

Table 1: Key units in Physical Chemistry

N_{Av} :	6.02214×10^{23}	mol^{-1}		
1 amu:	1.6605×10^{-27}	kg		
k_{B} :	1.38065×10^{-23}	J K^{-1}	8.61734×10^{-5}	eV K^{-1}
R :	8.314472	$\text{J K}^{-1} \text{mol}^{-1}$	8.2057×10^{-2}	$\text{l atm mol}^{-1} \text{K}^{-1}$
σ_{SB} :	5.6704×10^{-8}	$\text{J s}^{-1} \text{m}^{-2} \text{K}^{-4}$		
c :	2.99792458×10^8	m s^{-1}		
h :	6.62607×10^{-34}	J s	4.13566×10^{-15}	eV s
\hbar :	1.05457×10^{-34}	J s	6.58212×10^{-16}	eV s
hc :	1239.8	eV nm		
e :	1.60218×10^{-19}	C		
m_e :	$9.10938215 \times 10^{-31}$	kg	1: 0.5109989	MeV c^{-2}
ϵ_0 :	8.85419×10^{-12}	$\text{C}^2 \text{J}^{-1} \text{m}^{-1}$	5.52635×10^{-3}	$e^2 \text{\AA}^{-1} \text{eV}^{-1}$
$e^2/4\pi\epsilon_0$:	2.30708×10^{-28}	J m	14.39964	eV \AA
a_0 :	0.529177×10^{-10}	m	0.529177	\AA
E_{H} :	1	Ha	27.212	eV

1 The Classical Foundations

1.1 Lecture 0: Introduction

1. Burning lighter
2. Foundations of Physical Chemistry
 - (a) Quantum mechanics
 - (b) Statistical mechanics
 - (c) Thermodynamics, kinetics, spectroscopy
 - (d) Physical and chemical properties of matter

1.2 Lecture 1: Basic statistics

1. Discrete probability distributions—Coin flip
 - (a) Example of Bernoulli trial, 2^n possible outcomes from n flips
 - (b) Number of ways to get i heads in n flips, ${}_nC_i = n!/i!(n-i)!$
 - (c) Probability of i heads $P_i \propto {}_nC_i$
 - (d) Normalized probability, $\tilde{P}_i = P_i / \sum_i P_i = {}_nC_i / 2^n$
 - (e) Expectation value $\langle i \rangle = \sum_i i \tilde{P}_i$
2. Continuous distributions—temperature
 - (a) Probability density $P(x)$ has units $1/x$
 - (b) Normalized $\tilde{P}(x) = P(x) / \int P(x) dx$
 - (c) (Unitless) probability $a < x < b = \int_a^b \tilde{P}(x) dx$

(d) Expectation value $\langle f(x) \rangle = \int f(x) \tilde{P}(x) dx$

(e) Mean = $\langle x \rangle$

(f) Mean squared = $\langle x^2 \rangle$

(g) Variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

(h) Standard deviation $\Delta x = \sigma$

3. Boltzmann distribution

(a) $P(E) \propto e^{-E/k_B T}$, in some sense the definition of temperature

(b) Energy and its units

(c) Absolute temperature and its units

(d) $k_B T$ as an energy scale, 0.026 eV at 298 K

(e) Gravity example

i. $E(h) = mgh$, linear, continuous energy spectrum

ii. molecule vs car in a gravitational field (Table 2)

iii. Barometric law for gases, $P = P_0 e^{-mgh/k_B T}$

(f) Kinetic energy in 1-D example

i. $KE = \frac{1}{2} m v_x^2$

ii. $P_{1D}(v_x) = \left(\frac{m}{2\pi k_B T} \right)^{1/2} \exp \left(-\frac{m|v_x|^2}{2k_B T} \right)$

iii. Gaussian distribution, mean μ , variance σ^2

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left(-\frac{(x - \mu)^2}{2\sigma^2} \right)$$

iv. By inspection, $\mu = \langle v_x \rangle = 0$, $\sigma^2 = \langle v_x^2 \rangle = k_B T/m$

v. Molecule vs car again

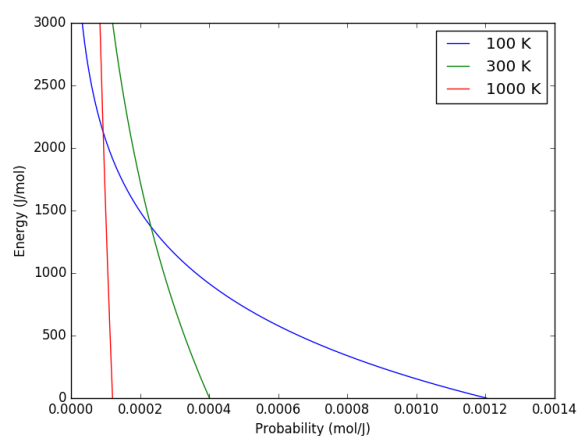
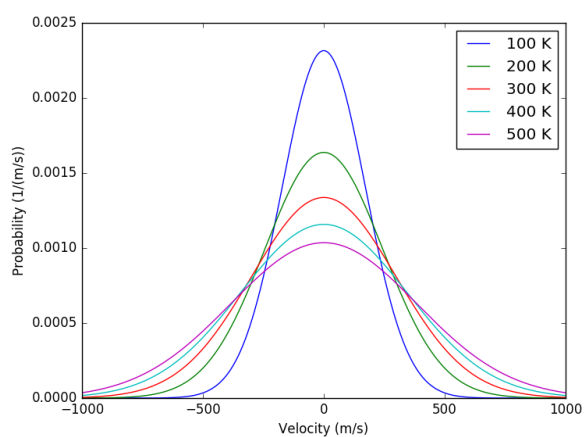
(g) Equipartition – energy freely exchanged between all degrees of freedom

Table 2: Car vs gas molecule at the earth's surface

	car	gas molecule
m	1000 kg	1×10^{-26} kg
h	1 m	1 m
mgh	9800 J	9.8×10^{-26} J
	6.1×10^{22} eV	6.1×10^{-7} eV
T	298 K	298 K
$k_B T$	0.026 eV	0.026 eV
$mgh/k_B T$	2.4×10^{24}	2.3×10^{-5}
$P(1 \text{ m})/P(0)$	$e^{-2.4 \times 10^{-24}}$	0.99998
$\langle h \rangle$	0 m	42 km
$\langle v_x \rangle^{1/2}$	2×10^{-12} m/s	640 m/s

Table 3: Energy conversions and correspondences

	J	eV	Hartree	kJ mol^{-1}	cm^{-1}
1 J =	1	6.2415×10^{18}	2.2937×10^{17}	6.0221×10^{20}	5.0340×10^{22}
1 eV =	1.6022×10^{-19}	1	0.036748	96.485	8065.5
1 Ha =	4.3598×10^{-18}	27.212	1	2625.6	219474.6
1 kJ mol^{-1} =	1.6605×10^{-21}	0.010364	3.8087×10^{-4}	1	83.5935
1 cm^{-1} =	1.986410^{-23}	1.23984×10^{-4}	4.55623×10^{-6}	0.011963	1

**Figure 1:** Boltzmann distribution at various temperatures**Figure 2:** One-dimensional (Gaussian) velocities of N_2 gas

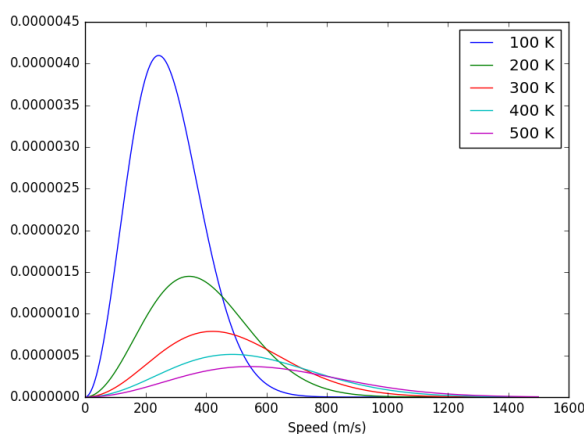


Figure 3: Maxwell-Boltzmann speed distribution of N_2 gas

1.3 Lecture 2: Kinetic theory of gases

1. Postulates

- (a) Gas is composed of molecules in constant random, thermal motion
- (b) Molecules only interact by perfectly elastic collisions
- (c) Volume of molecules is \ll total volume

