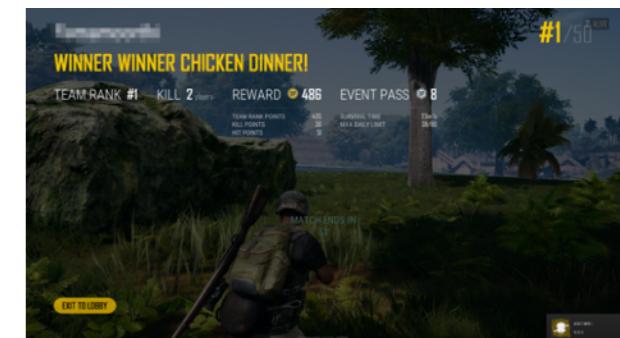


# Advertisement

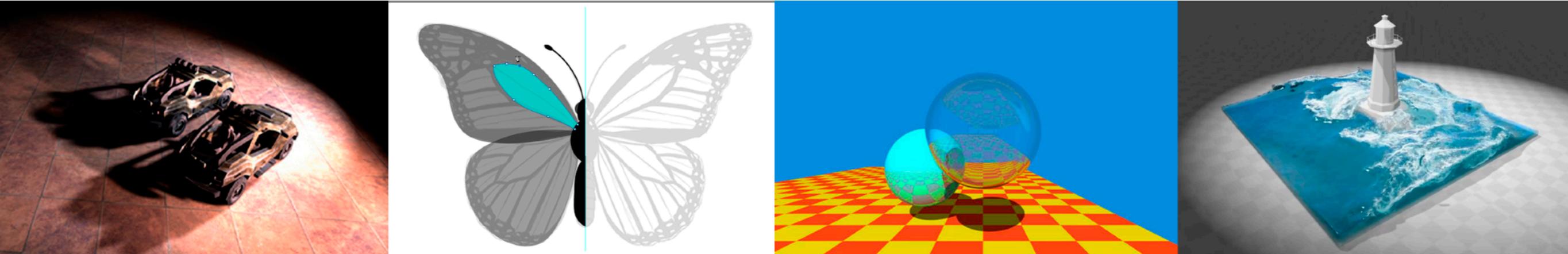
- Prof. Qi Sun from NYU
  - VR / AR / perception / RT graphics, <http://qisun.me>
- Fall 2020 - Spring 2021 (remote is fine)
  - 2-3 research interns
  - 1 postdoc / visiting scholar
- See details on GAMES website / WeChat group
- Send resume to [qisun@nyu.edu](mailto:qisun@nyu.edu) now!



# Introduction to Computer Graphics

GAMES101, Lingqi Yan, UC Santa Barbara

## Lecture 16: Ray Tracing 4 (Monte Carlo Path Tracing)



# Announcements

- Regarding the difficulty of the last lecture
  - Modern Graphics does require it
- We are working on final project ideas
  - But again, welcome to come up with your own
- Today's lecture is ~~easy normal~~ a little bit hard  
**(Next lectures will be much easier!)**

# Last Lecture

- Radiometry cont.
- Light transport
  - The reflection equation
  - The rendering equation
- Global illumination
- Probability review

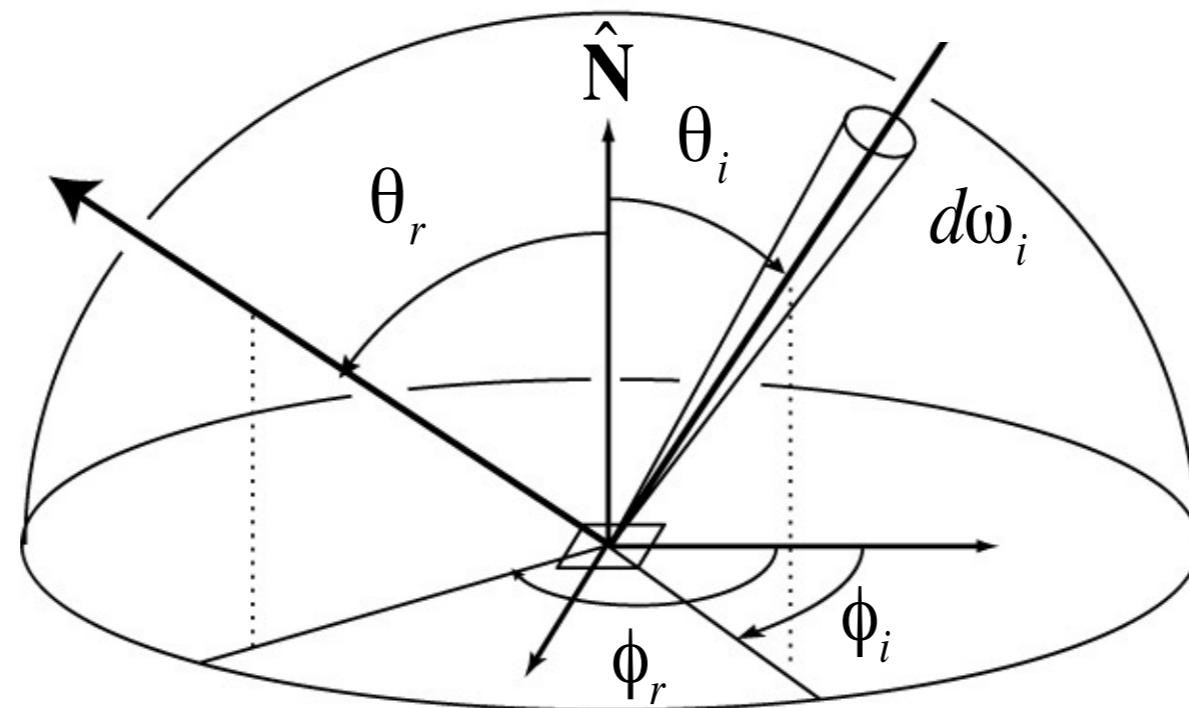
# Today

- A Brief Review
- Monte Carlo Integration
- Path Tracing

# Review - The Rendering Equation

- Describing the light transport

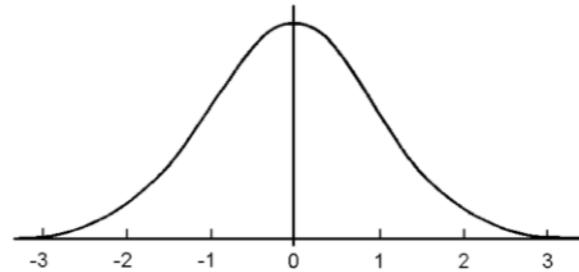
$$L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{\Omega^+} L_i(p, \omega_i) f_r(p, \omega_i, \omega_o) (n \cdot \omega_i) d\omega_i$$



# Review - Probabilities

- Continuous Variable and Probability Density Functions

$$X \sim p(x)$$



- Understanding: randomly pick an  $X \rightarrow$  more likely to be a number closer to 0 (in this case)

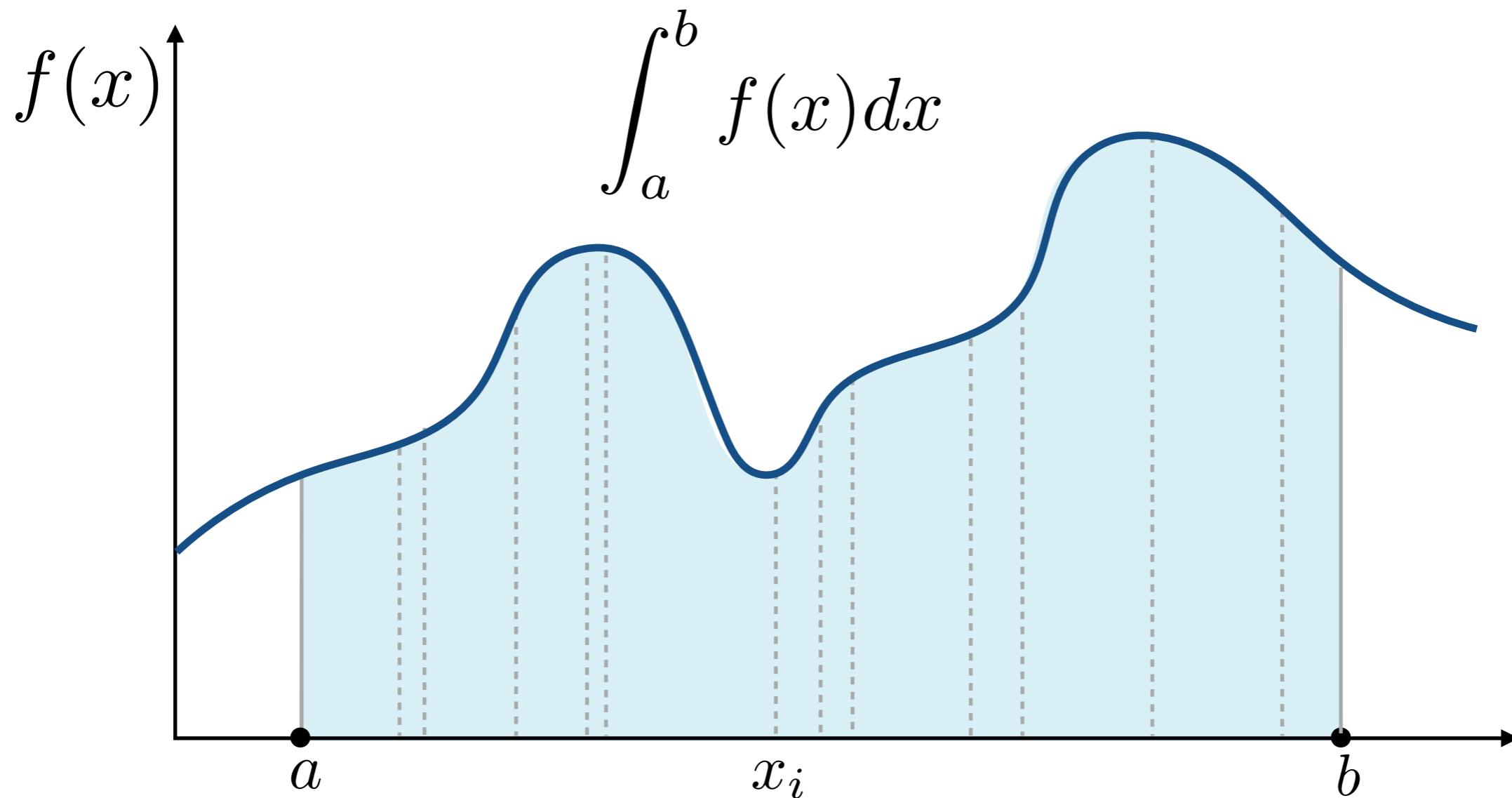
Conditions on  $p(x)$ :  $p(x) \geq 0$  and  $\int p(x) dx = 1$

