Transformation of RV

Theorem

Let X be a RV dfd on (Ω, β, P). Let g be a measurable fn $g: \mathbb{R} \to \mathbb{R}$, then $g(X)$ is also a RV.

$\boxed{\text{Ex 1}}$ Let $Y = g(x) = ax + b$, $a \neq 0$, $b \in \mathbb{R}$

$$G_Y(y) = P(Y \leq y) = P(g(x) \leq y)$$

$$= P(ax + b \leq y)$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) & \text{if } a > 0 \\ P\left(X \geq \frac{y-b}{a}\right) & \text{if } a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & \text{if } a > 0 \\ 1 - P\left(X \leq \frac{y-b}{a}\right) + P\left(X = \frac{y-b}{a}\right) & \text{if } a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) + P\left(X = \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$\boxed{\text{Ex } 2}$ $\quad Y = g(X) = |X|$

$$G_Y(y) = P(Y \le y) = 0 \quad \text{if} \quad y < 0$$

if $y > 0$

$$P(|X| \le y) = P(-y \le X \le y)$$
$$= F_X(y) - F_X(-y) + P(X = -y)$$

$\boxed{\text{Ex } 3}$ $\quad Y = g(X) = X^2$

$$G_Y(y) = \begin{cases} 0 & \text{if} \quad y < 0 \\ P(X^2 \le y) & \text{if} \quad y > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad y < 0 \\ P(-\sqrt{y} \le X \le \sqrt{y}) & \text{if} \quad y > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) + P(X = -\sqrt{y}) & \text{if} \quad y > 0 \end{cases}$$

$\boxed{\text{Ex 4}}$ Given $f_X(x) = \begin{cases} .05 & -10 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

Find cdf & pdf of $Y = 16X^2$.

$$F_Y(y) = P(Y \leqslant y)$$

$$= \int_{-\sqrt{y}/4}^{\sqrt{y}/4} .05\, dx \qquad \text{as, } 16x^2 = y$$

$$\text{or, } x = \pm\sqrt{\frac{y}{4}}$$

$$= \frac{\sqrt{y}}{40} \quad \text{where } 0 \leqslant y \leqslant 1600$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{.y^{-1/2}}{80}, \quad 0 \leqslant y \leqslant 1600$$

$\boxed{\text{Ex 5}}$ Given the pmf of X, find pmf of $y = X^2$.

$P(X=-2) = \frac{1}{5}$ $\quad P(X=-1) = \frac{1}{6}$ $\quad P(X=0) = \frac{1}{5}$

$P(X=1) = \frac{1}{15}$ $\quad P(X=2) = \frac{11}{30}$.

Here Y can take values 0, 1, 4. so pmf of Y

$P(Y=0) = \frac{1}{5}$ $\quad P(Y=1) = \frac{7}{30}$ $\quad P(Y=4) = \frac{17}{30}$.

Ex 6

Given $f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Find cdf and pdf of $Y = \max(X, 0)$

cdf

$$F_Y(y) = P(Y \leq y) = \begin{cases} P(X \leq 0) = \frac{1}{2} & y = 0 \\ P(X \leq 0) + P(X \in [0, 1]) = \frac{1}{2} + \frac{y}{2} & 0 < y \leq 1 \\ 1 & y > 1 \end{cases}$$

$$f_Y(y) = \frac{1}{2} \qquad 0 < y < 1$$

Ex 7

Given $f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

find $f_Y(y)$ where $Y = 10 + 500X$.

An.

$$f_Y(y) = \begin{cases} \dfrac{y - 10}{125000} & 10 \leq y \leq 510 \\ 0 & \text{otherwise} \end{cases}$$

Theorem

Let X be a continuous RV with pdf $f_X(x)$. Let $Y = g(X)$ be a differentiable function for all x and either $g'(x) > 0 \ \forall x$ or $g'(x) < 0 \ \forall x$. Then $Y = g(X)$ is continuous RV with pdf

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right)\left|\dfrac{d}{dy} g^{-1}(y)\right| & \alpha < y < \beta \\ 0 & \text{Otherwise} \end{cases}$$

