Backstepping

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2}u$$

} Idealized description

Energy-based (nominal) controller:

$$u \longleftarrow -k E_{tot} x_2, \quad k>0$$

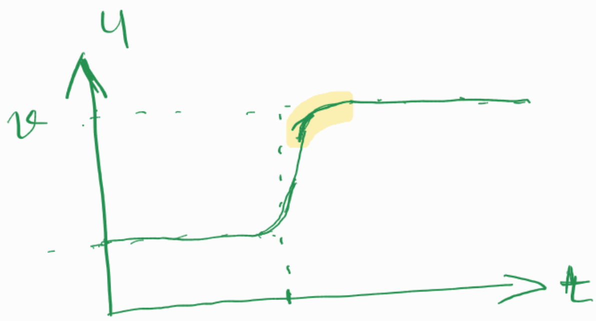
$$E_{tot} = mg\ell(1-\cos x_1) + \frac{1}{2}m\ell^2 x_2^2$$

$$L := \frac{1}{2}E_{tot}^2$$

$$\dot{L} = E_{tot}\, x_2\, u$$

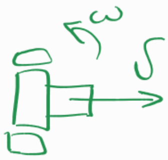
A more realistic description would include the dynamics of the actuator (say, an electr. motor):

$$\dot{x}_1 = x_2 \qquad (x_1 \text{ angle, } x_2 \text{ angular speed})$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2}u$$
$$\dot{u} = \frac{1}{\tau}(v-u)$$



$$\dot{u} = v \; ?$$

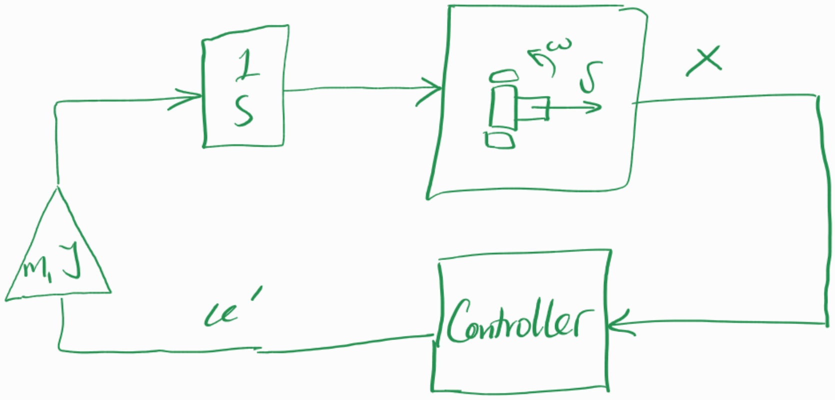
Example:



$$\dot{x}_c = v\cos\alpha_c \qquad u = (v, \omega)$$
$$\dot{y}_c = v\sin\alpha_c$$
$$\dot{\alpha}_c = \omega$$

If mass m, moment of inertia J,
$$\dot{v} = \frac{1}{m}F, \qquad F \text{ - force}, \; M \text{ - torque}$$
$$\dot{\omega} = \frac{1}{J}M$$

$$u' = (F, M)$$



$$\dot x_1 = x_2$$

$$\dot x_2 = -\frac{g}{\ell}\sin x_1 + \frac{1}{m\ell^2}u \quad \xrightarrow{?} \quad \begin{aligned}\dot x &= f(x,u)\\ \dot u &= v\end{aligned}$$

$$\dot u = \frac{1}{\tau}(v - u)$$

new control input

$$v \leftarrow \frac{1}{\tau}(v' - u)$$

$$v' = \tau v + u$$

Suppose we have a system

$$\dot x = f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

Suppose there is a policy ρ with a corresponding Lyapunov function L:

$$\mathcal{D}_{f(x)+g(x)\rho(x)} L(x) < 0 \quad \Rightarrow$$

$$\underbrace{\langle \nabla L, f(x) \rangle}_{\underbrace{\nabla L f(x)}_{\text{Lie derivative } \mathcal{L}_f L}} + \langle \nabla L, g(x)\rho(x) \rangle < 0$$

Now:
$$\dot x = f(x) + g(x)u$$
$$\dot u = v \leftarrow \text{new control input}$$

What to do?

$$L_c := L + \frac{1}{2}\|u - \rho(x)\|^2$$

Let's work it out

$$\dot{L}_c = \dot{L} + (u - \rho(x))^T(v - \nabla\rho\,\dot{x})$$

$$= \mathcal{L}_f L + \mathcal{L}_g L\, u + (u - \rho(x))^T(v - \mathcal{L}_f \rho - \mathcal{L}_g \rho\, u)$$

Recall: $\mathcal{L}_f L + \mathcal{L}_g L\, \rho(x) < 0$

Let's figure out the new policy:

$$v \leftarrow -K(u - \rho(x)),\ K > 0$$

that'd give a term $-K\|u - \rho(x)\|^2$

Next, cancel out the bad stuff like this:

$$v \leftarrow -K(u - \rho(x)) + \mathcal{L}_f \rho + \mathcal{L}_g \rho\, u \qquad (*)$$

Can we do even better?

So far, under $(*)$, we have:

$$\dot{L}_c = \mathcal{L}_f L + \mathcal{L}_g L\, u - K\|u - \rho(x)\|^2$$

Something missing

Want: $v \leftarrow -K(u - \rho(x)) + \mathcal{L}_f \rho + \mathcal{L}_g \rho\, u + \bullet$

???

$$\dot{L}_c = \mathcal{L}_f L + \mathcal{L}_g L\, u - K\|u - \rho(x)\|^2 + (u - \rho(x))^T \bullet$$

Want to recover this:

$$\mathcal{L}_f L + \mathcal{L}_g L\, \rho(x)$$

$$\bullet \leftarrow -\mathcal{L}_g L$$

Then, $(u-\rho(x))^T \quad = -\mathcal{L}_g L u + \mathcal{L}_g L \rho(x)$

Backstepping policy reads:

$$v \leftarrow -K(u-\rho(x)) - \mathcal{L}_g L + \mathcal{L}_{f+gu}\rho$$