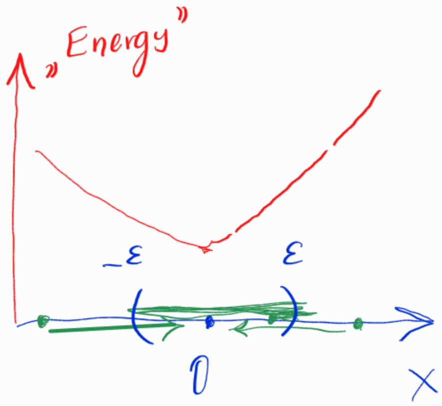
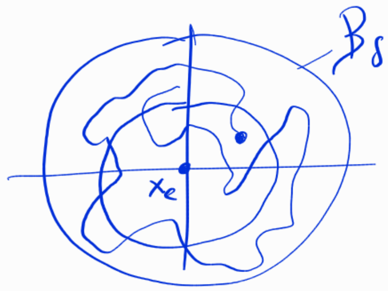
Fundamentals of stability theory.



An equilibrium $x_e = 0$ is called stable if

$$\forall \varepsilon > 0 \quad x(0) \in B_\varepsilon \quad \exists \delta > 0$$

$$\forall t \geqslant 0 \quad x(t) \in B_\delta$$



$$\delta = \delta(\varepsilon)$$

and asymptotically stable if also

$$\lim_{t\to\infty} x(t) = 0$$

A (strict) Lyapunov function for a system

$$\dot{x} = f(x)$$

is (assuming $x_e = 0$) a positive-definite function $L : \mathbb{X} \longrightarrow \mathbb{R}$ (to mean: $L(0)=0$,

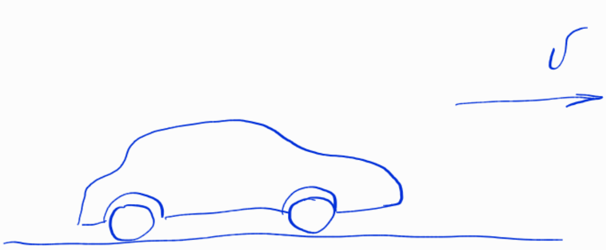
[\mathbb X]
state space

$L(x) > 0 \Leftrightarrow x \neq 0$) and decreasing

$$\dot{L} < 0$$

$$\left(\nabla_x L \cdot \dot{x} < 0 \Rightarrow \langle \nabla L, f(x) \rangle < 0 \right)$$

Recalling cruise control example.



State $x = v$

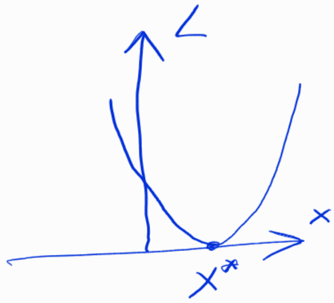
$$\dot{x} = \frac{1}{m}(F - F_w)$$
$$= \frac{1}{m}(F - cx^2)$$

Control $F \leftarrow k(x^* - x)$

$$\Rightarrow \dot{x} = \frac{1}{m}\left(k(x^* - x) - cx^2\right)$$

Lyapunov function candidate

$$L = \frac{1}{2}(x - x^*)^2$$



Need $\dot{L} < 0$

$$\dot{L} = \cancel{2} \cdot \cancel{\frac{1}{2}} (x - x^*)'(x - x^*) = (x - x^*)\underbrace{\dot{x}}_{f(x)} = (x - x^*)f(x)$$
$$= (x - x^*)f(x)$$
$$= (x - x^*)\frac{1}{m}\left(k(x^* - x) - cx^2\right)$$
$$= \frac{1}{m}(x - x^*)\left(k(x^* - x) - cx^2\right)$$
$$= -\frac{k}{m}(x - x^*)^2 - \frac{c}{m}x^2(x - x^*) \overset{?}{<} 0$$

Control Lyapunov function (CLF)

System $\dot{x} = f(x, u)$

1. L pos.-def.
2. $\forall x \quad \inf_u \langle \nabla L, f(x, u) \rangle < 0$

$$\dot{x} = \frac{1}{m}\left(u - cx^2\right)$$

$$L = \frac{1}{2}\left(x - x^*\right)^2$$ is (locally) a CLF

Sontag's formula!

(If you got a smooth CLF!)