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Interest Rate and Credit Risk Models

Exercise 1 Using the formulae on slide 645-646, we have:

a)
$$V_0^{prem}(x; \underbrace{\gamma^{Q}}_{=\nu}) = \overbrace{\mathbb{1}_{\{\tau>0\}}}^{=1} \sum_{i=1}^{n} x(t_i - t_{i-1}) e^{-\int_0^{t_i}(r+\nu)ds}$$

$$= \sum_{i=1}^{n} x(t_i - t_{i-1}) e^{-(r+\nu)t_i}$$

$$V_0^{def}(\underbrace{\gamma^{Q}}_{=\nu}) = \mathbb{1}_{\{\tau>0\}} \delta \int_0^{\tau} \nu e^{-\int_0^{s}(r+\nu)du} ds$$

$$= \nu\delta \int_0^{\tau} e^{-(r+\nu)s} ds = -\frac{\nu\delta}{r+\nu}\left(e^{-(r+\nu)\tau} - 1\right)$$

$$= \frac{\nu\delta}{r+\nu}\left(1 - e^{-(r+\nu)\tau}\right)$$

b) We solve $V_0^{prem}(x^*; \nu) = V_0^{def}(\nu)$

Using a), we get $\iff$
$$x^* = \frac{\frac{\nu\delta}{r+\nu}\left(1 - e^{-(r+\nu)\tau}\right)}{\sum_{i=1}^{F} \underbrace{(t_i - t_{i-1})}_{=\frac{1}{2}} e^{-(r+\nu)t_i}}$$

We plug with the given values in jupyter which yield a fair CDS spread of roughly

122 bps

c) The value of the contract to a buyer of a CDS spread is equal to

$$V_{contract} = \left(V_0^{def}(N) - V_0^{prem}(x_y ; N) \right) \cdot 100'000'000$$

Plugging in Python gives a value of the contract of

$$V_{contract} = 1.56 \text{ million}$$

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Exercise 2 IRMCRM

a) We have $V_0^{Prem}(x;\gamma^Q) = x\sum_{i=1}^{n}(t_i - t_{i-1})e^{-\int_0^{t_i} R(u)\,du}$

and $V_0^{def}(\gamma^Q) = \delta\int_0^{t_n}\gamma(s)e^{-\int_0^{s}R(u)du}\,ds$

So in this case

$$V_0^{Prem}(x;\gamma^Q) = \frac{x}{2}e^{-\int_0^{\frac{1}{2}}(2a+r)du} = \frac{x}{2}e^{-\frac{(2a+r)}{2}}$$

$$V_0^{def}(\gamma^Q) = \delta\int_0^{\frac{1}{2}} 2a\,e^{-(2a+r)s}\,ds$$

$$= \frac{\delta 2a}{2a+r}\left(1 - e^{-\frac{(2a+r)}{2}}\right)$$

Since we have that the contract is fairly priced, we have that the following relationship must hold

$$V_0^{Prem}(x;\gamma^Q) = V_0^{def}(\gamma^Q)$$

$$\Leftrightarrow \quad 0 = \frac{\delta 2a}{2a+r}\left(1 - e^{-\frac{(2a+r)}{2}}\right) - \frac{x}{2}e^{-\frac{(2a+r)}{2}}$$

Plugging into a Python solver this yields

$$a = 0.0246314$$

$(\delta = 0.4,\ r = 0.01,\ x = 0.02)$

b) again, starting from the same equations as in a),
we have now

$$V_0^{prem}(x;\gamma^{Q}) = \frac{x}{2}\left(e^{-\int_0^{\frac{1}{2}} R(u)du} + e^{-\int_0^{1} R(u)du}\right)$$

$$= \frac{x}{2}\left(e^{-\frac{(2a+r)}{2}} + e^{-\left(\frac{2a+b}{2} + \frac{(2a)}{2} + r\right)}\right.$$

$$V_0^{def} = \delta\left(\int_0^{\frac{1}{2}} \gamma(s) e^{-\int_0^{s} R(u)du} ds + \int_{\frac{1}{2}}^{1} \gamma(s) e^{-\int_0^{\frac{1}{2}} R(u)du - \int_{\frac{1}{2}}^{s} R(u)du}\right)$$

