# Random Variable

- Discrete, continuous and mixed random variable
- probability mass, probability density and cumulative distribution function
- mathematical expectation, moments, moment generating function
- median and quantiles
- Chebyshev's inequality
- Problems

4 lectures

## Example 1 (Tossing a coin)

$\Omega = \{H, T\}$ , $X(H) = 1$ $X(T) = 0$

$P$ assigns equal masses to $\{H\}$ & $\{T\}$

$$P(X=0) = \frac{1}{2} = P(X=1)$$

$$F(x) = Q(-\infty, x] = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x. \end{cases}$$

## Example 2 (Tossing two dice)

$$F(x) = Q(-\infty, x] = P(X \leq x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{1}{36} & \text{if } 2 \leq x < 3 \\ \frac{3}{36} & \text{if } x < 4 \\ \frac{6}{36} & \text{if } x < 5 \\ \vdots & \\ \frac{35}{36} & \text{if } x < 12 \\ 1 & \text{if } 12 \leq x \end{cases}$$

**Example 3** Rolling two dice, yielding $X$ and $Y$. Note combined score $Z = X + Y$.

$$2 \leq Z(\omega) \leq 12.$$

$$\{\omega : Z(\omega) \leq 12\} = \begin{cases} \phi & Z(\omega) < 2 \\ \{(1,1)\} & 2 \leq Z(\omega) < 3 \\ \{(1,2),(2,1)\} & 3 \leq Z(\omega) < 4 \\ \vdots & \\ \{(6,6)\} & Z(\omega) = 12 \end{cases}$$

**Example 4** Devise experiment that selects a pt. $P$ randomly from the interval $[0,2]$, where any pt. may be chosen. Let us write formally,

$$\Omega = \{\omega : \omega \in [0,2]\}$$

Define $X$ and $Y$

$$X(\omega) = \begin{cases} 0 & 0 \leq \omega \leq 1 \\ 1 & 1 < \omega \leq 2 \end{cases}$$

and $\quad Y(\omega) = \omega^2.$

$X$ can take only 2 values $\rightarrow$ discrete

$Y$ can take uncountable no. in $[0, 4] \rightarrow$ continuous.

Example 5 . Let $\Omega = [0, 1]$

Define $X$ on $\Omega$ as follows:

$$X(\omega) = \begin{cases} \omega & \text{if } 0 \leq \omega \leq \frac{1}{2} \\ \omega - \frac{1}{2} & \text{if } \frac{1}{2} < \omega \leq 1 \end{cases}$$

Define $Y : \{\omega : X(\omega) \in (\frac{1}{4}, \frac{1}{2})\}$

then $Y(\omega)$ takes values $(\frac{1}{4}, \frac{1}{2}) \cup (\frac{3}{4}, 1)$

Example 6 You ask people whether they approve the present government. The sample space could be

$\Omega = \{$approve strongly, approve, indifferent, disapprove, disapprove strongly$\}$

You may put in numerical scale

$$X = \{-2, -1, 0, 1, 2\}$$

or

$$Y = \{0, 1, 2, 3, 4\}$$

X, Y etc. are Random Variable.

# Introduction to Random Variable.

Experimental outcoms are not always numerical

$\Downarrow$

Assign a number to every outcome $\omega \in \Omega$ and denote by $X(\omega)$

(Advantage : Convenient to work)

**Example 1** Roll a fair die. $\Omega = \{1, 2, 3, 4, 5, 6\}$

X: Number shown

$$X(\omega) = \omega \quad , 1 \leqslant \omega \leqslant 6$$

Y : Number of six~~es~~

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega = 6 \\ 0 & \text{Otherwise} \end{cases}$$

**Example 2** Flip three coins.

$$\Omega = \{HHH, HHT, HTT, HTH, THH, TTH, THT, TTT\}$$

X : Number of heads

$$X(\omega) \text{ takes } \{0, 1, 2, 3\}$$

as $X(HHH) = 3 \qquad X(TTT) = 0$

Y : Signed difference betn no. of heads & tails

$$Y(\omega) \text{ takes } \{-3, -\cancel{2}, -1, \cancel{0}, 1, \cancel{2}, 3\}$$

e.g. $Y(TTH) = -1$

# Definition

A RV is a variable whose possible values are numerical outcomes of a random phenomenon.

