f) Recall the two formulae of an ATM caplet under Black and Bachelier dynamics derived in a).

$$\text{Caplet}_{\text{Black}}(t,T_0,T_1) = \delta P(t,T_1)F(t,T_0,T_1)\left(\Phi\left(\tfrac{1}{2}\sigma_{\text{Black}}\sqrt{T_0-t}\right) - \Phi\left(-\tfrac{1}{2}\sigma_{\text{Black}}\sqrt{T_0-t}\right)\right)$$

$$= \delta P(t,T_1)F(t,T_0,T_1)\left(2\Phi\left(\tfrac{1}{2}\sigma_{\text{Black}}\sqrt{T_0-t}\right) - 1\right)$$

$$\text{Caplet}_{\text{Bach}}(t,T_0,T_1) = \delta P(t,T_1)\sigma_{\text{Bach}}\sqrt{T_0-t}\,\phi(0)$$

Now, consider the Taylor expansion of $\Phi(x)$ around 0:

$$\Phi(x) = \Phi(0) + \phi(0)x + \quad + O(x^2)$$

Plugging this in $\text{Caplet}_{\text{Black}}$'s formula gives

$$\text{Caplet}_{\text{Black}}(t,T_0,T_1) = \delta P(t,T_1)F(t,T_0,T_1)\Big(2\big(\underbrace{\Phi(0)}_{=\frac{1}{2}}+\phi(0)\tfrac{1}{2}\sigma_{\text{Black}}\sqrt{T_0-t} + O\big(\big(\tfrac{\sigma_{\text{Black}}}{2}\sqrt{T_0-t}\big)^2\big)\big) - 1\Big)$$

$$= \delta P(t,T_1)F(t,T_0,T_1)\left(\phi(0)\sigma_{\text{Black}}\sqrt{T_0-t} + O\left(\left(\frac{\sigma_{\text{Black}}\sqrt{T_0-t}}{2}\right)^2\right)\right)$$

Now, equating Caplet Black with Caplet Bach yields

$$\text{CapletBlack}(t,T_0,T_1) = \text{CapletBach}(t,T_0,T_1)$$

$$\iff \delta P(t,T_1)F(t,T_0,T_1)\Big(\phi(0)\sigma_{\text{Black}}\sqrt{T_0-t}\Big) + \underbrace{O\left(\left(\frac{\sigma_{\text{Black}}\sqrt{T_0-t}}{2}\right)^2\right)}_{=\sigma_{\text{Black}}\sqrt{T_0-t}\,h(\sigma_{\text{Black}}\sqrt{T_0-t})}$$

$$= \delta P(t,T_1)\sigma_{\text{Bach}}\sqrt{T_0-t}\,\phi(0)$$

$$\Rightarrow \sigma_{\text{Bach}}\sqrt{T_0-t} = \sigma_{\text{Black}}\sqrt{T_0-t}\,F(t,T_0,T_1) + \sigma_{\text{Black}}\sqrt{T_0-t}\,h(\sigma_{\text{Black}}\sqrt{T_0-t})$$

with $h \longrightarrow 0$ as $\sigma_{\text{Black}}\sqrt{T_0-t} \longmapsto 0$