Expected value of  $X$ :  $E[X] = \int x p(x) dx$

# Monte Carlo Integration

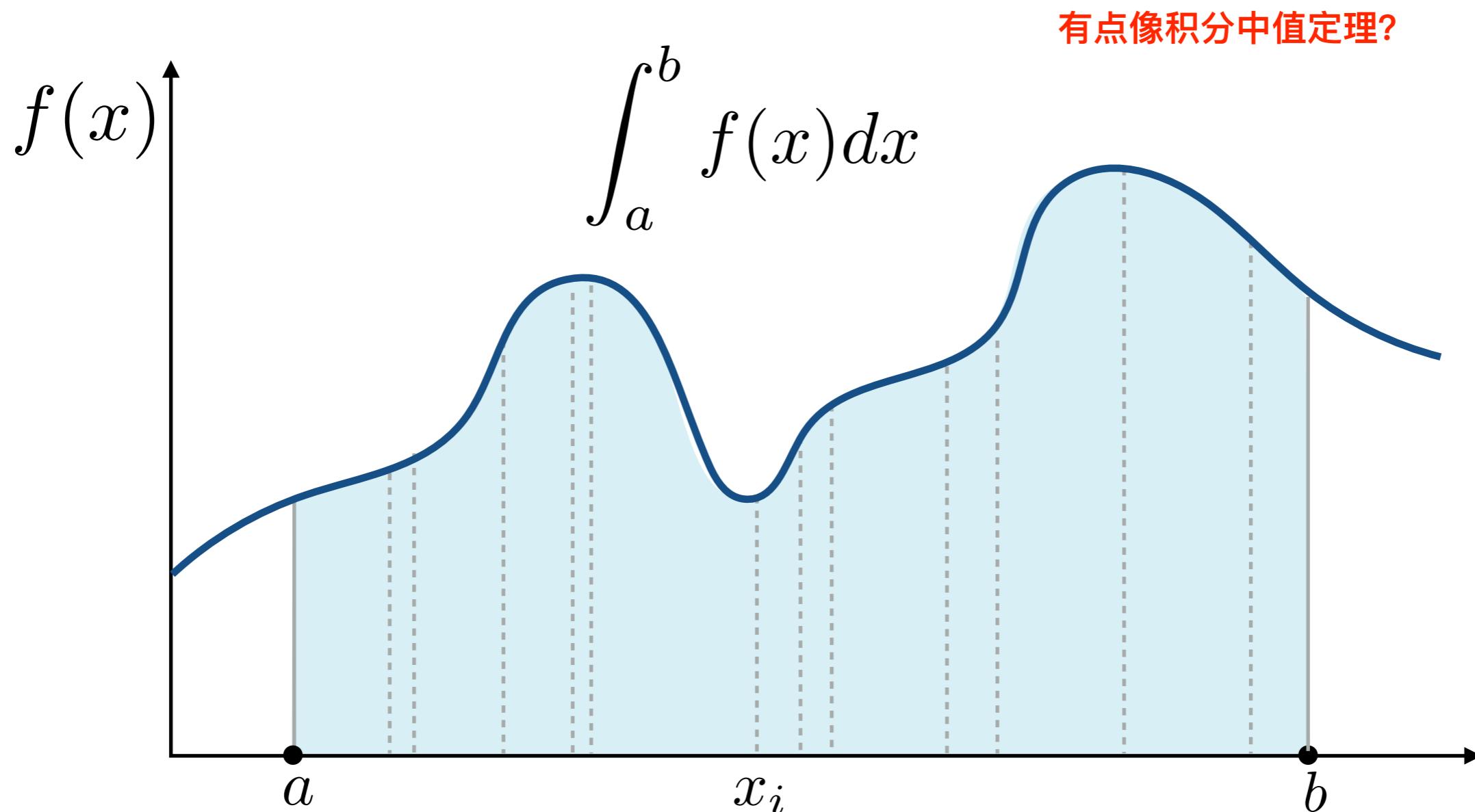
# Monte Carlo Integration

**Why:** we want to solve an integral, but it can be too difficult to solve analytically. 目的是算定积分，但是是数值方法



# Monte Carlo Integration

**What & How:** estimate the integral of a function by averaging random samples of the function's value.



# Monte Carlo Integration

Let us define the Monte Carlo estimator for the definite integral of given function  $f(x)$

Definite integral

$$\int_a^b f(x)dx$$

Random variable

$$X_i \sim p(x)$$

Monte Carlo **estimator**

$$F_N = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}$$

# Example: Uniform Monte Carlo Estimator

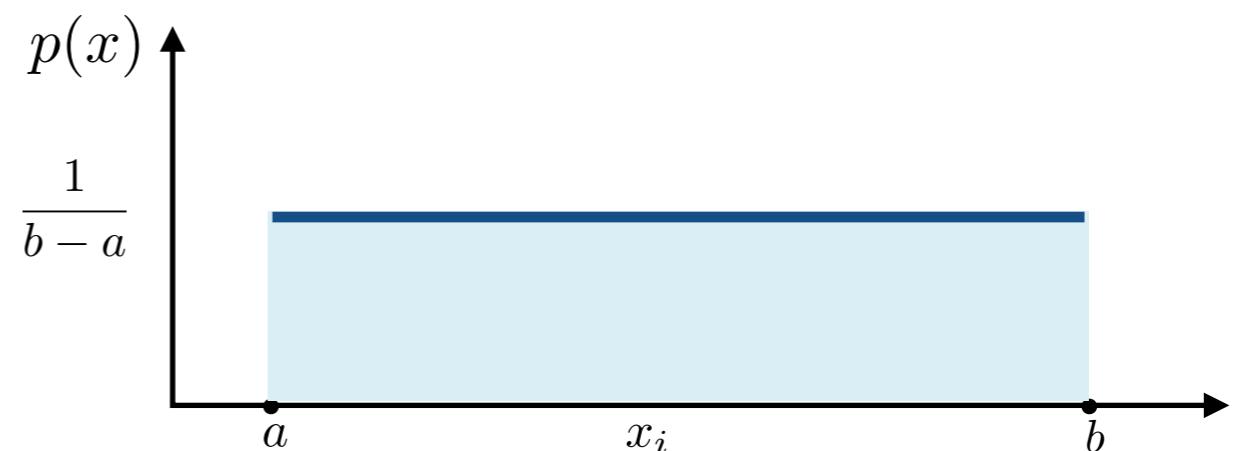
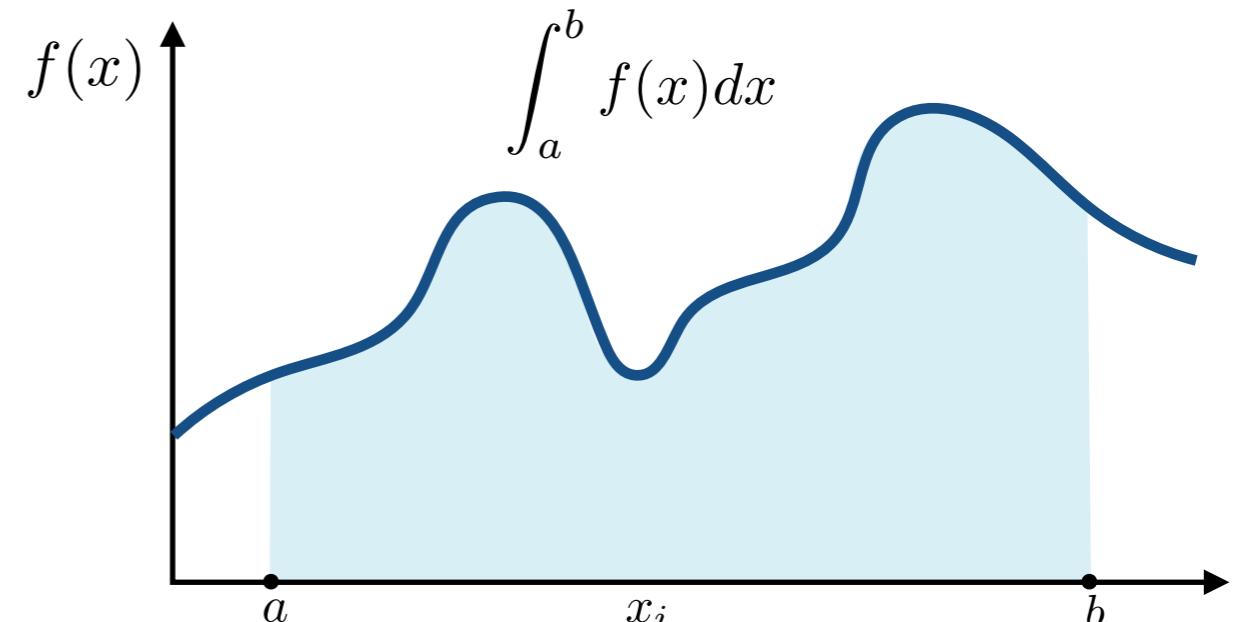
Uniform random variable

$$X_i \sim p(x) = C \text{ (constant)}$$

$$\int_a^b p(x) dx = 1$$

$$\Rightarrow \int_a^b C dx = 1$$

$$\Rightarrow C = \frac{1}{b-a}$$



# Example: Uniform Monte Carlo Estimator

Let us define the Monte Carlo estimator for the definite integral of given function  $f(x)$

Definite integral

$$\int_a^b f(x)dx$$

**Uniform** random variable

$$X_i \sim p(x) = \frac{1}{b-a}$$

Basic Monte Carlo estimator

$$F_N = \frac{b-a}{N} \sum_{i=1}^N f(X_i)$$

# Monte Carlo Integration

$$\int f(x) dx = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}$$

$$X_i \sim p(x)$$

积分域已经包含在PDF中了

Some notes:

- The more samples, the less variance.
- Sample on x, integrate on x.

