Advanced Derivatives
Problem Set 10

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Exercise 1

Suppose the short rate follows the CIR model:

$$dr = k(\theta - r_t)dt + \sigma_r \sqrt{r_t} dW_t,$$

The PDE for a derivative V is given by:

$$\frac{\partial V(t,r)}{\partial t} + \frac{\partial V(t,r)}{\partial r} k(\theta - r_t) + \frac{1}{2}\frac{\partial^2 V(t,r)}{\partial r^2}\sigma_r^2 r_t = r_t V(t,r) \tag{1}$$

If $V \equiv P(t,T)$ is a zero coupon bond, this PDE takes the folowing boundary condition:

$$P(T,T) = 1.$$

We try to solve the PDE using the ansatz:

$$V(\tau, r_t) = A(\tau)e^{-B(\tau)r_t}$$

where $\tau \equiv T - t$. Introducing this Ansatz in (1), we obtain:

$$\begin{cases} V\left(B'(\tau)r_t - \frac{A'(\tau)}{A(\tau)}\right) - BVk(\theta - r_t) + \frac{1}{2}B^2 V \sigma_r^2 r_t = r_t V \\ V(0) = 1 \end{cases} \tag{2}$$

where prime indices denotes derivative with respect to τ. We used the fact that

$$\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau} = -A'(\tau)e^{-B(\tau)r_t} + B'(\tau)r_t V = V\left(B'(\tau)r_t - \frac{A'(\tau)}{A(\tau)}\right)$$

$$\frac{\partial V}{\partial r} = -BV$$

$$\frac{\partial^2 V}{\partial r^2} = B^2 V$$

Using the fact that $V \geq 0$ and can be simplified away, (2) becomes:

$$\begin{cases} r_t \left(B'(\tau) + B(\tau)k + \frac{1}{2}B(\tau)^2\sigma_r^2 - 1 \right) - \frac{A'(\tau)}{A(\tau)} - Bk\theta = 0 \\ V(0) = 1 = A(0)e^{-B(0)r_0} \end{cases} \tag{3}$$

From the boundary condition, we deduce $A(0) = 1$ and $B(0) = 0$. Furthermore by separation of variable (first equation must be valid $\forall r_t$), we obtain the following system of ODE:

$$\begin{cases} B'(\tau) + B(\tau)k + \frac{1}{2}B(\tau)^2\sigma_r^2 - 1 = 0, \quad B(0) = 0, \\ -\frac{A'(\tau)}{A(\tau)} - B(\tau)k\theta = 0, \quad A(0) = 1. \end{cases} \tag{4}$$

First equation of (4), is a Riccati equation with explicit solution given by:

$$B(\tau) = \frac{2(e^{\gamma\tau} - 1)}{(\gamma + k)(e^{\gamma\tau} - 1) + 2\gamma},$$

where $\gamma = \sqrt{k^2 + 2\sigma_r^2}$. Indeed, by setting $H(\tau) \equiv B(\tau)^{-1}$ for $B(\cdot)$ satisfying the first ODE in (4), one has

$$H'(\tau) = -\frac{1}{B(\tau)^2}\left(-kB(\tau) - \frac{1}{2}B(\tau)^2\sigma_r^2 + 1\right) = -H(\tau)^2 + kH(\tau) + \frac{1}{2}\sigma_r^2. \tag{5}$$

with $H(0) = \infty$. It suffices to verify that

$$H(\tau) = \left(\frac{2(e^{\gamma\tau} - 1)}{(\gamma + k)(e^{\gamma\tau} - 1) + 2\gamma}\right)^{-1} = \frac{1}{g(\tau)} + \frac{\gamma + k}{2}, \quad g(\tau) \equiv \frac{1}{\gamma}\left(e^{\gamma\tau} - 1\right),$$

solves equation (5). First of all, the boundary condition clearly holds true for $B(\cdot)$ and $H(\cdot)$ respectively. Moreover, observing that $g'(\tau) = \gamma g(\tau) + 1$, the derivative of the claimed solution for (5) reads

$$H'(\tau) = -\frac{g'(\tau)}{g(\tau)^2} = -\left(\frac{\gamma}{g(\tau)} + \frac{1}{g(\tau)^2}\right) = -\left(H(\tau) - \frac{k}{2}\right)^2 + \frac{\gamma^2}{4}. \tag{6}$$

On the other hand, the transformed Riccati equation (5) can be rearranged as

$$\begin{aligned} H'(\tau) &= -H(\tau)^2 + kH(\tau) + \frac{1}{2}\sigma_r^2 \\ &= -H(\tau)^2 + kH(\tau) - \frac{k^2}{4} + \frac{\gamma^2}{4} \quad \left(\sigma_r^2 = \frac{\gamma^2 - k^2}{2}\right) \\ &= -\left(H(\tau) - \frac{k}{2}\right)^2 + \frac{\gamma^2}{4}, \end{aligned}$$

which is precisely (6) and the result follows.

