# Mathematics for AI

Adams Wai Kin Kong
School of Computer Science and Engineering
Nanyang Technological University, Singapore
adamskong@ntu.edu.sg

# Vectors

# Vectors

- The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.

- A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

- In fact, any point in $\Re^n$ can be considered a vector from origin.

# Vectors

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

# Vectors

- More generally, we will consider the set $V_n$ of all $n$-dimensional vectors.

- An $n$-dimensional vector is an ordered $n$-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\mathbf{a}$.

# Components

- Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

**Properties of Vectors** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $V_n$ and $c$ and $d$ are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Note that **0** is the zero vector <0, 0, …., 0 >

# Components

- Three vectors in $V_3$ play a special role. Let

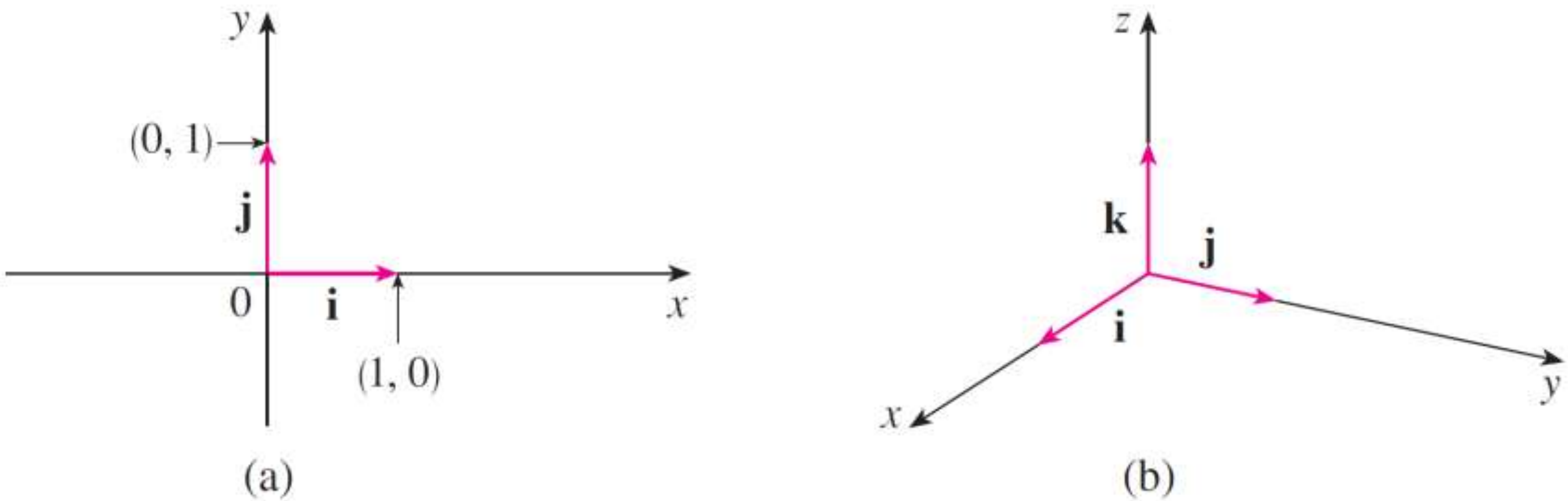
$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

- These vectors **i**, **j**, and **k** are called the **standard basis vectors**.

- They have length 1 and point in the directions of the positive *x*-, *y*-, and *z*-axes.

# Components

- Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.



# Components

- In an n-dimensional vector space, **standard basis vectors** are $e_1, e_2, \cdots e_n$, where $e_i =< 0, \cdots, 1, \cdots, 0 >$. More clearly, all the elements in $e_i$ are zero, except for the $i^{th}$ element, which is one.

- $e_i$ is commonly considered as a column vector.

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

# Components

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$
$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Thus any vector in $V_3$ can be expressed in terms of **i**, **j**, and **k**. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, for a n-dimensional vector $\mathbf{a} = \langle a_1, a_2, \ldots a_n \rangle$, **a** can be rewritten as $\boldsymbol{a} = \sum_{i=1}^{n} a_i e_i$.