2. Maxwell-Boltzmann distribution of molecular speeds

- (a) Speed $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$
- (b) $P_{MB}(v)dv = P_{1D}(v_x)P_{1D}(v_y)P_{1D}(v_z) * \text{degeneracy}(v)dv$
- (c) mean speeds $\propto \sqrt{T}$
- (d) mean energy $U = \frac{3}{2}RT$ and heat capacity $C_v = \frac{3}{2}R$

3. Flux and pressure

- (a) Velocity flux $j(v_x)dv_x = v_x \frac{N}{V} P(v_x)dv_x$, molecules /area /time / v_x
- (b) Wall collisions, J_w , total collisions /area /time
- (c) Momentum exchange, pressure, ideal gas law

4. Collisions and mean free path

- (a) Collision cross section $\sigma = \pi d^2$, size of molecule
- (b) Molecular collisions, z per molecule and z_{AA} per volume
- (c) Mean free path, λ , mean distance between collisions

Table 4: Kinetic theory of gases key equations

Boltzmann distribution ($g(E)$: degeneracy of E)	$P(E) = g(E)e^{-E/k_B T}$
Maxwell-Boltzmann distribution	$P_{\text{MB}}(v) = 4\pi v^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right)$
Mean and RMS speeds	$\langle v \rangle = \left(\frac{8k_B T}{\pi m}\right)^{1/2} \quad \langle v^2 \rangle^{1/2} = \left(\frac{3k_B T}{m}\right)^{1/2}$
Pressure	$\langle P \rangle = \frac{\Delta p}{\Delta t} = m \frac{N}{V} \frac{1}{3} \langle v^2 \rangle = \frac{N k_B T}{V} = \frac{n R T}{V}$
Wall collision frequency	$J_W = \frac{1}{4} \frac{N}{V} \langle v \rangle = \frac{P}{(2\pi m k_B T)^{1/2}}$
Molecular collision frequency	$z = \sqrt{2} \sigma \langle v \rangle \frac{N}{V} = \frac{4\sigma P}{(\pi m k_B T)^{1/2}}$
Total collisions	$z_{AA} = \frac{1}{2} \frac{N}{V} z$
Mean free path	$\lambda = \frac{\langle v \rangle}{z} = \frac{V}{\sqrt{2} \sigma N}$
Graham's effusion law	$\frac{dN}{dt} = \text{Area} \cdot J_w \propto 1/m^{1/2}$
Effusion from a vessel	$P = P_0 e^{-t/\tau}, \tau = \frac{V}{A} \left(\frac{2\pi m}{k_B T}\right)^{1/2}$
Self-diffusion constant	$D_{11} = \frac{1}{3} \langle v \rangle \lambda$
Diffusion rate	$\langle x^2 \rangle^{1/2} = \sqrt{2Dt} \quad \langle r^2 \rangle^{1/2} = \sqrt{6Dt}$
Einstein-Smoluchowski equation	$D_{11} = \frac{\delta^2}{2\tau}$
Stokes-Einstein equation for liquids	$D_{11} = \frac{k_B T}{4\pi\eta r}$ "Slip" boundary
	$D_{\text{Brownian}} = \frac{k_B T}{6\pi\eta r}$ "Stick" boundary

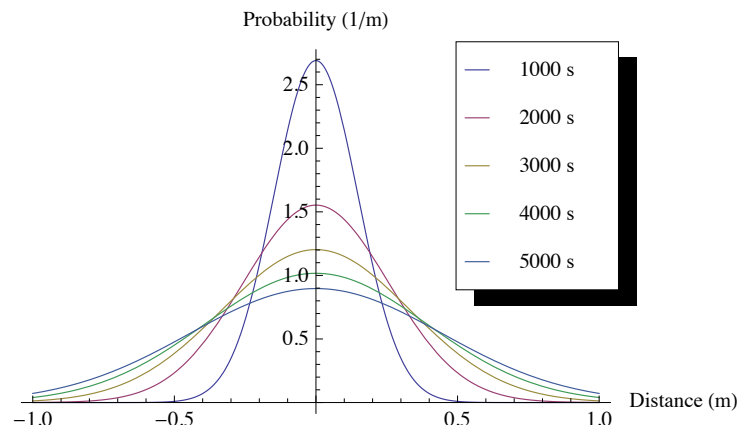


Figure 4: Diffusional spreading, $\sqrt{\langle x^2 \rangle} = \sqrt{2Dt}$

1.4 Lecture 3: Transport

1. Effusion and Graham's law, effusion rate $\propto MW^{-1/2}$
2. Fick's first law: net flux proportional to concentration gradient
 - (a) $j_x = -D \frac{dc}{dx}$
 - (b) Self-diffusion constant, $D = \frac{1}{3} \lambda \langle v \rangle$
3. Knudsen diffusion, $D = \frac{1}{3} l \langle v \rangle$
4. Fick's second law: time evolution of concentration gradient
 - (a) Continuity with no advection: $\frac{\partial c}{\partial t} = -\nabla \cdot \vec{j} + \text{gen}$
 - (b) One-dimension: $\frac{dc}{dt} = D \frac{d^2 c}{dx^2}$
 - (c) Diffusion has Gaussian probability distribution: $c(x, t)/c_0 = [2\sqrt{\pi Dt}]^{-1} \exp(-x^2/4Dt)$
5. Seeing is believing—Brownian motion
 - (a) Seemingly random motion of large particles (“dust”) due to “kicks” from invisible molecules
 - (b) Einstein receives Nobel Prize for showing:
 - i. Motion follows same Gaussian diffusion behavior
 - ii. From steady-state arguments in a field, diffusion constant is ratio of Boltzmann energy, $k_B T$, to mobility
 - iii. Mobility inversely related to viscosity
 - (c) Stokes-Einstein equation
 - (d) Allows measurement of Avogadro's number, final proof of kinetic theory
 - (e) Similar model for diffusion of liquid molecules, slip boundary
6. Random walk model of diffusion
 - (a) Binomial distribution
 - (b) Large N and Stirling approximation
 - (c) Einstein-Smoluchowski relation

Table 5: Classical waves

The wave equation	$\frac{\partial^2 \Psi(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x, t)}{\partial t^2}$
General solution	$\Psi(x, t) = A \sin(kx - \omega t)$
Wavelength (distance)	$\lambda = 2\pi/k$
Frequency (/time)	$\nu = \omega/2\pi$
Speed	$v = \lambda\nu$
Amplitude (distance)	A
Energy	$E \propto A^2$
Standing wave	$\Psi(x, t) = A \sin(kx) \cos(\omega t), \quad k = n\pi/a$

2 Quantum Mechanics: Blurred Lines Between Particles and Waves

2.1 Lecture 4: Duality and demise of classical physics

2.1.1 Properties of waves

1. Traveling waves, standing waves
2. interference, diffraction
3. Expected energy of a classical oscillator, $\langle \epsilon \rangle_\nu = k_B T$ for all ν

2.1.2 Blackbody radiation

1. Hohlraum spectrum (like the sun) empirically observed to obey:
 - (a) Stefan-Boltzmann law, total irradiance
 - (b) Wien's displacement law
2. Rayleigh-Jeans predicts spectrum using classical physics
 - (a) standing waves + classical oscillators \rightarrow ultraviolet catastrophe
3. Planck model
 - (a) Energy spectrum of oscillators are *quantized*, $\epsilon_\nu = nh\nu$
 - (b) Expected energy of a quantized oscillator, $\langle \epsilon \rangle_\nu = h\nu / (e^{h\nu/k_B T} - 1)$
 - (c) Correctly reproduces Stefan-Boltzmann and Wien Laws!

