SSY191 - Sensor Fusion and Nonlinear Filtering
Project report

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Task 1

For this project, the state vector is equal to the quaternion vector for representing orientation and the inputs to the system are chosen to be the angular velocity measurements from the gyroscope (which are assumed to be rather accurate). Other options would be to consider linear acceleration as input (which is much harder to do) or to have no input at all, in which case the changes in the phone's orientation should be modelled by increasing process noise. On the other hand, using the gyroscope measurements as system inputs assumes that the sensor is perfect, i.e. that it measures the true angular velocities. If the measurements are not accurate, the angular velocity should be included in the state vector and estimated during filtering.

Task 2

Measurement signals

Figure 1 shows measurements coming from the sensors while the phone is lying flat on the table. Judging by the plots, the sensors are relatively accurate - there is practically no drift (at least during the 200 seconds of the test) and the noise amplitude is rather small. This indicates that the measurements can be trusted, i.e. the assumed measurement noise covariance should not be too large.

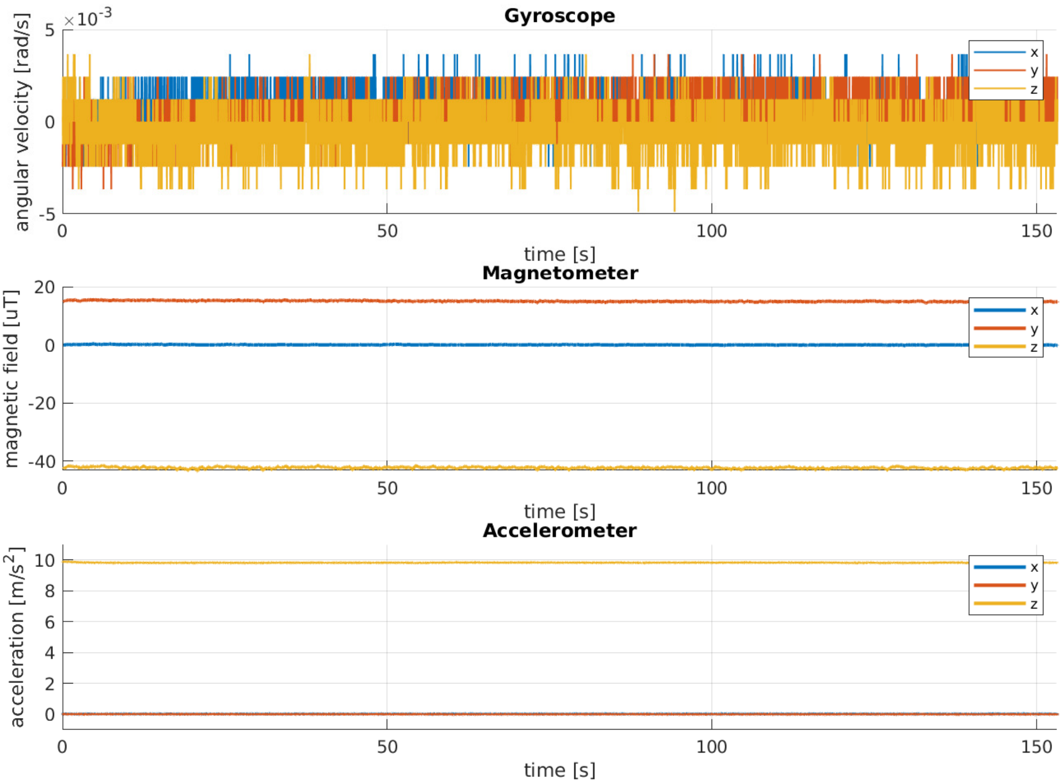
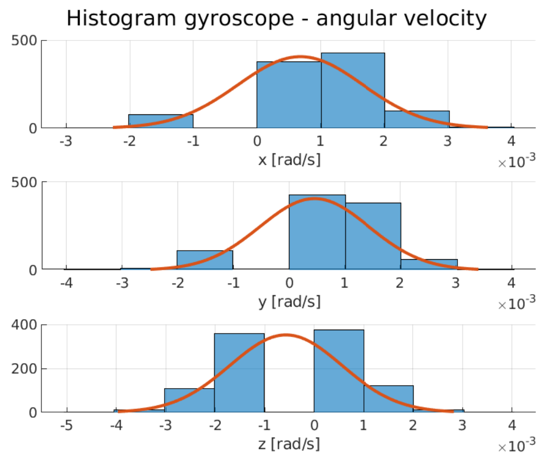