# Components

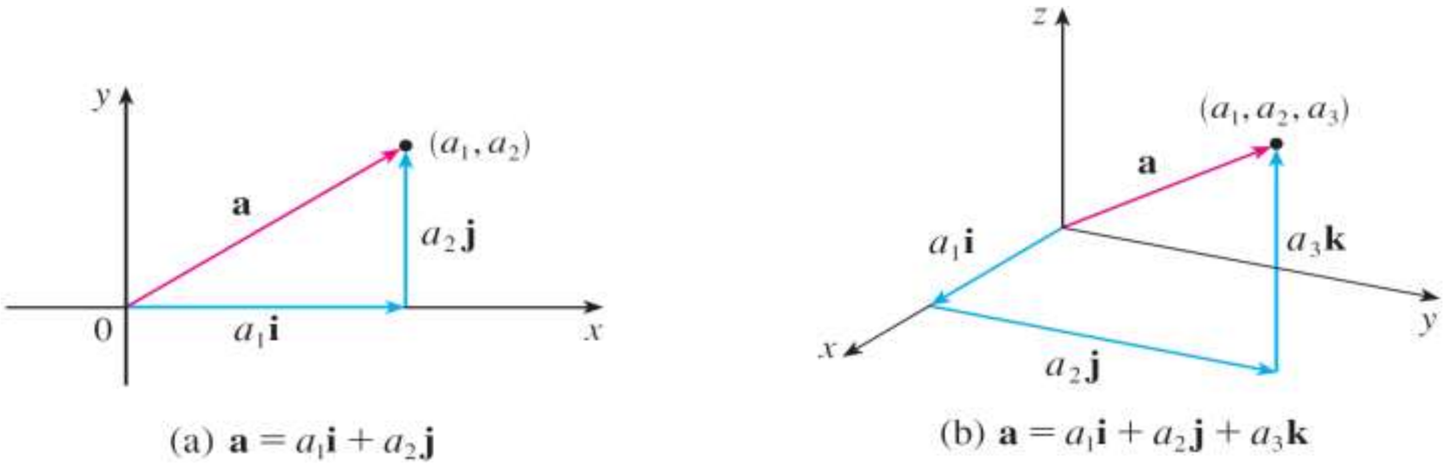



**Figure 18**

# Unit vector

- A **unit vector** is a vector whose length is $1$. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

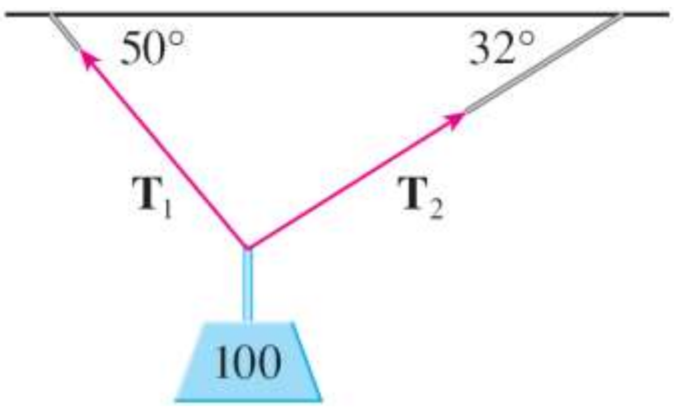
$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

- In order to verify this, we let $c = \dfrac{1}{\|\mathbf{a}\|}$. Then $\mathbf{u} = c\mathbf{a}$ and $c$ is a positive scalar, so **u** has the same direction as **a**. Also

$$\|\mathbf{u}\| = \|c\mathbf{a}\| = c\|\mathbf{a}\| = 1$$

# Applications

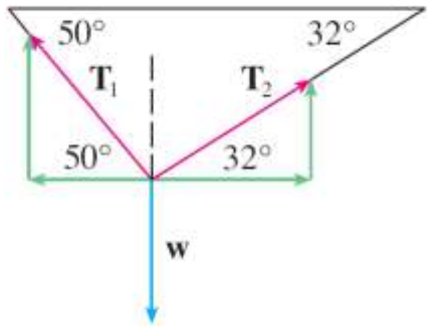
- A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.



# Applications

- We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From the figure we see that

$$\mathbf{T}_1 = -|\mathbf{T}_1| \cos 50° \mathbf{i} + |\mathbf{T}_1| \sin 50° \mathbf{j}$$
$$\mathbf{T}_2 = |\mathbf{T}_2| \cos 32° \mathbf{i} + |\mathbf{T}_2| \sin 32° \mathbf{j}$$




The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w} = -100\,\mathbf{j}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\,\mathbf{j}$$

# Applications

Thus

$$(-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32°)\,\mathbf{i} + (|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32°)\,\mathbf{j}$$

$$= 100\,\mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32° = 0$$

$$|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32° = 100$$

# Applications

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1|\sin 50^\circ + \frac{|\mathbf{T}_1|\cos 50^\circ}{\cos 32^\circ}\sin 32^\circ = 100$$

$$|\mathbf{T}_1|\left(\sin 50^\circ + \cos 50^\circ\,\frac{\sin 32^\circ}{\cos 32^\circ}\right) = 100$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb}$$

and $$|\mathbf{T}_2| = \frac{|\mathbf{T}_1|\cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb}$$

# Applications

Considering an image as a vector $I$, what does the equation below do?

$$I_r = aI + b\mathbb{1}$$

where $a, b \in \Re$ and $\mathbb{1} =< 1, 1, \cdots 1 >$.

