0.7741$

(c) If two students are selected randomly from the batch and it is noted that one of them has scored EX while the other has failed course. What is the probability that both the students belong to the same section? [2 marks]

Answer: $\frac{1}{3} = 0.3333$

Baye's theorem

a) $$\frac{\frac{11}{180} \cdot \frac{180}{580}}{\frac{11}{180} \cdot \frac{180}{580} + \left(\frac{9}{200} + \frac{13}{200}\right) \frac{200}{580}} = \frac{11}{11+9+13} = \boxed{0.}$$

b) $$\frac{\frac{12}{200} \cdot \frac{200}{580} + \frac{12}{200} \cdot \frac{200}{580}}{\frac{7}{180} \cdot \frac{180}{580} + \frac{12}{180} \cdot \frac{180}{580} + \frac{12}{200} \cdot \frac{200}{580}} = \frac{24}{31} = \boxed{0.77}$$

c) $$\frac{11 \cdot 7 + 9 \cdot 12 + 13 \cdot 12}{33 \cdot 31} = \frac{341}{1023} = \frac{1}{3} = \boxed{0.333}$$

3. Suppose n balls are distributed at random into r boxes. Find the probability that there are exactly k balls in the first $r_1 (< r)$ boxes. [2 marks]

Answer: $\boxed{\binom{n}{k}\left(\frac{r_1}{r}\right)^k\left(1-\frac{r_1}{r}\right)^{n-k}}$

Probability that a randomly distributed ball will be in first $r_1 (< r)$ boxes out of r boxes is $\left(\frac{r_1}{r}\right)$.

There are n balls

P(k balls will be distributed in first r_1 boxes out of r boxes)

$$= \binom{n}{k}\left(\frac{r_1}{r}\right)^k\left(1-\frac{r_1}{r}\right)^{n-k}$$

4. Tattoo bubble gums are on sale for Rs. 5 each. Each bubble gum contains exactly one tattoo, which can be one of five types with equal probability. Suppose you keep on buying bubble gums and stop when you collect all the five types of tattoos. What will be your expected expenditure?

[5 marks]

Answer: $\boxed{57.08}$

There are 5 categories of tattoos with equal probability (1/5)

Let x_1 : number of bubble gums purchased to get any category.

$\Rightarrow$ $X_1 = 1$ with prob. 1. $\Rightarrow E(X_1) = 1$

X_2 : number of bubble gums purchased to get a new category than the 1st one purchased

$\Rightarrow X_2 = 1 + Y_2$ where $Y_2 \sim geo\left(\frac{4}{5}\right)$.

$\Rightarrow E(X_2) = 1 + \frac{1/5}{4/5} = \frac{5}{4}$

Similarly

$X_3 = 1 + Y_3$ where $Y_3 \sim geo\left(\frac{3}{5}\right)$

$\Rightarrow E(X_3) = \frac{5}{3}$.

$X_4 = 1 + Y_4$ where $Y_4 \sim geo.\left(\frac{2}{5}\right)$.

$\Rightarrow E(X_4) = \frac{5}{2}$

where $Y_5 \sim geo\left(\frac{1}{5}\right)$

Expected number of purchase.

$$E\left(\sum_{i=1}^{5} X_i\right) = \sum_{i=1}^{5} E(X_i)$$

$$= 11{\cdot}4166.$$

Expected ~~number~~ expenditure

$$5 \times 11{\cdot}4166 = 57{\cdot}083$$

5. Let X be a discrete random variable with the properties $E(X) = 0$, $E(X^2) = 2$ and $E(X^4) = 4$.

(a) Find the moment generating function of X. [3 marks]

Answer: $\boxed{\frac{1}{2}\left(e^{\sqrt{2}t} + e^{-\sqrt{2}t}\right)}$

(b) Compute $E(X+1)^3$. [2 marks]

Answer: $\boxed{7}$

$$E(x^2) = 2$$

$$Var(x^2) = E(x^4) - (E(x^2))^2 = 4 - 4 = 0$$

$\Rightarrow x^2 = 2$ with prob. 1.

