Adaptive control basics

Inverted pendulum

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{u}{m\ell^2}$$

$$E_{tot} = mg\ell(1 - \cos x_1) + \frac{1}{2}m\ell^2 x_2^2$$

Energy-based controller

$$u \leftarrow -k x_2 E_{tot},$$

$$k > 0$$

$$L \leftarrow \frac{1}{2}E_{tot}^2 \Rightarrow$$

$$\dot{L} = E_{tot}\, x_2\, u$$

Now, pendulum with friction:

$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{u}{m\ell^2} - c x_2^2,$$

$$c > 0$$

$$L = \frac{1}{2} E_{tot}^2$$

$$\dot{L} = E_{tot} \cdot \dot{E}_{tot}$$

$$= E_{tot} X_2 \left(u - c m l^2 X_2^2 \right)$$

$$u \leftarrow \underbrace{u_{old}}_{-k X_2 E_{tot}} + c m l^2 x_2^2$$

? what if c is unknown ?

$\hat{c}$ — estimate

$$\tilde{c} := \hat{c} - c$$

$$L_c := L + \frac{1}{2\alpha} \tilde{c}^2$$

$\alpha > 0$, learning rate

$$\dot{L}_c = E_{tot} x_2 \left(u - c m l^2 x_2^2 \right) + \frac{1}{\alpha} \tilde{c} \dot{\hat{c}}$$

Observe that $c = \hat{c} - \tilde{c}$

$\Longrightarrow$

$$\dot{L}_c = E_{tot} x_2 \left(u - \hat{c} m l^2 x_2^2 \right) + E_{tot} x_2^3 \tilde{c} m l^2 + \frac{1}{\alpha} \tilde{c} \underline{\dot{\hat{c}}}$$

$$\dot{\hat{c}} \leftarrow -\alpha E_{tot} x_2^3 m l^2$$

General rule:

$$\dot{x} = f(x) + g(x) u, \quad \text{(control-affine system)}$$

$x \in \mathbb{R}^n$,

$$\dot{x} = f(x) + g(x) u + \theta^T \varphi(x)$$

θ: unknown, $\varphi(x)$: known

If θ were known, let's say that we came up with a controller $\rho(x,\theta)$ and a corresponding LF L:

$$\langle \nabla L, f(x) + g(x)\rho(x,\theta) + \theta^T \varphi(x) \rangle < 0$$

Now, θ is unknown

$$L_c := L + \frac{1}{2\alpha} \operatorname{tr}\{\tilde{\theta}\tilde{\theta}^T\}$$

$$\dot{L}_c = \dot{L} + \frac{1}{\alpha} \operatorname{tr}\{\tilde{\theta}^T \dot{\hat{\theta}}\}$$

$$= \langle \nabla L, f(x) + g(x)\rho(x, \hat{\theta}) + \theta^T \varphi(x) \rangle + \frac{1}{\alpha} \operatorname{tr}\{\tilde{\theta}^T \dot{\hat{\theta}}\}$$

$$(\theta = \hat{\theta} - \tilde{\theta}) \quad = \langle \nabla L, f(x) + g(x)\rho(x, \hat{\theta}) + \hat{\theta}^T \varphi(x) \rangle - \langle \nabla L, \tilde{\theta}^T \varphi(x) \rangle + \frac{1}{\alpha} \operatorname{tr}\{\tilde{\theta}^T \dot{\hat{\theta}}\}$$

$$= \left(\dot{\hat{\theta}} \leftarrow -\alpha \langle \nabla L, \varphi(x) \rangle \right)$$

$$= \langle \nabla L, f(x) + g(x) \rho(x, \hat{\theta}) + \hat{\theta}^T \varphi(x) \rangle < 0$$