MSAI-6124
Neuro Evolution & Fuzzy Intelligence

Week 4 – Part 1
Fuzzy Set, Fuzzy Logic, Fuzzy Rule Based System

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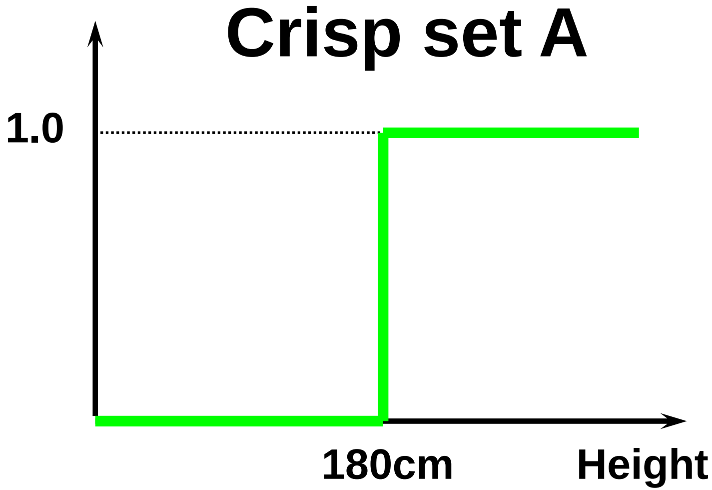
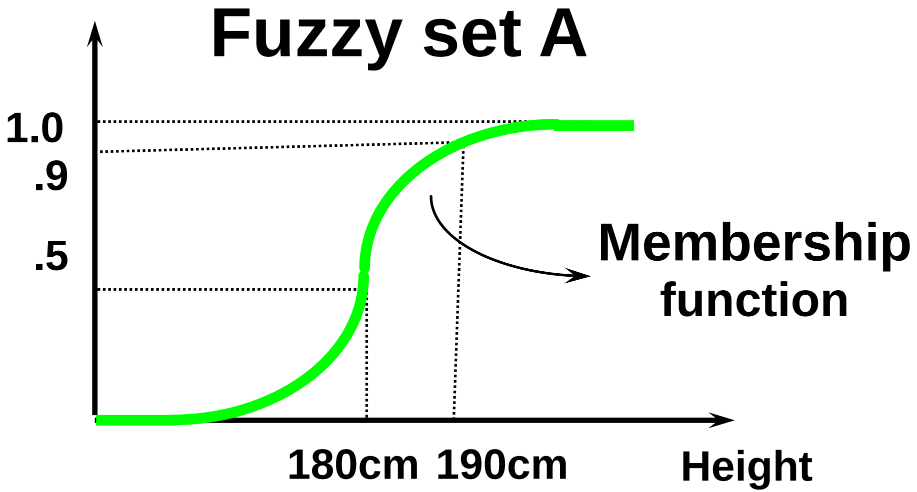
Fuzzy Sets

- Sets with fuzzy (non-crisp) boundaries (by Zadeh in 1965)

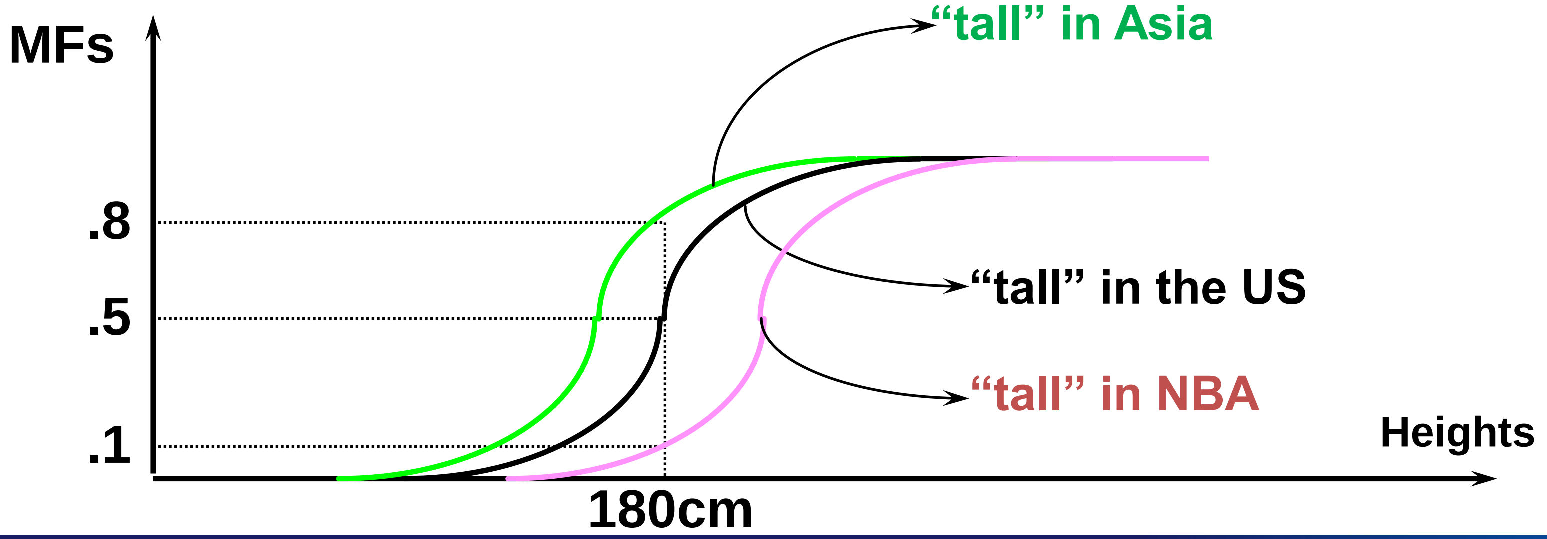
A = Set of tall people





Membership Functions (MFs)

- Characteristics of MFs:
  - Subjective measures
  - Not probability functions



Fuzzy Sets

- Formal definition:
  - A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$\underbrace{A}_{\text{Fuzzy set}} = \{(x, \underbrace{\mu_A(x)}_{\text{Membership function (MF)}}) \mid x \in \underbrace{X}_{\text{Universe or universe of discourse}}\}$$

$$\mathbf{0 \leq \mu_A(X) \leq 1}$$

A fuzzy set is totally characterized by a membership function (MF)

Fuzzy Sets with Discrete Universes

- Fuzzy set C = “desirable city to live in”

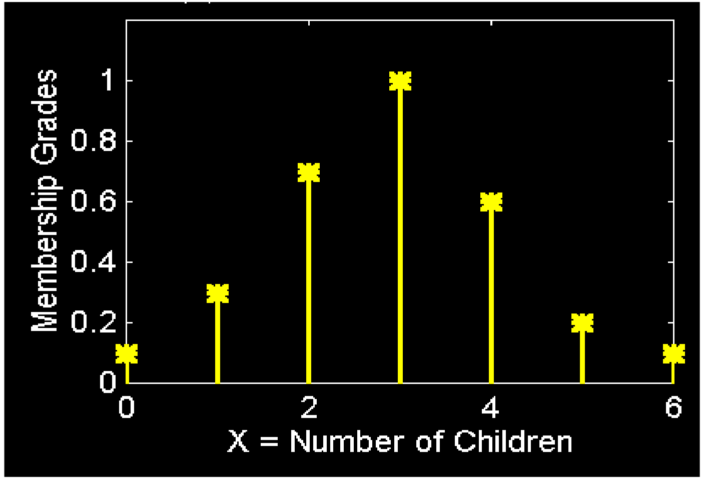
  X = {SF, Boston, LA} (discrete and nonordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

- Fuzzy set A = “sensible number of children”

  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



Fuzzy Sets with Continuous Universes

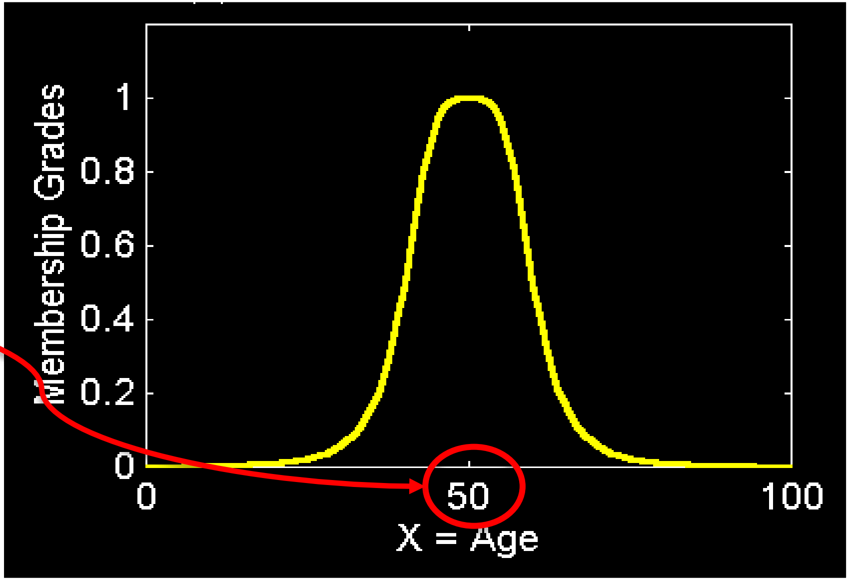
- Fuzzy set B = “about 50 years old”

  X = Set of positive real numbers (continuous)

  B = {(x, μ_B(x)) | x ∈ X}

$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^2}$$

centroid



Bell shape membership function

Alternative Notation

- A fuzzy set A can be alternatively denoted as follows:

This is Union not summation

X is discrete ⇨ $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)
A = {(0, 0.1), (1, 0.3), (2, 0.7), (3, 1.0), (4, 0.6), (5, 0.2), (6, 0.1)}
A = {0.1/0, 0.3/1, 0.7/2, 1.0/3, 0.6/4, 0.2/5, 0.1/6}

This is not actual division, just notation like 0.1/0 mean that Membership of 0 is 0.1.

