Gamma Distribution

pdf $\Rightarrow$ $$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0,\ 0 < x < \infty \\ 0 & \text{Otherwise} \end{cases}$$

cdf $\Rightarrow$ $$F_X(x) = \begin{cases} \int_0^x f(u)\,du = \frac{1}{\Gamma\lambda} \int_0^x e^{-u} u^{\lambda-1}\,du \\ 0 & \text{Otherwise} \end{cases}$$

Mgf $\Rightarrow$ $$M_X(t) = E\left(e^{tX}\right)$$

$$= \int_0^\infty e^{tx} f(x)\,dx = \frac{1}{\Gamma\lambda} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1}\,dx$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-(1-t)x} x^{\lambda-1}\,dx$$

$$= \frac{1}{\Gamma\lambda} \cdot \frac{\Gamma\lambda}{(1-t)^\lambda} \qquad |t| < 1$$

$$= (1-t)^{-\lambda}, \quad |t| < 1.$$

Gamma distn. with two parameters

$$X \sim \gamma(a, \lambda) \quad \text{or} \quad G(a, \lambda)$$

pdf $\Rightarrow$

$$f(x) = \begin{cases} \dfrac{a^{\lambda} e^{-ax} x^{\lambda-1}}{\Gamma\lambda} & a>0,\ \lambda>0,\ 0<x<\infty \\ 0 & \text{Otherwise} \end{cases}$$

mgf $\Rightarrow$ $M_X(t) = (1-at)^{-\lambda}$, $t < \frac{1}{a}$.

Mean

$$E(X) = M_X'(t)\Big|_{t=0} = \lambda \quad \text{or} \quad a\lambda$$

$$E(X^2) = M_X''(t)\Big|_{t=0} = \lambda(\lambda+1) \quad \text{or} \quad \lambda(\lambda+1)a^2$$

Variance

$$Var(X) = \lambda \quad \text{or} \quad \lambda a^2.$$

Note For $\lambda = 1$, Gamma distn. reduces to Expo distn.

1. The mean yield for one-acre plot is 662 kilos with s.d. 32 kilos. Assuming normal distribution, how many one-acre plots in a batch of 1000 plots would you expect to have yield (i) over 700 kilos (ii) below 650 kilos and (iii) what is the lowest yield of the best 100 plots?

2. There are 600 Economics students in PG class of a university, the probababiliity for any student to need a copy of a particular book from university library on any day is 0.05.
How many copies of book should be kept in university library so that the probability may be greater than ·90 that none of the students needing a copy from the library has to come back disappointed?

3. The local authorities in a certain city install 10,000 electric lamps in the streets of the city. If these lamps have an average life 1000 burning hours with a s.d. 200 hours. Assuming normality, what number of lamps might be expected to fail
(i) in the first 800 burning hours? (ii) between 800 and 1200 burning hours?
After what period of burning hours would you expect that (a) 10% lamps would fail (b) 10% lamps would be ~~sti~~ still burning?

4. The marks obtained by a number of students in a certain subject are assumed to be approximately normally distributed with mean value 65 and s.d. 5. If 3 students are taken at random for this set, what is the probability that exactly 2 of them will have marks over 70?

▨ The mean yield for one-acre plot is 662 kilos with s.d. 32 kilos. Assuming normal distribution, how many one-acre plots in a batch of 1000 plots would you expect to have yield (i) over 700 kilos (ii) below 650 kilos and (iii) what is the lowest yield of the best 100 plots?

→ (i) $P(X \geq 700) = P(Z > 1{\cdot}19) = {\cdot}1170$

Ans. No. of plots 117

(ii) $P(X < 650) = P(Z < -{\cdot}38) = {\cdot}352$

Ans. 352 plots.