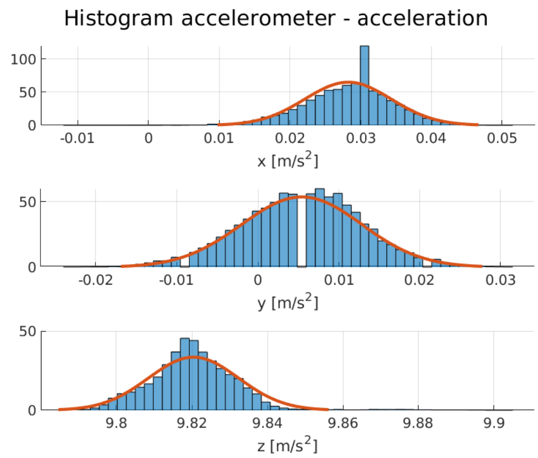
Figure 1: *Plots of measurement signals while the phone is lying still on the table.*

Measurement histograms

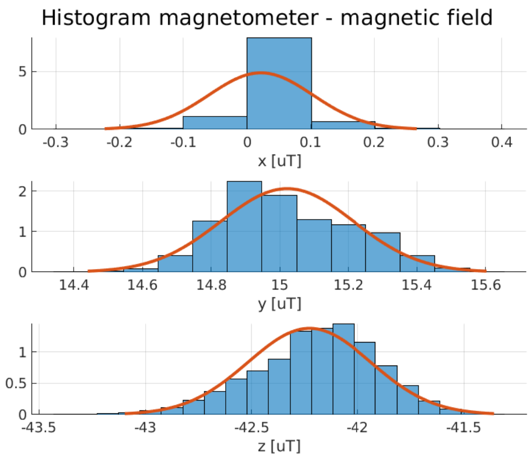
Figure 2 shows histograms of different axis measurements for the three sensors, together with a fitted Gaussian distribution. Although there are some gaps in the histograms due to the quantization in the ADC, the measurements seem to be normally distributed since the Gaussian distribution generated using the first two moments of the data fits the normalized histogram fairly well.



(a) Gyroscope



(b) Accelerometer



(c) Magnetometer

Figure 2: *Normalized histograms of sensor measurements when cellphone is lying still and flat on the table. Orange curve is the fitted Gaussian distribution*

Computed means and covariances

The estimated means and covariances for the acceleration $[m/s^2]$, angular velocity $[rad/s]$ and magnetic field $[\mu T]$ while the phone is lying flat on the table are:

$$\mu_\omega = \begin{bmatrix} 0.6829 & 0.4561 & -0.5690 \end{bmatrix}^\top \cdot 10^{-3} \qquad R_\omega = \begin{bmatrix} 0.0881 & 0.00397 & 0.00447 \\ 0.00397 & 0.0885 & 0.0175 \\ 0.00447 & 0.0175 & 0.123 \end{bmatrix} \cdot 10^{-5} \tag{1}$$

$$\mu_m = \begin{bmatrix} 0.0216 & 15.0216 & -42.2272 \end{bmatrix}^\top \qquad R_m = \begin{bmatrix} 0.0122 & -0.0010 & -0.0014 \\ -0.0010 & 0.0151 & 0.0114 \\ -0.0014 & 0.0114 & 0.0887 \end{bmatrix} \tag{2}$$

$$\mu_a = \begin{bmatrix} 0.0282 & 0.0054 & 9.8204 \end{bmatrix}^\top \qquad R_a = \begin{bmatrix} 0.0358 & -0.00608 & 0.0183 \\ -0.00608 & 0.0867 & -0.0234 \\ 0.0183 & -0.0234 & 0.176 \end{bmatrix} \cdot 10^{-3} \tag{3}$$

As expected, the gyroscope mean and covariance are almost zero, which confirms the hypothesis that the gyroscope is a very precise sensor. The accelerometer indicates the acceleration caused by gravity in the z-direction. The magnetic field vector mean depends on the orientation of the phone during the data generation. These values have been used later in the project to tune the filter.

