1.4 Matrices and Matrix Operations

Rectangular arrays of real numbers arise in contexts other than as augmented matrices for linear systems. In this section we will begin to study matrices as objects in their own right by defining operations of addition, subtraction, and multiplication on them (not on their rows).

The following rectangular array with three rows and seven columns might describe the number of hours that a student spent studying three subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

If we suppress the headings, then we are left with the following rectangular array of numbers with three rows and seven columns, called a "matrix":

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Definition: *A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.*

Examples:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

Note: *A matrix with only one column is called a **column vector** or a **column matrix**, and a matrix with only one row is called a **row vector** or a **row matrix**.*

A matrix A with n rows and n columns is called a square matrix of order n, and the shaded entries are said to be on the main diagonal of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Matrix Operations

1. Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

Consider the matrices

$$A=\begin{bmatrix}2 & 1\\ 3 & x\end{bmatrix},\quad B=\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix},\quad C=\begin{bmatrix}2 & 1 & 0\\ 3 & 4 & 0\end{bmatrix}$$

If $x=5$, then $A=B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are equal. There is no value of x for which $A=C$ since A and C have different sizes.

2. If A and B are matrices of the same size, then the sum A+B is the matrix obtained by adding the entries of B to the corresponding entries of A, and the difference A – B is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

Consider the matrices

$$A=\begin{bmatrix}2 & 1 & 0 & 3\\ -1 & 0 & 2 & 4\\ 4 & -2 & 7 & 0\end{bmatrix},\quad B=\begin{bmatrix}-4 & 3 & 5 & 1\\ 2 & 2 & 0 & -1\\ 3 & 2 & -4 & 5\end{bmatrix},\quad C=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$$

Then

$$A+B=\begin{bmatrix}-2 & 4 & 5 & 4\\ 1 & 2 & 2 & 3\\ 7 & 0 & 3 & 5\end{bmatrix}\quad\text{and}\quad A-B=\begin{bmatrix}6 & -2 & -5 & 2\\ -3 & -2 & 2 & 5\\ 1 & -4 & 11 & -5\end{bmatrix}$$

The expressions $A+C$, $B+C$, $A-C$, and $B-C$ are undefined.

3. If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a scalar multiple of A.

For the matrices

$$A=\begin{bmatrix}2 & 3 & 4\\ 1 & 3 & 1\end{bmatrix},\quad B=\begin{bmatrix}0 & 2 & 7\\ -1 & 3 & -5\end{bmatrix},\quad C=\begin{bmatrix}9 & -6 & 3\\ 3 & 0 & 12\end{bmatrix}$$

we have

$$2A=\begin{bmatrix}4 & 6 & 8\\ 2 & 6 & 2\end{bmatrix},\quad (-1)B=\begin{bmatrix}0 & -2 & -7\\ 1 & -3 & 5\end{bmatrix},\quad \tfrac{1}{3}C=\begin{bmatrix}3 & -2 & 1\\ 1 & 0 & 4\end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$.

4. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j

from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \boxed{26} & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

And hence,

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

5. Matrix multiplication has an important application to systems of linear equations. Consider a system of m linear equations in n unknowns:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = b_m \end{array}$$

The matrix on the left side of this equation can be written as a product to give

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate these matrices by A, **x**, and **b**, respectively, then we can replace the original system of m equations in n unknowns by the single matrix equation as

$$A\boldsymbol{x} = \boldsymbol{b}$$

The matrix A in this equation is called the coefficient matrix of the system. The **augmented matrix** for the system is obtained by adjoining b to A as the last column; thus the augmented matrix is

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

6. If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

2.1 Determinants

WARNING: It is important to keep in mind that det (A) is a number, whereas A is a matrix.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the expression $ad - bc$ is called the ***determinant*** of the matrix A. This determinant is denoted by writing

$$\det(A) = ad - bc \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Cofactors of a matrix

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

Definition

If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

[cofactor expansion along the jth column]

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

[cofactor expansion along the ith row]

Example 3:

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$
$$= 3(-4) - (1)(-11) + 0 = -1$$

Let A be the matrix in Example 3, and evaluate $\det(A)$ by cofactor expansion along the first column of A.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$
$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

Remark:

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22} \cdots a_{nn}$.

Basic Properties of Determinants

Suppose that A and B are $n \times n$ matrices and k is any scalar. Then:

1. $\det(A + B) \neq \det(A) + \det(B)$
2. $\det(kA) = k^n \det(A)$

3. $\det(AB) = \det(A)\det(B)$
4. A square matrix A is invertible if and only if $\det(A) \neq 0$.
5. $\det(A^{-1}) = \frac{1}{a_{11}}\frac{1}{a_{22}}\dots\frac{1}{a_{nn}}$
6. The points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2),\ P_3(x_3, y_3, z_3),\ P_4(x_4, y_4, z_4)$, are not coplanar if and only if the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0$$

Remarks:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A can be expressed as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix b.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix b.

(g) $\det(A) \neq 0$.

Work to do:

Q1(a)

Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$.

(b) Check whether the points (0,4,1), (4,0,0), (3,5,2) and (2,2,5) lie in the same plane or not.

Q2. For any square matrices A, B , P with P invertible, Complete the following

$\det A^{-1}$= ……………… det(AB)=………………

det(PAP^{-1}) =…………

Q3. Let A and B be 3 × 3 matrices with detA= 4 and det B = -3. Use properties of determinants to compute:

a. $\det AB$ b. $\det 5A$ c. $\det B^T$

d. $\det A^{-1}$ e. $\det A^3$