Augmented Matrices and Elementary Row Operations

As the number of equations and unknowns in a linear system increases so does the complexity of the algebra involved in finding solutions. So we write system of equations in the form of matrices and solve it by **row echelon form** (Gauss Elimination Method) or **reduced row echelon form** (Gauss Jorden Method). These methods are very useful to find the solutions of system linear equations.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad \cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

The ***augmented matrix*** for the system is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : b_2 \\ \vdots & & \ddots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : b_m \end{bmatrix}$$

The basic method for solving a linear system is to perform appropriate algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are as follows:

1. Multiply an equation through by a nonzero constant.

2. Interchange two equations.

3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.

2. Interchange two rows.

3. Add a constant times one row to another.

These are called **elementary row operations** on a matrix.

Gauss Elimination Method

Example 1 Use elementary row operations to solve the linear system:

$$x_1 + x_2 + 2x_3 = 9$$
$$2x_1 + 4x_2 - 3x_3 = 1$$
$$3x_1 + 6x_2 - 5x_3 = 0$$

Solution: The Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 2 & 4 & -3 & :1 \\ 3 & 6 & -5 & :0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 0 & 2 & -7 & :17 \\ 3 & 6 & -5 & :0 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 0 & 2 & -7 & :17 \\ 0 & 3 & -11 & :-27 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 0 & 1 & \frac{-7}{2} & :-\frac{17}{2} \\ 0 & 3 & -11 & :-27 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 0 & 1 & \frac{-7}{2} & :-\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & :-\frac{3}{2} \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 0 & 1 & \frac{-7}{2} & :-\frac{17}{2} \\ 0 & 0 & 1 & :3 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad -2R_3$$

Echelon Form

$$\begin{cases} x_3 = 3 \\ x_2 - \frac{7}{2}x_3 = -\frac{17}{2} \\ x_1 + x_2 + 2x_3 = 9 \end{cases}$$

Putting $x_3 = 3$ in equation (2), we get $x_2 = 2$

Using $x_3 = 3, x_2 = 2$ in equation (3), we get $x_1 = 1$

So (1, 2, 3) is solution of above system.

Gauss Jordan Method

Finding the solution of linear system by converting the augmented matrix into reduced echelon form is called Gauss Jorden Method. For this I take the matrix of echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 0 & 1 & \frac{-7}{2} & : -\frac{17}{2} \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & :9 \\ 0 & 1 & 0 & :2 \\ 0 & 0 & 1 & :3 \end{bmatrix} \text{-------------} R_2 + \frac{7}{2}R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & :3 \\ 0 & 1 & 0 & :2 \\ 0 & 0 & 1 & :3 \end{bmatrix} \text{-------------} R_1 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & :1 \\ 0 & 1 & 0 & :2 \\ 0 & 0 & 1 & :3 \end{bmatrix} \text{-------------} R_1 - R_2$$

Reduced row echelon form

$$x_1 = 1, x_2 = 2, x_3 = 3$$

So (1, 2, 3) is solution of above system.

Example 2: Solve the linear system of equation:

$$\begin{cases} x + z + 2w = 6 & \text{-----}\rightarrow (1) \\ y - 2z = -3 & \text{------}\rightarrow (2) \\ x + 2y - z = -2 & \text{-----}\rightarrow (3) \\ 2x + y + 3z - 2w = 0 & \text{----}\rightarrow (4) \end{cases}$$

Solution:

Using (1) and (4), we get

$$3x + y + 4z = 6 \text{-----}\rightarrow (5)$$

Using equation (3) and (5)

$$5y - 7z = -12 \text{-----}\rightarrow (6)$$

Using equation (2) and (6)

$$-3z = -3 \quad \text{or} \quad z = 1$$

Put $z = 1$ in equation (6) implies

$$y = -1$$

Putting value of y and z in equation (3), we get $x = 1$

Finally, using value of x, y and z in equation (1) we have $w = 2$

So $(1, -1, 1, 2)$ is solution of above system.

Repeat Example 1 Solve the linear system of equations using Gauss-Jordan method:

$$\begin{cases} x+z+2w=6 & \dashrightarrow (1) \\ y-2z=-3 & \dashrightarrow (2) \\ x+2y-z=-2 & \dashrightarrow (3) \\ 2x+y+3z-2w=0 & \dashrightarrow (4) \end{cases}$$

Note: Do and match your answer with the solution obtained above.