$= 1.2 \times$

$+100\mathbb{1}$

# The Dot Product

# The Dot Product

- So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows.

> **1 Definition** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by
>
> $$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Thus, to find the dot product of **a** and **b**, we multiply corresponding components and add.

# Example

$$\langle 2, 4\rangle \cdot \langle 3, -1\rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4\rangle \cdot \langle 6, 2, -1/2\rangle = (-1)(6) + 7(2) + 4(-1/2) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$

# The Dot Product

- The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

**2 Properties of the Dot Product** If **a**, **b**, and **c** are vectors in $V_3$ and $c$ is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Remarks

1) |a| is the norm of a. It is also denoted as ||a||.
2) **0** is a zero vector and 0 is a number.
3) The properties are valid for n-dimensional vectors.
4) $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b}$ if a and b are column vectors, where T represents transpose.

# The Dot Product

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3. $$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3 \\ &= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}\end{aligned}$$

# The Dot Product

- The dot product **a** · **b** can be given a geometric interpretation in terms of the **angle** $\theta$ **between a and b**, which is defined to be the angle between the representations of **a** and **b** that start at the origin, where $0 \leq \theta \leq \pi$.

- In other words, $\theta$ is the angle between the line segments *OA* and *OB* in Figure 1. Note that if **a** and **b** are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

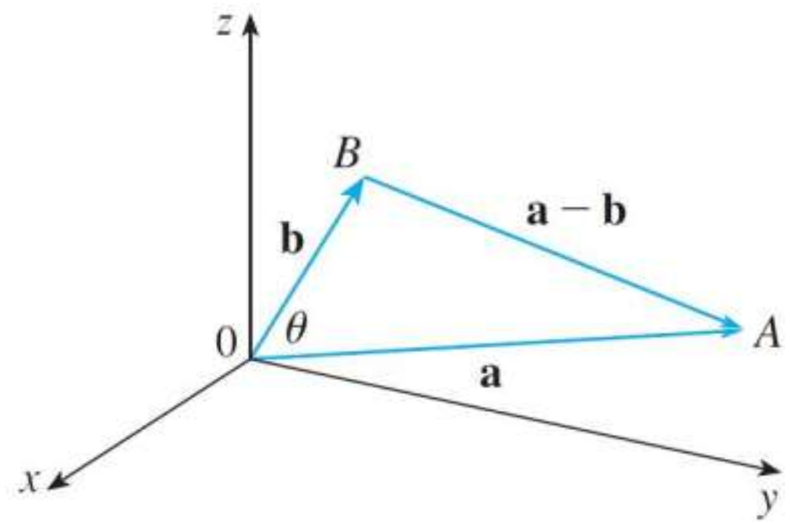



**Figure 1**

# The Dot Product

- The formula in the following theorem is used by physicists as the *definition* of the dot product

**3 Theorem** If $\theta$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

Example: If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$,

$$\mathbf{a} \bullet \mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos(\pi/3) = 4 \bullet 6 \bullet \mathbf{1/2=12}$$

# The Dot Product

- The formula in Theorem 3 also enables us to find the angle between two vectors.

**6 Corollary** If $\theta$ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

Example: Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1\rangle$ and $\mathbf{b} = \langle 5, -3, 2\rangle$.

Since $|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \quad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$

and $\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{2}{3\sqrt{38}} \qquad \theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84°)$$

# The Dot Product

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos \theta = 0$, so $\theta = \pi/2$. The zero vector **0** is considered to be perpendicular to all vectors.

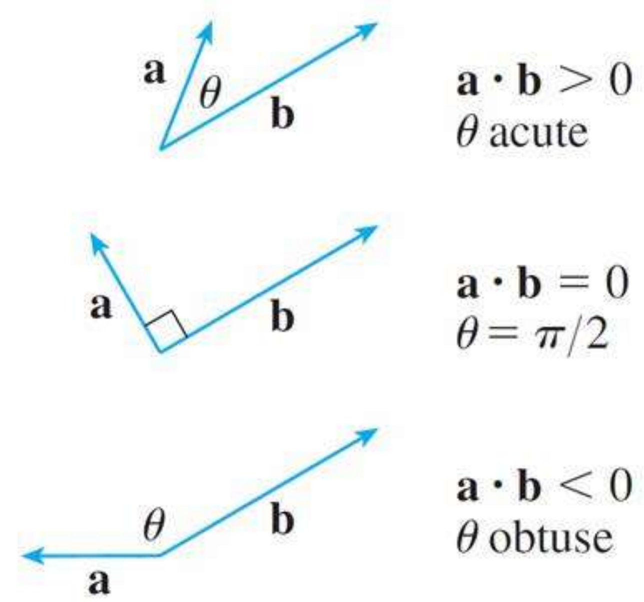
Therefore we have the following method for determining whether two vectors are orthogonal.

| | |
|---|---|
| 7 | Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. |

# The Dot Product

Because cos $\theta > 0$ if $0 \leq \theta < \pi/2$ and cos $\theta < 0$ if $\pi/2 < \theta \leq \pi$, we see that **a** · **b** is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of **a** · **b** as measuring the extent to which **a** and **b** point in the same direction.

