Special Probability Distributions

DISCRETE

- Discrete uniform
- Binomial
- Geometric
- Negative binomial
- Hypergeometric
- Poisson
- Multinomial

CONTINUOUS

- Uniform
- Exponential
- Gamma
- Weibull
- Pareto
- Beta
- Normal
- Cauchy
- Log-Normal

Reliability of Series & Parallel systems.

PROBLEMS

6 lectures

Rectangular or Uniform distribution

A RV is said to have continuous rectangular or uniform distribution over the interval (a, b) $(-\infty < a < b < \infty)$ if pdf is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

→ a & b are parameters

→ is written as $X \sim U(a, b)$

The cumulative distribution fn is.

$$F(x) = \begin{cases} 0 & \text{if } x \leq a \\ \dfrac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b. \end{cases}$$

$F(x)$ is continuous & not diff at $x = a$ & b

Trams on a certain line run every half hour between midnight and six in the morning. What is the probability that a man entering the station at random time during this period will have to wait atleast 20 min?

$\rightarrow$ $X \sim U(0, 30)$, $X \equiv$ waiting time

$$\Rightarrow P(X \geq 20) = \int_{20}^{30} f(x)\,dx = \frac{1}{30}\int_{20}^{30} dx = \frac{1}{3}.$$

$X \sim U(0,1)$ Find pdf of $Y = -2\log X$.

We know, $F(y) = P(Y \leq y) = P(-2\log x \leq y)$

$$= P\left(\log X \geq -\frac{y}{2}\right)$$

$$= 1 - P\left(X \leq e^{-y/2}\right)$$

$$= 1 - \int_{0}^{e^{-y/2}} dx = 1 - e^{-y/2}$$

Hence pdf of $Y = \frac{d}{dy}F(y) = \frac{1}{2}e^{-y/2}$

$0 < y < \infty.$

Exponential distⁿ.

A RV X is said to have exponential distⁿ. with parameter $\theta > 0$, if its pdf is given by

$$f(x,\theta) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

cdf is

$$F(x) = \int_0^x f(t)\,dt = \theta \int_0^x e^{-\theta t}\,dt$$

$$= \begin{cases} 1 - e^{-\theta x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Moment generating fⁿ.

$$M_X(t) = E\left(e^{tx}\right) = \theta \int_0^\infty e^{tx} \cdot e^{-\theta x}\,dx.$$

$$= \frac{\theta}{\theta - t} = \left(1 - \frac{t}{\theta}\right)^{-1}$$

$$= \sum_{r=0}^{\infty} \left(\frac{t}{\theta}\right)^r, \quad \theta > t.$$

Moments

$$\mu_r' = E(x^r) = \text{Coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

$$= \frac{r!}{\theta^r}. \quad r = 1, 2, \ldots$$

$$\text{Mean} = \mu_1' = \frac{1}{\theta}.$$

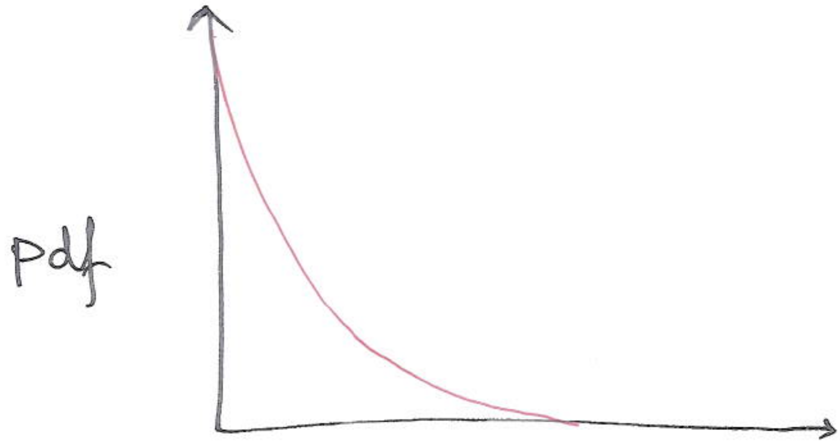
$$\text{Variance} = \mu_2' - \mu_1'^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}.$$

