Multi-Controlled U-Gate

Anshuman Bhardwaj

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We discuss how to create quantum circuits in Qiskit that implement a multi-controlled U-gate. In particular, we discuss two methods that achieve this and the resource cost in both cases.

I. METHOD 1: NIELSEN & CHUANG

The first method comes from the book by Nielsen & Chuang [1]. The idea is that since we wish to create a U-gate C^nU acting on a target qubit controlled by many control qubits, we are basically entangling them and so we entangle the qubits in groups of three and then spread this entanglement across other qubits. Below in Fig. 1 is shown the quantum circuit that effectively creates a C^5U.

We have 5 control qubits $|c_i\rangle$, 4 work qubits or ancillas $|w_i\rangle = |0\rangle$, and 1 target qubit $|t_1\rangle$. We use the work qubits to transfer the controlling behavior from the control to target qubits. Let us suppose that the unitary gate $U = X$. Then the action of the circuit below ξ can be written as follows:

$$\xi = A^\dagger \cdot (CU)_{w_4 t_1} \cdot A, \tag{1}$$

$$A = CCX_{c_5 w_3 w_4} CCX_{c_4 w_2 w_3} CCX_{c_3 w_1 w_2} CCX_{c_1 c_2 w_1}, \tag{2}$$

and the initial state in $|\Psi_0\rangle = |1111100001\rangle$ where we have colored the working qubit labels as red to help distinguish them from the control and target qubits.

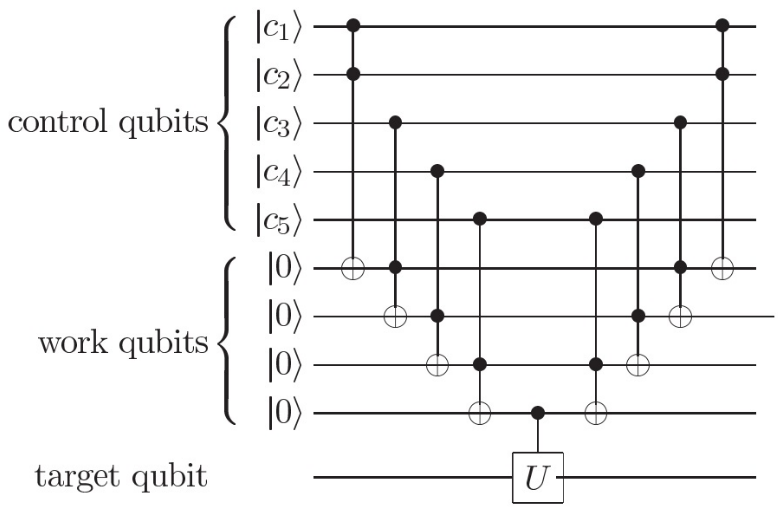

FIG. 1: A quantum circuit implementing C^5U. Taken from [1].

When A acts on $|\Psi_0$, then notice that it is imperative that the control qubits are set to $|1\rangle$ otherwise the work qubits will not get activated. As a result of our initial state all work or ancillas are activated which leads to the control action $(CU)_{w_4 t_1}$ between the last working and target qubits, which sets the target qubit to $|1\rangle$. After this, we apply $A^\dagger$ which basically reverses the action of A, and we end up implementing a multi-controlled X-gate with the final state being $|\Psi_{\rm f}\rangle = |1111100000\rangle$.

To count the resources, we use $2(n-1)$ Toffoli gates (CCX) and 1 CU gate. Since Toffoli gate can be further decomposed into single and two-qubit gates, we say that each CCX gate requires $O(1)$ gates whereas CU gate only depends on U so its resource requirement is constant. So in total, one needs $O(n)$ gates due to CCX to implement this circuit. Also, the action of each Toffoli gate here depends on the previous ones and thus the circuit depth is equal to the number of CCX gates, i.e., $O(n)$. Since we use $(n-1)$ ancillas as working qubits, therefore the resources used here scale as $O(n)$. So in conclusion, the resources used by this circuit scale linearly with the number of qubits.

II. METHOD 2: BARENCO ET AL. (1995)

The next method is due to Barenco et al. discussed in Ref. [2]. Here one first decomposes the required U-gate into its fractional power matrices. For example, if we want to implement a C^2U as in Fig. 2, then one must first find the square root $V = \sqrt{U}$. For the case of $U = X$, the result is $V = \sqrt{X} = (1-i)(I+iX)/2$ [1].