We can thus introduce $B(\cdot)$ in the other ODE and compute $A(\cdot)$ by integrating:

$$\int_0^\tau \frac{dA(s)}{A(s)} = -k\theta \int_0^\tau B(s)ds = \ldots = \left(\frac{2\gamma e^{(\gamma+k)\frac{\tau}{2}}}{(\gamma + k)(e^{\gamma\tau} - 1) + 2\gamma}\right)^{\frac{2k\theta}{\sigma_r^2}}.$$

Exercise 2

Suppose the short rate follows the Vasicek model:

$$dr = k(\theta - r_t)dt + \sigma dW_t,$$

The PDE for a derivative V is given by:

$$\frac{\partial V(t,r)}{\partial t} + \frac{\partial V(t,r)}{\partial r}k(\theta - r_t) + \frac{1}{2}\frac{\partial^2 V(t,r)}{\partial r^2}\sigma^2 = r_t V(t,r) \tag{7}$$

If $V \equiv P(t,T)$ is a zero coupon bond, this PDE takes the folowing boundary condition:

$$P(T,T) = 1.$$

We try to solve the PDE using the ansatz:

$$V(\tau, r_t) = A(\tau)e^{-B(\tau)r_t}$$

where $\tau \equiv T - t$. Introducing this guess solution in (7), we obtain:

$$\begin{cases} V\left(B'(\tau)r_t - \frac{A'(\tau)}{A(\tau)}\right) - B(\tau)Vk(\theta - r_t) + \frac{1}{2}B(\tau)^2V\sigma^2 = r_t V \\ V(0) = 1 \end{cases} \tag{8}$$

where prime indices denotes derivative with respect to τ. We used the fact that

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\frac{\partial V}{\partial \tau} = -A'(\tau)e^{-B(\tau)r_t} + B'(\tau)r_t V = V\left(B'(\tau)r_t - \frac{A'(\tau)}{A(\tau)}\right) \\ \frac{\partial V}{\partial r} &= -BV \\ \frac{\partial^2 V}{\partial r^2} &= B^2 V \end{aligned}$$

Using the fact that $V \geq 0$ and can be simplified away, (8) becomes:

$$\begin{cases} r_t\left(B'(\tau) + B(\tau)k - 1\right) + \frac{1}{2}B(\tau)^2\sigma^2 - \frac{A'(\tau)}{A(\tau)} - Bk\theta = 0 \\ V(0) = 1 = A(0)e^{-B(0)r_0} \end{cases} \tag{9}$$

From the boundary condition, we deduce $A(0) = 1$ and $B(0) = 0$. Furthermore by separation of variable (first equation must be valid $\forall r_t$), we obtain the following system of ODE:

$$\begin{cases} B'(\tau) + B(\tau)k - 1 = 0, \quad B(0) = 0 \\ -\frac{A'(\tau)}{A(\tau)} - B(\tau)k\theta + \frac{1}{2}B(\tau)^2\sigma^2 = 0, \quad A(0) = 1 \end{cases} \tag{10}$$

We start by solving the first ODE:

$$B'(\tau) + B(\tau)k - 1 = 0$$

A solution of the homogeneous equation $B'(\tau) + Bk - 1 = 0$ is:

$$B(\tau) = Ce^{-k\tau}$$

where C is a constant. A particular solution (if we suppose $B(\tau) = B$) is given by:

$$Bk = 1 \quad \Leftrightarrow B = \frac{1}{k}$$

Therefore the general solution has the form:

$$B(\tau) = Ce^{-k\tau} + \frac{1}{k}$$

The constant C can be determined using the initial condition:

$$B(0) = 0 = C + \frac{1}{k} \quad \Leftrightarrow C = -\frac{1}{k}$$

And the solution is given by

$$B(\tau) = \frac{1}{k}(1 - e^{-k\tau}) \tag{11}$$

Then introducing this solution for $B(\tau)$ in the other ODE, we can solve for $A(\tau)$ by integrating:

