Fixed Income Analysis
Solution 2

Teacher: Prof. Damir Filipović
Assistant: Lotfi Boudabsa

This solution sheet only contains hints for solving the exercises and should not be taken as a reference for deserving full grades at an exam.

Exercise 1

a) Plugging in $R_{swap}(t) = \frac{P(t,T_0)-P(t,T_n)}{\delta \sum_{i=1}^n P(t,T_i)}$ yields

$$\Pi_p(t) = N\delta \left(\frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^n P(t,T_i)} - K \right) \sum_{i=1}^n P(t,T_i) = N \left(P(t,T_0) - P(t,T_n) - K\delta \sum_{i=1}^n P(t,T_i) \right).$$

b) The swap rate is $R_{swap}(0) = 0.0924$.

Exercise 2 All values are approximate values.

a) $p = 103.7839$.

b) $y = 0.0870$.

c) $y(0, 1/4) = 0.0596$, $y(0, 1/2) = 0.0694$, $y(0, 3/4) = 0.0727$, $y(0, 1) = 0.0767$, $y(0, 5/4) = 0.0792$, $y(0, 3/2) = 0.0825$, $y(0, 7/4) = 0.0848$, $y(0, 2) = 0.0878$.

d) $D_{mac} \approx 1.8521$, $D \approx 1.8507$, $C \approx 3.5789$.

e) See Figure 1.

f) See Figure 1.

Exercise 3

a) We compute $H_1(0) = 92.1756$, $H_2(0) = 129.5137$, $\Pi(0) = 254.4093$. The following tabular collects the durations and convexities of the three instruments and the portfolio:

	Duration	Convexity
Bond 1	6.2121	41.4723
Bond 2	6.1602	44.0374
Portfolio	5.7614	41.6566

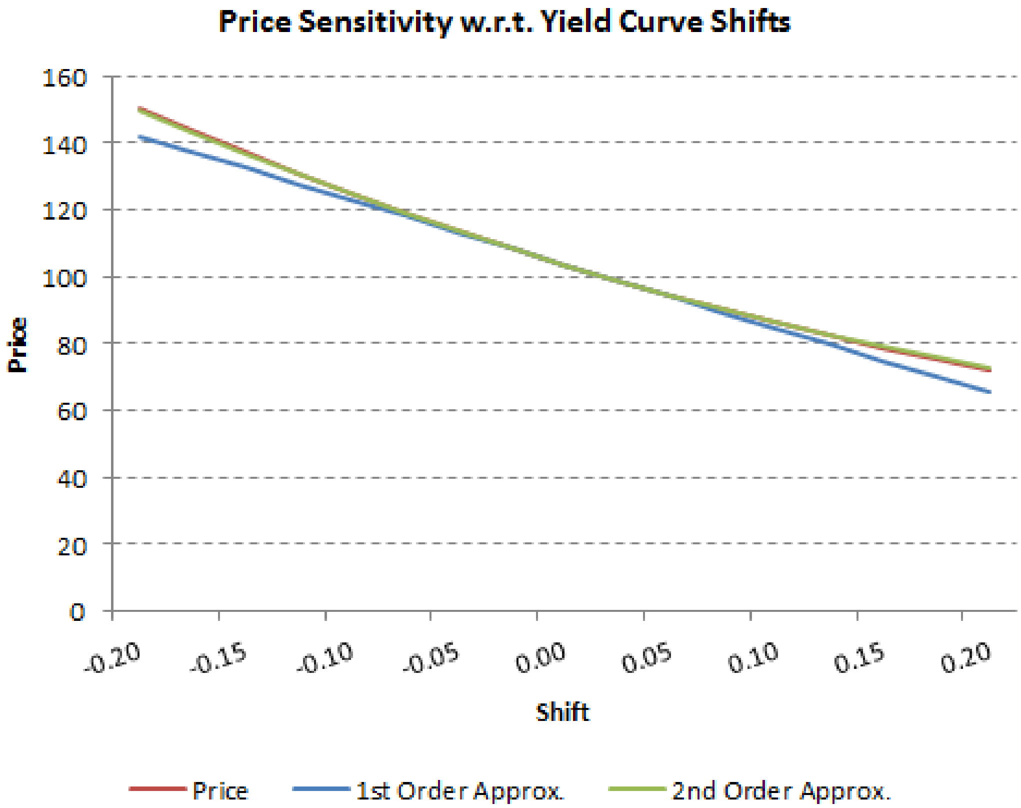

Figure 1: PV sensitivity w.r.t shift in Yield Curve

b) The convexity hedge is done by buying $q_1 = 0.4425$ shares of Bond 1 and going short by $q_2 = -2.1547$ shares of the Bond 2.

c) The plot is given in Figure 2.

