

High-Level Optimization of Abstract Data Types

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Programming With Abstract Data Types

- System specifications use sets and relations
 - Expressive and provable
- Semantics from mathematics

Syntax	Label	Syntax	Label
$set(T)$	Unordered, unique collection	$S \cup T$	Union
$S \leftrightarrow T$	Relation, $set(S \times T)$	$S \cap T$	Intersection
$S \rightarrow \mathbb{N}$	Bag/Multiset	$S \setminus T$	Difference
$\mathbb{N} \rightarrow S$	Sequence	$S \times T$	Cartesian Product
$\{x, y, \dots\}$	Set Enumeration	$dom(R)$	Domain
$\{x \cdot P \mid E\}$	Set Comprehension	$R[S]$	Image

Jean-Raymond Abrial. *Modeling in Event-B: system and software engineering*. Cambridge University Press, 2010.
David Gries and Fred Schneider. *A Logical Approach to Discrete Math*. Springer New York, 1993.

Example - Visitor Information System

- Academics attend workshops throughout CASCON 2025
- Every workshop must be held in one room
- Visitors may attend at most one workshop at a time

$Visitor = \{J. Nelson, Mark, Ehsan\}$
 $Room = \{1, 2, 3, 4\}$
 $Workshop = \{CDP, SENGEC, COGAI\}$

$location: Workshop \rightarrow Room$
 $attends: Visitor \rightarrow Workshop$

Total Injective Function

Partial Function

Then, the number of meals to prepare for a specific *room* would be:

$$card((location^{-1} \circ attends^{-1})[\{room\}])$$

Relational Image

Example - Warehouse Inventory System

- Furniture material catalogue
- Warehouse inventory
- Product recipes

$Material = \{2 \times 4 \text{ Plank}, \text{Hex Bolt}, 3 \text{ Inch Screw}, \dots\}$

$Price = \mathbb{N}_0$

$Product = \{Cabinet, Desk, Bookshelf, \dots\}$

$catalogue: Material \rightarrow Price$

$inventory: bag[Material]$

$recipes: Product \rightarrow bag[Material]$

Equivalent to: $Material \rightarrow \mathbb{N}$

Restocking price, with a target inventory $inventory_t$:

$\sum p \mapsto n_m \cdot$

$p \mapsto n_m \in catalogue^{-1} \circ (inventory_t - inventory) \mid p \times n_m$