Let $(\Omega, \beta, P)$ be a probability space. A function $X$ from $\Omega$ to $\mathbb{R}$ is a RV if it is measurable.

Let $\mathcal{C}$ be a class of subsets of $R$ so that $\mathcal{C}$ be a $\sigma$-field

i.e. for any $B \in \mathcal{C}$, $X^{-1}(B) \in \beta$

## Probability distn of RV

The RV $X$ defined on the probability space $(\Omega, \beta, P)$ induces a probability space $(\mathbb{R}, \mathcal{C}, Q)$ s.t.

$$Q(B) = P\{X^{-1}(B)\} = P\left\{\omega : X(\omega) \in B, \begin{smallmatrix} \omega \in \beta \\ B \in \mathcal{C} \end{smallmatrix}\right\}$$

Then $Q = PX^{-1}$ is the probability distn of $X$.

* Prove that $Q$ or $PX^{-1}$ is a probability measure

Proof $\quad P_1 : Q(B) \geq 0 \quad \forall B \in \mathcal{C}$

$$P_2 : Q(\mathbb{R}) = P(\Omega) = 1$$

$P_3$ : Let $B_i \in \mathcal{C}$ are pairwise disjoint

$$Q\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left\{X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right\}$$

$$= \sum_{i=1}^{\infty} PX^{-1}(B_i).$$

$$= \sum_{i=1}^{\infty} Q(B_i)$$

$\Rightarrow (\mathbb{R}, \mathcal{C}, Q)$ is a probability space.

# Cumulative distribution function

$F_X(x)$ is called cumulative distⁿ fⁿ. of RV X

if $F_X(x) = P(X \leq x)$

where $P(X \leq x) = P\{\omega : X(\omega) \leq x\}$

## Properties

1. $\lim\limits_{x \to -\infty} F_X(x) = 0$

2. $\lim\limits_{X \to +\infty} F_X(x) = 1$

3. If $x_1 < x_2$ then $F_X(x_1) < F_X(x_2)$

   (i.e. F is nondecreasing)

4. $\lim\limits_{h \to 0} F_X(x+h) = F_X(x)$

   (F is right continuous at every pt.)

1. Let $\{x_n\}$ be a decreasing sequence s.t. $\lim x_n = -\infty$

Let $A_n = \{\omega : X(\omega) \leq x_n\}$

$$\Rightarrow \lim_{n \to \infty} A_n = \phi$$

$$\Rightarrow \lim_{n \to \infty} P(A_n) = P(\phi) \Rightarrow \lim_{n \to \infty} F(x_n) = 0$$

Again,

$$\Updownarrow$$

$$P\left(\lim_{n \to \infty} A_n\right)$$

or, $\boxed{\lim_{x_n \to -\infty} F(x_n) = 0}$

2. Let $\{x_n\}$ be an increasing sequence such that $\lim x_n = +\infty$

Similarly, we can prove that

$$\lim P(A_n) = \lim P(\Omega) = 1.$$

So, Property 2 follows.

3. Given $x_1 < x_2$

$$\Rightarrow \{\omega : X(\omega) \leq x_1\} \subset \{\omega : X(\omega) \leq x_2\}$$

$$\Rightarrow P\{\omega : X(\omega) \leq x_1\} \leq P\{\omega : X(\omega) \leq x_2\}$$

$$\Rightarrow F(x_1) \leq F(x_2)$$

$\Rightarrow$ F is non-decreasing.

4. Let $\{x_n\}$ be a decreasing sequence such that $\lim_{n \to \infty} x_n = x$.

Take $A_n = \{\omega : x < X(\omega) \leq x_n\}$

$\Rightarrow A_n$ is decreasing sequence.

$$\Rightarrow \lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n = \phi$$

$$\Rightarrow \lim P(A_n) = P(\lim A_n)$$

$$= \lim_{n \to \infty} \left[F(x_n) - F(x)\right] = 0$$

$$\Rightarrow \lim_{n \to \infty} F(x_n) = F(x)$$

$$\Rightarrow \lim_{h \to 0} F(x + h) = F(x).$$

## Discrete and continuous RV

A RV $X$ defined on a probability space $(\Omega, \beta, P)$ is said to be discrete if $\exists$ a countable set $E \subset \mathbb{R}$ s.t $P(x \in E) = 1$.

