Chapter 4. Vector Space

Vector spaces with real scalars are called ***real vector spaces*** and those with complex scalars are called ***complex vector spaces***. For now, we will be concerned exclusively with real vector spaces.

4.1 Real Vector Spaces

Let V be a nonempty set of objects, on which two operations are defined:

a) Addition, b) Multiplication by scalars

With the following properties:

1. If $\vec{u}$ and $\vec{v}$ are elements in V, then $\vec{u}+\vec{v}$ is in V. (V is closed under addition)
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$, for all $\vec{u},\vec{v}$ in V. (holds Commutative Law)
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ (holds Associative Law)
4. There is an object $\vec{0}$ in V, called the zero vector for V such that $O+\vec{u}=\vec{u}+O=\vec{u}$, for each $\vec{u}$ in V. (have Additive Identity)
5. For each $\vec{u}$ in V, there is an object $-\vec{u}$ in V, called a negative of $\vec{u}$, such that $\vec{u}+(-\vec{u})=-\vec{u}+\vec{u}=O$. (have Additive Inverse)
6. If k is any scalar and $\vec{u}$ is any object in V, then $k\vec{u}$ is in V. (Closed under Scalar Multiplication).
7. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
8. $(k+m)\vec{u}=k\vec{u}+m\vec{u}$
9. $k(m\vec{u})=(km)\vec{u}$
10. $1\vec{u}=\vec{u}$ (have Multiplicative Identity)

then V is called a vector space and the objects in *V are* ***vectors***.

Example 1: Let $V=R^2=\{(x,y);x,y\in R\}$, prove that V is a vector space under the usual operations of addition and scalar multiplication defined by:

$$\vec{u}+\vec{v}=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,\vec{u}_2+\vec{v}_2)$$

$$k\vec{u}=k(\vec{u}_1,\vec{u}_2)=(k\vec{u}_1,k\vec{u}_2)$$

Solution:

1. V is closed under addition. (as defined)
2. Let $\vec{u}=(\vec{u}_1,\vec{u}_2),\vec{v}=(\vec{v}_1,\vec{v}_2)$

$$\vec{u}+\vec{v}=(\vec{u}_1,\vec{u}_2)+(\vec{v}_1,\vec{v}_2)$$

$$=(\vec{u}_1+\vec{v}_1,\vec{u}_2+\vec{v}_2)$$

$$=(\vec{v}_1+\vec{u}_1,\vec{v}_2+\vec{u}_2)$$

$$=(\vec{v}_1,\vec{v}_2)+(\vec{u}_1,\vec{u}_2)=\vec{v}+\vec{u}$$

3. Let $\vec{u}=(\vec{u}_1,\vec{u}_2),\vec{v}=(\vec{v}_1,\vec{v}_2),\vec{w}=(\vec{w}_1,\vec{w}_2)$

$$\begin{aligned}(\vec{u}+\vec{v})+\vec{w}&=(\vec{u}_1+\vec{v}_1,\vec{u}_2+\vec{v}_2)+(\vec{w}_1,\vec{w}_2)\\&=(\vec{u}_1+\vec{v}_1+\vec{w}_1,\vec{u}_2+\vec{v}_2+\vec{w}_2)\\&=(\vec{u}_1+(\vec{v}_1+\vec{w}_1),\vec{u}_2+(\vec{v}_2+\vec{w}_2))\\&=(\vec{u}_1,\vec{u}_2)+(\vec{v}_1+\vec{w}_1,\vec{v}_2+\vec{w}_2)\\&=\vec{u}+(\vec{v}+\vec{w})\end{aligned}$$

4. Let $\vec{u}=(\vec{u}_1,\vec{u}_2),\vec{0}=(0{,}0)$

$$\vec{u}+\vec{0}=(\vec{u}_1,\vec{u}_2)+(0,0)=(\vec{u}_1,\vec{u}_2)=\vec{u}$$

5. Let $\vec{u}=(\vec{u}_1,\vec{u}_2)$, then there exist $-\vec{u}=(-\vec{u}_1,-\vec{u}_2)$,

$$\vec{u}+(-\vec{u})=\left(\vec{u}_1+(-\vec{u}_1),\vec{u}_2+(-\vec{u}_2)\right)=(\vec{u}_1-\vec{u}_1,\vec{u}_2-\vec{u}_2)=(0,0)=\vec{0}$$