Solution: The Augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 2 : 6 \\ 0 & 1 & -2 & 0 : -3 \\ 1 & 2 & -1 & 0 : -2 \\ 2 & 1 & 3 & -2 : 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 : 6 \\ 0 & 1 & -2 & 0 : -3 \\ 0 & 2 & -2 & -2 : -8 \\ 0 & 1 & 1 & -6 : -12 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3-R_1, \; R_4-2R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 : 6 \\ 0 & 1 & -2 & 0 : -3 \\ 0 & 0 & 2 & -2 : -2 \\ 0 & 0 & 3 & -6 : -9 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3-2R_2, \; R_4-R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 : 6 \\ 0 & 1 & -2 & 0 : -3 \\ 0 & 0 & 1 & -1 : -1 \\ 0 & 0 & 3 & -6 : -9 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad \tfrac{1}{2}R_3$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 : 6 \\ 0 & 1 & 0 & -2 : -5 \\ 0 & 0 & 1 & -1 : -1 \\ 0 & 0 & 0 & -3 : -6 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim R_1-R_3, \; R_2+2R_3, \; R_4-3R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 : 7 \\ 0 & 1 & 0 & -2 : -5 \\ 0 & 0 & 1 & -1 : -1 \\ 0 & 0 & 0 & 1 : 2 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad -\tfrac{1}{3}R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 : 1 \\ 0 & 1 & 0 & 0 : -1 \\ 0 & 0 & 1 & 0 : 1 \\ 0 & 0 & 0 & 1 : 2 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_1-3R_4, \; R_2+2R_4, \; R_4-3R_3$$

Thus, $x=1, \quad y=-1, \quad z=1, \quad w=2.$

Some Facts About Echelon Forms

There are three facts about row echelon forms and reduced row echelon forms that are important to know but we will not prove:

1. Every matrix has a unique reduced row echelon form; that is, regardless of whether you use Gauss-Jordan elimination or some other sequence of elementary row operations, the same reduced row echelon form will result in the end.

2. Row echelon forms are not unique; that is, different sequences of elementary row operations can result in different row echelon forms.

3. Although row echelon forms are not unique, all row echelon forms of a matrix A have the same number of zero rows, and the leading 1's always occur in the same positions in the row echelon forms of A. Those are called the ***pivot positions*** of A. A column that contains a pivot position is called a ***pivot column*** of A.

Example 3

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{cases} z + w = 0 \\ y + z = 0 \\ x + y = 0 \\ x + w = 0 \end{cases}$$

Hint: Solution is (t,-t, t, -t).

Gauss-Jordan Method Continued….

Example Solve the linear system of equation using Gauss-Jordan method

$$\begin{cases} x+4y-z=12 \\ 3x+8y-2z=4 \end{cases}$$

Solution.

$$\begin{bmatrix} 1 & 4 & -1 & :12 \\ 3 & 8 & -2 & :4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & :12 \\ 0 & -4 & 1 & :-32 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2-3R_1$$

$$\begin{bmatrix} 1 & 4 & -1 & :12 \\ 0 & 1 & -\frac{1}{4} & :8 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad -\frac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & :-20 \\ 0 & 1 & -\frac{1}{4} & :8 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_1-4R_2$$

Let $z=t$,

$$y=8+\frac{1}{4}t$$

So Solution of this system is $\left(-20, 8+\frac{1}{4}t, t\right)$.

Example 2

Solve the following systems of linear equations using Gauss Jordan elimination method:

i) $$\begin{cases} x+2y+z=12 \\ 5x+2y-3z=1 \\ 2x+y-z=2 \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 1 & :12 \\ 5 & 2 & -3 & :1 \\ 2 & 1 & -1 & :2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & :12 \\ 0 & -8 & -8 & :-59 \\ 2 & 1 & -1 & :2 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2-5R_1$$

$$\begin{bmatrix} 1 & 2 & 1 & :12 \\ 0 & -8 & -8 & :-59 \\ 0 & -3 & -3 & :-22 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 1 & :12 \\ 0 & 1 & 1 & :\frac{59}{8} \\ 0 & -3 & -3 & :-22 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad -\frac{1}{8}R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & :12 \\ 0 & 1 & 1 & :\frac{59}{8} \\ 0 & 0 & 0 & :\frac{1}{8} \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3 + 3R_2$$

So the above system has no solution.

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ii) $\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases}$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & :6 \\ 2 & -3 & 2 & :14 \\ 3 & 1 & -1 & :-2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -4 & :2 \\ 3 & 1 & -1 & :-2 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -4 & :2 \\ 0 & -5 & -10 & :-20 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3 - 3R_1$$

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$$\begin{bmatrix} 1 & 0 & 0 & :1 \\ 0 & 1 & 0 & :-2 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

So the above system has solution (1, -2, 3).

(iii) $\begin{cases} x + 2y - 3z = -4 \\ 2x + y - 3z = 4 \end{cases}$

Solution: … The above system has solution (4+t, t-4, t).

Work to do:

Howard Anton (Exercise 1.2)

21.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w \quad - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

Answer:

$w = t,\ x = -t,\ y = t,\ z = 0$

22.

$$\begin{aligned} x_1 + 3x_2 \quad + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 \quad &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

23.

$$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 \quad - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$$

Answer:

$I_1 = -1,\ I_2 = 0,\ I_3 = 1,\ I_4 = 2$

Determine the values of a for which the system has no solutions, exactly one solution, or infinite many solutions.

25.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

Answer:

If $a = 4$, there are infinitely many solutions; if $a = -4$, there are no solutions; if $a \neq \pm 4$, there is exactly one solution.

Howard Anton (Exercise 1.2)

Q1. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

Q3. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Work to do:

Q5 Solve the linear system of equation by using Gauss elimination or Gauss-Jordan elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

11. In each part, find a system of linear equations corresponding to the given augmented matrix

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$