$$\begin{aligned}
&\int_0^\tau \frac{dA(s)}{A(s)} = -k\theta \int_0^\tau B(s)ds + \frac{1}{2}\sigma^2 \int_0^\tau B(s)^2 ds \\
&\Leftrightarrow \log(A(\tau)) = -\theta\left(\tau + \frac{e^{-ks}}{k}\Big|_0^\tau\right) + \frac{1}{2k^2}\sigma^2\left(\tau + 2\frac{e^{-ks}}{k}\Big|_0^\tau - \frac{e^{-2ks}}{2k}\Big|_0^\tau\right) \\
&\Leftrightarrow A(\tau) = \exp\left\{-\theta\left[\tau + \underbrace{\frac{e^{-k\tau}}{k} - \frac{1}{k}}_{-B(\tau)}\right] + \frac{\sigma^2}{2k^2}\left[\tau + \underbrace{\frac{e^{-k\tau}}{k} - \frac{1}{k}}_{=-B(\tau)} + \underbrace{\frac{2e^{-k\tau}}{2k} - \frac{e^{-2k\tau}}{2k} - \frac{1}{2k}}_{=-kB^2(\tau)}\right]\right\} \\
&\Leftrightarrow A(\tau) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)(B(\tau) - \tau) - \frac{\sigma^2}{4k}B(\tau)^2\right\}
\end{aligned} \tag{12}$$

It is possible to check that the solution $P(r_t, \tau) = A(\tau)e^{-B(\tau)r_t}$ with $A(\tau)$ and $B(\tau)$ respectively given by (12) and (11), verify (7) by plugging it into the PDE.

Using the solution $P(r_t, \tau) = A(\tau)e^{B(\tau)r_t}$ we are now able to derive the dynamics of $P(r_t, \tau)$ under the Q-measure from Ito's lemma:

$$\begin{aligned}
dP &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}dt \\
&= \underbrace{\frac{\partial P}{\partial r}\sigma}_{-\sigma B(\tau)P} dW_t^Q + \left(\underbrace{\frac{\partial P}{\partial t} + K(\theta - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}}_{rP}\right)dt \\
&= \sigma_P^T P dW_t^Q + rPdt
\end{aligned} \tag{13}$$

where $\sigma_P^T \equiv -\sigma B(\tau)$. Notice that if we define the money market as $dB = Brdt$ (hence $B(t) = e^{-\int_t^T r_s ds}$ with $B(0) = 1$), we find that $\frac{P(t,T)}{B(t)}$ is a martingale under the risk neutral $\mathbb{Q}$-measure. Indeed,

$$d\left(\frac{P(t,T)}{B(t)}\right) = \frac{dP}{B} - \frac{PdB}{B^2} = \frac{P}{B}\sigma_P^T dW_t^Q + r\frac{P}{B}dt - r\frac{PB}{B^2}dt = \frac{P}{B}\sigma_P^T dW_t^Q$$

and,

$$\frac{P(T,T)}{B(T)}\frac{B(0)}{P(0,T)} = e^{-\int_0^T(\sigma_P^T)^2 ds + \int_0^T \sigma_P^T dW_s^Q} \equiv Z_T^T$$

$Z_t^T = E_t^Q[Z_T^T]$ is a strictly positive Q-martingale satisfying $Z_0 = 1$. Hence, define an equivalent measure $Q^T \sim Q$ such that at time T, $\frac{dQ^T}{dQ} \equiv Z_T^T$. By Girsanov theorem:

$$dW_t^Q = dW_t^{Q^T} + \sigma_P^T dt \tag{14}$$

i) Using Bayes' theorem, the price of a call option with maturity T and stike K on a zero-coupon bond with expiration $T+\tau$ is given by:

$$\begin{aligned}
ZBC(t,T,T+\tau,K) &= E_t^Q[e^{-\int_t^T r_s ds}\left(P(T,T+\tau)-K\right)^+] \\
&= E_t^Q\left[\frac{B(t)}{B(T)}P(T,T+\tau)\mathbb{1}_{P(T,T+\tau)>K}\right] - KE_t^Q\left[\frac{B(t)}{B(T)}\mathbb{1}_{P(T,T+\tau)>K}\right] \\
&= E_t^{Q^{T+\tau}}\left[\frac{B(t)}{B(T)}P(T,T+\tau)\mathbb{1}_{P(T,T+\tau)>K}\frac{Z_t^{T+\tau}}{Z_T^{T+\tau}}\right] - KE_t^{Q^T}\left[\frac{B(t)}{B(T)}\mathbb{1}_{P(T,T+\tau)>K}\frac{Z_t^T}{Z_T^T}\right] \\
&= E_t^{Q^{T+\tau}}\left[\frac{B(t)}{B(T)}P(T,T+\tau)\mathbb{1}_{P(T,T+\tau)>K}\frac{P(t,T+\tau)B(T)}{B(t)P(T,T+\tau)}\right] \qquad (15) \\
&- KE_t^{Q^T}\left[\frac{B(t)}{B(T)}\mathbb{1}_{P(T,T+\tau)>K}\frac{P(t,T)B(T)}{B(t)P(T,T)}\right] \\
&= P(t,T+\tau)E_t^{Q^{T+\tau}}\left[\mathbb{1}_{P(T,T+\tau)>K}\right] - KP(t,T)E_t^{Q^T}\left[\mathbb{1}_{P(T,T+\tau)>K}\right] \qquad (16)
\end{aligned}$$

ii) Thanks to the Girsanov transformation (14), we can describe the dynamic of the zero coupon bond price $P(t,T+\tau)$ (13) under the different measures $\mathbb{Q}^T$ and $\mathbb{Q}^{T+\tau}$:

$$\begin{aligned}
\frac{dP(t,T+\tau)}{P(t,T+\tau)} = \sigma_P^{T+\tau}dW_t^Q + rdt &= \begin{cases}\sigma_P^{T+\tau}(dW_t^{Q^T} + \sigma_P^T dt) + rdt \\ \sigma_P^{T+\tau}(dW_t^{Q^{T+\tau}} + \sigma_P^{T+\tau}dt) + rdt\end{cases} \\
&= \begin{cases}\sigma_P^{T+\tau}dW_t^{Q^T} + (r + \sigma_P^{T+\tau}\sigma_P^T)dt \\ \sigma_P^{T+\tau}dW_t^{Q^{T+\tau}} + (r + (\sigma_P^{T+\tau})^2)dt\end{cases}
\end{aligned}$$

Therefore $P(t,T+\tau)$ follows a geometric Brownian motion,

$$P(T,T+\tau) = P(t,T+\tau)e^{\int_t^T r + \sigma_P^{T+\tau}\sigma_P^u - \frac{(\sigma_P^{T+\tau})^2}{2}ds + \int_t^T \sigma_P^{T+\tau}dW_s} \tag{17}$$

where $u = \mathbb{Q}^T, \mathbb{Q}^{T+\tau}$ and $\sigma_P^{T+\tau} = -\sigma\frac{1}{k}(1 - e^{-k(T+\tau-t)})$.
The argument of the exponential in (17) follows a normal distribution $N(\mu_u, \varsigma^2)$ with mean $\mu_u \equiv \int_t^T r + \sigma_P^{T+\tau}\sigma_P^u - \frac{(\sigma_P^{T+\tau})^2}{2} ds$ and variance $\varsigma^2 \equiv \int_t^T |\sigma_P^{T+\tau}|^2 ds$, we can therefore rewrite (17) as:

$$P(T, T+\tau) = P(t, T+\tau)e^{\varsigma z + \mu_u}$$

where $z \sim N(0, 1)$.
iii) Let us find the solution of (16). First we compute the following expectation for $u = \mathbb{Q}^T, \mathbb{Q}^{T+\tau}$:

$$\begin{aligned}
E_t^{Q^u}\left[\mathbb{1}_{P(T,T+\tau)>K}\right] &= Q^u\left(P(T, T+\tau) > K\right) \\
&= Q^u\left(P(t, T+\tau)e^{\varsigma z+\mu_u} > K\right) \\
&= Q^u\left(\varsigma z + \mu_u > -\log\left(\frac{P(t, T+\tau)}{K}\right)\right) \\
&= Q^u\left(z > \left(-\log\left(\frac{P(t, T+\tau)}{K}\right) - \mu_u\right)\frac{1}{\varsigma}\right) \\
&= \Phi(d_u)
\end{aligned}$$

where Φ is the cumulative distribution function of the standard distribution and,

$$d_u = \left(\log\left(\frac{P(t, T+\tau)}{K}\right) + \mu_u\right)\frac{1}{\varsigma}.$$