$\Rightarrow$ Range of $X$ is countable

The points of $E$ which have positive mass are called jump points and size of jump is the probability of RV taking at that point

## Probability Mass fn. (PMF) of discrete RV

$$P_X(x_i) = P(X = x_i) \quad , \quad x_i \in E$$

where,

(i) $P_X(x_i) > 0$

(ii) $\sum\limits_{x_i \in E} P_X(x_i) = 1$

## Relation betn. PMF & CDF

$$F_X(x) = \sum_{x_i \leq x} P_X(x_i)$$

Thus, $P_X(x_i) = P_X(x \leq x_i) - P_X(x < x_{i-1})$

$$= F_X(x_i) - F_X(x_{i-1}).$$

A computer store contains 10 computers of which 3 are defectives. A customer buy 2 at random. Find PMF and CDF.

Let The RV X be no. of defectives in purchase

$\Rightarrow X \in \{0, 1, 2\}$

$$P_X(X=0) = \frac{{}^7C_2}{{}^{10}C_2} = \frac{21}{45}$$

$$P_X(X=1) = \frac{{}^7C_1 \; {}^3C_1}{{}^{10}C_2} = \frac{21}{45}$$

$$P_X(X=2) = \frac{{}^3C_2}{{}^{10}C_2} = \frac{3}{45}$$

↑ PMF

$$F_X(X<0) = P(X<0) = 0$$

$$F_X(X\leq 0) = P(X\leq 0) = \frac{21}{45}$$

$$F_X(X\leq 1) = P(X\leq 0) + P(0<X\leq 1)$$

$$= \frac{42}{45}$$

$$F_X(X\leq 2) = P(X\leq 1) + P(1<X\leq 2)$$

$$= 1$$

↑ CDF

Let X be the number of heads in three tosses of a coin. What is $\Omega$? What are the probability of events $(X \leq 2.75)$ and $(0.5 \leq X \leq 1.72)$.

Write PMF and CDF of RV X.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

PMF

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{7}{8} & x = 2 \\ 1 & x \geq 3 \end{cases}$$

Find whether the following F are CDF or not.

(a) $$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$$

b) $$F(x) = \frac{1}{\pi} \tan^{-1} x \qquad -\infty < x < \infty$$

(c) $$F(x) = \begin{cases} 0 & x \leq 1 \\ 1 - \frac{1}{x} & x > 1 \end{cases}$$

(d) $$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

## Continuous RV

RVs associated with distribution fn. that have no jump points.

## Probability Density fn. (PDF)

Let $X$ be a RV defined on $(\Omega, \beta, P)$ with distribution fn. $F$. Then $X$ is said to be continuous type if $F$ is absolutely continous.

i.e.

if $\exists$ non-negative fn. $f_X(x)$ s.t.

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx \quad \forall \text{ real } x.$$

Then $f_X$ is called prob. density fn. of RV $X$

## Properties

- $P(a \leq x \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx$.
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.
- If $F$ is absolutely cont & $f$ is cont at $x$

$$\frac{d}{dx} F_X(x) = f_X(x).$$

# Note

For discrete RV $P(x=a) \Rightarrow$ prob that $x$ takes $a$

In cont. RV $f(a) \neq$ prob that $x$ takes $a$

Indeed, if $x$ is cont, it assums every value with probability 0.