Note that ∑ stands for the union of membership grades; and "/" stands for a marker and does not imply division

Alternative Notation

- A fuzzy set A can be alternatively denoted as follows:

X is continuous ⇒ $$A = \int_X \mu_A(x) / x$$

Note that the integral sign stands for the union of membership grades

Linguistic Hedge – Modifiers

- Linguistic hedges / modifiers are operations that modify the meaning of a term – fuzzy label (fuzzy set)
  - "*very tall*", the word *very* modifies *"tall"* which is a fuzzy set
- Other modifiers are:
  - "more or less" (morl), "possibly", and "definitely"

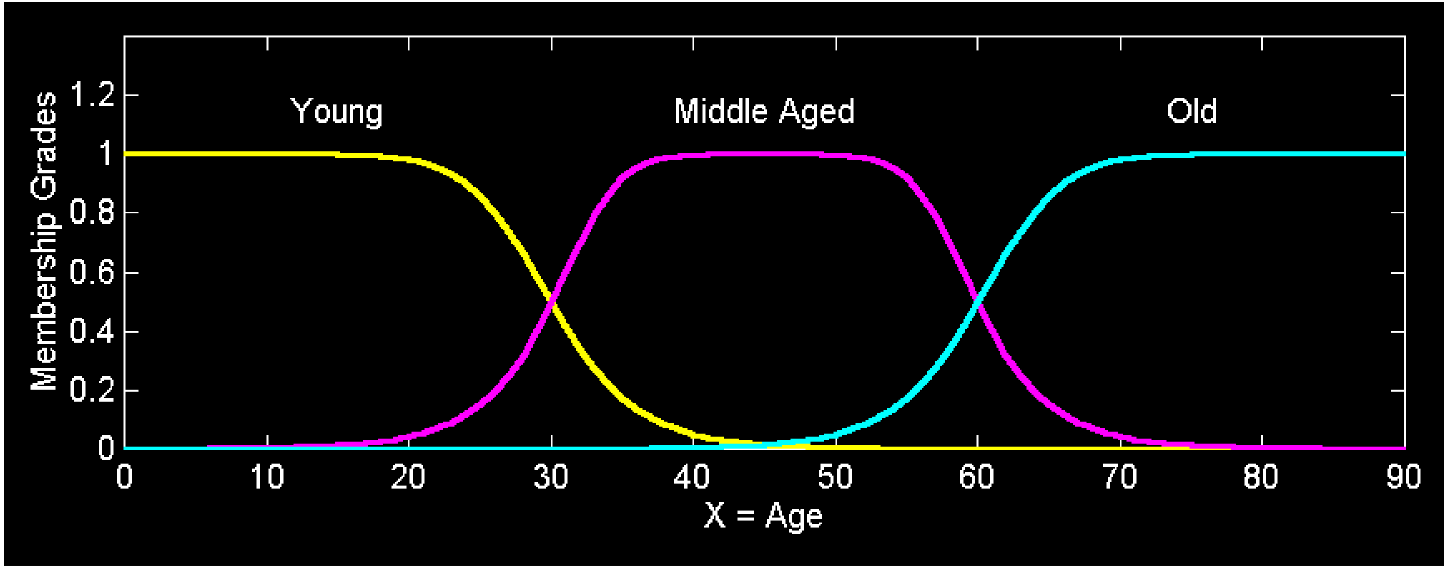
Linguistic Hedge – Modifiers

- very $\mathbf{a} = \mathbf{a}^2$
- morl $\mathbf{a} = \mathbf{a}^{0.5}$
- extremely $\mathbf{a} = \mathbf{a}^3$
- slightly $\mathbf{a} = \mathbf{a}^{0.333}$
- somewhat **a** = morl **a** AND not slightly **a**

E.g., young = {1/0, 0.6/20, 0.1/40, 0.0/60, 0.0/80}
very young = young^2
= {1/0, 0.36/20, 0.01/40, 0.0/60, 0.0/80}

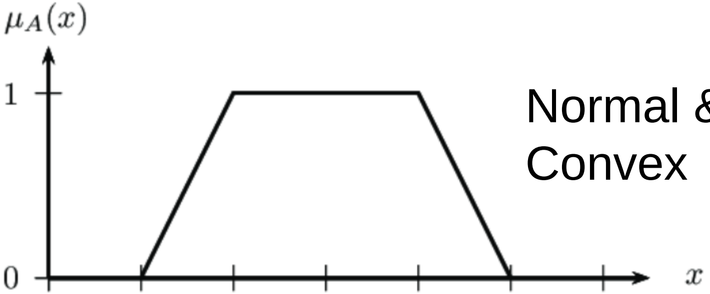
Fuzzy Partition

- Fuzzy partitions of “Age” formed by the linguistic values “young”, “middle aged”, and “old”:

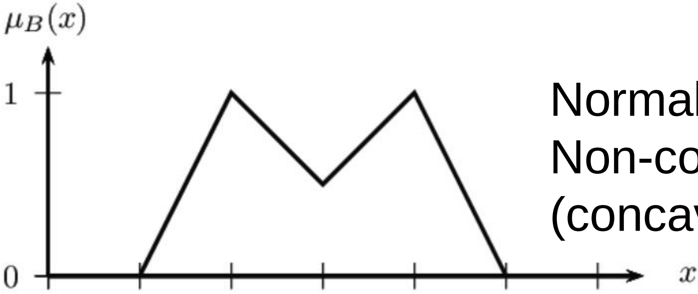


Normal and Convex Fuzzy Sets

- A **normal** fuzzy set has a height, i.e., maximal membership value, equal to one
- In a **convex** fuzzy set, the membership value of any element between two arbitrary elements is greater than or equal to the smaller membership value of the two arbitrary boundary elements



Normal &
Convex



Normal &
Non-convex
(concave)

Non-Pseudo Partitioning

- A fuzzy space is **not** pseudo-ly partitioned if:

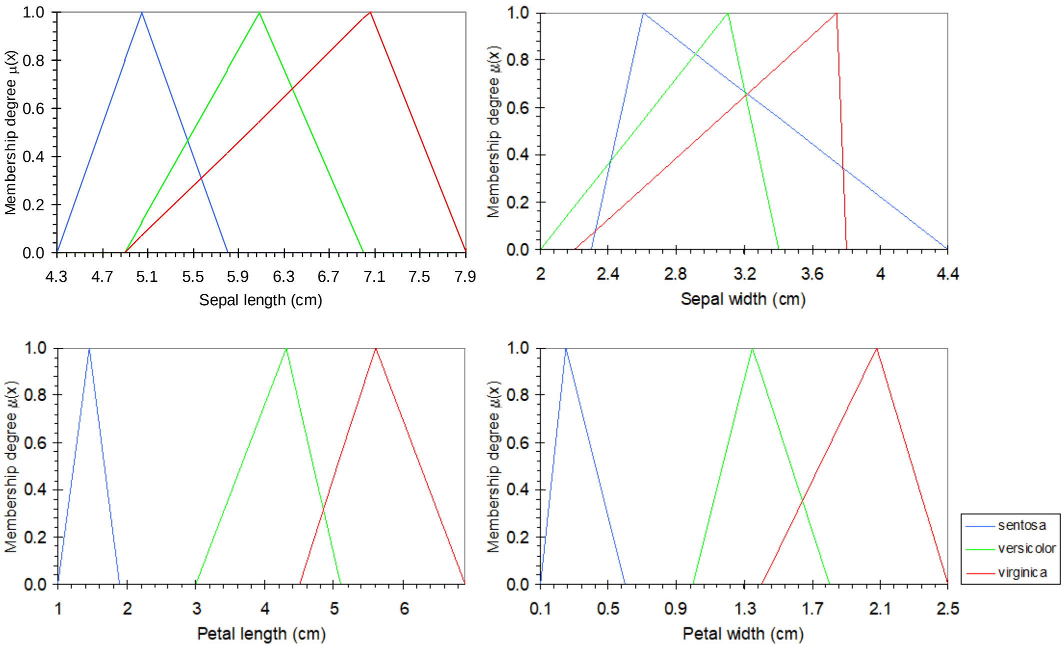
  Each mf is normal and convex

  $$\sup_{x}(\mu_{i,i\in c}(X))=1$$

  Summation of mf values at X is **not** 1

  $$\sum_{i=1}^{c}(\mu_{i,i\in c}(X)) \neq 1$$

  Ie the y values of each x does not sum to exactly 1.



Pseudo Partitioning

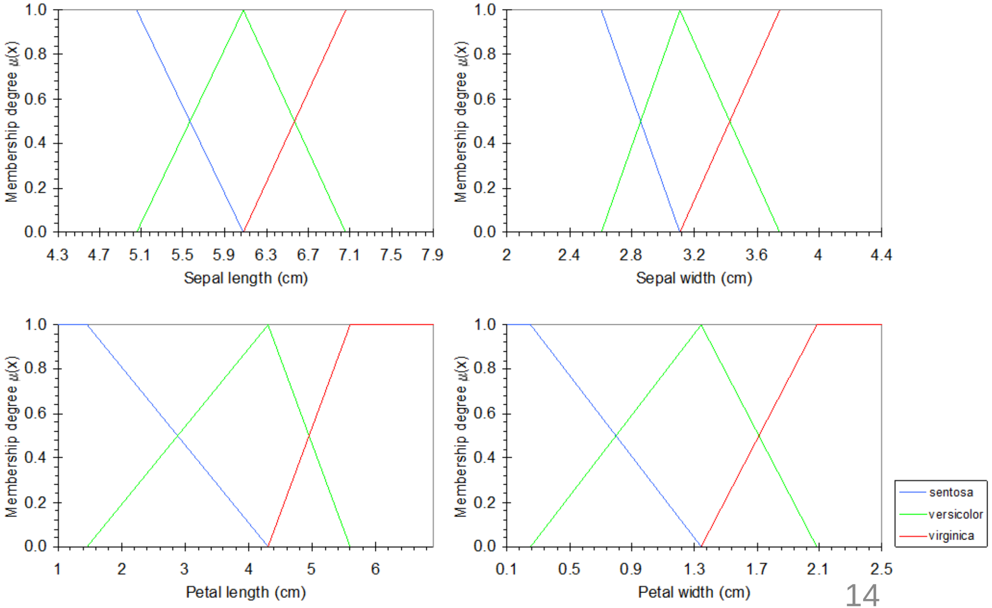
- A fuzzy space is pseudo-ly partitioned if:

