Partition & Law of Total Probability, Bayes' Theorem

Week 10

Instructor Name:
Rida Maryam

Introduction & Motivation

Imagine you know that, in a certain country there are three provinces, call them B1, B2, and B3 , We are interested in the total forest area in the country. Suppose that we know that the forest area in B1, B2, and B3 are 100km, 50km, and 150km squared, respectively.

How do you combine that information to find the overall probability of total forest area in the country?

Solution: Partitioning the sample space and applying the Law of Total Probability.

Partition of a Sample Space?

- **Definition**: A partition divides the sample space S into mutually exclusive (disjoint) and exhaustive events B1, B2,…, Bn.
- **Conditions**:
  - $B_i \cap B_j = \emptyset$ for $i \neq j$ (**mutually exclusive**: No two events occur together.)
  - $B_1 \cup B_2 \cup \cdots \cup B_n = S$ (**exhaustive:** Together they cover the entire sample space)

Example 1:

S = {1, 2, 3, 4, 5, 6} (rolling a die).

Partition B1= {1, 2}, B2= {3, 4}, B3= {5, 6}.

- This means that **every outcome** in the sample space belongs to **exactly one** of the events Bi.

Partitions: Example

✅ Example 2: Tossing a Coin Twice

Sample space:

$$S = \{HH, HT, TH, TT\}$$

Partition it into:

- $A_1 = \{HH\}$
- $A_2 = \{HT, TH\}$
- $A_3 = \{TT\}$

✔ No overlaps

✔ Together they make up the full sample space

Law of Total Probability

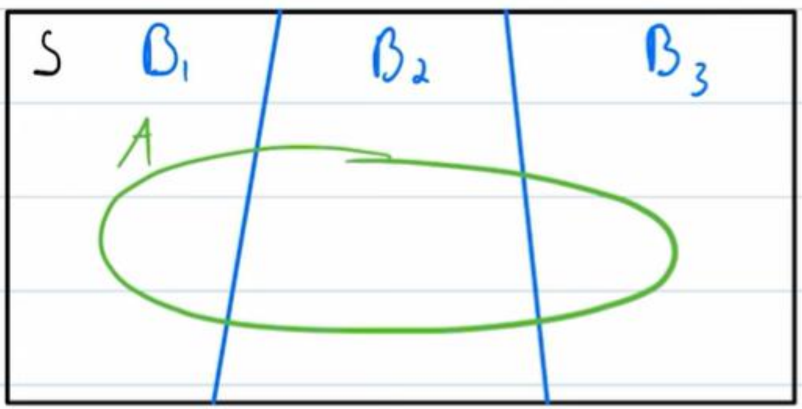
- If B1, B2, ..., Bn form a partition of the sample space and A is any event, then the probability of A can be calculated as:

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

- This law helps us calculate the probability of an event A by **summing over the different ways** A can happen

Law of Total Probability



$$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$$

$$P(A \cap B) = P(A \mid B)\, P(B)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$

$$P(A) = P(A \mid B_1) P(B_1) + P(A \mid B_2) P(B_2) + P(A \mid B_3) P(B_3)$$

Example scenario:

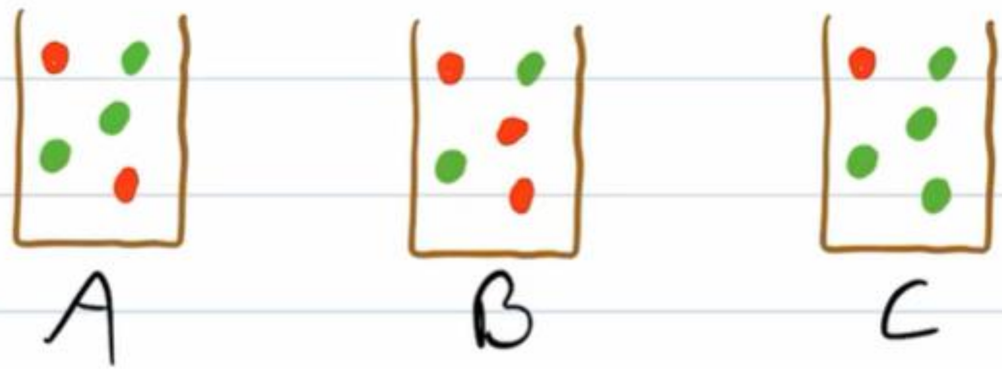
- You don’t know how likely someone is to **buy a product**, but you know:
- Which **store** they go to.
- The likelihood of a purchase **in each store**.