Bag Difference

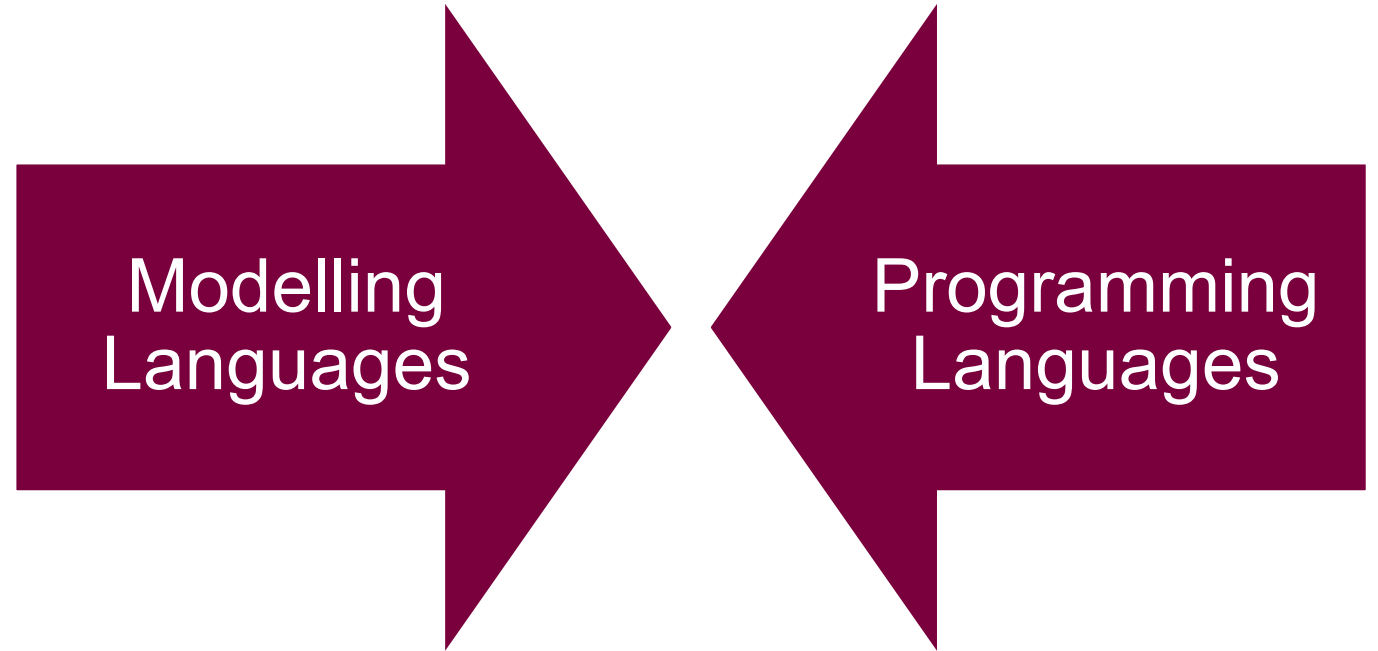
Abstract Data Types

In Programming and Modelling Languages

		Sets				Relations								Efficiently Executable	Implementation Free	
		\cup	\cap	\times	$\{x \cdot P \mid E\}$	dom	\triangleleft	\cup	\cap	\triangleleft	\circ	$R[S]$	R^{-1}			Multiplicity
Programming Languages	Python	✓	✓	✓	✓	✓	✗	✗	*	✓	✗	*	✗	✗	*	✗
	Haskell	✓	✓	✓	*	✓	*	✗	✓	✓	*	*	✓	*	✓	✗
	Rust	✓	✓	*	*	✓	✗	✗	✗	*	✗	✓	✗	✗	✓	✗
	C	*	*	*	✗	*	*	*	*	*	*	*	*	✗	✓	✗
Modelling Languages	SetL	✓	✓	*	✓	✓	✓	✓	*	*	✓	*	*	*	*	*
	UML	✓	✓	✓	✗	✓	✗	✗	✗	✓	✗	✓	✗	✓	*	✓
	Event-B	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	*	✓
	Alloy	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓

Goals

- Syntax close to discrete mathematics
- Simple, small, and usable
- High-level set-theory optimization
- Efficient in memory and time



Related Work

- ProB animation engine for Event-B [1]
- GHC rewrite rules [2]
- Rewrite systems for set-theory focused optimization [3,4,5]
- SETL data type *lowering* from abstract types to suitable concrete structures [6,7]

[1] Michael Leuschel. *Programming in B: Sets and Logic all the Way Down*. In: LPOP 2022.

[2] Simon Peyton Jones, Andrew Tolmach, and Tony Hoare. *Playing by the rules: rewriting as a practical optimisation technique in GHC*. In: 2001 Haskell Workshop. ACM SIGPLAN. Sept. 2001.

[3] Maximiliano Cristia and Gianfranco Rossi. *{log}: Programming and Automated Proof in Set Theory*. In: LPOP 2022.

[4] Elco Visser, Zine-el-Abidine Benaissa, and Andrew Tolmach. Building program optimizers with rewriting strategies. In: ICFP 1998.

[5] Douglas R. Smith and Stephen J. Westfold. *Transformations for Generating Type Refinements*. In: FM 2019.

[6] Edmond Schonberg, Jacob T. Schwartz, and Micha Sharir. 1979. *Automatic data structure selection in SETL*. In: POPL 1979.

[7] Stefan M. Freudenberger, Jacob T. Schwartz, and Micha Sharir. *Experience with the SETL Optimizer*. Association for Computing Machinery, 1983

Optimizing Abstract Data Type Operations (in Python)

$$(S \cup T) \cap V$$



$$(S \cap V) \cup ((T \setminus S) \cap V)$$

Python translation:

```
ret = set()
for x in S:
    ret.add(x)
# Step 1: len(ret) == len(S)
for x in T:
    ret.add(x)
# Step 2: len(ret) <= len(S) + len(T)
for x in ret:
    if x not in V:
        ret.remove(x)
# Step 3: len(ret) <= len(V)
```