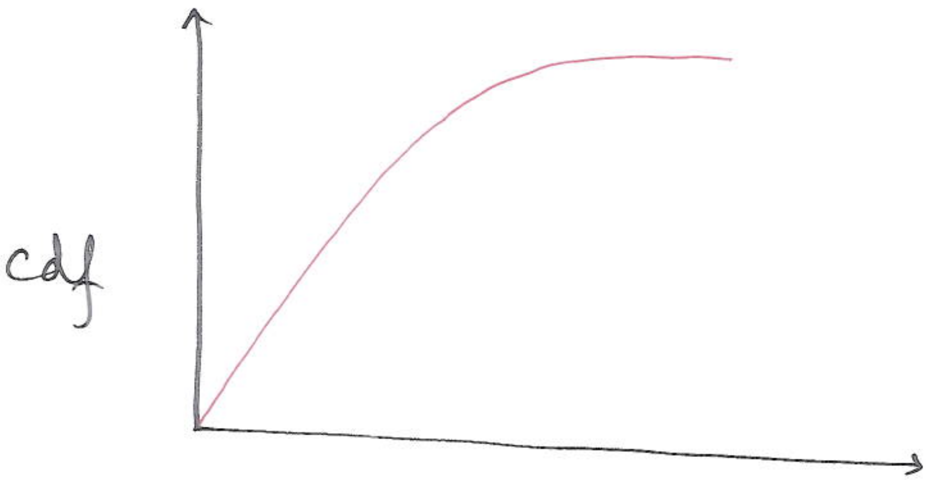
Note
Variance > Mean if $0 < \theta < 1$
Variance = Mean if $\theta = 1$
Variance < Mean if $\theta > 1$.

$$\boxed{\text{Variance} = \frac{\text{Mean}}{\theta}}$$

Exponential dist^n. has memoryless property.

Exponential dist^n:





An electronic component is known to have a useful life represented by an exponential distⁿ with failure rate 10^{-5} failure per hour. (i.e. $\theta = 10^{-5}$). The mean time to failure is $E(X)$ is thus 10^{5} hours. Suppose we want to determine the fraction of such components that would fail before the mean life or expected life is

$$P\left(T < \frac{1}{\theta}\right) = \int_0^{1/\theta} \theta e^{-\theta x}\,dx = 1 - \frac{1}{e} = \cdot 63.$$

Memoryless property of Exponential Distn.

$$P(X > s+t \,/\, X > t) = P(X > s)$$

$\Downarrow$ prob. that an item that is functioning at time t will continue to work an additional time s.

$\Downarrow$ Prob. that item is functioning s time unit.

$\Rightarrow$ No need to remember the age of functioning item As $\downarrow$

as long as it is functioning it is as good as 'new'.

LHS $= \dfrac{P(X > s+t \,,\, X > t)}{P(X > t)}$

$= \dfrac{P(X > s+t)}{P(X > t)}$

$= \dfrac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$

Suppose that an amount of time one spends in a bank is exponentially distributed with mean 10 minutes.

(i) What is the prob. that a customer will spend more than 15 minutes in the bank?

(ii) What is the probability that the customer will spend more than 15 mins. in the bank given that he is still in the bank after 10 mins.

Ans.

(i) $P(X > 15) = e^{-15\lambda} = e^{-3/2} = \cdot 2$

$\lambda = \frac{1}{10}$.

(ii) $P(X > 15 / X > 10)$

$= P(X > 5)$

$= e^{-5\lambda} = \cdot 6$.

Normal Distⁿ.

