Combinations with Repetition

On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the viewpoint of the restaurant)?

Let c, h, t, and f represent cheeseburger, hot dog, taco, and fish sandwich, respectively.

Table 1.6

(a)	(b)
1. c, c, h, h, t, t, f	1. x x | x x | x x | x
2. c, c, c, c, h, t, f	2. x x x x | x | x | x
3. c, c, c, c, c, c, f	3. x x x x x x | | | x
4. h, t, t, f, f, f, f	4. | x | x x | x x x x
5. t, t, t, t, t, f, f	5. | | x x x x x | x x
6. t, t, t, t, t, t, t	6. | | x x x x x x x |
7. f, f, f, f, f, f, f	7. | | | x x x x x x x

For a purchase in column (b) of Table 1.6 we realize that each x to the left of the first bar (|) represents a c, each x between the first and second bars represents an h, the x's between the second and third bars stand for t's, and each x to the right of the third bar stands for an f. The third purchase, for example, has three consecutive bars because no one bought a hot dog or taco; the bar at the start of the fourth purchase indicates that there were no cheeseburgers in that purchase.

KtuQbank

Once again a correspondence has been established between two collections of objects, where we know how to count the number in one collection. For the representations in

column (b) of Table 1.6, we are enumerating all arrangements of 10 symbols consisting of seven x's and three |'s, so by our correspondence the number of different purchases for column (a) is

$$\frac{10!}{7!\,3!} = \binom{10}{7}. \quad = \binom{4+7-1}{7}$$

This can be cosidered as the number of combinations of 4 objects taken 7 at a time

Consequently, the number of combinations of n objects taken r at a time, *with repetition*,

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}. \quad = \quad C(n+r-1, r).$$

3. Determine how many ways 20 coins can be selected from four large containers filled with pennies, nickels, dimes, and quarters. (Each container is filled with only one type of coin.)

$$\binom{4+20-1}{20} = \binom{23}{20}$$

4. A certain ice cream store has 31 flavors of ice cream available. In how many ways can we order a dozen ice cream cones if (a) we do not want the same flavor more than once? (b) a flavor may be ordered as many as 12 times? (c) a flavor may be ordered no more than 11 times?

(a) $\binom{31}{12}$ (b) $\binom{31+12-1}{12} = \binom{42}{12}$

(c) There are 31 ways to have 12 cones with the same flavor. So there are $\binom{42}{12} - 31$ ways to order the 12 cones and have at least two flavors.

8. In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among three student helpers if each helper wants at least one donut of each kind?

For the chocolate donuts there are $\binom{3+5-1}{5} = \binom{7}{5}$ distributions. There are $\binom{3+4-1}{4} = \binom{6}{4}$ ways to distribute the jelly donuts. By the rule of product there are $\binom{7}{5}\binom{6}{4}$ ways to distribute the donuts as specified.

In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

After giving each child one banana, consider the number of ways the remaining three bananas can be distributed among these four children.

$$C(4 + 3 - 1, 3) = \quad C(6, 3) = \quad 20$$

the number of ways we can distribute the six oranges among these four children is

$9!/(6!\,3!) = C(9, 6) = C(4 + 6 - 1, 6) = 84$ ways. [Here $n = 4$, $r = 6$.]

rule of product, there are $20 \times 84 = 1680$ ways to distribute the fruit

KtuQbank

A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

There are 12! ways to arrange the 12 different symbols, and for each of these arrangements there are 11 positions between the 12 symbols. Because there must be at least three spaces between successive symbols, we use up 33 of the 45 spaces and must now locate the remaining 12 spaces. This is now a selection, with repetition, of size 12 (the spaces) from a collection of size 11 (the locations), and this can be accomplished in $C(11 + 12 - 1, 12) = 646{,}646$ ways.

Consequently, by the rule of product the transmitter can send such messages with the required spacing in $(12!)\binom{22}{12} \doteq 3.097 \times 10^{14}$ ways.

Integer Solutions

Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \qquad \text{where } x_i \geq 0 \quad \text{for all } 1 \leq i \leq 4.$$

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$. (This is different from a solution such as $x_1 = 1, x_2 = 0, x_3 = 3, x_4 = 3$, even though the same four integers are being used.) A possible interpretation for the solution $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$ is that we are distributing seven pennies (identical objects) among four children (distinct containers), and here we have given three pennies to each of the first two children, nothing to the third child, and the last penny to the fourth child. Continuing with this interpretation, we see that each nonnegative integer solution of the equation corresponds to a selection, with repetition, of size 7 (the *identical* pennies) from a collection of size 4 (the *distinct* children), so there are $C(4 + 7 - 1, 7) = 120$ solutions.

