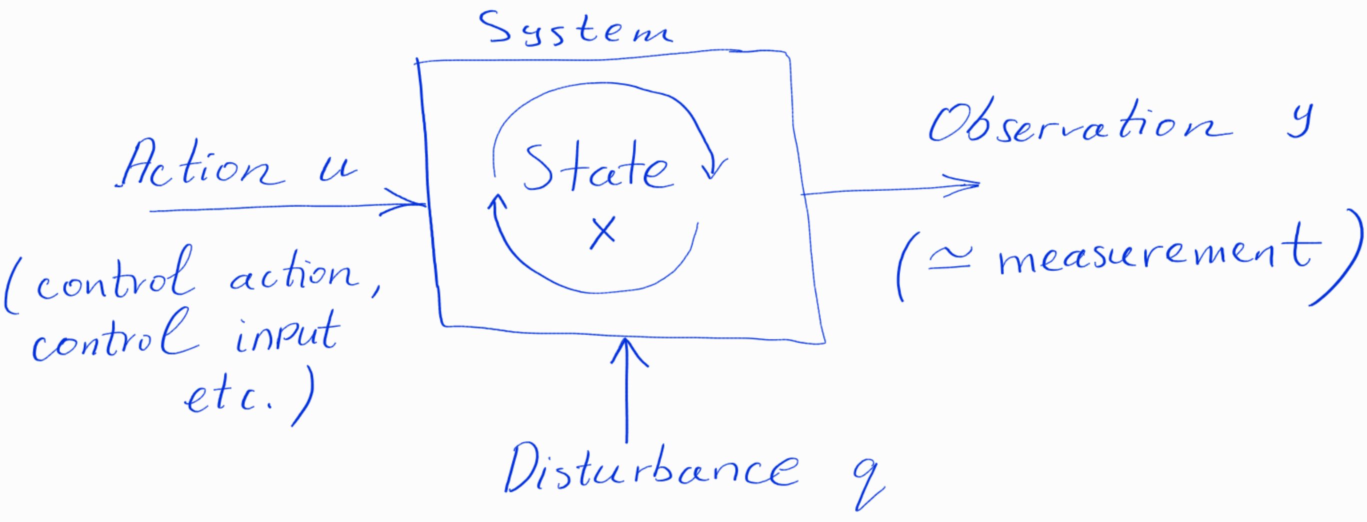
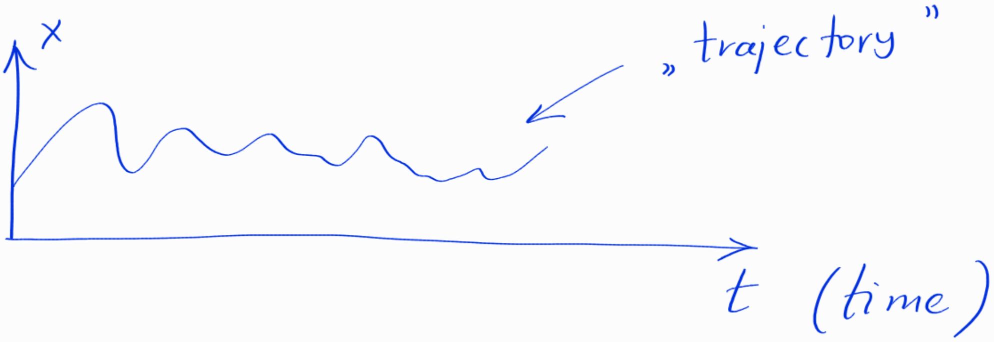
Introduction to nonlinear dynamical systems.

