Lyapunov-based controller design.
Sliding-mode control.

$$\dot{x} = f(x,u) \qquad u \leftarrow \rho(x) \qquad \longmapsto \qquad \dot{x} = f(x, \rho(x)) = f'(x)$$

$$\dot{x} = f(x) \quad , \quad x_e = 0$$

$$x_e \neq 0 \Rightarrow x_{new} := x - x_e$$

$$x \in \mathbb{R}^n$$

$$\|x(t)\| \longrightarrow 0 \qquad \sqrt{x^{(1)^2} + x^{(2)^2} + \dots}$$

LF $\quad L$

1. $\alpha_1(\|x\|) \leq L(x) \leq \alpha_2(\|x\|), \ \forall x$

$\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ Latex: \mathcal K

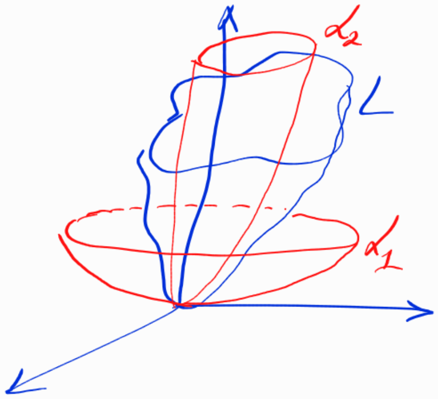
„kappa-infinity" class

$$\alpha \in \mathcal{K}_\infty \Rightarrow \alpha : \mathbb{R} \rightarrow \mathbb{R}$$

$\alpha(0) = 0$

α strictly increasing

$\lim_{x \to \infty} \alpha(x) = \infty$



\mathcal D   2. $\forall x \quad \mathcal{D}L(x) \leq -\alpha_3(\|x\|)$

$\alpha_3 \in \mathcal{K}_\infty$

Some facts

1. $\alpha_1(\|x\|) \leq L(x) \leq \alpha_2(\|x\|)$

$\Rightarrow \alpha_2^{-1}(L) \leq \|x\| \leq \alpha_1^{-1}(L), \quad L \in \mathbb{R}$

Start at x_0, $\quad L_0 := L(x_0)$

2. $\mathcal{D}L \leq 0 \Rightarrow \forall t \quad L(x(t)) \leq L_0$

$\Rightarrow \quad \forall t \quad \|x(t)\| \leq \alpha_1^{-1}(L_0)$

$\varepsilon - \delta$

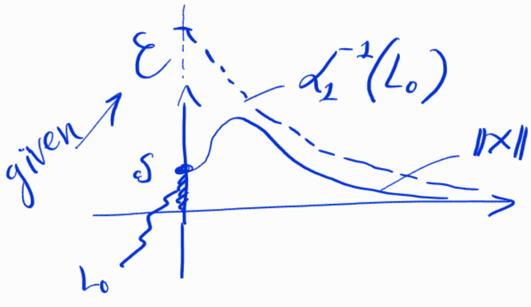
Recall: $\quad \forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $\|x_0\| \leq \delta \Rightarrow$
$\forall t \geq 0 \quad \|x(t)\| \leq \varepsilon$

Now, using our: L

Given $\varepsilon > 0$, define $\delta := \alpha_2^{-1}(\alpha_1(\varepsilon))$

So, if $\|x_0\| \leq \alpha_2^{-1}(\alpha_1(\varepsilon))$

$\Rightarrow \quad L_0 \leq \alpha_2(\|x_0\|) \leq \alpha_1(\varepsilon)$



$\Longrightarrow \forall t \geq 0 \ L(x(t)) \leq L_0 \Longleftarrow$

Apply α_1^{-1} both sides: $\|x(t)\| \leq \alpha_1^{-1}(L_0) \leq \varepsilon$

Student:

$\|x(t)\| \leq \alpha_1^{-1}(\underbrace{L(x(t)}_{\leq L_0}) \leq \alpha_1^{-1}(L(x(0)) \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|)) \leq$

$\leq \alpha_1^{-1}(\alpha_2(\varepsilon)) = \delta$

<u>Decay</u> $\quad \dot{L} \leq -\alpha_3(\|x\|)$!

$$\Longrightarrow \quad \dot{L} \leq -\alpha_3(\alpha_2(L_t))$$

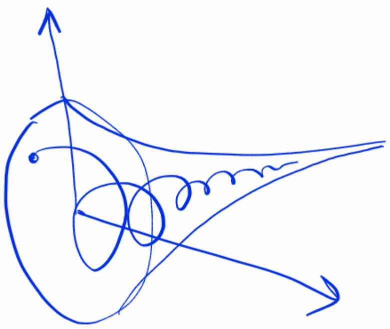
Consider an IVP: $\quad \dot{\ell} = -\alpha_3(\alpha_2(\ell))$

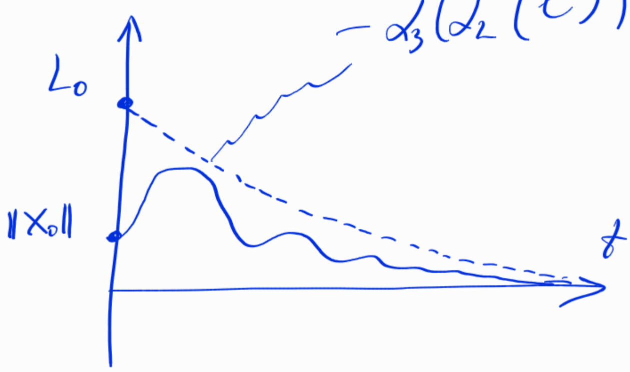
$\qquad\qquad\qquad\qquad\quad \ell(0) = L_0$