2.1.3 Heat capacities of solids

1. Law of DuLong and Pettite, $C_v = 3R$, fails at low T
2. Einstein model
 - (a) Atomic vibrations are *quantized*, $\epsilon_n = nh\nu$
 - (b) Heat capacity goes to zero at low T

2.1.4 Photoelectric effect

1. Stopping potential and work function, $E_{\text{kinetic}} = h\nu - W$
2. Kinetic energy varies with light frequency, number of electrons varies with light intensity

2.1.5 Compton effect

1. light scattering of electrons changes λ
2. Photon properties, $\epsilon = h\nu, p = h/\lambda$

2.1.6 Wave-particle duality**2.1.7 Rutherford, planetary model of atom**

1. Inconsistent with Maxwell's equations

2.1.8 Bohr model of H atom

1. Discrete H energy spectrum and Rydberg formula
2. Bohr model (the old quantum mechanics)
 - (a) Stable electron "orbits," quantized angular momentum
 - (b) Light emission corresponds to orbital jumps, $\nu = \Delta E/h$
 - (c) Bohr equations
 - (d) Comparison with Rydberg formula
 - (e) Failure for larger atoms

2.1.9 de Broglie relation

1. $\lambda = h/p$ *universally*
2. Relation to Bohr orbits
3. Davison and Germer experiment, e^- diffraction off Ni

```

1  import numpy as np
2  import matplotlib.pyplot as plt
3
4  hc = 1239.8          eV nm
5  c = 2.9979e8 * 1.e9  nm/s
6  k = 8.61734e5       eV /K
7  hck = hc/k          nm K
8
9  def Irrad(wl,T):
10     return (8. * np.pi * hc * c * wl**5) / (np.exp(hck/(wl*T))1)
11  def PlanckEnergy(wl,T):
12     return (hc/wl) / (np.exp(hck/(wl*T))1)
13
14  plt.figure()
15  wl=np.linspace(100,5000,1000)
16  for T in [1000.,2000.,3000.,4000.,5000.]:
17     Intensity = Irrad(wl,T)
18     plt.plot(wl,Intensity,label= K.format(T))

```

```

19
20 legend=plt.legend()
21 plt.xlabel(Wavelength (nm))
22 plt.ylabel(Irradiance (eV/nm3/s))
23 plt.title(Boltzmann distribution at various temperatures)
24 plt.savefig(./Images/BlackBody.png)
25
26 plt.figure()
27 color=[red,orange,green,blue,violet]
28 wl=np.linspace(100,20000,1000)
29 for T in [1000.,2000.,3000.,4000.,5000.]:
30     Energy = PlanckEnergy(wl,T)
31     plt.plot(wl,Energy,label= K.format(T),color=color[0])
32     kT = k*T
33     plt.plot([100,max(wl)], [kT,kT],ls=,color=color.pop(0))
34
35 legend=plt.legend()
36 plt.xlabel(Wavelength (nm))
37 plt.ylabel(Energy (eV))
38 plt.title(Boltzmann distribution at various temperatures)
39 plt.savefig(./Images/Planck.png)

```

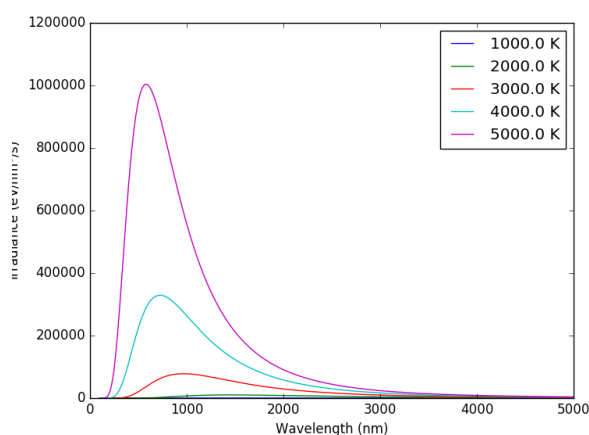


Figure 5: Blackbody irradiance

2.2 Lecture 5: Postulates of quantum mechanics

2.2.1 Schrödinger equation describes wave-like properties of matter

2.2.2 Born interpretation

1. wavefunction is a probability amplitude
2. wavefunction squared is probability density

2.2.3 Postulates

1. Wavefunction contains all information about a system
2. Operators used to extract that information
 - (a) QM operators are *Hermitian*

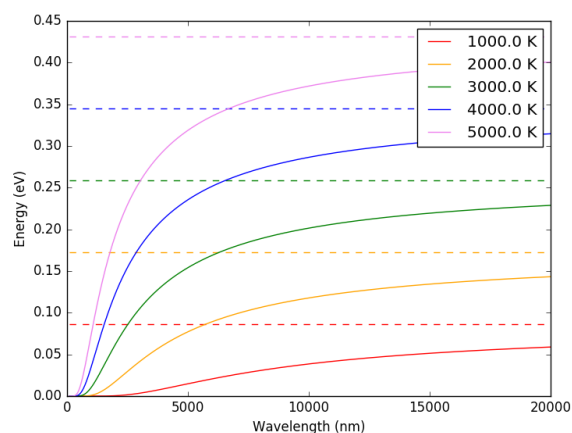


Figure 6: Average energy of a Planck quantized oscillator

Table 6: The new physics

Stefan-Boltzmann Law	$\int I(\lambda, T) d\lambda = \sigma_{\text{SB}} T^4$
Wien's Law	$\lambda_{\text{max}} T = 2897768 \text{ nm K}$
Rayleigh-Jeans eq	$I(\lambda, T) = \frac{8\pi}{\lambda^4} k_B T c$
Blackbody irradiance	$I(\lambda, T) = \frac{8\pi}{\lambda^5} \frac{hc^2}{e^{hc/\lambda k_B T} - 1}$
Einstein crystal	$C_v = 3R \left(\frac{h\nu}{k_B T} \right)^2 \frac{e^{h\nu/k_B T}}{(e^{h\nu/k_B T} - 1)^2}$
Photon energy	$\epsilon = h\nu = hc/\lambda$
Rydberg equation	$\nu = R_H c \left(1/n^2 - 1/k^2 \right)$
Bohr equations	$l_n = n\hbar$
$n = 1, 2, \dots$	$r_n = n^2 \left(\frac{4\pi\epsilon_0 \hbar^2}{e^2 m_e} \right) = n^2 a_0$
	$E_n = -\frac{m_e e^4}{8\epsilon_0^2 \hbar^2} \frac{1}{n^2} = -\frac{E_H}{2} \frac{1}{n^2}$
	$p_n = \frac{e^2}{4\pi\epsilon_0} \frac{m_e}{\hbar} \frac{1}{n} = p_0 \frac{1}{n}$
de Broglie equation	$\lambda = \frac{h}{p}$

- (b) Have eigenvectors and real eigenvalues, $\hat{O}\psi_i = o\psi_i$
 - (c) Are orthogonal, $\langle\psi_i|\psi_j\rangle = \delta_{ij}$
 - (d) Always observe an eigenvalue when making an observation
3. Expectation values
 4. Energy-invariant wavefunctions given by Schrödinger equation
 5. Uncertainty principle

2.2.4 Particle in a box illustrations

2.3 Lecture 6: Particle in a box model

2.3.1 Particle between infinite walls, electron confined in a wire

2.3.2 Classical solution, either stationary or uniform bouncing back and forth

2.3.3 One-dimensional QM solutions

1. Schrödinger equation and boundary conditions
2. discrete, quantized solutions
3. standing waves, $\lambda = 2L/n$, $n - 1$ nodes, non-uniform probability
4. [Ho paper](#), STM of Pd wire
5. zero point energy and uncertainty
6. correspondence principle
7. superpositions

2.3.4 Finite walls and tunneling

1. Potential well of finite depth V_0
2. Finite number of bound states
3. Classical region, $\psi(x) e^{ikx} + e^{-ikx}$, $k = \sqrt{2mE}/\hbar$
4. “Forbidden” region, $\psi(x) e^{\kappa x} + e^{-\kappa x}$, $\kappa = \sqrt{2m(V_0 - E)}/\hbar$
5. Non-zero probability to “tunnel” into forbidden region
6. Tunneling between two adjacent wells: chemical bonding, STM, nanoelectronics
7. H atom tunneling: NH_3 inversion, H transfer, kinetic isotope effect

2.3.5 Multiple dimensions

1. separation of variables, one quantum number for each dimension

Table 7: Postulates of Non-relativistic Quantum Mechanics

Postulate 1: The physical state of a system is completely described by its wavefunction Ψ . In general, Ψ is a complex function of the spatial coordinates and time. Ψ is required to be:

- I. Single-valued
- II. continuous and twice differentiable
- III. square-integrable ($\int \Psi^* \Psi d\tau$ is defined over all finite domains)
- IV. For bound systems, Ψ can always be normalized such that $\int \Psi^* \Psi d\tau = 1$

Postulate 2: To every physical observable quantity M there corresponds a Hermitian operator \hat{M} . **The only observable values of M are the eigenvalues of \hat{M} .**