6. V is closed under scalar multiplication. (as defined).

7.
$$\begin{aligned}k(\vec{u}+\vec{v})&=k\left((\vec{u}_1,\vec{u}_2)+(\vec{v}_1,\vec{v}_2)\right)\\&=k(\vec{u}_1+\vec{v}_1,\vec{u}_2+\vec{v}_2)\\&=(k\vec{u}_1+k\vec{v}_1,k\vec{u}_2+k\vec{v}_2)\\&=(k\vec{u}_1,k\vec{u}_2)+(k\vec{v}_1,k\vec{v}_2)\\&=k(\vec{u}_1,\vec{u}_2)+k(\vec{v}_1,\vec{v}_2)\\&=k\vec{u}+k\vec{v}\end{aligned}$$

8.
$$\begin{aligned}(k+m)\vec{u}&=(k+m)(\vec{u}_1,\vec{u}_2)\\&=(k\vec{u}_1+m\vec{u}_1,k\vec{u}_2+m\vec{u}_2)\\&=(k\vec{u}_1,k\vec{u}_2)+(m\vec{u}_1,m\vec{u}_2)\\&=k(\vec{u}_1,\vec{u}_2)+m(\vec{u}_1,\vec{u}_2)\\&=k\vec{u}+m\vec{u}\end{aligned}$$

9. $k(m\vec{u})=k\left(m(\vec{u}_1,\vec{u}_2)\right)=(km\vec{u}_1,km\vec{u}_2)$

$$=km(\vec{u}_1,\vec{u}_2)=km(\vec{u})$$

10. $1\vec{u}=1(\vec{u}_1,\vec{u}_2)=(\vec{u}_1,\vec{u}_2)=\vec{u}$

As the set V satisfies all the properties, so V is vector space.

Example 2: Let $V = R^3$, prove that V is a vector space under the usual operations of addition and scalar multiplication defined by:

$$\vec{u} + \vec{v} = (\vec{u}_1, \vec{u}_2, \vec{u}_3) + (\vec{v}_1, \vec{v}_2, \vec{v}_3) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2, \vec{u}_3 + \vec{v}_3)$$

$$k\vec{u} = k(\vec{u}_1, \vec{u}_2, \vec{u}_3) = (k\vec{u}_1, k\vec{u}_2, k\vec{u}_3)$$

Example 3: Let $V = R^2$, under the usual operations of addition defined by:

$$\vec{u} + \vec{v} = (\vec{u}_1, \vec{u}_2) + (\vec{v}_1, \vec{v}_2) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2)$$

And if k is any scalar number, then define

$$k\vec{u} = k(\vec{u}_1, \vec{u}_2) = (k\vec{u}_1, 0)$$

The addition operation is standard one on R^2, but the scalar multiplication is not.

Check whether V is vector space or not?

Solution: All properties of addition are satisfied. (Check it by yourself)

Let's check the properties of scalar multiplication.

6. Let $\vec{u} = (u_1, u_2)$ in V, then $k\vec{u} = k(u_1, u_2) = (ku_1, 0) \in V$.
7. Let $\vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2)$

$$\begin{aligned} k(\vec{u} + \vec{v}) &= k\big((u_1, u_2) + (v_1, v_2)\big) \\ &= k(u_1 + v_1, u_2 + v_2) \\ &= (ku_1 + kv_1, 0) \\ &= (k\vec{u}_1, 0) + (k\vec{v}_1, 0) \\ &= k(\vec{u}_1, 0) + k(\vec{v}_1, 0) \\ &\neq k\vec{u} + k\vec{v} \end{aligned}$$

As the 7th property does not satisfied So it's not a vector space.

Example 4:

Check whether V is vector space or not?

V =The set of all pairs of real numbers of the form $(x, 0)$. i.e. $\{(x, 0); x \in R\}$

with the standard operations on R^2.

$$\vec{u} + \vec{v} = (\vec{u}_1, \vec{u}_2) + (\vec{v}_1, \vec{v}_2) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2)$$

$$k\vec{u} = k(\vec{u}_1, \vec{u}_2) = (k\vec{u}_1, k\vec{u}_2)$$

Solution:

1. $$\vec{u} = (\vec{u}_1, 0), \vec{v} = (\vec{v}_1, 0) \in V$$

$$(\vec{u} + \vec{v}) = (\vec{u}_1 + \vec{v}_1, 0) \in V$$

V is closed under addition.