Task 3

The continuous time motion model is given by:

$$\dot{q}(t) = \frac{1}{2} S(\omega_{k-1} + v_{k-1}) q(t), \quad \text{for } t \in [t_{k-1}, t_k), \tag{4}$$

where $v_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_v)$. Since it is assumed that ω_{k-1} and v_{k-1} are piece-wise constant between the sampling times, the matrix $S(\cdot)$ will also be constant during that period. The analytical solution for a linear system on the form $\dot{x}(\tau) = Ax(\tau),\ \tau \in [t, t+T]$ is given by $x(t+T) = \exp\,(AT)x(t)$, where T denotes the sample time. This means that the solution to equation (4) is given by:

$$q(t+T) = \exp\,\left(\frac{1}{2} S(\omega_{k-1} + v_{k-1})T\right) q(t) \tag{5}$$

which can be approximated using the expression $\exp \mathbf{A} \approx \mathbf{I} + \mathbf{A}$ as:

$$q(t+T) = \left(I + \frac{1}{2} S(\omega_{k-1} + v_{k-1})T\right) q(t) \tag{6}$$

If $q(t_k)$ is denoted as q_k and $q(t_{k-1})$ as q_{k-1}, the discretized model can be derived by using the relations $S(\omega_1 + \omega_2) = S(\omega_1) + S(\omega_2)$ and $S(\omega)q = \bar{S}(q)\omega$:

$$q_k = \underbrace{\left(I + \frac{1}{2} S(\omega_{k-1})T\right)}_{F(\omega_{k-1})} q_{k-1} + \underbrace{\frac{1}{2}\bar{S}(q_{k-1})T}_{G(q_{k-1})}\, v_{k-1} \tag{7}$$

The motion model above is a nonlinear function of the Gaussian random variables q_{k-1} and v_{k-1} and therefore q_k will not be Gaussian. However, we can approximate q_k as a Gaussian random variable using the Extended Kalman Filter approach. To solve this problem, we resort to the Taylor expansion used in the EKF. The nonlinear term $\frac{1}{2}\bar{S}(q_{k-1})Tv_{k-1}$, let's call it $g(q_{k-1}, v_{k-1})$, can be linearized around the last estimation means $\hat{q}_{k-1}$ and $\hat{v}_{k-1}$:

$$g(q_{k-1}, v_{k-1}) \approx g(\hat{q}_{k-1}, \hat{v}_{k-1}) + \frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial q_{k-1}}(q_{k-1} - \hat{q}_{k-1}) + \frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial v_{k-1}}(v_{k-1} - \hat{v}_{k-1}) \tag{8}$$

whose mean is given by:

$$\begin{aligned} E[g(q_{k-1}, v_{k-1})] &\approx E\Big[g(\hat{q}_{k-1}, \hat{v}_{k-1}) + \frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial q_{k-1}}\cancelto{0}{(q_{k-1} - \hat{q}_{k-1})} + \frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial v_{k-1}}\cancelto{0}{(v_{k-1} - \hat{v}_{k-1})}\Big] \\ &= g(\hat{q}_{k-1}, \hat{v}_{k-1}) \\ &= \frac{1}{2}\bar{S}(\hat{q}_{k-1})\,T\,\cancelto{0}{\hat{v}_{k-1}} \\ &= 0 \end{aligned} \tag{9}$$

Similarly, using the definition of EKF for the covariance $Cov[g(x)] \approx g'(x)Pg'(x)^T$:

$$
\begin{aligned}
Cov[g_{k-1}] &= \cancelto{0}{\frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial q_{k-1}}} Cov[q_{k-1}] \cancelto{0}{\frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial q_{k-1}}^{\mathsf{T}}} + \frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial v_{k-1}} Cov[v_{k-1}] \frac{\partial g(\hat{q}_{k-1}, \hat{v}_{k-1})}{\partial v_{k-1}}^{\mathsf{T}} \\
&= \frac{T}{2} \bar{S}(\hat{q}_{k-1})\ R_w\ \bar{S}(\hat{q}_{k-1})^{\mathsf{T}} \frac{T}{2} \\
&= G(\hat{q}_{k-1})^{\mathsf{T}}\ R_w\ G(\hat{q}_{k-1})
\end{aligned} \tag{10}
$$