Physical quantity	Operator	Expression
Position x, y, z	$\hat{x}, \hat{y}, \hat{z}$	x, y, z
Linear momentum p_x, \dots	\hat{p}_x, \dots	$-i\hbar \frac{\partial}{\partial x}, \dots$
Angular momentum l_x, \dots	\hat{l}_x, \dots	$-i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \dots$
Kinetic energy T	\hat{T}	$-\frac{\hbar^2}{2m} \nabla^2$
Potential energy V	\hat{V}	$V(\mathbf{r}, t)$
Total energy E	\hat{H}	$-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t)$

Postulate 3: If a particular observable M is measured many times on many identical systems in a state Ψ , the average results will be the expectation value of the operator \hat{M} :

$$\langle M \rangle = \int \Psi^* (\hat{M} \Psi) d\tau$$

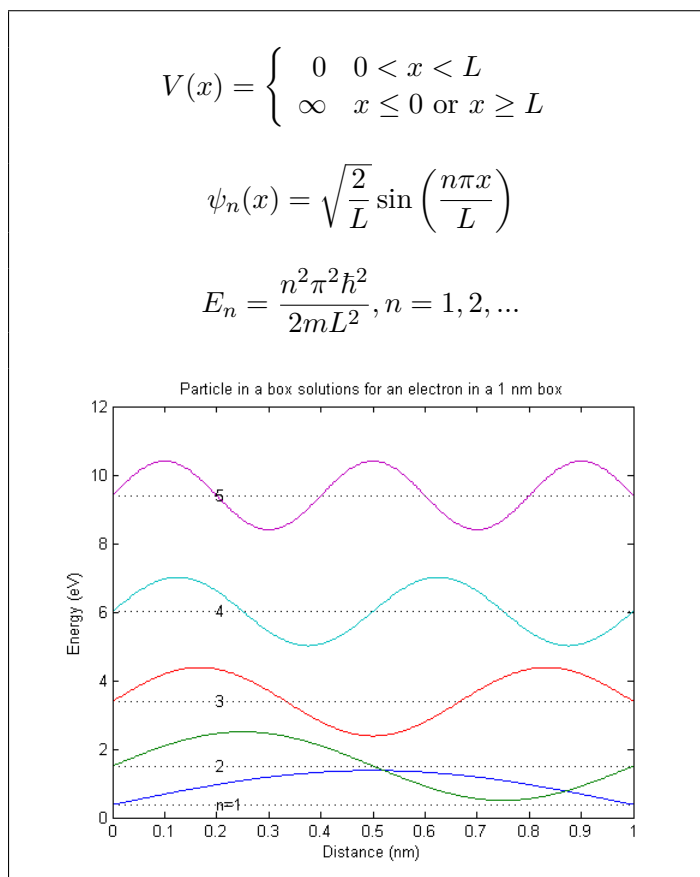
Postulate 4: The energy-invariant states of a system are solutions of the equation

$$\begin{aligned} \hat{H} \Psi(\mathbf{r}, t) &= i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) \\ \hat{H} &= \hat{T} + \hat{V} \end{aligned}$$

The time-independent, stationary states of the system are solutions to the equation

$$\hat{H} \Psi(\mathbf{r}) = E \Psi(\mathbf{r})$$

Postulate 5: (The **uncertainty principle**.) Operators that do not commute ($\hat{A}(\hat{B}\Psi) \neq \hat{B}(\hat{A}\Psi)$) are called *conjugate*. Conjugate observables cannot be determined simultaneously to arbitrary accuracy. For example, the standard deviation in the measured positions and momenta of particles all described by the same Ψ must satisfy $\Delta x \Delta p_x \geq \hbar/2$.

Table 8: Particle-in-a-box model**2.3.6 Introduce Pauli principle for fermions?****2.4 Lecture 7: Harmonic oscillator****2.4.1 Classical harmonic oscillator**

1. Hooke's law, $F = -k(x - x_0)$, k spring constant
2. Continuous sinusoidal motion
3. $x(t) = A \sin(\frac{k}{\mu})^{1/2} t$, $\nu = \frac{1}{2\pi}(\frac{k}{\mu})^{1/2}$, $E = \frac{1}{2}kA^2$
4. Exchanging kinetic and potential energies

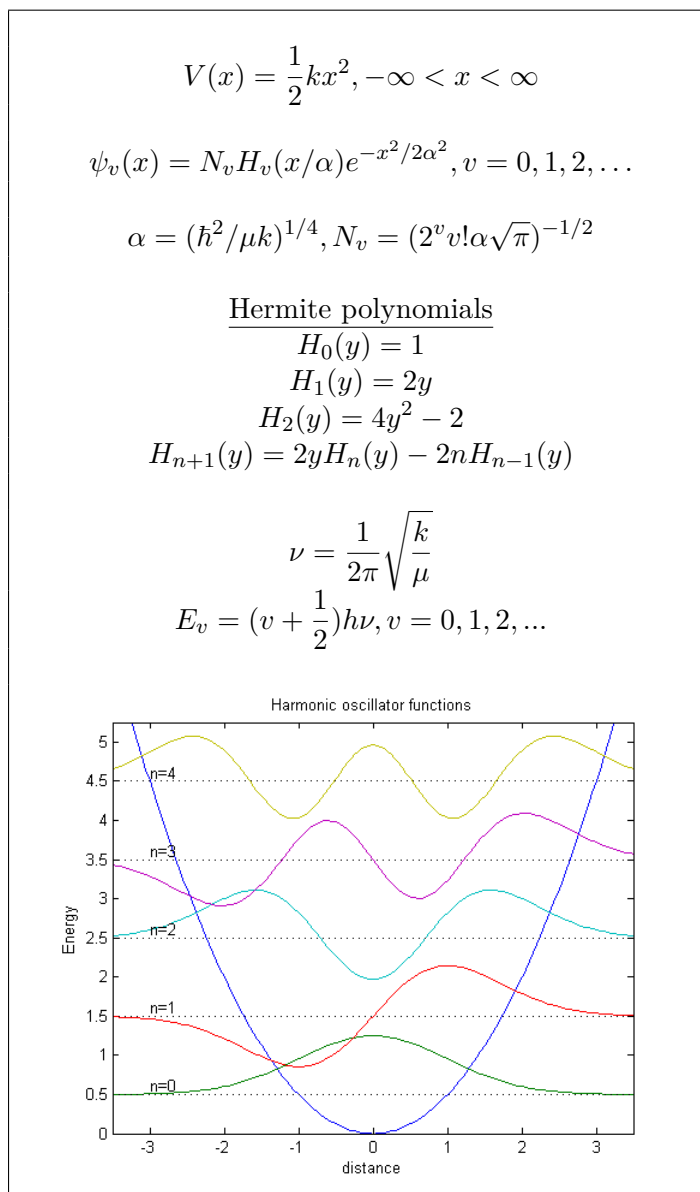
2.4.2 Quantum harmonic oscillator

1. Solutions like P-I-A-B, waves, nodes, even/odd symmetry
2. Zero-point energy
3. Expectation values: $\langle x \rangle = 0$, $\langle x^2 \rangle = \alpha^2(v + 1/2)$, $\langle V(x) \rangle = \frac{1}{2}h\nu(v + \frac{1}{2})$
4. Classical turning point and tunneling

5. Classical limiting behavior

2.4.3 HCl example

1. Reduced mass, $\frac{1}{\mu} = \frac{1}{m_A} + \frac{1}{m_B}$
2. ZPE, energy spacing in IR, Boltzmann probabilities

Table 9: Harmonic oscillator model

2.5 Lecture 8: Rigid Rotor

2.5.1 Classical rigid rotor

1. Compare rotation about an axis vs linear motion
2. Moment of inertia $I = \mu r^2$
3. Angular momentum, $\mathbf{l} = I\boldsymbol{\omega} = \mathbf{r} \times \mathbf{p}$, $T = l^2/2I$
 - (a) Angular momentum and energy continuous variables