$X \sim N(\mu, \sigma^2)$ or $N(\mu, \sigma)$

$\Rightarrow$ pdf $\boxed{f(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty}$

Standard Normal varite

$$Z \sim N(0, 1)$$

Where, $Z = \dfrac{X-\mu}{\sigma}$ & $X \sim N(\mu, \sigma)$

pdf $\boxed{\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty}$

cdf $\boxed{\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi(z)\,dz}$

Properties of Φ

$\rightarrow$ $\Phi(-z) = 1 - \Phi(z)$

$\rightarrow$ $P(a \leq \underset{\text{Normal}}{\underset{\downarrow}{X}} \leq b) = \Phi\left(\dfrac{b-\mu}{\sigma}\right) - \Phi\left(\dfrac{a-\mu}{\sigma}\right)$

Properties of Normal distⁿ.

→ The curve is bell-shaped and symmetrical about $x = \mu$

→ Mean is at the middle and divides the area into halves

→ It is completely determined by μ & σ

→ Mean, Mode and Median of distⁿ. coincide

→ skewness = 0 , Kurtosis $= \frac{\mu_4}{\sigma^4} = \frac{E(x-\mu)^4}{[E(x-\mu)^2]^2} = 3$

→ Points of inflexion are at $x = \mu+\sigma$ & $\mu-\sigma$

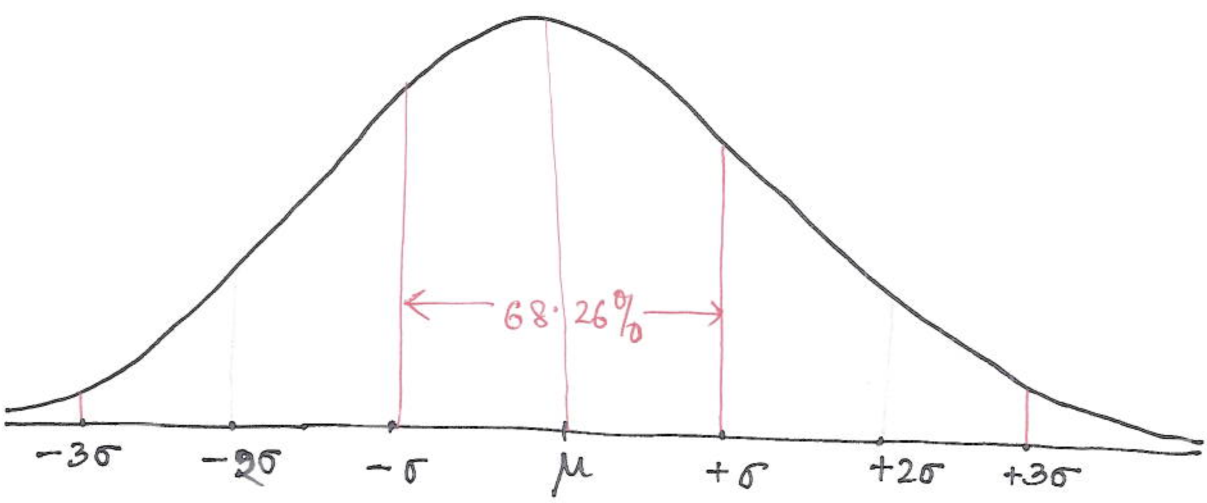
→ Area property :

$$P(\mu-\sigma \leq X \leq \mu+\sigma) = \cdot 6826$$

$$P(\mu-2\sigma \leq X \leq \mu+2\sigma) = \cdot 9544$$

$$P(\mu-3\sigma \leq X \leq \mu+3\sigma) = \cdot 9972$$

→ Total area under the curve is 1.



Moment generating function

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2 - 2tz\sigma\right)} dz$$

take $z = \frac{x-\mu}{\sigma}$.

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - \sigma t)^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \qquad \boxed{\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du}$$

$$\boxed{mgf = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

$X \sim N(70, 4)$. Find

(a) $P(68 \leq X \leq 74) = .817$

(b) $P(72 \leq X \leq 75) = .1525$

(c) $P(63 \leq X \leq 68) = .1584$

(d) $P(X \geq 73) = .0668$.