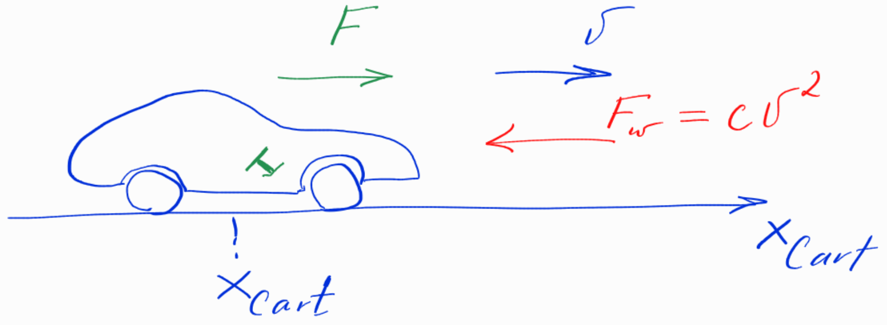


$$x_{next} \leftarrow f(x_{curr}, u, q)$$

$$y \leftarrow h(x_{curr})$$



$x \in \mathbb{R}^n, n \in \mathbb{N}$



$x = v$

Flow: $\Phi(\underset{\text{init. state}}{x_0}, u, q, t) = x(t)$

Diff. equation: $\dot{x} = \frac{1}{m}(F - F_w)$

$$= \frac{1}{m}\left(F - cx^2\right)$$

$$= \frac{1}{m}\left(\underset{\overset{\uparrow}{\text{controller gain}}}{k}\left(\underset{\overset{\uparrow}{\text{desired speed (say 5 m/s)}}}{x^*} - x\right) - cx^2\right)$$

$k \in \mathbb{Z}_{\geq 0}$

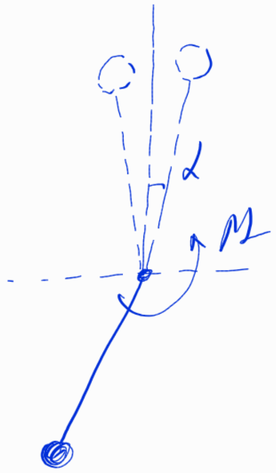
x_k – state at step k

$$x_{k+1} = f(x_k, u_k)$$

$$x_{k+1} = A x_k + B u_k , \quad x_k \in \mathbb{R}^n, \; u_k \in \mathbb{R}^m$$

matrices

$$x_{k+1} = x_k + \Delta t \frac{1}{m}\left(F_k - c \underset{\text{nonlinearity}}{x_k^2}\right)$$



$\sin \alpha \approx \alpha$

Recap on linear systems:

$$\dot{x} = Ax + Bu , \quad y = Cx , \quad x(0) = x_0$$

Flow:
$$x(t) = \underbrace{e}_{\text{matrix exponential}}{}^{At} x_0 + e^{At} \int_0^t e^{-A\tau} B u(\tau)\, d\tau$$

Free motion
(solution)

Forced motion

$$e^{At} = I + At + \frac{At^2}{2!} + \dots$$

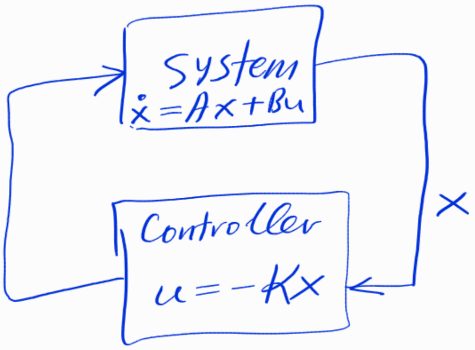
Goal: x → 0

$$\dot{x} = Ax + Bu$$

$$u := -\underset{\substack{\uparrow \\ \text{gain matrix}}}{K}x$$

(simple linear feedback)

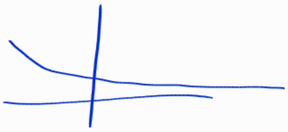
$$\Longrightarrow \quad \dot{x} = Ax - BKx$$
$$= \underbrace{(A - BK)}_{A'}x$$



$$\dot{x} = A'x$$

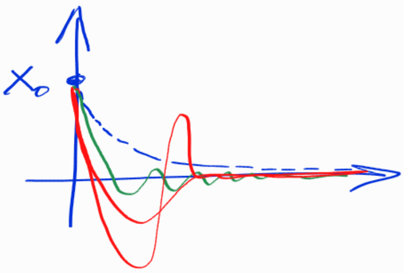
$$A' = \underset{\substack{\uparrow \\ \text{Matrix of} \\ \text{eigenvectors}}}{V} \; \underset{\substack{\nwarrow \\ \text{Matrix of} \\ \text{eigenvalues} \\ \text{diag}\{\lambda_1, \dots, \lambda_n\}}}{\Lambda} \; V^{-1}, \quad \text{if diagonalizable}$$

