Fixed Income Analysis
Exercise Sheet 6

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Please hand in your solutions on Wednesday 30.10.2019 at the beginning of the lecture.

Exercise 1 Consider the following dynamics for the short rate

$$dr(t) = \mu t\, dt + \sigma\, dW^*(t), \quad r(0) = \xi,$$

where $\mu, \sigma, \xi > 0$ are constants and W^* is a Brownian motion under the risk-neutral measure $\mathbb{Q}$.

a) Show that the T-bond price is

$$P(t,T) = \exp\left(\frac{\sigma^2}{6}(T-t)^3 - \frac{\mu}{6}T^3 + \frac{1}{2}\mu T t^2 - \frac{1}{3}\mu t^3 - (T-t)r(t)\right) \quad \text{for } t \leq T.$$

b) Find the $\mathbb{Q}$-dynamics of the forward rate $f(t,T)$, for $t \leq T$.

c) Check explicitly that the HJM drift condition is satisfied.

3 points

Exercise 2 Consider a Gaussian HJM model with deterministic forward rate volatility $\sigma(t,T)$. Derive a formula using the S-forward measure $\mathbb{Q}^S \sim \mathbb{Q}$ for

$$\pi(t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^S r(s)\mathrm{d}s}\left(R(T,S) - K\right)^+ \mid \mathcal{F}_t\right],$$

the price at time t of the caplet on the zero-coupon bond yield $R(T,S)$ with strike K, $0 < t \leq T \leq S$.

3 points

Exercise 3 Let the short rates follow a non-negative bounded Itô process $(r(t))_{0\geq t}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ where $\mathbb{Q}$ is the risk neutral measure. For a fixed finite time horizon $T > 0$, we consider the T-forward measure $\mathbb{Q}^T \sim \mathbb{Q}$.

a) Let $F(t;T,S)$ be the simple forward rate for $[T,S]$ prevailing at t. Show that $F(t;T,S)$, $t \leq T$, is a martingale with respect to some forward measure $\mathbb{Q}^u$; that is,

$$F(t;T,S) = E_{\mathbb{Q}^u}\left[F(T;T,S) \mid \mathcal{F}_t\right].$$

What is u?

b) Show that the discounted money market account $(B(t)/P(t,T))_{0\leq t\leq T}$ is a martingale under $\mathbb{Q}^T$.

3 points

Exercise 4 Consider the Vasicek short rate model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW^*(t), \quad r(0) = 0.01.$$

where $W^*(t)$ is a standard Brownian motion under the risk-neutral probability measure and $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.01$. Derive the price of an ATM cap with quarterly cashflows (first reset date in three months) and maturity in 30 years.

1 point