Fixed Income Analysis
Solution 4

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This solution sheet only contains hints for solving the exercises and should not be taken as a reference for deserving full grades at an exam.

Exercise 1

a) Solve the system of linear equations given in the Representer Theorem with the 8 maturities and yields from July 2015. Note the following functional form of the smoothed splines

$$h_k(u) = \begin{cases} 0 & u < 0 \\ T_k + T_k \min(u, T_k) - \frac{1}{2}\min(u, T_k)^2 & 0 \leq u \end{cases}$$

and the scalar products

$$\langle h_l, h_k \rangle_H = T_l T_k + \min(T_l, T_k) T_l T_k - \frac{1}{2}\min(T_l, T_k)^2 (T_l + T_k) + \frac{1}{3}\min(T_l, T_k)^3$$

The forward curve $f(u) = f(0) + \sum_{k=1}^{8} a_k h_k(u)$ is plotted in Figure 1 for $\alpha = 0.1, 1, 10$. Further the yield curves $Y_i = \frac{1}{T_i} f(0) + \sum_{k=1}^{8} a_k \int_0^{T_i} h_k(u)du$ are plotted, too. The higher the tuning parameter α the better the fit to the observed yield data.

b) The first three pricipal components, i.e. the ones corresponding to the eigenvalues 0.1480, 0.0354 and 0.0062 explain 98.61% of the variance. The loadings are plotted in Figure 2.

Exercise 2 The calibrated parameters for the N-S model are:

	β_0	β_1	β_2
$\alpha = 0.06$	-0.0198	0.0174	0.1193
$\alpha = 0.08$	0.0036	-0.0074	0.0799
$\alpha = 0.10$	0.0138	-0.0188	0.0600

Exercise 3

a) Obviously $\hat{Q}$ is symmetric. In order to verify the positive semi-definiteness we observe that for all $z \in \mathbb{R}^n$ we have

$$z^T \hat{Q} z = \frac{1}{N} \sum_{t=1}^{N} \sum_{i,j=1}^{n} z_i (\hat{x}_i(t) - \hat{\mu}_i)(\hat{x}_j(t) - \hat{\mu}_j) z_j = \frac{1}{N} \sum_{t=1}^{N} (z^T (x(t) - \hat{\mu}))^2.$$

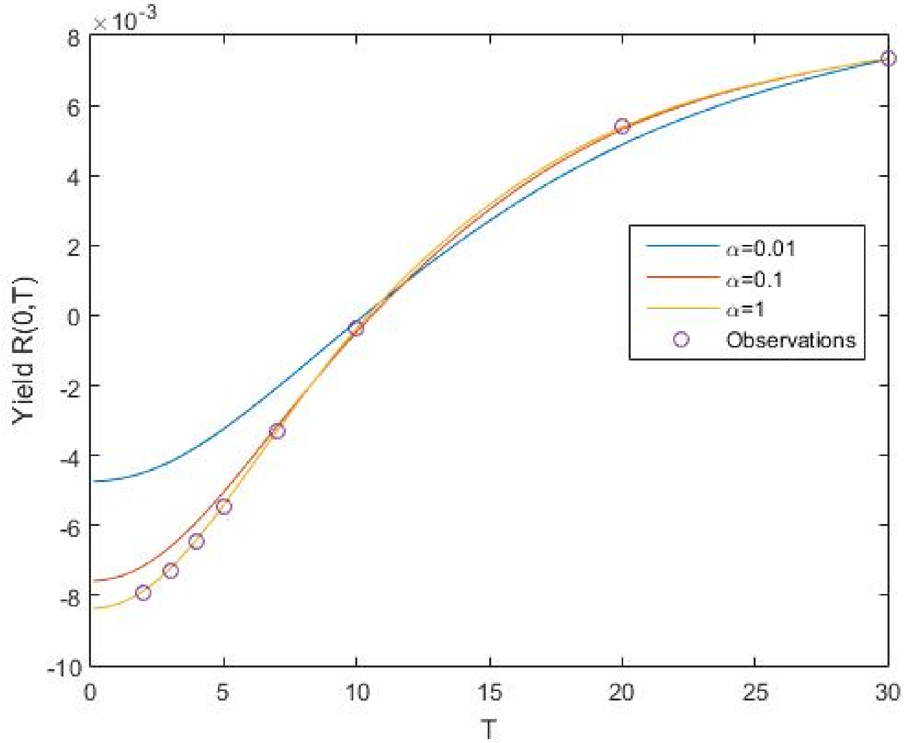

Figure 1: With $\alpha = 0.01, 0.1, 1$; we observe that the curve gets closer to the data as α increases.

