Variable

Week 13

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Continuous Random Variable

- In statistics and probability theory, a continuous random variable is a type of variable that can **take any value** within a given range.
- Continuous random variables can assume any value within an interval
- This makes them ideal for modelling a wide range of real-world phenomena, such as the **height of individuals**, the **time taken to complete a task**, or the **amount of rainfall** in a particular period.

Examples of CRV

- Continuous random variables can take any value within a given range and are commonly used in various fields to model and analyze real-world phenomena. Here are some examples:
  Height of Individuals: The height of people within a population can vary continuously. Measurements can be as precise as the measurement tool allows, such as 172.3 cm, 172.33 cm, etc.
- **Weight of Objects**: The weight of objects, such as fruits, animals, or packages, is another example. For instance, the weight of an apple can be 150.5 grams, 150.55 grams, and so on.
- **Temperature**: Temperature can be measured to a high degree of precision, such as 23.1°C, 23.12°C, and so on. It is a continuous variable because it can take any value within the thermometric scale.
- **Time**: The time it takes to complete a task or event, like running a marathon, is a continuous random variable. For instance, a marathon might be completed in 3 hours, 2 minutes, and 47.5 seconds.
- **Distance**: The distance between two points can vary continuously. For example, the distance someone runs can be 5.123 kilometers, 5.1234 kilometers, etc.

Characteristics of Continuous Random Variables

1. Infinite Outcomes:

- It can take any value within a given interval on the real number line.
- Example: Time taken to run a race (can be 12.32 seconds, 12.321, 12.3215, etc.).

2. Measured, Not Counted:

- Continuous variables are always measured, not counted.
- Example: Height, weight, temperature, age.

3. Probability of Exact Value is Zero:

- $P(X=x)= 0$ for any specific value x, because the number of possible values is infinite.
- Instead, we talk about the **probability over intervals**, like $P(a<X<b)$

Characteristics of Continuous Random Variables

4. **Probability Density Function (PDF):**
   - A continuous random variable is described by a **Probability Density Function**, denoted by $f(x)$.
   - The PDF defines the likelihood of the variable falling within a particular range.
   - The total area under the PDF curve equals 1:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

5. **Cumulative Distribution Function (CDF):**
   - The **Cumulative Distribution Function**, $F(x)$, gives the probability that the variable is less than or equal to a certain value:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

Continuous Sample Space

- A CONTINUOUS SAMPLE SPACE IS A SAMPLE SPACE THAT CONSISTS OF AN UNCOUNTABLE (AND OFTEN INFINITE) SET OF POSSIBLE OUTCOMES, TYPICALLY REPRESENTING MEASUREMENTS OR VALUES ALONG A CONTINUUM.
- UNLIKE A DISCRETE SAMPLE SPACE (WHICH HAS DISTINCT, COUNTABLE OUTCOMES), A CONTINUOUS SAMPLE SPACE INCLUDES ALL POSSIBLE VALUES WITHIN A GIVEN RANGE.

NOTE: THE CHARACTERISTICS OF CONTINUOUS SAMPLE SPACE ARE THE SAME AS CRV

PROBABILITY DENSITY FUNCTION (PDF)

- THE PROBABILITY DENSITY FUNCTION (PDF) IS A FUNCTION THAT DESCRIBES THE LIKELIHOOD OF A CONTINUOUS RANDOM VARIABLE TAKING ON A SPECIFIC VALUE WITHIN A RANGE.
- It defines how **probability is distributed** over the values of the variable.
- UNLIKE DISCRETE PROBABILITY DISTRIBUTIONS (WHERE WE HAVE A PROBABILITY MASS FUNCTION, OR PMF), THE PDF GIVES A DENSITY, NOT A DIRECT PROBABILITY.

PROBABILITY DENSITY FUNCTION (PDF)

1. **Non-negative:** $f(x) \geq 0$ for all x.
2. **Total area under the curve = 1:**

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

3. **Probability over an interval:**

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$$

4. **Probability at a single point is zero:**

$$P(X = c) = 0 \quad \text{(for any exact value } c\text{)}$$

PROBABILITY DENSITY FUNCTION (PDF)

Interpretation:

- The **value of f(x)**at a particular x is **not the probability**, but a **density**.
- The **area under the curve** of f(x) between two values **represents probability**.

Applications of PDF:

- **Engineering:** Noise signal measurement
- **Economics:** Modeling returns and risks
- **Biology:** Growth and decay models
- **Medicine:** Survival analysis

EXAMPLE 1

Example: Uniform Distribution (Simple PDF)

Suppose X is a random variable representing a random number chosen between 0 and 2, where every number in this range is equally likely.

Step 1: Define the PDF

For a uniform distribution over $[a, b]$, the PDF is:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

For our example ($a = 0, b = 2$):

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 1

Step 2: Check Total Probability

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{2} \frac{1}{2}\,dx = \frac{1}{2} \times (2-0) = 1$$

✅ The total probability is 1.

