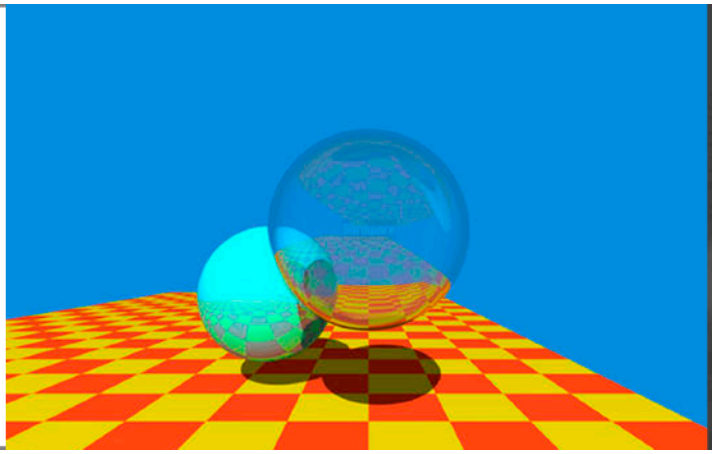
Introduction to Computer Graphics

GAMES101, Lingqi Yan, UC Santa Barbara

Lecture 11:

Geometry 2 (Curves and Surfaces)






http://www.cs.ucsb.edu/~lingqi/teaching/games101.html

Announcements

- Homework 3 deadline has been extended
  - To Thursday 23:59PM, Beijing time
- COVID-19 is getting worse in the US
  - Be careful, dude
  - Have to stream at home, network & lighting are worse

Last Lecture

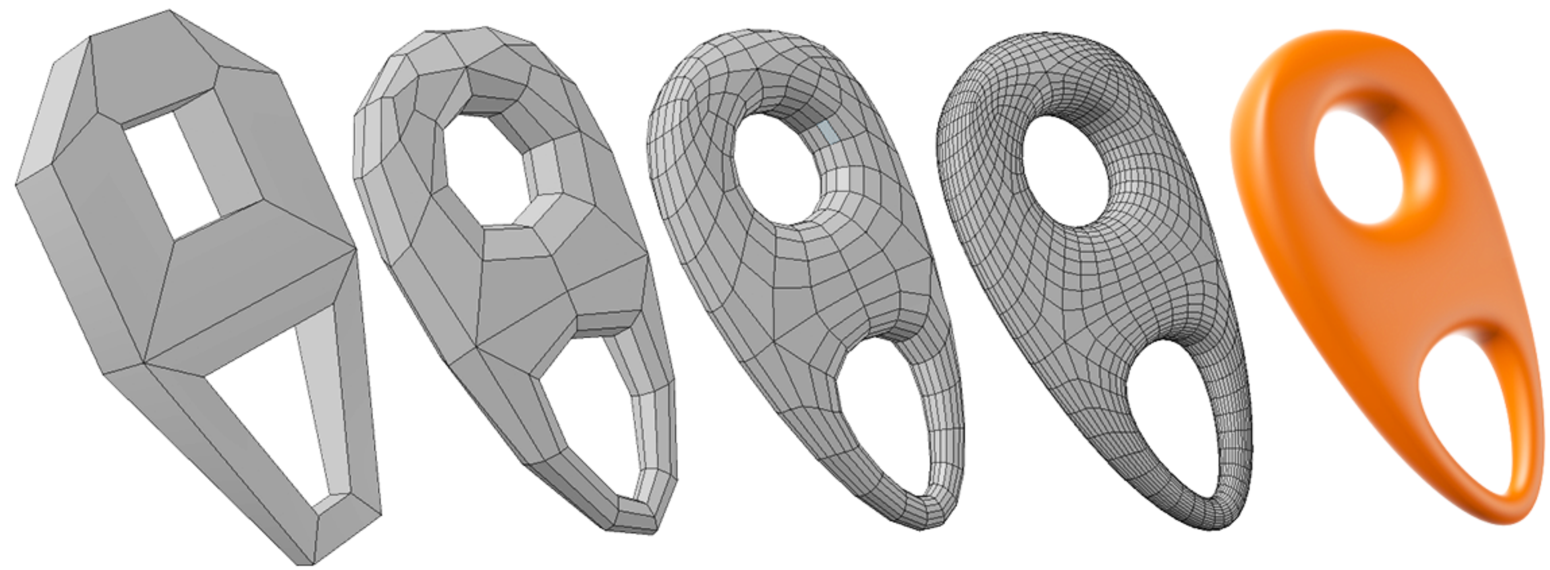
- Introduction to geometry
  - Examples of geometry
  - Various representations of geometry
    - Implicit
    - Explicit

Today

- Explicit Representations
- Curves
  - Bezier curves
  - De Casteljau's algorithm
  - B-splines, etc.
- Surfaces
  - Bezier surfaces
  - Triangles & quads
    - Subdivision, simplification, regularization

Explicit Representations in Computer Graphics

Many Explicit Representations in Graphics



triangle meshes

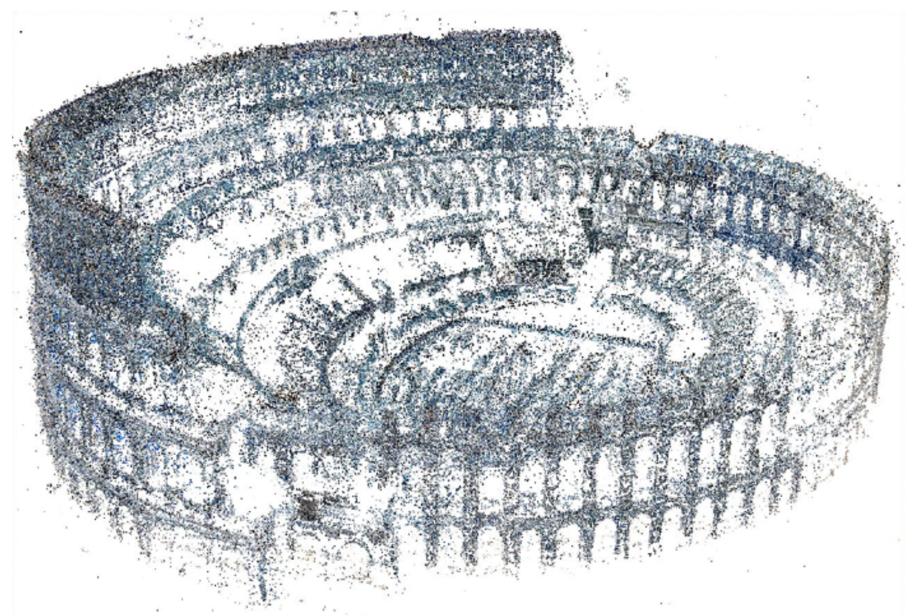
Bezier surfaces

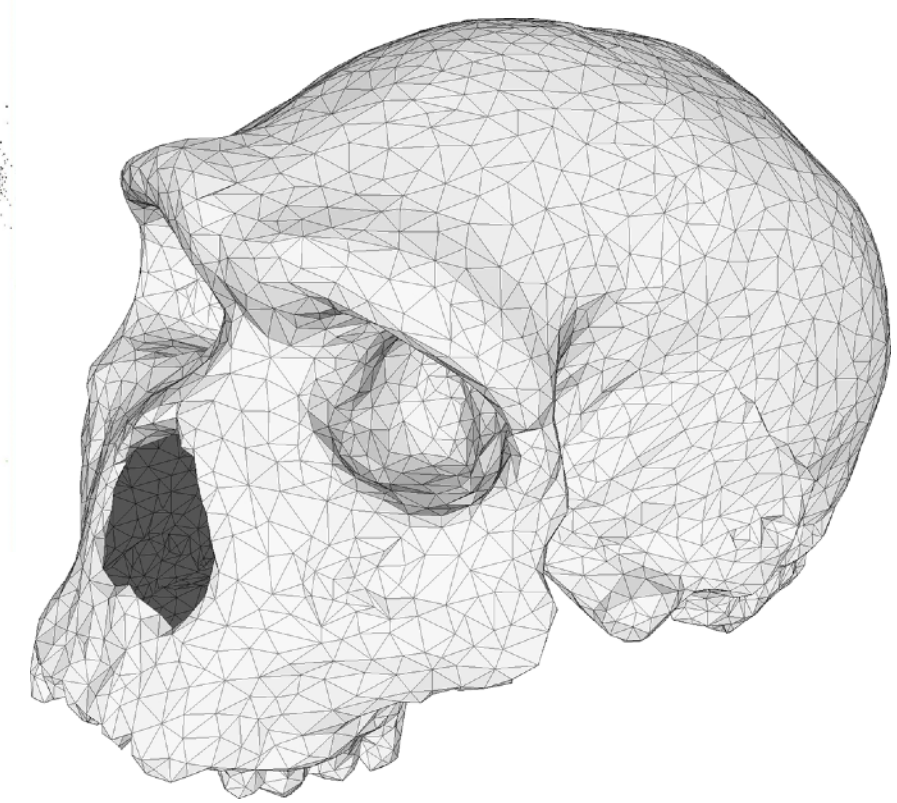
subdivision surfaces

NURBS

point clouds

…





Point Cloud (Explicit)

Easiest representation: list of points (x,y,z)

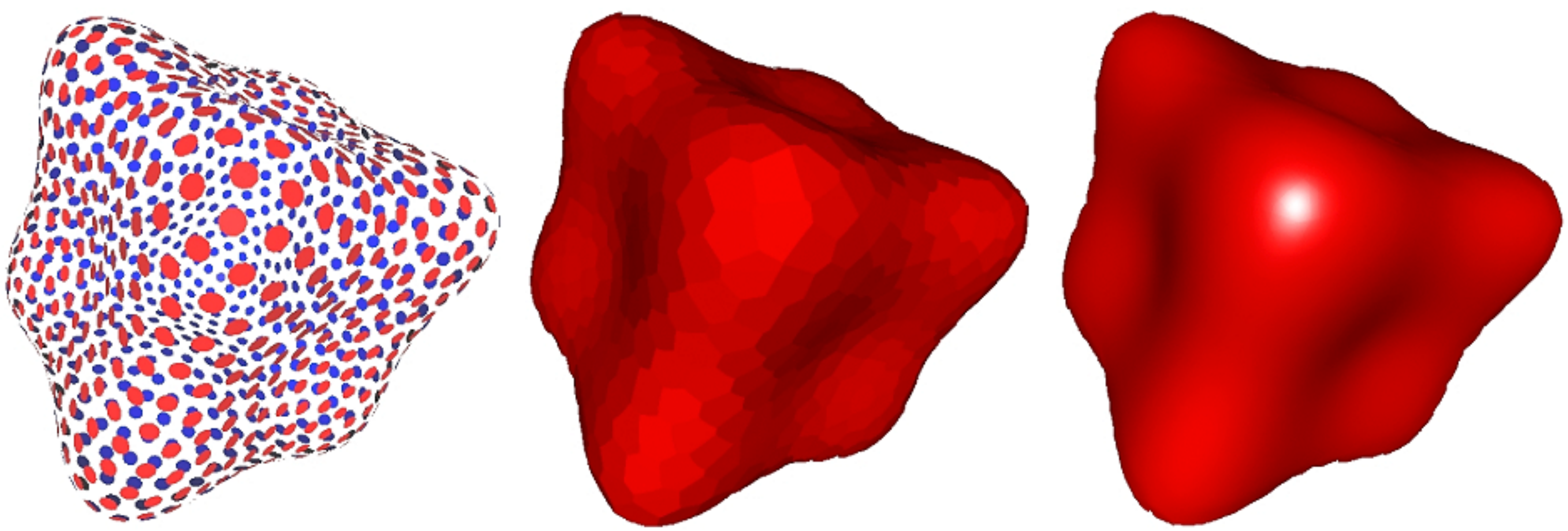
Easily represent any kind of geometry

Useful for LARGE datasets (>>1 point/pixel)

Often converted into polygon mesh

如何将离散点转化为三角形?

Difficult to draw in undersampled regions



Polygon Mesh (Explicit)

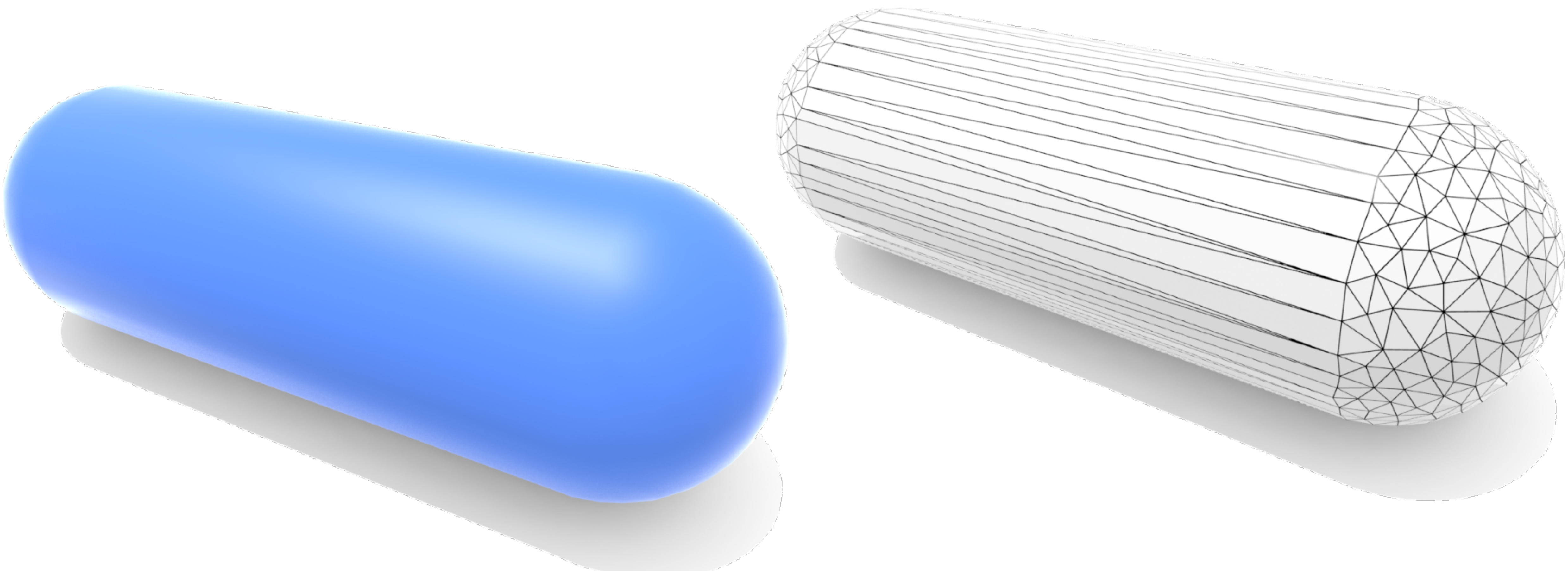
多边形面

Store vertices & polygons (often triangles or quads)