Where

$$\mu_u = \int_t^T r + \sigma_P^{T+\tau}\sigma_P^u - \frac{(\sigma_P^{T+\tau})^2}{2} ds$$

with $\sigma_P^u = -\sigma\frac{1}{k}(1 - e^{-k(u-t)})$, $u = T, T+\tau$.

$$\varsigma^2 = \int_t^T |\sigma_P^{T+\tau}|^2 ds = \frac{\sigma^2}{k^2}\left(\tau + \frac{2}{k}(e^{-k(T+\tau-t)} - e^{-k\tau}) - \frac{1}{2k}(e^{-2k(T+\tau-t)} - e^{-2k\tau})\right)$$

Finally we get the solution:

$$ZBC(t, T, T+\tau, K) = P(t, T+\tau)\Phi(d_{T+\tau}) - KP(t, T)\Phi(d_T).$$

Exercise 3

i) Consider a simple call option in the Black-Scholes model. The process for the stock price under the $\mathbb{Q}$-measure is:

$$\frac{dS}{S} = rdt + \sigma dW^{\mathbb{Q}}.$$

The time-t price of a call option with strike K and expiration T is given by:

$$\begin{aligned}
C_t &= E_t^Q[e^{-\int_t^T r_s ds}(S_T-K)^+] \\
&= E_t^Q\left[\frac{B(t)}{B(T)}S_T \mathbb{1}_{S_T>K}\right] - KE_t^Q\left[\frac{B(t)}{B(T)}\mathbb{1}_{S_T>K}\right] \\
&= S_t E_t^Q\left[\underbrace{\frac{S_T}{S_t}\frac{B(t)}{B(T)}}_{=\frac{Z_T}{Z_t}}\mathbb{1}_{S_T>K}\right] - Ke^{-r(T-t)}E_t^Q\left[\mathbb{1}_{S_T>K}\right] \\
&= S_t E_t^R\left[\mathbb{1}_{S_T>K}\right] - Ke^{-r(T-t)}E_t^Q\left[\mathbb{1}_{S_T>K}\right]
\end{aligned}$$

where $Z_T \equiv \frac{dR}{dQ} = \frac{S_T}{S_0}\frac{B_0}{B_T}$ is the Radon-Nykodym derivative characterizing the change of measure. Since

$$d\left(\frac{S_t}{B_t}\right) = \sigma dW_t^Q,$$

we have that,

$$Z_T = \frac{S_T}{S_0}\frac{B_0}{B_T} = e^{-\int_0^T \sigma^2 ds + \int_0^T \sigma dW_s^Q}$$

and therefore $Z_t = E_t^Q[Z_T]$ is a martingale and a stictly positive process with $Z_0 = 1$.
ii) By Girsanov theorem,

$$dW^Q = dW^R + \sigma dt$$

iii) The dynamic of the stock price under the equivalent measure $\mathbb{R} \sim \mathbb{Q}$ is given by:

$$\frac{dS}{S} = rdt + \sigma dW^Q = rdt + \sigma(dW^R + \sigma dt) = (r+\sigma^2)dt + \sigma dW^R$$

The solution of this SDE is a geometrical Brownian motion:

$$S_T = S_t e^{(r+\frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}\,z}, \quad z \sim \mathcal{N}(0,1).$$

Hence,

$$\begin{aligned}
E_t^R\left[\mathbb{1}_{S_T>K}\right] &= R\left(S_T > K\right) \\
&= R\left(S_t e^{(r+\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}z} > K\right) \\
&= R\left(\sigma\sqrt{T-t}z > -\log\left(\frac{S_t}{K}\right) - (r+\frac{\sigma^2}{2})(T-t)\right) \\
&= R\left(z > \left(-\log\left(\frac{S_t}{K}\right) - (r+\frac{\sigma^2}{2})(T-t)\right)\frac{1}{\sigma\sqrt{T-t}}\right) \\
&= \Phi(d_1)
\end{aligned}$$

where Φ is the cumulative distribution function of the standard distribution and,

$$d_1 \equiv \left(\log\left(\frac{S_t}{K}\right) + (r+\frac{\sigma^2}{2})(T-t)\right)\frac{1}{\sigma\sqrt{T-t}}.$$