Where $\alpha = \min\{g(-\infty), g(\infty)\}$

& $\beta = \max\{g(-\infty), g(\infty)\}$

Ex 7

$y = 10 + 500x$, $x \in [0,1]$

$\alpha = \min(10 + 500 \cdot 0,$ $10 + 500 \cdot 1$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= 2\,\frac{y-10}{500} \cdot \frac{1}{500}$$

$$= \begin{cases} \dfrac{y-10}{125000} & 10 \leq y \leq 510 \\ 0 & \text{Otherwise} \end{cases}$$

Case I $\quad g'(x) > 0 \quad \forall x$

$\Rightarrow y = g(x)$ is strictly monotonically increasing.

$\Rightarrow (X \leq x)$ is same as $(g(X) \leq g(x))$

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq g(x)) = P(X \leq x) = F_X(x)$$

Thus, $F_Y(y) = F_X(x)$

or, $\frac{d}{dx} F_X(x) = \frac{d}{dx} F_Y(y) = \frac{d}{dy} F_Y(y) \cdot \frac{dy}{dx}$.

or, $f_X(x) = f_Y(y) \cdot \frac{dy}{dx}$.

or, $f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$ as $\frac{dy}{dx} > 0$.

Case II

$g'(x) < 0 \quad \forall x \Rightarrow y = g(x)$ is m. decreasing

$\Rightarrow (X \leq x)$ is same as $(g(X) > g(x))$

$\Rightarrow P(X \leq x)$ is same as $1 - P(Y \leq y)$

i.e. $F_X(x) = 1 - F_Y(y)$

or, $\frac{d}{dx} F_X(x) = \frac{d}{dx} \{1 - F_Y(y)\}$

or, $f_X(x) = - f_Y(y) \cdot \frac{dy}{dx}$.

or, $f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$ as $\frac{dx}{dy} < 0$.

$\boxed{\text{Ex } 8}$ If $X \sim N(m, \sigma)$, find the distribution of $Y = aX + b$. Where a, b are constants.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad \left|\frac{dx}{dy}\right| = \frac{1}{|a|}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma\,|a|} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\;\sigma|a|} e^{-\frac{1}{2}\left(\frac{y-b-am}{a\sigma}\right)^2}$$

$$\Rightarrow Y \sim N\left(am + b,\ |a|\sigma\right)$$

$\boxed{\text{Ex } 9}$ X is a $B(\alpha, \beta)$ variate, then $Y = \frac{1}{X}$ is a $B(\beta, \alpha)$ variate.

Here, $y' = -\frac{1}{x^2} < 0$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{x^{\alpha-1}}{B(\alpha,\beta)\,(1+x)^{\alpha+\beta}} \cdot x^2$$

$$= \frac{y^{\beta-1}}{B(\beta,\alpha)\,(1+y)^{\alpha+\beta}} \sim B(\beta, \alpha)$$

$\boxed{\text{Ex } 10}$ $X \sim U(-1, 1)$ then find the distn of $|X|$.

$$Y = |X| \Rightarrow y = \begin{cases} x & x > 0 \\ -x & x < 0 \\ 0 & x = 0 \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 . \end{cases}$$

Now,

$$P(y \leq Y \leq y + dy)$$

$$= P(|x| \leq |X| \leq |x + dx|)$$

$$= \begin{cases} P(-(x+dx) \leq X \leq -x) + P(x \leq X \leq x + dx) & \text{if } x > 0 \\ P(x+dx \leq X \leq x) + P(-x \leq X \leq -(x+dx)) & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 2\, P(x \leq X \leq x + dx) & x > 0 \\ 2\, P(-x \leq X \leq -(x+dx)) & x < 0 \end{cases}$$

$$f_Y(y)\, dy = \begin{cases} 2\, f_X(x)\, dx & x > 0 \\ -2\, f_X(x)\, dx & x < 0 \end{cases} \Rightarrow f_Y(y) = 1$$

for $0 < y < 1$

Ex 11

If X is standard normal variate, then $Y = \frac{X^2}{2}$ is a $\gamma\left(\frac{1}{2}\right)$ variate