Easier to do processing / simulation, adaptive sampling

More complicated data structures

Perhaps most common representation in graphics



The Wavefront Object File (.obj) Format

Commonly used in Graphics research

Just a text file that specifies vertices, normals, texture coordinates **and their connectivities**

```
# This is a comment

v 1.000000 -1.000000 -1.000000
v 1.000000 -1.000000 1.000000
v -1.000000 -1.000000 1.000000
v -1.000000 -1.000000 -1.000000
v 1.000000 1.000000 -1.000000
v 0.999999 1.000000 1.000001
v -1.000000 1.000000 1.000000
v -1.000000 1.000000 -1.000000

vt 0.748573 0.750412
vt 0.749279 0.501284
vt 0.999110 0.501077
vt 0.999455 0.750380
vt 0.250471 0.500702
vt 0.249682 0.749677
vt 0.001085 0.750380
vt 0.001517 0.499994
vt 0.499422 0.500239
vt 0.500149 0.750166
vt 0.748355 0.998230
vt 0.500193 0.998728
vt 0.498993 0.250415
vt 0.748953 0.250920

```

12=6x2个纹理坐标

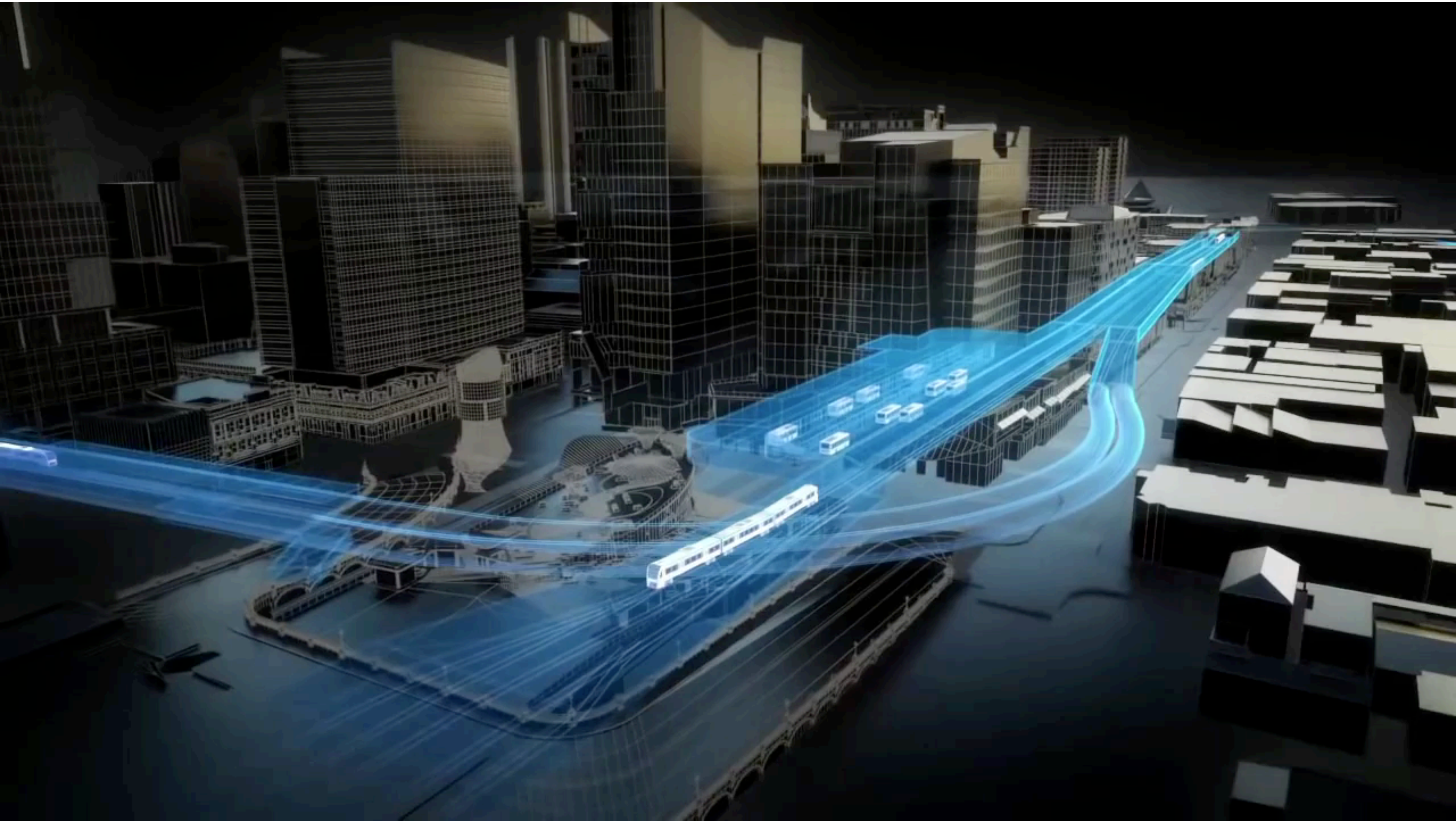
```

vn 0.000000 0.000000 -1.000000
vn -1.000000 -0.000000 -0.000000
vn -0.000000 -0.000000 1.000000
vn -0.000001 0.000000 1.000000
vn 1.000000 -0.000000 0.000000
vn 1.000000 0.000000 0.000001
vn 0.000000 1.000000 -0.000000
vn -0.000000 -1.000000 0.000000

f 5/1/1 1/2/1 4/3/1
f 5/1/1 4/3/1 8/4/1
f 3/5/2 7/6/2 8/7/2
f 3/5/2 8/7/2 4/8/2
f 2/9/3 6/10/3 3/5/3
f 6/10/4 7/6/4 3/5/4
f 1/2/5 5/1/5 2/9/5
f 5/1/6 6/10/6 2/9/6
f 5/1/7 8/11/7 6/10/7
f 8/11/7 7/12/7 6/10/7
f 1/2/8 2/9/8 3/13/8
f 1/2/8 3/13/8 4/14/8
```

6个法线（存在冗余）

连接关系

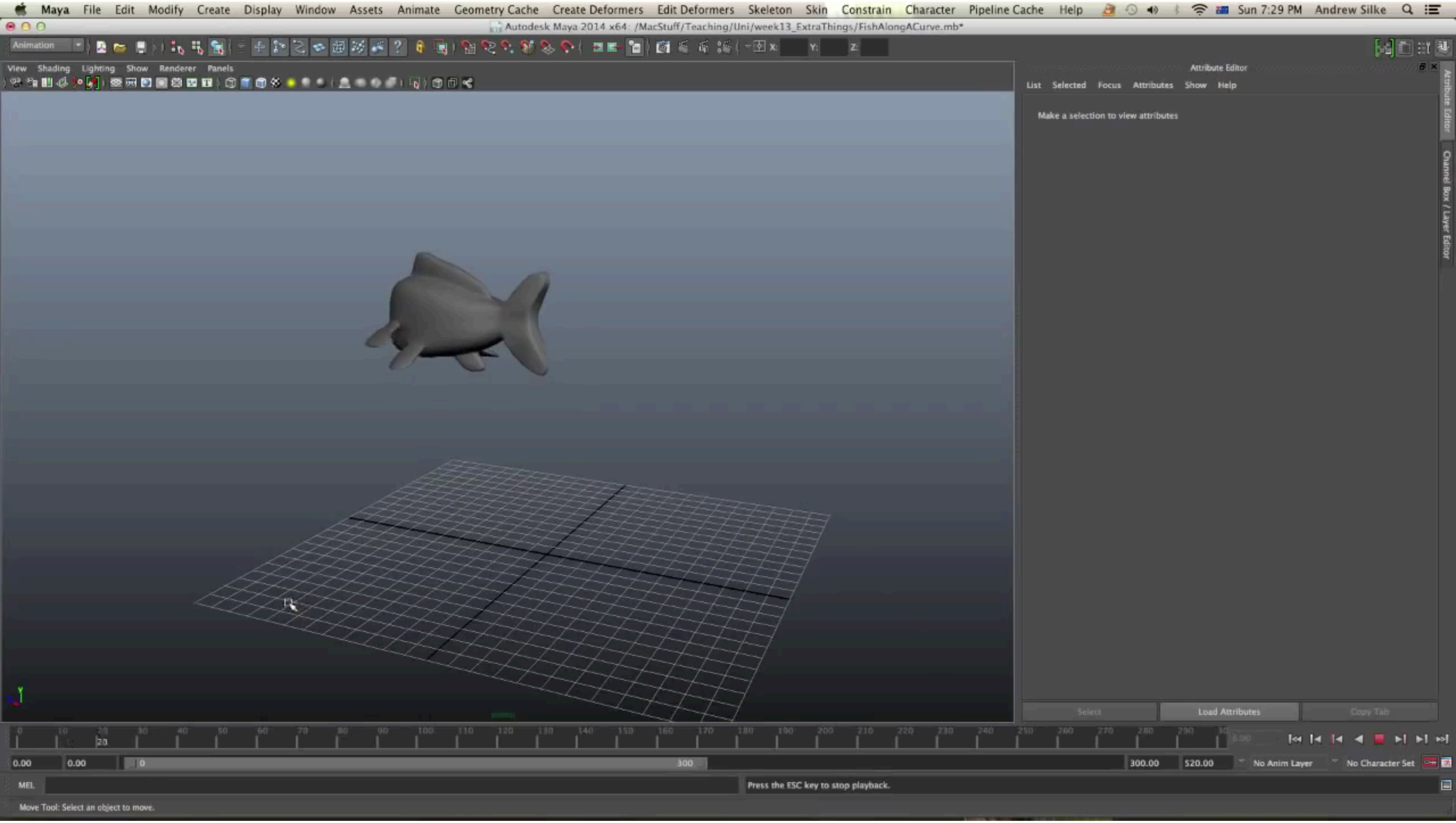
Curves

Camera Paths 相机的运动曲线



Flythrough of proposed Perth Citylink subway, https://youtu.be/rIJMuQPwr3E

Animation Curves



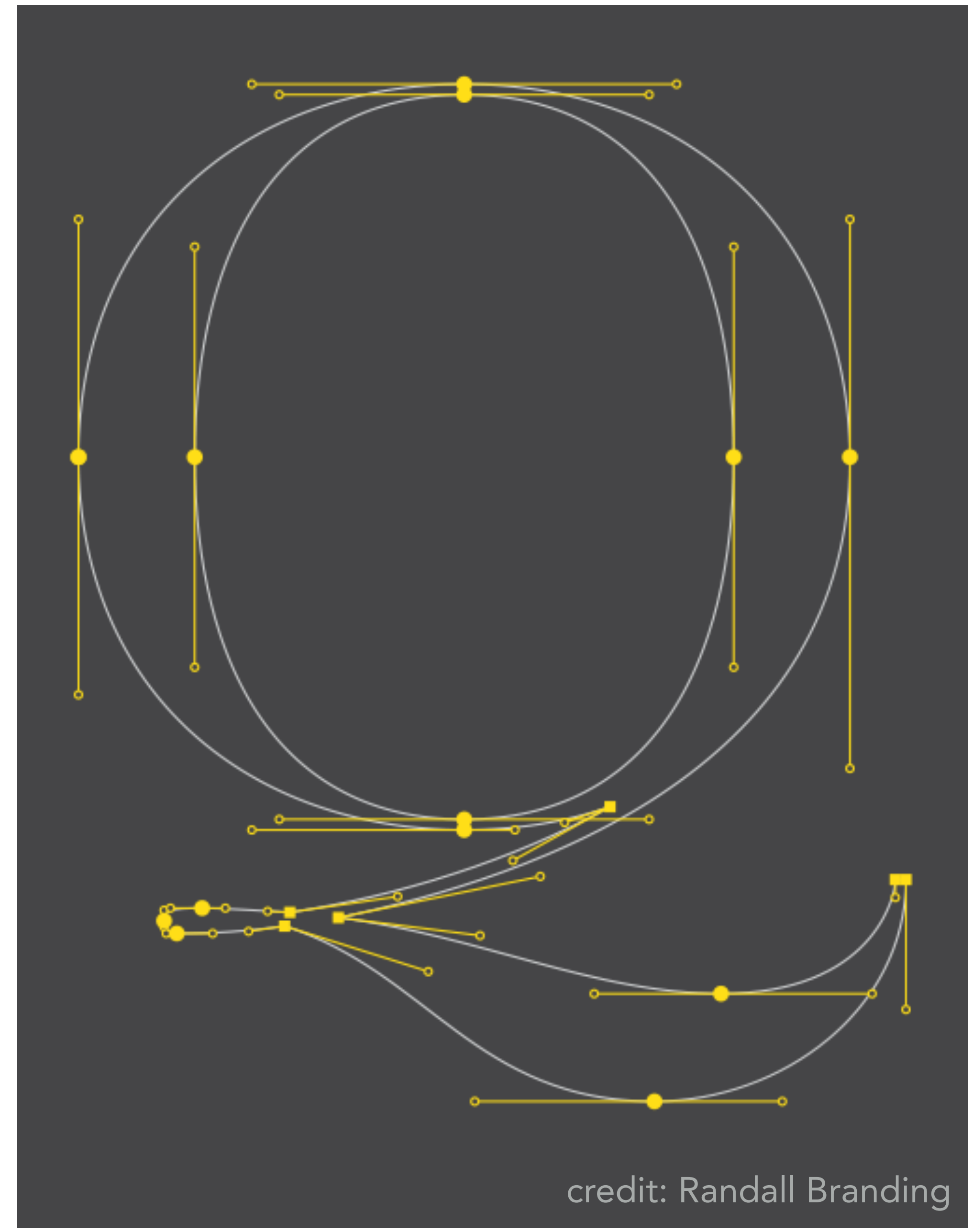
Maya Animation Tutorial: https://youtu.be/b-o5wtZlJPc

Vector Fonts

控制点

The Quick Brown Fox Jumps Over The Lazy Dog

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 0123456789

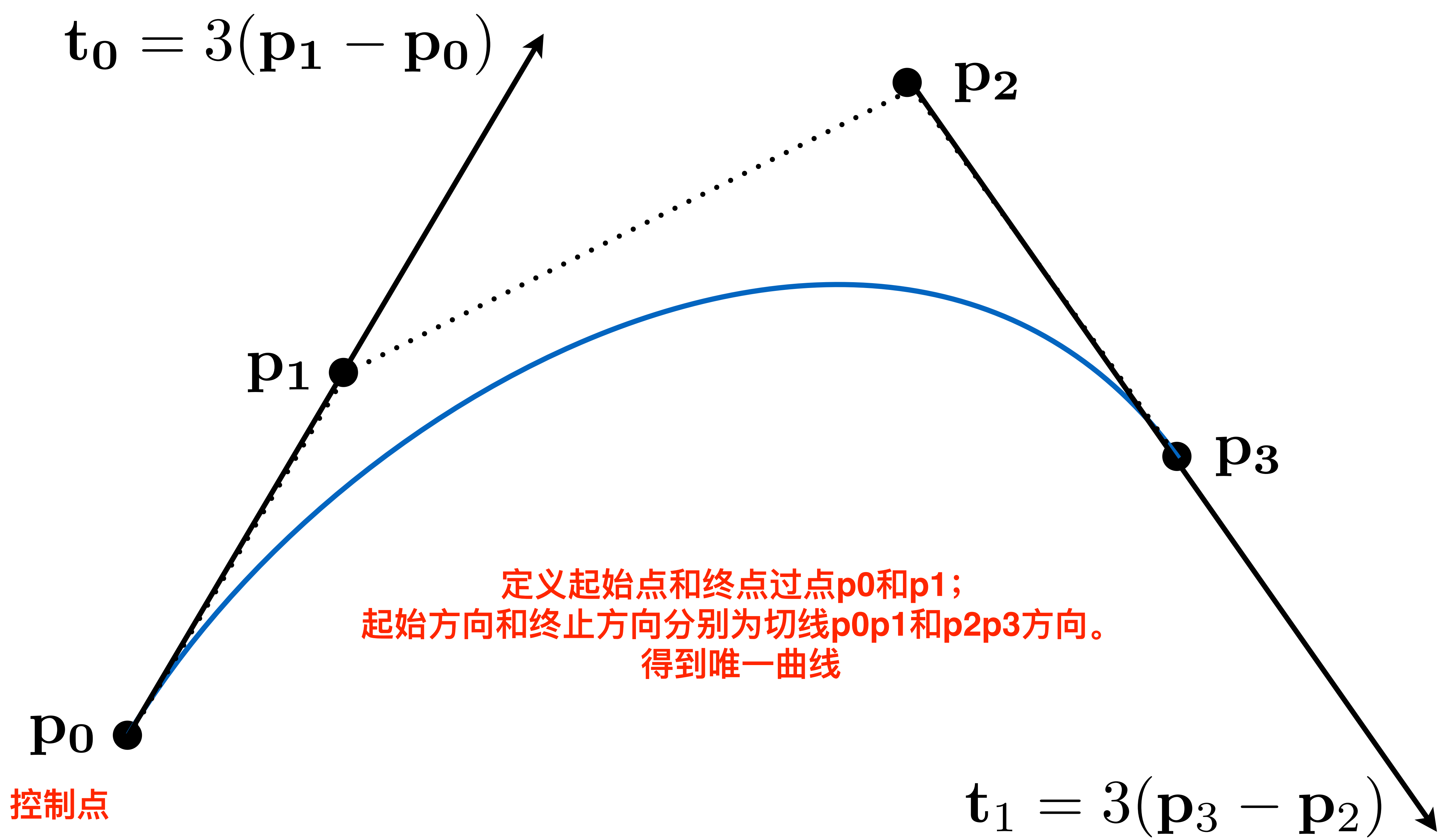


Baskerville font - represented as piecewise cubic Bézier curves

Bézier Curves

(贝塞尔曲线)