(iii) The lowest yield say x_1 of best 100 plots

$\Rightarrow P(X > x_1) = \frac{100}{1000} = {\cdot}1$

$\Rightarrow P\left(Z > \underbrace{\frac{x_1 - 662}{32}}_{z_1}\right) = {\cdot}1 \quad \Rightarrow z_1 = 1{\cdot}28$

$\Rightarrow x_1 = 1{\cdot}28 \times 32 + 662$

$= 702{\cdot}96.$

VI There are 600 Economics students in PG classes of a university, the prob. for any student to need a copy of a particular book from university library on any day is 0·05.
How many copies of book should be kept in the university library so that the prob. may be greater than ·90 that none of the students needing a copy from the library has to come back disappointed?

$n = 600,\ p = \cdot 05,\ q = \cdot 95$

$\mu = np = 30 \quad \& \quad \sigma^2 = npq = 28\cdot 5$

$$Z = \frac{X - np}{\sqrt{npq}} = \frac{X - 30}{\sqrt{28\cdot 5}} = \frac{X - 30}{5\cdot 34} \sim N(0, 1)$$

Find x such that

$P(X < x) > \cdot 9 \Rightarrow P(Z < z) > \cdot 9$

$\Rightarrow z \geq 1\cdot 28$

$\Rightarrow x > 36\cdot 84 \cong 37.$

▨ The local authorities in a certain city instal 10,000 electric lamps in the streets of the city. If these lamps have an average life of 1000 burning hours with a s.d. 200 hrs. Assuming normality, what number of lamps might be expected to fail

(i) in the first 800 burning hours?

(ii) between 800 and 1200 burning hours? After what period of burning hours would you expect that (a) 10% lamps would fail

(b) 10% of the lamps would be still burning?

i) $P(X < 800) = P(Z < -1) = P(Z > 1)$

$= \cdot 1587.$

$\Rightarrow$ out of 10,000 bulbs 1587 bulbs will fail in first 800 hrs.

ii) $P(800 < X < 1200) = \cdot 6826.$

(a) $P(X < x_1) = \cdot 1 \Rightarrow x_1 = 744$

(b) $P(X > x_2) = \cdot 1 \Rightarrow x_2 = 1256.$

The marks obtained by a number of students for a certain subject are assumed to be approximately normally distributed with mean value 65 and s.d. 5. If 3 students are taken at random from this set, what is the prob. that exactly 2 of them will have marks over 70?

→ Prob. that a randomly selected student gets marks over 70

$= P(X > 70) = P(Z > 1) = \cdot 1587. = p$ (say)

This is same for all.

Out of 3, 2 exactly will have 70 marks

the corresponding prob.

$${}^3C_2\, p^2 (1-p) = \cdot 06357.$$

Weibull Distribution

$$X \sim W(\alpha, \beta)$$

pdf

$$f_X(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} & x > 0 \\ & \alpha > 0 \\ & \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

cdf

$$F_X(x) = \int_0^x f_X(u)\,du = \begin{cases} 1 - e^{-\alpha x^{\beta}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

If we put $\beta = 1$, Weibull distn. reduces to Exponential distn.

Moments

$$E(X^k) = \int_0^\infty \alpha\beta\, x^{\beta+k-1}\, e^{-\alpha x^{\beta}}\, dx$$

Take $x^{\beta} = y$

or, $\beta x^{\beta-1}\,dx = dy$

$$E(X^k) = \int \alpha y^{k/\beta} e^{-\alpha y}\, dy$$

$$= \frac{\alpha \Gamma\left(\frac{k}{\beta} + 1\right)}{\alpha^{k/\beta + 1}}$$

$$= \frac{\Gamma\left(\frac{k+\beta}{\beta}\right)}{\alpha^{k/\beta}}.$$

$$\mu_1' = E(X) = \frac{\Gamma\left(\frac{\beta+1}{\beta}\right)}{\alpha^{1/\beta}}.$$

$$\mu_2' = \alpha^{-2/\beta}\, \Gamma\left(\frac{\beta+2}{\beta}\right)$$

Therefor,

$$\text{variance}\,(X) = \mu_2' - (\mu_1')^2$$

$$= \alpha^{-2/\beta} \left[\Gamma\left(\frac{\beta+2}{\beta}\right) - \Gamma\left(\frac{\beta+1}{\beta}\right)^2 \right]$$