$$x^{(\ell)} \propto e^{\lambda_\ell t}$$



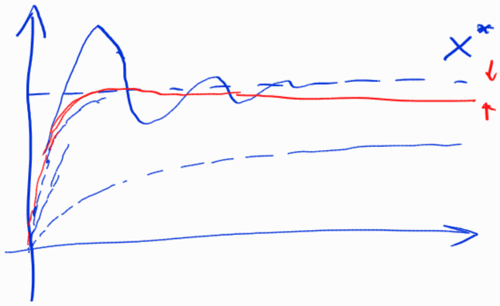
$$t^{\ell} \underset{\cos}{\sin}(Im(\lambda_i)) \cdot e^{Re(\lambda_i)t}$$

Pole placement



PID :

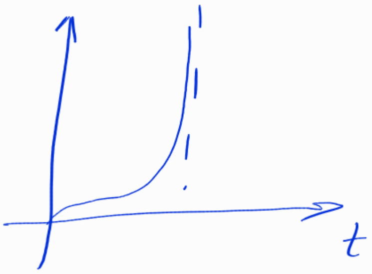
$$u = k_p(x^* - x) + k_d \frac{d}{dt}(x^* - x) + k_i \int_0^t (x^* - x)\,dt$$

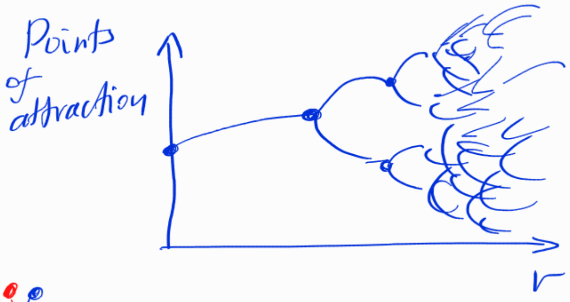


Phenomena in nonlinear systems:

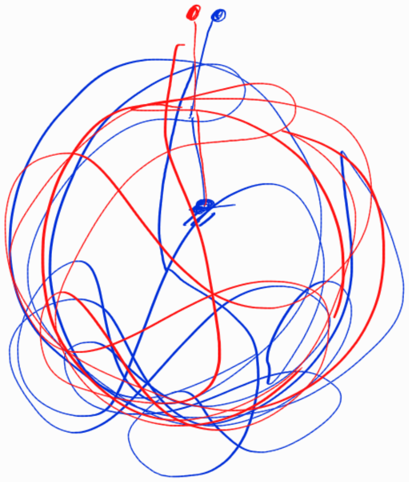
- solution blow-up
- bifurcation

$$x_{k+1} = x_k(1 - \underset{\nearrow}{r} x_k)$$



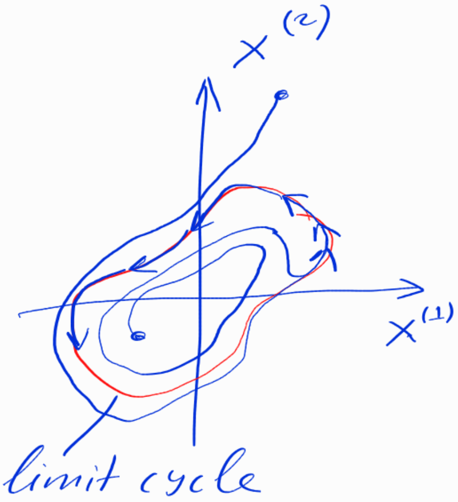


- chaos

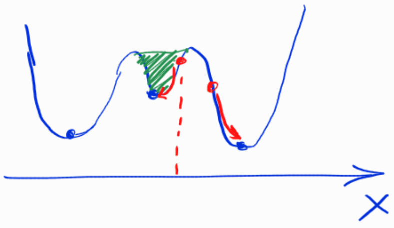


- strange attractors

- Basins of attraction



Van der Pol oscillators



Lorentz attractors

Types of control:

- simple feedback (PID, $-Kx$)
- sophisticated (complex) feedback
- adaptive control
- robust control
- optimal control
- [dynamical, time-varying control]

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