The Steane Code Under Random Pauli Noise

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Quantum information is extremely sensitive to errors from environmental noise, and so the area of Quantum Error Correction (QEC) was invented which is a method to detect and correct such errors. The Steane code is an example of a quantum error correction code (QECC) an is represented as $[[7, 1, 3]]$, i.e., it requires 7 physical qubits to prepare one logical qubit, and has the ability to correct any single-qubit error using stabilizer measurements. In this project, we implement the Steane code using Qiskit, apply a random Pauli noise model, and measure how effectively the code preserves the logical qubit.

I. MATHEMATICAL BACKGROUND

A. The Steane Code

The Steane code [1, 2] is a CSS code derived from the classical $[7, 4, 3]$ Hamming code. It encodes 1 logical qubit into 7 physical qubits and has distance 3, meaning it can detect and correct any single-qubit error. It is constructed using two classical codes: one to correct bit-flip (X) errors and one for phase-flip (Z) errors. The stabilizers are derived from the parity-check matrix of the Hamming code:

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This matrix defines both X and Z stabilizers for syndrome extraction. Now in order to encode the logical state (using the Steane QECC), we encode the logical $|0\rangle_L$ by using the following circuit:

- Apply Hadamard gates on qubits 0, 1, and 2.
- Use CNOT gates according to the Steane encoding to entangle the qubits based on the classical generator matrix.

The circuit prepares a $+1$ eigenstate of all stabilizers, which defines the logical $|0\rangle$. The next step is to define how syndrome measurements will be performed, where to detect errors, we measure the stabilizers using ancilla qubits as follows:

- Z-stabilizers are measured using X-basis ancilla prepared with Hadamards.
- X-stabilizers are measured using Z-basis ancilla (no pre-rotation).

The ancillas are entangled with subsets of the data qubits according to the parity-check matrix, and then measured to obtain the syndrome. This procedure requires usage of more qubits (as ancillas), but still is efficient because it measures the syndrome aspects of the circuit without directly disturbing the data qubits.

B. The Pauli Error Model

Till now we have described how to create the 'guard', i.e., in the even that an error occurs, how to have a mechanism that eliminates it, which is provided by the above Steane code. Now, in order to test this, we assume that each qubit has the ability to independently interact with a Markovian environment [1] and thus go through some kind of noisy channel. If each qubit undergoes an independent error described by:

$$\mathcal{E}(\rho) = (1 - 3p)\rho + p(X\rho X + Y\rho Y + Z\rho Z)$$

with probabilities:

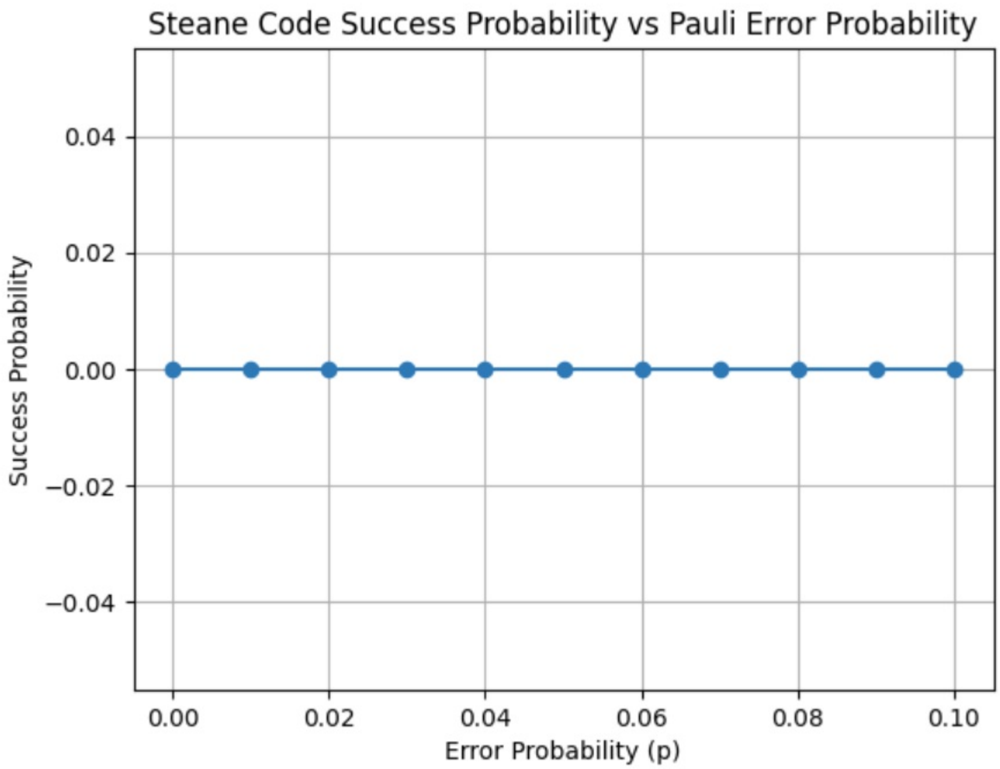
$$P(I) = 1 - 3p, \quad P(X) = P(Y) = P(Z) = p,$$

then we call this the random Pauli channel, and this what we will be testing for, i.e., when the qubits protected by the Steane code undergo such a noisy channel, then what are the chances that the quantum information stored in the form of logical states survives with increasing error probability p.

II. SIMULATION WITH NOISE

We simulate the circuit using Qiskit [3] and apply the noise model using the `NoiseModel` class. The noise is applied using identity gates that serve as placeholders for noisy operations. The simulation is run with a specified number of shots (e.g., 1024), and the result is decoded by counting how often the logical all-zero state is recovered.

We ran simulations for different values of the error probability p. The following figure shows the success probability:



My expectation was that the success probability should decrease as the error rate increases, which would have demonstrated how resilient the Steane code is to small errors bit it's also limited at higher noise levels. However, that is not the case here and the preliminary results shown by my code suggest a flat curve, i.e., the success probability has nothing to do with noise. This is of course incorrect and further investigation needs to be made in the code.

Once we achieve in correcting our code and getting a sensing graph for the success probability, in the future work, we could look into comparing our results with other codes (e.g., Shor, surface code), and probably extending this work to implementation of fault-tolerance in quantum circuits.

ACKNOWLEDGMENTS

[1] Michael A. Nielsen and Isaac L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 10th Anniversary Edition, 2010.
[2] A. M. Steane, "Error Correcting Codes in Quantum Theory," *Phys. Rev. Lett.* **77**, 793 (1996). doi:10.1103/PhysRevLett.77.793
[3] H. Abraham *et al.*, "Qiskit: An Open-source Framework for Quantum Computing," (2019). `https://qiskit.org`