# Path Tracing

# Motivation: Whitted-Style Ray Tracing

Whitted-style ray tracing:

光滑物体，比如玻璃

- Always perform **specular** reflections / refractions
- Stop bouncing at **diffuse** surfaces

Are these simplifications reasonable?

High level: let's progressively **improve** upon Whitted-Style Ray Tracing and lead to our path tracing algorithm!

改进了Ray Tracing中不正确的假设

# Whitted-Style Ray Tracing: Problem 1

Where should the ray be reflected for glossy materials?

不应该  
是完美的镜面反射



Mirror reflection  
**(Pure) Specular**



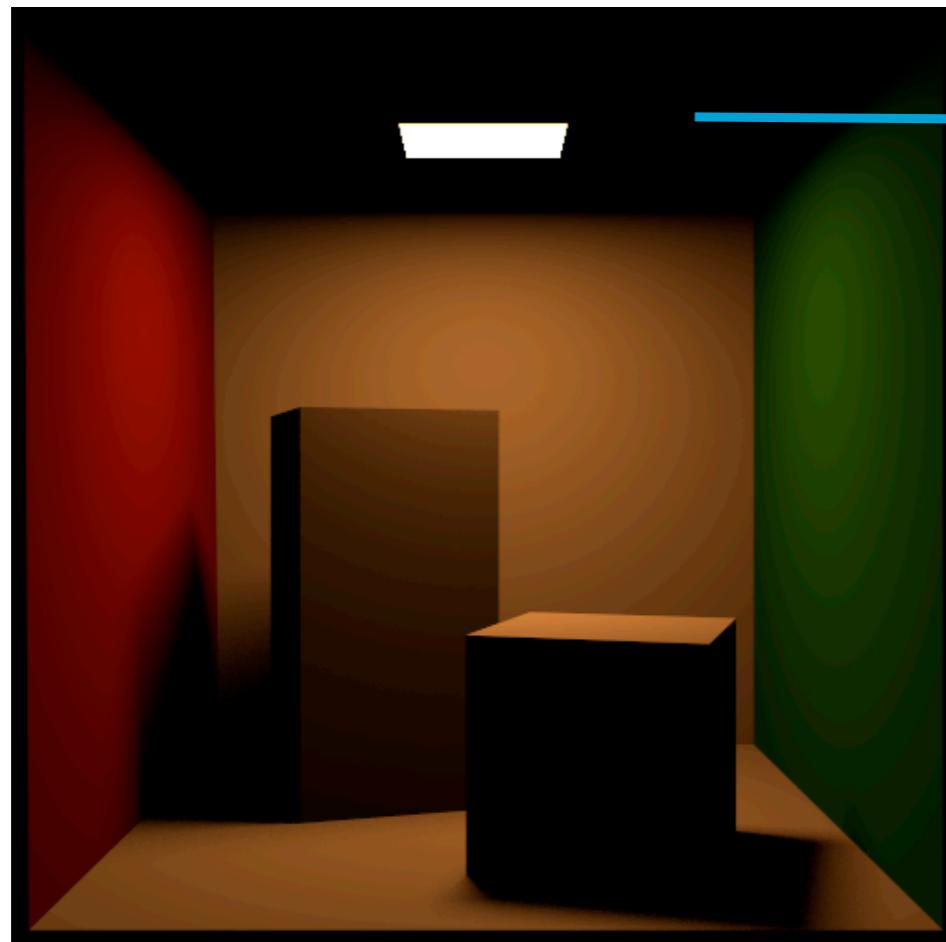
**Glossy** reflection  
没那么光滑

The Utah teapot

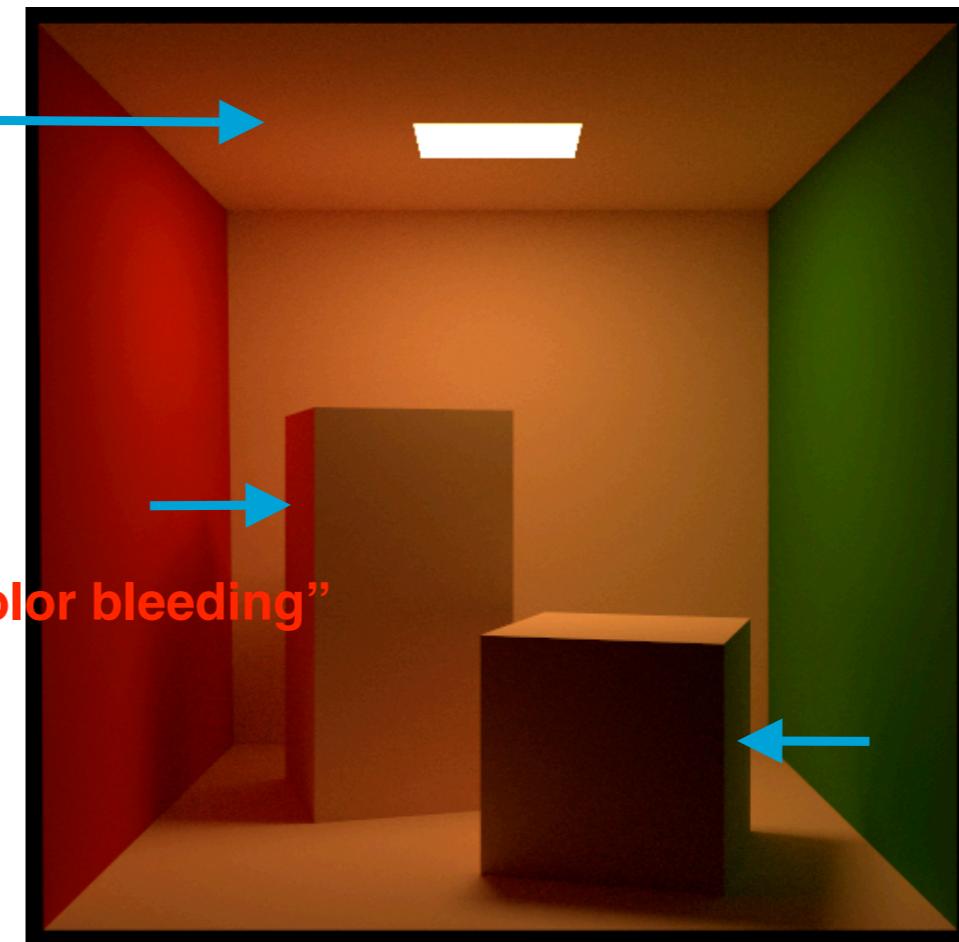
# Whitted-Style Ray Tracing: Problem 2

No reflections between diffuse materials?

并不合适，漫反射也会反射，反射到不同的方向上



Path traced:  
direct illumination



Path traced:  
global illumination

The Cornell box  
场景中全部是漫反射物体

# Whitted-Style Ray Tracing is Wrong

But the rendering equation is correct

$$L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{\Omega^+} L_i(p, \omega_i) f_r(p, \omega_i, \omega_o) (n \cdot \omega_i) d\omega_i$$

