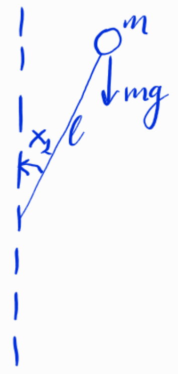
# Elements of control under uncertainty.




$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2}u + q$$

Energy-based (nominal) controller:

$$u \leftarrow -k E_{tot} x_2 \,, \quad k > 0$$

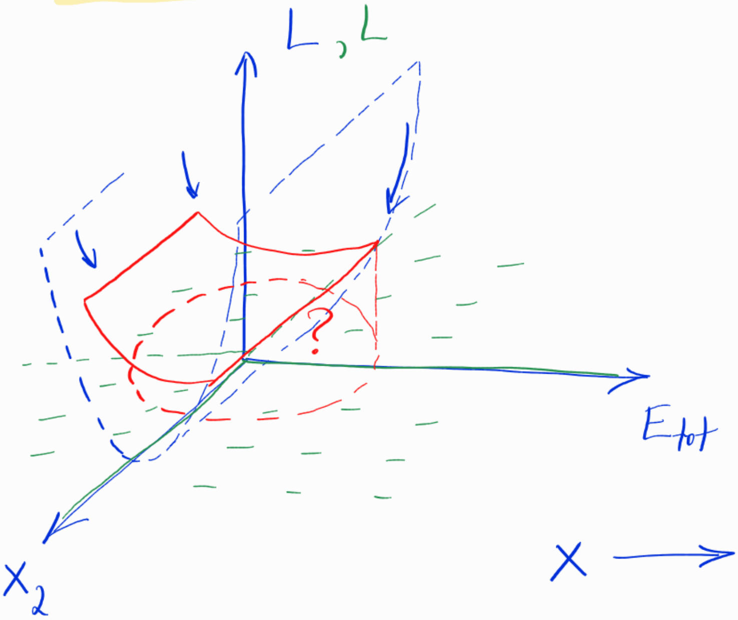
$$E_{tot} = mg\ell(1 - \cos x_1) + \frac{1}{2}m\ell^2 x_2^2$$

$$L := \frac{1}{2}E_{tot}^2$$

$$\dot{L} = E_{tot} x_2 u$$

Under uncertainty:

$$\dot{L} = E_{tot} x_2 (u + m\ell^2 q)$$
$$= -k E_{tot}^2 x_2^2 + E_{tot} x_2 m\ell^2 q$$





$$X \longrightarrow \{ k E_{tot} x_2 \le m\ell\bar{q} \}$$

bound on $q$

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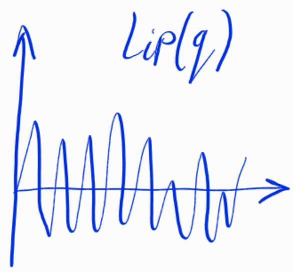
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2}u + q$$

$$u \leftarrow -kE_{tot}x_2 - q m l^2$$

$q$ - ?




$$\hat{q}, \quad \tilde{q} := \hat{q} - q$$

$$L_c := L + \frac{1}{2\alpha}\tilde{q}^2$$

$$\dot{L}_c = E_{tot}x_2(-kE_{tot}x_2 - \hat{q}ml^2 + ml^2 q) + \frac{1}{\alpha}\tilde{q}\dot{\hat{q}} + \frac{1}{\alpha}\tilde{q}\cdot\dot{q}$$

$$\leq E_{tot}x_2(-kE_{tot}x - \hat{q}ml^2 - ml^2\tilde{q} + \hat{q}ml^2) + \frac{1}{\alpha}\tilde{q}\dot{\hat{q}} + \frac{1}{\alpha}\tilde{q}\cdot Lip(q)$$

$$\leq -kE_{tot}^2x_2^2 - E_{tot}x_2 ml^2\tilde{q} + \frac{1}{\alpha}\tilde{q}\dot{\hat{q}} + \frac{1}{\alpha}\tilde{q}\,Lip(q)$$

(using $\dot{\hat{q}} \leftarrow \alpha E_{tot}x_2 ml^2$)

$$\leq -kE_{tot}^2x_2^2 + \frac{1}{\alpha}\tilde{q}\,Lip(q)$$

$$x \longrightarrow \{kE_{tot}^2x_2^2 \leq \frac{1}{\alpha}\tilde{q}_0 Lip(q)\}, \quad \tilde{q} \leq \tilde{q}(0) = \tilde{q}_0$$

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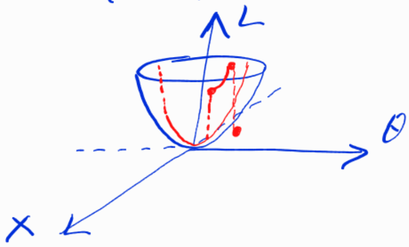
$$L(x,\theta), \quad \dot{L} \leq -k\|x\|^2, \quad \|\theta\| ?$$

$$z = \begin{pmatrix} x \\ \theta \end{pmatrix}, \quad z \longrightarrow 0$$

$\dot{L} \leq 0 \Rightarrow L$ is not growing

$\Rightarrow \|\theta(t)\|$ is always s.t. $L(x,\theta) \leq L_0$



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2} u \; (+q)$$

Say, malfunctioning controller (i.e. it has an actuator fault)

$$u \longmapsto u + d$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2}(u+d)$$
$$= -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2}u + \underbrace{\frac{1}{m\ell^2}d}_{=:q}$$

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$$\dot{x} = f(x,u)$$
$$\dot{x} = f(x, u+d) \longmapsto \dot{x} = f(x,u) + q$$
$$\dot{x} = f(x,u) + \underbrace{\langle \nabla f, d\rangle + \mathcal{O}(d^2)}_{q}$$

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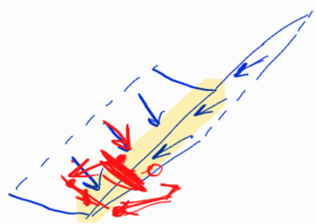
$$\dot{x} = f(x,u), \quad u \leftarrow \rho(x)$$
$$u \leftarrow \rho(x+e)$$



CLF: $\mathcal{D}_{f(x,u)} L < 0$

Smooth case:

$$\langle \nabla L, f(x, \rho(x))\rangle \leq -\alpha_3(\|x\|)$$

Meas. error:

$$\langle \nabla L, f(x, \rho(x+e))\rangle \leq$$

$$\langle \nabla L, f(x, \rho(x)) \rangle + \|\nabla L\| \cdot \mathrm{Lip}(f) \cdot \mathrm{Lip}(\rho) \cdot \bar{e}$$

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$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l} \sin x_1 + \frac{1}{ml^2} u$$

Model:

$$\underbrace{\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 \\ \dot{\hat{x}}_2 &= w_1 \sin \hat{x}_1 + w_2 u \end{aligned}}_{\hat{f}^w(\hat{x}, u)} \quad , \quad w \text{ — weights}$$

Model with filter

$$\dot{\hat{x}} = \hat{f}^w(\hat{x}, u) + C\tilde{x} \quad , \quad \tilde{x} = \hat{x} - x$$

$$u \leftarrow \rho(\hat{x})$$

We know $\exists\, w^*$ s.t. $\hat{f}^{w^*}(x, u) \equiv f(x, u)$

But if we don't know the structure of the system equations, we could use an RNN

$$\dot{\hat{x}} = \hat{f}^w(\hat{x}, u)$$

$$\Rightarrow \quad \hat{f}^{w^*}(x, u) + e(x, u) \equiv f(x, u)$$