  Each mf is normal and convex

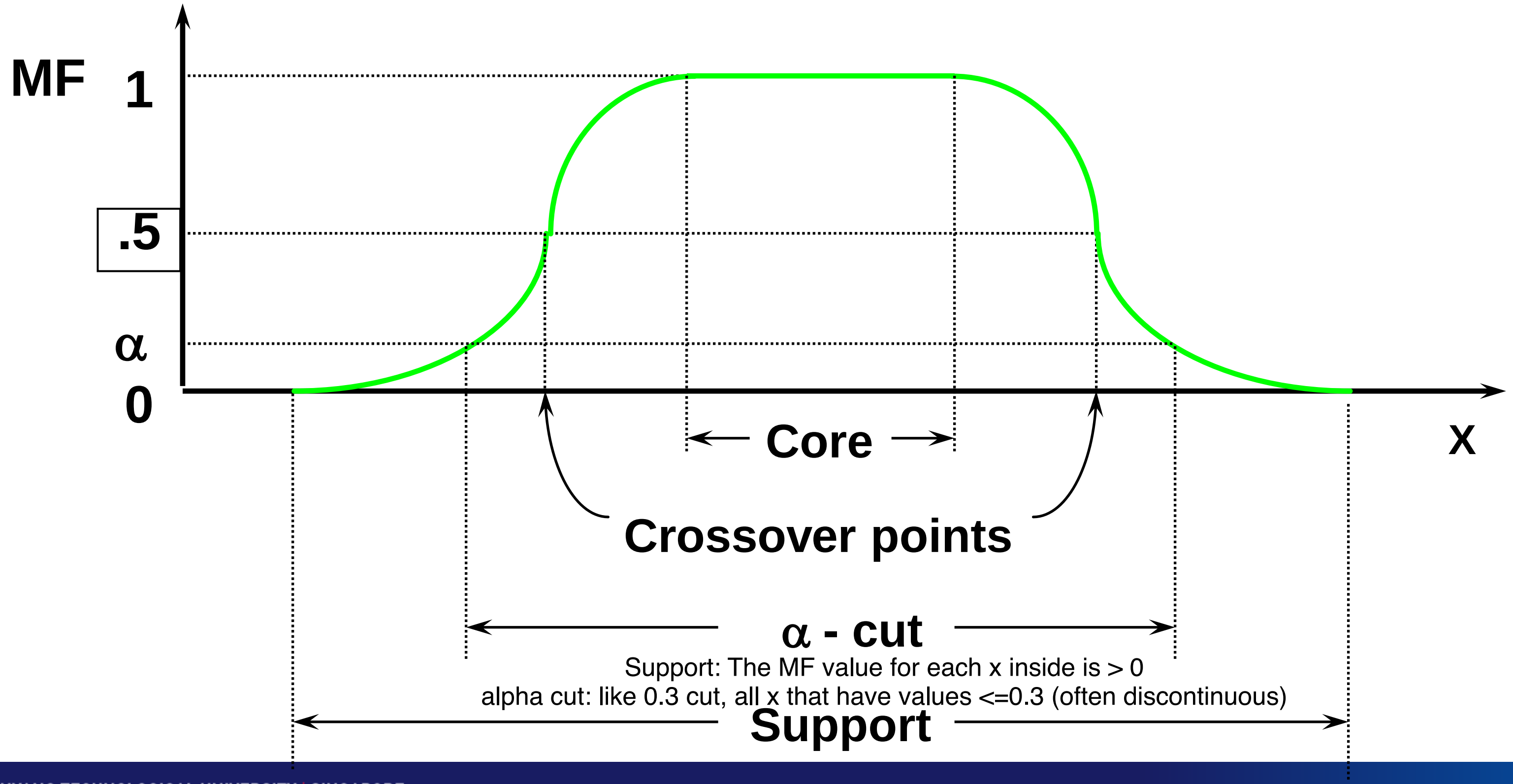
  $$\sup_{x}(\mu_{i,i\in c}(X))=1$$

  Summation of mf values at X is 1

  $$\sum_{i=1}^{c}(\mu_{i,i\in c}(X)) = 1$$



MF Terminologies



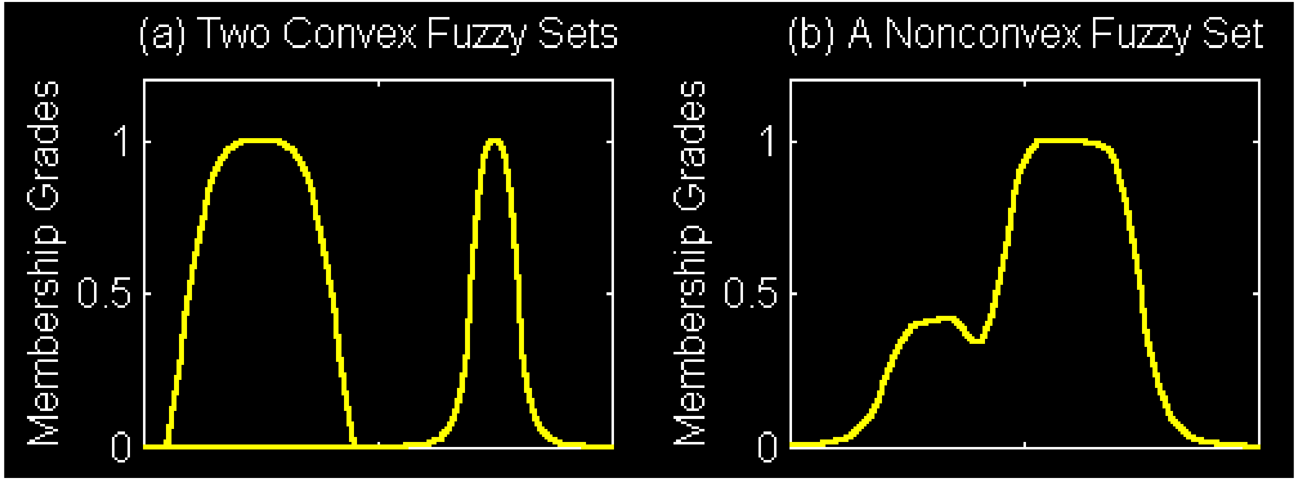
Support: The MF value for each x inside is > 0
alpha cut: like 0.3 cut, all x that have values <=0.3 (often discontinuous)

Formal Definition of Convexity of Fuzzy Sets

- A fuzzy set A is convex if for any λ in [0, 1], x1 and x2 are 2 randomly selected points

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

- Alternatively, A is convex if all its α-cuts are convex.



Widely Adopted MF Formulations

Triangular MF:

$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:

$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

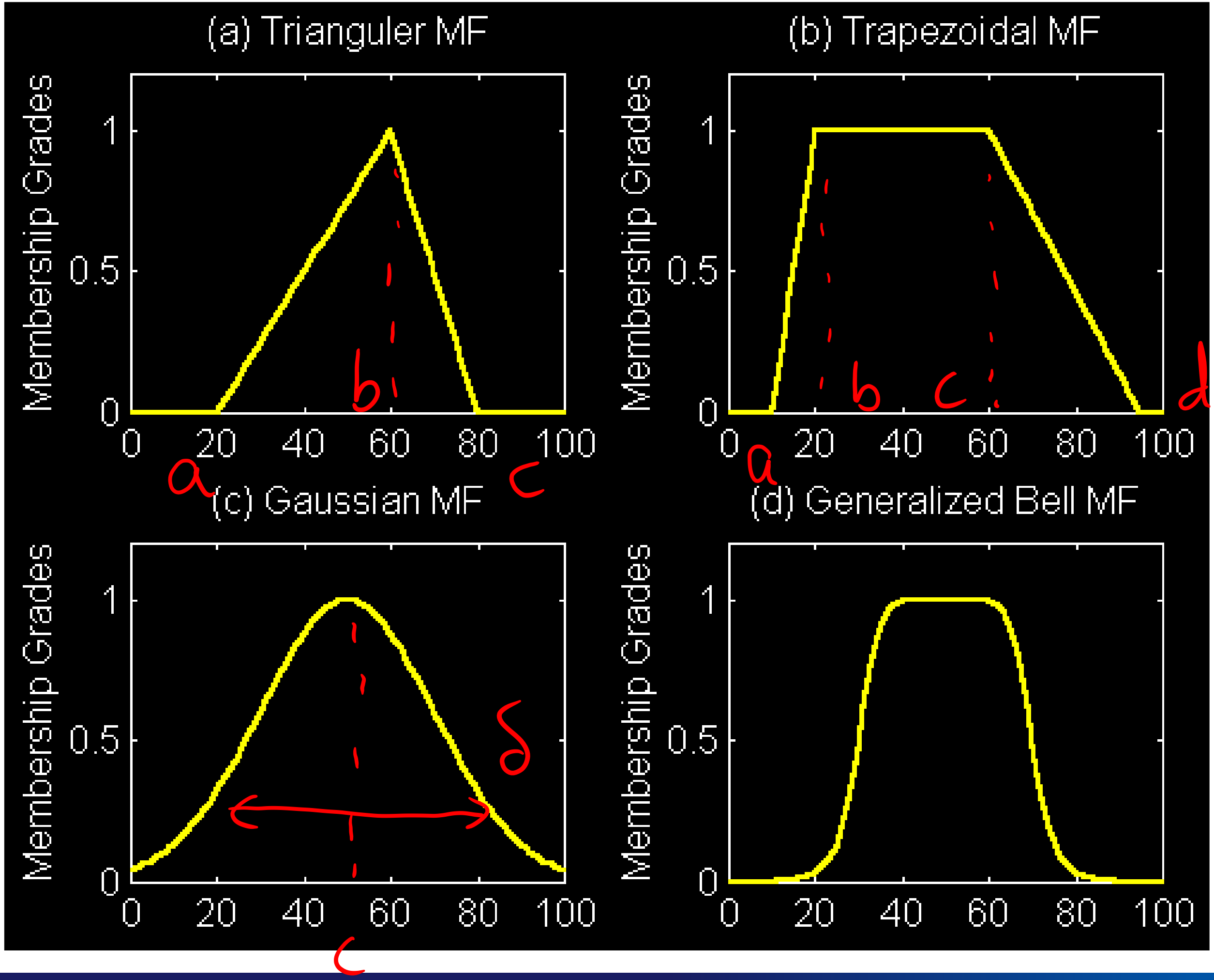
Gaussian MF:

$$gaussmf(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Generalized bell MF:

$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

Widely Adopted MF Formulations



Sigmoidal MF

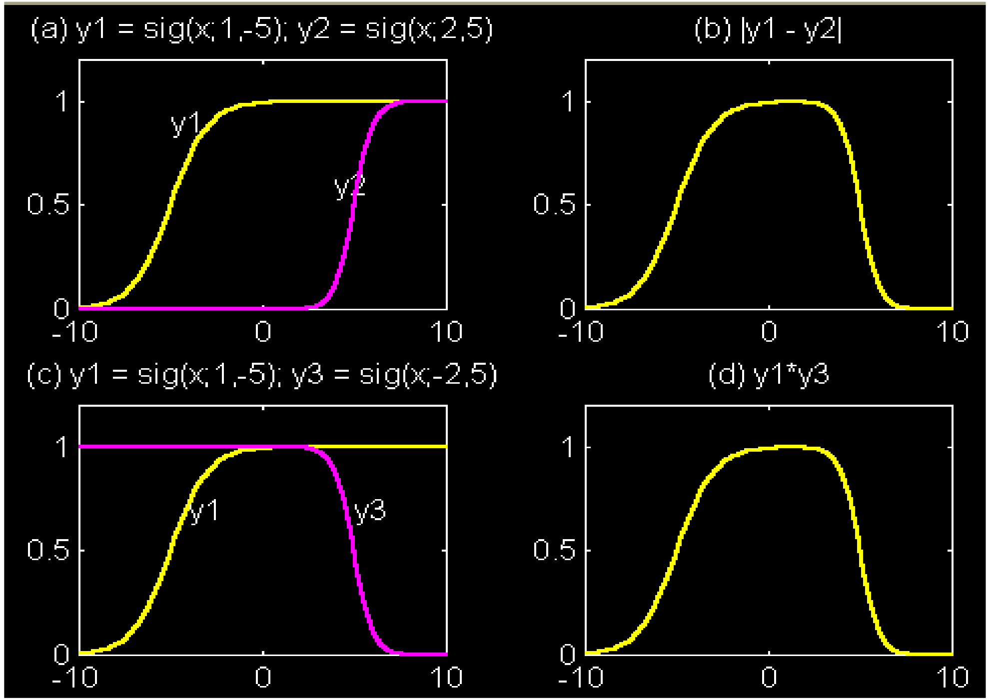
$$sigmf(x;a,b,c) = \frac{1}{1+e^{-a(x-c)}}$$

a: controls the slope at the crossover point; *b* = 1 is omitted in this case

Extensions:

Abs. difference of two sig. MF

Product of two sig. MF



Left-Right (L-R) MF: Asymmetric

$$LR(x;c,\alpha,\beta)=\begin{cases} F_L\left(\dfrac{c-x}{\alpha}\right), x<c \\ F_R\left(\dfrac{x-c}{\beta}\right), x\geq c \end{cases}$$

Another Example:

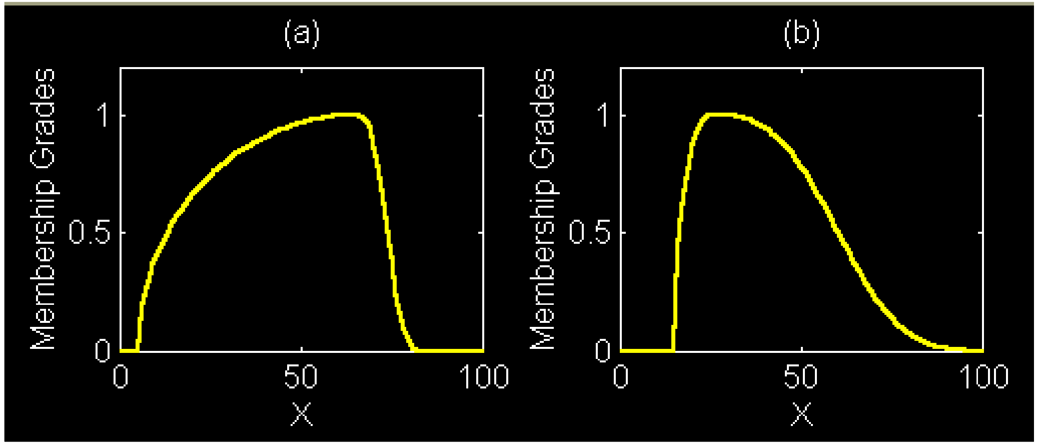
$$F_L(x)=\sqrt{\max(0,1-x^2)}$$

$$F_R(x)=\exp(-|x|^3)$$

c=65

α=60

β=10



c=25

α =10

β =40

Summary of Membership Functions

- Fuzzy sets allow the description of vague concepts (e.g., *SLOW, MEDIUM* and *FAST*) for a fuzzy variable (e.g., *SPEED*)
- This provides the **semantics** (concepts) to linguistic rules involving fuzzy variables:

  e.g., The *SPEED* is *FAST*
- The fuzzy set admits the possibility of partial memberships in it:
  e.g., The WEATHER is <u>rather</u> HOT

Set-Theoretic Operations

- Subset: $A \subseteq B \Leftrightarrow \mu_A \leq \mu_B$
- Complement: $\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$
- Union (OR – Disjunction):

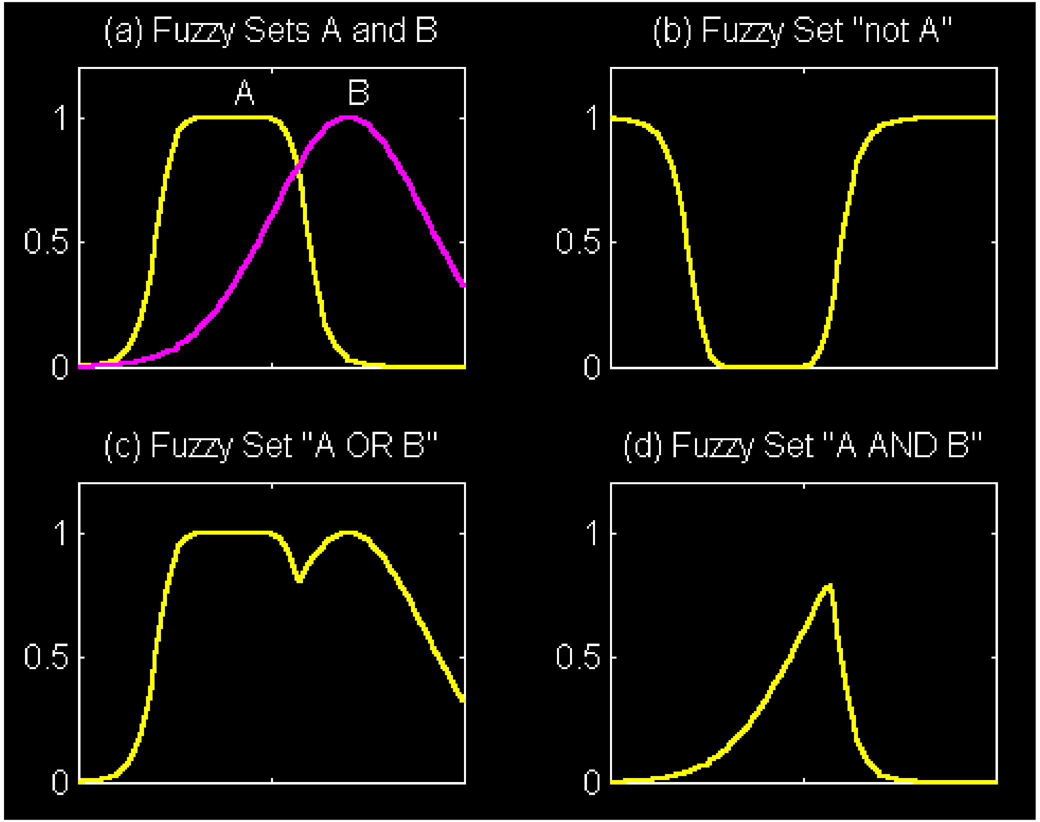
$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

- Intersection (AND – Conjunction):

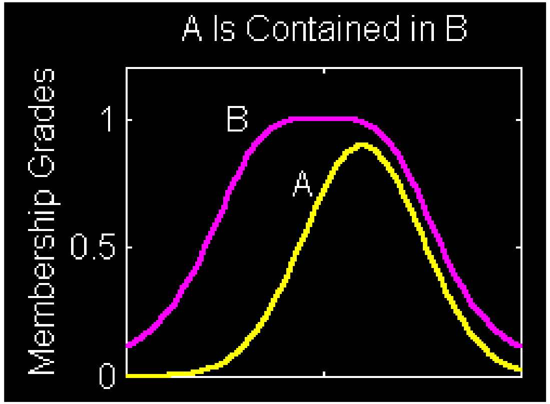
$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Set-Theoretic Operations

$$1-\mu_A(x)$$



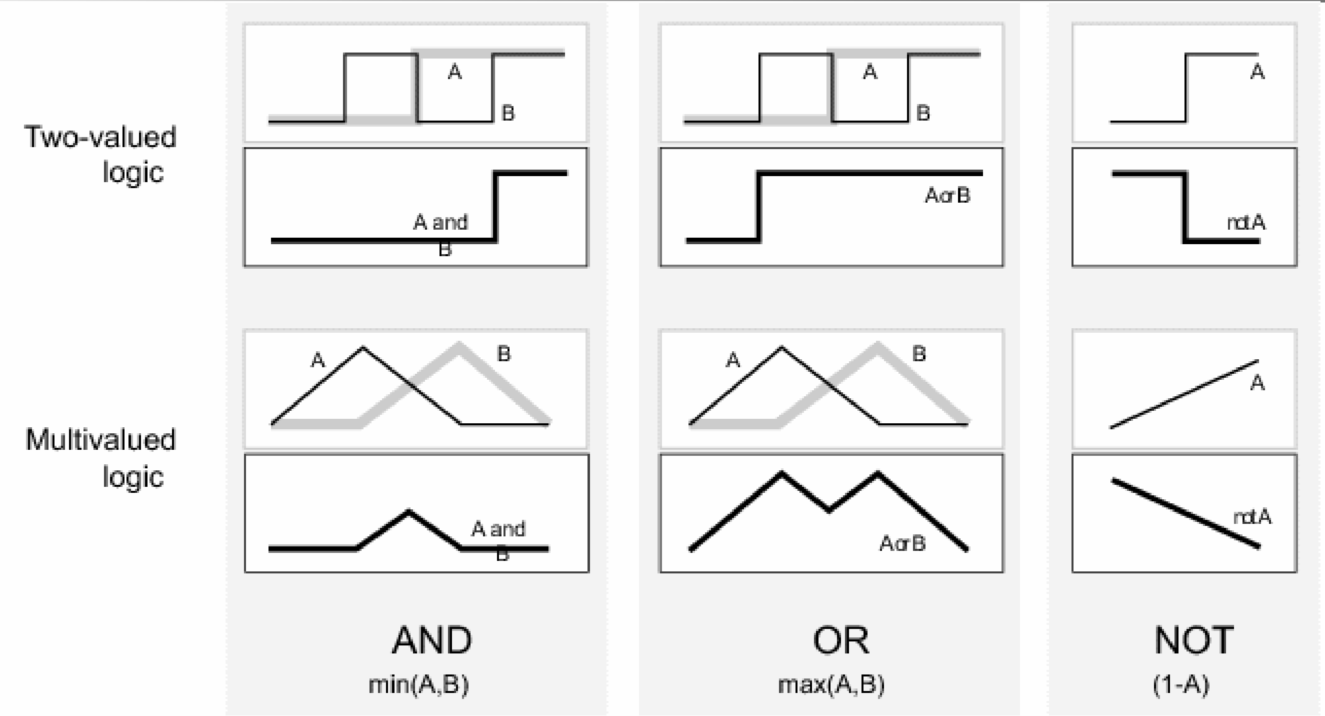
$$\mu_A \le \mu_B$$



$$\max(\mu_A(x), \mu_B(x))$$

$$\min(\mu_A(x), \mu_B(x))$$

Fuzzy Logical Operation



T-norm S-norm/T-conorm

Fuzzy Complement

- The fuzzy complement of A, denoted by $\bar{A}$ or NOT A (N(A)), is defined by the membership function $\mu_{\bar{A}}(X) = 1 - \mu_A(X)$
- Properties of fuzzy complement:
  - Boundary: N(0)=1 and N(1) = 0
  - Monotonicity: N(a) > N(b) if a < b
  - Involution: N(N(a)) = a
- Two types of fuzzy complements:
  - Sugeno's complement: $N_s(a) = \dfrac{1-a}{1+sa}$
  - Yager's complement: $N_w(a) = (1-a^w)^{1/w}$