(a) $P(68 \leq X \leq 74) = P\left(\frac{68-70}{2} \leq \frac{X-70}{2} \leq \frac{74-70}{2}\right)$

$= P(-1 \leq Z \leq 2)$

$= \Phi(2) - \Phi(-1)$

$= \Phi(2) + \Phi(1) - 1$

$= .97725 + .84134 - 1$

$= .8185$

(b) $P(72 \leq X \leq 75) = P\left(\frac{72-70}{2} \leq \frac{X-70}{2} \leq \frac{75-70}{2}\right)$

$= P(1 \leq Z \leq 2.5)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$f'(x) = -\frac{f(x)}{\sigma^2}(x-\mu)$$

$$f''(x) = -\frac{f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

At $x = \mu$, $f'(x) = 0$, $f''(x) < 0$

$\Rightarrow$ Mode

At $x = \mu \pm \sigma$ $\quad f''(x) = 0$, $f'''(x) \neq 0$

$\Rightarrow$ Point of inflexion

▨ Let $X_i \sim N(\mu_i, \sigma_i^2)$, then for n independent X_i's

$$\sum_{i=1}^{n} a_i X_i \sim N\left[\sum_{i=1}^{n} a_i \mu_i,\ \sum_{i=1}^{n} a_i^2 \sigma_i^2\right]$$

▨ $\bar{X} \sim N(\mu, \sigma^2/n)$, where $\bar{X} = \frac{1}{n}\sum X_i$

& $X_i \sim N(\mu, \sigma^2)$

Proof: As $M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$

$$M_{\sum a_i X_i}(t) = \prod_{i=1}^{n} M_{a_i x_i}(t)$$

$$= M_{X_1}(a_1 t)\, M_{X_2}(a_2 t) \cdots M_{X_n}(a_n t)$$

$$\left[\text{as, } M_{cX}(t) = M_X(ct)\right]$$

$$= \prod_{i=1}^{n} e^{\left(a_i \mu_i t + \frac{1}{2} a_i^2 \sigma_i^2 t\right)}$$

$$= e^{(a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n)}$$

$$\times$$

$$e^{\frac{1}{2}t^2\left(a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2\right)}$$

$$= e^{\mu t + \frac{1}{2}\sigma t^2}$$

Given $X \sim N(12, 4)$. Find probability of

(a) (i) $(X \geqslant 20)$ (ii) $(X \leqslant 20)$ (iii) $12 \leqslant X \leqslant 18$

$\cdot 00003$ $\cdot 99997$

(b) Find x' when $Prob(x > x') = \cdot 24$

$13 \cdot 42$

(c) Find x_0' and x_1' when $Prob(x_0' < X < x_1') = \cdot 5$

$10 \cdot 64$.

and $Prob(X > x_1') = \cdot 25$.

$13 \cdot 36$

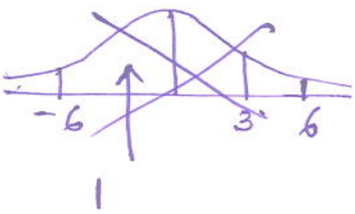
$$P(X \geq 20) = 1 - P(X \leq 20) = 1 - P\left(\frac{X-12}{2} \leq \frac{20-12}{2}\right)$$