2.5.2 Quantum rotor in a plane

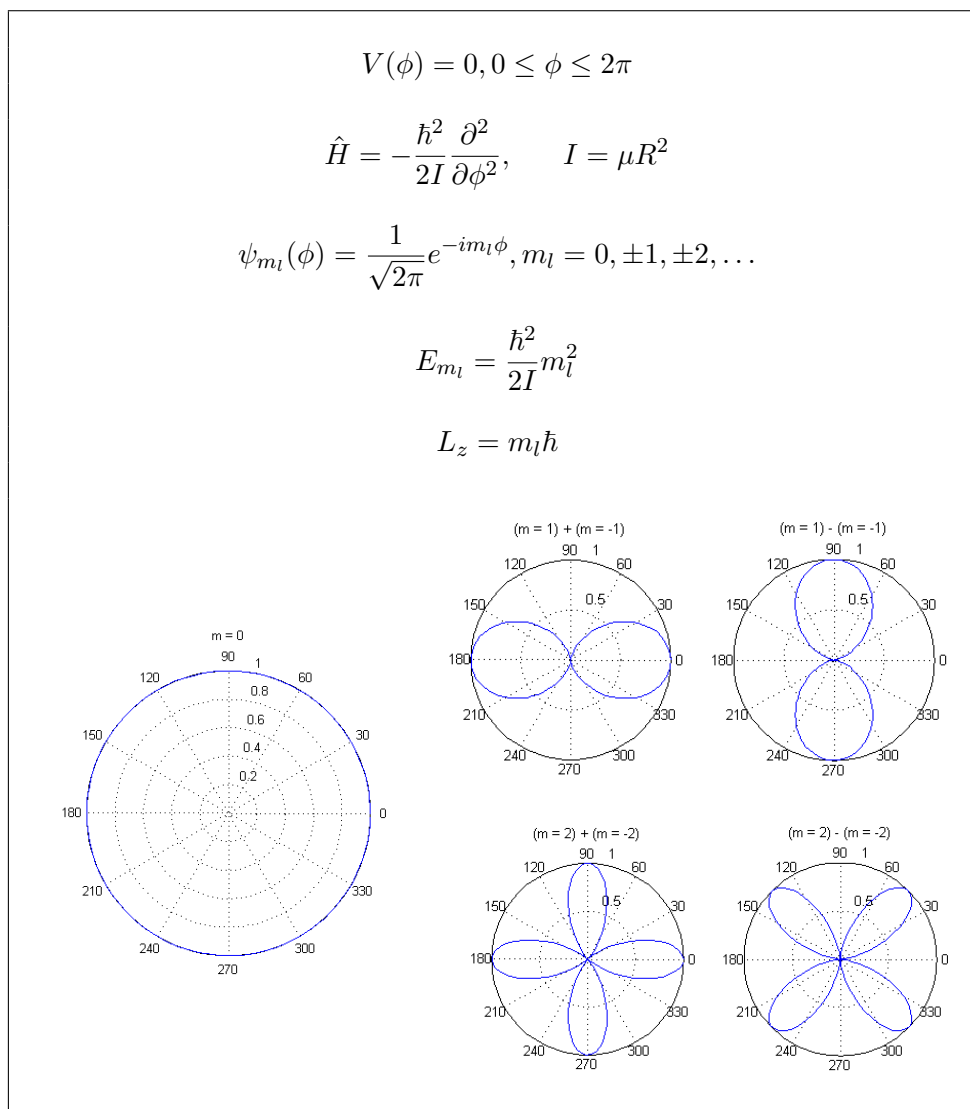
1. Angular momentum and kinetic energy operators in polar coordinates, $\hat{l}_z = -i\hbar \frac{d}{d\phi}$
2. Eigenfunctions degenerate, cw and ccw rotation
3. No zero point energy
4. Angular momentum eigenfunctions, $l_z = m_l \hbar$
5. Energy superpositions and localization

2.5.3 Quantum rotor in 3-D

1. Angular momentum and kinetic energy operators in spherical coordinates
2. Spherical harmonic solutions, Y_{lm_l}
3. Azimuthal QN $l = 0, 1, \dots$
4. Magnetic QN $m_l = -l, -l+1, \dots, l$
5. Energy spectrum, $2l+1$ degeneracy
6. Vector model - can only know total $|L|$ and L_z
7. Wavefunctions look like atomic orbitals, l nodes

```

1  import matplotlib.pyplot as plt
2  from matplotlib import cm, colors
3  from mpl_toolkits.mplot3d import Axes3D
4  import numpy as np
5  from scipy.special import sphharm
6
7  phi = np.linspace(0, np.pi, 100)
8  theta = np.linspace(0, 2*np.pi, 100)
9  phi, theta = np.meshgrid(phi, theta)
10
11  The Cartesian coordinates of the unit sphere
12  x = np.sin(phi) * np.cos(theta)
13  y = np.sin(phi) * np.sin(theta)
14  z = np.cos(phi)
15
16  m, l = 0, 0
17
```

Table 10: 2-D rigid rotor model

```

18  Calculate the spherical harmonic Y(l,m) and normalize to [0,1]
19  fcolors = sphharm(m, l, theta, phi).real
20  fmax, fmin = fcolors.max(), fcolors.min()
21  fcolors = (fcolors - fmin)/(fmax - fmin)
22
23  Set the aspect ratio to 1 so our sphere looks spherical
24  sfig = plt.figure(figsize=plt.figaspect(1.))
25  s = sfig.add_subplot(111, projection='3d')
26  s.plotsurface(x, y, z, rstride=1, cstride=1, facecolors=cm.seismic(fcolors))
27  Turn off the axis planes
28  s.setaxisoff()
29  plt.savefig('./Images/s.png')
30
31  m, l = 0, 1
32
33  Calculate the spherical harmonic Y(l,m) and normalize to [0,1]
34  fcolors = sphharm(m, l, theta, phi).real
35  fmax, fmin = fcolors.max(), fcolors.min()

```


Table 11: 3-D rigid rotor model

$$\begin{aligned}
 V(\theta, \phi) &= 0, 0 \leq \phi \leq 2\pi, 0 \leq \theta < \pi \\
 \hat{L}^2 &= -\hbar^2 \left[\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right] \\
 \hat{H}_{\text{rot}} &= \frac{1}{2I} \hat{L}^2 \\
 Y_{lm_l}(\theta, \phi) &= N_l^{|m_l|} P_l^{|m_l|}(\cos(\theta)) e^{im_l \phi} \\
 l &= 0, 1, 2, \dots, \quad m_l = 0, \pm 1, \dots, \pm l \\
 E_l &= \frac{\hbar^2}{2I} l(l+1) \\
 |L| &= \hbar \sqrt{l(l+1)}, L_z = m_l \hbar
 \end{aligned}$$

```

36 fcolors = (fcolors - fmin)/(fmax - fmin)
37
38 Set the aspect ratio to 1 so our sphere looks spherical
39 fig = plt.figure(figsize=plt.figaspect(1.))
40 pfig = plt.figure(figsize=plt.figaspect(1.))
41 p = pfig.add_subplot(111, projection='3d')
42 p.plot_surface(x, y, z, rstride=1, cstride=1, facecolors=cm.seismic(fcolors))
43 Turn off the axis planes
44 p.setaxisoff()
45
46 plt.savefig('./Images/p.png')
47
48 m, l = 1, 2
49
50 Calculate the spherical harmonic Y(l,m) and normalize to [0,1]
51 fcolors = sphharm(m, l, theta, phi).real
52 fmax, fmin = fcolors.max(), fcolors.min()
53 fcolors = (fcolors - fmin)/(fmax - fmin)
54
55 Set the aspect ratio to 1 so our sphere looks spherical
56 fig = plt.figure(figsize=plt.figaspect(1.))
57 dfig = plt.figure(figsize=plt.figaspect(1.))
58 d = dfig.add_subplot(111, projection='3d')
59 d.plot_surface(x, y, z, rstride=1, cstride=1, facecolors=cm.seismic(fcolors))
60 Turn off the axis planes
61 d.setaxisoff()
62
63 plt.savefig('./Images/d.png')

```

2.5.4 Particle angular momentum

1. Fermions, mass, half-integer spin

(a) Electron, $s = 1/2, m_s = \pm 1/2$



Figure 7: Pythonic s spherical harmonic



Figure 8: Pythonic p spherical harmonic

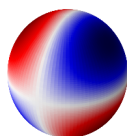


Figure 9: Pythonic d spherical harmonic

2. Bosons, force-carrying, integer spin

2.6 Lecture 9: Spectroscopy

2.6.1 Spectroscopy is quantitative measurement of interaction of light with matter

1. Observed $I(\nu)/I(\nu_0)$
2. Bohr condition, $|E_f - E_i|/h = \nu = c\tilde{\nu} = c/\lambda$
3. Intensities determined by state populations and transition probabilities

2.6.2 Einstein coefficients

1. Stimulated absorption, $dn_1/dt = -n_1 B \rho(\nu)$
2. Stimulated emission, $dn_2/dt = -n_2 B \rho(\nu)$
3. Spontaneous emission, $dn_2/dt = -n_2 A$, $A = \left(\frac{8\pi h \nu^3}{c^3}\right) B$
4. $1/A$ = lifetime

