Testing of Hypotheses

Main problem in statistical inference can be broadly classified into two areas

- Area of estimation of population parameter
- Tests of statistical hypothesis

Let population be $N(\mu, \sigma^2)$

$(x_1, x_2 \dots x_n)$ be a random sample.

$$\boxed{H : \mu = \mu_0} \longrightarrow \text{Test}$$

We know,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

If H is true

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

Let $\quad Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

For given α,

$$P\left(-Z_{\alpha/2} < Z < Z_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow \text{Prob}\left(-1.96 < \frac{\bar{X}-m_0}{\sigma/\sqrt{n}} < 1.96\right) = 1-\alpha$$
$$= .95$$

$$Z_{.025} = 1.96$$

$$\Rightarrow \text{Prob}\left(\left|\frac{\bar{x}-m_0}{\sigma/\sqrt{n}}\right| > 1.96\right) = .05$$

$\Rightarrow$ If hypothesis true $P(|Z| > 1.96)$ is very small
i.e. little chance to satisfy $|Z| > 1.96$

$\Rightarrow$ Reject H if $Z \in (-\infty, -1.96) \cup (1.96, \infty)$

or

Accept H if $Z \notin (-\infty, -1.96) \cup (1.96, \infty)$

$\alpha = .05$ is significance level
$\{Z : |Z| > 1.96\}$ is critical region

Null hypothesis & Alternative hypothesis

We write

$$H_0 : \theta = \theta_0 \longrightarrow \text{Null}$$

$$H_1 : \theta \neq \theta_0 \longrightarrow \text{Alternative.}$$

If we are more inclined to accept one hypothesis or reject it under consideration we say

Null hypothesis → value of parameter has not really changed, the sample values are simply due to chance

Alternative hypo → Really there has been a change in the value of the parameter, results are not due to chance.

Type I error : Rejecting H_0 when H_0 is true

Type II error : Accepting H_0 when H_0 is false.

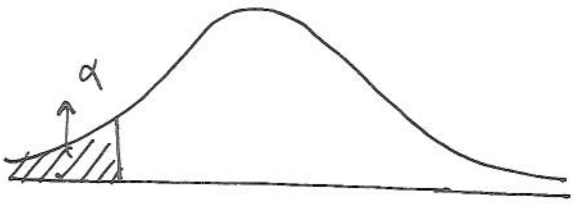
<u>Hypothesis test for mean</u> (σ known)

$$H_0 : \mu = \mu_0$$

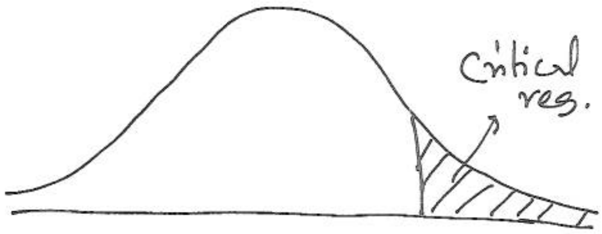
$$H_1 : \mu < \mu_0 \ \underline{\text{or}}\ \mu > \mu_0 \ \underline{\text{or}}\ \mu \neq \mu_0.$$

Compute test statistic $Z_0 = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$.

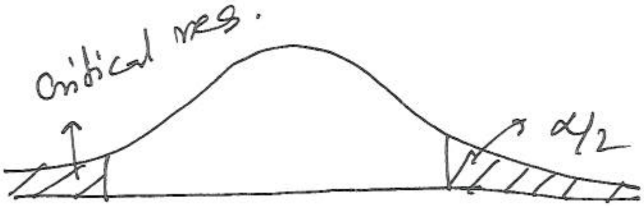
If sample value z of test statistic lies in the critical region then H_0 rejected or acceptence of H_1.



$H_1 : \mu < \mu_0$



$H_1 : \mu > \mu_0$

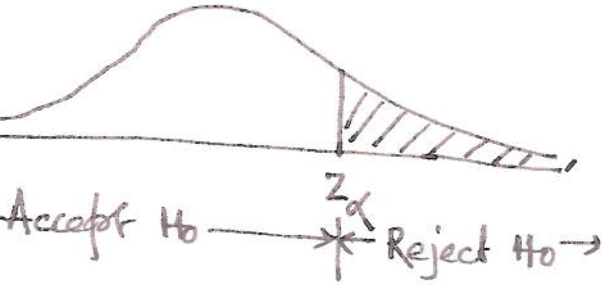


$H_1 : \mu \neq \mu_0$.