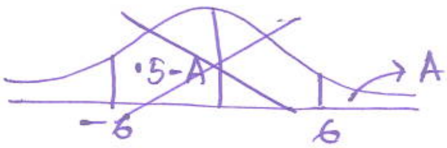
$$= 1 - P(Z \leq 4)$$

$$= 1 - .99997 = \boxed{.00003}$$

$$P(X \leq 20) = \boxed{.99997}$$

$$P(12 \leq X \leq 18) = P\left(\frac{12-12}{2} \leq x \leq \frac{18-12}{2}\right) = P(0 \leq x \leq 3)$$





$$= .99865 - .5$$

$$= .49865$$

$$P(x > x') = .24 \Rightarrow P(x < x') = 1 - .24 = .76$$

$$\Rightarrow P\left(Z < \frac{x'-12}{2}\right) = .76$$

$$\Rightarrow \frac{x'-12}{2} = .71$$

$$\text{or } x' = 12 + 1.42 = \boxed{13.42}$$

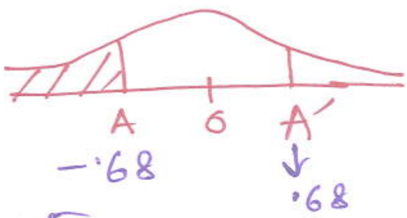
$$P(x > x_1') = .25 \Rightarrow 1 - P(x < x_1') = .25$$

$$\Rightarrow P(x < x_1') = .75 \Rightarrow P\left(Z < \frac{x_1'-12}{2}\right) = .75$$

$$\Rightarrow \frac{x_1'-12}{2} = .68 \quad \text{or, } \boxed{x_1' = 13.36}$$

$$P(x_0' < x < 13.36) = .5$$

$$\text{or } P\left(\frac{x_0'-12}{2} < \frac{x-12}{2} < \frac{13.36-12}{2}\right) = .5$$



$$\text{or } \phi(.68) - \phi\left(\frac{x_0'-12}{2}\right) = .5$$

$$\text{or, } .75 - \phi(\,) = .5 \quad \text{or } \phi\left(\frac{x_0'-12}{2}\right) = .25$$

$$\text{or, } x_0' - 12 = -1.36 \quad \text{or, } \boxed{x_0' = 10.64}$$

X is a normal variate with mean 30 and s.d. 5. Find the probability that

(i) $(26 \leq X \leq 40)$ (ii) $(X \geq 45)$ (iii) $|X - 30| > 5$.

$\cdot 7653$ $\cdot 00135$ $\cdot 68268$

$\Downarrow$

$1 - P(25 \leq X \leq 35$

(i) $P(26 \leq X \leq 40) = P\left(\frac{26-30}{5} \leq \frac{X-30}{5} \leq \frac{40-30}{5}\right)$

$= P(-{\cdot}8 \leq Z \leq 2) = P(2) - P(-{\cdot}8) = P(2) + P({\cdot}8) - 1$

$= {\cdot}97725 + {\cdot}78811 - 1$

$= \boxed{{\cdot}76539}$

(ii) $P(X \geq 45) = 1 - P(X \leq 45) = 1 - P\left(\frac{X-30}{5} \leq \frac{45-30}{5}\right)$

$= 1 - P(Z \leq 3) = 1 - {\cdot}99865$

$= \boxed{{\cdot}00135}$

(iii) $P(|X-30| > 5) = P(25 \leq X \leq 35)$.