$$= \delta\left(\frac{2a}{2a+r}\left(1 - e^{-\frac{(2a+r)}{2}}\right) + (2a+b)e^{-\frac{(2a+r)}{2}} \int_{\frac{1}{2}}^{1} e^{-\int_{\frac{1}{2}}^{s}(2a+b+r)du} ds\right.$$

$$= \delta\left(\frac{2a}{2a+r}\left(1 - e^{-\frac{(2a+r)}{2}}\right) + (2a+b)e^{-\frac{(2a+r)}{2}} \int_{\frac{1}{2}}^{1} e^{\frac{2a+b+r}{2} - (2a+b+r)s} ds\right.$$

$$= \delta\left(\frac{2a}{2a+r}\left(1 - e^{-\frac{(2a+r)}{2}}\right) + (2a+b) e^{\frac{b}{2}} \int_{\frac{1}{2}}^{1} e^{-(2a+b+r)s} ds\right.$$

$$= \delta\left(\frac{2a}{2a+r}\left(1 - e^{-\frac{(2a+r)}{2}}\right) - \frac{(2a+b)e^{\frac{b}{2}}}{(2a+b+r)}\left(e^{-(2a+b+r)} - e^{-\frac{(2a+b+r)}{2}}\right)\right)$$

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Exercise 2 IRMCRM

b) (suite) by using the same scheme as in a),

with $\delta = 0.4$, $r = 0.01$, $x = 0.04$),

and letting $V_0^{prem}(x;\delta^Q) = V_0^{def}(\delta^Q)$ and

solving using Python, we get

$$b = 0.10564099$$

3) a- From slide 674 we have.

$$P_1(t,T) = (1-\delta)\,P_0(t,T) + \mathbb{1}_{\{\tau > t\}}\,\delta\,\mathbb{E}^Q\left(e^{-\int_t^T R_s\,ds} \mid \mathcal{F}_t\right)$$

× We have $P_0(t,T) = \mathbb{E}^Q\left(e^{-\int_t^T r_s\,ds} \mid \mathcal{F}_t\right)$

We know that r_t is a Vasiček model therefore we have:

$$P_0(t,T) = e^{-A(t,T) - B(t,T)\,r(t)}$$

where $A(t,T) = \sigma_r^2\left[\dfrac{4e^{\beta_r(T-t)} - e^{2\beta_r(T-t)} - 2\beta_r(T-t) - 3}{4\beta_r^3}\right] + \dfrac{b_r}{\beta_r^2}\left[e^{\beta_r(T-t)} - 1 - \beta_r(T-t)\right]$

$$B(t,T) = \frac{1}{\beta_r}\left(e^{\beta_r(T-t)} - 1\right)$$

× We also have $R_s = r_s + y_s$, and since W_t^r and W_t^y are independent r_s and y_s are independent

Therefore:

$$\mathbb{E}^Q\left[e^{-\int_t^T r_s + y_s\,ds} \mid \mathcal{F}_t\right] = \underbrace{\mathbb{E}^Q\left(e^{-\int_t^T r_s\,ds} \mid \mathcal{F}_t\right)}_{=P_0(t,T)}\,\mathbb{E}^Q\left(e^{-\int_t^T y_s\,ds}\right)$$

Since y_t follows CIR model we have:

$$\mathbb{E}^Q\left[e^{-\int_t^T y_s\,ds} \mid \mathcal{F}_t\right] = e^{-A^*(t,T) - B^*(t,T)\,y(t)}$$

where $B^*(t,T) = \dfrac{2\left(e^{h(T-t)} - 1\right)}{(h-\beta_y)\left[e^{h(T-t)} - 1\right] + 2h}$ $\qquad h = \sqrt{\beta_y^2 + 2\sigma_y^2}$

$$A^*(t,T) = -\frac{2b_y}{\sigma_y^2}\log\left(\frac{2h\,e^{(h-\beta_y)(T-t)/2}}{(h-\beta_y)\left(e^{h(T-t)} - 1\right) + 2h}\right)$$