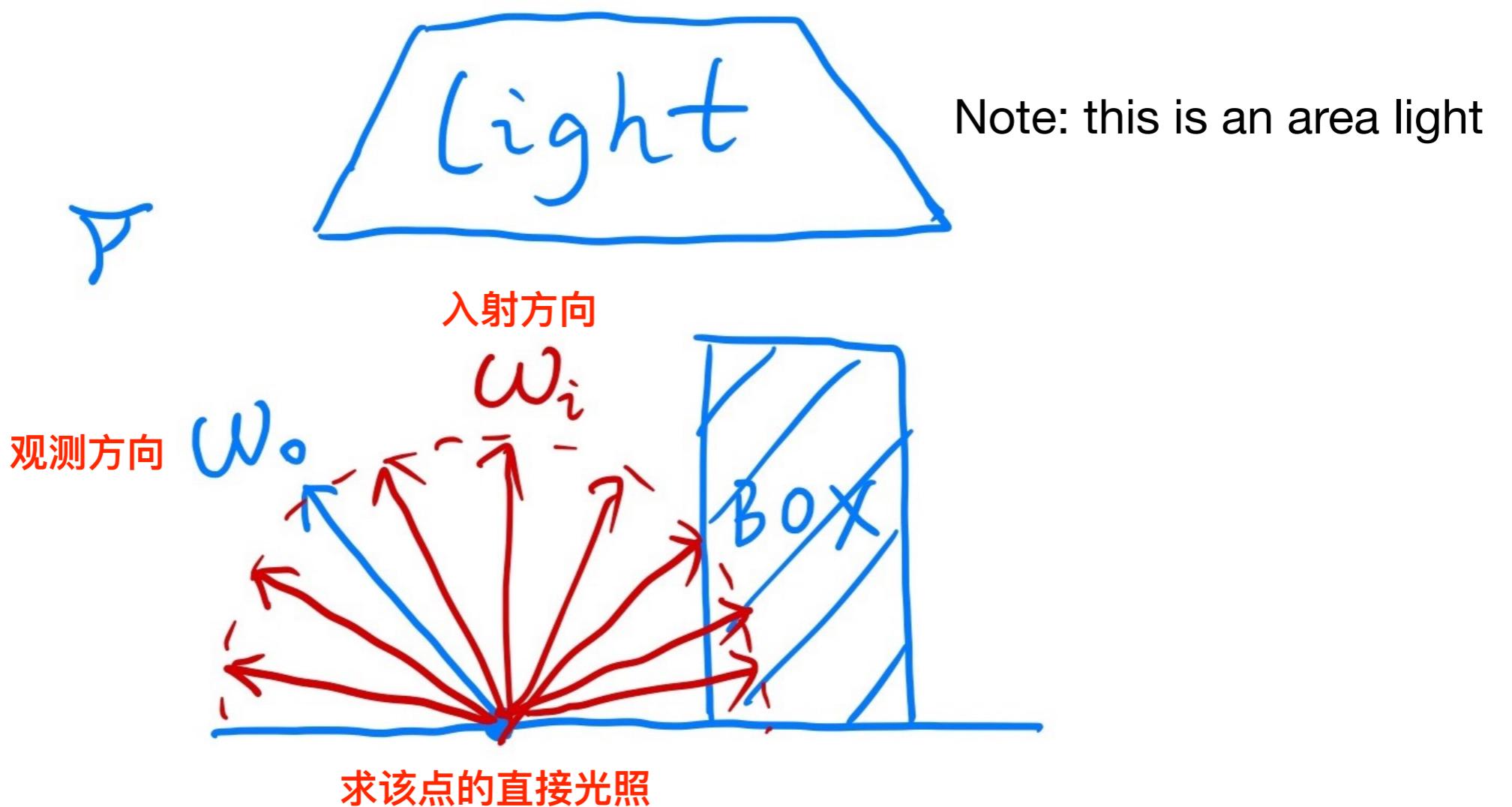
But it involves

- Solving an integral over the hemisphere, and
- Recursive execution

How do you solve an integral numerically?

# A Simple Monte Carlo Solution

Suppose we want to render **one pixel (point)** in the following scene for **direct illumination** only



# A Simple Monte Carlo Solution

Abuse the concept of Reflection Equation a little bit

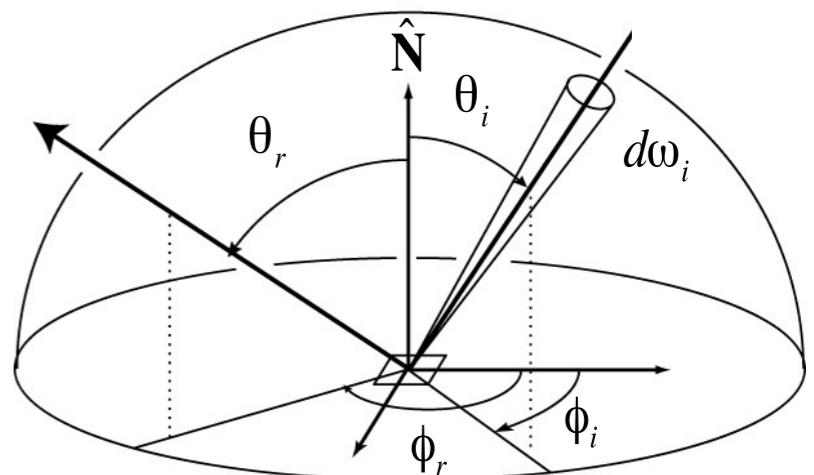
暂时忽略发光项, 变成反射方程

$$L_o(p, \omega_o) = \int_{\Omega^+} L_i(p, \omega_i) f_r(p, \omega_i, \omega_o) (\mathbf{n} \cdot \omega_i) d\omega_i$$

↑  
(again, we assume all directions are **pointing outwards**)  
不考虑间接光照, 所以只能来自光源

Fancy as it is, it's still just an integration over directions

So, of course we can solve it using  
Monte Carlo integration!



# A Simple Monte Carlo Solution

We want to compute the radiance at p towards the camera

$$L_o(p, \omega_o) = \int_{\Omega^+} L_i(p, \omega_i) f_r(p, \omega_i, \omega_o) (\mathbf{n} \cdot \omega_i) d\omega_i$$

Monte Carlo integration:  $\int_a^b f(x) dx \approx \frac{1}{N} \sum_{k=1}^N \frac{f(X_k)}{p(X_k)}$   $X_k \sim p(x)$

What's our "f(x)"?  $L_i(p, \omega_i) f_r(p, \omega_i, \omega_o) (\mathbf{n} \cdot \omega_i)$

What's our pdf?  $p(\omega_i) = 1/2\pi$

(assume uniformly sampling the hemisphere)

# A Simple Monte Carlo Solution

So, in general

$$\begin{aligned} L_o(p, \omega_o) &= \int_{\Omega^+} L_i(p, \omega_i) f_r(p, \omega_i, \omega_o) (\mathbf{n} \cdot \omega_i) d\omega_i \\ &\approx \frac{1}{N} \sum_{i=1}^N \frac{L_i(p, \omega_i) f_r(p, \omega_i, \omega_o) (\mathbf{n} \cdot \omega_i)}{p(\omega_i)} \end{aligned}$$

对着色点p，来自任何一个方向的直接光照贡献

(note: abuse notation a little bit for i)

What does it mean?

A correct shading algorithm for direct illumination!

# A Simple Monte Carlo Solution

$$L_o(p, \omega_o) \approx \frac{1}{N} \sum_{i=1}^N \frac{L_i(p, \omega_i) f_r(p, \omega_i, \omega_o)(n \cdot \omega_i)}{p(\omega_i)}$$

shade(p, wo)

此算法仅针对直接光照

Randomly choose N directions  $\omega_i \sim \text{pdf}$

$L_o = 0.0$

For each  $\omega_i$

Trace a ray  $r(p, \omega_i)$

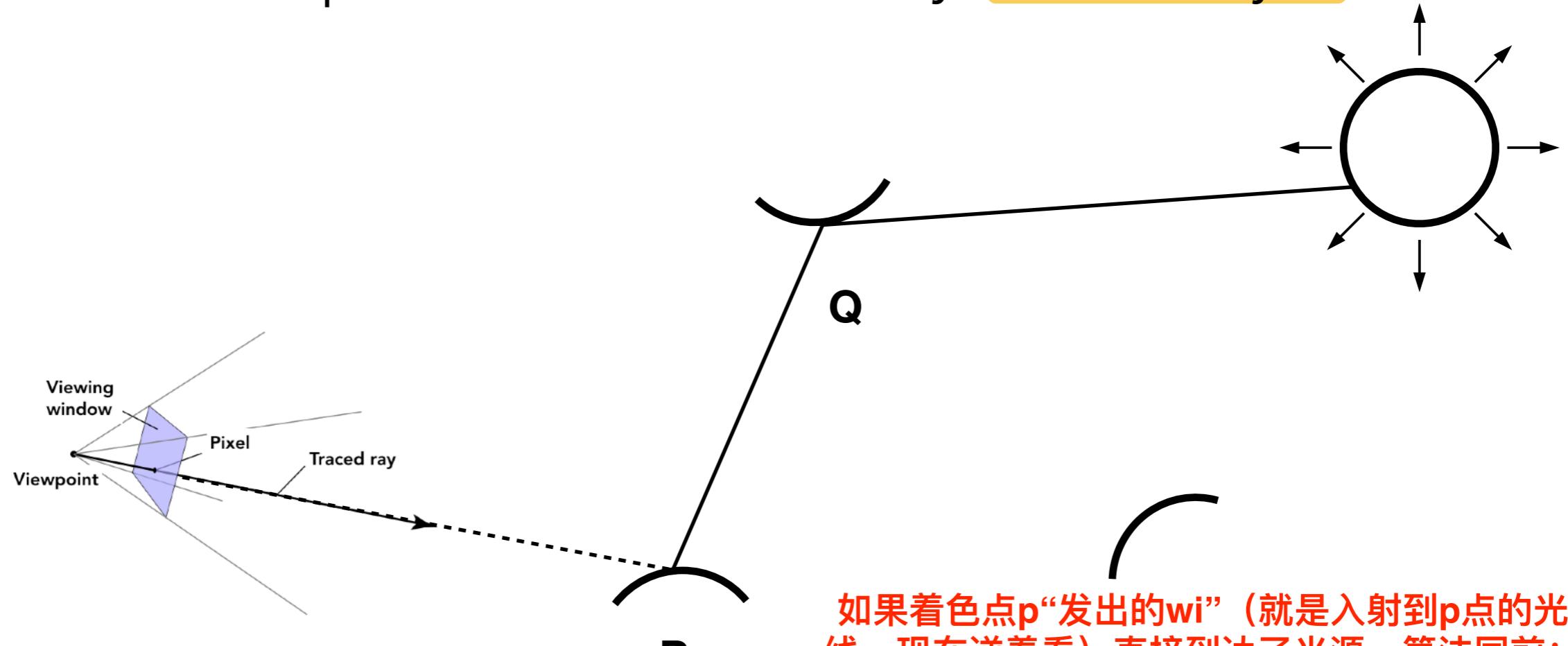
If ray r hit the light

$L_o += (1 / N) * L_i * f_r * \text{cosine} / \text{pdf}(\omega_i)$

Return  $L_o$

# Introducing Global Illumination

One more step forward: what if a ray hits an object?