Defining Cubic Bézier Curve With Tangents

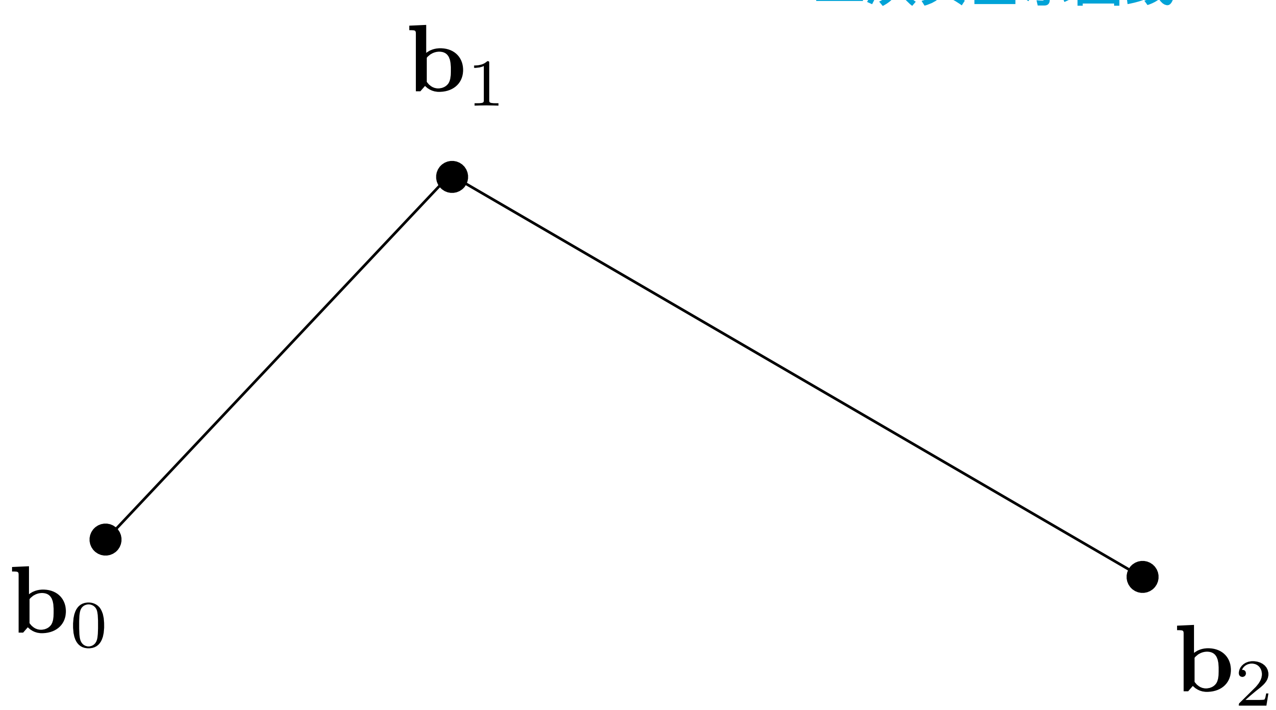


Evaluating Bézier Curves (de Casteljau Algorithm)

Bézier Curves – de Casteljau Algorithm

Consider three points (quadratic Bezier)

二次贝塞尔曲线

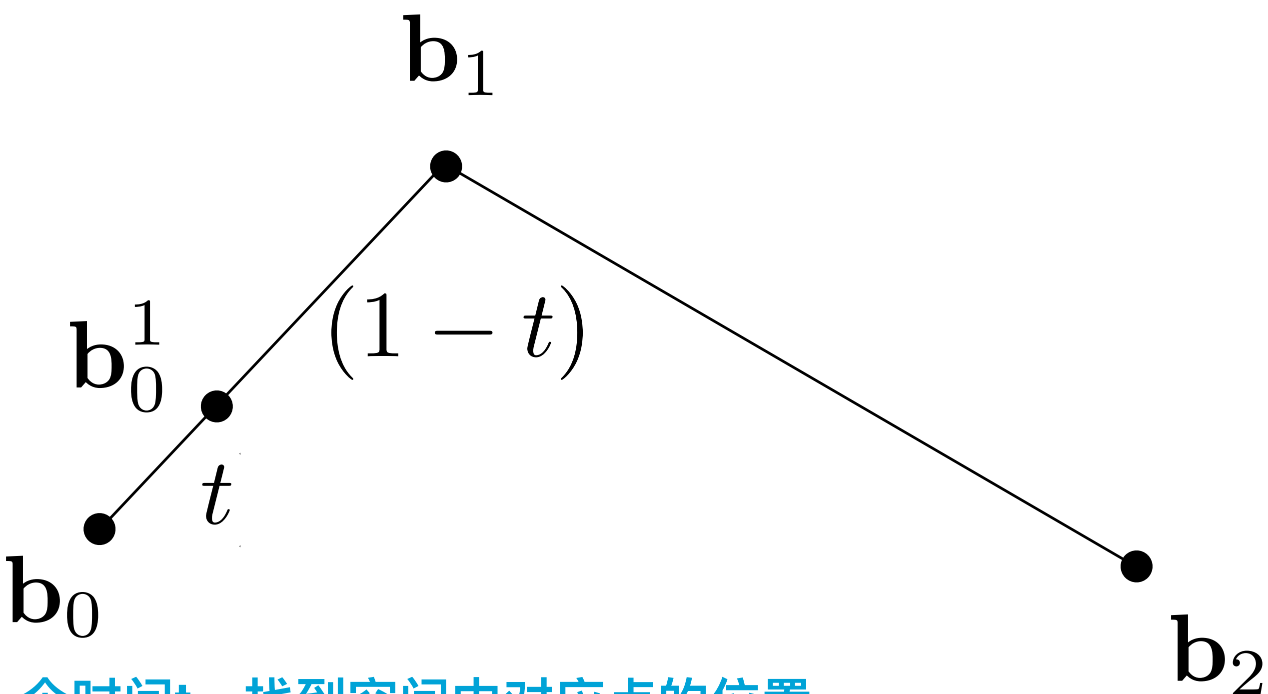


Pierre Bézier
1910 – 1999

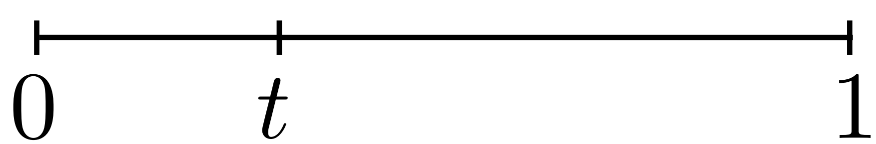
Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert a point using linear interpolation



给定任意一个时间t，找到空间中对应点的位置

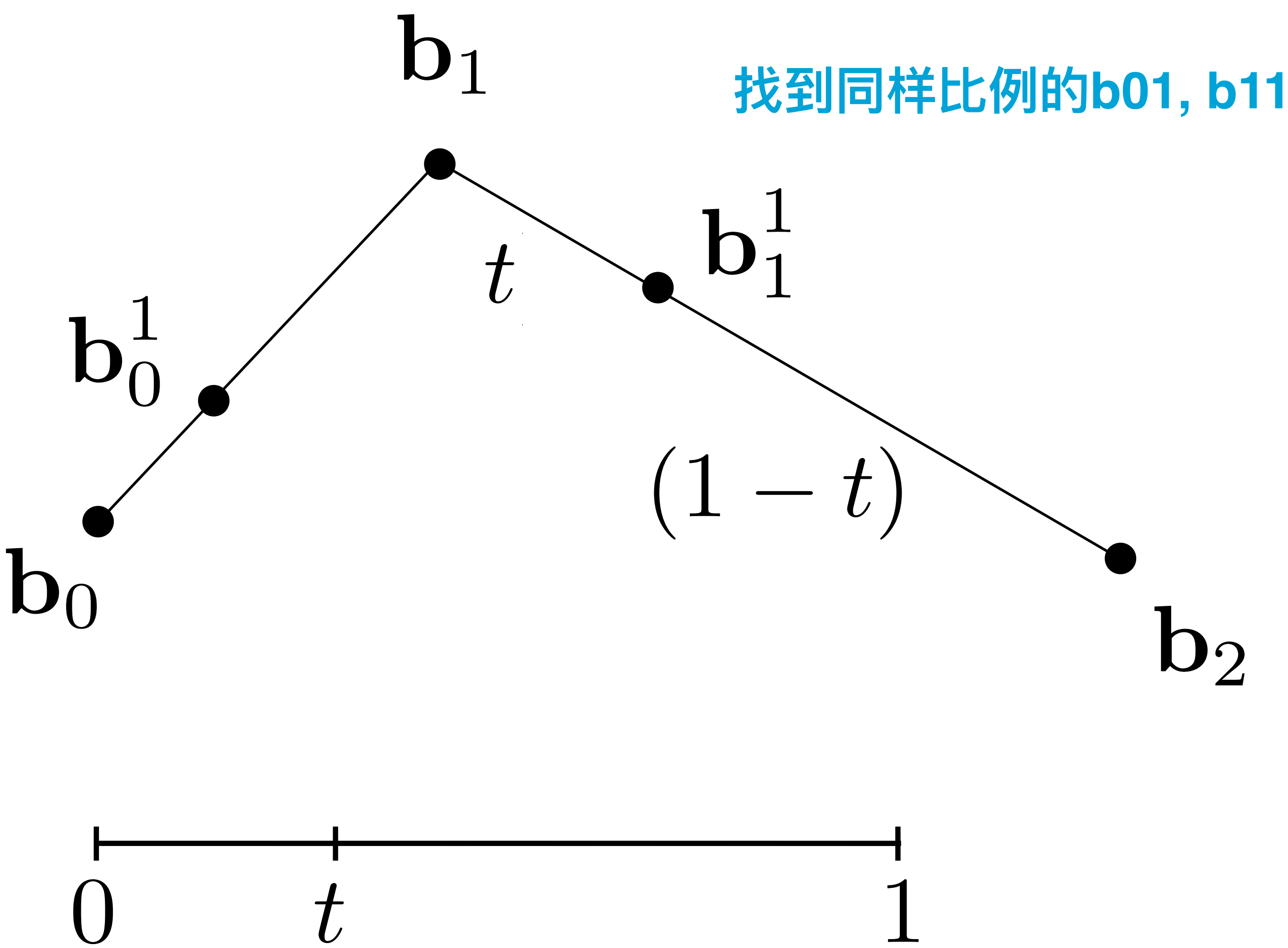


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert on both edges



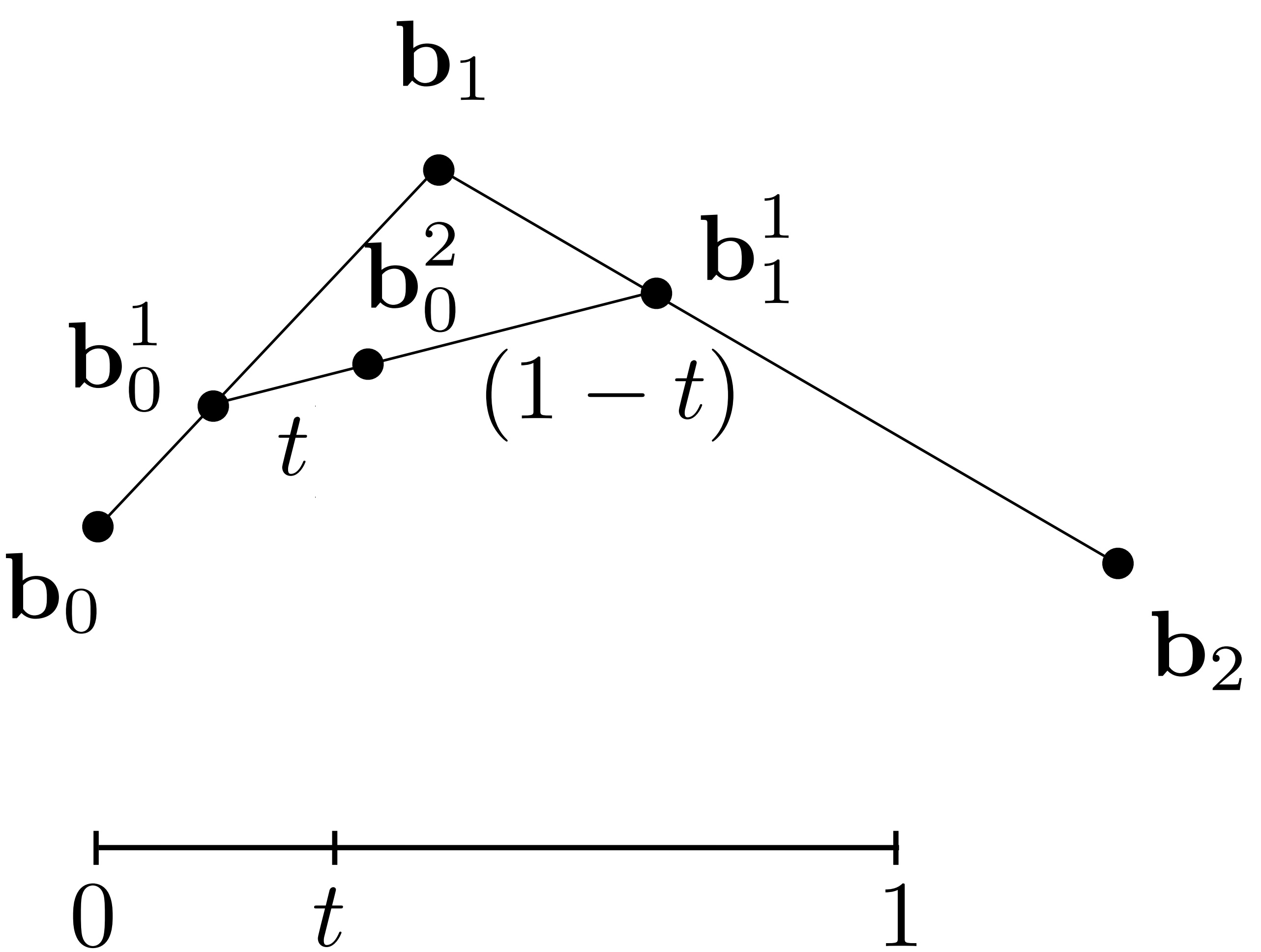
Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Repeat recursively