CDF given $F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$

$$F_X'(x) = f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

PDF given $f_X(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases}$

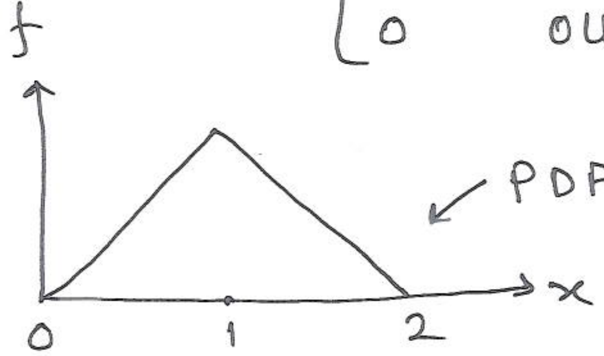
$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \begin{cases} 0 & x < 10 \\ \int_{10}^{x} \frac{10}{t^2}\,dt = \left(1 - \frac{10}{x}\right) & x \geqslant 10 \end{cases}$$

$$P(15 < X < 20) = \int_{15}^{20} \frac{10}{x^2}\,dx = \frac{1}{6}.$$

$$= F_X(20) - F_X(15)$$

check !!

Given PDF $f_X(x) = \begin{cases} x & 0 < x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{Otherwise} \end{cases}$




$F_X(x)$

$$= \begin{cases} 0 & x \leq 0 \\ \int_0^x t\,dt = \frac{x^2}{2} & 0 < x \leq 1 \\ \int_0^1 t\,dt + \int_1^x (2-t)\,dt & 1 < x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

Find $P(.3 \leq X \leq 1.5)$

$= .83.$

Given
$$f_X(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq 1 \\ \frac{1}{2} & 1 \leq x \leq 2 \\ \frac{3-x}{2} & 2 < x \leq 3 \end{cases}$$

Find $F_X(x)$. Hence find $P\left(\frac{1}{2} \leq X \leq \frac{5}{2}\right)$.

$$F_X(x) = \begin{cases} \frac{x^2}{4} & 0 \leq x < 1 \\ \frac{1}{4} + \frac{x-1}{2} & 1 \leq x < 2 \\ 1 - \frac{(3-x)^2}{4} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

The diameter of an electric cable, say X, is assumed to be a continuous RV with PDF

$$f(x) = 6x(1-x) \quad 0 \leq x \leq 1.$$

(i) check that $f(x)$ is PDF

(ii) Determine a number b such that $P(x < b) = P(x > b)$

(Ans. $b = \frac{1}{2}$)

Let X be a continuous RV with ~~following~~ pdf

$$f(x) = \begin{cases} ax & 0 \leq x < 1 \\ a & 1 \leq x \leq 2 \\ -ax + 3a & 2 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

Determine a. Compute $P(X \leq 1.5)$

(Ans. $a = \frac{1}{2}$ & $P = \frac{1}{2}$)

## Symmetric Distribution

A RV $X$ is symmetric about a point $\alpha$ if

$$\boxed{\begin{gathered} P(X \geq \alpha + x) = P(X \leq \alpha - x) \;\; \forall x \\ \text{or} \\ F(\alpha - x) = 1 - F(\alpha + x) + P(X = \alpha + x) \end{gathered}}$$

If $\alpha = 0$

$$F(-x) = 1 - F(x) + P(X = x)$$

If $X$ is continuous,

$$F(-x) = 1 - F(x)$$

$$\text{or } f(-x) = f(x)$$

$$\text{or, } f(\alpha - x) = f(\alpha + x)$$

$P(X = -1) = \frac{1}{4}$, $P(X = 0) = \frac{1}{2}$, $P(X = 1) = \frac{1}{4}$.

Symmetric about 0.

# Moments

## Mathematical Expectation

Let $X$ be discrete RV with pmf $P_X(x_i)$, $x_i \in \Omega$.
We define expected value of $X$ as

$$\boxed{E(X) = \sum_{x_i} x_i \, P_X(x_i)}$$

provided the series is absolutely convergent.

$$E(X) = 0 \cdot \frac{21}{45} + 1 \cdot \frac{21}{45} + 2 \cdot \frac{3}{45} = \frac{27}{45}$$

$P_X(0) = \frac{21}{45}$
$P_X(1) = \frac{21}{45}$
$P_X(2) = \frac{3}{45}$

$P_X(X=1) = P_X(X=2) = \frac{1}{4}$, $P_X(X=3) = \frac{1}{2}$

$$E(X) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} = \frac{9}{4}.$$

If $X$ is continuous RV with pdf $f_X(x)$

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx}$$

provided the integral is absolutely convergent

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^1 x\,dx = \frac{1}{2}$$

$$f_X(x) = \frac{10}{x^2} \qquad x > 10$$

$$E(X) = \int_{10}^{\infty} x \cdot \frac{10}{x^2}\,dx = 10 \ln x \Big|_{10}^{\infty}$$

$\longrightarrow$ divergent

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad -\infty < x < \infty$$

But $E(X)$ is divergent.

