

4.9 Matrix Transformations from \mathbb{R}^n to \mathbb{R}^m

A *matrix transformation* $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a mapping of the form

$$T(\vec{x}) = A\vec{x},$$

for all \vec{x} vectors in \mathbb{R}^n and an $m \times n$ matrix A .

The matrix transformations are precisely the *linear transformations* from \mathbb{R}^n to \mathbb{R}^m , that is, the transformations with the linearity properties

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{and} \quad T(k\mathbf{u}) = kT(\mathbf{u})$$

We will use these two properties as the starting point for defining more general linear transformations.

Remark: It is important to note that a linear transformation is a special kind of function. The input and output are both vectors.

If we denote the output vector $T(\vec{x})$ by \vec{y} we can write

$$\vec{y} = A\vec{x}$$

Finding the Standard Matrix for a Matrix Transformation

Step 1. Find the images of the standard basis vectors e_1, e_2, \dots, e_n for \mathbb{R}^n in column form.

Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

Types of linear Transformations

There are two types of linear transformations (defining from \mathbb{R}^2 to \mathbb{R}^2):

- 1- Euclidean Transformation
- 2- Affine Transformation

1 – Euclidean Transformations

A Euclidean Transformation is a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = A\vec{x} + \vec{a}, \quad \forall \vec{x} \in \mathbb{R}^2$$

Where A is an orthogonal 2×2 matrix and $\vec{a} \in \mathbb{R}^2$. These types of transformations always preserve distance/shape.

An orthogonal matrix holds the property $AA^T = I$ or $A^T = A^{-1}$

Types of Euclidean transformation:

1- Translation, 2- Reflection, 3- Rotation.

1- Translation

Translation is a transformation from $(\mathbb{R}^2 \text{ to } \mathbb{R}^2)$ or $(\mathbb{R}^3 \text{ to } \mathbb{R}^3)$ defined as:

$$\mathbf{T}(\vec{x}) = \vec{x} + \vec{a}, \quad \forall \vec{x} \in \mathbb{R}^2$$

where matrix A is the identity matrix.

Example 1: (Translation of a triangle)

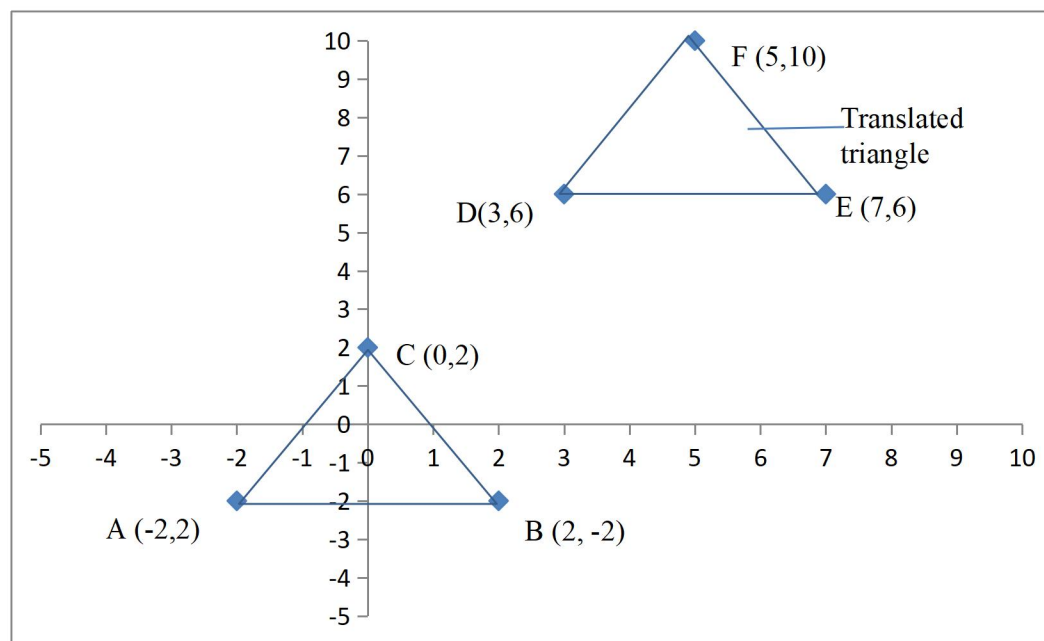
Let $A = (-2, -2)$, $B = (2, -2)$, $C = (0, 2)$ form a triangle. Find the translated triangle with vector $\vec{a} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

Solution: As the transformation of translation is $\mathbf{T}(\vec{x}) = \vec{x} + \vec{a}$.