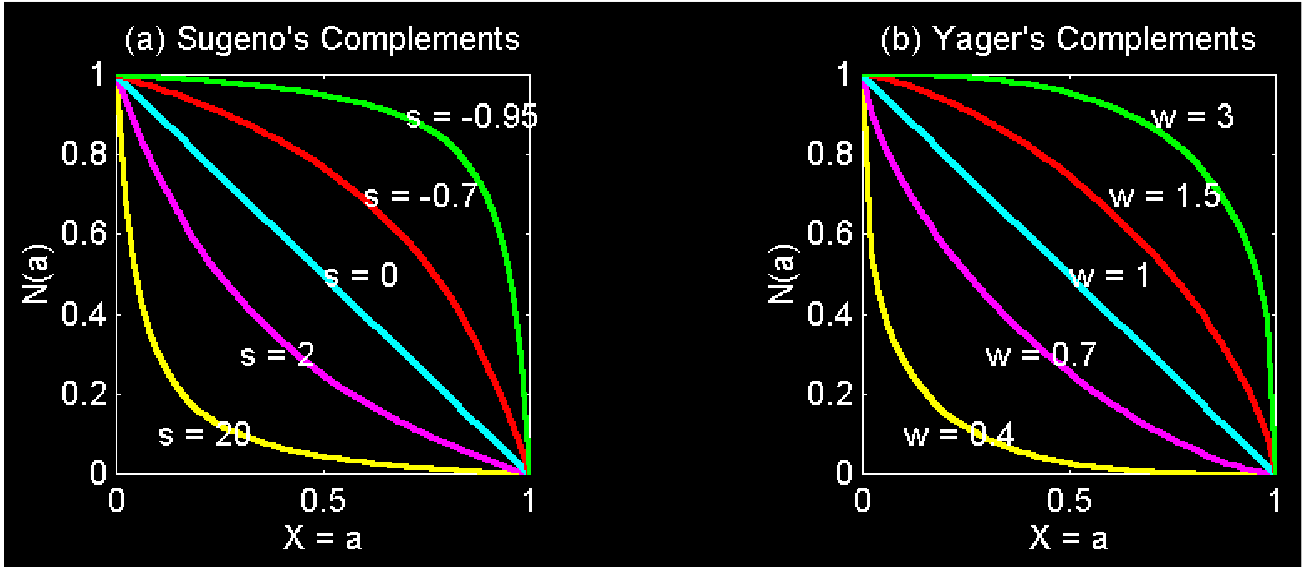
Fuzzy Complement

Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$



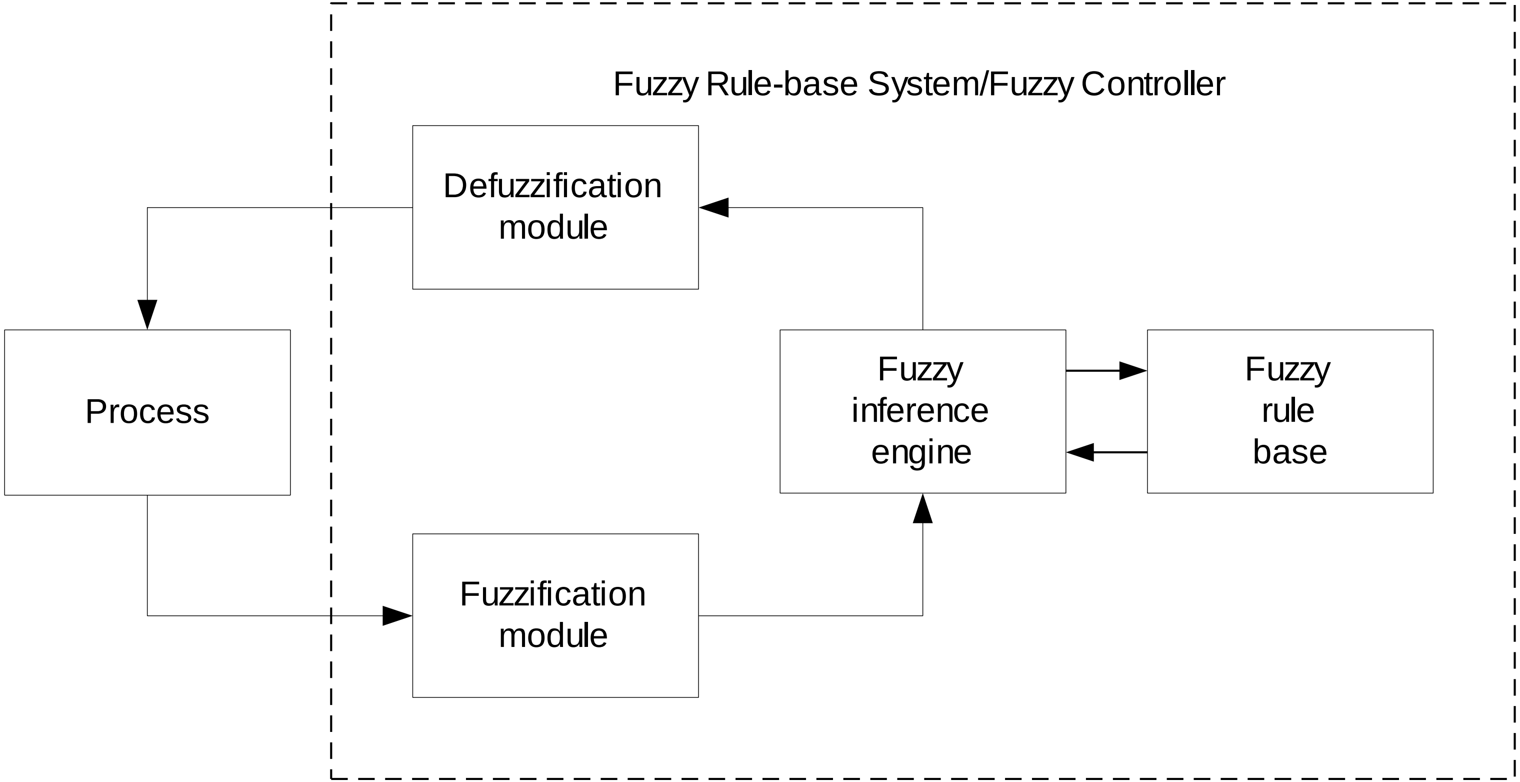
Fuzzy Intersection: T-norm

- Properties of T-norm:
  - Boundary: $T(0, 0) = 0$, $T(a, 1) = T(1, a) = a$
  - Monotonicity: $T(a, b) < T(c, d)$ if $a < c$ and $b < d$
  - Commutativity: $T(a, b) = T(b, a)$
  - Associativity: $T(a, T(b, c)) = T(T(a, b), c)$

Fuzzy Union: S-norm or T-conorm

- Properties of S-norm:
  - Boundary: $S(1, 1) = 1$, $S(a, 0) = S(0, a) = a$
  - Monotonicity: $S(a, b) < S(c, d)$ if $a < c$ and $b < d$
  - Commutativity: $S(a, b) = S(b, a)$
  - Associativity: $S(a, S(b, c)) = S(S(a, b), c)$

Fuzzy Rule Base (FRB) Systems



Dynamics of FRB Systems

1. First, measurements are taken of all variables from the process
2. Next, these measurements are converted into appropriate fuzzy sets to express measurement uncertainties – **fuzzification**
3. The fuzzified measurements are then used by the inference engine to evaluate the control rules stored in the **fuzzy rule base** – fuzzy rules defined with fuzzy (linguistic) variables using fuzzy labels (sets)
4. The result of this evaluation is one or several fuzzy sets defined on the universe of possible control actions. This fuzzy set is then converted, in the final step of the cycle, into a single crisp value or a vector of values which best represents the resultant fuzzy set(s) – **defuzzification**

Components of a Fuzzy Rule

- A single fuzzy if-then rule:
  - If *x* is *A* then *y* is *B*.

fuzzy antecedent fuzzy conclusion

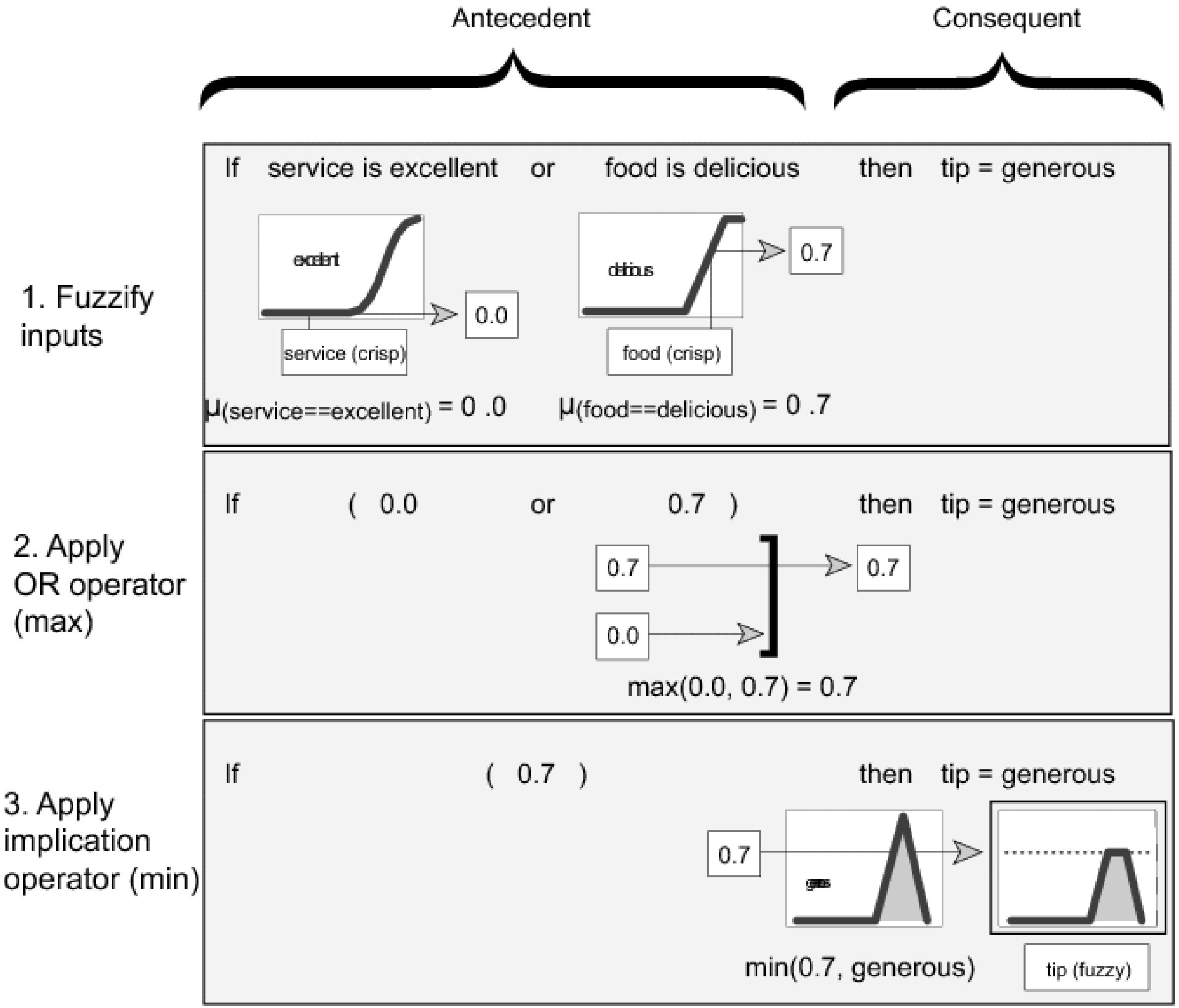
A and *B* are linguistic values/labels defined by fuzzy sets on the range of discourse for linguistic/fuzzy variables *x* and *y*, respectively