如果着色点p“发出的wi”（就是入射到p点的光线，现在逆着看）直接到达了光源，算法同前；如果着色点p“发出的wi”到达了另一物体表面，那么可以认为Q点是另一个光源，直接调用Q点发出的radianc即可，即shade(q,-wi)

Q also reflects light to P! How much? The dir. illum. at Q!

# Introducing Global Illumination

shade( $p$ ,  $\omega_0$ )

Randomly choose  $N$  directions  $w_i \sim \text{pdf}$

$L_o = 0.0$

For each  $w_i$

Trace a ray  $r(p, w_i)$

If ray  $r$  hit the light

$L_o += (1 / N) * L_i * f_r * \text{cosine} / \text{pdf}(w_i)$

Else If ray  $r$  hit an object at  $q$

$L_o += (1 / N) * \text{shade}(q, -w_i) * f_r * \text{cosine} / \text{pdf}(w_i)$

在 $q$ 点, 以 $-w_i$ 方向看 (对于 $p$ 点是 $+w_i$ ) ,  $q$ 点的直接光照结果  
Recursive!

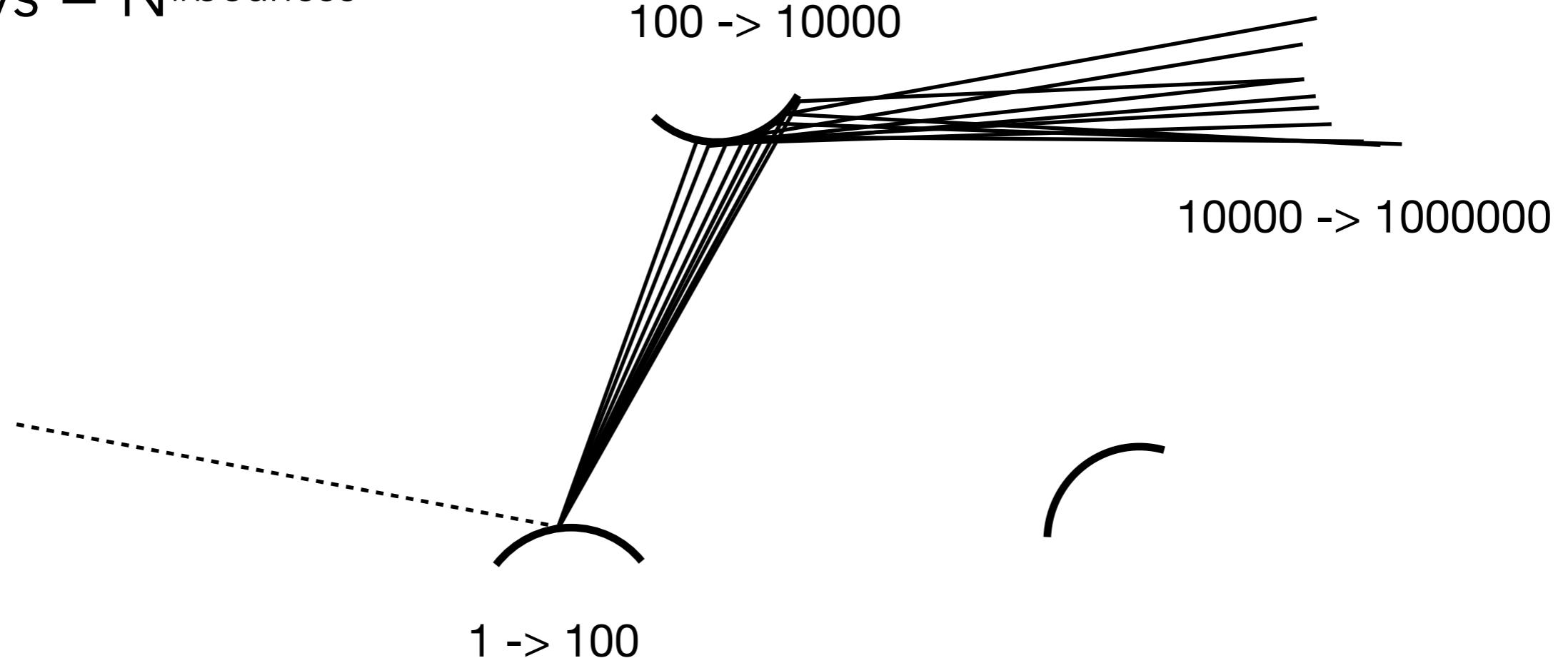
Return  $L_o$

Is it done? **No.**

# Path Tracing

**Problem 1:** Expllosion of #rays as #bounces go up:

$$\#rays = N^{\#bounces}$$



**Key observation:** #rays will not explode iff  $N = ????????$

# Path Tracing

From now on, we always assume that  
only 1 ray is traced at each shading point:

**shade(p, wo)**

Randomly choose **ONE** direction  $w \sim \text{pdf}(w)$

Trace a ray  $r(p, w_i)$

If ray r hit the light

Return  $L_i * f_r * \cosine / \text{pdf}(w_i)$  1/N \rightarrow 1

Else If ray r hit an object at q

```
Return shade(q, -wi) * f_r * cosine / pdf(wi)
```

This is **path tracing**! (FYI, Distributed Ray Tracing if N != 1)

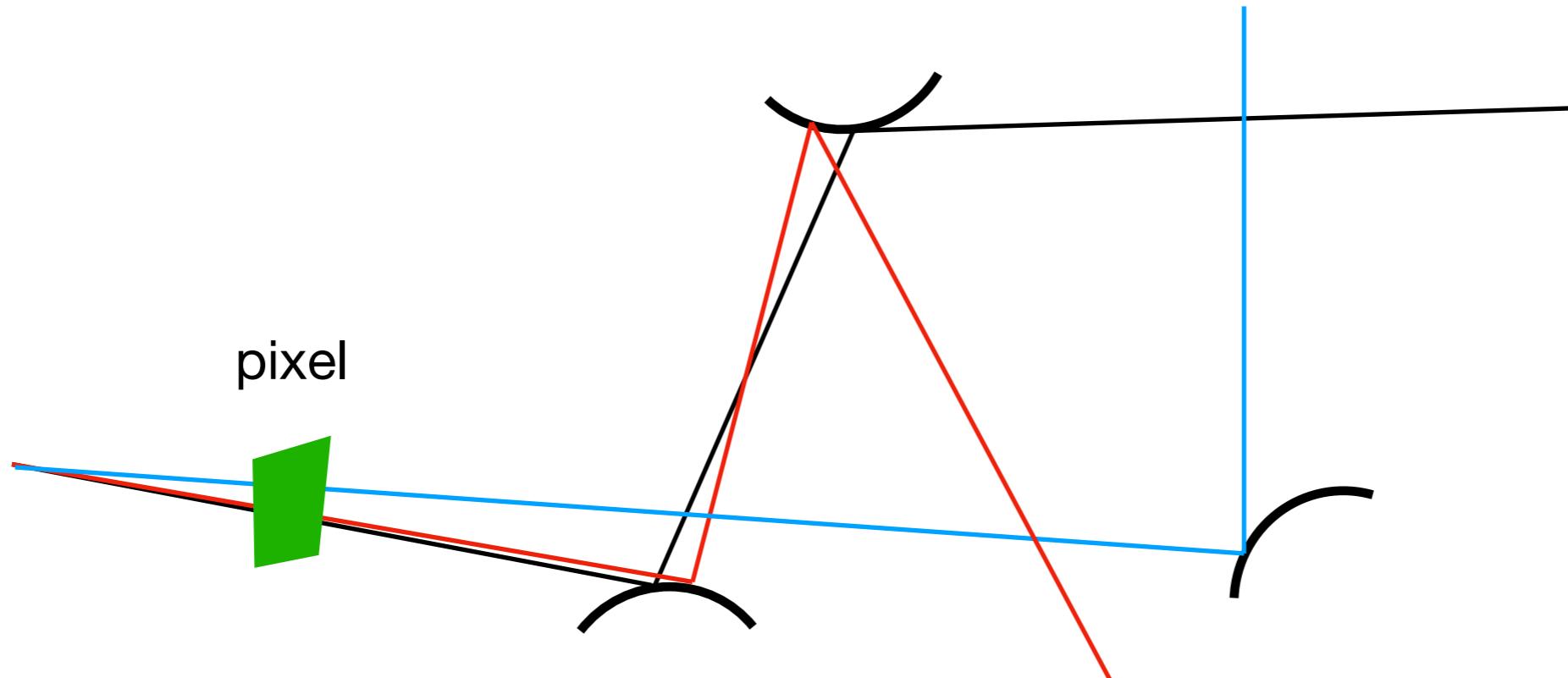
但方差大

分布追踪

# Ray Generation

But this will be **noisy!**

No problem, just trace more **paths** through each pixel and average their radiance! **path:** 随机选择一个方向反射，而不是一束光



# Ray Generation

Very similar to ray casting in ray tracing

对某一个像素，发出多组path

```
ray_generation(camPos, pixel)
    Uniformly choose N sample positions within the pixel
    pixel_radiance = 0.0
    For each sample in the pixel
        Shoot a ray r(camPos, cam_to_sample)
        If ray r hit the scene at p
            pixel_radiance += 1 / N * shade(p, sample_to_cam)
    Return pixel_radiance
                    对N条path取平均
                    出射到camera里
```

# Path Tracing

Now are we good? Any other problems in shade()?