Reliability of a System

If the RV T denotes survival of a system or life span of a system

Reliability of the system at time t

= Probability of functioning the system at time t

$$= P(T > t)$$

$$= 1 - P(T < t)$$

$$= 1 - F_T(t)$$

$$= 1 - \int_0^t f(x)\,dx.$$

Instantaneous Failure rate of system at time t

$$\text{Hazard rate } H(t) = \lim_{h \to 0} \frac{P(t \leq T \leq t+h \,/\, T > t)}{h}$$

Prob That The system worked upto t but failed immediatly after that

$$= \lim_{h \to 0} \frac{P(t \leq T \leq t+h \,,\, T > t)}{h\, P(T > t)}$$

$$= \lim_{h \to 0} \frac{P(t \leq T \leq t+h)}{h\, P(T > t)}$$

$$= \lim_{h \to 0} \frac{F_T(t+h) - F_T(t)}{h\, R(t)} = \frac{f_T(t)}{R(t)}$$

$$= \frac{f_T(t)}{1 - F_T(t)}$$

$$= -\frac{d}{dt}\left(\log\left(1 - F_T(t)\right)\right)$$

$$\boxed{\begin{aligned} R(t) &= 1 - F_T(t) \\ H(t) &= \frac{f_T(t)}{R(t)} \\ 1 - F_T(t) &= k\, e^{-\int H(t)\,dt} \end{aligned}}$$

Reliability of system with Weibull dist^n.

$$X \sim W(\alpha, \beta)$$

$$f_X(x) = \begin{cases} \alpha \beta x^{\beta-1} e^{-\alpha x^{\beta}} & x > 0 \\ & \alpha > 0 \ \beta > 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\alpha x^{\beta}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$R(t) = \begin{cases} e^{-\alpha t^{\beta}} & t > 0 \\ 1 & t \leqslant 0 \end{cases}$$

$$H(t) = \frac{f(t)}{R(t)} = \alpha\beta\, t^{\beta - 1}$$

$$R(t) = \begin{cases} e^{-\alpha t^{\beta}} & t > 0 \\ 1 & t \leq 0 \end{cases}$$

$$H(t) = \alpha\beta\, t^{\beta - 1}$$

<u>Note</u>: * If β is integer Hazard rate is polynomial

β real. $\left\{ \begin{array}{l} \text{If } \beta < 1 \Rightarrow \text{Hazard rate is decreasing with time} \\ \text{If } \beta > 1 \Rightarrow \text{Hazard rate is increasing with time} \end{array} \right.$

<u>Again</u>

$$R(t) = K e^{-\int H(t)\,dt}$$

$$= K e^{-\int \alpha\beta t^{\beta-1} dt}$$

$$= K e^{-\alpha\beta \frac{t^{\beta}}{\beta}} = K e^{-\alpha t^{\beta}}.$$

Initial condn. $R(0) = 1 \Rightarrow K = 1$

$$R(t) = e^{-\alpha t^{\beta}}$$

$$\left.\begin{array}{l} \Rightarrow F(t) = 1 - e^{-\alpha t^{\beta}} \\ \& \; f(t) = \alpha\beta x^{\beta-1} e^{-\alpha t^{\beta}} \end{array}\right\} \text{Weibull distn.}$$

Special case $\beta = 1$

$\Rightarrow$ Exponential dist^n

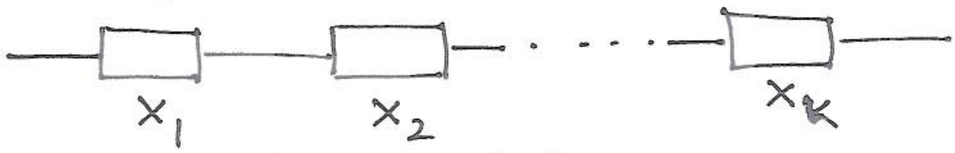
We get, for $X \sim \text{Expo}(\lambda)$

$$f_X(x) = \lambda e^{-\lambda x} \quad x > 0.$$

Here, $H(t) = \lambda \Rightarrow$ Hazard rate is constant.

