1.5 Inverse of a Matrix

Definition: If A is a square matrix, and if a matrix B of the same size can be found such that

$$AB = BA = I\ (Identity\ Matrix)$$

then A is said to be **invertible** (or **nonsingular**) and B is called an **inverse** of A. If no such matrix B can be found, then A is said to be **singular**.

The relationship $AB = BA = I$ is not changed by interchanging A and B, so if A is invertible and B is an inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when $AB = BA = I$,we say that A and B are inverses of one another.

Let

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

Remarks:

1. *In general, a square matrix with a row or column of zeros is singular.*
2. *An invertible matrix has exactly one inverse.*

Formula for the Inverse of a matrix:

If A is an $n \times n$ matrix and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)}(\operatorname{adj} A) = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\ \frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \cdots & \frac{A_{n2}}{\det(A)} \\ \vdots & \vdots & & \vdots \\ \frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \cdots & \frac{A_{nn}}{\det(A)} \end{bmatrix}$$

For a 2x2 matrix this formula is easy to use. For example

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1:

In each part, determine whether the matrix is invertible. If so, find its inverse.

(a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

Solution

(a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

(b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

Example 2:

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

We leave it for you to show that

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

and also that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Remark:

If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

For a matrix of order higher than 2x2 we use the inversion algorithm to find its inverse (based on steps used for the reduced row echelon form).

1.5 Inversion Algorithm

To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

Example 4:

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution We want to reduce A to the identity matrix by row operations and simultaneously apply these operations to I to produce A^{-1}. To accomplish this we will adjoin the identity matrix to the right side of A, thereby producing a partitioned matrix of the form

$$[A \mid I]$$

Then we will apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$\left[I \mid A^{-1}\right]$$

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \leftarrow$$

We added -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \leftarrow$$

We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow$$

We multiplied the third row by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We added 3 times the third row to the second and } -3 \text{ times the third row to the first.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We added } -2 \text{ times the second row to the first.}$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example 5:

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \leftarrow \text{We added } -2 \text{ times the first row to the second and added the first row to the third.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \leftarrow \text{We added the second row to the third.}$$

Since we have obtained a row of zeros on the left side, A is not invertible.

Work to do

Q1. Determine whether the statement is true or false, and justify your answer.

(i) For all square matrices A and B of the same size $(A+B)^2 = A^2 + 2AB + B^2$.

(ii) The product of two elementary matrices of the same size must be an elementary matrix.

(iii) If A is an $n \times n$ matrix that is not invertible, then the linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has infinitely many solutions.

(iv) It is impossible for a linear system of linear equations to have exactly two solutions.

(v) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(vi) Every elementary matrix is invertible.

(vii) If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible.

(viii) If the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution, then the linear system $\boldsymbol{Ax} = \boldsymbol{c}$ also must have a unique solution.

Q2. Use the inversion algorithm to find the inverses of the given matrices, if exist.

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$ $\quad (ii)$ $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 5 & 5 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$, $\quad$ (iv) $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$,

(v) $\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Q3. Find all values of c, if any, for which the given matrix is invertible.

(i) $\begin{bmatrix} c & c & 1 \\ 1 & 1 & c \\ 0 & 1 & c \end{bmatrix}$

(ii) $\begin{bmatrix} c & c & -1 \\ 1 & 1 & 2c \\ 0 & 1 & c \end{bmatrix}$

(iii) $\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$

Q4. Find det A and det B using row or column of your choice and using reduced matrix.

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$$

Q5. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, then $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = ?$