The dot product **a** · **b** is positive if **a** and **b** point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2).



# The Dot Product

In the extreme case where **a** and **b** point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$$

If **a** and **b** point in exactly opposite directions, then we have $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$.

# Direction Angles and Direction Cosines

- The **direction angles** of a nonzero vector **a** are the angles $\alpha$, $\beta$, and $\gamma$ (in the interval $[0, \pi]$) that **a** makes with the positive *x*-, *y*-, and *z*-axes, respectively. (See Figure 3.)

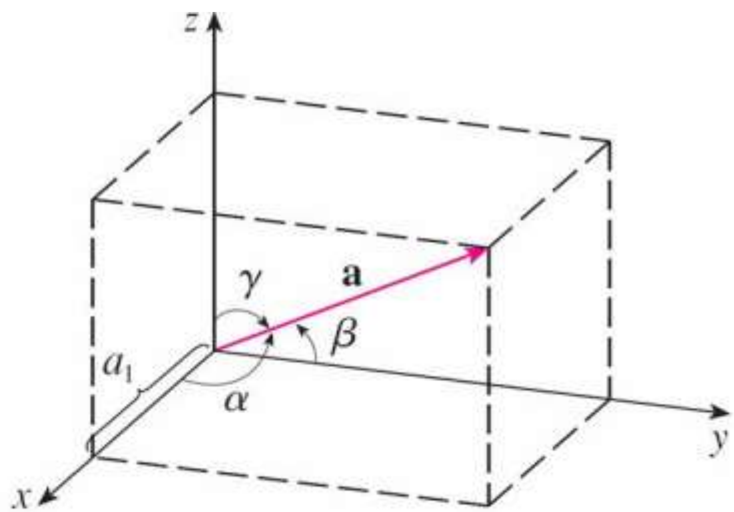



**Figure 3**

# Direction Angles and Direction Cosines

The cosines of these direction angles, cos $\alpha$, cos $\beta$, and cos $\gamma$, are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain

$$\boxed{8} \qquad \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}|\,|\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.) Similarly, we also have

$$\boxed{9} \qquad \cos \beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

Note $i =< 1, 0,0 >$, j$=< 0\ 1,0 >$, and k $=< 0, 0,1 >$

# Direction Angles and Direction Cosines

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{10}$$

We can also use Equations 8 and 9 to write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle$$

$$= |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

# Direction Angles and Direction Cosines

Therefore

$$\boxed{11} \qquad \frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

# Example

Find the direction angles of the vector **a** = ⟨1, 2, 3⟩.

## Solution:

Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

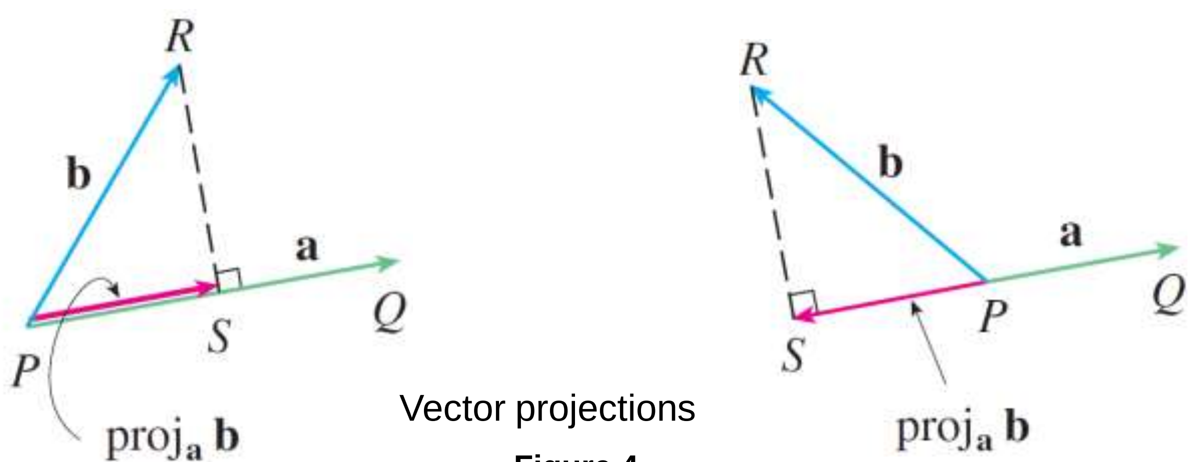
$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

and so

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ \qquad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ \qquad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

# Projections

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point $P$. If $S$ is the foot of the perpendicular from $R$ to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\, \mathbf{b}$.
(You can think of it as a shadow of **b**).