Thus,
system with exponential dist^n is <u>more stable</u>

Reliability of series system

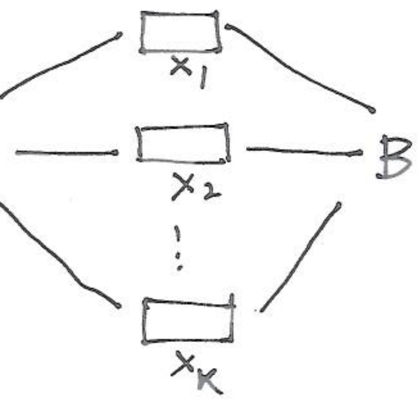


Reliability of the whole system X is

$$R_X(t) = P(X > t) = P(X_1 > t,\ X_2 > t \ \cdots\ X_K > t)$$
$$= \prod_{i=1}^{K} R_{X_i}(t)$$

X: compound system life

Reliability of parallel system



$$R_X(t) = P(X > t) = 1 - \underbrace{P(X \leq t)}_{\text{system fails before } t}$$

$$= 1 - \prod_{i=1}^{K} P(X_i < t)$$

$$= 1 - \left\{(1 - R_{X_1}(t))(1 - R_{X_2}(t)) \cdot (1 - R_{X_K}(t))\right\}$$

The lung cancer failure rate of a t-year old male smoker is given by $Z(t) = 0.027 + 0.00025\,(t-40)^2$, $t \geq 40$. Derive the density function of life.
Find the probability that he survives to age 50.
If he survives age 50, what is the probability that he will survive till age 60.

→ ~~Hazard~~ rate here is a shifted weibull distn.
→ Assume the person is alive till age 40.

Approach

$$H(t) \longrightarrow R(t) \longrightarrow F(t) \longrightarrow f(t)\,?$$

$$R(t) = e^{-\int_{40}^{t} H(t)\,dt} = e^{-0.027(t-40) + \frac{.00025}{3}(t-40)^3}$$

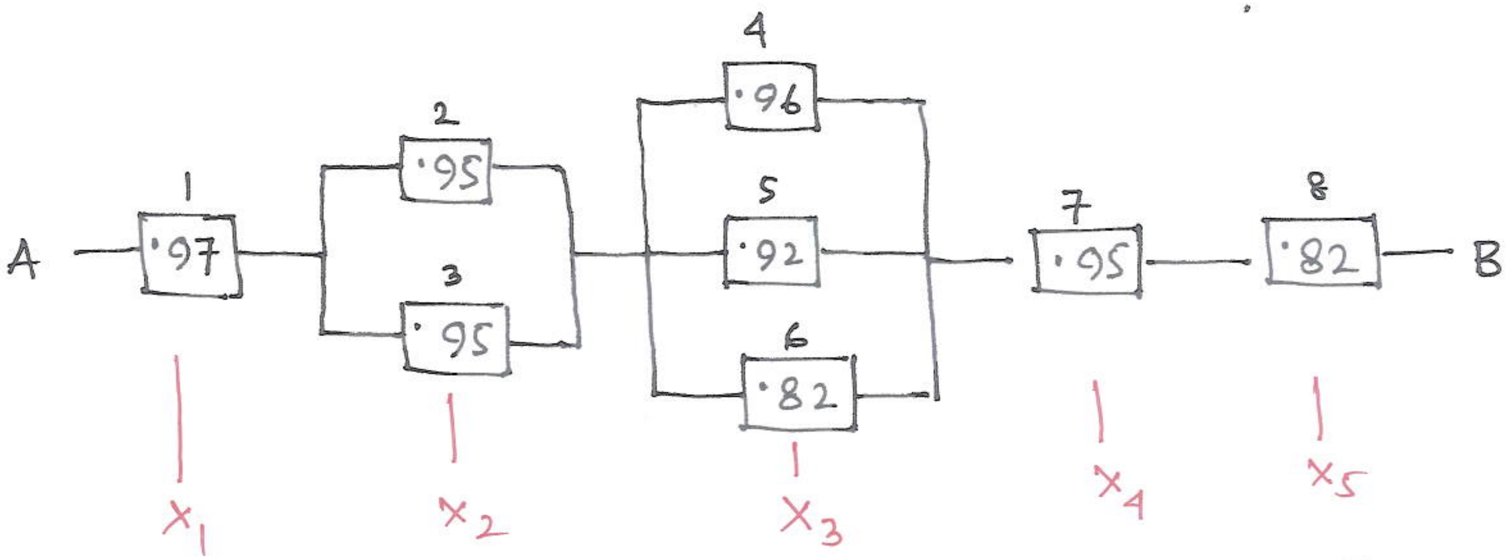
Prob that he survives till age 50.

