Fixed Income Analysis
Exercise Sheet 3

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Please hand in your solutions on Wednesday 09.10.2019 at the beginning of the lecture. Remember that the exercise session of Thursday 03.10.2019 will take place in room CH B3 31.

Exercise 1 Is the following statement true or false?

An out-of-the-money cap can contain in-the-money caplets.

Explain why it is true/false.

1 point

Exercise 2 Consider a swap, cap and floor determined by the sequence of reset/cash flow dates

$$0 < T_0 < T_1 < \cdots < T_n,$$

(T_0 is the first reset date) such that $T_i - T_{i-1} \equiv \delta$, a fixed rate $\kappa > 0$, and a nominal value N. Let $t \leq T_0$.

a) Show that the cash flow of the ith caplet

$$\delta(L(T_{i-1}, T_i) - \kappa)^+$$

at time T_i is equivalent to the cash flow

$$(1 + \delta\kappa) \left(\frac{1}{1 + \delta\kappa} - P(T_{i-1}, T_i) \right)^+$$

at maturity T_{i-1} of a put option on a T_i-bond price.

b) Prove the parity relation

$$Cp(t) - Fl(t) = \Pi_p(t), \tag{1}$$

where $\Pi_p(t)$ is the value at t of a payer swap with rate κ, nominal one and the same tenor structure as the cap and floor.

c) Verify that Black's formulas for caplets and floorlets satisfy the parity relations in (1).

2 points

Exercise 3 On the Moodle page you will find the file

`Bootstrap_data.xls`

containing data from the Cash, Futures and Swap Euro markets with spot date 03 October 2012. The first column contains the maturity dates for the different quoted instruments, the second column contains the market rate for each of these instruments and the third column gives the nature of the quoted instruments (either cash, future or swap). We to make the following assumptions:

- the day-count convention is Actual/360;
- the spot date (t_0) is 03 October 2012;
- we take the futures rate for the period $[T_{i-1}, T_i]$ as simple forward rate for that period;
- in this data set taken from Bloomberg the futures rate for period $[T_{i-1}, T_i]$ is quoted with respect to T_i, the end date of the accrual period of the underlying LIBOR. For example, the futures rate on the 19.06.2013 corresponds to the rate of the futures contract with settlement date on 20.03.2013;
- the swaps pay annual fixed coupons.

With these assumptions, estimate the corresponding discount curve for the tenor dates indicated in the Excel file:

a) Bootstrap;

b) Pseudo-inverse on increments (see Appendix);

Plot the simple forward curve (simple forward rates between consecutive tenor dates) for each method. Report moreover explicitly the simple forward rate between 04/10/2041 and 03/10/2042 in percentage points and round to 4 decimal places.

7 points

Appendix

Assume the standard setup of a market with n instruments available with cash flow dates $t_0 < T_1 < \cdots < T_N$, $x_i := T_i - t_0$ for $i = 1, \ldots, N$. Instead of estimating the vector of discount factors $d = (D(x_1), \ldots, D(x_N))'$, we would estimate the vector of weighted increments

$$\begin{aligned}\Delta &= \left(\frac{D(x_1) - 1}{\sqrt{x_1}}, \frac{D(x_2) - D(x_1)}{\sqrt{x_2 - x_1}}, \ldots, \frac{D(x_N) - D(x_{N-1})}{\sqrt{x_N - x_{N-1}}}\right)^\top \\ &=: W(Md - e_1),\end{aligned}$$

with $e_1 = \big(1, 0, \ldots, 0\big)^\top$ the first basis vector and

$$W = \operatorname{diag}\left(\tfrac{1}{\sqrt{x_1}}, \ldots, \tfrac{1}{\sqrt{x_N - x_{N-1}}}\right) \quad \text{and} \quad M = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & & \vdots \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

In consequence, Δ satisfies $d = M^{-1}\left(W^{-1}\Delta + e_1\right)$, where

$$W^{-1} = \operatorname{diag}\left(\sqrt{x_1}, \dots, \sqrt{x_N - x_{N-1}}\right) \quad \text{and} \quad M^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Hence the problem for estimating the discount curve reads $CM^{-1}\left(W^{-1}\Delta + e_1\right) = p$, that is

$$CM^{-1}W^{-1}\Delta = p - CM^{-1}e_1, \tag{2}$$

where p is the column vector of N market prices and C the related cash flow matrix. The pseudo inverse solution Δ^* of the problem (2)

$$\Delta^* = A^\top(AA^\top)^{-1}(p - CM^{-1}e_1), \quad A = CM^{-1}W^{-1},$$

will give the smoothest discount curve, in the sense that it minimizes

$$\sum_{i=1}^{N} \left| \frac{D(T_i) - D(T_{i-1})}{\sqrt{T_i - T_{i-1}}} \right|^2 \approx \int_{t_0}^{T_N} D'(x)^2 \mathrm{d}x,$$

where we set $T_0 = t_0$.