Fixed Income Analysis
Solution 7

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This solution sheet only contains hints for solving the exercises and should not be taken as a reference for deserving full grades at an exam.

Exercise 1 a) We consider a put option on a coupon bond with maturity T_0 given by

$$\left(K - \sum_{i=1}^{n} c_i e^{-A(T_0,T_i)-B(T_0,T_i)r_{T_0}}\right)^+$$

for some strike $K > 0$, coupon dates $T_0 < T_1 < \ldots < T_n$ and coupons c_i where c_n contains the nominal. Since the function p is strictly decreasing, there is a unique r^* such that $p(r^*) = K$. Hence, we obtain that the price computes as follows

$$\begin{aligned}
& E_{\mathbb{Q}}\left[e^{-\int_0^{T_0} r(s)ds}\left(K - \sum_{i=1}^{n} c_i e^{-A(T_0,T_i)-B(T_0,T_i)r_{T_0}}\right)^+\right] \\
= \; & E_{\mathbb{Q}}\left[e^{-\int_0^{T_0} r(s)ds}\left(K - \sum_{i=1}^{n} c_i e^{-A(T_0,T_i)-B(T_0,T_i)r_{T_0}}\right)1_{\{r_{T_0}\geq r^*\}}\right] \\
= \; & \sum_{i=1}^{n} c_i E_{\mathbb{Q}}\left[e^{-\int_0^{T_0} r(s)ds}\left(K_i - e^{-A(T_0,T_i)-B(T_0,T_i)r_{T_0}}\right)^+\right]
\end{aligned}$$

for $K_i := e^{-A(T_0,T_i)-B(T_0,T_i)r^*}$.
b) The price is 3.9198.

Exercise 2

a) The relation between the instantaneous forward rate and futures rate follows immediately by taking $\mathbb{Q}$ expectations of

$$f(T,T) = f(0,T) + \int_0^T \alpha(s,T)ds + \int_0^T \sigma(s,T)dW_s^{\mathbb{Q}},$$

where α must satisfy the HJM drift condition.

The relation between the simple forward and futures rate can be retrieved by recalling that under $\mathbb{Q}$ the discounted T-bond price satisfies for $t \leq T$:

$$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}_t(\nu(\cdot,T)\bullet W^*)$$

and similar for S. Therefore we get:

$$\frac{P(t,T)}{P(t,S)} = \frac{P(0,T)}{P(0,S)} \exp\left(\int_0^t \nu(s,T) - \nu(s,S) dW^*(s) - \frac{1}{2}\int_0^t (\|\nu(s,T)\|^2 - \|\nu(s,S)\|^2) ds\right)$$

Taking $\mathbb{Q}$-expectations of $\frac{P(T,T)}{P(T,S)}$ gives:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\frac{P(T,T)}{P(T,S)}\right] &= \frac{P(t,T)}{P(t,S)} \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\int_t^T \nu(s,T) - \nu(s,S) dW^*(s) - \frac{1}{2}\int_t^T (\|\nu(s,T)\|^2 - \|\nu(s,S)\|^2) ds\right)\right] \\
&= \frac{P(t,T)}{P(t,S)} \exp\left(\int_t^T \|\nu(s,S)\|^2 - \nu(s,T)\nu(s,S)^\top ds\right)
\end{aligned}$$

Using the definition of simple forward rates we now easily get:

$$\begin{aligned}
F(t,T,S) - \mathbb{E}_t^{\mathbb{Q}}[F(T,S)] &= \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P(T,T)}{P(T,S)}\right]\right) \\
&= \frac{1}{S-T}\frac{P(t,T)}{P(t,S)}\left(1 - \exp\left(\int_t^T (\nu(s,S) - \nu(s,T))\nu(s,S)^\top ds\right)\right).
\end{aligned}$$

Substituting $\nu(s,k) = -\int_s^k \sigma(s,u) du$, $k = T, S$, proves the lemma.

b) In the Ho-Lee model the volatility of the forward rate is constant, i.e. $\sigma(t,T) \equiv= \sigma$. Plugging this in the first result in Lemma 28.2 we get that the convexity adjustment equals

$$\frac{-\sigma^2 T^2}{2}.$$

c) In the Hull-White model $\sigma(t,T) = \exp(\beta(T-t))\sigma$. Let

$$B(t,T) = \frac{\sigma^2}{2\beta^3}\left(e^{2\beta(T+1/4-t)} - e^{\beta(2T+1/4-2t)} - 2e^{\beta(T+1/4-t)} + 2e^{\beta(T-t)} + 3e^{\beta/4} - e^{\beta/2} - 2\right)$$

Then

$$F(t;T,T+1/4) = L(t,T) - \frac{4P(t,T)}{P(t,T+1/4)}\left(e^{B(t,T)} - 1\right).$$

Exercise 3 Log-transforming the stock price process $x_t := \log(S_t)$ and adding the variance process $v_t := \sigma_t^2$ to the state vector we get the following 4 dimensional system of SDEs:

$$\begin{cases} dx_t &= (r_t - 0.5v_t)dt + \sigma_t dW_t^s \\ dr_t &= \lambda(\bar{r} - r_t)dt + \eta dW_t^r \\ d\sigma_t &= \kappa(\bar{\sigma} - \sigma_t)dt + \gamma dW_t^\sigma \\ dv_t &= (-2\kappa v_t + 2\kappa\bar{\sigma}\sigma_t + \gamma^2)dt + 2\sigma_t\gamma dW_t^\sigma \end{cases} \tag{0.1}$$

This system is affine in the state vector $X_t = [x_t, r_t, \sigma_t, v_t]^\top$ since the drift vector $b(X_t)$ and diffusion matrix $a(X_t)$ are given by:

$$b(X_t) = \begin{pmatrix} 0 \\ \lambda\bar{r} \\ \kappa\bar{\sigma} \\ \gamma^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & -0.5 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 2\kappa\bar{\sigma} & -2\kappa \end{pmatrix} X_t$$

$$a(X_t) = \begin{pmatrix} v_t & \sigma_t \eta \rho_{x,r} & \sigma_t \gamma \rho_{x,\sigma} & 2 v_t \gamma \rho_{x,v} \\ & \eta^2 & \eta\gamma\rho_{r,\sigma} & 2\eta\sigma_t\gamma\rho_{r,v} \\ & & \gamma^2 & 2\sigma_t\gamma^2 \\ & & & 4\gamma^2 v_t \end{pmatrix} = a + \sum_{i=1}^{4} \alpha_i X_t[i],$$

with $\alpha_1 = \alpha_2 = 0$ and

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & \eta^2 & \eta\gamma\rho_{r,\sigma} & 0 \\ & & \gamma^2 & 0 \\ & & & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & \eta\rho_{x,r} & \gamma\rho_{x,\sigma} & 0 \\ & 0 & 0 & 2\eta\gamma\rho_{r,v} \\ & & 0 & 2\gamma^2 \\ & & & 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 2\gamma\rho_{x,v} \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 4\gamma^2 \end{pmatrix}$$

The system of Riccati equations is then given by

$$\begin{aligned} \partial_t \phi(t,u) &= \frac{1}{2}\psi(t,u)^\top a\, \psi(t,u) + b^\top \psi(t,u) \\ \phi(0,u) &= 0 \\ \partial_t \psi_i(t,u) &= \frac{1}{2}\psi(t,u)^\top \alpha_i\, \psi(t,u) + \beta_i^\top \psi(t,u) \\ \psi(0,u) &= u. \end{aligned}$$