2. $$\begin{aligned}(\vec{u} + \vec{v}) &= (\vec{u}_1, 0) + (\vec{v}_1, 0)\\ &= (\vec{u}_1 + \vec{v}_1, 0)\\ &= (\vec{v}_1 + \vec{u}_1, 0)\\ &= (\vec{v}_1, 0) + (\vec{u}_1, 0)\\ &= \vec{v} + \vec{u}\end{aligned}$$

3. $$\begin{aligned}\vec{u} + (\vec{v} + \vec{w}) &= (\vec{u}_1, 0) + ((\vec{v}_1, 0) + (\vec{w}_1, 0))\\ &= (\vec{u}_1, 0) + (\vec{v}_1 + \vec{w}_1, 0)\\ &= (\vec{u}_1 + \vec{v}_1 + \vec{w}_1, 0)\\ &= (\vec{u}_1 + \vec{v}_1, 0) + (\vec{w}_1, 0)\\ &= (\vec{u} + \vec{v}) + \vec{w}\end{aligned}$$

4. $\vec{u} + \vec{0} = (\vec{u}_1, 0) + (0,0) = (\vec{u}_1, 0) = \vec{u}$
5. $\vec{u} + (-\vec{u}) = (\vec{u}_1, 0) + (-\vec{u}_1, 0)$

$$= (\vec{u}_1 - \vec{u}_1, 0) = (0,0) = \vec{0}$$

6. $\vec{u} = (\vec{u}_1, 0) \in V$

Then $k\vec{u} = (k\vec{u}_1, k0) = (ku_1, 0) \in V$

7. $$k(\vec{u} + \vec{v}) = k\big((u_1, 0) + (v_1, 0)\big) = k(\vec{u}_1 + \vec{v}_1, 0) = (k\vec{u}_1 + k\vec{v}_1, 0)$$

$$= (k\vec{u}_1, 0) + (k\vec{v}_1, 0) = k(\vec{u}_1, 0) + k(\vec{v}_1, 0)$$

$$= (k\vec{u} + k\vec{v})$$

8. $$(k + m)\vec{u} = (k + m)(\vec{u}_1, 0)$$

$$= \left((k+m)\vec{u}_1, 0\right) = (k\vec{u}_1 + m\vec{u}_1, 0)$$

$$= (k\vec{u}_1, 0) + (m\vec{u}_1, 0)$$

$$= k(\vec{u}_1, 0) + m(\vec{u}_1, 0) \ = k\vec{u} + m\vec{u}$$

9. $k(m\vec{u}) = k(m\vec{u}_1, 0) = (km\vec{u}_1, 0)$

$$= km(\vec{u}_1, 0) = (km)\vec{u}$$

10. $1\vec{u} = 1(\vec{u}_1, 0) = (\vec{u}_1, 0) = \vec{u}$

So V is a vector space.

Example 5: Check whether V is a vector space or not.

V = set of all pairs of real numbers of the form (x, y), where $x \geq 0$, i.e.

$$V = \{(x, y); x \geq 0, y \in R\}$$

With standard operations on R^2.

Solution:

As

$$V = \{(x, y); x \geq 0, y \in R\}$$

1. Let $\quad \vec{u} = (\vec{u}_1, \vec{u}_2), \quad \vec{v} = (\vec{v}_1, \vec{v}_2) \in V$

$$(\vec{u} + \vec{v}) = (\vec{u}_1 + \vec{v}_1, \vec{u}_2 + \vec{v}_2) \in V$$

Because $\vec{u}_1 + \vec{v}_1 \geq 0$. So, V is closed under addition.

2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Easy to verify)
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w})$ (Easy to verify)
4. Let $\vec{u} = (\vec{u}_1, \vec{u}_2)$, $\vec{u} + \vec{0} = (\vec{u}_1, \vec{u}_2) + (0,0) = (\vec{u}_1, \vec{u}_2) = \vec{u}$
5. Let $\vec{u} = (\vec{u}_1, \vec{u}_2)$, Then there doesn't exist $-\vec{u} = (-\vec{u}_1, -\vec{u}_2)$ because $\vec{u}_1$should be positive.
   5^{th} property fails, So V is not vector space.

