Fundamentals of Probability

- Probability is a branch of mathematics.
- It measures the likelihood or chance of a random event occurring.
- It quantifies how likely an event is to happen,
- with values ranging from 0 to 1
- where 0 means the event is impossible
- and 1 means the event is certain

Key Concepts in Probability

- **Experiment:** A process or action that leads to one or more outcomes. For example, tossing a coin or rolling a die.
- **Sample Space (S):** The set of all possible outcomes of an experiment. For tossing one coin, the sample space is {Heads, Tails}. For rolling a six-sided die, the sample space is {1, 2, 3, 4, 5, 6}
- **Event:** A group of outcomes you are interested in. **Example:** Getting an even number when rolling a die → A = {2,4,6}
- **Favorable Outcomes**: Outcomes of interest. $X>4$ means {5,6}
- **Probability of an Event**: The ratio of favorable outcomes to the total number of possible outcomes in the sample space.

Example 1

Consider the experiment of rolling a fair six-sided die:

- **Sample space** $S=\{1,2,3,4,5,6\}, |S|=6$.
- **Event E**: rolling a 4.
- **Number of favorable outcomes** $|E|=1$ (only the number 4).
- **Probability of rolling a 4**:

$$P(E)=1/6$$

This means there is a one in six chance of rolling a 4

Example 2

Consider the experiment of rolling a fair six-sided die:
Find the probability of rolling an even number, the event

- **Sample space** $S=\{1,2,3,4,5,6\}, |S|=6$.
- **Event E**: rolling an even number.
- **Number of favorable outcomes** $|E|=3$ (2,4,6).
- **Probability of rolling a 4**:

$$P(E)=3/6$$

This means there is a 50% chance of getting even number.

- The sample space is the set of all possible outcomes.
- An event is a subset of the sample space.
- A single outcome can be considered an elementary event, but usually, events are sets of outcomes.
- Probability is assigned to events, not just individual outcomes, allowing us to calculate the likelihood of various scenarios

Term	Definition	Example (Rolling a Die)
Outcome	A single possible result of the experiment	Rolling a 4
Event	A set of one or more outcomes	Rolling an even number {2, 4, 6}

Probability Axioms (Kolmogorov's Axioms)

The axioms of probability are foundational rules that any probability measure must satisfy. These were formalized by Andrey Kolmogorov in 1933 and are central to modern probability theory.

1. Non-negativity

For any event A, the probability is always a non-negative real number:

$$P(A) \geq 0$$

This means that the probability of any event cannot be less than zero

Probability Axioms (Kolmogorov's Axioms)

2. Normalization (Unit Measure)

The probability of the entire sample space S is 1:

$$P(S)=1$$

This reflects the certainty that something in the sample space will occur when the experiment is performed

3. Additivity

If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

Example

Die Roll:

- $P(1) = 1/6 \geq 0$ ✅ (Axiom 1)
- $P(S) = P(1)+P(2)+...+P(6) = 1$ ✅ (Axiom 2)
- $P(1 \text{ or } 2) = P(1) + P(2) = 1/6 + 1/6 = 1/3$ ✅ (Axiom 3)

Applying Set Theory to Probability

What is Set Theory in Probability?

- ➢Set theory is a branch of mathematical logic that deals with collections of objects, called **sets**, and their relationships.
- ➢In probability, set theory provides a structured way to represent and analyze events, sample spaces, and their interactions.

1. **Sample Space (S):** The set of all possible outcomes of an experiment.
2. **Event (A, B, etc.):** A subset of the sample space (e.g., rolling an even number on a die: {2, 4, 6}).
3. **Operations:** Union ($A \cup B$), Intersection ($A \cap B$), Complement (A^c), etc.

Core Connections Between Set Theory and Probability

1. **Sample Space as a Universal Set**
   - ➢The set of all possible outcomes of an experiment.
2. **Events as Subsets**
   - ➢Multiple outcomes (e.g., rolling an even number: {2,4,6})
3. **Set Operations for Combining Events**
   - ➢Union ($A \cup B$), Intersection ($A \cap B$), Complement (A^c), etc.