E.g., If *service* is ***good*** then *tips* is ***average***

Interpreting fuzzy if-then rule:

- *Evaluate antecedent (fuzzifying input and necessary fuzzy operators)*
- *Apply that result to the conclusion/consequent (known as implication)*

Fuzzy Inference – Example

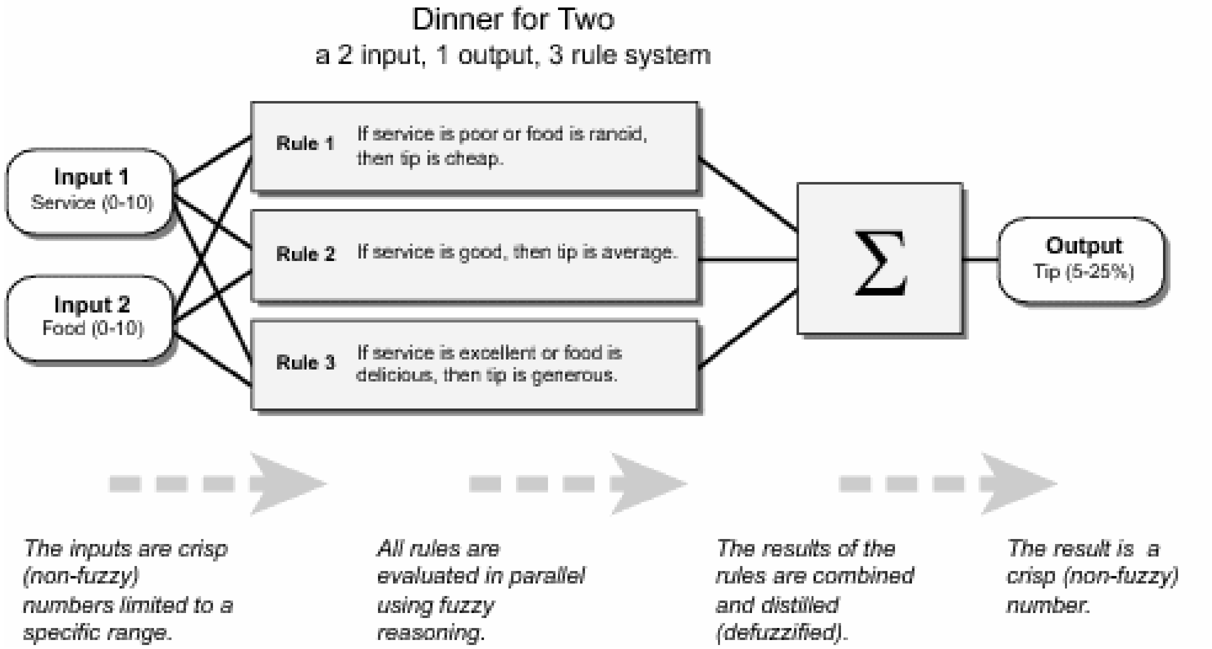


Steps in Fuzzy Inference

1. **Fuzzify inputs**: Resolve all fuzzy statements in the antecedent to a degree of membership between 0 and 1
2. **Apply fuzzy operator to multiple part antecedents**: If there are multiple parts to the antecedent, apply fuzzy logic operators and resolve the antecedent to a single number between 0 and 1
3. **Apply implication method**: Use the degree of support for the entire rule to shape the output fuzzy set. The consequent of a fuzzy rule assigns an entire fuzzy set to the output. This fuzzy set is represented by a membership function that is chosen to indicate the qualities of the consequent
4. **Aggregation** of the consequents across the rules, and
5. **Defuzzification**

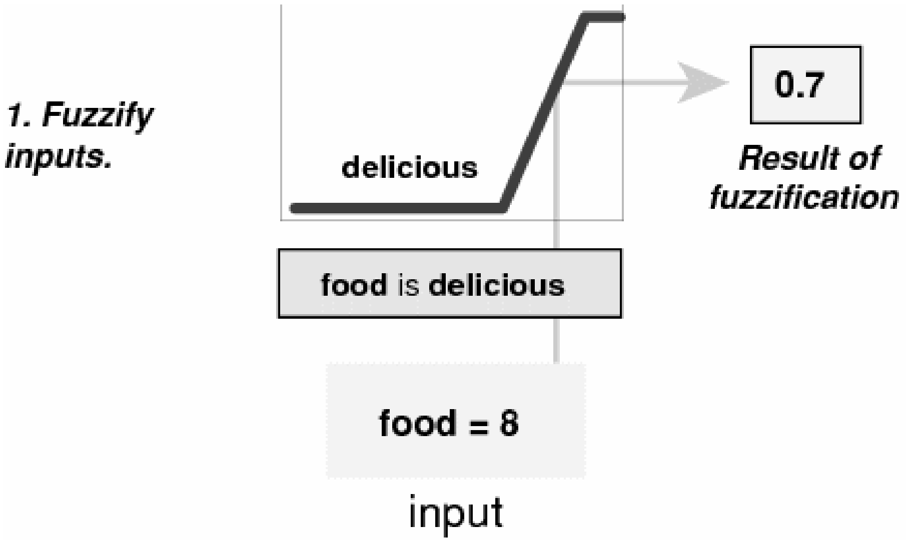
Example on Service (multiple rules)

- Consider the example: service for a dinner for two



Step 1 – Fuzzification

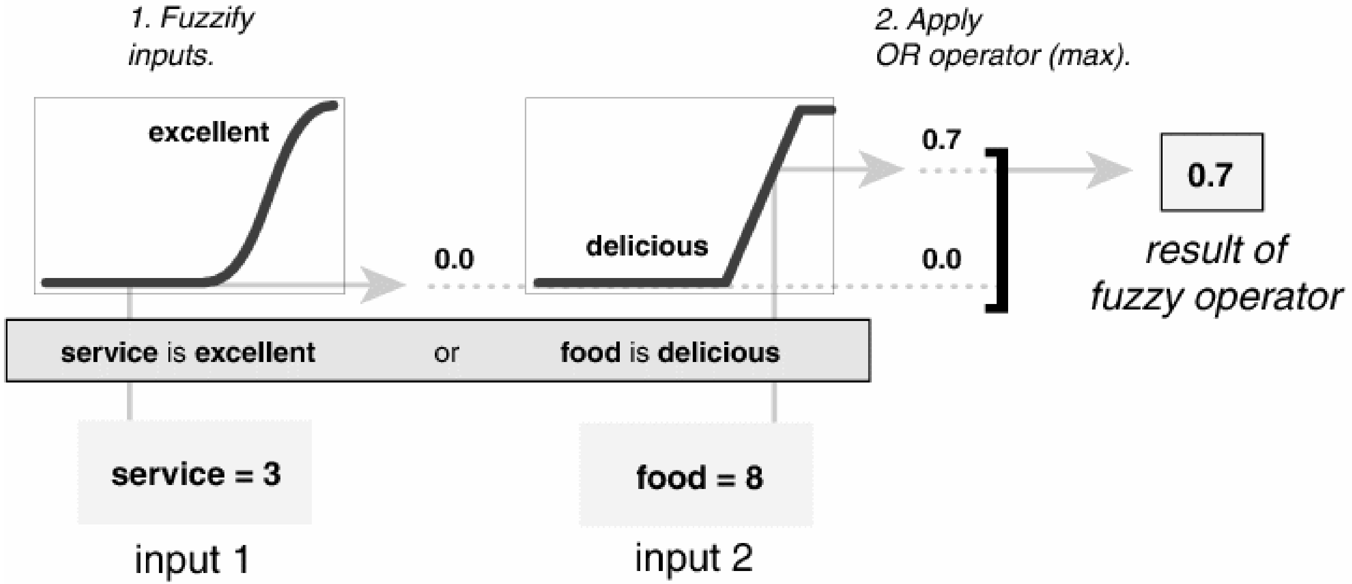
- Fuzzification of the input amounts to a function evaluation



- In this manner, each input is fuzzified over all the qualifying membership functions required by the rules