$$\therefore = R(50) \cong .70243.$$

$$P(x>60/x>50) = \frac{P(x>60)}{P(x>50)} = \frac{R(60)}{R(50)} \doteq .426.$$

Again $$f(t) = -\frac{d}{dt}\left(R(t)\right) = \left\{-.027 + .00025(t-40)^2\right\} \times e^{-.027(t-40) + \frac{.00025}{3}(t-40)^3}$$

Reliability information of following assembly system consists of several components is given



x_i : Life of i^{th} system

$$R_x(t) = \prod_{i=1}^{5} R_{x_i}(t) = R_{x_1}(t) \cdot R_{x_2}(t)\, R_{x_3}(t)\, R_{x_4}(t)\, R_{x_5}(t)$$

$$R_{x_1}(t) = \cdot 97$$

$$R_{x_2}(t) = 1 - (1 - \cdot 95)(1 - \cdot 95)$$

$$R_{x_3}(t) = 1 - (1 - \cdot 96)(1 - \cdot 92)(1 - \cdot 82)$$

$$R_{x_4}(t) = \cdot 95 \qquad R_{x_5}(t) = \cdot 82$$

$$= \cdot 7689$$

Note : Reliability of compound system is low though the individual reliabilities are high.

A system consists of two independent components connected in series. The lifespan of the first component follows a Weibull distribution with $\alpha = 0.006$ & $\beta = 0.5$. The second has a lifespan that follows the exponential distribution, with mean 25000 hrs.

(a) Find the reliability of the system at 2500 hrs.

(b) Find the probability that the system will fail before 2000 hrs.

(c) If the two components are connected in parallel, what is the system reliability at 2500 hrs?

$$f_{X_1}(x_1) = \alpha\beta\, x_1^{\beta-1} e^{-\alpha x_1^{\beta}}$$

$\alpha = .006$

$\beta = 0.5$

$$R_{X_1}(t) = e^{-\alpha t^{\beta}}$$

$$= e^{-.006\, t^{0.5}}$$

Reliability of X_1

$$f_{X_2}(x_2) = \frac{1}{25000} e^{-x_2/25000}$$

$x_2 > 0$

$$R_{X_2}(t) = e^{-t/25000}$$

Reliability of X_2

$R_X(t) =$ Reliability of X $= R_{X_1}(t)\ R_{X_2}(t)$

$$= e^{-.006 t^{.5}} \cdot e^{-t/25000}$$

$$\boxed{\text{(a) } R_X(2500) = 0.67}$$

(b) $P(X < 2000) = 1 - P(X > 2000)$

$= 1 - R_X(2000)$

$= \cdot 98$

(c) If X_1 ~~and~~ $\cdot$ X_2 are connected in parallel

$$R_X(t) = 1 - (1 - R_{X_1}(t))(1 - R_{X_2}(t))$$

$$R_X(2500) = \cdot 98$$

Beta Distribution

$$X \sim B(\alpha, \beta)$$

pdf

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} & 0 < x < 1 \\ & \alpha, \beta > 0 \\ 0 & \text{Otherwise} \end{cases}$$