Critical region

It is the region in the sample space when the null hypothesis H_0 is rejected.

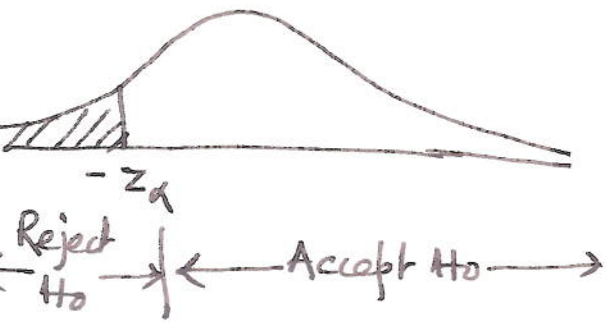
$\boxed{\text{Case 1}}$ $H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$ (σ known)



Criterion

Reject H_0 if $\bar{x} \geq \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}}$

$\boxed{\text{Case 2}}$ $H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$

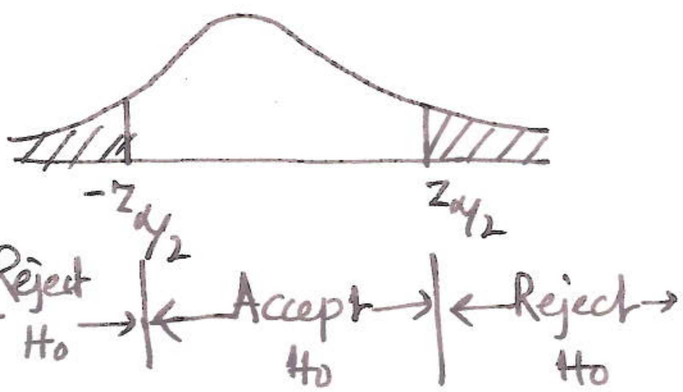


Criterion

Reject H_0 if

$\bar{x} < \mu_0 - Z_\alpha \frac{\sigma}{\sqrt{n}}$.

$\boxed{\text{Case 3}}$ $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$



Criterion

Reject H_0 if

$$\left| \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \right| \geq Z_{\alpha/2}.$$

	H_0 is true	H_0 is not true
Accept H_0	Correct decision	Type II error(β)
Reject H_0	Type I error(α)	Correct decision.

The daily consumption of milk in a particular township is assumed to be approximately exponentially distributed. Suppose that a hypothesis H_0: expected consumption is 10,000 gallons, is tested against the hypothesis that it is 20,000. Suppose that the criterion is as follows: A day is selected at random. If the consumption of the day is 16,000 gallon or more H_0 is rejected and H_1 accepted. Evaluate α and β.

$$H_0 : \theta = 10{,}000 \quad , H_1 : \theta = 20{,}000$$

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \quad x > 0$$

$$\alpha = P\left(\text{Reject } H_0 / H_0 \text{ is true}\right)$$
$$= P\left(x \geq 16000 / \theta = 10{,}000\right) = \int_{16000}^{\infty} \frac{1}{10000} e^{-\frac{x}{10000}} \, dx$$
$$= e^{-1.6}.$$

$$\beta = P\left(\text{Accept } H_0 / H_1 \text{ is true}\right)$$
$$= P\left(x \leq 16000 / \theta = 20000\right) = 1 - e^{-0.8}.$$

<u>Power of a test</u>

$$1 - \beta = P\left(\text{Reject } H_0 / H_1 \text{ is true}\right)$$
$$= P\left(x \in \text{Critical region} / H_1 \text{ is true}\right).$$

A population R.V $X \sim N(\mu, 4)$

$H_0 : \mu = 15$.

A random sample of size 25, drawn from population results in a sample mean $\underline{\bar{x} = 16}$. Test the null hypothesis at significance level $\alpha = \cdot 01$ against each of the following alternative hypothesis:

(1) $H_1 : \mu < 15$, (2) $H_1 : \mu > 15$ (3) $H_1 : \mu \neq 15$.

The value of test statistic $Z = \dfrac{16 - 15}{2/5} = 2 \cdot 5$.