**shade**( $p, wo$ )

Randomly choose ONE direction  $wi \sim pdf(w)$

Trace a ray  $r(p, wi)$

If ray  $r$  hit the light

    Return  $L_i * f_r * cosine / pdf(wi)$

Else If ray  $r$  hit an object at  $q$

    Return **shade**( $q, -wi$ ) \*  $f_r * cosine / pdf(wi)$

**Problem 2:** The recursive algorithm will never stop!

# Path Tracing

Dilemma: the light does not stop bouncing indeed!

Cutting #bounces == cutting energy!

3 bounces



# Path Tracing

Dilemma: the light does not stop bouncing indeed!

Cutting #bounces == cutting energy!

17 bounces



# Solution: Russian Roulette (RR)

(俄罗斯轮盘赌)

Russian Roulette is all about probability

With probability  $0 < P < 1$ , you are fine

With probability  $1 - P$ , otherwise

用一定的概率停止继续追踪



Example: two bullets,  
Survival probability  $P = 4 / 6$

# Solution: Russian Roulette (RR)

Previously, we always shoot a ray at a shading point and get the shading result  $\text{Lo}$

Suppose we manually set a probability  $P$  ( $0 < P < 1$ )

With probability  $P$ , shoot a ray and return the **shading result divided by  $P$** :  $\text{Lo} / P$

With probability  $1-P$ , don't shoot a ray and you'll get  $0$

In this way, you can **still expect** to get  $\text{Lo}$ !

$$E = P * (\text{Lo} / P) + (1 - P) * 0 = \text{Lo}$$

# Solution: Russian Roulette (RR)

**shade(p, wo)**

Manually specify a probability P\_RR

Randomly select ksi in a uniform dist. in [0, 1]

If (ksi > P\_RR) return 0.0;

图形学中不强调边界

Randomly choose ONE direction wi~pdf(w)

Trace a ray r(p, wi)

If ray r hit the light

Return L\_i \* f\_r \* cosine / pdf(wi) / P\_RR

Else If ray r hit an object at q

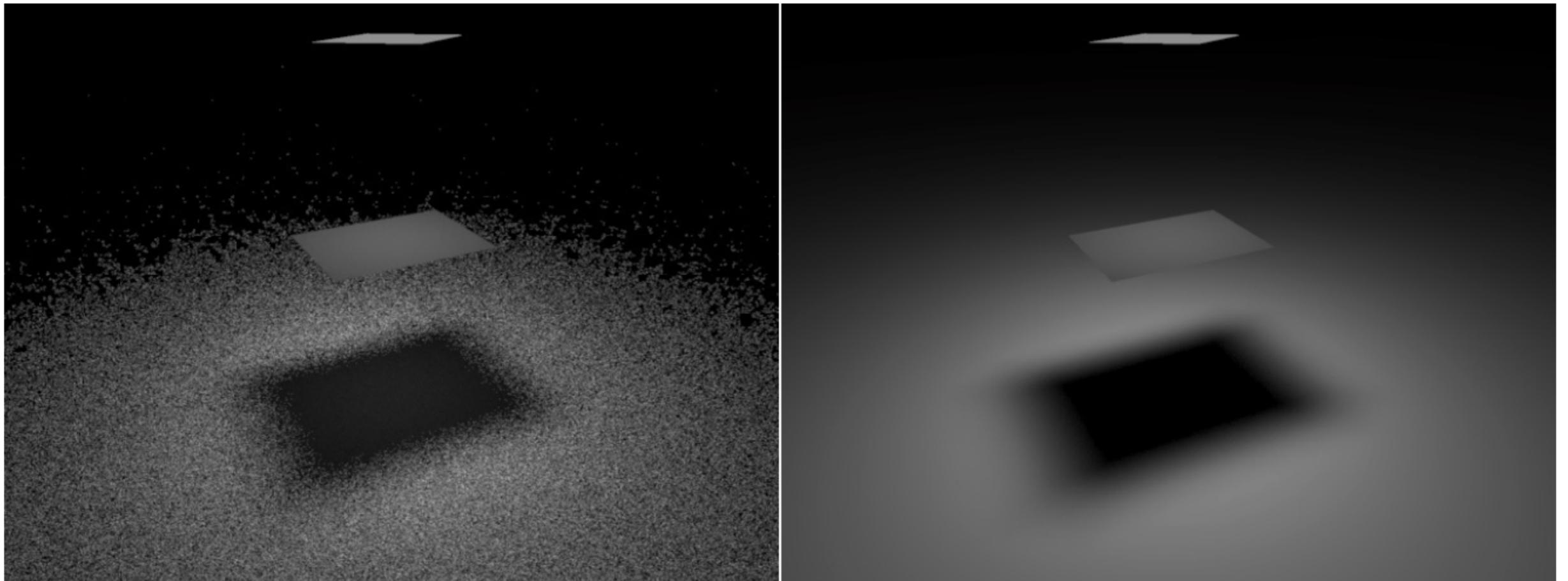
除以生存概率

Return shade(q, -wi) \* f\_r \* cosine / pdf(wi) / P\_RR

# Path Tracing

Now we already have a **correct** version of path tracing!

But it's **not really efficient.**

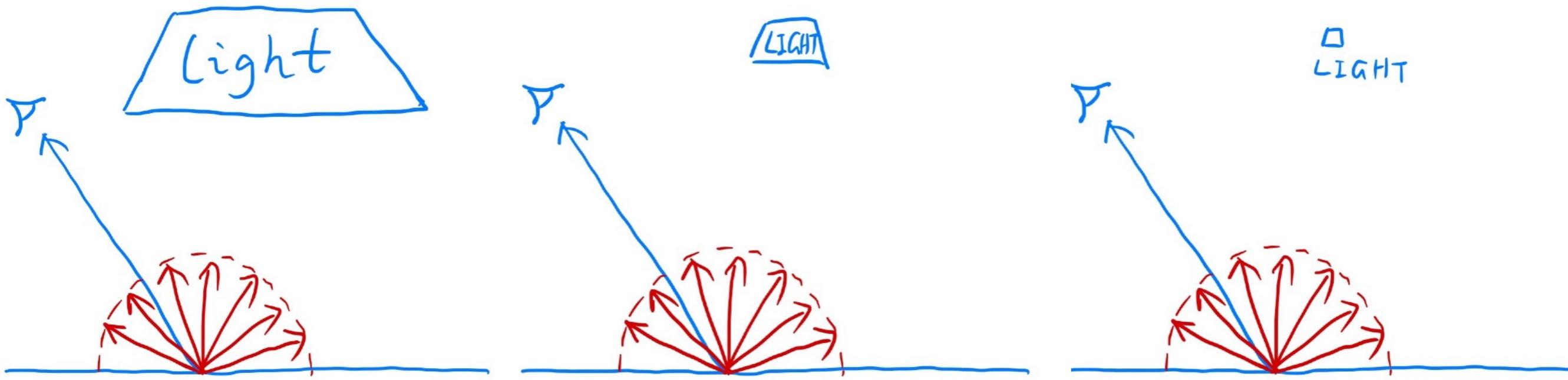


Low SPP (samples per pixel)  
noisy results

High SPP

# Sampling the Light

Understanding the reason of being inefficient



Every 5 rays,

500 rays,

50000 rays,

there will be 1 ray hitting the light. So a lot of rays are “wasted” if we uniformly sample the hemisphere at the shading point.

# Sampling the Light (pure math)

Monte Carlo methods allows any sampling methods, so we can sample the light (therefore no rays are “wasted”)

Assume uniformly sampling on the light:

$$\text{pdf} = 1 / A \text{ (because } \int \text{pdf } dA = 1)$$

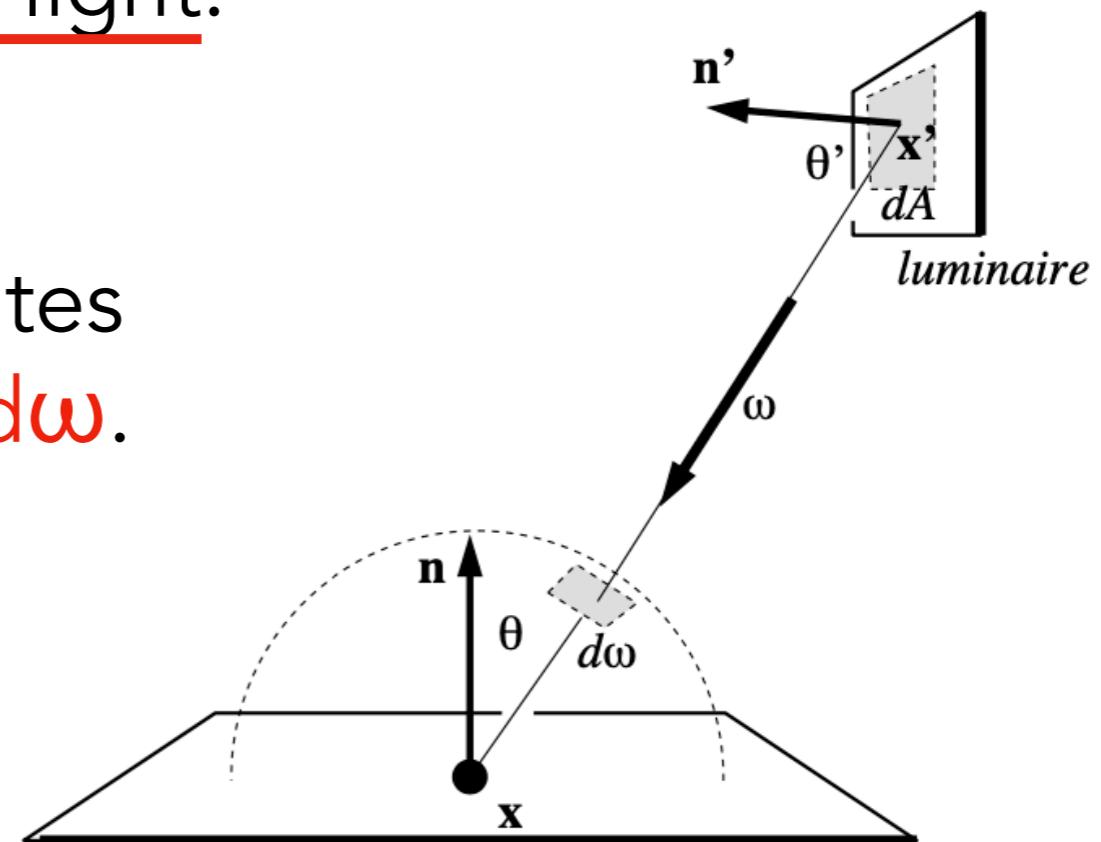
But the rendering equation integrates on the solid angle:  $L_o = \int L_i f_r \cos d\omega$ .