Bezier表示是显式表示，参数是t

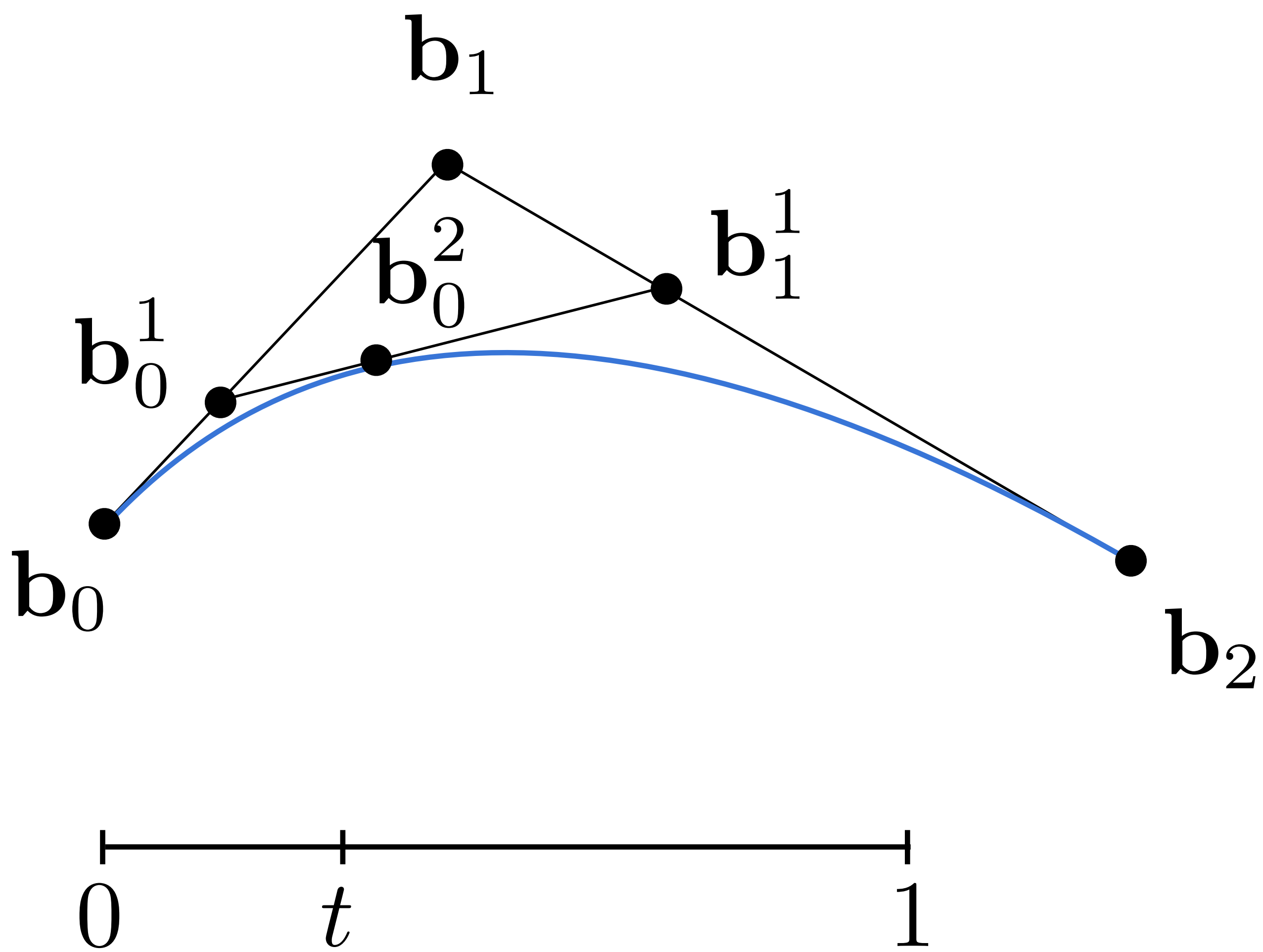


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Run the same algorithm for every t in [0,1]



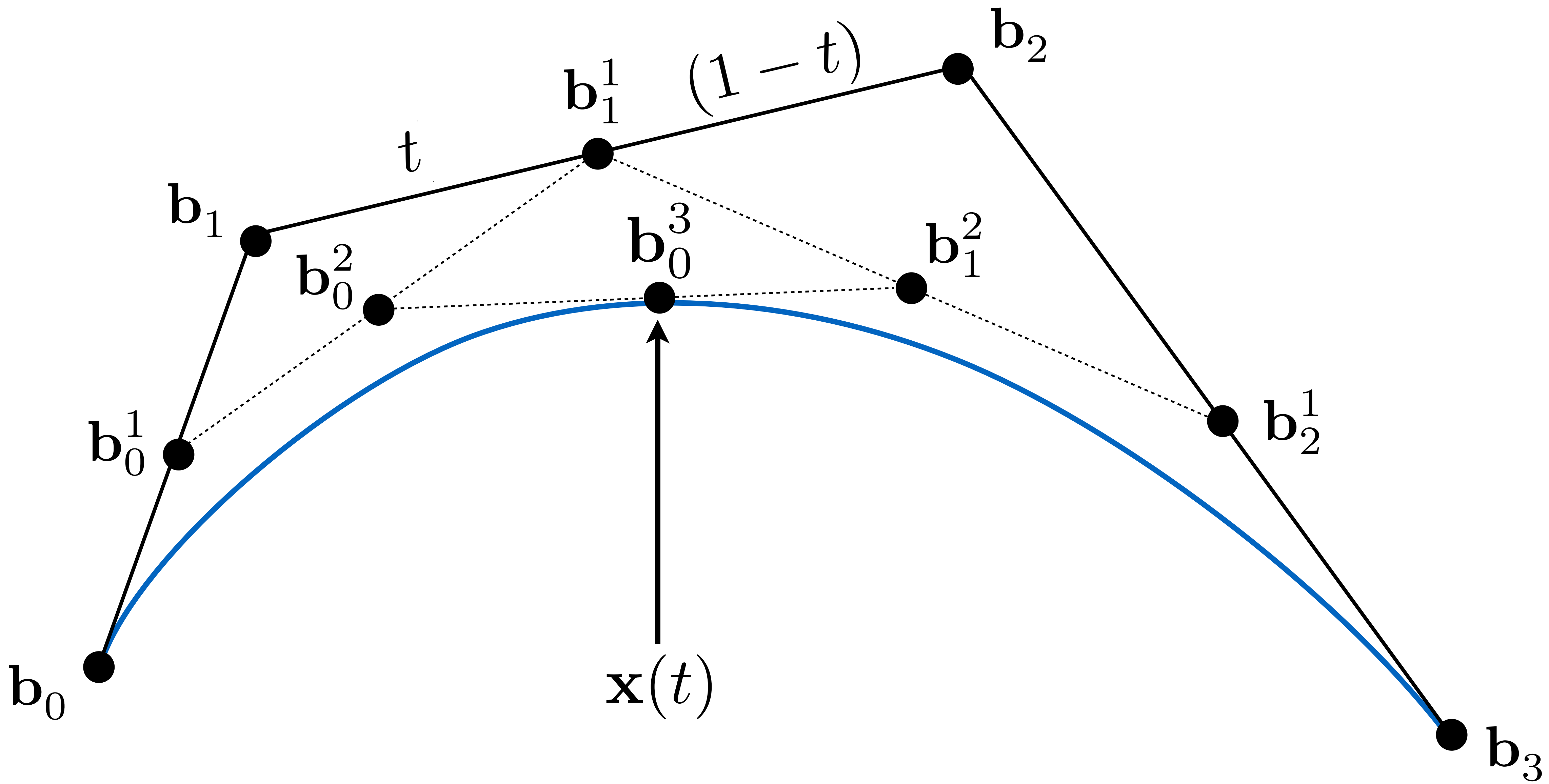
Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

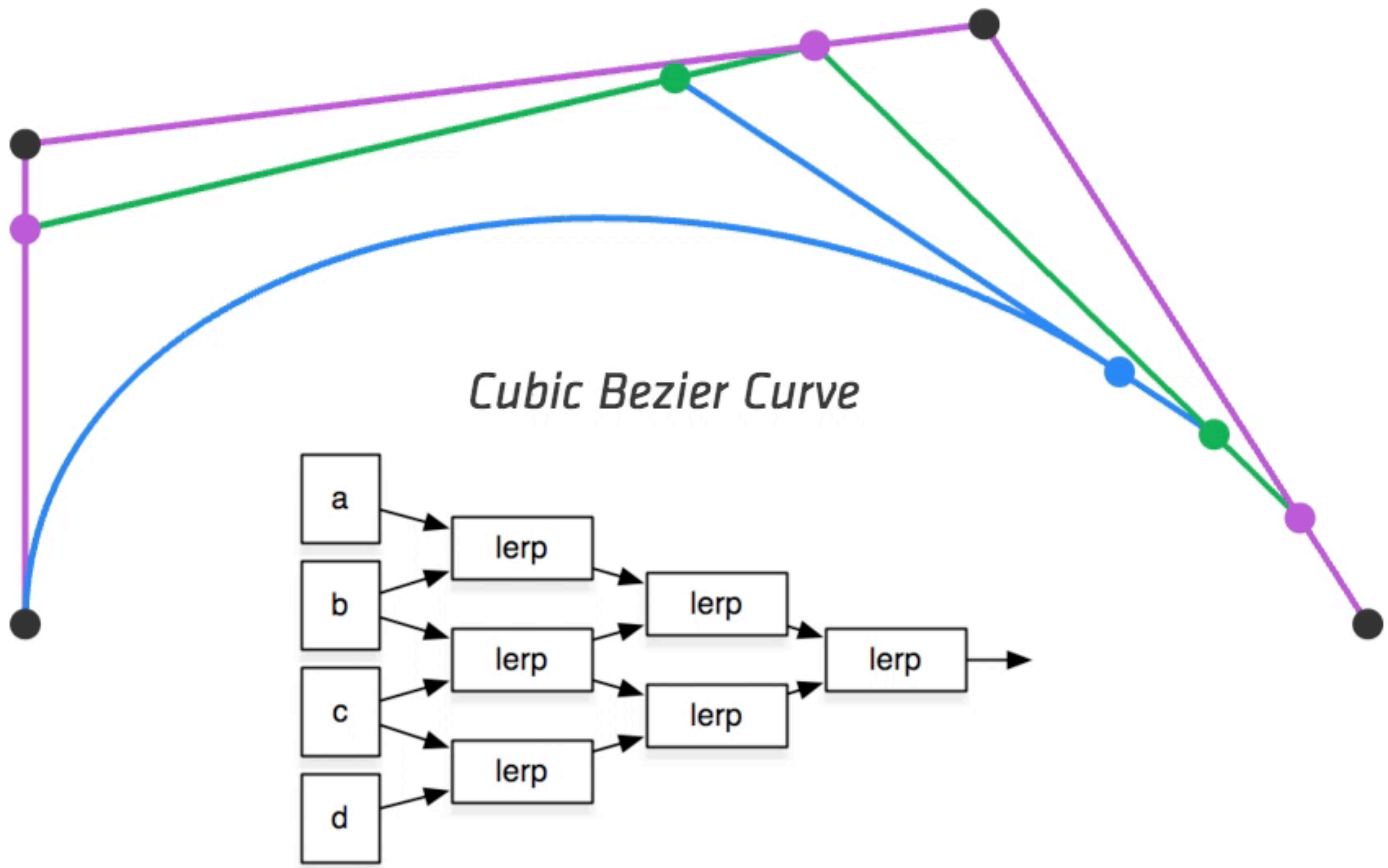
Cubic Bézier Curve – de Casteljau

Four input points in total

Same recursive linear interpolations



Visualizing de Casteljau Algorithm



Animation: Steven Wittens, Making Things with Maths, http://acko.net

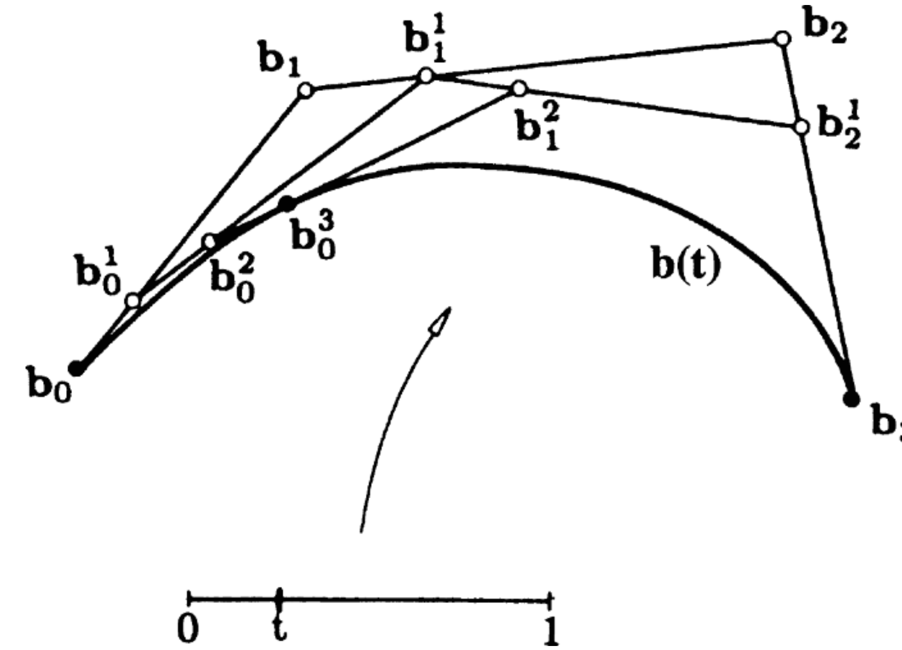
Evaluating Bézier Curves

Algebraic Formula

Bézier Curve – Algebraic Formula

de Casteljau algorithm gives a pyramid of coefficients

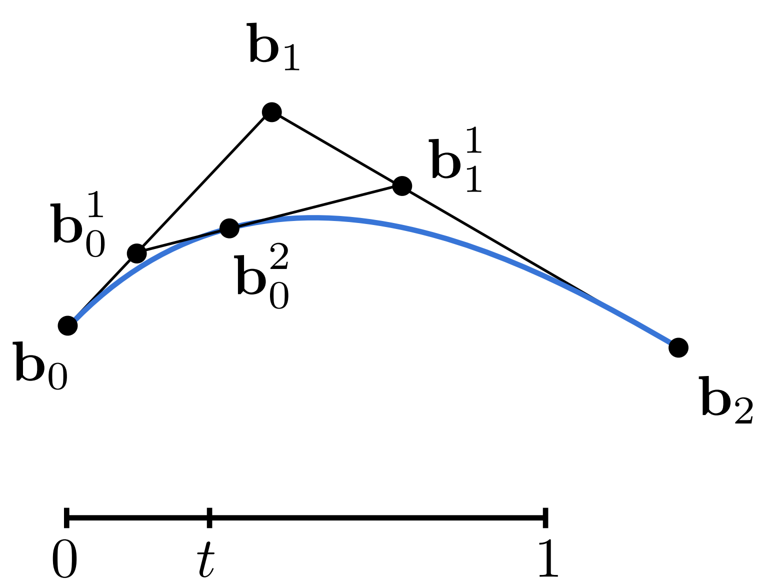
$$
\begin{tikzcd}
 & & & \mathbf{b}_0^3 & & & \mathbf{x}(t) \arrow[lll] \\
 & & \mathbf{b}_0^2 \arrow[ur] & & \mathbf{b}_1^2 \arrow[ul] & & \\
 & \mathbf{b}_0^1 \arrow[ur] & & \mathbf{b}_1^1 \arrow[ul] \arrow[ur] & & \mathbf{b}_2^1 \arrow[ul] & \\
\mathbf{b}_0 \arrow[ur, "t"] & & \mathbf{b}_1 \arrow[ul, "(1-t)"'] \arrow[ur] & & \mathbf{b}_2 \arrow[ul] \arrow[ur] & & \mathbf{b}_3 \arrow[ul]
\end{tikzcd}
$$



