IRCRM Pb 8

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Exercise 1

$E = \mathbb{R}_+^m \times \mathbb{R}^n \qquad d = m+n$

$I = \{1, \dots, m\} \quad , \quad J = \{m+1, \dots, m+n\}$

$a, \alpha_i \in S_+^d$

$a_{II} = 0 \quad (\text{and} \quad a_{IJ} = a_{JI}^T = 0)$

$\alpha_j = 0 \quad \forall j \in J$

$\alpha_{i,kl} = \alpha_{i,lk} = 0 \quad \forall k \in I \setminus \{i\} \quad \forall 1 \leq i, l \leq d$

$b \in \mathbb{R}_+^m \times \mathbb{R}^n$

$B_{IJ} = 0$

$B_{II} \geq 0$

$\Rightarrow \quad dX_t = (b + \beta X_t)\,dt + \sigma_t\, dW_t$

where $\quad \sigma_t \sigma_t^T = a_t + \alpha X_t$

a) $(m, n) = (1, 2)$

$$b = \begin{pmatrix} + \\ * \\ * \end{pmatrix} \quad \text{and} \quad B = \left(\begin{array}{c|cc} * & + & + \\ \hline * & * & * \\ * & * & * \end{array}\right)$$

$$a = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline & + & * \\ & & * \end{array}\right) \qquad \alpha_1 = \left(\begin{array}{c|cc} + & * & * \\ \hline & + & * \\ & & + \end{array}\right) \quad , \quad \alpha_2 = \alpha_3 = 0$$

Therefore :

$$\sigma_t \sigma_t^T = \begin{pmatrix} \sigma_1^2 X_1 & \sigma_1 \sigma_2 \rho_{12} X_1 & \sigma_1 \sigma_3 \rho_{13} X_1 \\ \sigma_1 \sigma_2 \rho_{12} X_1 & \sigma_4^2 + \sigma_2^2 X_1 & \sigma_4 \sigma_5 \rho_{45} + \sigma_2 \sigma_3 \rho_{23} X_1 \\ \sigma_1 \sigma_3 \rho_{13} X_1 & \sigma_4 \sigma_5 \rho_{45} + \sigma_2 \sigma_3 \rho_{23} X_1 & \sigma_5 + \sigma_3^2 X_1 \end{pmatrix}$$

By the Cholesky decomposition we have:

$$\sigma_{11} = \sqrt{a_{11}} = \sigma_1\sqrt{X_1}$$

$$\sigma_{j1} = \frac{a_{1j}}{\sigma_{11}} \Rightarrow \sigma_{21} = \frac{a_{12}}{\ell_{11}} = \sigma_2\rho_{12}\sqrt{X_1}$$

$$\sigma_{31} = \frac{a_{13}}{\ell_{11}} = \sigma_3\rho_{13}\sqrt{X_1}$$

$$\sigma_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1}\sigma_{ik}^2} \Rightarrow \sigma_{22} = \sqrt{\sigma_4^2 + \sigma_2^2 X_1 - \sigma_2^2\rho_{12}^2 X_1}$$

$$\sigma_{22} = \sqrt{\sigma_4^2 + \sigma_2^2 X_1(1-\rho_{12}^2)}$$

$$\sigma_{33} = \sqrt{\sigma_5^2 + \sigma_3^2 X_1 - \sigma_{32}^2 - \sigma_3^2\rho_{13}^2 X_1}$$

$$\sigma_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1}\sigma_{ik}\sigma_{jk}}{\sigma_{ii}} \Rightarrow \sigma_{32} = \frac{a_{23} - \sigma_{21}\sigma_{31}}{\sigma_{22}} = \frac{\sigma_4\sigma_5\rho_{45} + \sigma_2\sigma_3\rho_{23}X_1 - \sigma_2\rho_{12}\rho_{13}X_1}{\sqrt{\sigma_4^2 + \sigma_2^2 X_1(1-\rho_{12}^2)}}$$

where $\sigma_t = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$

Therefore we have

$$dX_t = (b + \beta X_t)dt + \sigma_t\, dW_t \qquad \left(dW_t = \begin{pmatrix} dW_{1t} \\ dW_{2t} \\ dW_{3t} \end{pmatrix} \right)$$