$$\Rightarrow x = \pm\sqrt{2}$$

$$E(x) = 0 \Rightarrow P(X = x) = \begin{cases} 1/2 & \text{if} \\ 1/2 & \text{if } x = \\ 0 & \text{o} \end{cases}$$

(a) $$M_X(t) = \frac{1}{2}\left(e^{\sqrt{2}t} + e^{-\sqrt{2}t}\right)$$

(b) $$E(x+1)^3 = E(x^3) + 3E(x^2) + 3E(x) + 1$$

$$= 0 + (3)\cdot(2) + (3)\cdot(0) + 1$$

$$= 7$$

because $E(x^3) = 0$

6. Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ be a discrete random vector with the joint probability mass function given as follows:

$$f_{X,Y}(x, y) = \begin{cases} \frac{(1-p)^x p}{s} & \text{for } x = 0, 1, \ldots; \quad y = 1, 2, \ldots, s \\ 0 & \text{otherwise} \end{cases}$$

for some $0 < p < 1$.

(a) Compute $E(X)$. [1 mark]

Answer: $\boxed{\dfrac{1-p}{p}}$

(b) Compute $Var(Y|X)$. [1 mark]

Answer: $\boxed{\dfrac{s^2-1}{12}}$

(c) Compute $Var(X)$. [1 mark]

Answer: $\boxed{\dfrac{1-p}{p^2}}$

Note that

$$f_{xy}(x,y) = (1-p)^x p \cdot \frac{1}{s}$$

$$= f_x(x)\, f_y(y)$$

where $f_x(x) = p(1-p)^x$ $\quad x = 0, 1, \ldots$

is a geometric distribution with parameter p

and $f_y(y) = \frac{1}{s}$ $\quad y = 1, 2, \ldots, s$

density

Thus x and Y are independent.

(Joint is product of marginals)

$\text{var}(Y|X) = \text{var}(Y)$

7. Let X be a discrete random variable which follows geometric distribution with parameter $p = \frac{1}{3}$. Define a new random variable

$$Y = \frac{2^X}{X!}$$

(a) Compute $P(Y \leqslant 2)$. [1 mark]

Answer: $\boxed{1}$

(b) Compute $E(Y)$. [2 marks]

Answer: $\boxed{\frac{e^{4/3}}{3} = 1.2646}$

Note that $\frac{2^x}{x!} \leq 2 \quad x = 0, 1, \cdots$

$$Y \leq 2 \quad \forall\, x = 0, 1, \cdots$$

$$\Rightarrow P(Y \leq 2) = 1$$

To compute

$$E(Y) = \sum_y y\, P(Y = y)$$

$$= \sum_x \frac{2^x}{x!} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^x$$

$$= \frac{1}{3} \sum_{x=0}^{\infty} \left(\frac{4}{3}\right)^x \frac{1}{x!}$$

$$= \frac{e^{4/3}}{3} = \boxed{1.2646}$$

8. For what value of k, a continuous random variable X will have the following function $f(x)$ as its probability density function? [2 marks]

$$f(x) = ke^{-|x|} \quad \text{for} \quad -1 < x < 1$$

Answer: $\frac{e}{2(e-1)} = \boxed{0.791}$

$$\int_{-1}^{1} f(x)\,dx = 1$$

in order that $f(x)$ is a density

$$\Rightarrow \int_{-1}^{1} k e^{-|x|}\,dx = 1$$

$$\Rightarrow 2\int_{0}^{1} k e^{-(x)}\,dx = 1$$

$$\Rightarrow \frac{k\left[e^{-x}\right]_0^1}{-1} = \frac{1}{2}$$

$$\Rightarrow k\left[1 - \frac{1}{e}\right] = \frac{1}{2}$$

$$\Rightarrow k = \frac{1}{2(1-\frac{1}{e})} = \frac{e}{2(e-1)} = \boxed{0.791}$$