Finally, the final expression for the covariance $P_k = Cov[q_k]$ and the mean $\hat{q}_k = E[q_k]$ can be given using equations 7, 9, and 10:

$$
\begin{aligned}
\hat{q}_k =& F(\omega_{k-1})\ \hat{q}_{k-1} \\
P_k =& F(\omega_{k-1})\ P_{k-1} F(\omega_{k-1})^{\mathsf{T}} + G(\hat{q}_{k-1})\ R_w\ G(\hat{q}_{k-1})^{\mathsf{T}}
\end{aligned} \tag{11}
$$

which is equivalent to consider the motion model as:

$$
q_k = F(\omega_{k-1})\ \hat{q}_{k-1} + G(\hat{q}_{k-1})\ v_{k-1} \tag{12}
$$

Task 4

In case gyroscope measurements are available, we predict using the motion model described in equations 11. Otherwise, in case the angular velocity is not available, a reasonable approach would be to use a random walk motion model:

$$
q_k = q_{k-1} + e_k^q \tag{13}
$$

where $e_k^q \sim (\mathbf{0}, \mathbf{R}_q)$ with the covariance matrix as a tuning parameter.

Therefore, we switch between two motion models, which seems a reasonable approach, since the sensor is subjected to failure.

Task 5

The gyroscope only measures angular velocity, which does not provide an absolute orientation measurement. However, the Kalman filter can estimate the orientation based on the motion model, which explicitly provides a relation between the state space variables and the gyroscope readings.

The problem is that, since we only measure angular velocity, we are subjected to drift in the orientation due to approximations when performing the numerical integration described in the motion model. Another problem is that we do not know the initial orientation when using only the gyroscope and this error will not be compensated for.

These problems can be verified when doing some experiments. If we start the filter with the phone on the side instead of laying face up on the desk, our orientation will always keep an offset in comparison to the Google estimate.

Another problem happens when we shake the phone rapidly. High angular accelerations increase the numerical integration error and this error will be added to the offset we had before. This phenomenon is called *drift*.

Task 6 & Task 9

The expression for the accelerometer measurements is:

$$y_k^a = Q^\top(q_k)(g^0 + f_k^a) + e_k^a \tag{14}$$

where g^0 is the nominal gravity vector, f_k^a are additional external forces and e_k^a is the measurement noise. In this case, the measurement model is nonlinear in the states since $h(q) = Q^\top(q)g^0$ (if we assume that $f^a = 0$). In the EKF procedure, the model should be linearized, which can be done by the following equation:

$$h'(x) = \left[\frac{\partial Q^\top}{\partial q_1} g_0 \ , \ \frac{\partial Q^\top}{\partial q_2} g_0 \ , \ \frac{\partial Q^\top}{\partial q_3} g_0 \ , \ \frac{\partial Q^\top}{\partial q_4} g_0 \right] \tag{15}$$

Then, the linear Kalman update equations can be used to calculate the innovation covariance:

$$S_k = h'(\hat{x}_{k|k-1})P_{k|k-1}h'(\hat{x}_{k|k-1})^\top + R_a \tag{16}$$

and the Kalman gain:

$$K_k = P_{k|k-1}h'(\hat{x}_{k|k-1})^\top S_k^{-1} \tag{17}$$

Finally, the estimation update can be done by using:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left(y_k^a - h(\hat{x}_{k|k-1}) \right) \tag{18}$$

$$P_{k|k} = P_{k|k-1} - K_k S_k K_k^{-1} \tag{19}$$

The procedure for updating the estimates using the magnetometer measurements is very similar, with the measurement model given by:

$$y_k^m = Q^\top(q_k)(m^0 + f_k^m) + e_k^m \tag{20}$$

where m^0 is the earth magnetic field in world coordinates, f_k^m denotes additional magnetic fields and e_k^m is the measurement error. The estimated value for the earth magnetic field is $m_0 = [0 \ 17.7 \ -45.4]^\top \mu T$.

Task 7 & Task 8

If the accelerometer readings are used to update the orientation estimation, there should be no specific forces other than gravity affecting the sensor. If they are introduced, the assumption about the orientation of the reference vector is no longer valid and the estimates will be incorrect.

