

Problem set 9

Due on November 19, 2019, at 11:15am

Exercise 1

Consider a 3-month put option on a 5-year zero-coupon bond, with strike $K = 0.805$. The short-term rate follows a Vasicek process

$$dr = k(\theta - r(t))dt + \sigma_r dW_t \quad (1)$$

Use $k = 0.15$, $\sigma_r = 0.01$, $\theta = 0.05$ and $r_0 = 0.042$

Price the option in 3 different ways

- Using the analytical formula

$$ZBP(0, T_1, T_2, K) = KP(0, T_1; r_0)N(-h + \tilde{\sigma}) - P(0, T_2; r_0)N(-h), \quad (2)$$

where T_1 is the option expiration (3-months in our case), T_2 is the bond expiration (5 years and 3 months), $P(0, T; r_0)$ is the bond price at time 0 with expiration T , given the initial short rate value r_0 (see formula below, eq(8)-(10)), $N(x)$ is the standard normal cdf, and finally

$$\tilde{\sigma} = \sigma_r \sqrt{\frac{1 - e^{-2kT_1}}{2k}} B(T_1, T_2) \quad (3)$$

$$h = \frac{1}{\tilde{\sigma}} \ln \frac{P(0, T_2; r_0)}{P(0, T_1; r_0)K} + \frac{\tilde{\sigma}}{2} \quad (4)$$

$$B(T_1, T_2) = \frac{1}{k}(1 - e^{-k(T_2 - T_1)}) \quad (5)$$

- Solving the option PDE

$$\frac{\partial V(t, r)}{\partial t} + \frac{\partial V(t, r)}{\partial r} k(\theta - r(t)) + \frac{1}{2} \frac{\partial^2 V(t, r)}{\partial r^2} \sigma_r^2 = r(t)V(t, r) \quad (6)$$

using an explicit method

- Solving the option PDE (7) using a Crank-Nicholson method.

Remember that the boundary condition $V(T_1, r(T_1))$ is

$$V(T_1, r(T_1)) = \max(0, P(T_1, T_2; r(T_1)) - K) \quad (7)$$

where the bond price $P(T_1, T_2; r(T_1))$ as a function of the short-term interest rate at expiration $r(T_1)$ is equal to

$$A(T_1, T_2) \exp(-B(T_1, T_2)r(T_1)) \quad (8)$$

with

$$A(T_1, T_2) = \exp\left(\left(\theta - \frac{\sigma^2}{2k^2}\right)(B(T_1, T_2) - (T_2 - T_1)) - \frac{\sigma^2}{4k}B(T_1, T_2)^2\right) \quad (9)$$

$$B(T_1, T_2) = \frac{1}{k}(1 - e^{-k(T_2 - T_1)}) \quad (10)$$

Although in theory r could become smaller than 0 in the Vasicek model, set $r_{min} = 0$ on the grid, and pick an appropriately large r_{max} . What are appropriate boundary conditions for $V(t, r_{min})$ and $V(t, r_{max})$?

Experiment with different time and “space” intervals δ_t and δ_r . What intervals do you need in the explicit and implicit method for the final price to be within 1% of the analytical price?