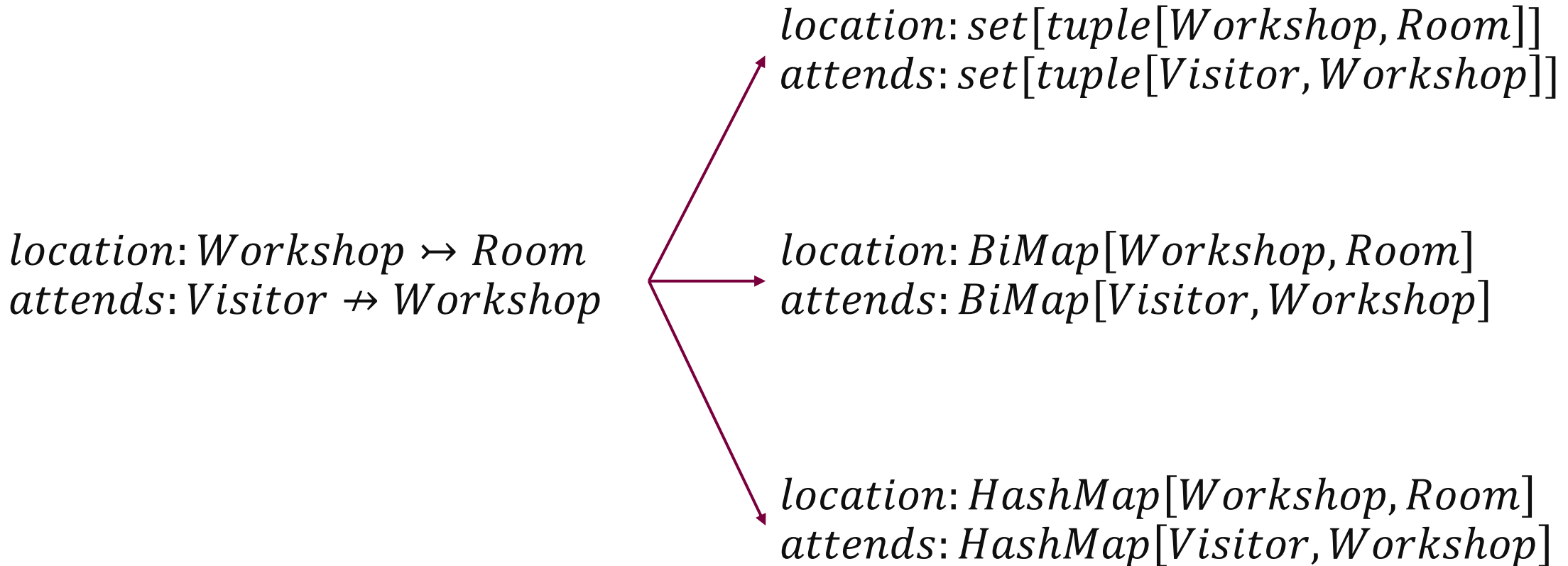
New Python translation:

```
ret = set()
for x in S:
    if x in V:
        ret.add(x)
# Step 1: len(ret) <= min(len(S), len(V))
for x in T:
    if x not in S and x in V:
        ret.add(x)
# Step 2: len(ret) <= min(len(S) + len(T), len(V))
```

What if $\text{len}(S) + \text{len}(T) > \text{len}(V)$?

ret never grows larger than needed

Concrete Data Types of the Visitor Information System



Naïve Implementation of the Visitor Information System

Concrete Types

$location: set[tuple[Workshop, Room]]$
 $attends: set[tuple[Visitor, Workshop]]$

Imperative Interpretation

$card((location^{-1} \circ attends^{-1})[\{room\}])$



$l := inverse(location)$
 $a := inverse(attends)$
 $c := compose(l, a)$

...

Library

```
proc inverse( $r$ ):  
   $r' := \{\}$   
  for  $k, v \in r$  do  
     $r' := r' \cup \{(v, k)\}$   
  return  $r'$ 
```

Runtime: $O(n)$
Memory: up to $O(n)$

```
proc compose( $r_1, r_2$ ):  
   $r' := \{\}$   
  for  $k, v \in r_1$  do  
    for  $v', t \in r_2$  do  
      if  $v = v'$  then  
         $r' := r' \cup \{(k, t)\}$   
  return  $r'$ 
```

Runtime: $O(n^2)$
Memory: $O(n)$

Designing Optimizations

- Set theory semantics for optimizations
- Strong type system with refinements
- Rewrite system:
 - Simplify operations
 - Select concrete data representations from refined types
 - Generate efficient code for every concrete representation and operation sequence

Term Rewriting for Hash-based Sets and Relations

1. Comprehension Construction
2. Generator Selection
3. Loop Lowering
4. Relational Subtyping Loop Simplification
5. Loop Code Generation
6. Replace and Simplify