If outlier rejection is implemented, additional specific forces will not have a large effect on the measurements since they will be disregarded and other sensors will be used for estimation during those time steps.

Task 10

Without outlier rejection, introducing a magnetic disturbance results in an estimation error since the filter will try to modify the estimates using the dominant magnetic field (which is not the earth's magnetic field anymore) and the assumption about the reference orientation is no longer valid.

Task 11

To detect outliers, we assume that the measured reference, m^0, is the true value of the earth's magnetic field. This is reasonable if it has been measured away from other magnetic sources. Otherwise, it could cause rejection of all the measurements.

After implementing outlier rejection, adding a magnetic disturbance does not have a significant effect since the measurements affected by it are disregarded in the filtering algorithm.

Task 12

All three sensors

Using all three sensors, the implemented EKF gives results comparable to those of the reference (Google) filter, which is considered as the ground truth.

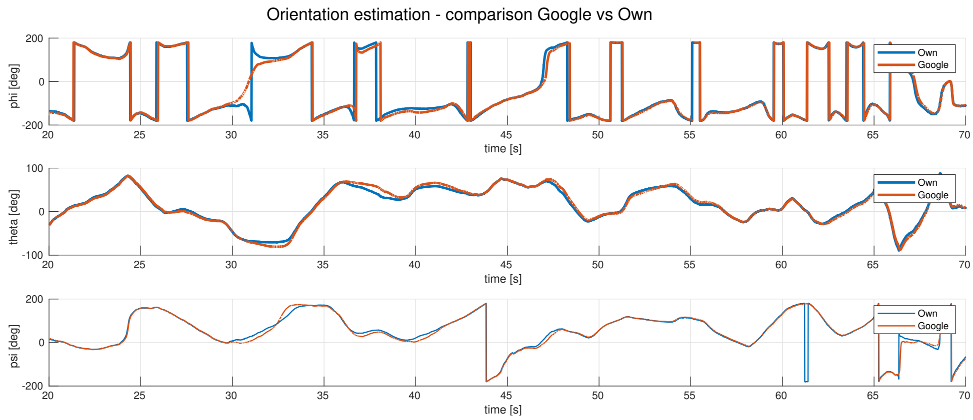


Figure 3: *Orientation estimation using all three sensors.*

Accelerometer and magnetometer

Figure 4 shows the orientation estimation when the angular velocity measurements are not available and the random walk motion model is used in the prediction step of the filter. The estimated process noise for this model was $R_q = 0.1 I_4$, which resulted in a fairly good (but clearly a bit noisy) estimation. Some deviations seen in the plots (e.g. around the 20th second in the bottom subplot) are due to quaternion representation ambiguity and the implementation of the `q2euler` function.

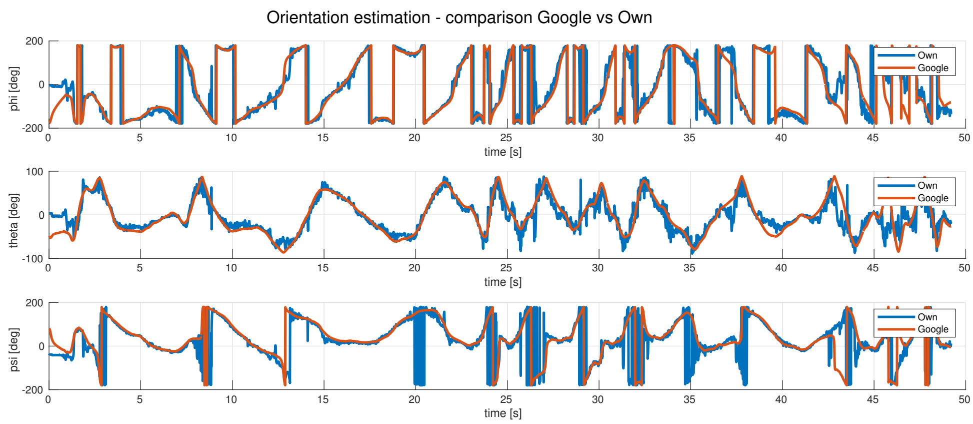


Figure 4: *Orientation estimation without using the gyroscope.*

Gyroscope and magnetometer

When not using the accelerometer, the filter performance deteriorates since one of the reference directions is removed. By doing this, a degree of freedom is lost because the gyroscope provides only relative orientation information.