## Properties

= If $Y = aX + b$, then $E(Y) = a\,E(X) + b$.

= If $Y = g(X)$, $E(Y) = E(g(X))$

$$= \begin{cases} \sum_{x_i} g(x_i)\, P_X(x_i) \\ \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx \end{cases}$$

= variance $= E(X-\mu)^2 = E(X^2) - \{E(X)\}^2$

## Moments

$$\boxed{\mu_k' = E(X^k)}$$ $\longrightarrow$ $k^{th}$ ~~mo~~ moment about origin or non-central moment

$$\boxed{\mu_k = E(X-\mu)^k}$$ $\longrightarrow$ $k^{th}$ central moment

$\mu_2 = E(X-\mu)^2 =$ variance of $X$.

$$\boxed{\begin{aligned}\mu_k &= E(x-\mu)^k \\ &= E\left[x^k - {}^kc_1 x^{k-1}\mu + {}^kc_2 x^{k-2}\mu^2 + \dots + (-1)^k \mu^k\right] \\ &= \mu_k' - {}^kc_1 \mu_{k-1}' \mu + {}^kc_2 \mu_{k-2}' \mu^2 + \dots + (-1)^k \mu^k\end{aligned}}$$

$\Rightarrow$

$$\mu_2 = \mu_2' - 2\mu_1'\mu + \mu^2$$

$$= \mu_2' - \mu_1'^2 = E(x^2) - \{E(x)\}^2$$

$\Rightarrow$ As $\mu_2 \geq 0$, $E(x^2) \geq \{E(x)\}^2$ Note

## Absolute Moment

$$\boxed{\begin{aligned}\beta_k' &= E|x|^k \\ \beta_k &= E|x-\mu|^k\end{aligned}}$$

X has uniform distribution of first N natural numbers

$$P(X = k) = \frac{1}{N}, \quad k = 1, 2 \ldots N$$

Find mean and variance

Hint : $E(X) = \sum_{k=1}^{N} k \cdot \frac{1}{N} = \frac{N+1}{2}$

$$E(X^2) = \sum_{k=1}^{N} k^2 \cdot \frac{1}{N} = \frac{(N+1)(2N+1)}{6}$$

Find mean and variance of the following

$$f_X(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Hint : $E(X) = \int_1^{\infty} x \cdot \frac{2}{x^3} \, dx = 2$

$$E(X^2) = \int_1^{\infty} x^2 \cdot \frac{2}{x^3} \, dx \longrightarrow \text{divergent}$$

$\Rightarrow$ Lower order moment may exist.

Properties

⇒ If X and Y are two RVs then

$$\boxed{E(X \pm Y) = E(X) \pm E(Y)}$$

(Addition Rule)

⇒ If X and Y are RVs

(Multiplication Rule)

$$\boxed{\begin{aligned} E(XY) &= E(X)\,E(Y) \quad \text{if } X \,\&\, Y \text{ are ind.} \\ &= E(X)\,E(Y) + Cov(X,Y) \quad \text{if } X \,\&\, Y \text{ are dep.} \end{aligned}}$$

Covariance (Defn.)

$$\begin{aligned} Cov(X,Y) &= E\left[\left(X - E(X)\right)\left(Y - E(Y)\right)\right] \\ &= E\left(XY - X\,E(Y) - Y\,E(X) + E(X)\,E(Y)\right) \\ &= E(XY) - E(X)\,E(Y) \end{aligned}$$

$Cov(X,Y) > 0 \Rightarrow$ Y increases as X increases

$Cov(X,Y) < 0 \Rightarrow$ Y decreases as X increases

# Moment Generating Function (mgf)

The mgf about origin is defined as

$$\boxed{M_X(t) = E\left(e^{tX}\right)} = \begin{cases} \sum_{x_i} e^{tx_i} \, p_X(x_i) & X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx & X \text{ is continuous} \end{cases}$$

$$M_X(t) = E\left[1 + tX + \frac{t^2}{2!}X^2 + \cdots + \frac{t^r}{r!}X^r + \cdots\right]$$

$$= 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

Thus,

$$\boxed{\mu_r' = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0}}$$

Find the mgf of the exponential distⁿ:

$$f(x) = a e^{-ax}, \quad 0 \leq x \leq \infty, \quad a > 0$$

Hence find its means and SD.