Phase 1: Set Comprehension Construction

Set-typed variables and literals are decomposed into set comprehensions

Rewrite Rules	
Predicate Operations	$S \cup T \rightsquigarrow \{x \cdot x \in S \vee x \in T \mid x\}$
	$S \cap T \rightsquigarrow \{x \cdot x \in S \wedge x \in T \mid x\}$
	$S \setminus T \rightsquigarrow \{x \cdot x \in S \wedge x \notin T \mid x\}$
Membership Collapse	$x \in \{E \mid P\} \rightsquigarrow \exists y \cdot P \wedge x = E$
Image	$R[S] \rightsquigarrow \{y \cdot \exists x \cdot x \mapsto y \in R \wedge x \in S \mid y\}$
Composition	$x \mapsto z \in R \circ Q \rightsquigarrow x, z \cdot \exists y, y' \cdot x \mapsto y \in R \wedge y' \mapsto z \in Q \wedge y = y'$
Inverse	$x \mapsto y \in R^{-1} \rightsquigarrow y \mapsto x \in R$
Cardinality	$card(S) \rightsquigarrow \sum x \cdot x \in S \mid 1$

Post-condition:

- All set-like terms must be comprehensions

$$\begin{array}{l}
 \text{Workshop} \mapsto \text{Room} \quad \text{Visitor} \mapsto \text{Workshop} \\
 numMeals = \bigvee card((location^{-1} \circ attends^{-1})[\{room\}]) \\
 \quad \quad \quad \downarrow \text{Image} \\
 numMeals = card(\{v \cdot \exists r \cdot r \mapsto v \in (location^{-1} \circ attends^{-1}) \wedge r \in \{room\} \mid v\}) \\
 \quad \quad \quad \downarrow \text{Composition, Inverse} \\
 numMeals = card(\{v, r \cdot \exists p, p' \cdot v \mapsto p' \in attends \wedge p \mapsto r \in location \wedge p = p' \wedge r \in \{room\} \mid v\}) \\
 \quad \quad \quad \downarrow \text{Cardinality, Membership} \\
 numMeals = \sum v, r \cdot \exists p, p' \cdot v \mapsto p' \in attends \wedge p \mapsto r \in location \wedge p = p' \wedge r = room \mid 1
 \end{array}$$

Phase 2: Generator Selection

Selecting element generators for use as iterables in generated for-loops

Rewrite Rules

Generator Selection

$$\bigwedge P_i \rightsquigarrow P_g \wedge \bigwedge_{P_i \neq P_g} P_i$$

- P_g is of the form $x \in S$
- x is the quantifier's bound variable
- S is a set-like term

- In the case of multiple candidate generators, selection is based on heuristics

$$\text{numMeals} = \sum v, r \cdot \exists p, p' \cdot v \mapsto p' \in \text{attends} \\ \wedge p \mapsto r \in \text{location} \wedge p = p' \wedge r = \text{room} \mid 1$$



Generator Selection

Post-condition:

- All set-like terms must be comprehensions
- Quantifier predicates are in DNF
 - Guaranteed top-level-v operation
- One bound variable per generator
- All predicate v-clauses have an assigned generator

$$\text{numMeals} = \sum \mathbf{v}, \mathbf{r} \cdot \exists p, p' \cdot \mathbf{v} \mapsto \mathbf{p}' \in \mathbf{attends} \\ \wedge \mathbf{p} \mapsto \mathbf{r} \in \mathbf{location} \wedge p = p' \wedge r = \text{room} \mid 1$$

Phase 3: Loop Lowering

Start lowering expressions into imperative-like loops

Rewrite Rules

Quantifier Generation	$\oplus E \mid P$	\rightsquigarrow	$a := identity(\oplus)$ loop P do $a := a \oplus E$
Chained Generators	loop $G_1 \wedge G_2$ do $a := a \oplus E$	\rightsquigarrow	loop G_1 do loop G_2 do $a := a \oplus E$
Disjunct Generators	loop $G_1 \vee G_2$ do $a := a \oplus E$	\rightsquigarrow	loop G_1 do $a := a \oplus E$ loop $G_2 \wedge \neg G_1$ do $a := a \oplus E$

Post-condition:

- No quantifiers exist within the AST
- No \vee -operators exist within a **loop**'s predicate
- All predicate \vee -clauses have an assigned generator

- \oplus is any of $\{\dots\}, \Sigma, \Pi, \cup, \cap$

$numMeals =$
 $\Sigma \mathbf{v}, \mathbf{r} \cdot \exists p', p \cdot \mathbf{v} \mapsto \mathbf{p}' \in \textit{attends}$
 $\wedge \mathbf{p} \mapsto \mathbf{r} \in \textit{location} \wedge r = room \wedge p = p' \mid 1)$