For point A: $D = \mathbf{T}(A) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

For point B: $E = \mathbf{T}(B) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

For point C: $F = \mathbf{T}(C) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$



Example 2: (Translation of a line)

For a line $3x - 4y = 2$, find the equation of line translated through vector $\vec{a} = (2, 3)$.

Solution: The transformation of translation is:

$$T(\vec{x}) = \vec{x} + \vec{a}$$

$$\text{So } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{Or } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2 \\ y + 3 \end{bmatrix}$$

Which implies $x' = x + 2$ and $y' = y + 3$.

Then, $x = x' - 2$ and $y = y' - 3$.

Put these in our given equation of line that is

$$3(x' - 2) - 4(y' - 3) = 2$$

$$3x' - 4y' = -4 \quad \text{is the required translated line.}$$

To draw original line $3x - 4y = 2$

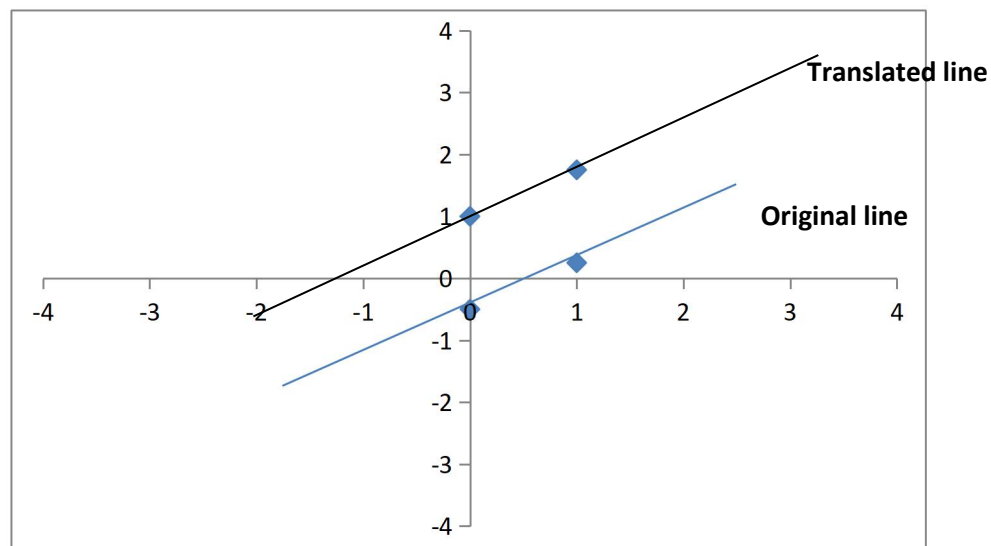
put $x = 0$ implies $y = -1/2$, so **A** $(0, -1/2)$ is a point on this line.

Similarly $x = 1$ implies $y = 1/4$ and **B** $(1, 1/4)$ is another point on it.

In the same manner to draw the Translated line $3x' - 4y' = -4$

Putting $x = 0$ gives $y' = 1$ and **C** $(0, 1)$.

Putting $x = 1$ provides $y' = 7/4$ and **D** $(1, 7/4)$.



Example 3: (Translation of circle)

Let $(x - 4)^2 + (y - 3)^2 = 9$ be a circle. Find the equation of the translated circle using vector $(2, 3)$.

Note: As equation of circle: $(x - a)^2 + (y - b)^2 = r^2$ with Centre = (a, b) and Radius = r . While $x^2 + y^2 = r^2$ is circle with Center = $(0, 0)$ and Radius = r .

