Fixed Income Analysis
Exercise Sheet 2

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Please hand in your solutions on Wednesday 02.10.2019 at the beginning of the lecture. Remember that the exercise session of Thursday 03.10.2019 will take place in room CH B3 31.

Exercise 1 Consider a swap with reset and cash flow dates

$$0 < T_0 < T_1 < \ldots < T_n$$

(T_0 is the first reset date) such that $T_i - T_{i-1} \equiv \delta$, and coupons $c_i = K\delta N$, for some fixed rate K and nominal N.

a) Let $t \leq T_0$. Show that the time t value of the payer swap equals

$$\Pi_p(t) = N\delta(R_{swap}(t) - K)\sum_{i=1}^{n} P(t, T_i).$$

We now consider a numerical example: today is $t = 0$, first reset date is $T_0 = 1/4$, cash flow dates are $T_i = T_{i-1} + 1/4$, maturity is $T_7 = 2$. The forward curve is given by

$$\begin{pmatrix} F(0;0,1/4) \\ \vdots \\ \vdots \\ F(0;7/4,2) \end{pmatrix} = \begin{pmatrix} 0.06 \\ 0.08 \\ 0.08 \\ 0.09 \\ 0.09 \\ 0.1 \\ 0.1 \\ 0.11 \end{pmatrix}. \tag{1}$$

b) Find the corresponding swap rate $R_{swap}(0)$.

2 point

Exercise 2 Consider the forward curve from Exercise 1. Today is $t = 0$. Consider a coupon bond with maturity in two years, semiannual coupon payments $c = 5.5$ and nominal 100.

a) What is its price p?

b) What is its continuously compounded yield-to-maturity y?

c) Compute the yield curve $y_i = y(0, i/4)$, $i = 1, \dots, 8$.

d) Compute the Macaulay duration D_{Mac}, the duration D and convexity C of the bond.

e) Consider a parallel shift of the yield curve

$$y_i \to \tilde{y}_i = y_i + s, \quad i = 1, \dots, 8$$

by $s \in [-0.2, 0.2]$. Plot the price $p(s)$ of the coupon bond with shifted yield curve (same maturity, coupons and nominal) as a functions of s.

f) Plot the first- and second-order approximations

$$p - D_{Mac} ps, \quad p - Dps, \quad p - Dps + \frac{1}{2} Cps^2$$

as functions of $s \in [-0.2, 0.2]$. Are there any differences?

2 points

Exercise 3 Today is time 0. Consider the following term-structure

Maturity	$y(0,T)$	Maturity	$y(0,T)$
1	6.000	6	5.250
2	5.800	7	5.200
3	5.620	8	5.160
4	5.460	9	5.125
5	5.330	10	5.100

where $y(0,T)$ is the continuously compounded spot rate at date 0 for maturity T expressed in percentage. Given is a bond portfolio with the following cash-flows at times $1, \dots, 10$ years:

Year	Cash-flow	Year	Cash-flow
1	6	6	102
2	8	7	3
3	106	8	3
4	7	9	3
5	0	10	110

Moreover, there are two coupon bonds in the market which will serve as hedging instruments: The Bond 1 is a 7-Year coupon bond with annual coupons of 4%, and the Bond 2 is 8-Year coupon bond with annual coupons of 10%. These two coupon bonds have nominal \$100.

a) Compute the prices, duration, and convexity of the bond portfolio and the two coupon bonds at time 0.

b) Implement a convexity hedge of the portfolio by means Bonds 1 and 2.

c) Plot the value change of the hedged portfolios you obtain in questions e) and d) as a function of parallel shifts $s \in [-0.2, 0.2]$ of the yield curve.

2 points

Exercise 4 In Table 1 you find a set of caplet prices (quoted in Black implied volatility) for

Strike (%):	1.00	1.50	2.00	3.00	3.50	4.00
IV (%):	64.00	59.65	56.30	51.95	50.50	49.45
Strike (%):	5.00	6.00	7.00	8.00	9.00	
IV (%):	47.95	47.00	46.35	45.95	45.70	

Table 1: Prices of caplets on 28/04/2012 with time to maturity 5 years, quoted in implied Black volatility. Source: Bloomberg

different strikes with maturity 5 years from now (today is time 0). These caplets give the owner at the maturity date $T = 5$ the option to pay a certain strike K in order to receive the simple spot rate $L(4, 5)$ (i.e. $\delta = 1$). Assume that the continuously compounded spot rate corresponding with this maturity is $y(0, 5) = 0.03$.

a) Suppose the current forward rate $F(0, 4, 5)$ is equal to 0.04. Looking at the table this means that the ATM caplet has an implied Black volatility of 49.45%. Calibrate the Bachelier volatilty to ATM prices, i.e. find a value for σ_{bach} such that the the Bachelier formula gives exactly the price that we observe in the table for the ATM caplet.

b) Using the calibrated Bachelier volatility parameter from a), compute with the Bachelier model the caplet prices for all the strikes in the table. Next, convert these prices to Black implied volatilities. Finally make a plot with three curves: 1) the implied Black volatilities coming from the Bachelier model you just obtained, 2) the original quotes from the table and 3) a horizontal line with the implied Black volatility of the ATM caplet (i.e. 49.45%). What do you observe? Are the Black or Bachelier models able to explain the prices observed in the market?

c) Something in between the Black and Bachelier model is the so called *shifted log-normal* model. The dynamics of the forward rate in this model are given by:

$$\mathrm{d}F_t = \sigma[\beta F_t + (1-\beta)F_0]\mathrm{d}B_t,$$

with $0 < \beta \leq 1$ and short notation $F_t := F(t, 4, 5)$, $0 \leq t \leq 4$. Show that the price of a caplet in the Displaced Diffusion model at time $t = 0$ is given by the Black caplet formula with redefined parameters:

$$Cpl_{DD}(F_0, K, \sigma, T) = Cpl_{Black}(F_0/\beta, K + F_0(1/\beta - 1), \beta\sigma, T).$$

d) Compute the caplet prices for the range of strikes in Table 1 using the shifted log-normal model (choose $\sigma = 49.45\%$ and $\beta = 0.75$). Next derive the implied Black volatility from these prices and add this curve to the plot you made in b).

e) In the Black model the forward rates are positive by construction, is this still true in the shifted log-normal model? If not, is there another lower bound?

f) Consider an ATM caplet. Prove that

$$\sigma_{Bach}\sqrt{T_0 - t} = \sigma_{Black}\sqrt{T_0 - t}\ F(t, T_0, T_1) + \sigma_{Black}\sqrt{T_0 - t}\ h(\sigma_{Black}\sqrt{T_0 - t}), \quad (2)$$

where h is a function that converges to 0 when $\sigma_{Black}\sqrt{T_0 - t} \to 0$.

This approximation is useful for traders since they can translate the implied Black volatility of an ATM caplet, when $\sigma_{Black}\sqrt{T_0 - t} \to 0$, into implied Bachelier volatility straightforwardly.

4 points