Here, $\frac{dy}{dx} = x \Rightarrow \frac{dy}{dx}$ changes sign with x.

$$P\left(y \leq Y \leq (y + dy)\right) = P\left(x^2 \leq X \leq (x+dx)^2\right)$$

$$= P\left(-(x+dx) \leq X \leq -x\right) + P\left(x \leq X \leq (x+dx)\right)$$

$$= 2\ P\left(x \leq X \leq (x+dx)\right)$$

due to symmetry

$$f_Y(y) = 2\, f_X(x) \cdot \frac{dx}{dy}$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \cdot \frac{1}{x}$$

$$= \frac{e^{-y} \cdot y^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} \qquad 0 < y < \infty$$

Ex 12 If $X \sim U(0,1)$, then find the distn. of $Y = -2\log_e X$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$Y = -2\log_e X \Rightarrow \frac{dy}{dx} = -\frac{2}{x} < 0 \quad \forall\ x \in [0,1]$$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{2}e^{-y/2}, \quad 0 < y < \infty.$$

Ex 13 Given
$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 < x \leq 1 \\ \frac{1}{2x^2} & 1 < x < \infty \end{cases}$$

Find $f_Y(y)$ where $Y = \frac{1}{x}$.

Here, $\frac{dy}{dx} = -\frac{1}{x^2}$ or, $\frac{dx}{dy} = -\frac{1}{y^2}$.

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2}\cdot\frac{1}{y^2} & 1 \leq y \\ \frac{y^2}{2}\cdot\frac{1}{y^2} & 0 < x < 1 \end{cases} = \begin{cases} 0 & y < 0 \\ \frac{1}{2} & 0 < y < 1 \\ \frac{1}{2y^2} & 1 < y < \infty \end{cases}$$

Hence X and Y have same distribution.

Suppose a car showroom has 10 cars of a brand, out of which 5 are good (G), 2 have defective transmission (DT) and 3 have defective steering (DS). If 2 cars are selected at random find the probability distribution of (X, Y) where

X: no. of cars with DT

Y: no. of cars with DS.

Here $X \in \{0, 1, 2\}$ & $Y \in \{0, 1, 2\}$

$P_{XY}(x, y)$

X \ Y	0	1	2
0	$\frac{10}{45}$	$\frac{15}{45}$	$\frac{3}{45}$
1	$\frac{10}{45}$	$\frac{6}{45}$	0
2	$\frac{1}{45}$	0	0

$$P_{X,Y}(0,0) = P(X=0, Y=0) = {}^5C_2 / {}^{10}C_2 = \frac{10}{45}$$

$$P_{X,Y}(1,1) = P(X=1, Y=1) = \frac{{}^2C_1 \, {}^3C_1}{{}^{10}C_2} = \frac{6}{45} \text{ etc.}$$

Now to find $P(X \leq 1, Y \leq 1)$

$$= P_{XY}(0,0) + P_{XY}(0,1) + \cancel{P_{XY}(0,2)} + P_{XY}(1,0) + P_{XY}(1,1)$$

$$= 41/45.$$

X \ Y	0	1	2	Marginal dist^n of X $P(X=x)$
0	$10/45$	$15/45$	$3/45$	$28/45$
1	$10/45$	$6/45$	0	$16/45$
2	$1/45$	0	0	$1/45$
$P(Y=y)$	$21/45$	$21/45$	$3/45$	

$$P(X<2) = P(X=0) + P(X=1) = \frac{44}{45}.$$

Conditional dist^n.

$$P(X=x' / Y=y') = \frac{P(X=x', Y=y')}{P(Y=y')}$$

Thus,

$$P(y=0/x=0) = \frac{P(x=0, y=0)}{P(x=0)} = \frac{10/45}{28/45}$$

$$P(Y=0/x=2) = ?$$ Ans. 1.