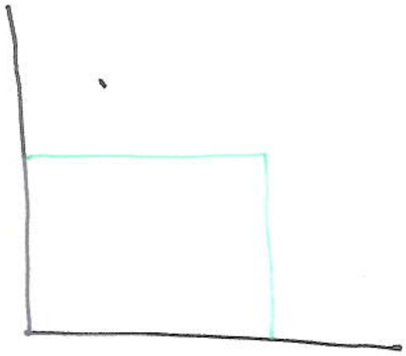
cdf

$$F_X(x) = \begin{cases} \dfrac{1}{B(\alpha,\beta)} \displaystyle\int_0^x u^{\alpha-1}(1-u)^{\beta-1}\,du & \\ & 0 < u < 1 \\ 1 & u \geq 1 \end{cases}$$

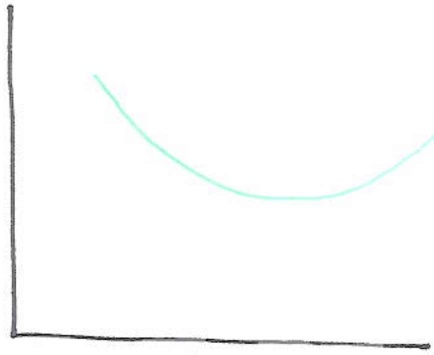
Moments

$$E(X^r) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+r-1} (1-x)^{\beta-1}\,dx$$

$$= \frac{B(\alpha+r, \beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+r)\,\Gamma\beta}{\Gamma(\alpha+\beta+r)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\,\Gamma\beta}$$

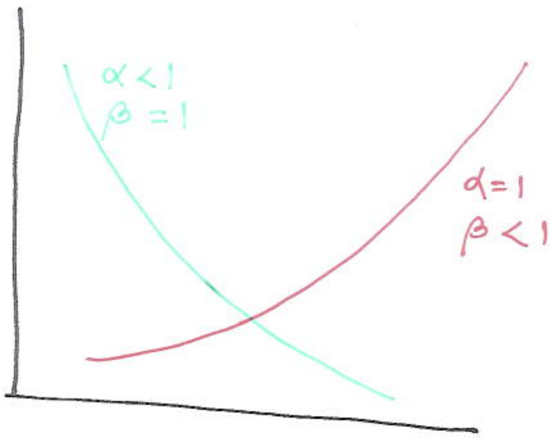


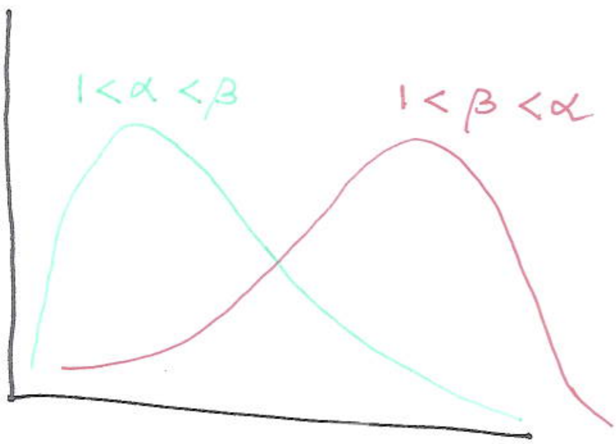
$\alpha = \beta = 1$



$\alpha = \beta < 1$

$\alpha = \beta > 1$





$$\text{Mean} = \frac{\alpha}{\alpha+\beta}$$

$$\text{Variance} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Find $P(\cdot 2 < X < \cdot 5)$ if X is distributed with pdf

$$f(x) = \begin{cases} \frac{1}{12} x^2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence find mean and variance also.

$\Rightarrow X \sim B(3, 2)$

$P(\cdot 2 < X < \cdot 5)$

$$= \frac{1}{12} \int_{\cdot 2}^{\cdot 5} x^2(1-x)\, dx = \cdot 023$$

$$\text{Mean} = \frac{3}{5} \qquad \text{Variance} = \frac{1}{25}$$