Every rightward arrow is multiplication by t,
Every leftward arrow by (1–t)

Bézier Curve – Algebraic Formula

Example: quadratic Bézier curve from three points



$$\mathbf{b}_0^1(t) = (1-t)\mathbf{b}_0 + t\mathbf{b}_1$$

$$\mathbf{b}_1^1(t) = (1-t)\mathbf{b}_1 + t\mathbf{b}_2$$

$$\mathbf{b}_0^2(t) = (1-t)\mathbf{b}_0^1 + t\mathbf{b}_1^1$$

$$\mathbf{b}_0^2(t) = (1-t)^2\mathbf{b}_0 + 2t(1-t)\mathbf{b}_1 + t^2\mathbf{b}_2$$

Bézier Curve – General Algebraic Formula

给出n+1个控制点，得到n阶Bezier曲线

Bernstein form of a Bézier curve of order n:

$$\mathbf{b}^n(t) = \mathbf{b}_0^n(t) = \sum_{j=0}^{n} \mathbf{b}_j B_j^n(t)$$

控制点的线性组合

Bézier curve order n
(vector polynomial of degree n)

Bernstein polynomial
(scalar polynomial of degree n)

Bézier control points
(vector in R^N)

Bernstein polynomials:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Bézier Curve – Algebraic Formula: Example

Bernstein form of a Bézier curve of order n:

$$\mathbf{b}^n(t) = \sum_{j=0}^{n} \mathbf{b}_j B_j^n(t)$$

Example: assume n = 3 and we are in R^3

i.e. we could have control points in 3D such as:

$$\mathbf{b}_0 = (0, 2, 3), \;\; \mathbf{b}_1 = (2, 3, 5), \;\; \mathbf{b}_2 = (6, 7, 9), \;\; \mathbf{b}_3 = (3, 4, 5)$$

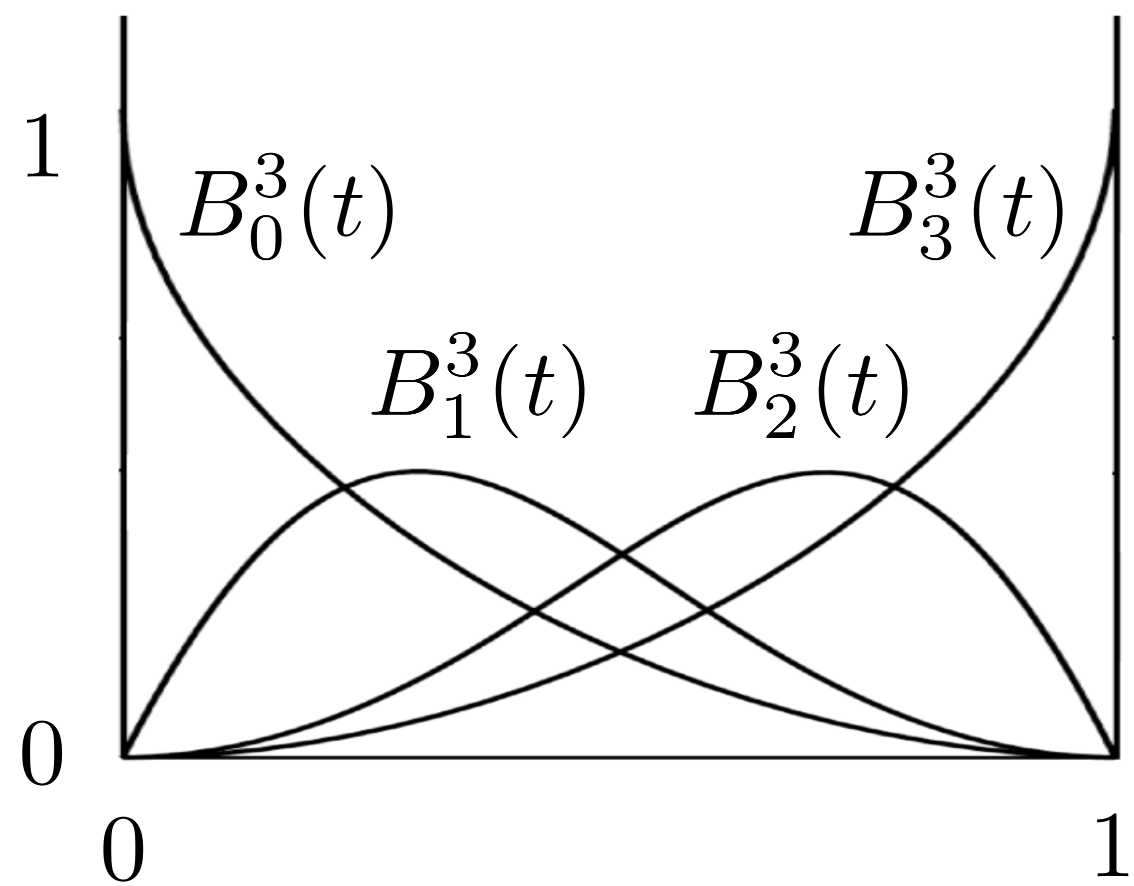
These points define a Bezier curve in 3D that is a cubic polynomial in t:

$$\mathbf{b}^n(t) = \;\; \mathbf{b}_0 \, (1-t)^3 + \mathbf{b}_1 \, 3t(1-t)^2 + \mathbf{b}_2 \, 3t^2(1-t) + \mathbf{b}_3 \, t^3$$

Cubic Bézier Basis Functions

Bernstein Polynomials

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



Sergei N. Bernstein
1880 – 1968

Properties of Bézier Curves

Interpolates endpoints

- For cubic Bézier: $\mathbf{b}(0) = \mathbf{b}_0; \quad \mathbf{b}(1) = \mathbf{b}_3$

Tangent to end segments

- Cubic case: $\mathbf{b}'(0) = 3(\mathbf{b}_1 - \mathbf{b}_0); \quad \mathbf{b}'(1) = 3(\mathbf{b}_3 - \mathbf{b}_2)$

Affine transformation property

- Transform curve by transforming control points

只需要对控制点做仿射变换，重新画一下曲线就行

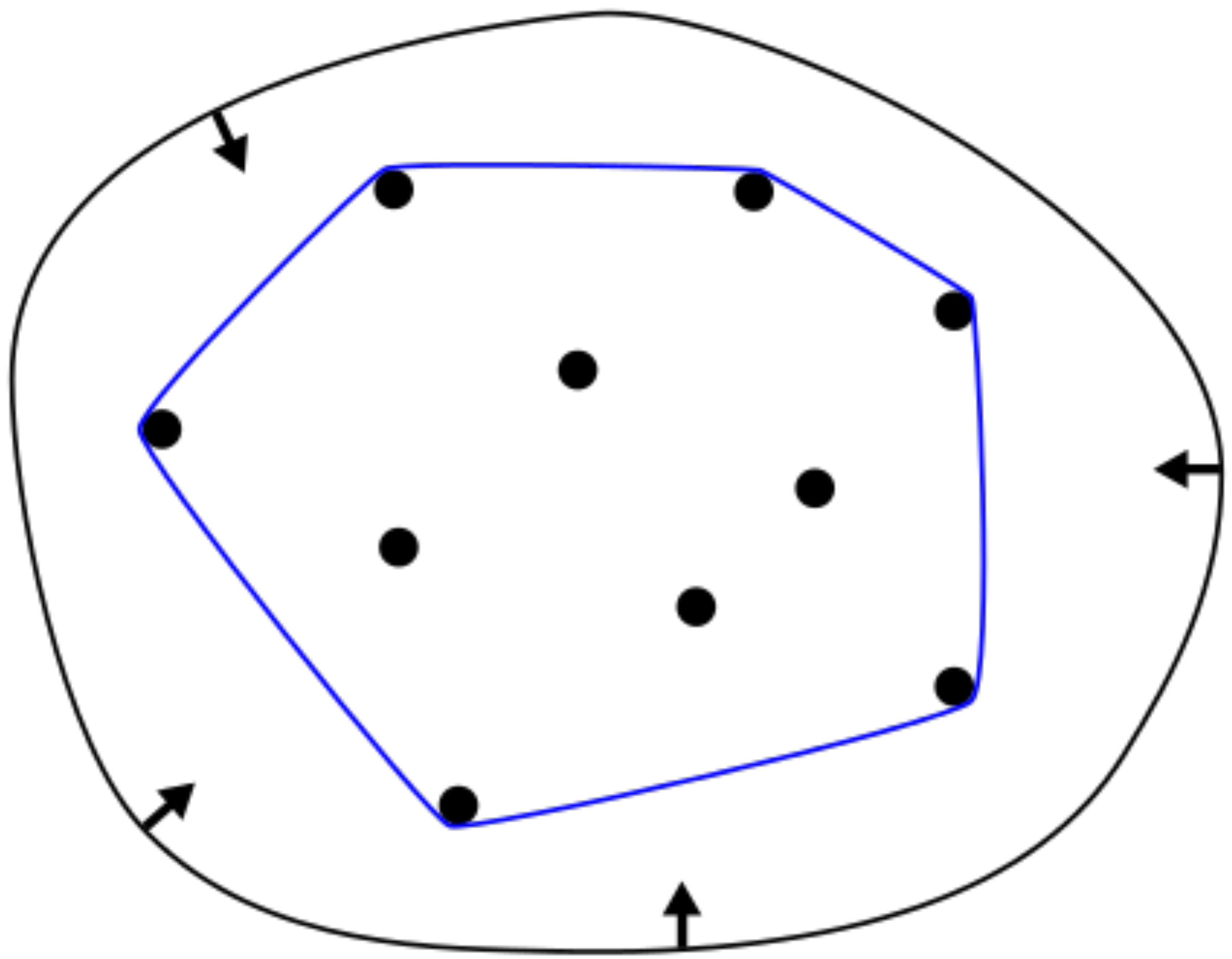
对其他变换不一定是这样

Convex hull property

凸包性质

- Curve is within convex hull of control points

BTW: What's a Convex Hull



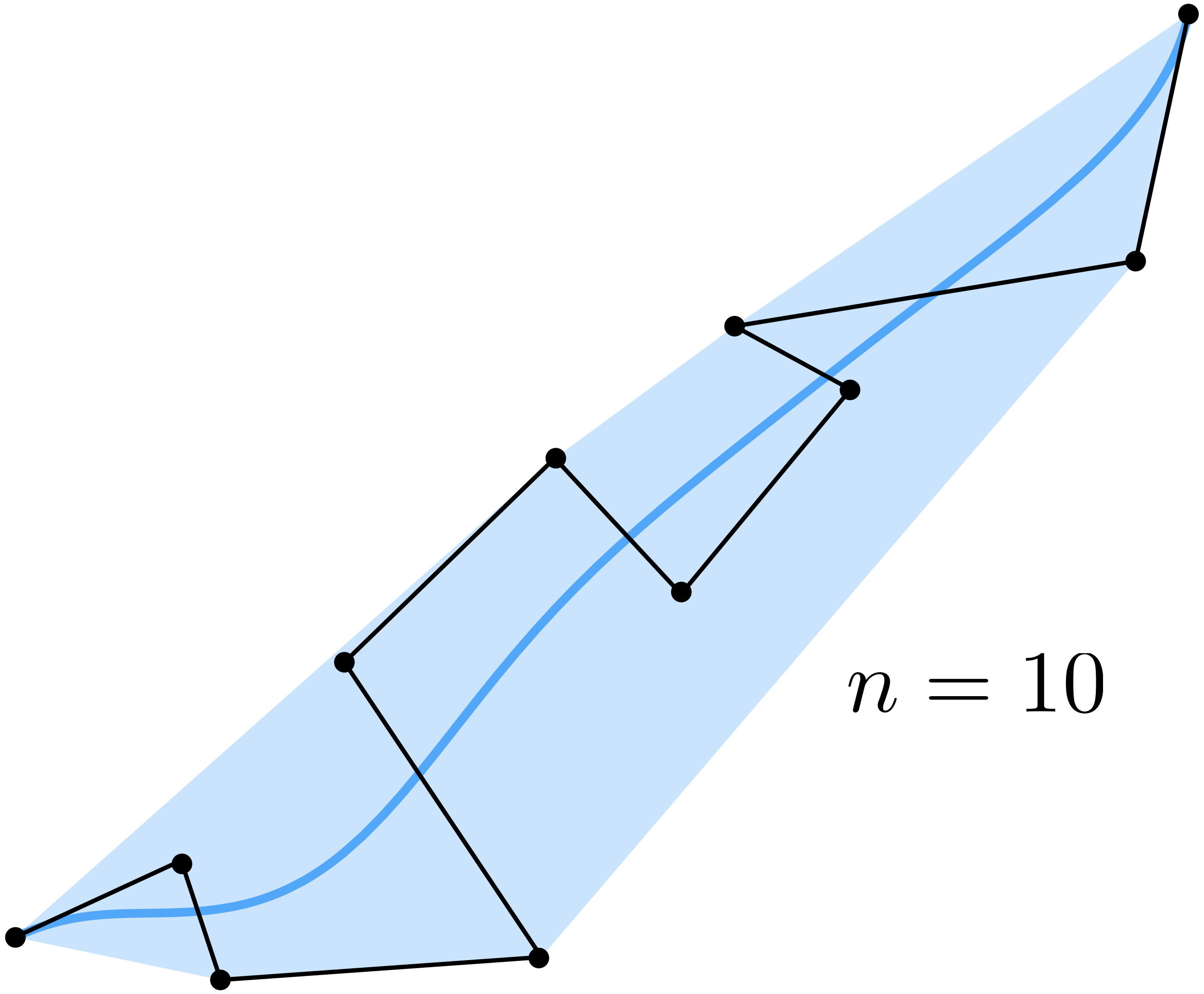
Bezier曲线一定在Convex Hull内部

[from Wikipedia]

Piecewise Bézier Curves

逐段Bezier曲线

Higher-Order Bézier Curves?