Given $f_{X,Y}(x,y) = \begin{cases} 10xy^2 & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$

Find marginal and conditional distribution.

$$f_X(x) = \int_x^1 f_{X,Y}(x,y)\,dy = \begin{cases} \frac{10}{3}x(1-x^3) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,dx = \begin{cases} 5y^4 & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(X=x/Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{10xy^2}{5y^4} = \begin{cases} \frac{2x}{y^2} & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(Y=y/X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \begin{cases} \frac{3y^2}{1-x^3} & 0<x<y<1 \\ 0 & \text{Otherwise} \end{cases}$$

Hence find

$$P\left(X<\tfrac{1}{4}\right) = \int_0^{1/4} f_X(x)\,dx$$

$$P\left(Y>\tfrac{3}{4}\right) = \int_{3/4}^1 f(y)\,dy$$

$$P\left(0<X+Y<\tfrac{1}{2}\right)$$

$$P\left(X<\tfrac{1}{2} / Y=\tfrac{3}{4}\right)$$

$$P\left(\tfrac{1}{4}<Y<\tfrac{3}{4}\right)$$

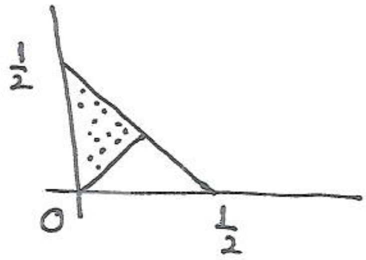
Ans.

- $P(X < \frac{1}{4}) = \int_0^{1/4} \frac{10}{3}(x - x^3)\, dx$

- $P(Y > \frac{3}{4}) = \int_{3/4}^{1} 5y^4\, dy$

- $P(0 < X + Y < \frac{1}{2})$

$$= 10 \int_0^{1/4} \int_0^{\frac{1}{2} - x} xy^2\, dy\, dx$$



- $P(X < \frac{1}{2} / Y = 3/4)$

$$= \frac{P(X < \frac{1}{2},\ Y = \frac{3}{4})}{P(Y = \frac{3}{4})}$$

$$= \frac{\int_0^{1/2} \frac{10}{3} x \left(\frac{3}{4}\right)^2 dx}{5 \cdot \left(\frac{3}{4}\right)^4}$$

- $P(\frac{1}{4} < Y < \frac{3}{4}) = \int_{1/4}^{3/4} 5y^4\, dy.$

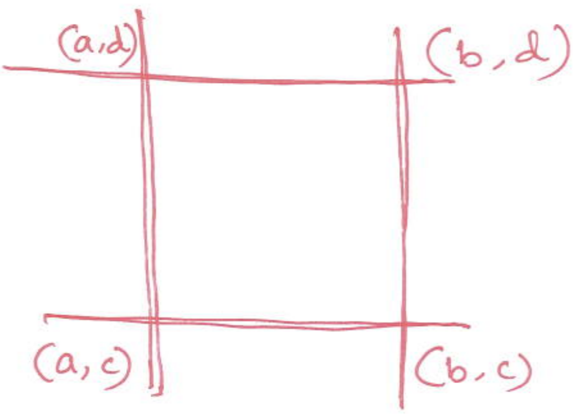
Distribution fn. in two dimension

Let X and Y are RV defined on sample space S.
Joint distribution

$$F_{X,Y}(x,y) = F(x,y) = P(-\infty < X \leq x, -\infty < Y \leq y)$$

When $(-\infty < X \leq x, -\infty < Y \leq y)$ means joint occurance of $(-\infty < X \leq x)$ and $(-\infty < Y \leq y)$

Properties.



$a \leq x \leq b, \; c \leq Y \leq d.$

- $F(x,y)$ is monotone non-decreasing in both variables x and y.

$$\Rightarrow F(b,c) \geq F(a,c) \text{ if } b > a$$
$$F(a,d) \geq F(a,c) \text{ if } d > c$$

