Fixed Income Analysis
Exercise Sheet 5

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Please hand in your solutions on Wednesday 23.10.2019 at the beginning of the lecture.

Exercise 1 Consider the SDE of the Vasicek model:

$$\mathrm{d}r(t) = [b + \beta r(t)]\,\mathrm{d}t + \sigma \mathrm{d}W^*(t),$$

where b, β and σ are constants and W^* is a $\mathbb{Q}$-Brownian motion as defined in the lecture notes.

a) Find the solution to this SDE with initial condition $r(0) = r_0 > 0$ and derive expressions for $\mathrm{E}^{\mathbb{Q}}[r(t)]$ and $\mathrm{Var}^{\mathbb{Q}}[r(t)]$.

b) What is the (risk-neutral) probability of having a negative short rate?

c) Derive an expression for $f(0,T)$ and find a sufficient and necessary condition on $r(0)$ that makes $T \mapsto f(0,T)$ decreasing.

3 points

Exercise 2 Consider the Vasicek model

$$\mathrm{d}r(t) = [b + \beta r(t)]\,\mathrm{d}t + \sigma \mathrm{d}W^*(t),$$

with $\beta = -0.86, b/|\beta| = 0.09, \sigma = 0.0148$ and $r(0) = 0.095$. Write an exact[1] Monte Carlo simulation scheme in order to simulate the short rate $(r_t)_{t\geq 0}$ at

[1] We call *exact* a Monte Carlo simulation scheme that is such that the distribution of the $r(t_1), \ldots, r(t_n)$ it produces is precisely that of the underlying stochastic model at times $t_1, \ldots, t_n$ for the same value of $r(0)$.

times $t_i = i/252$, $i = 0, 1, \dots, 2 \times 252$. Compute the Monte Carlo price of the zero-coupon bond maturing at $T = 2$ by approximating the the pathwise value of the bank account at time T as

$$B(T) \approx \exp\left(\frac{1}{252} \sum_{i=0}^{\lfloor T \cdot 252 \rfloor - 1} r_{t_i}\right).$$

Compare your result to the closed form solution for $N = 10'000$ simulations.

3 points

Exercise 3 Assume that $r_0 \in [0, c]$ and that the dynamics of the short rate are given by

$$dr_t = r_t\Big[ac - b + (1-a)r_t\Big]dt + r_t(c - r_t)dW_t$$

where (a, b, c) are nonnegative constants such that $c \leq b$ and W is a Brownian motion under *the real world probability measure.*

a) Give the *risk neutral* dynamics of the short rate under the assumption that the market price of risk is constant and equal to a.

b) Show that the price at time $t \leq T$ of a zero coupon bond with maturity date T is explicitly given by

$$P(t, T) = 1 + D(T - t)r_t$$

for some deterministic function $D(\cdot)$ to be determined.

4 points