Anshul Notes ①

Cummulative Distribution Function. (CDF)

$$F_x(x) = P(X \le x)$$

Properties :-

1) $0 \le F_x(x) \le 1$; $-\infty < x < \infty$

2) $\lim_{x \to \infty} F_x(x) \longrightarrow 1$

$\lim_{x \to -\infty} F_x(x) \longrightarrow 0$

3) Non-decreasing Function.

if $x_1 > x_2 \Rightarrow F_x(x_1) \ge F_x(x_2)$

4) Function is Right Continuous.

$$\lim_{\delta \to 0} \left[F_x(x+\delta) - F_x(x) \right] = 0$$

Lebesgue Decomposition Theorem :-

$$F_x(x) = \underbrace{G_x(x)}_{\text{Continuous part}} + \underbrace{H_x(x)}_{\text{Right handed Continuous Step function.}}$$

if $(G_x(x) = 0) \Rightarrow$ discrete Random Variable

if $(H_x(x) = 0) \Rightarrow$ Continuous Random Variable

else $\Rightarrow$ Mixed type Random Variable

Probability Mass Function (PMF)

↳ {only for discrete r.v.}

$$\boxed{P_x(x_i) = F_x(x_i) - F_x(x_{i-1})} \quad \forall\, x_{i-1} < x_i$$

Probability density Function (Pdf)

↳ {Cont. r.v.}

$$\boxed{f_x(x) = \frac{d}{dx} F_x(x)}$$

$$F_x(x) = \int_{-\infty}^{x} f_x(t)\, dt$$

Properties :- Range Space

1) $f_x(x) \ge 0 \quad \forall\, x \in R_x$

2) $\int_{R_x} f_x(x)\, dx = 1$

3) $f_x(x)$ is piecewise Continuous.

4) $f_x(x) = 0$ if x is not in R_x.

$\boxed{P_x(X = x) = 0} \quad \forall\, x \in R_x$

Mean (First Moment $M_x'(0)$) :- (μ)

$$\boxed{\mu = \sum_i x_i P_x(x_i)}$$

↳ for discrete r.v.

$$\boxed{\mu = \int_{-\infty}^{\infty} x f_x(x)\, dx}$$

↳ for Continuous r.v.

Variance (σ^2) :-

↳ Measure of dispersion of the probability

$$\sigma^2 = \sum_i (x_i - \mu)^2 P_x(x_i) \quad \forall\, d.r.v.$$

$$= \int_{-\infty}^{\infty} (x_i - \mu)^2 f_x(x)\, dx \quad \forall\, c.r.v.$$

Standard Deviation (σ) :-

$$\boxed{\sigma = \sqrt{\text{Variance}}}$$

③

Moments :—

→ Moments about the origin (raw Moments) are called Origin Moments (μ'_k)

$$\mu'_k = \sum_i x_i^k P_X(x_i) \quad \forall \text{ d.r.v.}$$

$$\mu'_k = \int_{-\infty}^{\infty} x^k f_X(x)\,dx \quad \forall \text{ c.r.v.}$$

→ Moments about the mean are called Central Moments (μ_k)

$$\mu_k = \sum_i (x_i-\mu)^k P_X(x_i) \quad \forall \text{ d.r.v.}$$

$$\mu_k = \int_{-\infty}^{\infty} (x-\mu)^k f_X(x)\,dx \quad \forall \text{ c.r.v}$$

Note :— Mean(μ) = μ'_1

Variance (σ^2) = μ_2

Chebyshev's Inequality :—

$$\boxed{P_X(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}}$$

Conversion for function of Random variable (X) :—

eg: {$y = ax+b$}

Put $x = \frac{y-b}{a}$

$$\boxed{f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|}$$

Expectation :—

$$E(H(X)) = \sum_i H(x_i) P_X(x_i) \quad \forall \text{ d.r.v.}$$

$$E(H(X)) = \int_{-\infty}^{\infty} H(x) f_X(x)\,dx \quad \forall \text{ c.r.v.}$$

④

Note :—

- $E(X) = \mu$ (Mean)
- $\underbrace{E((X-\mu)^2)}$ $= \sigma^2$ (Variance)

  $V(X) = E(X^2) - (E(X))^2$
- $\mu'_k = E(X^k)$

  $\mu_k = E((X-\mu)^k)$
- $E(aX+b) = a\,E(X) + b$
- $V(aX+b) = a^2 V(X)$

$E(X+Y) = E(X)+E(Y)$

Moment - Generating Function :— ($M_X(t)$)

$$M_X(t) = E(e^{tX})$$

$$M_X(t) = \sum_i e^{tx_i} P_X(x_i) \quad \forall \text{ d.r.v}$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \quad \forall \text{ c.r.v.}$$

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots$$

$$\rightarrow E(e^{tX}) = 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$\boxed{\left.\frac{d^r}{dt^r} M_X(t)\right|_{t=0} = E(X^r)} = \mu'_r$$

if $Y = X_1 + X_2 + X_3 + \cdots + X_N$

$$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot M_{X_3}(t) \cdots M_{X_N}(t)$$

Some Important Distributions

(5)

[A] Discrete

1> Bernoulli Trials and Distribution :—

↳ only two possible outcomes.

$$P(x_j) = \begin{cases} p & x_j = 1, \; j = 1,2,\dots,n \\ 1-p = q & x_j = 0, \; j = 1,2,\dots,n \\ 0 & o.w. \end{cases}$$