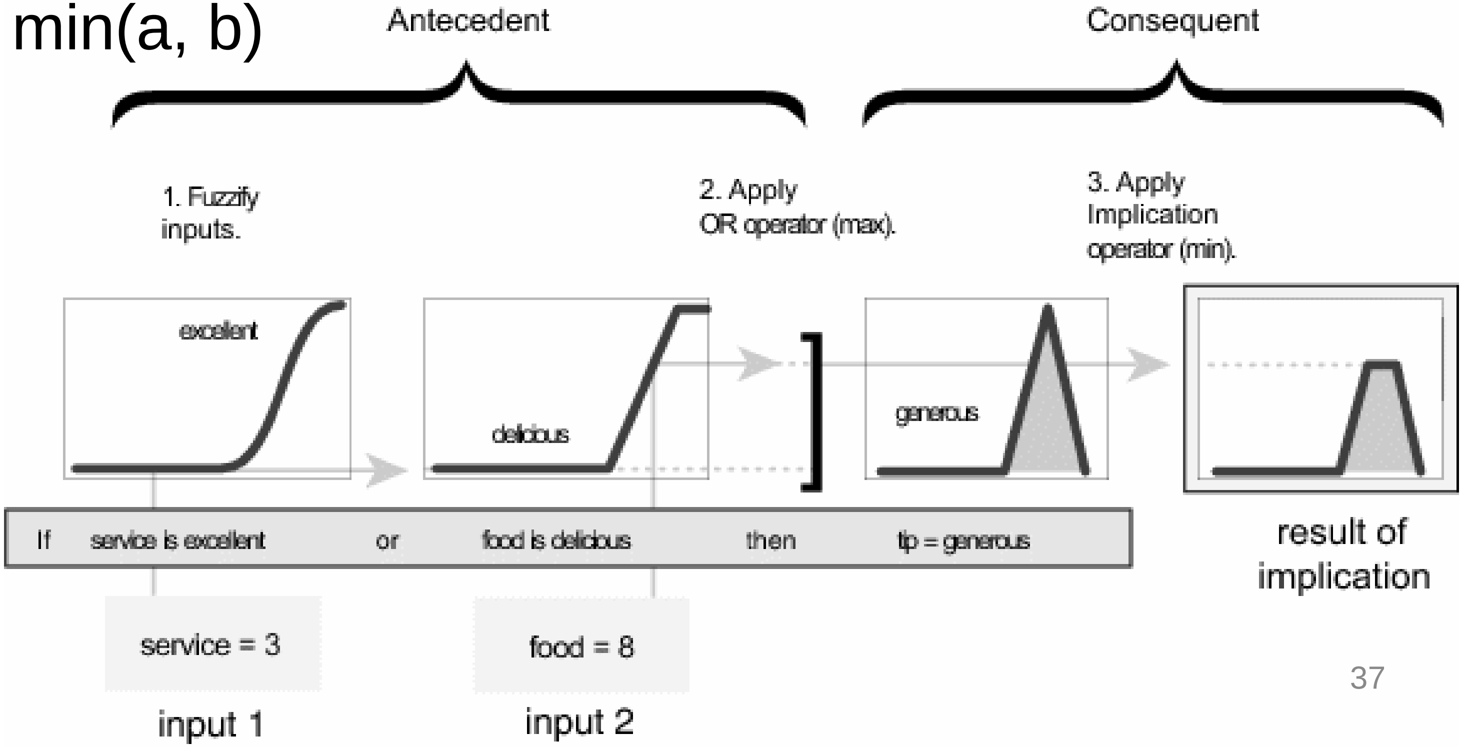
Step 2 – Applying Fuzzy Operators

- Example using Fuzzy Rule 3
- OR-max(a,b) – S-norm



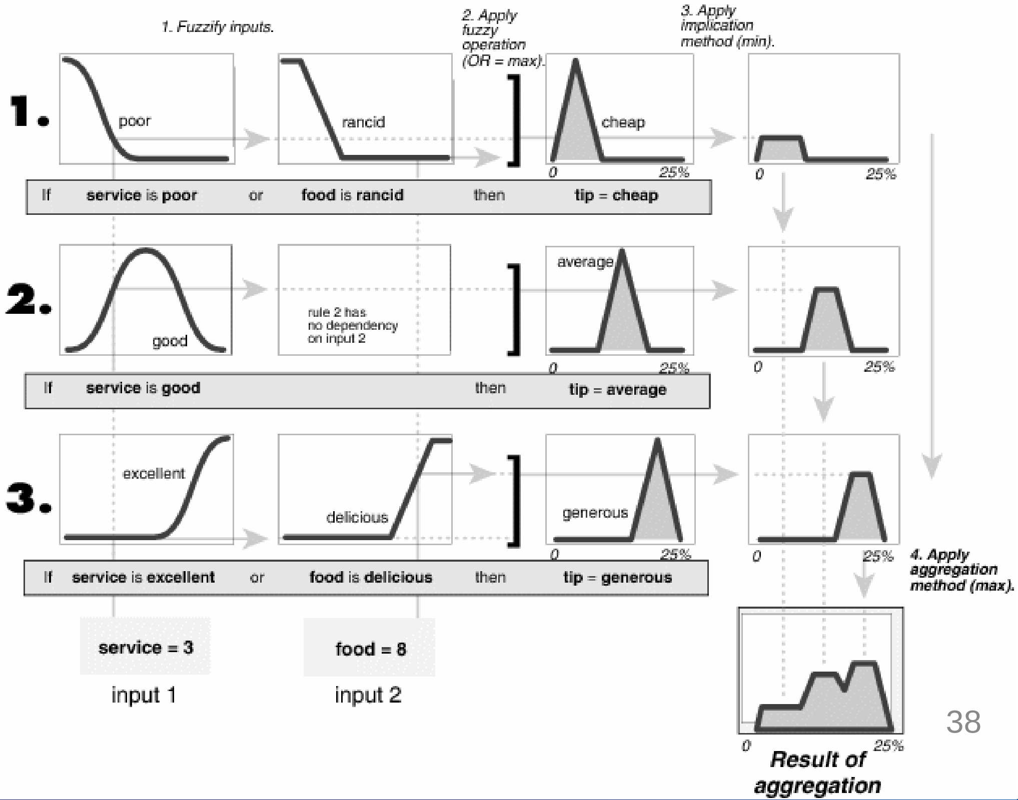
Step 3 – Applying Implication Method

- T-norm - min(a, b)



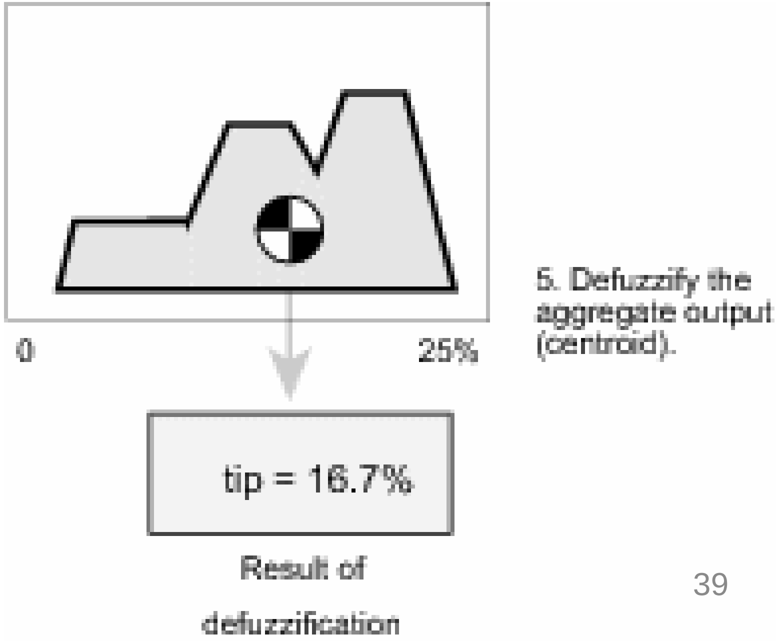
Step 4 – Aggregating All Outputs

- Aggregation is the process by which the fuzzy sets that represent the outputs of each rule are combined into a single fuzzy set
- E.g., max operator



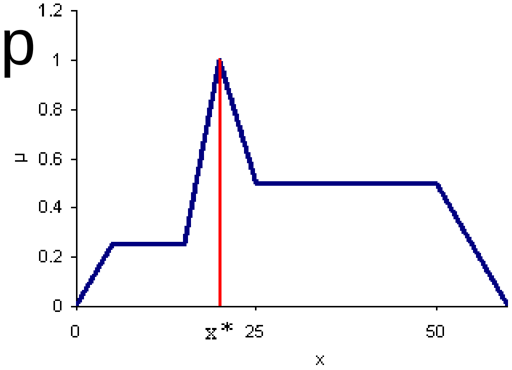
Step 5 – Defuzzification

- The input to the defuzzification process is a fuzzy set (the aggregated output fuzzy set)
- The output is a single number



Types of Defuzzifications

- Max-membership defuzzification



- Centroid defuzzification (Center of gravity)

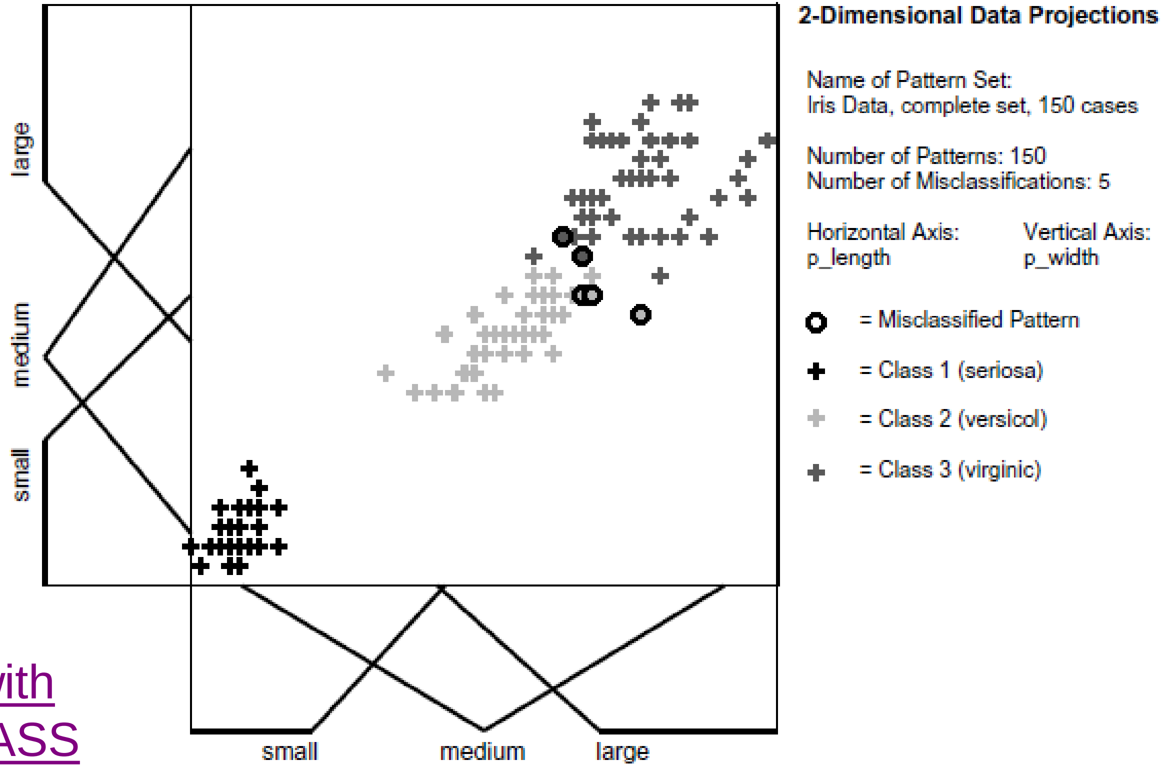
$$x^* = \frac{\int \mu_i(x)\, x\, dx}{\int \mu_i(x)\, dx}$$