(1) $P(Z \leq 2 \cdot 5) = \cdot 9938 \nless \cdot 01$

$\Rightarrow$ Accept H_0

Critical reg.
$Z < -2 \cdot 33$

(2) $P(Z \geq 2 \cdot 5) = \cdot 0062 < \cdot 01$

$\Rightarrow$ Reject H_0

Critical reg
$Z > 2 \cdot 33$

(3) $P(|z| < 2 \cdot 5) = \cdot 0062 \times 2 = \cdot 0124 \nless \cdot 01$

$\Rightarrow$ Accept H_0.

$$\boxed{\begin{array}{l} -2 \cdot 58 \geq z \\ 2 \cdot 58 \leq z \end{array}}$$

Steps for hypothesis testing (for μ when σ is known)

Step 1 State null $H_0: \mu = \mu_0$ and alternative H_1 hypo.

Step 2 Compute test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

Assuming H_0 is true, compute $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$.

Step 3 Determine critical region (using statistical table for given α)

OR

Determine p-value.

For $H_1: \mu < \mu_0$: $p$value $= P(Z \leq z)$

For $H_1: \mu > \mu_0$: p value $= P(Z \geq z)$

For $H_1: \mu \neq \mu_0$: p value $= P(Z \leq -|z|) + P(Z \geq |z|)$

or $= 2P(Z > |z|)$

Step 4 Draw conclusion.

Critical region: If z_0 lies in critical region reject H_0, Otherwise accept.

p value: If p value $\leq \alpha \Rightarrow H_0$ reject

p value $> \alpha \Rightarrow H_0$ accept.

Hypothesis test for μ when σ is unknown

- Set $H_0 : \mu = \mu_0$ & alternative as follows

cases

(i) $H_1 : \mu < \mu_0$ (ii) $H_1 : \mu > \mu_0$ (iii) $H_1 : \mu \neq \mu_0$.

- Compute the test statistic assuming H_0 is true

$$t^0_{n-1} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

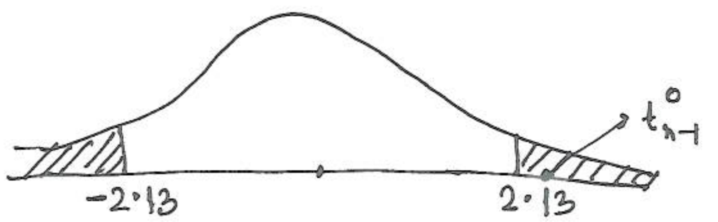
where $s = \sqrt{\frac{1}{n-1} \left(\sum (x_i - \bar{x})^2\right)}$

- Determine critical region using t table with $(n-1)$ d.f.

- Draw conclusion
  - For $H_1 : \mu < \mu_0$, given c.r. $P(t \leq t^*) = \alpha$
    if t^0_{n-1} lies in critical region reject H_0.
  - For $H_1 : \mu > \mu_0$, c.r. $P(t \geq t^*) = \alpha$
    if t^0_{n-1} lies in c.r. reject H_0
  - For $H_1 : \mu \neq \mu_0$
    critical region : $2P(t \geq t^*) = \alpha$
    same as above.

▨ A population R.V. X is normally distributed with unknown mean and standard deviation. A random sample of size 16 yields a sample mean $\bar{x} = 110$ and s.d. 18·18.
Test the null hypothesis $H_0 : \mu = 100$ against the alternative hypothesis $H_1 : \mu \neq 100$ at the significant level $\alpha = \cdot 05$.

Here, $t^0_{15} = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} = \dfrac{110 - 100}{18 \cdot 18/4} = 2 \cdot 2.$



Hence,
reject H_0.

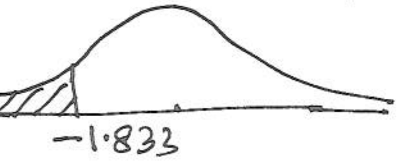
▨ The following cholesterol levels were found in a random sample of 10 women aged 20 to 24 engaged in a low-fat diet program:

176, 180, 175, 186, 182, 188, 180, 186, 168, 184

The null hypothesis is that average cholesterol level of all women who maintain the diet is normal with mean 184.
Test H_0 against $H_1 : \mu < 184$ with $\alpha = \cdot 05$

$t^0_9 = \dfrac{180 \cdot 5 - 184}{6 \cdot 133/\sqrt{10}} = -1 \cdot 8.$

critical region
$(-\infty, -1 \cdot 833)$



~~Reject H_0~~ ~~Accept H_0~~.

▨ Test the same at ·01 significance level.

critical region: $(-\infty, -2.821)$

$\Rightarrow$ Accept Ho

▨ Suppose the cholesterol levels of a random sample of 10 women in the low-fat diet program have sample mean 180·5 and $S = 5.2$. Test $H_0 : \mu = 184$ against $H_1 : \mu < 184$
(i) at ·05 significance level (ii) $\alpha = .01$.

$$t_9^0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}} = \frac{180.5 - 184}{5.2/\sqrt{10}} \cong -\overset{2.13}{\cancel{2.433}}.$$

$\Rightarrow$ (i) Ho rejected

(ii) Ho accepted.