- Mean $= E(X_j) = 0.q + 1.p = p$
- $V(X_j) = (0^2 q + 1^2 p) - p^2 = p.q$
- $\boxed{M_{X_j}(t) = q + pe^t}$

2> Binomial Distribution :—

$$P(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0,1,2,\dots,n \\ 0 & o.w. \end{cases}$$

- Mean $= E(X) = np$
- $V(X) = npq$
- $M_X(t) = (pe^t + q)^n$

$$F_X(x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$

3> Geometric Distribution :—

↳ number of trials is not fixed.
↳ do until first success is achieved

$$P(x) = \begin{cases} q^{x-1} p, & x = 1,2, \\ 0, & o.w. \end{cases}$$

- Mean $= E(X) = \frac{1}{p}$
- $V(X) = q/p^2$
- $M_X(t) = \frac{pe^t}{1-qe^t}$

↳ This is the only discrete distribution having memoryless property ↓

$$P(X > x+s \mid X > s) = P(X > x)$$

4> Hypergeometric Distribution :—

$N \to$ items available
$D \to$ fall in class of interest
$n \to$ sample size

$$P(x) = \begin{cases} \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}, & x = 0,1,2,\dots, \min(n, D) \\ 0, & o.w. \end{cases}$$

- Mean $= E(X) = n.\left[\frac{D}{N}\right]$ — Not GIF
- $V(X) = n.\left[\frac{D}{N}\right]\left[1-\frac{D}{N}\right].\left[\frac{N-n}{N-1}\right]$

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5) Poisson Distribution :-

$$P(x) = \begin{cases} \dfrac{c^x e^{-c}}{x!} & , x = 0,1,2,\ldots \\ 0 & , \text{o.w.} \end{cases}$$

$\forall\ c = \lambda t$

Value of outcome considering uniform distribution (Mean)

from the binomial,

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$x = 0,1,2,\ldots,n$

- Mean $= E(X) = c$
- $E(X^2) = c^2 + c$
- $V(X) = c$
- $\boxed{M_X(t) = e^{c(e^t - 1)}}$

[B] Continuous :-

1) Uniform Distribution :-

$$f(x) = \begin{cases} \dfrac{1}{b-a} & ; a \le x \le b \\ 0 & ; \text{otherwise} \end{cases}$$

- Mean $= E(X) = \dfrac{a+b}{2}$
- $V(X) = \dfrac{(b-a)^2}{12}$
- $M_X(t) = \dfrac{e^{tb} - e^{ta}}{t(b-a)}$

$$F_X(x) = \begin{cases} 0 & ; x < a \\ \displaystyle\int_a^x \frac{dx}{b-a} = \frac{x-a}{b-a} & ; a \le x < b \\ 1 & ; x \ge b \end{cases}$$

2) Exponential Distribution :-

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$

- Mean $= E(X) = 1/\lambda$
- $V(X) = 1/\lambda^2$; $E(X^m) = \dfrac{m!}{\lambda^m}$
- $M_X(t) = \left(1 - \dfrac{t}{\lambda}\right)^{-1}$

$$F_X(x) = \begin{cases} 0 & ; x < 0 \\ 1 - e^{-\lambda x} & ; x \ge 0 \end{cases}$$

follows Memory-less property :-

$$P(X > x+s \mid X > x) = P(X > s)$$

3) Gamma Distribution :-

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \quad \forall x > 0$$

$$\boxed{\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)}$$

$$\boxed{\Gamma(\alpha) = (\alpha-1)!} \ \forall\ \alpha > 0$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & , x > 0 \\ 0 & , \text{o.w.} \end{cases}$$

- Mean $= E(X) = \alpha/\lambda$
- $V(X) = \alpha/\lambda^2$
- $M_X(t) = \left(1 - \dfrac{t}{\lambda}\right)^{-\alpha}$

4) THE NORMAL DISTRIBUTION :-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; -\infty < x < \infty$$

Properties :-

(a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(b) $f(x) \geq 0 \quad \forall x$

(c) $\lim_{x \to +\infty} f(x) = 0$ & $\lim_{x \to -\infty} f(x) = 0$

(d) $f(\mu + x) = f(\mu - x)$

↳ Density is symmetric about μ.

(e) Max value at μ,

(f) point of inflection of f are $\mu \pm \sigma$.

$\left\{ y = \frac{x-\mu}{\sigma} \text{ gen. used} \right\}$

- Mean = $E(X) = \mu$
- $V(X) = \sigma^2$
- $\boxed{M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$

Standard Normal Distribution :-

$z = \frac{X-\mu}{\sigma}$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad , -\infty < z < \infty$$

- Mean = 0 = Median
- Variance = 1
- $M_Z(t) = e^{t^2/2}$

Note :- if $Z \sim N(0,1) \rightarrow$ we say Z has a Standard Normal Distribution

$Z \rightarrow$ Standard Normal Random Variable

$$\boxed{Z = \frac{X-\mu}{\sigma}}$$

$\phi(z) \rightarrow$ Corresponding distribution function

$$\boxed{F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right) = \phi(z)}$$

Cauchy Density :-

$$g(x) = \frac{1}{\pi(1+x^2)}$$

Mode = 3 × Median − 2 × Mean

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