Quantifier Generation

$numMeals := 0$
loop $\mathbf{v} \mapsto \mathbf{p}' \in \textit{attends}$
 $\wedge \mathbf{p} \mapsto \mathbf{r} \in \textit{location} \wedge p = p' \wedge r = room$ **do**
 $numMeals := numMeals + 1$

Chained Generators

$numMeals := 0$
loop $\mathbf{v} \mapsto \mathbf{p}' \in \textit{attends}$ **do**
loop $\mathbf{p} \mapsto \mathbf{r} \in \textit{location} \wedge r = room \wedge p = p'$ **do**
 $numMeals := numMeals + 1$

Phase 4: Relation Subtyping Loop Simplification

Eliminate unnecessary loops

Rewrite Rules

Restricted
Generator

loop $x \mapsto y \in R$
 $\wedge x = x' \wedge y = y'$ **do** **if** $R(x') = y'$ **then**
 body \rightsquigarrow body

- R is total
- R is one-to-one

One-to-one Total Function

$numMeals := 0$
loop $v \mapsto p' \in attends$ **do**
 loop $p \mapsto r \in location \wedge r = room \wedge p = p'$ **do**
 $numMeals := numMeals + 1$



$numMeals := 0$
loop $v \mapsto p' \in attends$ **do**
 if $location(p') = room$ **then**
 $numMeals := numMeals + 1$

Post-condition:

- No quantifiers exist within the AST
- No v-operators exist within a *loop*'s predicate
- All predicate v-clauses have an assigned generator

Phase 5: Loop Code Generation

Lower into imperative *loop* structures

Rewrite Rules

Conjunct
Conditional

$$\begin{array}{l} \text{loop } P_g \\ \wedge \bigwedge P_i \text{ do } \rightsquigarrow \\ \text{body} \end{array} \quad \rightsquigarrow \quad \begin{array}{l} \text{if } \bigwedge_{free(P_i)} P_i \text{ then} \\ \text{for } P_g \text{ do} \\ \text{if } \bigwedge_{\neg free(P_i)} P_i \text{ then} \\ \text{body} \end{array}$$

Post-condition:

- Code is imperatively executable

```
numMeals := 0
loop v ↦ p' ∈ attends do
  if location(p') = room then
    numMeals := numMeals + 1
```



```
numMeals := 0
for v, p' ∈ attends do
  if location(p') = room then
    numMeals := numMeals + 1
```

Visitor Information System - Translation

$location: Workshop \mapsto Room$
 $attends: Visitor \mapsto Workshop$

One-to-One Total Function

$numMeals$
 $= card((location^{-1} \circ attends^{-1})[\{room\}])$

$numMeals$
 $= \sum v, r \cdot \exists p, p' \cdot v \mapsto p' \in attends$
 $\wedge p \mapsto r \in location \wedge \mathbf{p} = \mathbf{p'} \wedge \mathbf{r} = \mathbf{room} \mid 1$

$numMeals$
 $= \sum v \cdot \exists p' \cdot v \mapsto p' \in attends \wedge location(p') = room \mid 1$

$numMeals := 0$
for $v, p' \in attends$ **do**
 if $location(p') = room$ **then**
 $numMeals := numMeals + 1$

Warehouse Inventory System – Translation

$catalogue: Material \rightarrow Price$
 $inventory: bag[Material]$
 $recipes: Product \rightarrow bag[Material]$

Many-to-One Total Function

Many-to-One Function

Restocking price, with a target inventory $inventory_t$:

$$price = \sum p \mapsto n_m \cdot p \mapsto n_m \in catalogue^{-1} \circ (inventory_t - inventory) \mid p \times n_m$$

$$price = \sum m \mapsto n_m \cdot \exists n, m, p \cdot m \mapsto n_m \in inventory_t \wedge n = n_m - inventory(m) \wedge n \geq 0 \wedge p = catalogue(m) \mid p \times n_m$$