Exercise 4

a) The ATM caplet has an implied Black volatility of 49.45%, by plugging this volatility in the Black formula we get a price of 0.0130. The Bachelier formula for ATM caplets, $F = K$, becomes:

$$Cpl = P(t,T)\sqrt{T-1}\sigma_{Bach}\phi(0).$$

Therefore we can easily solve for the σ_{Bach} that makes the caplet price equal to 0.0130:

$$\sigma_{Bach} = \frac{0.0130}{\exp(-0.03 \cdot 5)\sqrt{4}\phi(0)} = 0.0190.$$

b) See Figure 3. The Black model gives by construction of course a constant Black volatility, while the quotes we observe in the market actually show a downward sloping skew. The Bachelier model manages to create a downward sloping skew, but it is too severe. In other words, both models are not able to explain the prices we observe in the market.

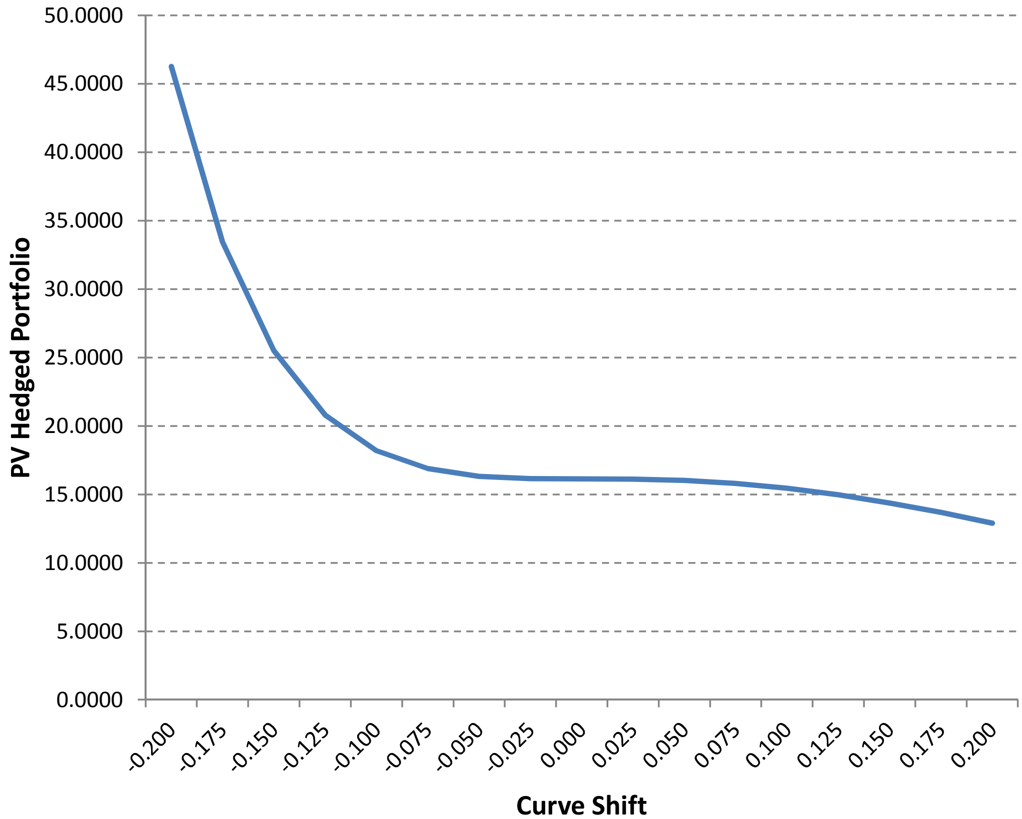

Figure 2: PV sensitivity w.r.t shift in Yield Curve for hedged portfolio with minimal cost

c) Define the following auxiliary stochastic process $Y_t := \beta F_t + (1-\beta)F_0$. One can easily show that Y follows log-normal dynamics:

$$dY_t = \sigma\beta Y_t dB_t \quad \Rightarrow \quad Y_t = Y_0 \exp(-0.5\sigma^2\beta^2 t + \sigma\beta B_t).$$