Proof: $F(b,c) - F(a,c)$

$$= P(-\infty < X \leq b, -\infty < Y \leq c) - P(-\infty < X \leq a, -\infty < Y \leq c)$$
$$= P(a \leq X \leq b, -\infty < Y \leq c)$$

non-~~positive~~ negative

- $P(a \leq X \leq b, c \leq Y \leq d)$
$$= F(b,d) + F(a,c) - F(a,d) - F(b,c)$$

- $F(-\infty, y) = 0 \qquad F(x, -\infty) = 0$

- $F(\infty, \infty) = 1.$

Marginal distributions

$$F_X(x) = F(x, \infty)$$

$$F_Y(y) = F(\infty, y)$$

If X and Y are independent

$$F_{X,Y}(x, y) = F_X(x) \; F_Y(y)$$

$$P(a \leq X \leq b, \; c \leq Y \leq d)$$

$$= P(a \leq X \leq b) \; P(c \leq Y \leq d)$$

$$= \Big(F_X(b) - F_X(a)\Big)\Big(F_Y(d) - F_Y(c)\Big)$$

$$= F(b,d) + F(a,c) - F(b,c) - F(a,d)$$

Discrete distribution

$$P(X = x_i, \; Y = y_j) = f_{ij}$$

$$P(a \leq X \leq b, \; c \leq Y \leq d) = \sum_{c \leq y_j \leq d} \; \sum_{a \leq x_i \leq b} f_{ij}$$

Marginal $\quad f_{i\cdot} = \sum_{j=-\infty}^{\infty} f_{ij}$

$$f_{\cdot j} = \sum_{i=-\infty}^{\infty} f_{ij}$$

Method of Transformation

- Let $X \sim f_X(x)$ and $Y = g(X)$ is either increasing or decreasing in X then

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

- If $y_1 = g_1(x_1, x_2)$ & $y_2 = g_2(x_1, x_2)$ and $(X_1, X_2) \sim f_{X_1, X_2}(x_1, x_2)$ then

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \, |J|$$

Where, $J = \dfrac{\partial(x_1, x_2)}{\partial(y_1, y_2)}$

$$= \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_2}{\partial y_1} \\ \dfrac{\partial x_1}{\partial y_2} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$$

The joint distribution fn. of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, \quad 0 < y < \infty \\ 0 & \text{Otherwise} \end{cases}$$

Find (a) $P(X > 1, Y < 1)$ (b) $P(X < Y)$

(c) $P(X < a)$

(a) $P(X > 1, Y < 1) = \displaystyle\int_{y=0}^{1} \int_{x=1}^{\infty} 2e^{-x}e^{-2y}\, dx\, dy$

$$= \boxed{\frac{1}{e}\left(1 - \frac{1}{e^2}\right)}$$

(b) $P(X < Y) = \displaystyle\int_{y=0}^{\infty} \int_{0}^{y} 2e^{-x}e^{-2y}\, dx\, dy$

$$= \boxed{\frac{1}{3}}$$

(c) $P(X < a) = \displaystyle\int_{0}^{a} \int_{0}^{\infty} 2e^{-x}e^{-2y}\, dx\, dy$

$$= \boxed{1 - \frac{1}{e^a}}.$$

If X and Y are two RV with joint density fn.

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y) & 0 \leqslant x < 2,\ 2 \leqslant y < 4 \\ 0 & \text{Otherwise} \end{cases}$$

Find (i) $P((X < 1) \cap (Y < 3))$ (ii) $P(X + Y < 3)$

and $P(X < 1 / Y < 3)$.

$\rightarrow$ $$P((X < 1) \cap (Y < 3)) = \int_0^1 \int_2^3 \frac{1}{8}(6 - x - y)\, dx\, dy$$

$$= \frac{3}{8}.$$

$\rightarrow$ $$P(X + Y < 3) = \int_0^1 \int_2^{3-x} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{5}{24}$$

$\rightarrow$ $$P(X < 1 / Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)}$$

$$= \frac{3/8}{\int_0^2 \int_2^3 \frac{1}{8}(6 - x - y)\, dx\, dy} = \frac{3}{5}.$$

Suppose 15% families in a certain community have no children, 20% have 1, 35% have 2 and 30% have 3 children. Suppose further that each child is equally likely to be a boy or a girl. If the family is choosen at random from this community, then B, the number of boys, and G, the number of girls, in this family will have the joint probability mass fn. P(B, G).

(i) Hence find prob that a family choosen will have atleast 1 girl.