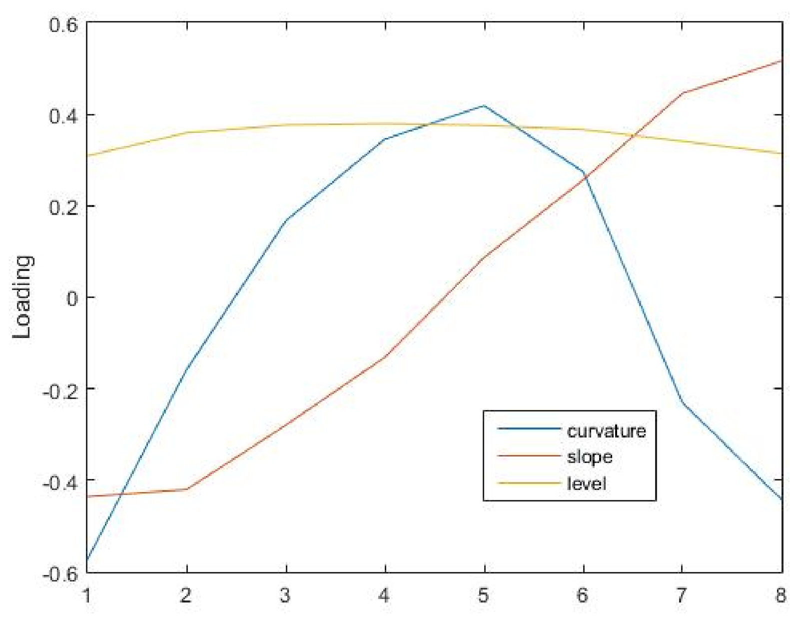

Figure 2: PCA of monthly yield curve changes of Swiss Confederation bonds from 2005-2015.