By a comparison principle,

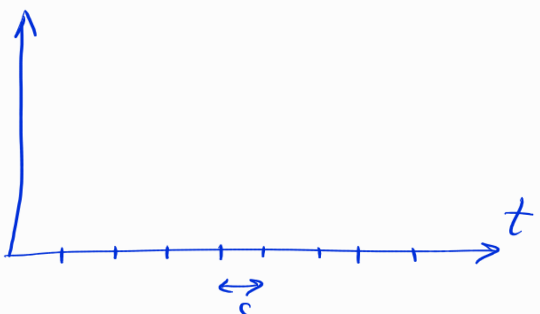
$$\forall t \quad L_t \leq \ell(t)$$

$$\Longrightarrow \forall t \ \|x(t)\| \leq \alpha_1^{-1}(\ell(t))$$





Imagine we analize L_t in δ-steps



$t_1 := (k+1)\delta, \quad k \in \mathbb{Z}_{\geq 0}$

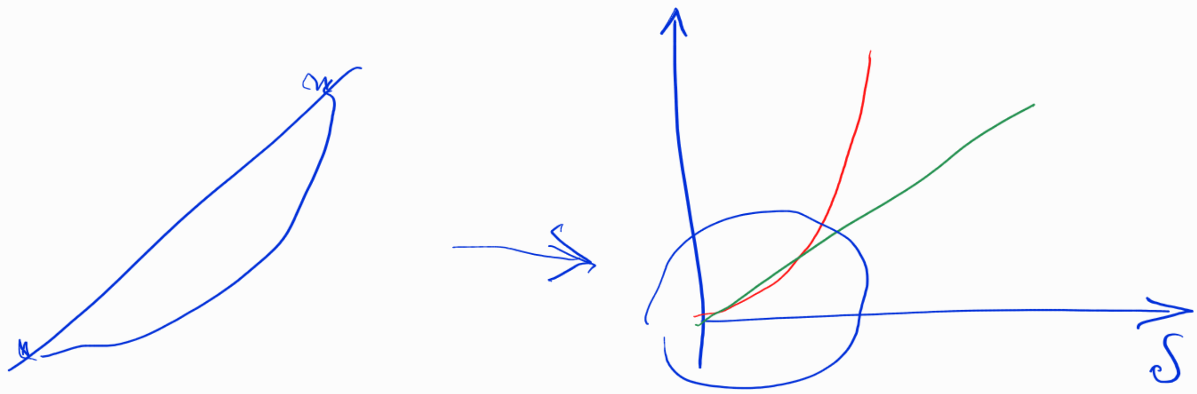
$t_0 := k\delta$

What happens from one step to another?

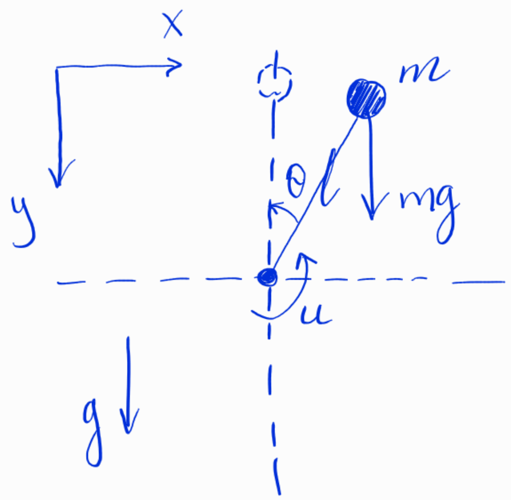
$$L_{t_1} - L_{t_0} = \delta \langle \nabla L_{t_0}, f(x(t_0)) \rangle + \mathcal{O}(\delta^2)$$

$$\leq \delta \dot{L}_{t_0} + \mathcal{O}(\delta^2)$$

$$\leq -\delta \alpha_3(\|x(t_0)\|) + \mathcal{O}(\delta^2)$$



Example: energy-based control of an inverted pendulum



$$J = ml^2$$

$$J\ddot{\theta} = -mgl \sin\theta + u \Rightarrow$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{u}{ml^2}$$

State vector: $x := \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$

So,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{u}{ml^2}$$

Energy:

$$\left| \frac{mv^2}{2} \right. \qquad V = \dot{\Theta}\ell \qquad \frac{m\dot{\Theta}\ell}{2}$$

$$E_{tot} = E_{pot} + E_{kin}$$

$$= mg(\ell - \ell \cdot \cos\Theta) + \frac{m(\dot{\Theta}\ell)^2}{2}$$

$$= mg(\ell - \ell\cos x_1) + \frac{m(x_2\ell)^2}{2}$$

$$= mg\ell(1 - \cos x_1) + \frac{1}{2}m\ell^2 x_2^2$$

Target energy: $E^* = 0$

Candidate Lyapunov function:

$$L := \frac{1}{2} E_{tot}^2$$

$$\Rightarrow \dot{L} = E_{tot}\dot{E}_{tot}$$

$$= E_{tot} x_2 u$$

$\Rightarrow$ a control policy choice could be

$$u \leftarrow -k x_2 E_{tot}, \quad k > 0 \text{ control gain}$$

Then, $\dot{L} = -k x_2^2 E_{tot}^2$