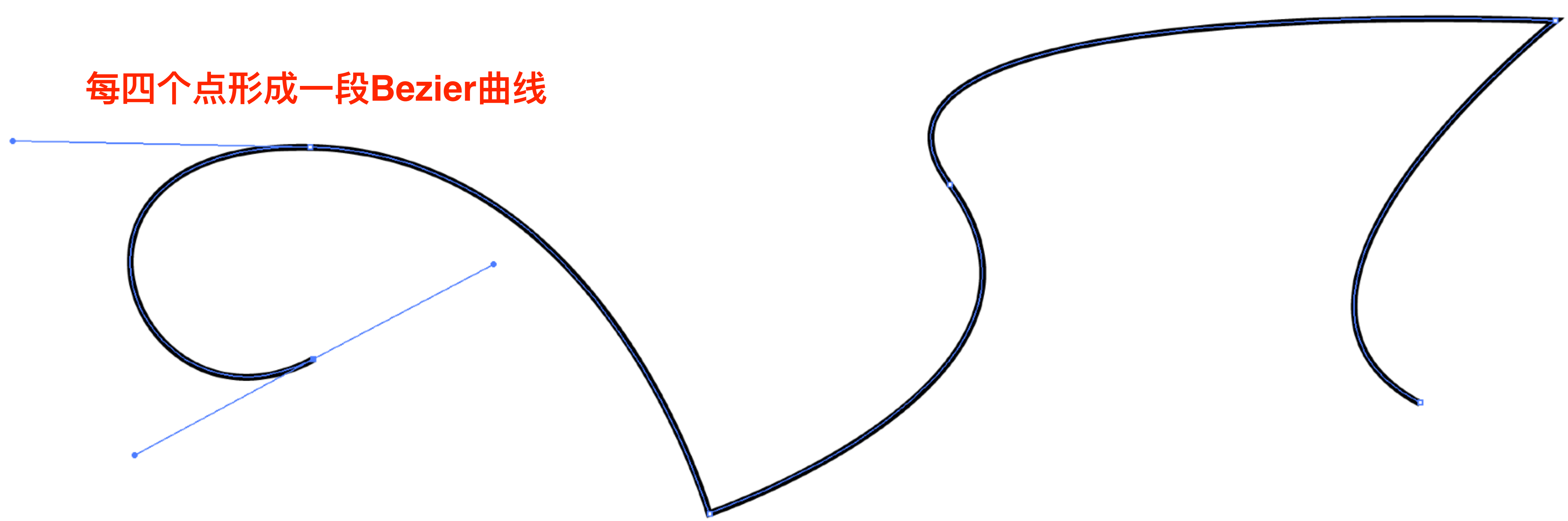
Very hard to control!

Uncommon

Piecewise Bézier Curves

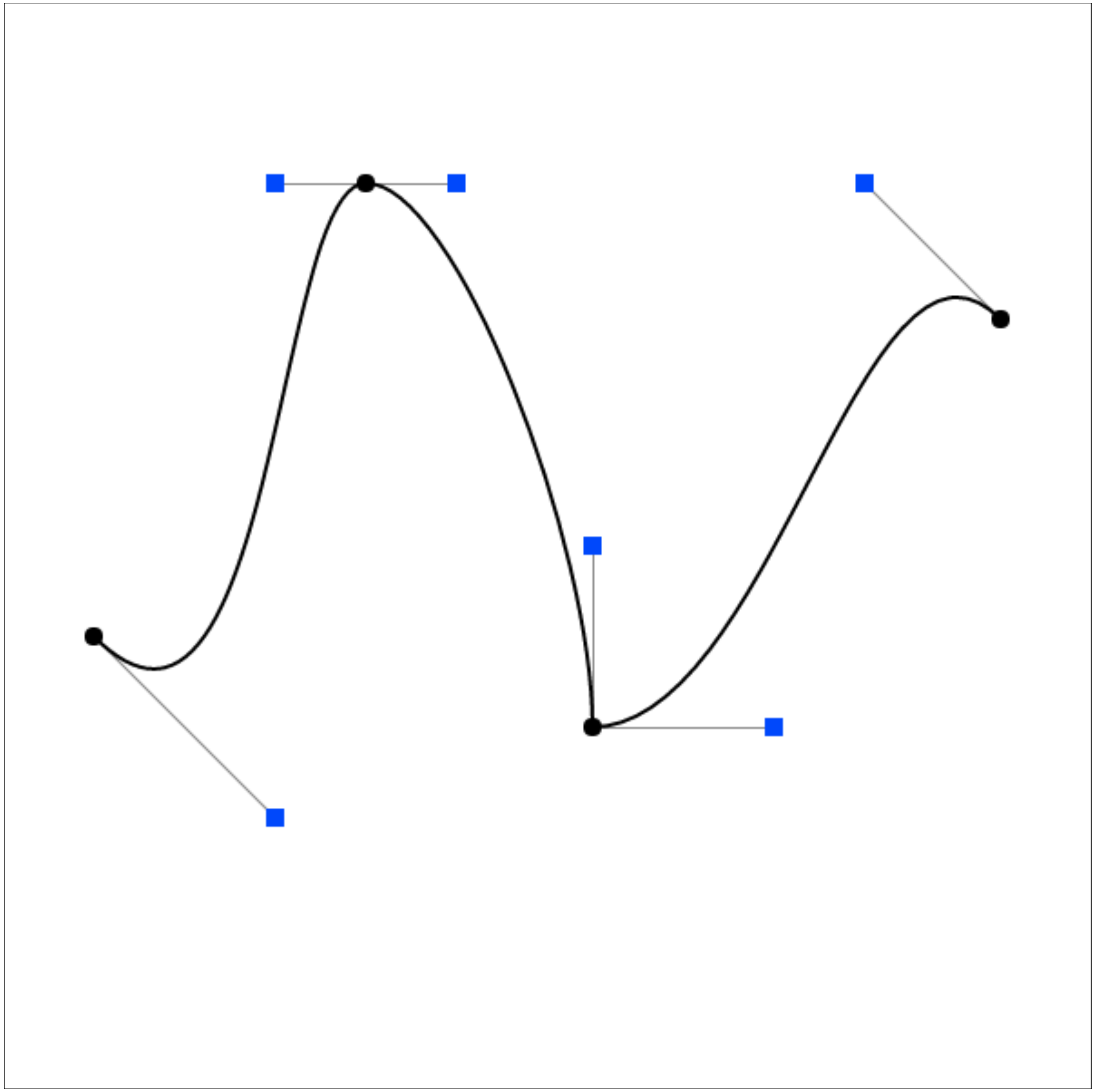
Instead, chain many low-order Bézier curve

Piecewise cubic Bézier the most common technique



Widely used (fonts, paths, Illustrator, Keynote, …)

Demo – Piecewise Cubic Bézier Curve



David Eck, http://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html

Piecewise Bézier Curve – Continuity

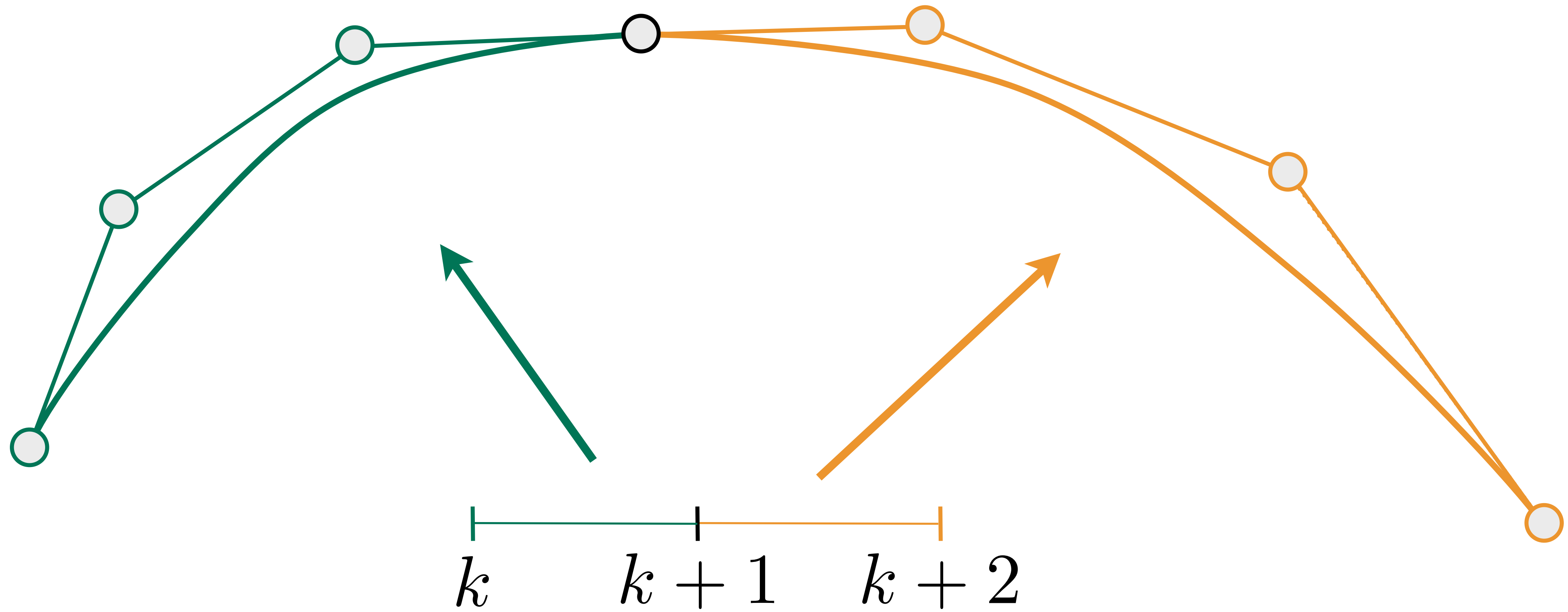
Two Bézier curves

$$\mathbf{a} : [k, k+1] \to \mathbb{R}^N$$

$$\mathbf{b} : [k+1, k+2] \to \mathbb{R}^N$$

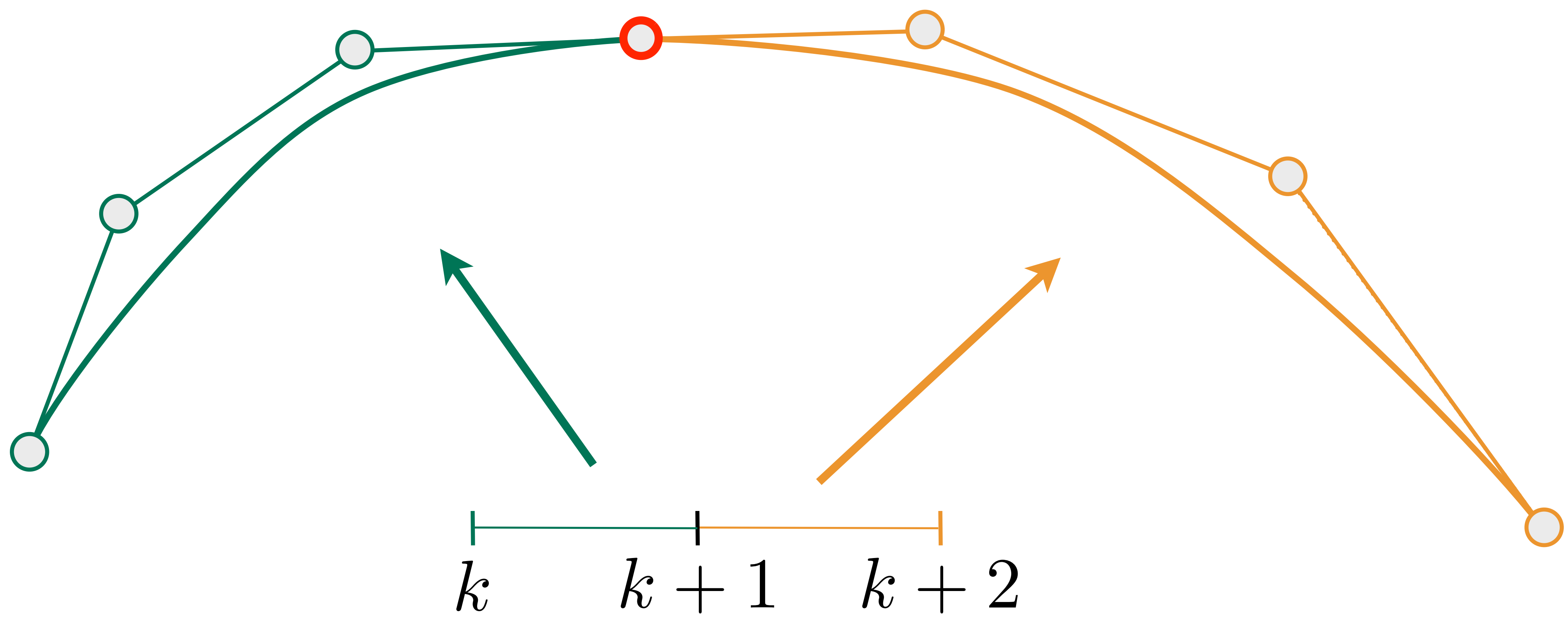
Assuming integer partitions here,
can generalize