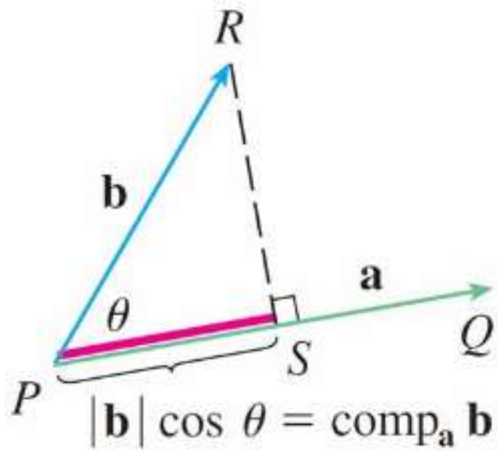



Vector projections

**Figure 4**

# Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}| \cos \theta$, where $\theta$ is the angle between **a** and **b**.
(See Figure 5.)




Scalar projection

**Figure 5**

# Projections

This is denoted by $\text{comp}_{\mathbf{a}}\, \mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta = |\mathbf{a}|(|\mathbf{b}| \cos \theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

# Projections

We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

# Example

Find the scalar projection and vector projection of **b** = ⟨1, 1, 2⟩ onto **a** = ⟨–2, 3, 1⟩.

## Solution:

Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}}\, \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

$$= \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}$$

# Example

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\,\frac{\mathbf{a}}{|\mathbf{a}|}$$

$$= \frac{3}{14}\,\mathbf{a}$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

# Projections

The work done by a constant force $F$ in moving an object through a distance $d$ as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6.

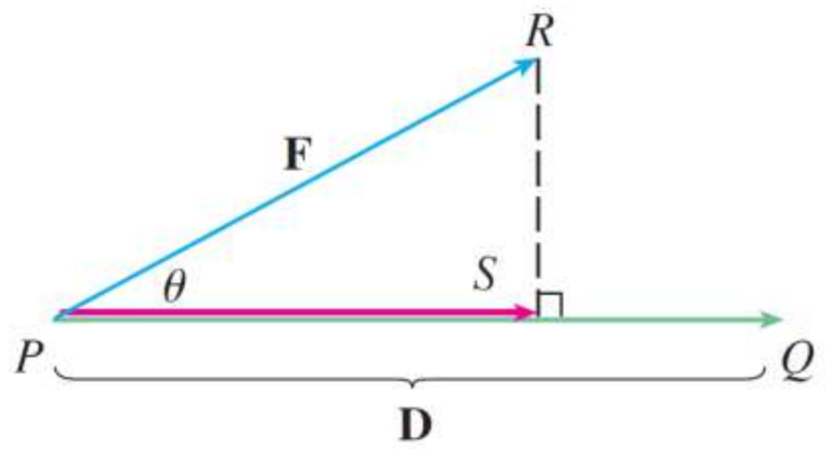



Figure 6

# Projections

If the force moves the object from $P$ to $Q$, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:

$$W = (|\mathbf{F}| \cos\theta)\ |\mathbf{D}|$$

But then, from Theorem 3, we have

12 $$W = |\mathbf{F}|\ |\mathbf{D}| \cos\theta = \mathbf{F} \cdot \mathbf{D}$$

Thus the work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D}$ is the displacement vector.

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# Example

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

Solution:

If **F** and **D** are the force and displacement vectors, as pictured in Figure 7, then the work done is

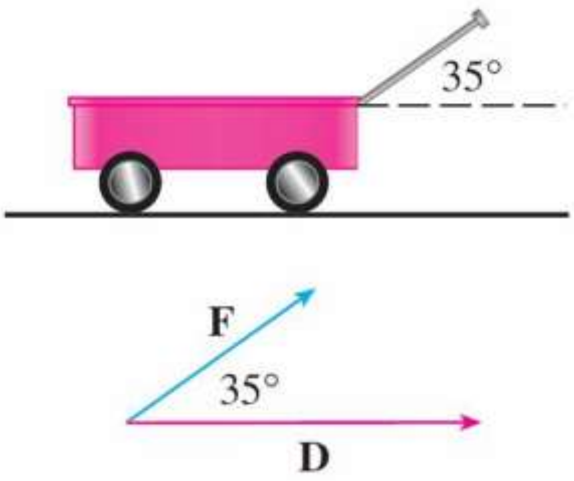


Figure 7

$$W = \mathbf{F} \cdot \mathbf{D}$$
$$= |\mathbf{F}|\,|\mathbf{D}| \cos 35°$$

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# Example

$$= (70)(100)\cos 35^\circ$$

$$\approx 5734\ \mathrm{N{\cdot}m}$$

$$= 5734\ \mathrm{J}$$

# Lines and Planes

# Lines

A line in the *xy*-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given.