b) $(m, n) = (2, 1)$

$$b = \begin{pmatrix} + \\ + \\ \hline * \end{pmatrix}, \quad \beta = \left(\begin{array}{cc|c} * & + & 0 \\ + & * & 0 \\ \hline * & * & * \end{array}\right)$$

$$a = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ & 0 & 0 \\ \hline & & + \end{array}\right) \quad \alpha_1 = \left(\begin{array}{cc|c} + & 0 & * \\ & 0 & 0 \\ \hline & & + \end{array}\right), \quad \alpha_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ & + & * \\ \hline & & + \end{array}\right), \quad \alpha_3 = 0$$

$$\Rightarrow \sigma_t\sigma_t^T = \begin{pmatrix} \sigma_1^2 X_1 & 0 & \rho_{12}\sigma_1\sigma_2 X_1 \\ 0 & \sigma_3^2 X_2 & \rho_{34}\sigma_3\sigma_4 X_2 \\ \rho_{12}\sigma_1\sigma_2 X_1 & \rho_{34}\sigma_3\sigma_4 X_2 & \sigma_4^2 X_2 + \sigma_2^2 X_1 \end{pmatrix}$$

By the Cholesky decomposition we have:

$$\sigma_{11} = \sqrt{a_{11}} = \sigma_1 \sqrt{X_1}$$

$$\sigma_{j1} = \frac{a_{1j}}{\sigma_{11}} \Rightarrow \sigma_{21} = 0$$

$$\sigma_{31} = \rho_{12}\, \sigma_2 \sqrt{X_1}$$

$$\sigma_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} \sigma_{ik}^2} \Rightarrow \sigma_{22} = \sqrt{\sigma_3^2 X_2 - 0} = \sigma_3 \sqrt{X_2}$$

$$\sigma_{33} = \sqrt{\sigma_5^2 + \sigma_4^2 X_2 + \sigma_2^2 X_1 - \sigma_{32} - \rho_{12}^2 \sigma_2^2 X_1}$$

$$= \sqrt{\sigma_5^2 + (1-\rho_{34}^2)\sigma_4^2 X_2 + (1-\rho_{12}^2)\sigma_2^2 X_1}$$

$$\sigma_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} \sigma_{ik}\sigma_{jk}}{\sigma_{ii}} \Rightarrow \sigma_{32} = \frac{\rho_{34}\sigma_3 \sigma_4 X_2}{\sigma_3 \sqrt{X_2}} = \rho_{34}\sigma_4 \sqrt{X_2}$$

$$\Rightarrow \begin{cases} dX_{1t} = (b_1 + B_{11} X_{1t} + B_{12} X_{2t})dt + \sigma_1 \sqrt{X_1}\, dW_{1t} \\ dX_{2t} = (b_2 + B_{21} X_{1t} + B_{22} X_{2t})dt + \sigma_3 \sqrt{X_2}\, dW_{2t} \\ dX_{3t} = (b_3 + B_{31} X_{1t} + B_{32} X_{2t} + B_{33} X_{3t})dt + \sigma_{33}\, dW_{3t} + \sigma_{33}\, dW_{2t} + \sigma_{31}\, dW_{1t} \end{cases}$$

c) $(m, n) = (3, 0)$

$$b = \begin{pmatrix} + \\ + \\ + \end{pmatrix}, \quad B = \begin{pmatrix} * & + & + \\ + & * & + \\ + & + & * \end{pmatrix}$$

$$a = 0, \quad \alpha_1 = \begin{pmatrix} + & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ & + & 0 \\ & & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & + \end{pmatrix}$$

We have directly here:

$$\begin{cases} dX_{1t} = (b_1 + B_{11} X_{1t} + B_{21} X_{2t} + B_{31} X_{3t})dt + \sigma_1 \sqrt{X_{1t}}\, dW_1^t \\ dX_{2t} = (b_2 + B_{12} X_{1t} + B_{22} X_{2t} + B_{32} X_{3t})dt + \sigma_2 \sqrt{X_{2t}}\, dW_2^t \\ dX_{3t} = (b_3 + B_{13} X_{1t} + B_{23} X_{3t} + B_{33} X_{3t})dt + \sigma_3 \sqrt{X_{3t}}\, dW_3^t \end{cases}$$