切线方向和大小连续（大小也要连续）



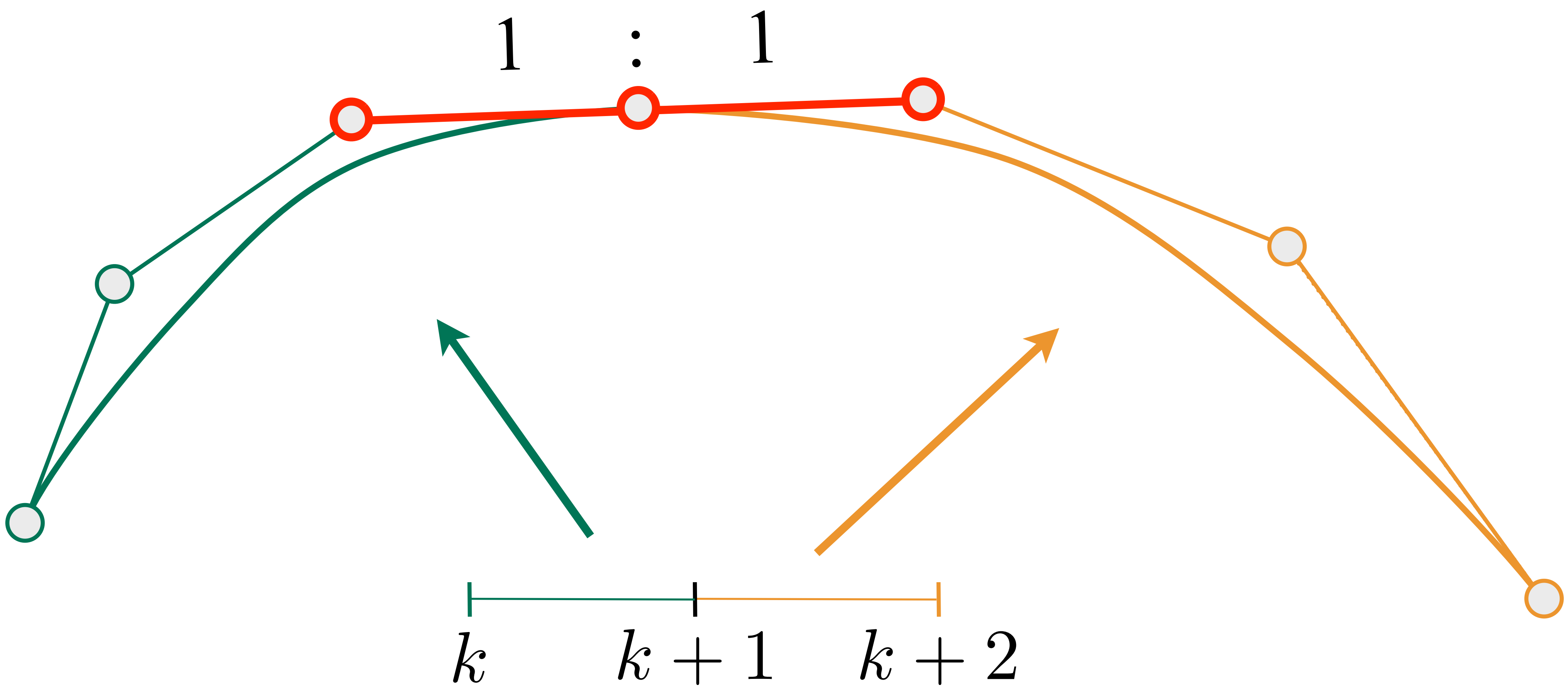
Piecewise Bézier Curve – Continuity

C^0 continuity: $\mathbf{a}_n = \mathbf{b}_0$



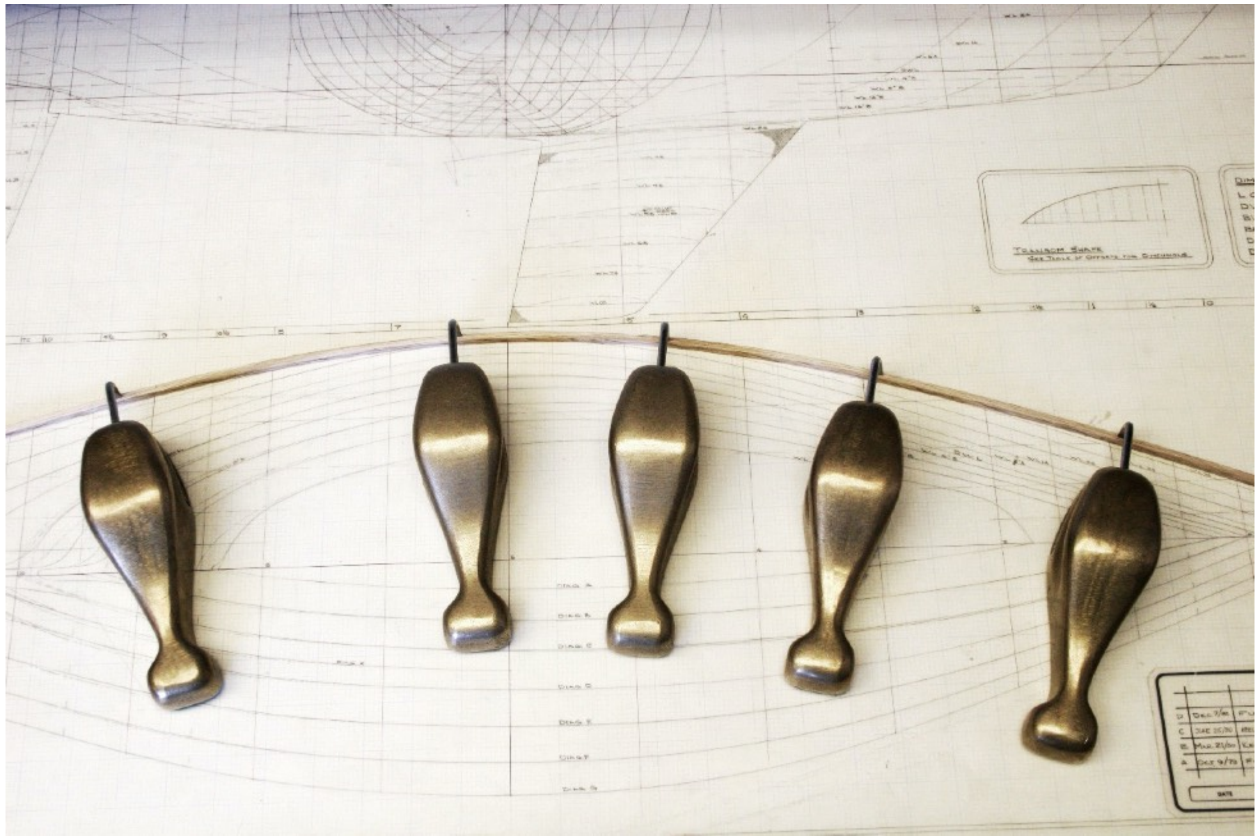
Piecewise Bézier Curve – Continuity

C^1 continuity: $$\mathbf{a}_n = \mathbf{b}_0 = \frac{1}{2}\left(\mathbf{a}_{n-1} + \mathbf{b}_1\right)$$



Other types of splines

- Spline
  - a continuous curve constructed so as to pass through a given set of points and have a certain number of continuous derivatives.
  - In short, a curve under control 样条：一个可控的曲线



A Real Draftsman's Spline
http://www.alatown.com/spline-history-architecture/

Other types of splines

- B-splines
  - Short for basis splines 基函数样条
  - Require more information than Bezier curves
  - Satisfy all important properties that Bézier curves have (i.e. superset) **Bezier曲线不具有局部性质，B-Spline具有局部性质**

https://en.wikipedia.org/wiki/B-spline

Important Note

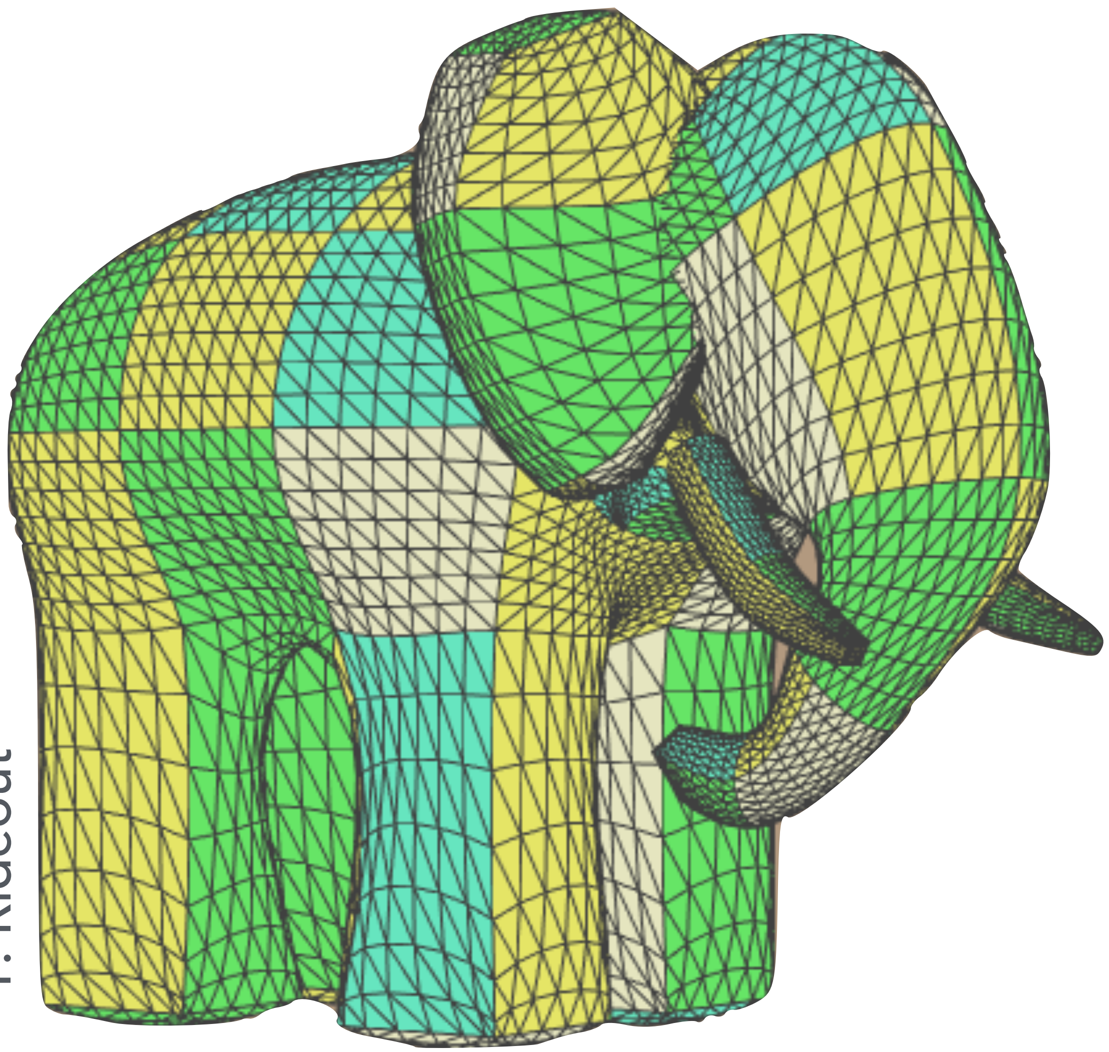
- In this course
    - We do not cover B-splines and NURBS
    - We also do not cover operations on curves (e.g. increasing/decreasing orders, etc.)
    - To learn more / deeper, you are welcome to refer to Prof. Shi-Min Hu's course: https://www.bilibili.com/video/av66548502?from=search&seid=65256805876131485

Today

- Curves
  - Bezier curves
  - De Casteljau's algorithm
  - B-splines, etc.
- Surfaces
  - Bezier surfaces
  - Subdivision surfaces (triangles & quads)

Bézier Surfaces

Extend Bézier curves to surfaces



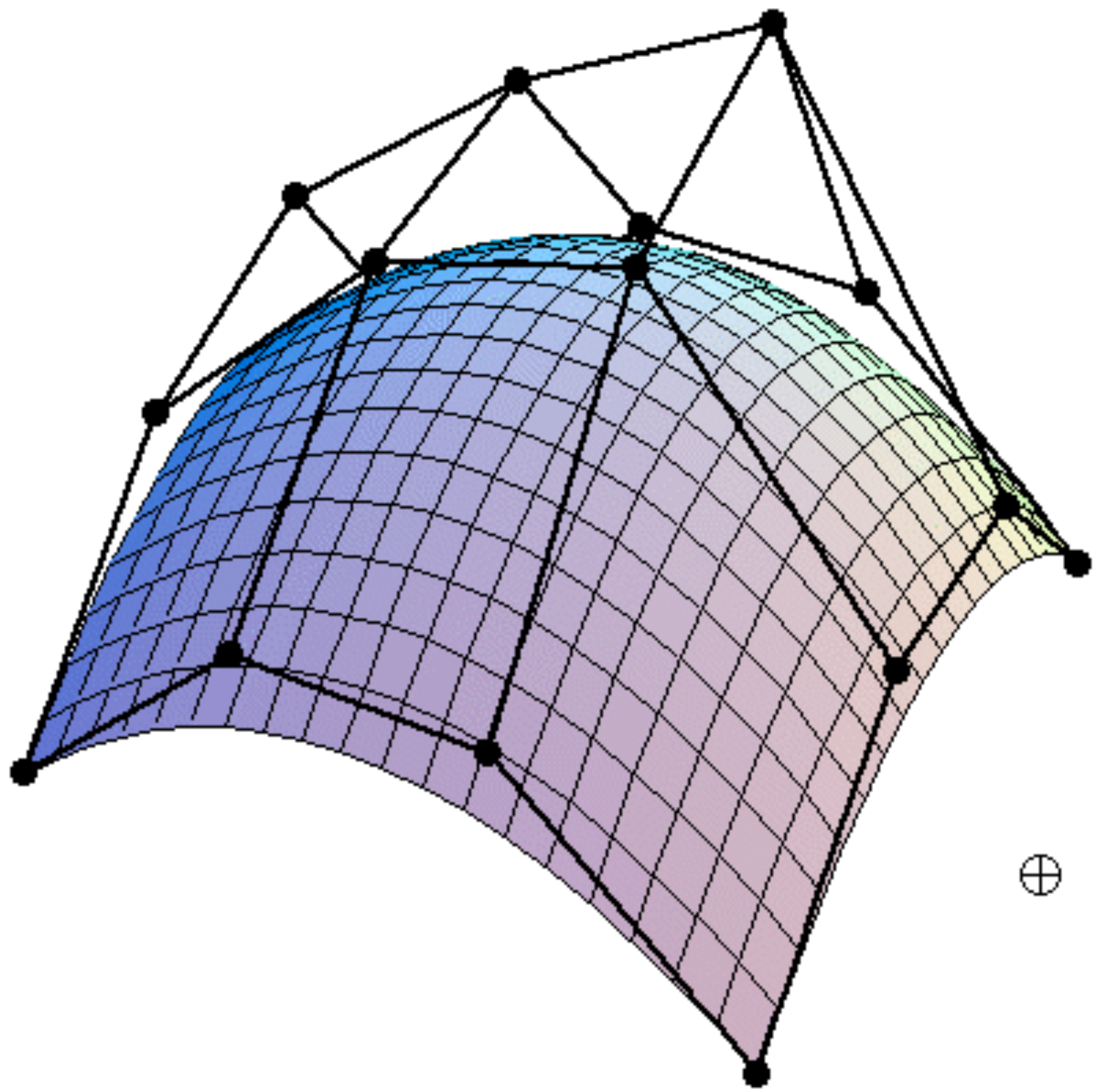
P. Rideout

Ed Catmull's "Gumbo" model

renderspirit.com

Utah Teapot

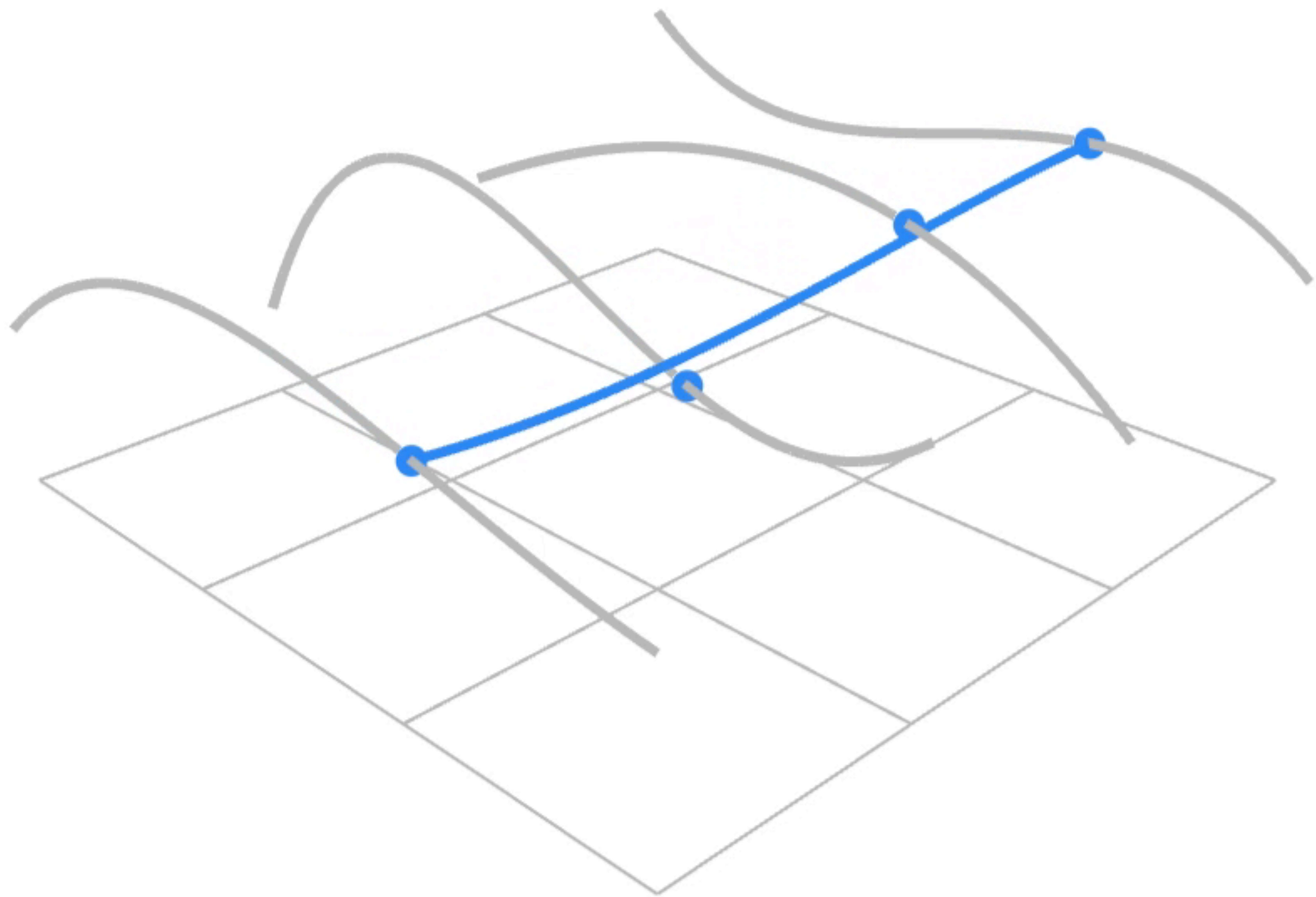
Bicubic Bézier Surface Patch



Bezier surface and 4 x 4 array of control points

双线性插值!