2.6.3 Transition probability

1. Einstein coefficient $B_{if} = \frac{|\mu_{if}|^2}{6\epsilon_0 \hbar^2}$
2. Classical electric dipole, $\vec{\mu} = q \cdot \vec{r}$, quantum dipole operator $\hat{\mu} = e \cdot \vec{r}$
3. Transition dipole moment, $\mu_{if} = \left(\frac{d\mu}{dx}\right) \langle \psi_i | \hat{\mu} | \psi_f \rangle$
4. Selection rules—conditions that make μ_{if} non-zero, “allowed” vs “forbidden” transitions

2.7 Lecture 10: Vibrational and rotational spectroscopy

2.7.1 Diatomic rotational spectroscopy

1. Rotational constant $B = \hbar/4\pi I c \text{ cm}^{-1}$, $I = \mu R^2$
2. Gross selection rule: dynamic dipole moment non-zero (heteronuclear, not homonuclear)
3. Specific selection rule: $\Delta l = \pm 1$, $\Delta m_l = 0, \pm 1$
4. $\Delta \tilde{E}_l = 2B(l+1) \text{ cm}^{-1}$
5. Rotational state populations

2.7.2 Diatomic vibrational transitions

1. Gross selection rule: dynamic dipole $d\mu/dx$ non-zero
2. Homo- vs. heteronuclear
3. Specific selection rule: dipole integral $\langle \psi_v | \hat{\mu} | \psi_{v'} \rangle = 0$ unless $\Delta v = \pm 1$
4. Allowed $\Delta E = h\nu$
5. Boltzmann distribution implies $v = 1$ states dominate at normal T

2.7.3 Raman spectroscopy

1. Shine in light of arbitrary frequency $\tilde{\nu}_0$, mostly get out the same
2. Some light comes out at $\tilde{\nu}_0 - \tilde{\nu}$ (Stoke's line)
3. Some light comes out at $\tilde{\nu}_0 + \tilde{\nu}$ (anti-Stoke's line)
4. Gross selection rule: dynamic polarizability non-zero (homonuclear, not heteronuclear)

2.7.4 Anharmonicity, Morse potential

2.7.5 Vibration-rotation spectroscopy

1. Harmonic oscillator + rigid rotor
2. Selection rules: $\Delta v = \pm 1, \Delta l = \pm 1$
3. R branch: $\Delta \tilde{E} = \tilde{\nu} + 2B(l+1), \Delta l = 1$
4. P branch: $\Delta \tilde{E} = \tilde{\nu} - 2B(l), \Delta l = -1$

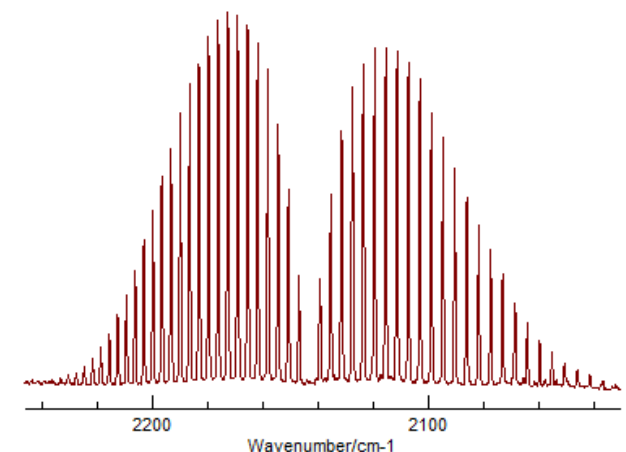


Figure 10: Rovibrational spectrum of carbon monoxide

2.7.6 Polyatomic vibrational spectroscopy

1. Polyatomics, $3n - 6$ ($3n - 5$ for linear polyatomic) vibrational modes
2. Selection rules and degeneracies affect number of observed features
3. CO₂ example

2.7.7 Polyatomic rotational spectroscopy

1. Three distinct moments of inertia (I_x, I_y, I_z)
2. Spectra more complex

2.8 Lecture 11: Hydrogen atom

2.8.1 Schrödinger equation

1. Spherical coordinates and separation of variables
2. Coulomb potential $v_{\text{Coulomb}}(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$
3. Centripetal potential $v = \hbar^2 \frac{l(l+1)}{2\mu r^2}$

2.8.2 Solutions

1. $\psi(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$
2. Principle quantum number $n = 1, 2, \dots$
 - (a) K, L, M, N, \dots shells
 - (b) $n - 1$ radial nodes
3. Azimuthal quantum number $l = 0, 1, \dots, n - 1$
 - (a) s, p, d, \dots orbital sub-shells
 - (b) l angular nodes
4. Magnetic quantum number $m_l = -l, -l + 1, \dots, l$
5. Spin quantum number $m_s = \pm 1/2$
6. Energy spectrum and populations
7. Electronic selection rules
 - (a) $\Delta l = \pm 1 \quad \Delta m_s = 0 \quad \Delta m_l = 0, \pm 1$
8. Wavefunctions = “orbitals”
9. Radial probability function $P_{nl}(r) = r^2 R_{nl}^2(r)$
 - (a) $\langle r \rangle = \int r P_{nl}(r) dr = \left(\frac{3}{2} n^2 - l(l+1) \right) a_0$

2.8.3 Variational principle

1. Solutions of Schrödinger equation always form a complete set
2. True wavefunction energy is therefore lower bound on energy of any trial wavefunction

$$\langle \psi_{\text{trial}}^\lambda | \hat{H} | \psi_{\text{trial}}^\lambda \rangle = E_{\text{trial}}^\lambda \geq E_0$$

1. Optimize wavefunction with respect to variational parameter

$$\left(\frac{\partial \langle \psi_{\text{trial}}^\lambda | \hat{H} | \psi_{\text{trial}}^\lambda \rangle}{\partial \lambda} \right) = 0 \rightarrow \lambda_{\text{opt}}$$

Table 12: Hydrogen atom

$$\begin{aligned}
 V(r) &= -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}, 0 < r < \infty \\
 \hat{H} &= -\frac{\hbar^2}{2m_e} \frac{1}{r^2} \left[\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \hat{L}^2 \right] + V(r) \\
 \psi(r, \theta, \phi) &= R(r) Y_{l, m_l}(\theta, \phi) \\
 \left\{ -\frac{\hbar^2}{2m_e} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{\hbar^2 l(l+1)}{2m_e r^2} - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r} \right\} R(r) &= E R(r) \\
 R_{nl}(r) &= N_{nl} e^{-x/2} x^l L_{nl}(x), \quad x = \frac{2r}{na_0} \\
 P_{nl}(r) &= r^2 R_{nl}^2 \\
 n &= 1, 2, \dots, \quad l = 0, \dots, n-1 \quad m_l = 0, \pm 1, \dots, \pm l \\
 N_{nl} &= \sqrt{\left(\frac{2}{na_0} \right)^3 \frac{(n-l-1)!}{2n(n+l)!}} \\
 L_{10} = L_{21} = L_{32} = \dots &= 1 \quad L_{20} = 2 - x \quad L_{31} = 4 - x \\
 E_n &= -\frac{1}{2} \frac{\hbar^2}{m_e a_0^2} \frac{1}{n^2} = -\frac{E_H}{2} \frac{1}{n^2} \\
 |L| &= \hbar \sqrt{l(l+1)}, L_z = m_l \hbar
 \end{aligned}$$

2.9 Lecture 12: Many-electron atoms

2.9.1 Many-electron problem, Schrödinger equation not exactly solvable

1. $e^- - e^-$ interaction terms prevent separation of variables
2. Independent electron model basis of all solutions, describes each electron by its own wavefunction, or “orbital”

2.9.2 Qualitative solutions

1. ψ_i look like H atom orbitals, labeled by same quantum numbers
2. *Aufbau principle*: “Build-up” electron configuration by adding electrons into H-atom-like orbitals, from bottom up

3. *Pauli exclusion principle*: Every electron in atom must have a unique set of quantum numbers, so only two per orbital (with opposite spin)
4. *Pauli exclusion principle (formally)*: The wavefunction of a multi-particle system must be anti-symmetric to coordinate exchange if the particles are fermions, and symmetric to coordinate exchange if the particles are bosons
5. *Hund's rule*: Electrons in degenerate orbitals prefer to be spin-aligned. Configuration with highest *spin multiplicity* is the most preferred