The equation of the line can then be written using the point-slope form.

Likewise, a line $L$ in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on $L$ and the direction of $L$. In three dimensions the direction of a line is conveniently described by a vector, so we let $\mathbf{v}$ be a vector parallel to $L$.

# Lines

Let $P(x, y, z)$ be an arbitrary point on $L$ and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of $P_0$ and $P$ (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$ ).

If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$.

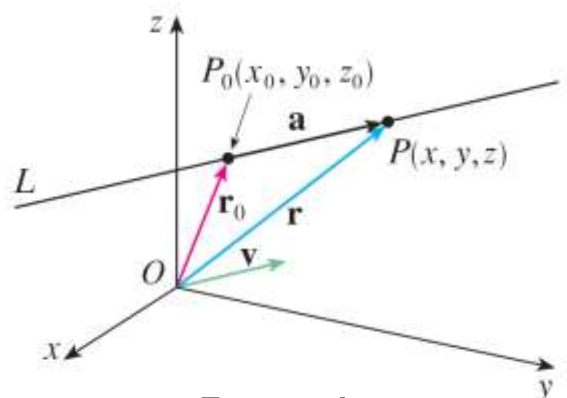


Figure 1

# Lines

But, since **a** and **v** are parallel vectors, there is a scalar $t$ such that $\mathbf{a} = t\mathbf{v}$. Thus

1 $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of $L$.

Each value of the **parameter** $t$ gives the position vector $\mathbf{r}$ of a point on $L$. In other words, as $t$ varies, the line is traced out by the tip of the vector $\mathbf{r}$.

# Lines

As Figure 2 indicates, positive values of $t$ correspond to points on $L$ that lie on one side of $P_0$, whereas negative values of $t$ correspond to points that lie on the other side of $P_0$.

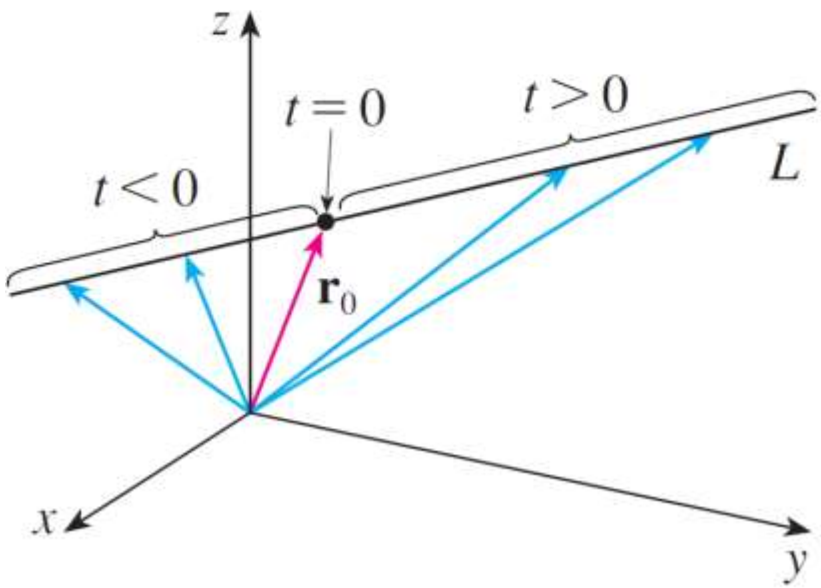


Figure 2

# Lines

If the vector **v** that gives the direction of the line $L$ is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$.

We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal.

# Lines

Therefore we have the three scalar equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

where $t \in \mathbb{R}$.

These equations are called **parametric equations** of the line $L$ through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$.

Each value of the parameter $t$ gives a point $(x, y, z)$ on $L$.

# Lines

**2** Parametric equations for a line through the point $(x_0, y_0, z_0)$ and parallel to the direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

# Example

**(a)** Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

**(b)** Find two other points on the line.

Solution:

Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or $$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

# Example

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

**(b)** Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line.

Similarly, $t = -1$ gives the point $(4, -3, 5)$.

# Lines

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change.

For instance, if, instead of (5, 1, 3), we choose the point (6, 5, 1), then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

# Lines

Or, if we stay with the point (5, 1, 3) but choose the parallel vector 2**i** + 8**j** – 4**k**, we arrive at the equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector **v** = $\langle a, b, c \rangle$ is used to describe the direction of a line $L$, then the numbers $a$, $b$, and $c$ are called **direction numbers** of $L$.

Since any vector parallel to **v** could also be used, we see that any three numbers proportional to $a$, $b$, and $c$ could also be used as a set of direction numbers for $L$.

# Lines

Another way of describing a line *L* is to eliminate the parameter *t* from Equation 2.