Example:

- **Experiment:** Roll a die.
  - ✓ **Sample Space (S):** {1, 2, 3, 4, 5, 6}
  - ✓ **Event A (Even number):** {2, 4, 6}
  - ✓ **Event B (Number > 4):** {5, 6}
  - ✓ **$A \cup B$:** {2, 4, 5, 6} → $P(A \cup B) = 4/6$
  - ✓ **$A \cap B$:** {6} → $P(A \cap B) = 1/6$

Example 1: Rolling a Die - Union and Intersection of Events

- Sample space: $S=\{1,2,3,4,5,6\}$ Define two events:
  - $A = \{2,4,6\}$ (rolling an even number)
  - $B = \{1,2,3\}$ (rolling a number less than or equal to 3)
- Union $A \cup B$: Outcomes in A or B $A \cup B = \{1,2,3,4,6\}$
- Intersection $A \cap B$: Outcomes in both A and B $A \cap B = \{2\}$
- Probability calculation (assuming a fair die):

$$P(A \cup B)=P(A)+P(B)-P(A \cap B)=3/6+3/6-1/6 = 5/6$$

- This uses set union, intersection, and the inclusion-exclusion principle from set theory to find probabilities

Example 2:

We are given:

- ➢P(Internet)=0.6
- ➢P(TV)=0.8
- ➢P(Internet ∩ TV)=0.5

We are asked to find the probability that a household subscribes to **at least one** service.

Solution: Use the Addition Rule

$$P(\text{Internet} \cup \text{TV}) = P(\text{Internet}) + P(\text{TV}) - P(\text{Internet} \cap \text{TV})$$

$$P(\text{Internet} \cup \text{TV}) = 0.6 + 0.8 - 0.5 = 0.9$$

So, **90%** of households subscribe to **at least one** of the two services.

Questions?

1. In a survey of readers:
"60% of people read the Daily News, 50% read The Times, and 30% read both."
Task for Students:
- What is the probability that a person reads **at least one** of the newspapers?
- What is the probability that a person reads **neither**?

2. A health club notes:
"70% of the members attend yoga classes, 60% attend gym sessions, and 50% attend both."
Task for Students:
- What percentage of members attend only yoga?
- What percentage attend neither yoga nor gym?

3. A college report says:
"Out of all students, 65% are part of the Photography Club, 55% are in the Debate Club, and 35% are in both."
Task for Students:
- How many students are in exactly one club?
- What is the probability that a student is in none of the clubs?

Conditional Probability

Suppose I told you 60 out of 100 students in this school are girls, and 30 girls take science. What is the chance that a randomly selected girl studies science?

What is your Answer?

Conditional Probability

Definition:

- Conditional probability is the likelihood of an event **A** occurring **given** that another event **B** has already occurred
- Conditional Probability is the probability of an event occurring given that another related event has already happened.
- It answers the question: *"What is the chance of event A happening if event B is known to have occurred?"*

Conditional Probability

In regular probability,

We ask: What is the chance something will happen?

But in **conditional probability**, we already **know something happened**, and we want to know the **chance of another thing happening based on that knowledge**.

<u>For example</u>

1. What is the chance someone is tall, given that they play basketball?
2. What is the chance it rains, given that the sky is cloudy?

Conditional Probability

Formula:

$$P(A|B) = P(A \cap B)/P(B)$$

Symbol	Meaning
$A \mid B$	Probability of A **given** B
$P(A \cap B)$	Probability A **and** B both happen
$P(B)$	Probability B happens (since we know B happened)

Conditional Probability

Example: Suppose I told you 60 out of 100 students in this school are girls, and 30 girls take science. What is the chance that a randomly selected girl studies science?

- Total students = 100
- 60 are girls
- 30 girls study science
- What is P(Science | Girl

$$P(\text{Science} \mid \text{Girl}) = \frac{30}{60} = 0.5$$

Because we already know the student is a girl (60 total girls), we don’t consider all 100 students.

Example

- 80 students play sports
- 50 students play cricket
- 30 play **both** sports and cricket

Ask:

> "What is the probability that a student plays cricket **given** they already play sports?"

$$P(\text{Cricket} \mid \text{Sports}) = \frac{P(\text{Cricket and Sports})}{P(\text{Sports})} = \frac{30}{80} = 0.375$$

Example

<u>DATA to solve:</u>

- ➢ 100 students
- ➢ 60 take math
- ➢ 40 take physics
- ➢ 25 take both math and physics

Find

What is the probability that a student takes **physics given** they take math?

<u>DATA to solve:</u>

Over the past 100 days:

It was cloudy on 60 days.

It rained on 50 days.

On 40 of those days, it was both cloudy and rainy.

Q: What is the probability that it rained given that it was cloudy?

What is Independence in Probability?

Independence means that **the occurrence of one event does NOT affect the probability of another event**.

Two events **A** and **B** are said to be **independent** if:

$$P(A \cap B) = P(A) \times P(B)$$

This means:

> If A and B are independent, then the chance of both happening is simply the product of their individual probabilities.

What is Independence in Probability?

1. Coin Toss and Dice Roll

- Tossing a coin (Head or Tail)
- Rolling a die (1 to 6)

These are **independent** because the outcome of the coin **does not affect** the die roll.

If:

- $P(\text{Head}) = \frac{1}{2}$
- $P(4 \text{ on die}) = \frac{1}{6}$

Then:

$$P(\text{Head and } 4) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$