2.9.3 Structure of the periodic table

1. Electrons in different subshells experience different effective nuclear charge $Z_{\text{eff}} = Z - \sigma_{nl}$
2. Inner (“core”) shells not shielded well at
3. Inner shell electrons “shield” outer electrons well
4. Within a shell, *s* shielded less than *p* less than *d* ..., causes degeneracy to break down
5. Electrons in same subshell shield each other poorly, causing ionization energy to increase across the subshell

2.9.4 Quantitative solutions

1. Schrödinger equation

$$\hat{H}\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots) = E\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots)$$

$$\hat{H} = \sum_i \hat{h}_i + \frac{e^2}{4\pi\epsilon_0} \sum_i \sum_{j>i} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\hat{h}_i = -\frac{\hbar^2}{2m_e} \nabla_i^2 - \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}_i|}$$

2. Construct candidate many-electron wavefunction Ψ from one electron wavefunctions (mathematical details vary with exact approach)

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots) \approx \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)\dots\psi_n(\mathbf{r}_n)$$

3. Calculate expectation value of E of approximate model and apply *variational principle* to find equations that describe “best” (lowest total energy) set of ψ_i

$$\frac{\partial E}{\partial \psi_i} = 0 \quad \forall i$$

$$\hat{f}\psi = \left\{ \hat{h} + \hat{v}_{\text{Coul}}[\psi_i] + \hat{v}_{\text{ex}}[\psi_i] + \hat{v}_{\text{corr}}[\psi_i] \right\} \psi = \epsilon\psi$$

$$E = \sum_i \epsilon_i - \frac{1}{2} \langle \Psi | \hat{v}_{\text{Coul}}[\psi_i] + \hat{v}_{\text{ex}}[\psi_i] + \hat{v}_{\text{corr}}[\psi_i] | \Psi \rangle$$

4. Motivate as equation for an electron moving in a “field” of other electrons, adding an electron to a known set of ψ_i

2.9.5 Electron-electron interactions

1. Coulomb (\hat{v}_{Coul}): classical repulsion between distinguishable electron “clouds”
2. Exchange (\hat{v}_{ex}): accounts for electron indistinguishability (Pauli principle for fermions). Decreases Coulomb repulsion because electrons of like spin intrinsically avoid one another
3. Correlation (\hat{v}_{corr}): decrease in Coulomb repulsion due to dynamic ability of electrons to avoid one another; “fixes” orbital approximation
4. General form of exchange potential is expensive to calculate; general form of correlation potential is unknown

2.9.6 Popular models

1. *Hartree model*: Include only classical Coulomb repulsion \hat{v}_{Coul}
2. *Hartree-Fock model*: Include Coulomb and exchange
3. *Density-functional theory* (DFT): Include Coulomb and approximate expressions for exchange and correlation
4. All the potential terms \hat{v} depend on the solutions, so equations must be solved *iteratively* to *self-consistency*

2.9.7 DFT calculations on atoms

1. See <http://www.chemsoft.ch/qc/fda.htm>

2.10 Lecture 13: Molecular orbital theory of molecules

2.10.1 Clamped nucleus (“Born-Oppenheimer”) approximation

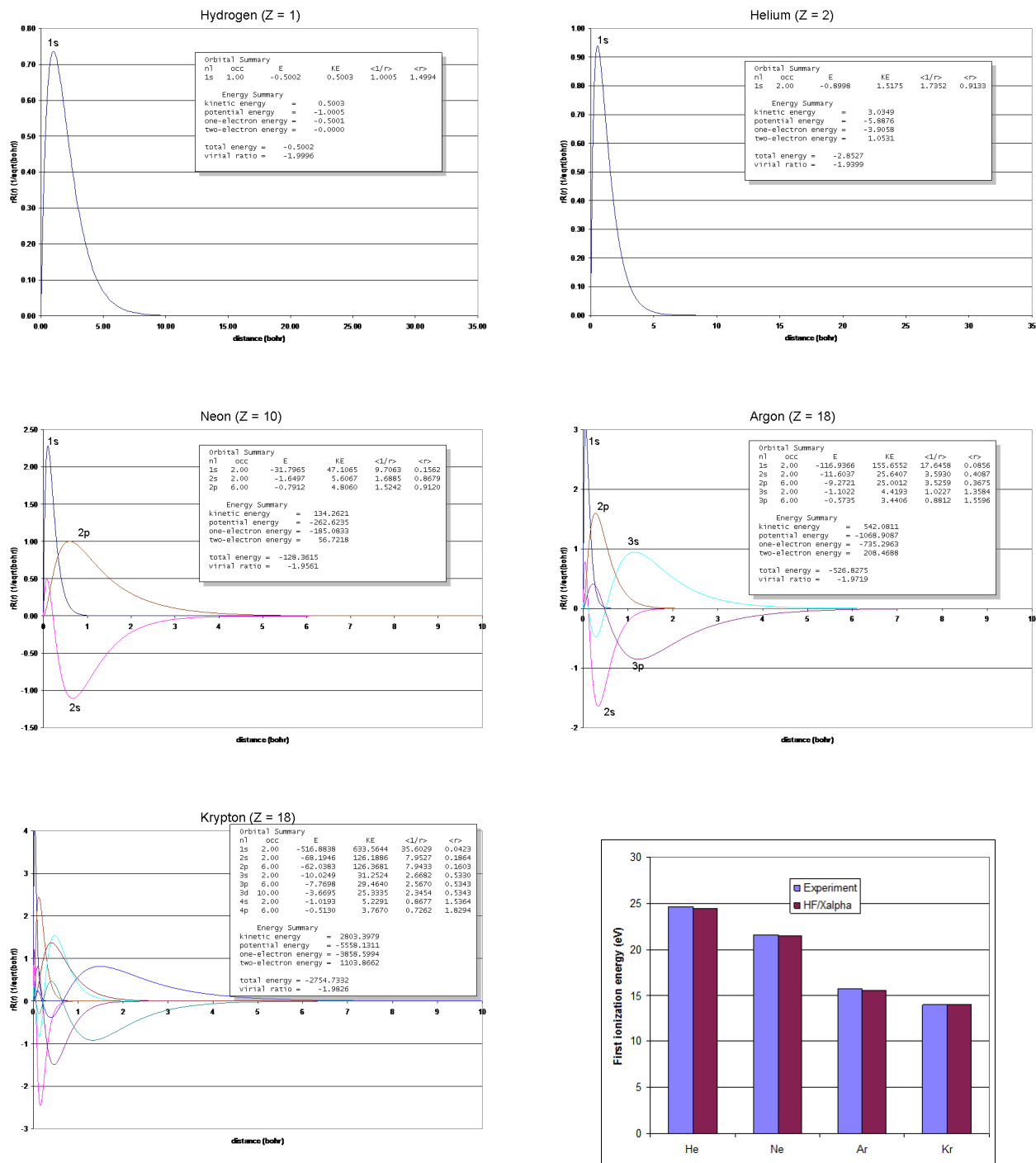
1. Write one-electron equations parametrically in terms of positions of all atoms

$$\hat{h} = -\frac{\hbar^2}{2m_e}\nabla^2 - \sum_{\alpha} \frac{Z_{\alpha}e^2}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{R}_{\alpha}|} \quad (1)$$

$$\hat{f}\psi = \left\{ \hat{h} + \hat{v}_{\text{Coul}}[\psi_i] + \hat{v}_{\text{ex}}[\psi_i] + \hat{v}_{\text{corr}}[\psi_i] \right\} \psi = \epsilon\psi \quad (2)$$

1. Solve as for atoms, using some model for electron-electron interactions
2. Potential energy surface (PES)

$$E(\mathbf{R}_{\alpha}, \mathbf{R}_{\beta}, \dots) = E_{\text{elec}} + \frac{e^2}{4\pi\epsilon_0} \sum_{\alpha} \sum_{\beta > \alpha} \frac{Z_{\alpha}Z_{\beta}}{|\mathbf{R}_{\alpha} - \mathbf{R}_{\beta}|}$$

Table 13: Numerical DFT Solutions for Atoms

2.10.2 H₂ molecule as perturbation on two H atoms brought from infinite distance

1. “Bonding” orbital, $\sigma_g(\mathbf{r}) = 1s_A + 1s_B$
2. “Anti-bonding” orbital, $\sigma_u(\mathbf{r}) = 1s_A - 1s_B$
3. Interaction scales with “overlap $\langle 1s_A | 1s_B \rangle$ ”
4. Ground configuration = σ_g^2
5. Bond order = $\frac{1}{2}(n - n^*)$