If none of *a*, *b*, or *c* is 0, we can solve each of these equations for *t*:

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \tag{3}$$

These equations are called **symmetric equations** of *L*.

# Lines

- Notice that the numbers *a*, *b*, and *c* that appear in the denominators of Equations 3 are direction numbers of *L*, that is, components of a vector parallel to *L*.

- If one of *a*, *b*, or *c* is 0, we can still eliminate *t*. For instance, if *a* = 0, we could write the equations of *L* as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- This means that *L* lies in the vertical plane $x = x_0$.

# Lines

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \le t \le 1$.

$\boxed{4}$ The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

# Lines

The line equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is valid for high dimensional space.

Although we can derive an equation similar Eq 3 for the high dimensional case, we don't use it. We normally use the equation in vector form i.e., $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

# Planes

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe.

A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction.

Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector **n** that is orthogonal to the plane. This orthogonal vector **n** is called a **normal vector**.

# Planes

Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of $P_0$ and $P$. Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. (See Figure 6.)

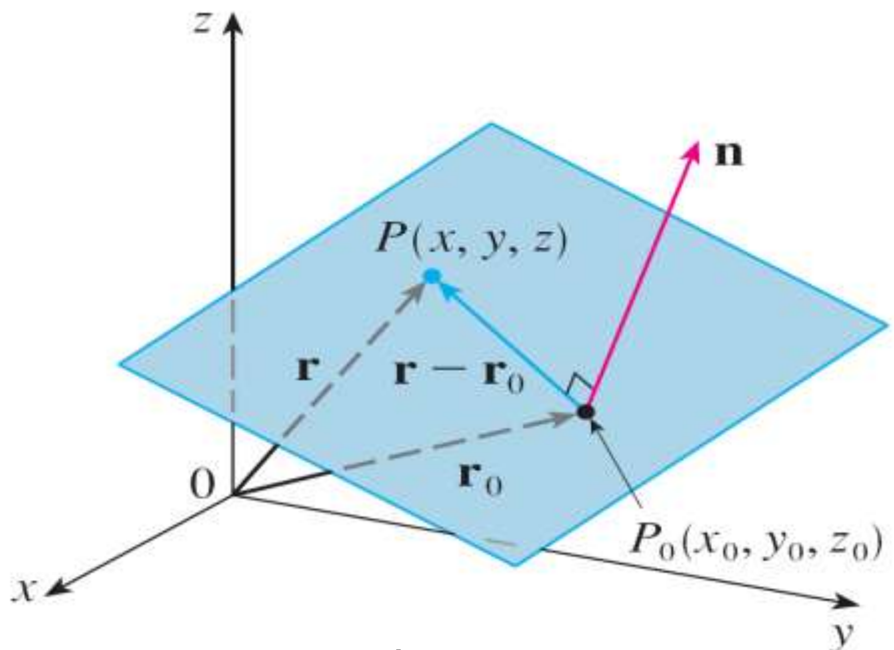


**Figure 6**

# Planes

The normal vector **n** is orthogonal to every vector in the given plane. In particular, **n** is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

**5** $$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be rewritten as

**6** $$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

# Planes

To obtain a scalar equation for the plane, we write

$\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$.

Then the vector equation (5) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

**7** A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

# Example

Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Solution:

Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

To find the $x$-intercept we set $y = z = 0$ in this equation and obtain $x = 6$.

# Example

Similarly, the $y$-intercept is 4 and the $z$-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7).

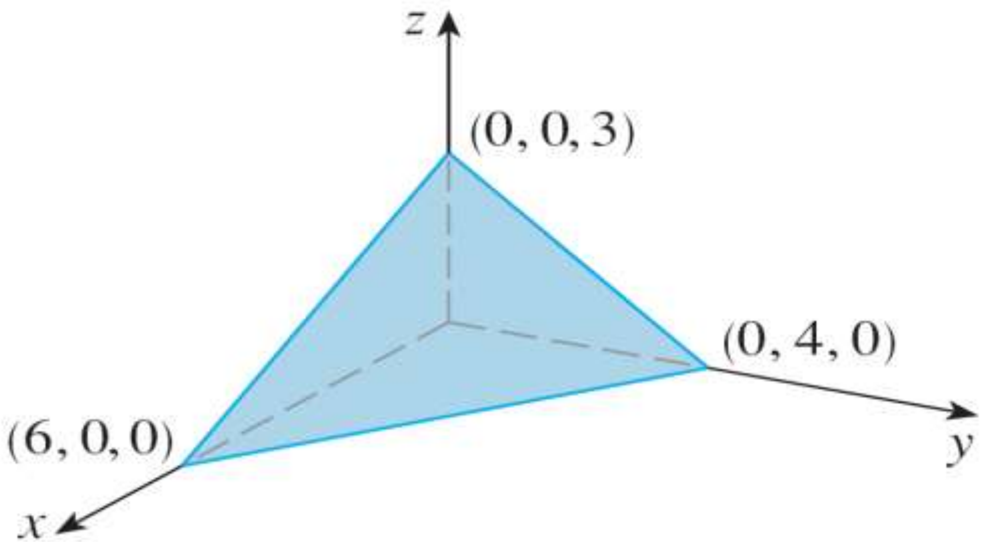


Figure 7

# Planes

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

**8**

$$\boxed{ax + by + cz + d = 0}$$

where $d = -(ax_0 + by_0 + cz_0)$.