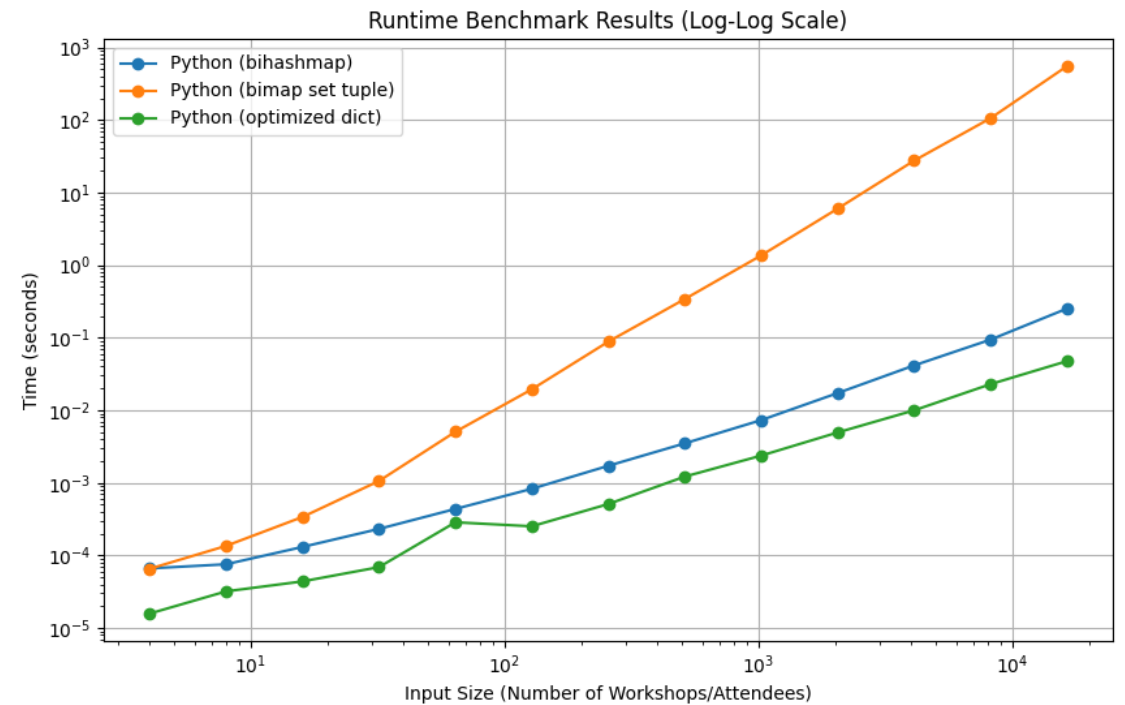
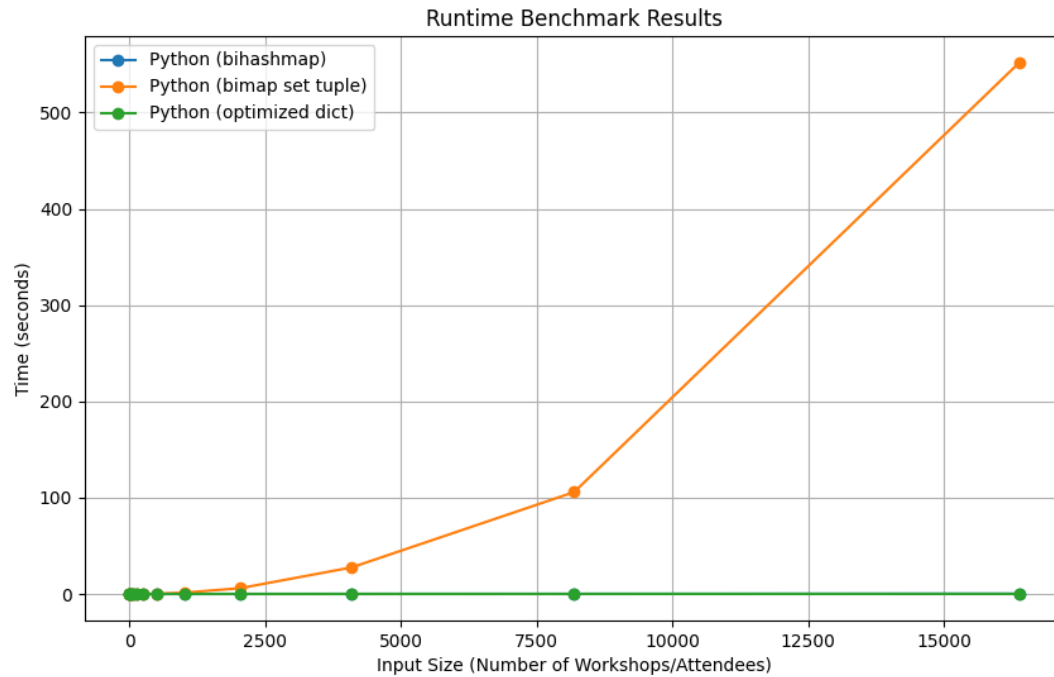
```

price := 0
for m, n_m in inventory_t do
    n := n_m - inventory(m)
    if n ≥ 0 then
        price := price + catalogue(m) × n
    
```