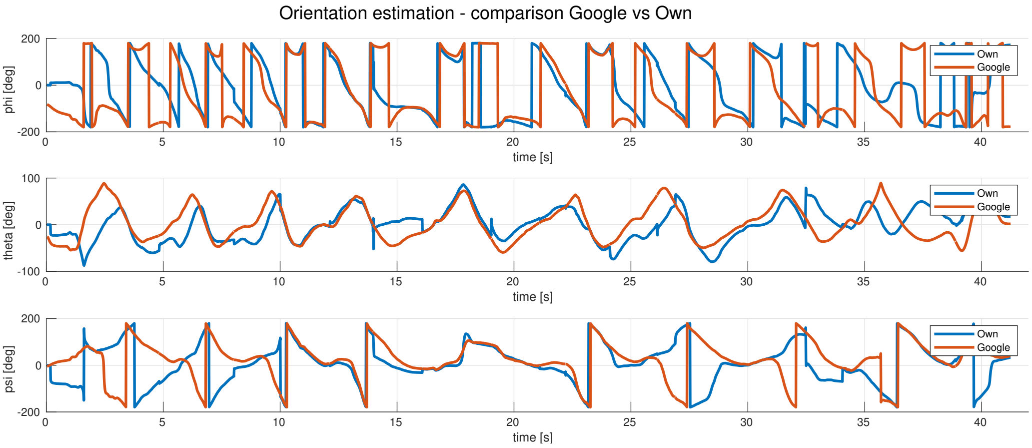


Figure 5: *Orientation estimation without using the gyroscope.*

Gyroscope and accelerometer

Similar to the previous case, using the filter without the magnetometer measurements does not give satisfactory results. Again, a reference point is lost and ambiguity in the absolute orientation is introduced.

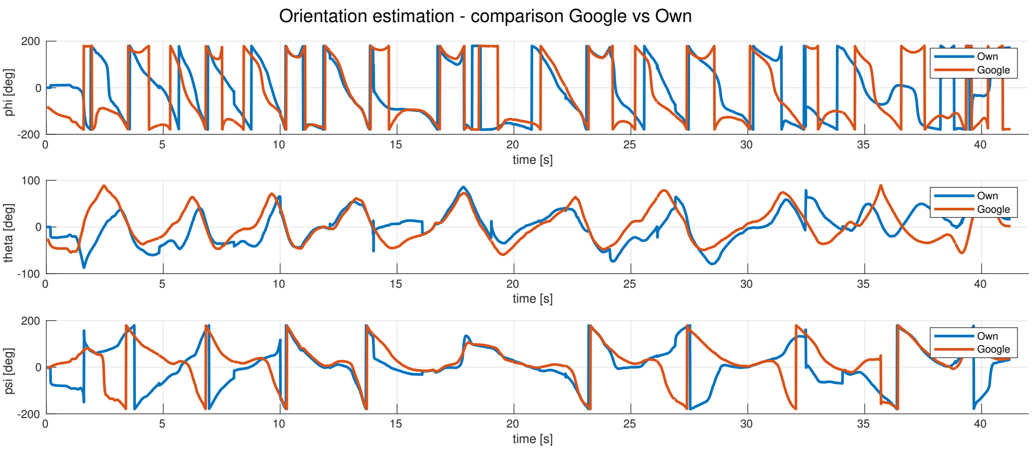


Figure 6: *Orientation estimation without using the magnetometer.*

General conclusions

Based on the performed experiments, it was concluded that, out of the three sensors, the gyroscope is the least important. Although it provides correct measurements, it does not give information about the absolute orientation of the sensor. Additionally, using a simple motion model (random walk) with the other two sensors gives better results than using the gyroscope with only the accelerometer or the magnetometer.

Since the sensors are rather accurate, adding their bias in the state vector and estimating it was not necessary. This resulted in a simpler model and implementation.

Finally, if all three sensors are used, the obtained filter performance is almost the same as that of the Google filter. By estimating the sensor characteristics carefully, the need for additional tuning of the filter parameters was reduced.