Tree Diagram, Counting Methods, Independent Trials

Week 11
Instructor Name:
Rida Maryam

Tree Diagram

Definition: A **tree diagram** is a visual tool used to list all possible outcomes of a sequence of events.

Structure:

- **Branches**: Represent different possible outcomes of an event.
- **Levels**: Each level of branches represents a step or stage in the sequence of events.
- **Probabilities**: Each branch is labeled with the probability of that outcome occurring.
- **Paths**: A path from the start to an end point represents a complete sequence of events.

- Multiply probabilities **along branches** to get the total probability of an outcome

For example – a fair coin is toss twice

1st 2nd

H HH

H

T HT

Possible Outcomes

H TH

T

T TT

Attach probabilities

1^{st} 2^{nd}

½ H HH P(H,H)=½x½=¼

½ H

½ T HT P(H,T)=½x½=¼

½ H TH P(T,H)=½x½=¼

½ T

½ T TT P(T,T)=½x½=¼

INDEPENDENT EVENTS – 1^{st} spin has no effect on the 2^{nd} spin

Calculate probabilities

1st   2nd

$\frac{1}{2}$ H   HH $P(H,H)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ *

$\frac{1}{2}$ H

$\frac{1}{2}$ T   HT $P(H,T)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ *

$\frac{1}{2}$ H   TH $P(T,H)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ *

$\frac{1}{2}$ T

$\frac{1}{2}$ T   TT $P(T,T)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

Probability of *at least* one Head?

For example

10 coloured balls in a bag – 3 Red, 2 Blue, 5 Green. One taken, colour noted, returned to bag, then a second taken.

1st	2nd	
	R	RR
	B	RB
R	G	RG
	R	BR
B	B	BB
	G	BG
	R	GR
G	B	GB
	G	GG

INDEPENDENT EVENTS

Probabilities

1st		2nd			
0.3	R	0.3	R	RR	P(RR) = 0.3x0.3 = 0.09
		0.2	B	RB	P(RB) = 0.3x0.2 = 0.06
		0.5	G	RG	P(RG) = 0.3x0.5 = 0.15
0.2	B	0.3	R	BR	P(BR) = 0.2x0.3 = 0.06
		0.2	B	BB	P(BB) = 0.2x0.2 = 0.04
		0.5	G	BG	P(BG) = 0.2x0.5 = 0.10
0.5	G	0.3	R	GR	P(GR) = 0.5x0.3 = 0.15
		0.2	B	GB	P(GB) = 0.5x0.2 = 0.10
		0.5	G	GG	P(GG) = 0.5x0.5 = 0.25

All ADD UP to 1.0

Choose a meal

Main course

Salad 0.2

Egg & Chips 0.5

Pizza 0.3

Pudding

Ice Cream 0.45

Apple Pie 0.55

IC P(S,IC) = 0.2 x 0.45 = 0.09

0.45

S 0.55

AP P(S,AP) = 0.2 x 0.55 = 0.110

0.2

0.45 IC P(E,IC) = 0.5 x 0.45 = 0.225

0.5 E

0.55

AP P(E,AP) = 0.5 x 0.55 = 0.275

0.3 IC P(P,IC) = 0.3 x 0.45 = 0.135

P 0.45

0.55

AP P(P,AP) = 0.3 x 0.55 = 0.165

Counting Techniques

For experiments with large number of outcomes the following methods are used to count the number of outcomes of the experiment

Fundamental Principle Of Counting (Rule of Multiplication)

Permutations

Combinations

Counting Techniques : Multiplication Rule

- The multiplication principle states that if one event can occur in **m** ways and another event can occur in **n** ways, then the total number of ways both events can occur together is m × n times

Example

- If a coin is flipped (**2** outcomes: Heads or Tails) and a die is rolled (**6** outcomes: 1 to 6), the total sample space size is **2×6=12**

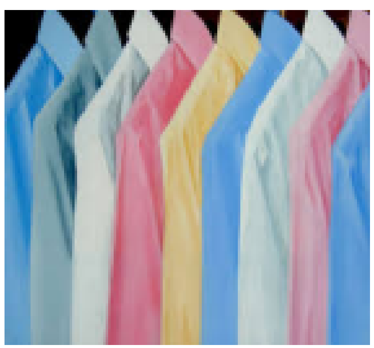
So, the sample space consists of

{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

Product Rule

- Suppose you have 9 shirts, 4 pairs of pants and 6 ties. How many choices do you have for an outfit?

- 9x4



Problem

- For the following list of flavors, toppings and sauce, how many variants of ice creams can you have? Will you apply sum rule or product rule? Why?

Flavours
Vanilla
Chocolate
Banana

Toppings
Flake
Sprinkles
Marshmallows

Sauce
Toffee
Raspberry
Lemon

Counting Techniques : Multiplication Rule

- **Example.** How many sample points are there in the sample space when a pair of dice is thrown once?

: The first die can land face-up in any one of $n_1 = 6$ ways. For each of the
the second die can also land face-up in $n_2 = 6$ ways. Therefore, the p
can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

Counting Techniques

Generalized Multiplication Rule

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Example. Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

***Solution*:** Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are

$$n_l \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

different ways to order the parts.