渲染方程不是定义在光源上的，而是在立体角上的

Recall Monte Carlo Integration:  
Sample on  $x$  & integrate on  $x$

所以要将渲染方程写成在光源上的微元进行积分

Since we sample on the light, can we integrate on the light?



# Sampling the Light

Need to make the rendering equation as an integral of  $dA$

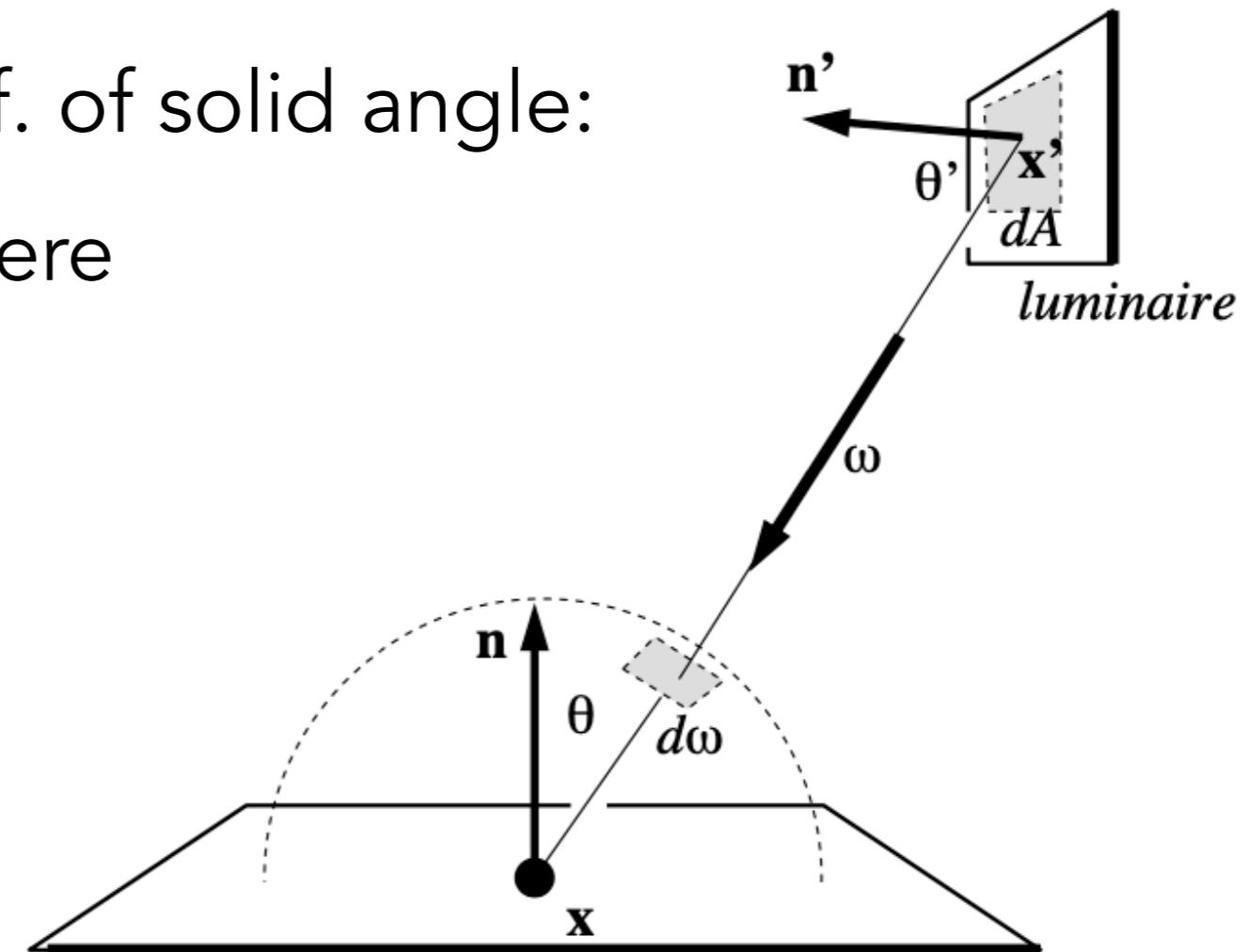
Need the relationship between  $d\omega$  and  $dA$

Easy! Recall the alternative def. of solid angle:

Projected area on the unit sphere

$$d\omega = \frac{dA \cos \theta'}{\|x' - x\|^2}$$

(Note:  $\theta'$ , not  $\theta$ )



# Sampling the Light

Then we can rewrite the rendering equation as

$$\begin{aligned} L_o(x, \omega_o) &= \int_{\Omega^+} L_i(x, \omega_i) f_r(x, \omega_i, \omega_o) \cos \theta d\omega_i \\ &= \int_A L_i(x, \omega_i) f_r(x, \omega_i, \omega_o) \frac{\cos \theta \cos \theta'}{\|x' - x\|^2} dA \end{aligned}$$

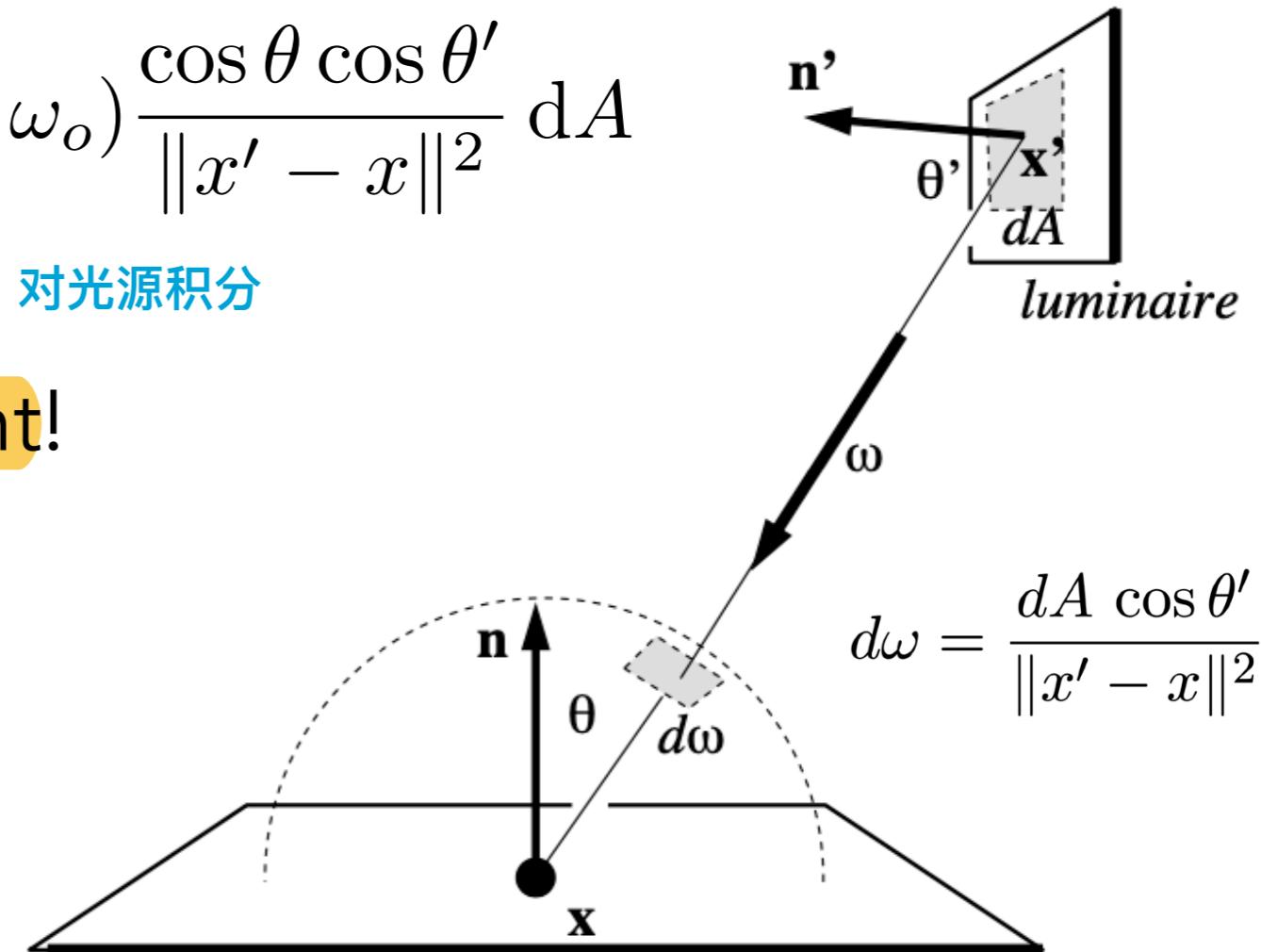
对光源采样，对光源积分

Now an integration on the light!