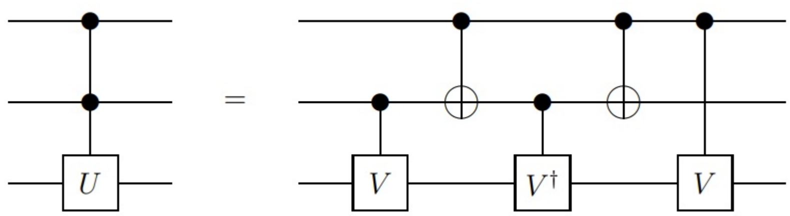

FIG. 2: A quantum circuit implementing C^2U. Taken from [2].

Notice that this decomposition does not require ancillas and only uses two-qubit gates as compared to the method we discussed in the last section. Now let us look at the case where $U = X$ starting with the initial state $|\Psi_0\rangle = |1\rangle|1\rangle|0\rangle$ with the circuit operator ξ having the following mathematical form:

$$\xi = C\sqrt{X}_{c_1t_1}CX_{c_1c_2}C\sqrt{X}^{\dagger}_{c_2t_1}CX_{c_1c_2}C\sqrt{X}_{c_2t_1}, \tag{3}$$

which when acting on the initial state $|\Psi_0\rangle$, gives us the following results step by step (from right to left side of ξ):

$$\begin{aligned}
|1\rangle|1\rangle|0\rangle &\longrightarrow |1\rangle|1\rangle\sqrt{X}|0\rangle && (4)\\
&\longrightarrow |1\rangle|0\rangle\sqrt{X}|0\rangle && (5)\\
&\longrightarrow |1\rangle|0\rangle\sqrt{X}|0\rangle && (6)\\
&\longrightarrow |1\rangle|1\rangle\sqrt{X}|0\rangle && (7)\\
&\longrightarrow |1\rangle|1\rangle\sqrt{X}\sqrt{X}|0\rangle = |1\rangle|1\rangle|1\rangle. && (8)
\end{aligned}$$

Notice, that in the second line the bit-value of second control qubit becomes 0 and thus in the next step where a $\sqrt{X}^{\dagger}$ was going to be implemented on target conditional on second control qubit, doesn't get affected. This is important because according to $\sqrt{X}^{\dagger}\sqrt{X} = I$, we would have otherwise lost our $\sqrt{X}$ needed in the last step. On the other hand, if we had instead started with $|100\rangle$, then $\sqrt{X}^{\dagger}$ would have acted in the third line and cancelled the $\sqrt{X}$ from the first line leading to no change, i.e., $|100\rangle \rightarrow |100\rangle$. Similarly, one can implement C^3U by finding the fourth root of the unitary gate U as shown in Fig.3 below.

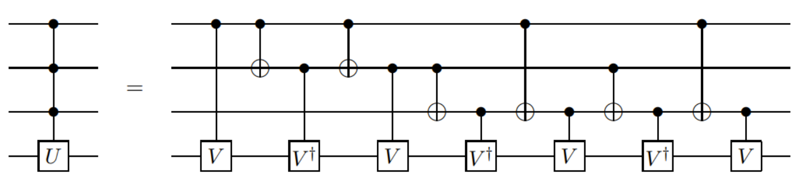

FIG. 3: A quantum circuit implementing C^3U. Taken from [2]

To count the resources, firstly we don't use any ancillas here so there is no scaling here. As for the number of gates required, Ref. [2] (page 22) say that this scales quadratically, i.e., $O(n^2)$, whereas the circuit depth scales linearly due to the same reasons as the method discussed in the previous section. It is possible that these scalings could be further improved by having better circuits [3] that simulate controlled U-gates more efficiently.

ACKNOWLEDGMENTS

[1] Michael A. Nielsen and Isaac L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 10th Anniversary Edition, 2010.
[2] A. Barenco et al., "Elementary gates for quantum computation," Phys. Rev. A **52**, 3457 (1995).
[3] Adenilton J. da Silva1 and Daniel K. Park, "Linear-depth quantum circuits for multiqubit controlled gates," Phys. Rev. A **106**, 042602 (2022).