Permutations

Permutation is a concept in mathematics that refers to an arrangement of a set of items in a specific order. It focuses on **ordering**, meaning the sequence in which items are arranged matters.

Key Characteristics of Permutations:

1. **Order Matters**: For example, the arrangements ABC and BAC are considered different permutations.

2. **No Repetition (usually)**: Unless specified otherwise, items in a permutation are unique and are not repeated.

Counting Techniques : Permutation

- Consider the three letters **a, b**, and **c**. The possible permutations are **abc, acb, bac, bca, cab, and cba**. Thus, we see that there are **6** distinct arrangements.
- we could arrive at the answer 6 without actually listing the different orders by the following arguments: There are **n_1 = 3 choices** for the first position. No matter which letter is chosen, there are always **n_2 = 2 choices** for the second position. No matter which two letters are chosen for the first two positions, there is only **n_3 = 1** choice for the last position, giving a total of

  $n_1 n_2 n_3 = (3)(2)(1) = 6$ permutations

Counting Techniques : Permutation

- In general, n distinct objects can be arranged in

  $n(n - 1)(n - 2)\cdots(3)(2)(1)$ ways.

For any non-negative integer n, $n!$, called "n factorial," is defined as

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

Theorem: The number of permutations of n objects is n!.

Counting Techniques : Permutation

- The number of permutations of the four letters **a, b, c, and d** will be **4! = 24**.
- Now consider the number of permutations that are possible by taking **two letters** at a time from **four**. These would be **ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, and dc.**
- we have **two positions** to fill, with **n_1 = 4 choices** for the first and then **n_2 = 3 choices** for the second, for a total of

$$n_1 n_2 = (4)(3) = 12$$

permutations. In general, n distinct objects taken r at a time can be arranged in

$$n(n-1)(n-2)\cdots(n-r+1)$$

ways. We represent this product by the symbol

$$n!$$

- **Permutation (Ordered Arrangements) :** The arrangements of n objects in a specific order using r objects at a time is called permutation of n objects taking r objects at a time

- **Example:** In one year, **three awards** (research, teaching, and service) will be given to a class of **25 graduate students** in a statistics department. If each student can receive **at most one award**, how many possible selections are there?

Since the awards are distinguishable, it is a permutation problem. number of sample points is

$$_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800.$$

Example. How many different letter arrangements can be made from the letters in the word **PEN**?

n! = 3! = 3*2*1 = **6**

Example. How many different letter arrangements can be made from the letters in the word **STATISTICS**?

$$\binom{10}{3,3,2,1,1} = \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!} = 50,400.$$

Example: In a college football training session, the defensive coordinator needs to have **10 players** standing in a row. Among these 10 players, there are **1 freshman, 2 sophomores, 4 juniors, and 3 seniors**. How many different ways can they be arranged in a row if only their class level will be distinguished?

Solution: we find that the total number of arrangements is

$$\frac{10!}{1!\ 2!\ 4!\ 3!} = 12,600.$$

Combination Problems

The number of ways to choose a sample of ***r*** **elements** from a set of ***n*** **distinct objects** where **order does not matter.**

Counting Techniques: Combinations

Combinations (Unordered Selections) : A selection of distinct objects without regard to order i.e. the order or arrangement is not important

Example

Organizing Tournaments

***n* teams** are participating in Round 1 of a soccer tournament

Every team plays every other team **exactly once**

Each game is refereed by a professional, charging $1 per game

How many games are played?

1. **Understanding the Problem**:
   1. Each team plays against every other team exactly once.
   2. The number of games between two teams is 1.

      2. Combinatorial Calculation:
   3. For *n* teams, the number of unique matches (games) is equivalent to choosing 2 teams from *n* teams.
   4. This is given by the combination formula

      n= population and r is the subset of n that we have to choose

Example: A young boy asks his mother to get 5 Game-BoyTM cartridges from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

***Solution*:** The number of ways of selecting 3 cartridges from 10 is

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = 120.$$

The number of ways of selecting 2 cartridges from 5 is

$$\binom{5}{2} = \frac{5!}{2!\,3!} = 10.$$

sing the multiplication ru$_1 = 120$ and $n_2 = 10$, w 20)(10) = 1200 ways

Permutation – Combination

Example : How many three digit numbers can be formed from the digits 1, 2, 4, 5, and 9 when each digit is used only once?

Solution

Example : In how many ways can an instructor select five students for a group project out of an class of 12?

Solution

Independent trials

- Trials in an experiment are independent if the probability of each possible outcome does not change from trial to trial.

- Independent trials refer to a sequence of experiments or observations where the outcome of one trial does not affect the outcome of another.

For example

If you toss a coin fifty times, the coin tosses are independent trials,

because the outcome of one toss (heads or tails) does not affect the likelihood of getting a heads or tails on the next toss.

Example

You buy a coffee from a vending machine, and your phone receives a random message from a friend.

Are these events related?