Law of Total Probability

Bag A has 2 red balls and 3 green balls
Bag B has 3 red balls and 2 green balls
Bag C has 1 red ball and 4 green balls
A ball is randomly selected from a random bag; what is the probability that the ball is red?

Law of Total Probability

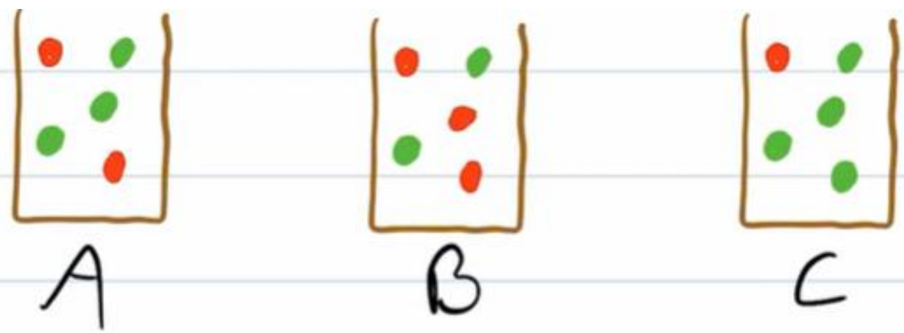
R: ball is red



$$P(R) = P(A \cap R) + P(B \cap R) + P(C \cap R)$$

Law of Total Probability

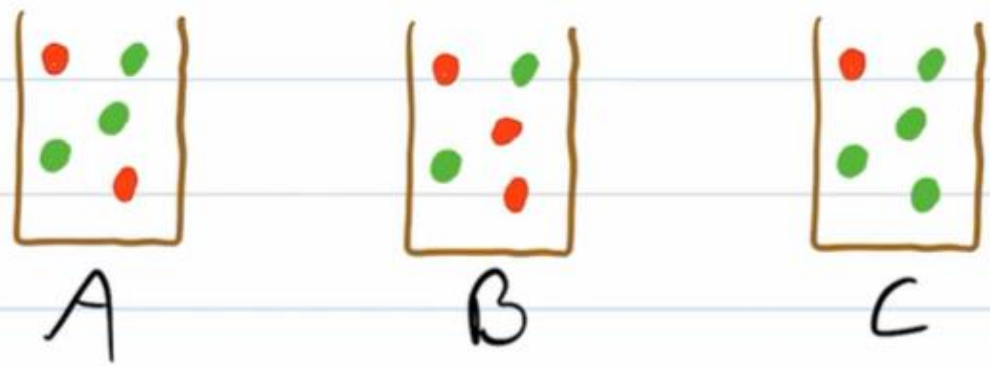
R: ball is red



$$P(R) = P(R|A)P(A) + P(R|B)P(B) + P(R|C)P(C)$$

Law of Total Probability

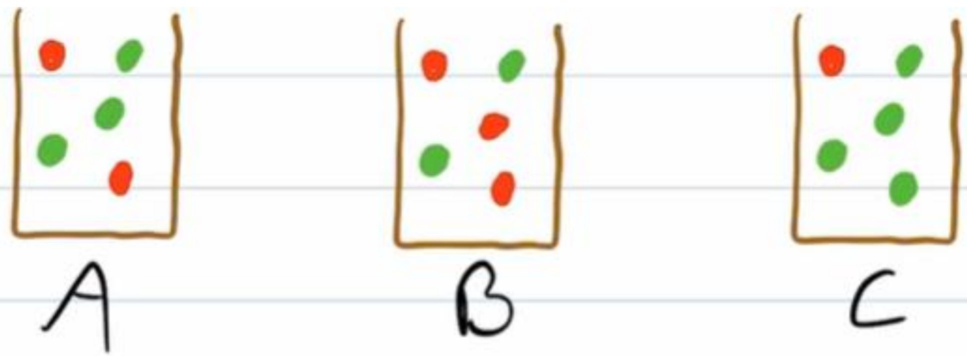
R: ball is red



$$P(R) = \frac{2}{5} \cdot \frac{1}{3} + P(R|B)P(B) + P(R|C)P(C)$$

Law of Total Probability

R: ball is red



$$P(R) = \frac{2}{5} \cdot \frac{1}{3} + \frac{3}{5} \cdot \frac{1}{3} + \frac{1}{5} \cdot \frac{1}{3}$$

Example Problem

- A student can attend one of three batches:
- Morning (40% of students), Afternoon (35%), Evening (25%)
- The probability of passing the final test in each batch is:
- Morning: 80%
- Afternoon: 60%
- Evening: 70%
- **Find the overall probability that a student passes the test.**
- $P(Pass) = 0.8*0.4 + 0.6*0.35 + 0.7*0.25 = 0.705$

Solution:

Step 1: Define Events

Let:

B1: Morning batch → $P(B1)=0.4$

B2: Afternoon batch → $P(B2)=0.35$

B3: Evening batch → $P(B3)=0.25$

A: Event that student passes the test

$P(A|B1)=0.8$

$P(A|B2)=0.6$

$P(A|B3)=0.7$

Step 2: Apply the Law of Total Probability

$$P(A)=P(A|B1)\cdot P(B1)+P(A|B2)\cdot P(B2)+P(A|B3)\cdot P(B3)$$

$$P(A)=(0.8)(0.4)+(0.6)(0.35)+(0.7)(0.25)$$

$$P(A)=0.32+0.21+0.175=0.705$$

Interpretation:

There is a **70.5% chance** that a randomly selected student passes the test.

Real-Life Example 1: Product Purchase Across Stores

A customer chooses between Store A (30%), B (50%), and C (20%).
The purchase probabilities at each store are 40%, 70%, and 60%, respectively.
What is the total probability the customer makes a purchase?

Solution:

P(Purchase) = 0.4*0.3 + 0.7*0.5 + 0.6*0.2 = 0.59

Class Participation Question 1

A person chooses one of three restaurants (P, Q, R):

$P(P)=0.5$, $P(Q)=0.3$, $P(R)=0.2$

Probability of ordering dessert:

- $P(A|P)=0.6$, $P(A|Q)=0.8$, $P(A|R)=0.4$

Q: What is the total probability the person orders dessert?

Class Participation Question 2

A student uses one of two internet providers:

Provider A (60%), Provider B (40%)

Probability of disconnection:

A: 10%, B: 20%

Q: What is the probability that the student faces a disconnection?

Class Participation Question 3

A company recruits employees from three universities: **U1**, **U2**, and **U3**.

30% of the employees are from U1,

50% are from U2,

20% are from U3.

After training, the probability that an employee performs well is:

70% for U1 graduates,

60% for U2 graduates,

90% for U3 graduates.

1. What is the probability that a randomly selected employee performs well?
2. If a randomly selected employee did perform well, what is the probability that they graduated from U3?

Bayes’ Theorem (Bayes' Rule)

Bayes' Theorem, helps us reverse conditional probabilities using background information.

Bayes’ Theorem is a fundamental concept in probability theory used to **update the probability of an event based on new evidence**.