Solution: The transformation of translation is

$$T(\vec{x}) = \vec{x} + \vec{a}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x + 2 \\ y + 3 \end{bmatrix}$$

$$x' = x + 2 \text{ then } x = x' - 2 \text{ and } y' = y + 3 \text{ then } y = y' - 3$$

Putting these equations in the equation of circle

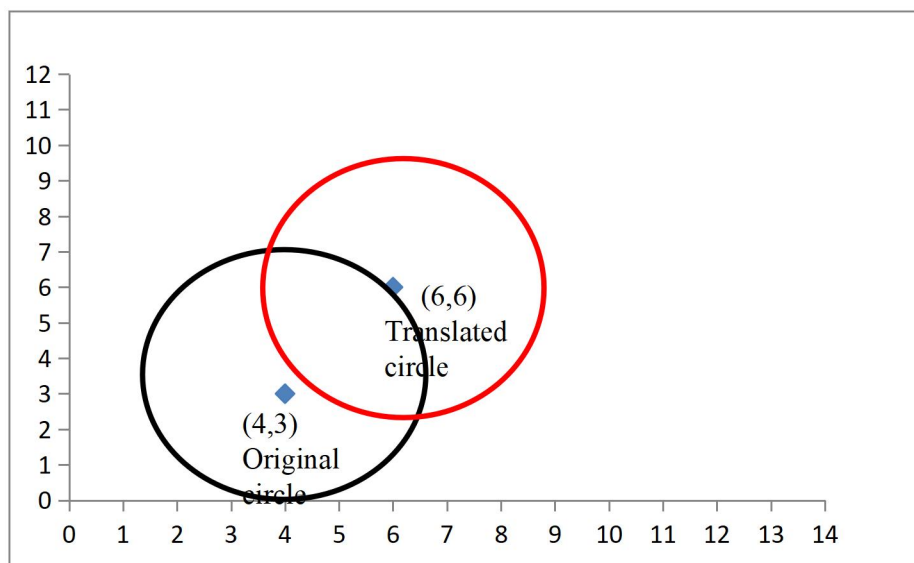
$$(x - 4)^2 + (y - 3)^2 = 9$$

$$(x' - 2 - 4)^2 + (y' - 3 - 3)^2 = 9$$

$$(x' - 6)^2 + (y' - 6)^2 = 9$$

Hence, Original circle is $(x - 4)^2 + (y - 3)^2 = 9$ with Center = $(4, 3)$, Radius = 3.

While Translated circle is $(x' - 6)^2 + (y' - 6)^2 = 9$ with Center = $(6, 6)$, Radius = 3



Note: As we said earlier that **Euclidean transformations** are distance/shape preserving. So in all above examples we can see that translation transformation being a Euclidean transformation preserves the shape of each object and just translated or moved the object.

Work to do:

Q1. Consider the letter L in figure, made up of the vectors $(1, 0)$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $(0, 2)$ or $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, show that the effect of the linear transformation on the matrices and describe each of the transformation in words.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution: As, $T(\vec{x}) = A\vec{x}$

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

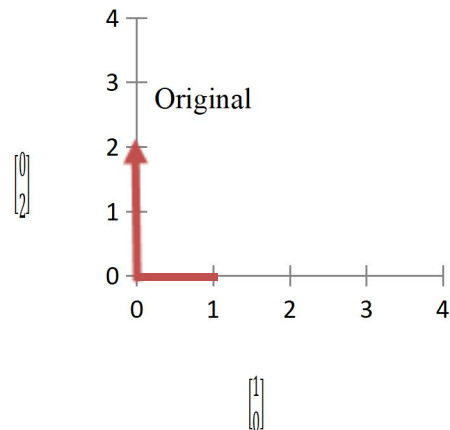
For

$$T(\vec{x}) = C\vec{x}$$

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

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Q2. Let $A = (3, 4)$, $B = (3, 2)$, $C = (6, 2)$ and $D = (6, 4)$ form a rectangle. Find its translation thorough vector $(3, 5)$ and verify your translated rectangle from the figure below.

Original rectangle