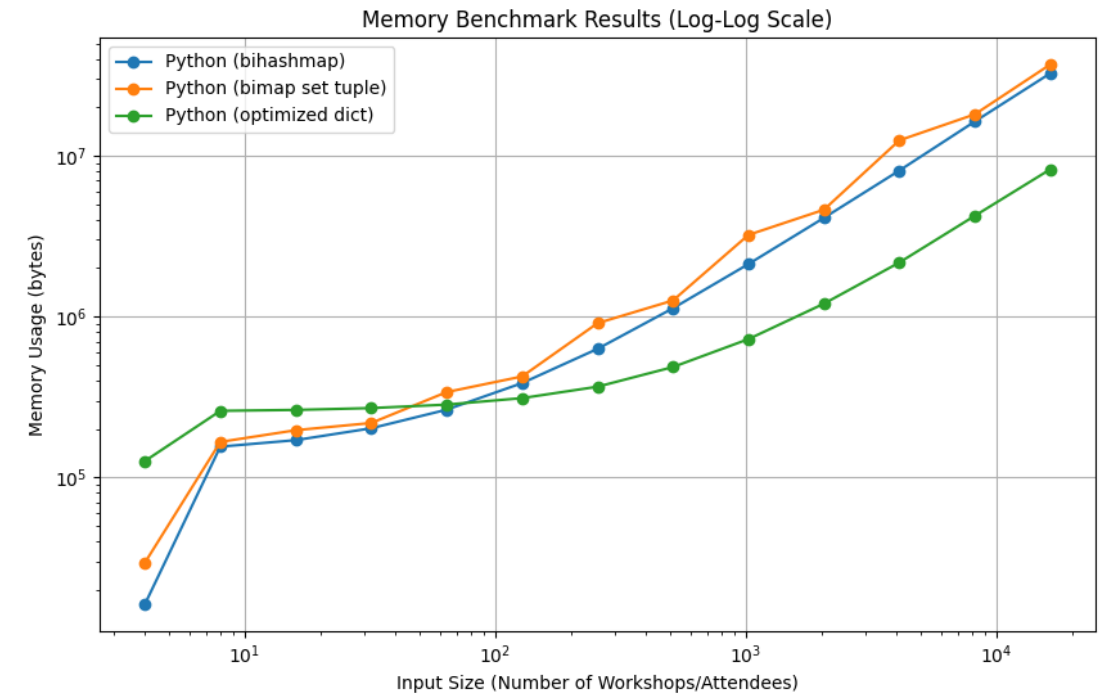
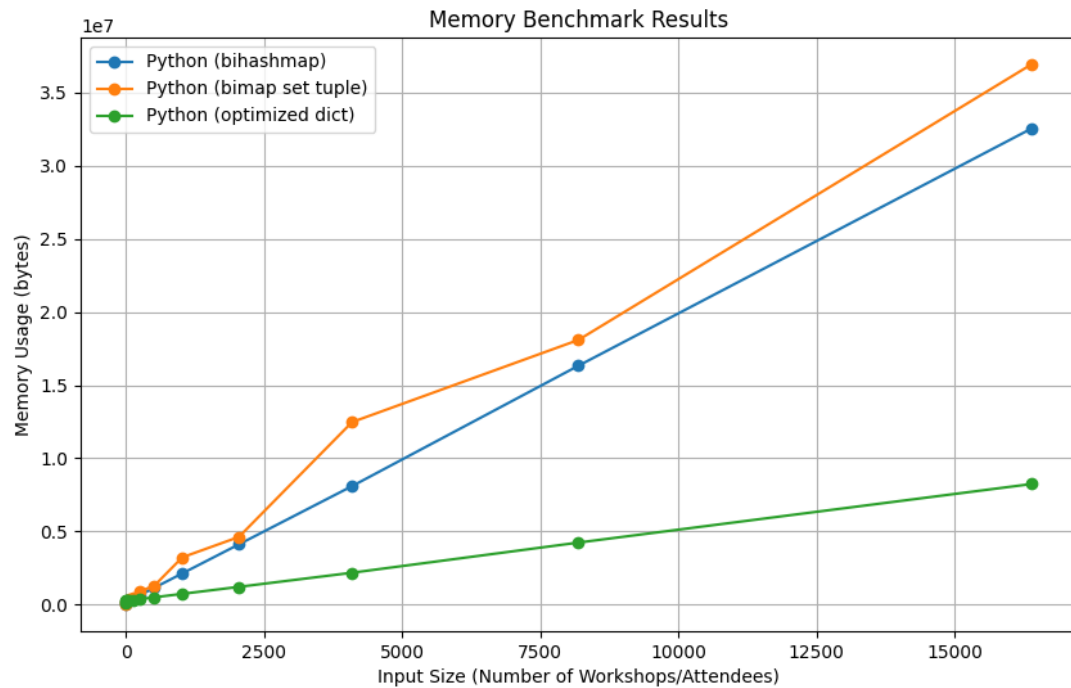
Preliminary Results