Exercise 2

a)

We have $X_i(t) = \left[\sqrt{x_i} + B(t)\right]^2 \quad \forall i \in \{1,2\}$

We note $Y_i(t) = \sqrt{X_i(t)}$ using Itô lemma we have:

$$dY_i^2 = 2Y_i\,dY_i + \frac{1}{2}\,2\,d\langle Y_i, Y_i\rangle$$
$$= 2Y_i\,dB_t + dt$$
$$= dX_i$$

So we have $\forall i$ $dX_i = 2\sqrt{X_i}\,dB_t + dt$

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} dt + 2\begin{bmatrix} \sqrt{X_1} & 0 \\ \sqrt{X_2} & 0 \end{bmatrix} dB_t \qquad \text{and } X(0) = 0.$$
$$= b(X(t))\,dt + \rho(X(t))\,dB_t$$

We define $a(X_t) = \rho(X_t)\,\rho(X_t)^T$

$$a(X_t) = 2\begin{bmatrix} \sqrt{X_1} & 0 \\ \sqrt{X_2} & 0 \end{bmatrix} 2\begin{bmatrix} \sqrt{X_1} & \sqrt{X_2} \\ 0 & 0 \end{bmatrix} = 4\begin{bmatrix} X_1 & \sqrt{X_1X_2} \\ \sqrt{X_1X_2} & X_2 \end{bmatrix}$$

Based on the definition of $a(x) = a + \sum_{i=1}^{2} \alpha_i X_i$ it's clear that X_t is not an affine process.

$u \in i\mathbb{R}^2$ b) $E_x\left(\exp(u^T X_t)\right) = E_x\left[\exp\left(i\left[u_1X_1 + u_2X_2\right]\right)\right]$

$$= E_x\left[\exp\left(iu_1(\sqrt{x_1} + B_t)^2 + iu_2(\sqrt{x_2} + B_t)^2\right)\right]$$

$$= \exp\left[i\left(-\frac{(u_1\sqrt{x_1} + u_2\sqrt{x_2})^2}{u_1+u_2} + u_1x_1 + u_2x_2\right)\right] E_x\left[\exp\left(i(u_1+u_2)t\left[\underbrace{\frac{1}{\sqrt{t}}B_t + \frac{1}{\sqrt{t}}\frac{u_1\sqrt{x_1} + u_2\sqrt{x_2}}{u_1+u_2}}_{(1)}\right]^2\right)\right]$$

if we define $\mathcal{E} = \left[\frac{u_1\sqrt{x_1} + u_2\sqrt{x_2}}{u_1+u_2}\right]^2 \frac{1}{t}$

$\tilde{u} = i(u_1+u_2)t$ we have:

$$(1) = E_x\left[\exp\left(i(u_1+u_2)t\left[\frac{1}{\sqrt{t}}B_t + \frac{1}{\sqrt{t}}\frac{u_1\sqrt{x_1}+u_2\sqrt{x_2}}{u_1+u_2}\right]^2\right)\right]$$

$$(1) = \frac{\exp\left[\frac{\tilde{\mu}\,\tilde{u}}{1-2\tilde{x}}\right]}{(1-2\tilde{x})^{1/2}}$$ using the formula given in the pb set.

So we have:

$$E_x\left(e^{u^T X_t}\right) = \exp\left[\frac{-2\left(u_1\sqrt{x_1}+u_2\sqrt{x_2}\right)^2}{1-2ti(u_1+u_2)}\right]\frac{\exp(i(u_1x_1+u_2x_2))}{\left[1-2ti\,(u_1+u_2)\right]^{1/2}}$$

Based on the formula we just obtain we can conclude that we can't find 2 functions $\phi(t,u)$ and $\Psi(t,u)$ such that

$E_x\left[e^{u^T X_t}\right] = e^{\phi(t,u)+\Psi(t,u)^T x}$ so we conclude that X is not an affine process.