- Weighted average defuzzification

$$x^* = \frac{\sum_{i=1}^{n} m^i w_i}{\sum_{i=1}^{n} m^i}$$

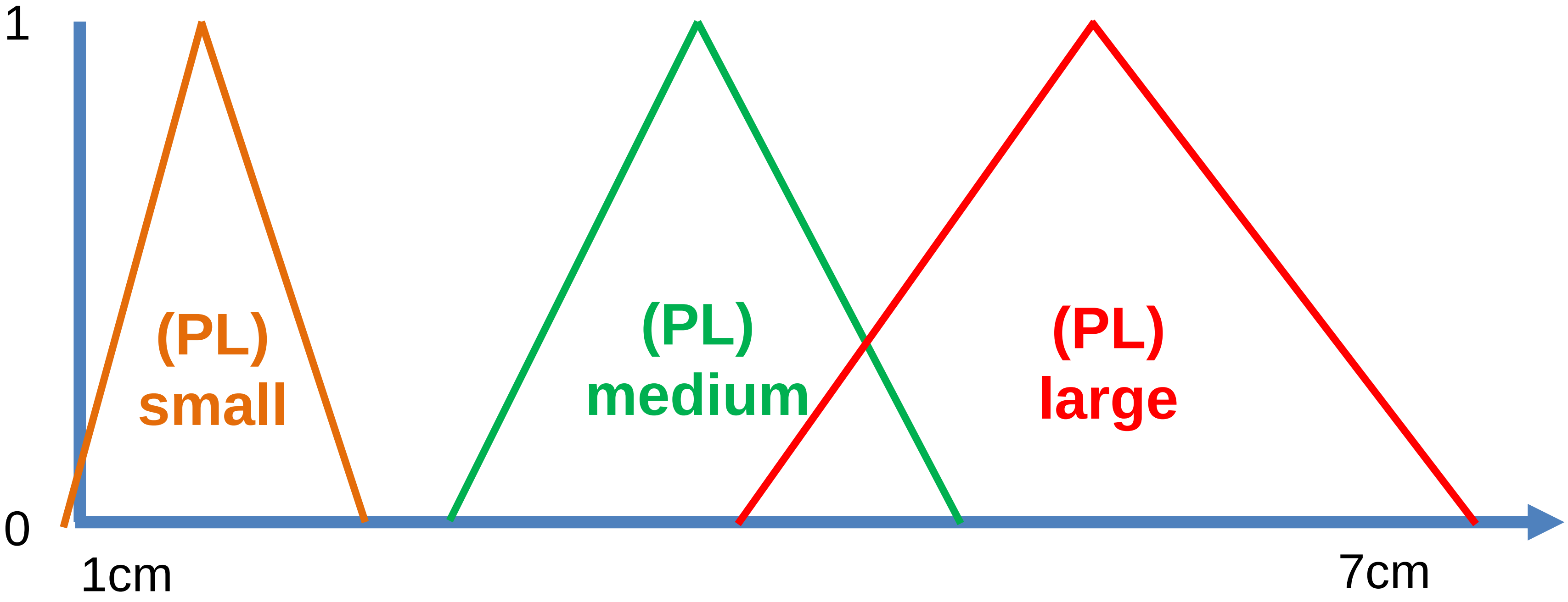
Formulating FMF

- The Iris dataset is highly distinguishable in 2D:
  - ➢ Petal length
  - ➢ Petal width

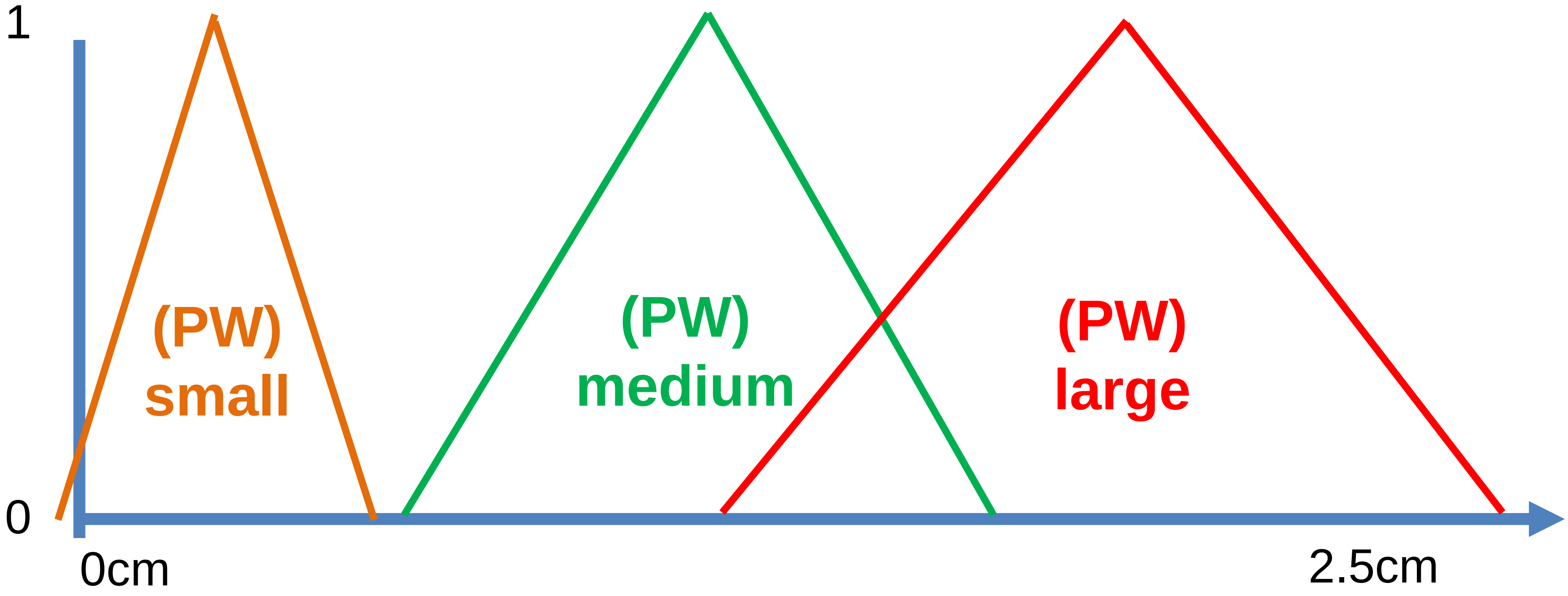


Source:
[Generating classification rules with the neuro-fuzzy system NEFCLASS]

FMF on Petal Length (PL)



FMF on Petal Width (PW)



Formulating Fuzzy Rules

- Based on the 3 MFs on PL and PW, respectively, we can formulate the following 3 rules
  - Rule 1:
    If PL is small and PW is small, then class = 1
  - Rule 2:
    If PL is medium and PW is medium, then class = 2
  - Rule 3:
    If PL is large and PW is large, then class = 3

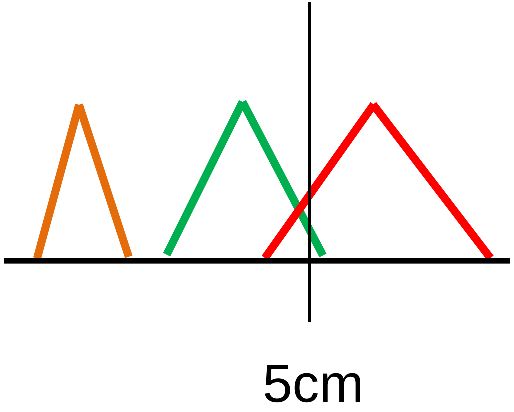
Zero-order TS rules

Inference Using Fuzzy Rules

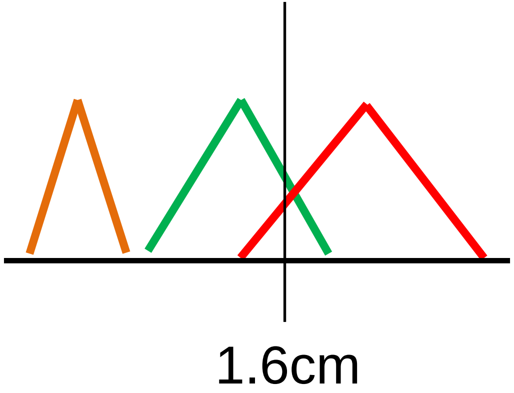
- Given a new sample:

  PW = 5cm
  &
  PL = 1.6cm,

  class=?

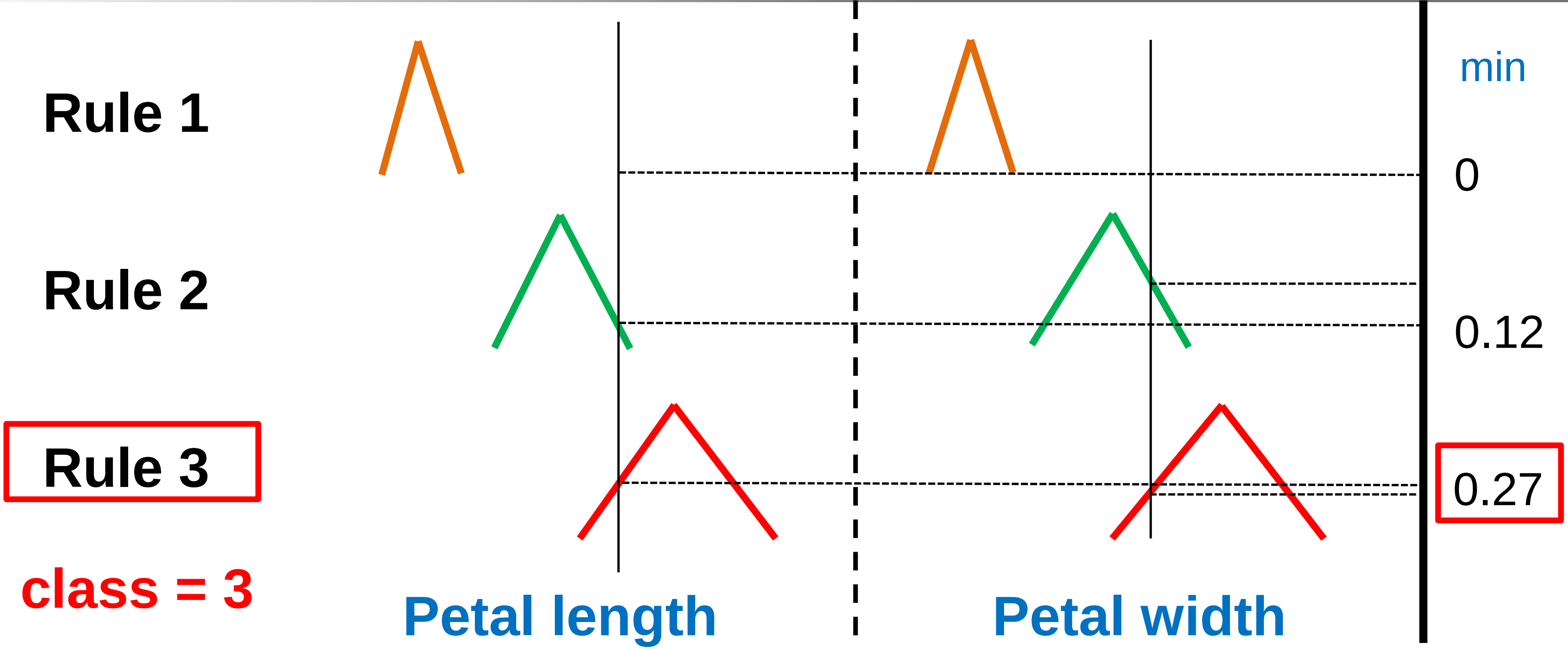


Petal length



Petal width

Obtaining Rule-firing Strength



Open-Source Fuzzy Logic Codes

- in Java:

http://jfuzzylogic.sourceforge.net/html/index.html

- In Python:

https://pythonhosted.org/scikit-fuzzy/overview.html