2.10.3 Secular equations

1. Expand molecular orbitals in “basis” of atomic-like orbitals

$$\psi_{\text{MO}} = \sum_a c_a \phi_a(\mathbf{r}) \quad (3)$$

2. Problem reduces to finding set of c_a that give best molecular orbitals (MOs)
3. Substituting into Fock equation and integrating yields set of linear equations for the c_a for each MO

$$\begin{pmatrix} F_{11} - \epsilon S_{11} & F_{12} - \epsilon S_{12} & \dots \\ F_{21} - \epsilon S_{21} & F_{22} - \epsilon S_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = 0$$

- (a) $F_{ij} = F_{ji} = \langle \phi_i | \hat{f} | \phi_j \rangle$ are Fock “matrix elements”
 - (b) $S_{ij} = S_{ji} = \langle \phi_i | \phi_j \rangle$ are overlaps
 - (c) Typically basis functions normalized such that $S_{ii} = 1$
 - (d) ϵ are molecular orbital energies (to be solved for, as many as there are equations)
4. From linear algebra, only possible solutions are those that make the determinant vanish

$$\begin{vmatrix} F_{11} - \epsilon S_{11} & F_{12} - \epsilon S_{12} & \dots \\ F_{21} - \epsilon S_{21} & F_{22} - \epsilon S_{22} & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} = 0$$

5. Solve for ϵ s and back-substitute to find correspond c_i s

2.10.4 Qualitative solutions of secular equations

1. Lot’s of insight into chemical bonding can be obtained from approximate solutions to secular equations, basis of “molecular orbital theory”
2. Two general assumptions
 - (a) Diagonal Fock elements are approximately equal to energies of corresponding atomic orbitals: $F_{ii} \approx \epsilon_{i,\text{ao}}$
 - (b) Off-diagonal elements proportional to overlap and inversely proportional to energy difference:

$$F_{ij} \propto \frac{S_{ij}}{\epsilon_{i,\text{ao}} - \epsilon_{j,\text{ao}}}$$

- (c) (Often) set differential overlap $S_{ij} = 0$

2.10.5 H₂ example, again

1. Assign one 1s atomic orbital (“basis function”) to each atom

$$F_{11} = F_{22} = \epsilon_{1s} = \alpha$$

$$F_{12} = F_{21} = \beta$$

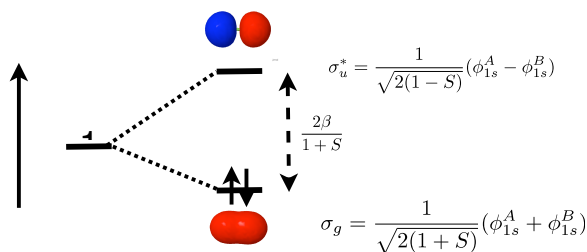
$$\alpha < \beta < 0 \text{ typically}$$

2. Set-up and solve secular matrix

$$\begin{vmatrix} \alpha - \epsilon & \beta - \epsilon S \\ \beta - \epsilon S & \alpha - \epsilon \end{vmatrix} = 0$$

$$\epsilon_+ = \frac{\alpha + \beta}{1 + S}, \quad c_1 = c_2 = \frac{1}{\sqrt{2(1 + S)}}$$

$$\epsilon_- = \frac{\alpha - \beta}{1 - S}, \quad c_1 = -c_2 = \frac{1}{\sqrt{2(1 - S)}}$$



3. From Taylor expansion get picture of atomic orbitals destabilized by electron repulsion βS and split by interaction β

$$\epsilon_+ \approx \alpha - \beta S + \beta$$

$$\epsilon_- \approx \alpha - \beta S - \beta$$

4. Makes clear that bonding stabilization < anti-bonding destabilization

2.10.6 Heteronuclear diatomic: LiH, HF, BH example

1. Only AOs of appropriate symmetry, overlap, and energy match can combine to form MOs

$$\epsilon_+ \approx \alpha_1 - \beta S - \beta^2/|\alpha_1 - \alpha_2|$$

$$\epsilon_- \approx \alpha_2 - \beta S + \beta^2/|\alpha_1 - \alpha_2|$$

2. LiH: H 1s + Li 2s, bond polarized towards H
3. HF: H 1s + F 2p, bond polarized towards F, lots of non-bonding orbitals
4. BH: H 1s, B 2s and 2p_z → bonding, non-bonding, anti-bonding orbitals

2.10.7 Homonuclear diatomic: O₂

1. Assign aos, 1s, 2s, 2p for each atom (10 total)
2. In principle, solve 10×10 secular matrix
3. In practice, matrix elements rules mean only a few off-diagonal elements survive
 - (a) 1s + 1s do nothing
 - (b) 2s + 2s form σ bond and anti-bond
 - (c) $2p_z + 2p_z$ form second bond and anti-bond
 - (d) $2p_{x,y} + 2p_{x,y}$ form degenerate π bonds and anti-bonds
 - (e) O₂ is a triplet, consistent with experiment!

2.10.8 The Hückel/tight binding model

1. $F_{ii} = \alpha, S_{ij} = \delta_{ij}, F_{ij} = \beta$ iff i adjacent to j
2. Ethylene example
3. Butadiene example
4. Benzene example
5. Infinite chain example

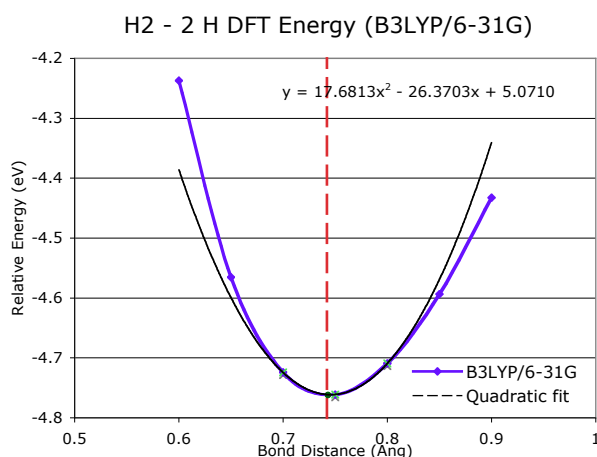
2.10.9 Band structure of solids**2.11 Lecture 14: Computational chemistry****2.11.1 Numerical solvers of Schrödinger equation for molecules readily available today****2.11.2 Have to specify:**

1. Identity of atoms
2. Positions of atoms (distances, angles, ...)
3. (spin multiplicity)
4. exact theoretical model (how are Coulomb, exchange, and correlation described?)
 - (a) Hartree, Hartree-Fock, DFT (various flavors), ...
5. basis set to express wavefunctions in terms of
6. initial guess of wavefunction coefficients (often guessed for you)

2.11.3 Secular equations solved iteratively until input coefficients = output coefficients

1. “self-consistent field”
2. Output
 - (a) energies of molecular orbitals
 - (b) occupancies of molecular orbitals
 - (c) coefficients describing molecular orbitals
 - (d) total electron wavefunction, total electron density, dipole moment, ...
 - (e) total molecular energy
 - (f) derivatives (“gradients”) of total energy w.r.t. atom positions
3. Plot total energy vs. \ internal coordinates: potential energy surface (PES)
4. Search iteratively for minimum point on PES (by hand or using gradient-driven search): equilibrium geometry
5. Find second derivative of energy at minimum point on PES: harmonic vibrational frequency
6. Find energy at minimum relative to atoms (or other molecules): reaction energy

2.11.4 H₂ example



Bond length

H-H exp't:	0.742 Å
B3LYP opt:	0.743

Bond energy

H-H exp't	= 4.478 eV
2 H: 2(-0.4969) au = -27.041 eV	
H ₂ : -1.1687 au = -31.803 eV	
Dissociation E	= 4.762 eV
ZPE corrected	= 4.484 eV

Vibrational frequencies

Experiment:	4401 cm ⁻¹
B3LYP harmonic:	4487 cm ⁻¹
ZPE:	0.278 eV

2.11.5 Polyatomic molecules

1. Gradient-driven optimizations, $3n - 6$ degrees of freedom
2. Hessian matrix for frequencies

2.12 Lecture 15: Electronic spectroscopy

3 Statistical Mechanics: The Bridge from the Tiny to the Many