Using the definition of Y_t we can now easily recover an expression for the forward rate F_t:

$$F_t = \frac{F_0}{\beta}(\exp(-0.5\sigma^2\beta^2 t + \sigma\beta B_t) - (1-\beta)).$$

The payoff at time T can therefore be written as follows:

$$\left(\frac{F_0}{\beta}\exp(-0.5\sigma^2\beta^2 T + \sigma\beta B_T) - F_0\frac{1-\beta}{\beta} - K\right)^+.$$

In a regular Black model we model the dynamics of the forward rate F_t with a log-normal process, therefore the payoff at T in a regular Black model is given by:

$$\left(F_0\exp(-0.5\sigma^2 T + \sigma B_T) - K\right)^+.$$

We can now easily see that the price of a caplet in the Displaced Diffusion model is given by the Black formula where the initial forward rate F_0 is replaced by F_0/β, the strike K is replaced by $K + F_0(1/\beta - 1)$ and the volatility is $\beta\sigma_{DD}$.

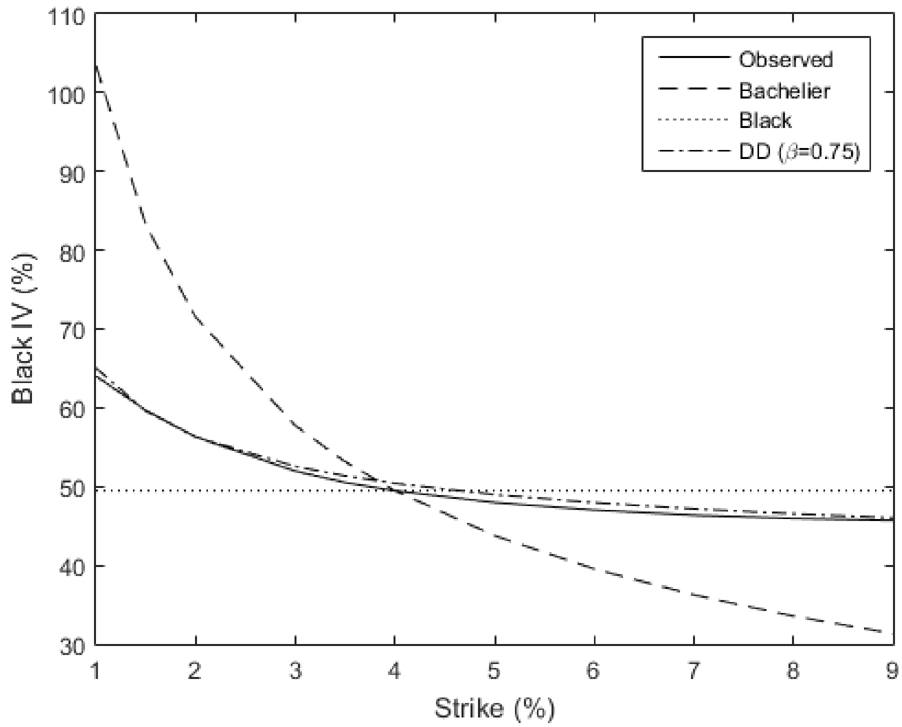

Figure 3: Implied Black volatility observed in the market together with the implied Black volatility produced by a Black, Bachelier and Displaced Diffusion model with $\beta = 0.75$.

d) See Figure 3.

e) In the Displaced Diffusion model the process Y follows log-normal dynamics and is therefore bounded below by 0. From this we can get a lower bound on F_t:

$$Y_t > 0 \quad \Leftrightarrow F_t > -F_0(1/\beta - 1).$$

In this model the forward rate can therefore take on negative values $0 < \beta < 1$ (which is the case in our model), although the probability is very small if β is sufficiently close to 1.

f) A Taylor expansion of order 2 in the Black's formula gives the result.