Fixed Income Analysis
Solution 5

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This solution sheet only contains hints for solving the exercises and should not be taken as a reference for deserving full grades at an exam.

Exercise 1

a) By applying Itô's lemma to $\exp(-\beta t) r_t$, one can easily find:

$$r(t) = e^{\beta t} r(0) + \frac{b}{-\beta}\left(1 - e^{\beta t}\right) + \sigma \int_0^t e^{\beta(t-s)} \mathrm{d}W^*(s).$$

This is a Gaussian random variable with mean and variance respectively given by:

$$\mathbb{E}^{\mathbb{Q}}[r(t)] = e^{\beta t} r(0) - \frac{b}{-\beta}(e^{\beta t} - 1), \quad \mathrm{Var}^{\mathbb{Q}}[r(t)] = \frac{\sigma^2}{-2\beta}(1 - e^{2\beta t}).$$

b)

$$\mathbb{Q}[r(t) < 0] = \Phi\left(\frac{-\mathbb{E}^{\mathbb{Q}}[r(t)]}{\sqrt{\mathrm{Var}^{\mathbb{Q}}[r(t)]})}\right)$$

c)

$$\begin{aligned} f(0,T) &= -\frac{\partial \log P(0,T)}{\partial T} \\ &= r(0) e^{\beta T} + \frac{b}{\beta}\left(e^{\beta T} - 1\right) - \frac{\sigma^2}{2\beta^2}\left(e^{\beta T} - 1\right)^2. \end{aligned}$$

If $\beta < 0$, then $T \mapsto f(0,T)$ will be decreasing iff

$$r(0) > -\frac{b}{\beta}.$$

If $\beta > 0$, then then $T \mapsto f(0,T)$ will be decreasing iff

$$r(0) < -\frac{b}{\beta}.$$

Exercise 2

a) Recall that the explicit solution of the Vasicek model is given by

$$r_t = r_0 e^{\beta t} + \frac{b}{\beta}(e^{\beta t} - 1) + \sigma e^{\beta t} \int_0^t e^{-\beta s} \mathrm{d}W_s^*,$$

so that r_t, given r_s, $s \leq t$, is normally distributed with mean

$$e^{\beta(t-s)} r(s) + \frac{b}{\beta}(e^{\beta(t-s)} - 1) =: e^{\beta(t-s)} r(s) + \mu(s,t)$$

and variance

$$\frac{\sigma^2}{2\beta}(e^{2\beta(t-s)} - 1) =: \sigma^2(s,t).$$

Therefore, to simulate r at times $0 = t_0 < t_1 < \cdots < t_n$, we do $i = 0, \ldots, N-1$

$$r_{t_{i+1}} = e^{\beta(t_{i+1} - t_i)} r_{t_i} + \mu(t_i, t_{i+1}) + \sigma(t_i, t_{i+1}) Z_{i+1},$$

where $Z_1, \ldots, Z_N$ are i.i.d. $\mathcal{N}(0,1)$.

b) The price of the 2-year zero-coupon bond equals 0.8314.

Exercise 3 a) We have that

$$W^*(t) = W(t) + at$$

is a standard BM under the risk neutral probability measure $\mathbb{Q}$. Hence, the risk neutral dynamics of the short rate are given by

$$\begin{aligned} dr(t) &= r(t)(ac - b + (1-a)r(t))dt + r(t)(c - r(t))(dW^*(t) - adt) \\ &= r(t)(ac - b + (1-a)r(t) - a(c - r(t)))dt + r(t)(c - r(t))dW^*(t) \\ &= r(t)(r(t) - b)dt + r(t)(c - r(t))dW^*(t). \end{aligned}$$

b)Recall that the discounted zero-coupon bond price

$$\frac{P(t,T)}{B(t)}, \quad \text{where} \quad B(t) = e^{-\int_0^t r(s)ds},$$

is a $\mathbb{Q}$-martingale, hence, without drift. Assuming that $P(t,T) = 1 + D(T-t)r(t)$ we obtain by applying Itô-Formula:

$$\begin{aligned} d\frac{P(t,T)}{B(t)} &= \frac{1}{B(t)} \mathbf{d}(\mathbf{D}(\mathbf{T} - \mathbf{t})\mathbf{r}(\mathbf{t})) + (1 + D(T-t)r(t))\mathbf{d}\frac{\mathbf{1}}{\mathbf{B}(\mathbf{t})} \\ &= \frac{r(t)}{B(t)} \left(-1 - bD(T-t) - D'(T-t)\right) \mathbf{dt} + \frac{r(T)}{B(t)} D(T-t)(c - r(t))\mathbf{dW}^*(\mathbf{T}). \end{aligned}$$

According to the introductory remark we conclude that D must satisfy

$$D'(\tau) = -1 - bD(\tau), \quad D(0) = 0. \tag{0.1}$$

Solving this differential equation we obtain

$$D(\tau) = \frac{1}{b}\left(e^{-b\tau} - 1\right).$$

In order to verify whether our assumption is correct we plug

$$P(t,T) = 1 + \frac{1}{b}\left(e^{-b(T-t)} - 1\right) r(t)$$

into the term–structure equation and see that it is indeed satisfied.

An alternative approach is to assume that $P(t,T) = 1 + D(T-t)r(t)$ and directly plugging the corresponding partial derivatives into the term–structure equation. Obviously, also in this case we derive that D must satisfy (0.1)…