Bayes’ Theorem is a way of finding a probability when we know certain other probabilities

Formula

$$P(A|B) = \frac{P(A)\,P(B|A)}{P(B)}$$

Bayes' Theorem

$$P(A|B) = \frac{P(A)\, P(B|A)}{P(B)}$$

Which tells us : How often A happens *given that B happens*, written P(A|B)

When we know : How often B happens *given that A happens*, written P(B|A)

And how likely A is on its own, written P(A)

And how likely B is on its own, written P(B)

Bayes’ Theorem Proof

The proof of Bayes's, Theorem is given as, according to the conditional probability formula,

$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)}$ **.....(i)**

Then, by using the multiplication rule of probability, we get

$P(E_i \cap A) = P(E_i) \cdot P(A \mid E_i)$**......(ii)**

Now, by the total probability theorem,

$P(A) = \sum_{k=1}^{n} P(E_k) \cdot P(A \mid E_k)$**.....(iii)**

Substituting the value of $P(E_i \cap A)$ and P(A) from eq (ii) and eq(iii) in eq(i) we get,

$$P(E_i \mid A) = \frac{P(E_i) \cdot P(A|E_i)}{\sum_{k=1}^{n} P(E_k) \cdot P(A|E_k)}$$

Bayes' Theorem

- **More Generalized Definition**

- Let E_1, E_2, E_3… E_K is a collection of mutually exclusive and exhaustive events with probability P(E_I), I = 1, 2, 3…k. Then for any event B for which P(B) > 0

$$\boldsymbol{P(Ei|B)} = \frac{\boldsymbol{P(E_I)P(B|E_I)}\ (joint\ probability\ of\ desireable\ event)}{\boldsymbol{P(E_1)P(B|E_1) + P(E_2)P(B|E_2) + P(E_3)P(B|E_3) + \cdots + P(E_K)P(B|E_K)}}$$

$$sum\ of\ joint\ probabilities\ of\ all\ possible\ events$$

Bayes' Theorem

- **More Generalized Definition**

- In simple words we can say that

- $P(Ei|B) \ = \frac{joint\ probabiliy\ of\ desireable\ event}{sum\ of\ probabilities\ of\ all\ joint\ events}$

- Joint probability for two events A and B mean $P(A \cap B) = P(A)P(B|A)$

Example

A disease affects 1% of a population. A test detects the disease 95% of the time when present, but has a 5% false positive rate.

Let:

- D: person has the disease
- T: test is positive

Using Bayes' Theorem:

$$P(D|T) = \frac{P(T|D) \cdot P(D)}{P(T)}$$

This lets us calculate the **real chance of having the disease given a positive test**, not just the test accuracy.

Solution:

- Let D = person has disease, $P(D) = 0.01$
- D' = person has not disease, $P(D') = 0.99$

- Define the positive test event as T

- Conditional probabilities
- $P(T/D) = 0.95$ and $P(T/D') = 0.05$

Solution:

- Revised probabilities

Event	Prior Probability	Conditional Probability	Joint Probability	Revised Probability
D (Disease)	0.01	0.95	0.01*0.95 = 0.0095	0.0095/**0.059** = 0.161
D’ (No Disease)	0.99	0.05	0.99*0.05 = 0.0495	0.0495/**0.059** = 0.838
Total			**0.059**	

Solution:

✅ Step-by-Step Using Bayes' Theorem:

$$P(D|T) = \frac{P(T|D) \cdot P(D)}{P(T)}$$

First calculate $P(T)$ using **total probability**:

$$P(T) = P(T|D)P(D) + P(T|\neg D)P(\neg D)$$

$$= (0.95)(0.01) + (0.05)(0.99) = 0.0095 + 0.0495 = 0.059$$

Now plug back into Bayes' formula:

$$P(D|T) = \frac{(0.95)(0.01)}{0.059} = \frac{0.0095}{0.059} \approx \boxed{0.161}$$

Class Participation Questions

Q1: 2% of packages are faulty. A test correctly identifies faulty packages 95% of the time and wrongly identifies good packages as faulty 10% of the time.

What is the probability a package is actually faulty if it fails the test?

Class Participation Questions

- **Q2:** In a city, 1 in 500 drivers is drunk. A breathalyzer has a 99% accuracy.
- What is the probability a person is actually drunk if the test is positive?