Ktu Q bank

At this point it is crucial that we recognize the equivalence of the following:

a) The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \qquad x_i \geq 0, \qquad 1 \leq i \leq n.$$

b) The number of selections, with repetition, of size r from a collection of size n.

c) The number of ways r identical objects can be distributed among n distinct containers.

In how many ways can one distribute 10 (identical) white marbles among six distinct containers?

Solving this problem is equivalent to finding the number of nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. That number is the number of selections of size 10, with repetition, from a collection of size 6. Hence the answer is $C(6 + 10 - 1, 10) = 3003$.

From Example 1.34 we know that there are 3003 nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. How many such solutions are there to the inequality $x_1 + x_2 + \cdots + x_6 < 10$?

One approach that may seem feasible in dealing with this inequality is to determine the number of such solutions to $x_1 + x_2 + \cdots + x_6 = k$, where k is an integer and $0 \leq k \leq 9$. Although feasible now, the technique becomes unrealistic if 10 is replaced by a somewhat larger number, say 100. In Example 3.12 of Chapter 3, however, we shall establish a combinatorial identity that will help us obtain an alternative solution to the problem by using this approach.

For the present we transform the problem by noting the correspondence between the nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_6 < 10 \tag{1}$$

and the integer solutions of

$$x_1 + x_2 + \cdots + x_6 + x_7 = 10, \qquad 0 \leq x_i, \qquad 1 \leq i \leq 6, \qquad 0 < x_7. \tag{2}$$

The number of solutions of Eq. (2) is the same as the number of nonnegative integer solutions of $y_1 + y_2 + \cdots + y_6 + y_7 = 9$, where $y_i = x_i$ for $1 \leq i \leq 6$, and $y_7 = x_7 - 1$. This is $C(7 + 9 - 1, 9) = 5005$.

7. Determine the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 32,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 4$ b) $x_i > 0, \quad 1 \leq i \leq 4$

c) $x_1, x_2 \geq 5, \quad x_3, x_4 \geq 7$

d) $x_i \geq 8, \quad 1 \leq i \leq 4$ e) $x_i \geq -2, \quad 1 \leq i \leq 4$

f) $x_1, x_2, x_3 > 0, \quad 0 < x_4 \leq 25$

(a) $\binom{4+32-1}{32} = \binom{35}{32}$

(b) $\binom{4+28-1}{28} = \binom{31}{28}$

(c) $\binom{4+8-1}{8} = \binom{11}{8}$

(d) 1

(e) $x_1 + x_2 + x_3 + x_4 = 32$, $x_i \geq -2$, $1 \leq i \leq 4$. Let $y_i = x_i + 2$, $1 \leq i \leq 4$. The number of solutions to the given problem is then the same as the number of solutions to $y_1 + y_2 + y_3 + y_4 = 40$, $y_i \geq 0$, $1 \leq i \leq 4$. This is $\binom{4+40-1}{40} = \binom{43}{40}$.

(f) $\binom{4+28-1}{28} - \binom{4+3-1}{3} = \binom{31}{28} - \binom{6}{3}$, where the term $\binom{6}{3}$ accounts for the solutions where $x_4 \geq 26$.

10. In how many ways can Lisa toss 100 (identical) dice so that at least three of each type of face will be showing?

Here we want the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100$, $x_i \geq 3$, $1 \leq i \leq 6$. (For $1 \leq i \leq 6$, x_i counts the number of times the face with i dots is rolled.) This is equal to the number of nonnegative integer solutions there are to $y_1+y_2+y_3+y_4+y_5+y_6 = 82$, $y_i \geq 0$, $1 \leq i \leq 6$. Consequently the answer is $\binom{6+82-1}{82} = \binom{87}{82}$.

KtuQbank

12. Determine the number of integer solutions for

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 5$

b) $x_i \geq -3, \quad 1 \leq i \leq 5$

13. In how many ways can we distribute eight identical white balls into four distinct containers so that (a) no container is left empty? (b) the fourth container has an odd number of balls in it?

Ktu Q bank