(ii) Again find $P(B=0/G=1)$, $P(B=1/G=1)$, $P(B=2/G=1)$ and $P(B=3/G=1)$.

B \ G	0	1	2	3	M(B)
0	·15	·10	·0875	·0375	·375
1	·10	·175	·1125	0	·3875
2	·0875	·1125	0	0	·2
3	·0375	0	0	0	·0375
M(G)	·375	·3875	·2	·0375	1

$P(B=0, G=0) = P(\text{no ch.}) = \cdot 15$

$P(B=0, G=1) = P(1 \text{ child})\, P(1G/1 \text{ chi})$
$= \cdot 20 \times \cdot 5 = \cdot 1$

$P(B=0, G=2) = P(2c)\, P(2G/2c) = \cdot 35 \times \cdot 5 \times \cdot 5 = \cdot 0875$

so on..

(i) $P(\text{atleast one girl}) = P(1G) + P(2G) + P(3G) = \cdot 625$

(ii) $P(B=0/G=1) = \dfrac{P(B=0, G=1)}{P(G=1)} = \dfrac{\cdot 1}{\cdot 3875} = \cdot 258$

$P(B=1/G=1) = \dfrac{P(B=1, G=1)}{P(G=1)} = \dfrac{\cdot 175}{\cdot 3875} = \cdot 4516$

$P(B=2/G=1) = \dfrac{P(B=2, G=1)}{P(G=1)} = \dfrac{\cdot 1125}{\cdot 3875} = \cdot 29$

$P(B=3/G=1) = P(B=3, G=1)/P(G=1) = 0$

A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between 12 noon to 1 PM, then what is the probability that first person to arrive has to wait longer than 10 minutes?

X and Y are independent RVs which is uniformly over (0, 60).

Required prob $= P(X+10<Y) + P(Y+10<X)$

$= 2\,P(X+10<Y)$ or $2\,P(Y+10<X)$

$$= 2 \iint\limits_{X+10<Y} f(x,y)\,dx\,dy$$

$$= 2 \int_{y=10}^{60} \int_{x=0}^{Y-10} f_X(x)\, f_Y(y)\,dx\,dy$$

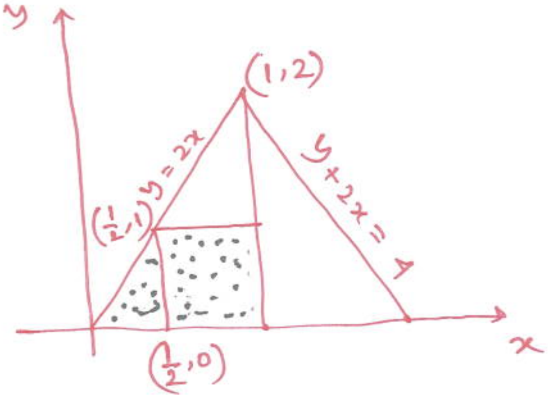
$$= 2 \int_{10}^{60} \int_{0}^{y-10} \left(\frac{1}{60}\right)^2 dx\,dy$$

$$= \frac{25}{36}.$$

The joint distribution of X and Y is uniform over the interior of a triangle with vertices (0,0), (2,0) and (1,2). Find the density fn. of (X, Y) and $P(X \le 1, Y \le 1)$

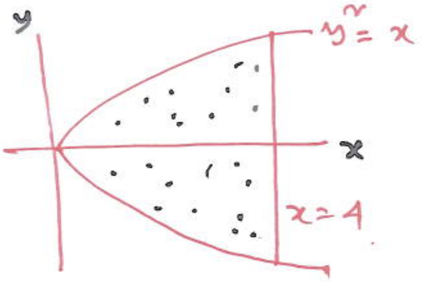
Here pdf of (X, Y) is $f(x,y) = \begin{cases} \frac{1}{2} & (x,y) \in R \\ 0 & \text{elsewhere} \end{cases}$

R: interior of given triangle with area 2 sq units.



$$P(X \le 1, Y \le 1)$$
$$= \int_{-\infty}^{1} \int_{-\infty}^{1} f(x,y)\, dx\, dy$$
$$= \int_{0}^{1} \int_{0}^{2x} \left(\tfrac{1}{2}\, dy\right) dx + \int_{1/2}^{1} \int_{0}^{1} \left(\tfrac{1}{2}\, dy\right) dx$$
$$= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

Suppose (X, Y) is uniformly distributed over the area bounded by $y^2 = x$ and $x = 4$. Find the joint distribution of X and Y and $P(X < 3, Y < 0)$.