Monte Carlo integration:

“ $f(x)$ ”: everything inside

Pdf:  $1 / A$

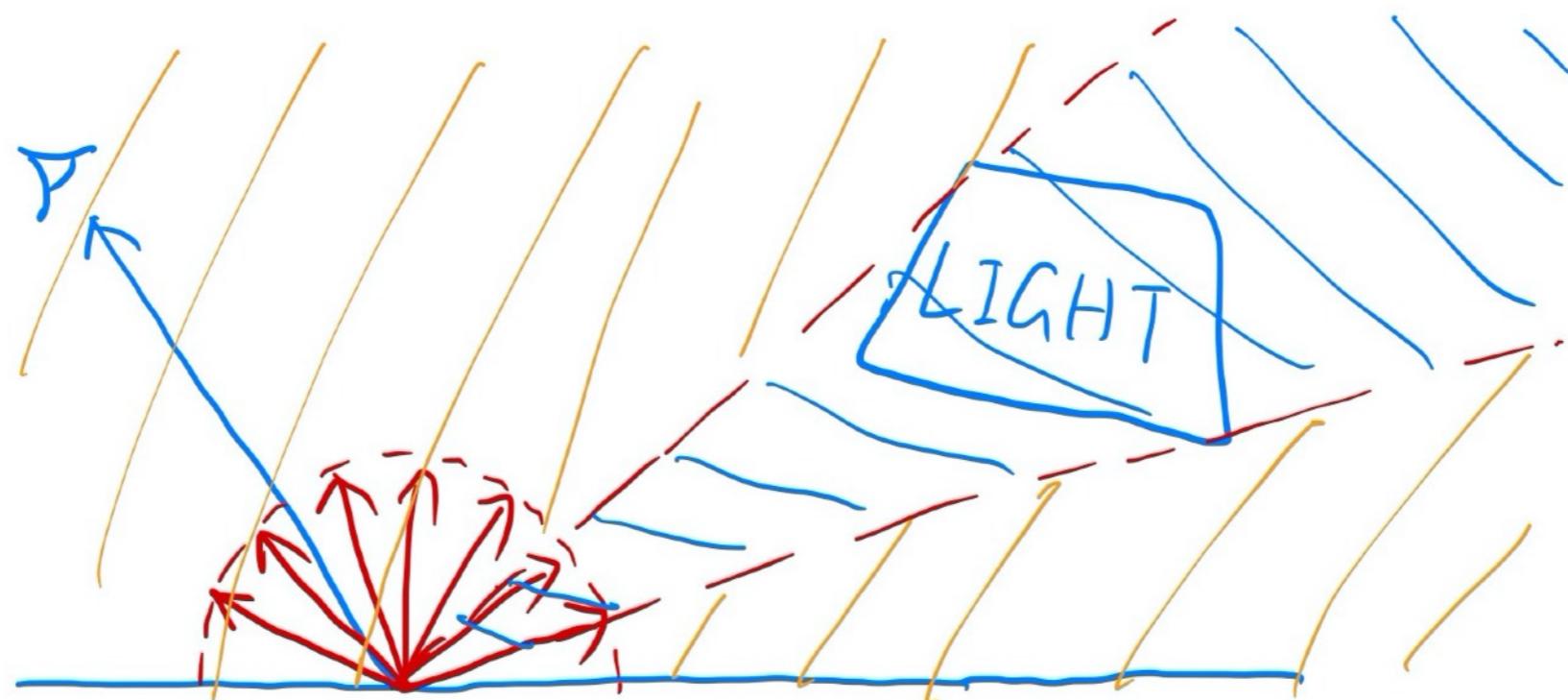


# Sampling the Light

Previously, we assume the light is “accidentally” shot by uniform hemisphere sampling

Now we consider the radiance coming from two parts:

- 光源一定会对着色点贡献，不发生弹射
1. light source (direct, no need to have RR)
  2. other reflectors (indirect, RR) 涉及到多次弹射，才要考虑轮盘赌



# Sampling the Light

`shade(p, wo)`

# Contribution from the light source.

**Uniformly** sample the light at  $x'$  (`pdf_light = 1 / A`)

$L_{dir} = L_i * f_r * \cos \theta * \cos \theta' / |x' - p|^2 / pdf\_light$

# Contribution from other reflectors.

$L_{indir} = 0.0$

Test **Russian Roulette** with probability `P_RR`

**Uniformly** sample the hemisphere toward  $w_i$  (`pdf_hemi = 1 / 2pi`)

Trace a ray `r(p, wi)`

If ray `r` hit a **non-emitting** object at  $q$

$L_{indir} = shade(q, -wi) * f_r * \cos \theta / pdf\_hemi / P_{RR}$

Return  $L_{dir} + L_{indir}$

# Sampling the Light

One final thing: how do we know if the sample on the light is not blocked or not?

# Contribution from the light source.

L\_dir = 0.0

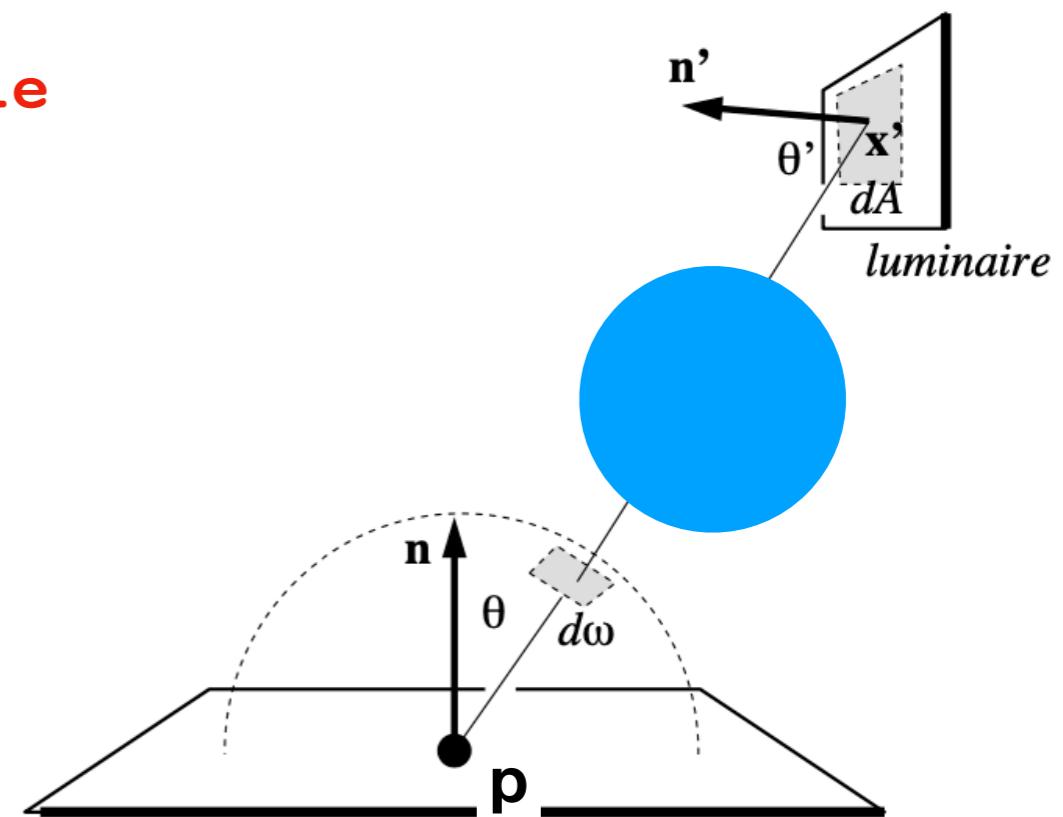
Uniformly sample the light at  $x'$  ( $\text{pdf\_light} = 1 / A$ )

Shoot a ray from  $p$  to  $x'$

If the ray is **not blocked in the middle**

L\_dir = ...

**Now path tracing is finally done!**



# Some Side Notes

- Path tracing (PT) is indeed difficult
  - Consider it the most challenging in undergrad CS
  - Why: physics, probability, calculus, coding
  - Learning PT will help you understand deeper in these
- Is it still “Introductory”?
  - Not really, but it’s “modern” :)
  - And so learning it will be rewarding also because ...

# Is Path Tracing Correct?

Yes, almost 100% correct, a.k.a. **PHOTO-REALISTIC**



**Photo**



Path traced:  
**global** illumination

The Cornell box – <http://www.graphics.cornell.edu/online/box/compare.html>

# Ray tracing: Previous vs. Modern Concepts

- Previous
  - Ray tracing == Whitted-style ray tracing
- Modern (my own definition) **path tracing就属于ray tracing的一种**
  - The **general solution** of light transport, including
  - (Unidirectional & bidirectional) path tracing
  - Photon mapping 生成一张图: 要么Rasterization, 要不ray tracing
  - Metropolis light transport
  - VCM / UPBP...

# Things we haven't covered / won't cover

- Uniformly sampling the hemisphere
  - How? And in general, how to sample any function?  
(sampling)
- Monte Carlo integration allows arbitrary pdfs
  - What's the best choice? (importance sampling)  
如何针对性地选择概率密度分布函数      “重要性采样”理论
- Do random numbers matter?
  - Yes! (low discrepancy sequences)

随机数也有质量

# Things we haven't covered / won't cover

- I can sample the hemisphere **and** the light
  - Can I combine them? Yes! (multiple imp. sampling)  
**multiple important sampling**
- The radiance of a pixel is the average of radiance on all paths passing through it
  - Why? (pixel reconstruction filter) 像素平均的时候要不要加权?
- Is the radiance of a pixel the color of a pixel?
  - No. (gamma correction, curves, color space)  
**gamma校正**
- Asking again, is path tracing still “Introductory”?
  - This time, yes. **Fear the science**, my friends.

# Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)