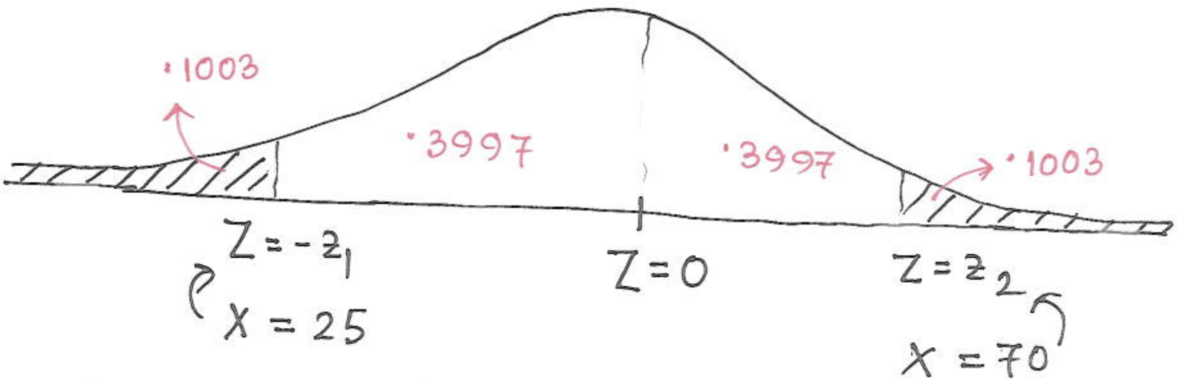
$= P(-1 \leq Z \leq 1) =$

$= 2P(Z \leq 1) - 1 = 2 \times {\cdot}84134 - 1$

$= \boxed{{\cdot}68268}$

In a distribution, which is given normal, 10·03% of the items are under 25 kg weight and 89·97% items are under 70 kg weight. What are the mean and s.d. of the distribution?

[$P(0 < Z < 1\cdot 28) = \cdot 3997$.



$\mu = 47\cdot 5$ $\sigma = 17\cdot 578$

$$\frac{25-\mu}{\sigma} = 1\cdot 28$$

$$\frac{70-\mu}{\sigma} = 1\cdot 28$$

Mean deviation about mean

$$= \int_{-\infty}^{\infty} |x-\mu| \, f(x) \, dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} |x-\mu| \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Take $\frac{x-\mu}{\sigma} = Z$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| \, e^{-z^2/2} \, dz$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} |z| \, e^{-z^2/2} \, dz$$

Take $\frac{z^2}{2} = t$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \, dt$$

$$= \sqrt{\frac{2}{\pi}} \, \sigma \left[-e^{-t}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}} \, \sigma = \frac{4}{5} \, \sigma \text{ (Approx.)}$$

Two independent RVs X and Y are both normally distributed with means 1 & 2 and s.d. 3 & 4 resp. If $Z = X - Y$, write the pdf of Z. And Find mean, median and s.d. of Z. Also find $P(Z+1 \leq 0)$.

$$Z = X - Y \sim N(1-2, \; 9+16)$$

$$f(Z) = \frac{1}{\sqrt{2\pi} \times 5} \, e^{-\frac{1}{2}\left(\frac{Z+1}{5}\right)^2}$$

$\Rightarrow$ Mean = Median = -1 & s.d. 5.

$P(Z \leq -1) = \cdot 5.$

Log-normal distribution

The positive RV X is said to have a log-normal distribution if $\log_e X$ is normally distributed.

$$Y = \log_e X \sim N(\mu, \sigma^2) \quad \text{for } x > 0$$

OR

If $X \sim N(\mu, \sigma^2)$, $Y = e^X$ is log-normal.

$$F_X(x) = P(X \leq x) = P(\log_e X \leq \log_e x)$$

$$= P(Y \leq \log_e x)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\log_e x} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{0}^{u} e^{-\frac{1}{2}\left(\frac{\log u - \mu}{\sigma}\right)^2} \frac{du}{u} \quad [y = \log u]$$

$$\Rightarrow \boxed{f(x) = \frac{1}{x\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2}, \quad x > 0.}$$

r^{th} moment $= \mu_r' = E(X^r) = E\left(e^{rY}\right)$

(since $Y = \log X$, or $X = e^Y$)

As $Y \sim N(\mu, \sigma^2)$, X is log-normal

$E\left(e^{rY}\right) = \text{mgf of } Y = e^{\mu r + \frac{1}{2} r^2 \sigma^2}$

Mean $= \mu_1' = e^{\mu + \frac{1}{2}\sigma^2}$

$\mu_2' = e^{2\mu + 2\sigma^2}$

Variance $= \mu_2' - \mu_1'^2 = e^{2(\mu + \sigma^2)} - e^{2\mu + \sigma^2}$

$= e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$

$= \mu_1'\left(e^{\sigma^2} - 1\right)$