Visualizing Bicubic Bézier Surface Patch

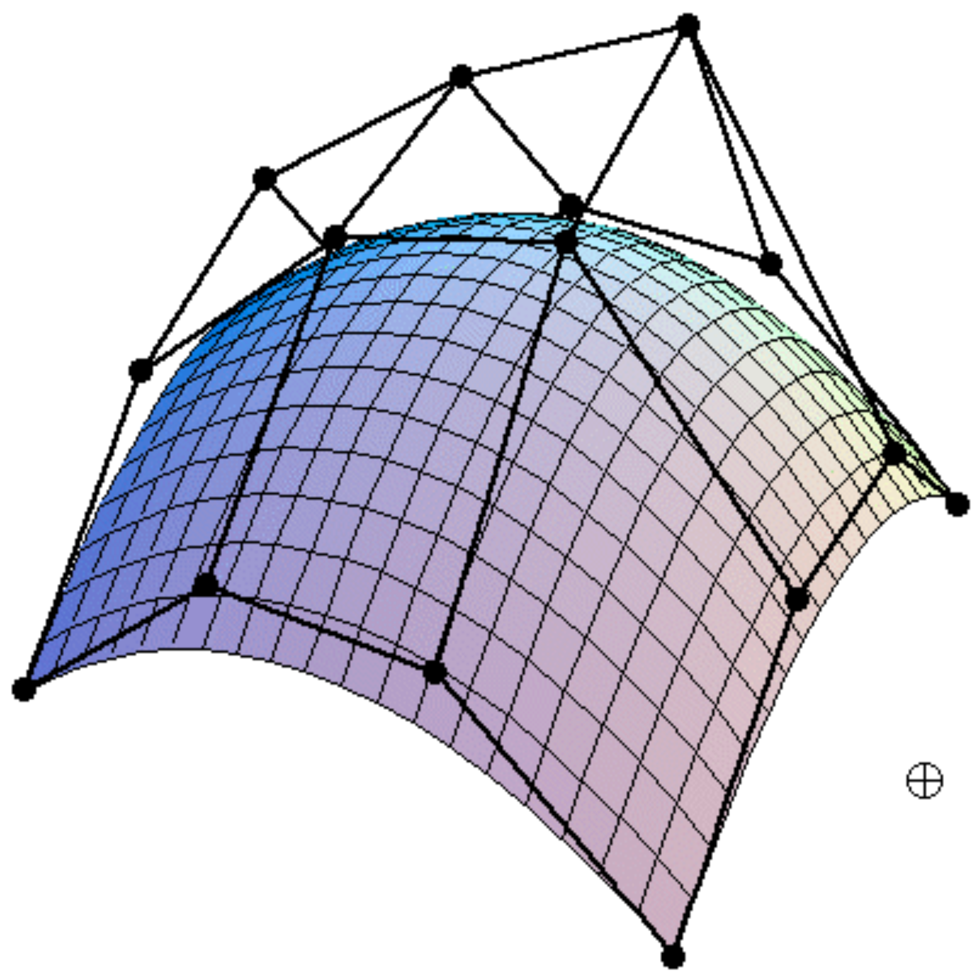


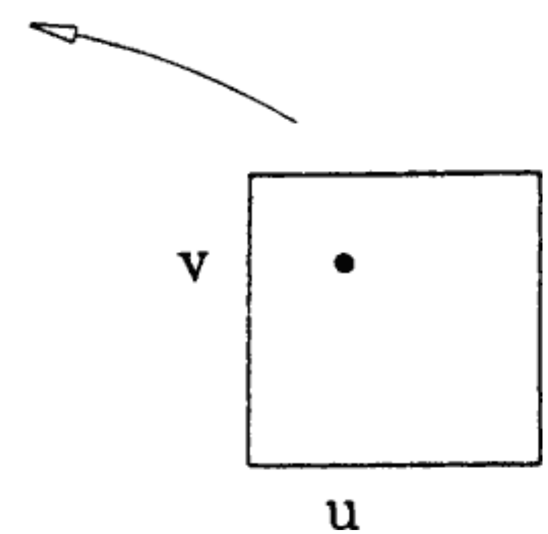
Animation: Steven Wittens, Making Things with Maths, http://acko.net

Evaluating Bézier Surfaces

Evaluating Surface Position For Parameters (u,v)

For bi-cubic Bezier surface patch,
- Input: 4x4 control points
- Output is 2D surface parameterized by (u,v) in $[0,1]^2$





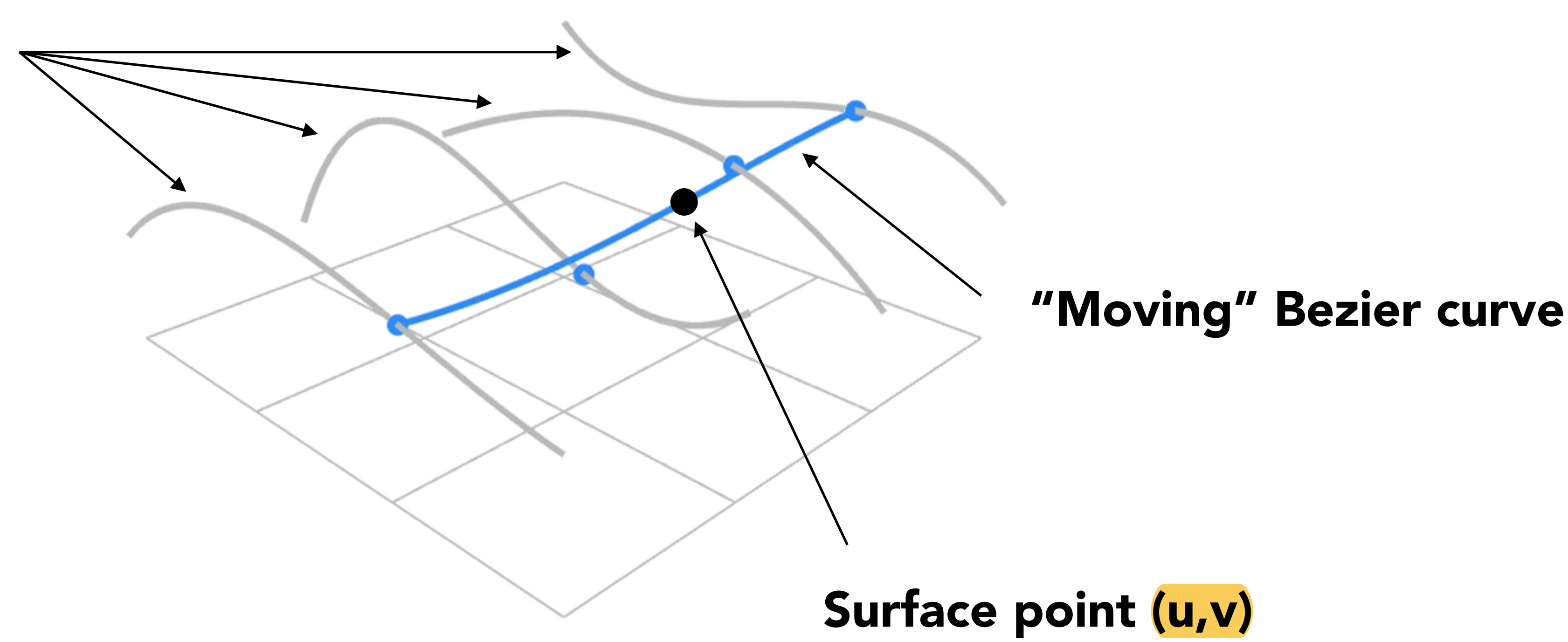
需要两个时间t

Method: Separable 1D de Casteljau Algorithm

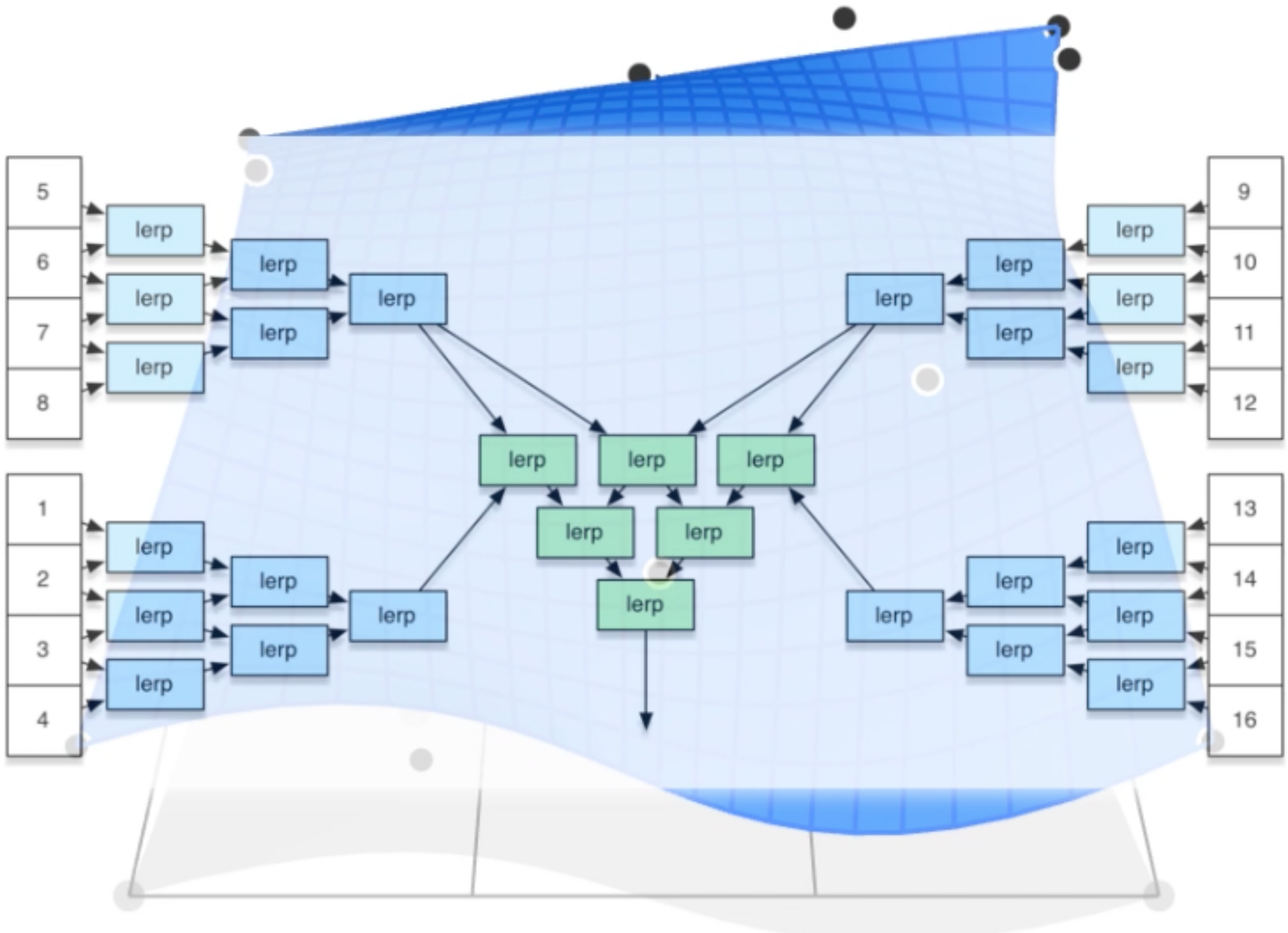
Goal: Evaluate surface position corresponding to (u,v)

(u,v)-separable application of de Casteljau algorithm

- Use de Casteljau to evaluate point u on each of the 4 Bezier curves in u. This gives 4 control points for the “moving” Bezier curve
- Use 1D de Casteljau to evaluate point v on the “moving” curve

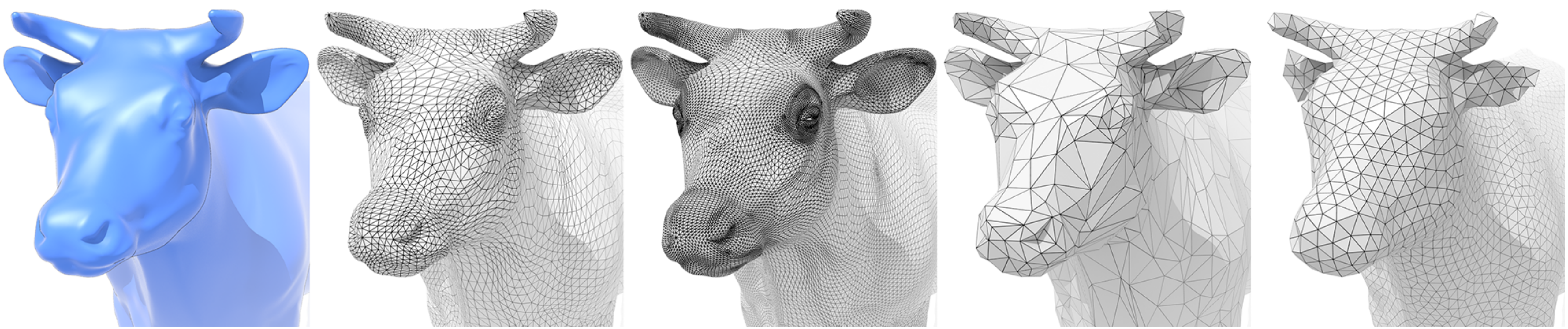


Method: Separable 1D de Casteljau Algorithm



Mesh Operations: Geometry Processing

- Mesh subdivision 网格细分
- Mesh simplification 网格简化
- Mesh regularization 规范化：三角形形状都近似正三角形



Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)