Example 6: Show that the set of all pairs of real numbers of the form $(x, 1)$ with the operations

$(x, 1) + \left(x^{'}, 1\right) = (x + x^{'}, 1)$ & $k(x, 1) = (k^2 x, 1)$ is not a vector space.

Example 7: Determine whether the set of all triples of real numbers with standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$

is a vector space or not.

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Axiom 8 fails.

Example 8: Determine whether the set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y')$$

$$k(1, y) = (1, ky)$$

is a vector space or not.

Example 9: Determine whether V is a vector space or not.

V= the set of all triples of the form (x, y, z) with the operations

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$$

& $$k(x, y, z) = (kx, y, z)$$

Example 10: Determine whether V is a vector space or not.

Let V be the set of all 2×2 matrices with real entries and take the vector space operations on V to be usual operations of matrix addition and scalar multiplication i.e.

$$\vec{u} + \vec{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$k\vec{u} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

Solution:

1. V is closed under addition.
2. $$\vec{u}+\vec{v}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$
$$=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$
$$=\begin{bmatrix} v_{11}+u_{11} & v_{12}+u_{12} \\ v_{21}+u_{21} & v_{22}+u_{22} \end{bmatrix}$$
$$=\vec{v}+\vec{u}$$
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. $\vec{u}+0=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\vec{u}$
5. $\vec{u}+(-\vec{u})=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$
$$=\begin{bmatrix} u_{11}+(-u_{11}) & u_{12}+(-u_{12}) \\ u_{21}+(-u_{21}) & u_{22}+(-u_{22}) \end{bmatrix}$$
$$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\vec{0}$$

Similarly, you can prove all the properties of scalar multiplication. (Prove it by yourself).

So, V is a vector space.

Example 11: Let $V=R^n$ and define operations on V to be the usual operations of addition and scalar multiplication.

$$\vec{u}+\vec{v}=(u_1,u_2,u_3,\dots,u_n)+(v_1,v_2,v_3,\dots,v_n)$$

$$=(u_1+v_1,u_2+v_2,\dots,u_n+v_n)$$

& $$k\vec{u}=(ku_1,ku_2,ku_3,\dots,ku_n)$$

Then V is vector space.

Example 12: Let V be the set of polynomials of the form

$$P(t)=a_nt^n+a_{n-1}t^{n-1}+\dots+a_1t+a_0.$$

Determine whether V is a vector space or not under the usual operations of addition and scalar multiplication?

THEOREM 4.1.1

Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:

(a) $0\,\boldsymbol{u} = 0$

(b) $k\,\mathbf{0} = 0$

(c) $(-1)\boldsymbol{u} = -\boldsymbol{u}$

(d) If $k\,\boldsymbol{u} = 0$, then $k = 0$ or $\boldsymbol{u} = \mathbf{0}$.

Exercise 4.1

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2),\ k\mathbf{u} = (0, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and ku for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$ and $k = 3$.

(b) In words, explain why V is closed under addition and scalar multiplication.

(c) Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

(d) Show that Axioms 7, 8, and 9 hold.

(e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

Answer:

(a) $\mathbf{u} + \mathbf{v} = (2, 6)$, $3\mathbf{u} = (0, 6)$

(c) Axioms 1–5

2. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1),\quad k\mathbf{u} = (ku_1, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and ku for $u = (0, 4)$, $v = (1, -3)$, and $k = 2$.

(b) Show that $(0, 0) \neq \mathbf{0}$.

(c) Show that $(-1, -1) = 0$.

(d) Show that Axiom 5 holds by producing an ordered pair $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$ for $\mathbf{u} = (u_1, u_2)$.

(e) Find two vector space axioms that fail to hold.

True-False Exercises

In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) A vector is a directed line segment (an arrow).

(b) A vector is an n-tuple of real numbers.

(c) A vector is any element of a vector space.

(d) There is a vector space consisting of exactly two distinct vectors.

(e) The set of polynomials with degree exactly 1 is a vector space under the operations defined in Exercise 12.