b) Since $y(t) = \hat{A}^T(x(t) - \hat{\mu})$ and $A^{-1} = A^T$ we have that

$$x(t) = \hat{\mu} + \hat{A}y(t) = \hat{\mu} + \sum_{i=1}^{n} y_i(t)\hat{a}_i.$$

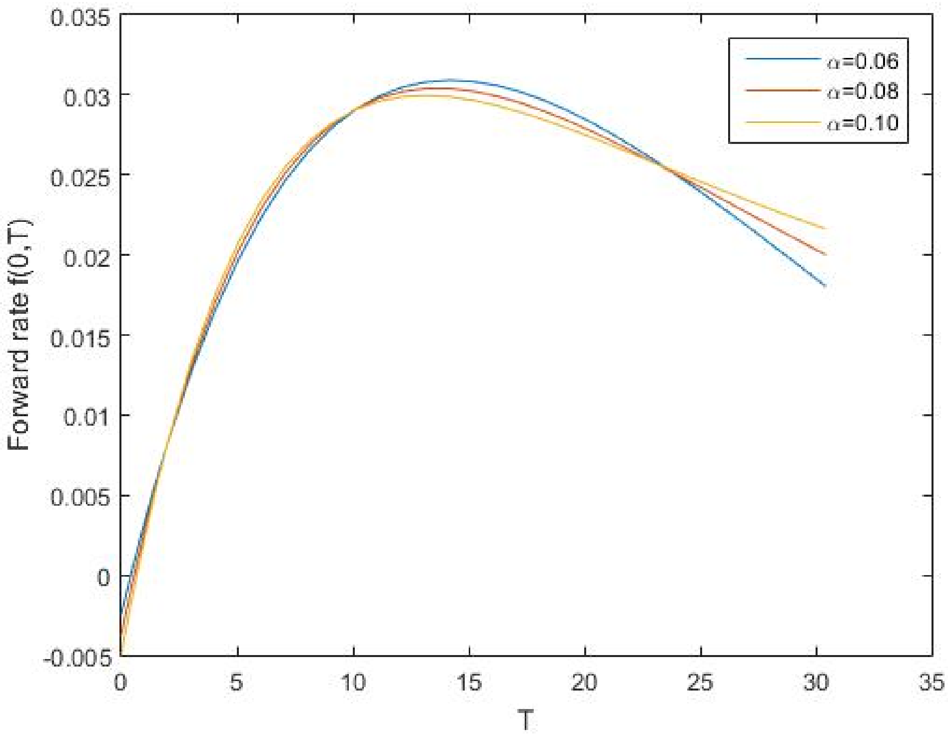

Figure 3: Instantaneous forward rate of calibrated N-S model.

Moreover,

$$\begin{aligned}
Cov(y_i, y_j) &= \frac{1}{N}\sum_{t=1}^{N} y_i(t) y_j(t) \\
&= \frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=1}^{n} \hat{a}_{ki}(x_k(t) - \hat{\mu}_k)\right)\left(\sum_{l=1}^{n} \hat{a}_{lj}(x_l(t) - \hat{\mu}_l)\right) \\
&= \sum_{k,l=1}^{n} \hat{a}_{ki}\hat{a}_{lj} Cov(x_k, x_l) \\
&= (\hat{A}^T \hat{Q} \hat{A})_{ij} \\
&= L_{ij}.
\end{aligned}$$

c) $\hat{Q}$ being degenerate means that at least one $\hat{\lambda}_i$ is zero. But then for this i, we have $y_i = 0$ as well (since expectation and variance of y_i are zero). Consequently, we have that

$$0 = y_i = \sum_{j=1}^{n} \hat{a}_{ji}(x_j - \hat{\mu}_j),$$

and for k such that $a_{ki} \neq 0$ we obtain $x_k = \hat{\mu}_k + \sum_{j=1, j\neq k}^{n} \frac{a_{ji}}{a_{ki}}(x_j - \hat{\mu}_j)$.

d) Since $\hat{Q}$ is non-degenerate, all eigenvalues $\hat{\lambda}_i$ are strictly positive. Hence, we can define

$$w_i := \frac{1}{\sqrt{\hat{\lambda}_i}} y_i, \quad i = 1, \dots, n.$$

Exercise 4 An instrument with fixed cashflows C_i in x_i years has value

$$V = \sum_{i=1}^{n} C_i e^{-x_i y(0,x_i)},$$

where

$$y(0, x_i) = \beta_0 + \beta_1 \left(\frac{1 - e^{-ax_i}}{ax_i} \right) + \beta_2 \left(\frac{1 - (1 + ax_i)e^{-ax_i}}{ax_i} \right).$$

We obtain the following partial derivatives:

$$\begin{aligned}
\frac{\partial V}{\beta_0} &= \sum_{i=1}^{n} C_i e^{-x_i y(0,x_i)} (-x_i) \\
\frac{\partial V}{\beta_1} &= \sum_{i=1}^{n} C_i e^{-x_i y(0,x_i)} (-x_i) \left(\frac{1 - e^{-ax_i}}{ax_i} \right) \\
\frac{\partial V}{\beta_1} &= \sum_{i=1}^{n} C_i e^{-x_i y(0,x_i)} (-x_i) \left(\frac{1 - (1 + ax_i)e^{-ax_i}}{ax_i} \right).
\end{aligned}$$

We now compute the quantities q_1, q_2, q_3 of the zero-coupon bonds with maturity in 1year, 2years, and 3 years, respectively, such that the portfolio composed of the coupon bond and the zero-coupon bonds has $\frac{\partial V}{\beta_0} = \frac{\partial V}{\beta_1} = \frac{\partial V}{\beta_2} = 0$. The solution is given by:

$$q_1 = -1.5099, \quad q_2 = 3.9662, \quad a_3 = -5.6231.$$