Equation 8 is called a **linear equation** in $x$, $y$, and $z$. Conversely, it can be shown that if $a$, $b$, and $c$ are not all 0, then the linear equation (8) represents a plane with normal vector $\langle a, b, c \rangle$.

# Planes

Two planes are **parallel** if their normal vectors are parallel.

For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$.

If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle $\theta$ in Figure 9).

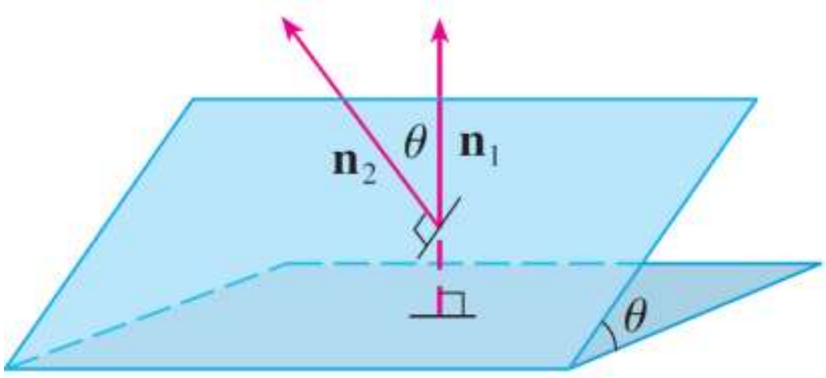


Figure 9

# Hyperplanes

If **n, r** and $\mathbf{r}_0$ are n-dimensional vectors, equations 5 and 6 are still valid. If these vectors are column vectors,

$$\mathbf{n}\cdot(\mathbf{r}-\boldsymbol{r}_{\mathbf{0}}) = \mathbf{n}^{T}(\mathbf{r}-\boldsymbol{r}_{\mathbf{0}}) = 0$$

Let $\mathbf{n}^T = [\mathrm{a}_1, \cdots, \mathrm{a}_\mathrm{n}]$, $\mathbf{r}^T = [x_1, \cdots, x_\mathrm{n}]$ and $\boldsymbol{r}_{\mathbf{0}}^{T} = [c_1, \cdots, c_\mathrm{n}]$.

$$\boxed{\mathbf{n}^{T}(\mathbf{r}-\boldsymbol{r}_{\mathbf{0}}) = \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} a_i x_i + d = 0}$$

where $d = -\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} a_i c_i$. When $\boldsymbol{r}_{\mathbf{0}}$=**0,** $d$=0 and the plane equation becomes $\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} a_i x_i = 0$. Note that $[\mathrm{a}_1, \cdots, \mathrm{a}_\mathrm{n}]$ is the normal vector.

# Example

Find a formula for the distance $D$ from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

## Solution:

Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let **b** be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance $D$ from $P_1$ to the plane is equal to the absolute value of the scalar projection of **b** onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$.

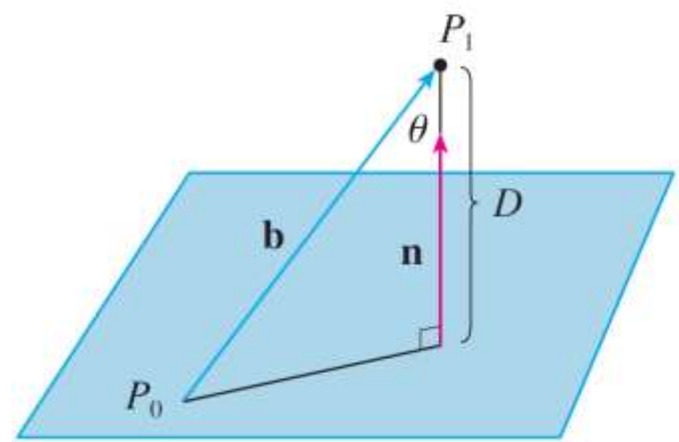


Figure 12

# Example

Thus

$$D = |\mathrm{comp}_{\mathbf{n}}\,\mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

# Example

Since $P_0$ lies in the plane, its coordinates satisfy the equation of the plane and so we have

$$ax_0 + by_0 + cz_0 + d = 0.$$

Thus the formula for $D$ can be written as

9
$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$