▨ For a sample of size 5, $\bar{x} = 4.8$, $S = .3$

Test $H_0 : \mu = 5$ against $H_1 : \mu < 5$.

Are the test results statistically significant at $\alpha = .05$?

Ans. $t_4^0 = -1.49$.

$\Rightarrow$ There is not enough evidence that the null hypothesis be rejected.

Hypothesis test for variances (μ unknown)

$H_0 : \sigma^2 = \sigma_0^2$ against (i) $H_1 : \sigma^2 < \sigma_0^2$

(ii) $H_1 : \sigma^2 > \sigma_0^2$

(iii) $H_1 : \sigma^2 \neq \sigma_0^2$.

Test statistic $\chi^2_{n-1} = \dfrac{(n-1)s^2}{\sigma^2}$.

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Determine critical region and draw conclusion.

Over the years the grades in a mathematics Professor's calculus classes have been normally distributed with mean 75 and s.d. 8. Recently the grades seem to have fallen and show more variation. A sample of 41 recent grades has mean $\bar{x} = 73$ and s.d. 9.6.

Assuming the grades are still normally dist$^{\text{n}}$. test $H_0 : \sigma^2 = 64$ against $H_1 : \sigma^2 > 64$ at ·05 significance level.

▨ The amount of soda in 96 oz bottles is N (92, 1.44). A new bottling procedure is designed to decrease the variability of the amount of soda in the bottles. A sample of 101 bottles has a s.d. ·98 oz.

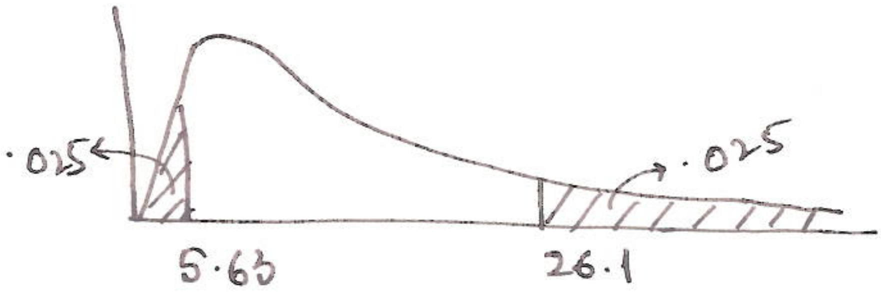
Test the null hypothesis Ho: $\sigma^2 = 1.44$ against the alternative hypothesis H_1: $\sigma^2 < 1.44$ at ·025 level.

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2} = \frac{100 \times (\cdot 98)^2}{1\cdot 44} = 66\cdot 69.$$

Again $\chi^2_{n-1,1-\cdot 025} = 74\cdot 2$

$\Rightarrow$ Ho is rejected at ·025 significance level.

▨ The number of hours spent sleeping by an undergraduate college student is a normal random variable with mean $\mu = 7\cdot 5$ and $\sigma^2 = 1\cdot 25$.
In graduate school the student's sleep pattern changes. A sample of 15 days gives an average of $\bar{x} = 6.25$ hrs and $s^2 = 1\cdot 5$. Assuming that the sleeping hours are normally distributed, test the null hypothesis Ho : $\sigma^2 = 1.25$ against the alternative hypothesis H_1: $\sigma^2 \neq 1.25$ at ·05 significance level.



d.f = 14.

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{14 \times 1.5}{1.25} = 16.8$$

$\Rightarrow$ Ho is accepted.

The weight of a 16 oz bag potato chips is a R.V. X with $\mu = 16$ oz and s.d $\sigma = .5$ oz. A new quality control procedure is introduced to reduce the variability of X. The weights of a random sample of 25 bags are as follows.

15.8	15.4	15.9	16.5	16.3	15.9	16.0	15.9
16.6	15.5	16.4	15.2	16.6	16.2	15.8	16.6
15.7	15.4	15.9	16.1	15.5	16.4	15.4	15.5
16.4							

Assuming that the mean of all bags produced under the new system is still 16 oz, test the null hypothesis $H_0: \sigma^2 = .25$ against alternative $:\sigma^2 < .25$ at .01 level.

Ans. Null hypothesis is accepted.