Here, $f(x,y) = \begin{cases} c & \text{if } x \in R \\ 0 & \text{Otherwise} \end{cases}$

R: shaded region

$$\int_{-2}^{2} \int_{y^2}^{4} (c\, dx)\, dy = 1 \quad \Rightarrow \quad c = \frac{3}{32}$$

Then, $P(X < 3, Y < 0) = \int_{0}^{3} \int_{-\sqrt{x}}^{0} \{f(x,y)\, dy\}\, dx$

$$= \frac{3}{32} \int_{0}^{3} \int_{-\sqrt{x}}^{0} dy\, dx = \frac{3\sqrt{3}}{16}.$$

Expectation of joint distribution

Let $g(X,Y)$ be a measurable fn. of X and Y.
And if $P_{X,Y}(x,y)$ is the pmf then

$$\boxed{E\{g(X,Y)\} = \sum_{x_i} \sum_{y_j} g(x_i, y_j)\, p_{XY}(x_i, y_j)}$$

provided the series is absolutely conv.

And if $f_{X,Y}(x,y)$ is the pdf (joint) then

$$\boxed{E\{g(X,Y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy}$$

provided the integral is absolutely convergent.

Product Moments

$$\boxed{\mu'_{r,s} = E(X^r Y^s)}$$

$$\Rightarrow \mu'_{1,1} = E(x,y) \quad , \quad \mu'_{1,0} = E(X) = \mu_X$$

$$\mu'_{0,1} = E(Y) = \mu_Y$$

Central product moments

$$\mu_{r,s} = E(X-\mu_X)^r (Y-\mu_Y)^s$$

Covariance betn X and Y

Putting $r=1$, $s=1$

$$\mu_{1,1} = E(X-\mu_X)(Y-\mu_Y)$$

$$= E(XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y)$$

$$= E(XY) - \mu_X \mu_Y$$

$$= E(XY) - E(X)\,E(Y)$$

If X and Y are independent

$$E(X^r Y^s) = E(X^r)\,E(Y^s)$$

$$E(X-\mu_X)^r (Y-\mu_Y)^s = E(X-\mu_X)^r\,(Y-\mu_Y)^s$$

Covariance $= 0$.

Correlation betn. X and Y

correlation coefficient $= \boxed{\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sigma_X \ \sigma_Y}}$.

Where, $\sigma_X^2 = var(X)$, $\sigma_Y^2 = var(Y)$

$$\boxed{-1 \leq \rho_{X,Y} \leq 1}$$

Let X, Y are discrete random variables, values pmf of (X, Y) is discrete, then

$$Cov(X,Y) = E(X-\mu_X)(Y-\mu_Y) = \frac{1}{n}\sum_{x_i}\sum_{Y_i}(x_i-\bar{x})(y_i-\bar{y})$$

$$\sigma_X^2 = \frac{1}{n}\sum(x_i-\bar{x})^2 \qquad \sigma_Y^2 = \frac{1}{n}\sum(y_i-\bar{y})^2$$

$$\rho_{x,y} = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\left\{\sum(x_i-\bar{x})^2\sum(y_i-\bar{y})^2\right\}^{1/2}} = \frac{\sum a_i b_i}{\left\{\sum a_i^2 \sum b_i^2\right\}^{1/2}} \text{(say)}$$

Schwartz inequality : $\left(\sum a_i b_i\right)^2 \leq \sum a_i^2 \sum b_i^2$

Equality holds for $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

$$\Rightarrow \rho_{X,Y}^2 \leq 1$$

$$\Rightarrow -1 \leq \rho_{X,Y} \leq 1.$$