Step 3: Calculate Probabilities

- **What is $P(0.5 \leq X \leq 1.5)$?**

$$P(0.5 \leq X \leq 1.5) = \int_{0.5}^{1.5} \frac{1}{2}\,dx = \frac{1}{2} \times (1.5-0.5) = 0.5$$

- **What is $P(X=1)$?**

$$P(X=1) = 0 \quad \text{(since exact points have zero probability)}$$

EXAMPLE 2

📌 Example 1: **Uniform Distribution**

If a random variable X is uniformly distributed over the interval [2, 6]:

$$f(x) = \begin{cases} \frac{1}{6-2} = \frac{1}{4}, & \text{for } 2 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

- Find $P(3 \leq X \leq 5)$:

$$P(3 \leq X \leq 5) = \int_3^5 \frac{1}{4}\, dx = \frac{1}{4}(5-3) = \frac{1}{2}$$

EXAMPLE 3

Imagine you have a **fair spinner** that can land anywhere between **0 and 4** on a circular scale. Every position on the spinner is equally likely.

Step 1: Define the Continuous Sample Space

- The **sample space** S is all real numbers between 0 and 4:

$$S = [0, 4]$$

Step 2: Define the Continuous Random Variable

- Let X be the **random variable** representing the spinner's stopping position.
- Since every point in $[0, 4]$ is equally likely, X follows a **uniform distribution**.

EXAMPLE 3

Step 3: Write the Probability Density Function (PDF)

For a uniform distribution over $[a, b]$, the PDF is:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $a = 0$ and $b = 4$, so:

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq x \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

Step 4: Verify the PDF

The total probability must be 1:

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{4} \frac{1}{4}\, dx = \frac{1}{4} \times (4 - 0) = 1.$$

EXAMPLE 3

Step 5: Compute Probabilities

1. **What is $P(1 \leq X \leq 3)$?**

$$P(1 \leq X \leq 3) = \int_1^3 \frac{1}{4}\, dx = \frac{1}{4} \times (3 - 1) = \frac{2}{4} = 0.5.$$

   Interpretation: There's a **50% chance** the spinner stops between 1 and 3.

2. **What is $P(X = 2)$?**

$$P(X = 2) = 0.$$

   Interpretation: The probability of landing on **exactly 2** is **0** (as expected for continuous variables).

EXAMPLE 4

A **battery** is designed to last anywhere between **100 and 150 hours** of continuous use. The failure time T (in hours) is **uniformly distributed** over this interval.

Tasks

1. **Define the sample space S and the PDF $f(t)$.**
2. **Verify that the total probability is 1.**
3. **Calculate the following probabilities:**
   - (a) The battery fails **between 120 and 140 hours.**
   - (b) The battery lasts **longer than 130 hours.**
   - (c) The battery fails **exactly at 125 hours.**

Cumulative Distribution Function

The **Cumulative Distribution Function (CDF)** of a continuous random variable X gives the **probability that X takes a value less than or equal to a given number** x.

It is denoted as:

$$F(x) = P(X \leq x)$$

For a continuous random variable with PDF $f(x)$, the CDF is obtained by **integrating the PDF** from the lower bound up to x:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

Key Properties of CDF:

Key Properties of CDF:

1. **Monotonic:** Always non-decreasing (since probabilities accumulate).
2. **Range:** $0 \leq F(x) \leq 1$ for all x.
3. **Limits:**
   - $\lim_{x \to -\infty} F(x) = 0$ (no probability at $-\infty$).
   - $\lim_{x \to \infty} F(x) = 1$ (total probability = 1).

EXAMPLE: UNIFORM DISTRIBUTION CDF

Let X be a continuous random variable with the following PDF:

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

EXAMPLE: UNIFORM DISTRIBUTION CDF

✅ Step 1: Find the CDF $F(x)$

We integrate the PDF from 0 to x:

$$F(x) = \int_0^x 2t\,dt = 2\left[\frac{t^2}{2}\right]_0^x = x^2$$

So the **CDF is:**

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

EXAMPLE: UNIFORM DISTRIBUTION CDF

- Example 1: What is $P(X \leq 0.8)$?

$$F(0.8) = (0.8)^2 = 0.64$$

- Example 2: What is $P(0.3 \leq X \leq 0.7)$?

$$P(0.3 \leq X \leq 0.7) = F(0.7) - F(0.3) = (0.7)^2 - (0.3)^2 = 0.49 - 0.09 = 0.40$$

- Example 3: What is $P(X > 0.5)$?

$$P(X > 0.5) = 1 - F(0.5) = 1 - (0.5)^2 = 1 - 0.25 = 0.75$$

📘 Question: CDF Practice

Let a continuous random variable X have the following probability density function (PDF):

$$f(x) = \begin{cases} 4x^3, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

📌 Tasks:

a. Find the cumulative distribution function (CDF), $F(x)$, for X.

b. Use the CDF to find $P(X \le 0.5)$

c. Use the CDF to find $P(0.3 \le X \le 0.8)$

d. What is the value of $P(X > 0.9)$?