- Intermediate sets never grow larger than $\max(\text{starting sets}, \text{resulting set})$
- Unions, Relation Overriding, and Cartesian Products are limited in growth
- Other operators only decrease the size of a resulting set
- What about running time?...

Visitor Information System Benchmark – Runtime



Visitor Information System Benchmark – Memory Consumption



Examples Running Time

	Core Expression	Naïve Running Time	Optimized Running Time
Visitor Information System	$(location^{-1} \circ attends^{-1})[\{room\}]$	$O(location attends)$	$O(attends)$
Warehouse Inventory System	$\sum p \mapsto n_m \cdot p \mapsto n_m$ $\in (inventory_t - inventory) \circ catalogue^{-1}$ $ p \times n_m$	$O(inventory_t catalogue)$	$O(inventory_t)$

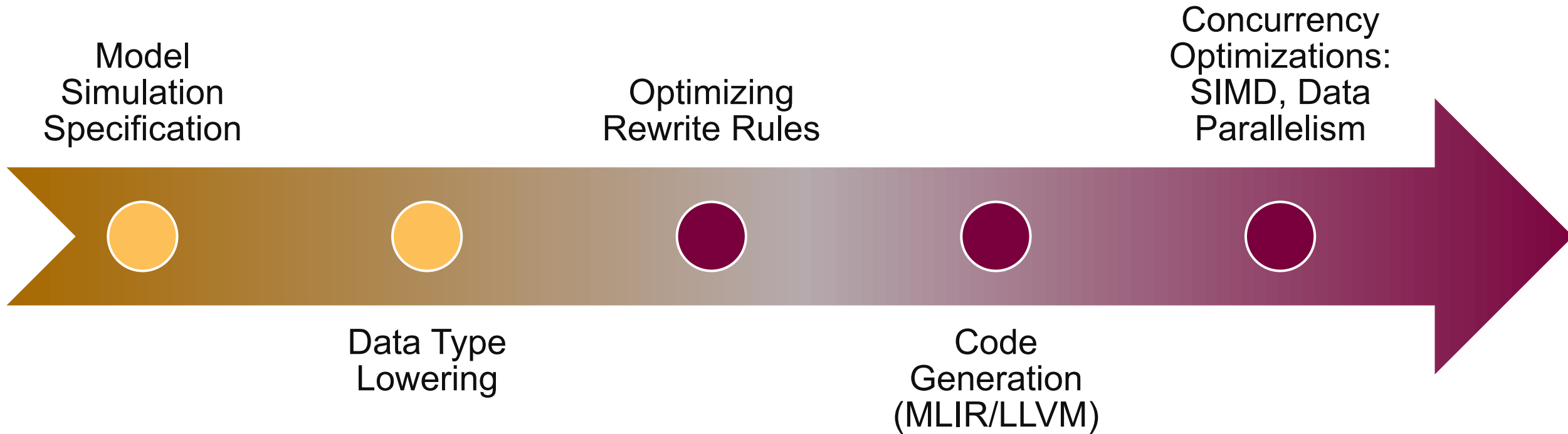
Operation Running Time With (Bi-directional) HashMaps

Operation	Expected Running Time
$S \cup T$	$O(S + T)$
$S \cap T$	$O(\min(S , T))$
$R[S]$	$O(\min(S , R))$
$R \circ Q$	$O(R Q)$
$(S \cup T) \cap U$	$O(\min(S + T , U))$
$S \cap R[T]$	$O(\min(S , R , T))$
$(R \circ Q)[S]$	$O(\min(S , R) + \min(R[S] , Q))$

- Simplify large groups of \wedge -predicates
- Eagerly apply conditions on sequential \vee -predicates

Current Compiler State and Roadmap

Development of a High-Level, Efficient, Set-Based Language



References

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- Stefan M. Freudenberger, Jacob T. Schwartz, and Micha Sharir. 1983. *Experience with the SETL Optimizer*. ACM Trans. Program. Lang. Syst. 5, 1 (Jan. 1983), 26–45. <https://doi.org/10.1145/357195.357197>