By defⁿ:

$$M_X(t) = E(e^{tX}) = \int_0^\infty a e^{(t-a)x} dx$$

$$= \left(1 - \frac{t}{a}\right)^{-1}$$

$$= 1 + \frac{t}{a} + \frac{t^2}{a^2} + \frac{t^3}{a^3} + \cdots$$

$$\mu_1' = \frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{1}{a}$$

$$\mu_2' = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = \frac{2}{a^2}$$

$$\Rightarrow \text{Mean} = \frac{1}{a}, \quad SD = \sqrt{\mu_2' - (\mu_1')^2} = \frac{1}{a}.$$

Find mgf of a Bernoulli Variable $X$ with pmf

$$P(X=1) = p \qquad P(X=0) = q \quad , \; p+q=1 .$$

Hence find its mean and variance.

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx_i} \, P_X(x_i)$$

$$= pe^t + q$$

$\Rightarrow \mu_1' = p$ and $\mu_2' = p$

$\Rightarrow$ Mean $= p$

Variance $= \mu_2' - \mu_1'^2 = p(1-p) = pq .$

| Distribution | Moment Generating fn. |
|---|---|
| Binomial $(n, p)$ | $(1-p+pe^t)^n$ |
| Poisson $(\lambda)$ | $e^{\lambda(e^t-1)}$ |
| Uniform $(a, b)$ | $\dfrac{e^{tb} - e^{ta}}{t(b-a)}$ |

# Chebyshev's Inequality

If $X$ is RV with mean $\mu$ and variance $\sigma^2$ then for any value $k > 0$

$$\boxed{P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}}$$

Proof.

$$\sigma^2 = \operatorname{var}(X) = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$\geq \int_{|x-\mu| \geq k} (x-\mu)^2 f_X(x)\,dx$$

$$\geq k^2 \int_{|x-\mu| \geq k} f_X(x)\,dx$$

$$\geq k^2 P(|X-\mu| \geq k)$$

Replacing $k$ by $k\sigma$

$$\boxed{P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}}$$

Another form

$$\boxed{P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}}$$

# Markov Inequality

If X is a RV that takes non-negative values, then for any value $a > 0$

$$\boxed{P(X \geq a) \leq \frac{E(X)}{a}.}$$

Proof.

$$E(X) = \int_0^\infty x f(x)\,dx = \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx$$

$$\geq a \int_a^\infty f(x)\,dx$$

$$= a\, P(X \geq a).$$

Suppose that it is known that the number of items produced in a factory during a week is RV with mean 50.

(a) What can be said about the probability that this weak's production will exceed 75?

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60?

Ans.

(a) $$P(X \geq 75) \leq \frac{E(X)}{75} = \frac{50}{75} = \frac{2}{3}.$$

using Markov's inequality

(b) $$P(|X-50| < 10) \geq 1 - \frac{1}{4} = \frac{3}{4}$$

as

$$P(|X-5| \geq 10) \leq \frac{\sigma^2}{10^2} = \frac{1}{4}$$

using Chebyshev's inequality

## Weak Law of Large Numbers

Let $X_1, X_2 \cdots X_n$ be a sequence of iid RVs, each having mean $\mu$ (i.e. $E(x_i) = \mu$). Then for any $\epsilon > 0$

$$\boxed{P\left\{\left|\frac{X_1 + X_2 \cdots + X_n}{n} - \mu\right| > \epsilon\right\} \to 0 \text{ as } n \to \infty}$$

$$E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \mu \qquad Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{\sigma^2}{n}$$

By Chebyshev's inequality

$$P\left\{\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

$\Rightarrow$ as $n \to \infty$, LHS $\to 0$.