Fixed Income Analysis
Solution 7

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This solution sheet only contains hints for solving the exercises and should not be taken as a reference for deserving full grades at an exam.

Exercise 1 Note that the stated dynamics are a Ho-Lee model with linear drift $b(t) = \mu t$.

a) The stated model is affine, yielding the following Riccati equations

$$\partial_t A(t,T) = \frac{\sigma^2}{2} B^2(t,T) - \mu t B(t,T), \quad A(T,T) = 0$$

$$\partial_t B(t,T) = -1, \quad B(T,T) = 0,$$

which have the following solutions

$$A(t,T) = -\frac{\sigma^2}{6}(T-t)^3 + \frac{\mu}{6}T^3 - \frac{\mu}{2}Tt^2 + \frac{\mu}{3}t^3$$

$$B(t,T) = T - t$$

This provides the T-bond price

$$P(t,T) = \exp\left(\frac{\sigma^2}{6}(T-t)^3 - \frac{\mu}{6}T^3 + \frac{1}{2}\mu T t^2 - \frac{1}{3}\mu t^3 - (T-t)r_t\right) \quad \text{for } T \geq t.$$

b) By definition we have $f(t,T) = \partial_T A(t,T) + \partial_T B(t,T) r_t$, which gives us the forward rates

$$f(t,T) = -\frac{\sigma^2}{2}(T-t)^2 + \frac{\mu}{2}T^2 - \frac{\mu}{2}t^2 + r_t$$

The dynamics are, by plugging in dr_t,

$$df(t,T) = \sigma^2(T-t)dt + \sigma dW_t^*$$

c) The drift is $\alpha(t,T) = \sigma^2(T-t)$ and volatility is $\sigma(t,T) = \sigma$. Hence the HJM drift condition

$$\alpha(t,T) = \sigma^2(T-t) = \sigma(t,T)\int_t^T \sigma(t,u)du,$$

is satisfied.

Exercise 2 Note that, from the martingale property of $(P(t,S)/B(t))_{t\geq 0}$ under $\mathbb{Q}$ and the definition of $\mathbb{Q}^S$, we have

$$\begin{aligned}
\pi &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^S r(s)\mathrm{d}s}\left(f(T;T,S)-K\right)^+ \mid \mathcal{F}_t\right] \\
&= P(t,S)\mathbb{E}_{\mathbb{Q}}\left[\frac{P(T,S)B(t)}{P(t,S)B(T)}\left(f(T;T,S)-K\right)^+ \mid \mathcal{F}_t\right] \\
&= P(t,S)\mathbb{E}_{\mathbb{Q}^S}\left[\left(f(T;T,S)-K\right)^+ \mid \mathcal{F}_t\right].
\end{aligned}$$

Now, note that for $t < u \leq T$

$$\begin{aligned}
f(u;T,S) &= -\frac{1}{S-T}\log\left(P(u,S)/P(u,T)\right) \\
&= f(t;T,S) - \frac{1}{S-T}\left(\int_t^u (v(s,S)-v(s,T))\mathrm{d}W^*(s) - \frac{1}{2}(v(s,S)^2 - v(s,T)^2)\mathrm{d}s\right) \\
&= f(t;T,S) - \frac{1}{S-T}\left(\int_t^u (v(s,S)-v(s,T))\mathrm{d}W^S(s) + \frac{1}{2}(v(s,S) - v(s,T))^2\mathrm{d}s\right),
\end{aligned}$$

where $(W^S(t))_{t\geq 0}$ is a Brownian motion under $\mathbb{Q}^S$. Under the assumption that bond volatility processes are deterministic, $f(u;T,S)$ has a Gaussian distribution given $\mathcal{F}_t$ with conditional mean

$$\mu_S(u) = f(t;T,S) + \frac{1}{2(S-T)}\int_t^u (v(s,S)-v(s,T))^2\mathrm{d}s$$

under $\mathbb{Q}^S$ and variance

$$\sigma_S^2(u) = \frac{1}{(S-T)^2}\int_t^u (v(s,S)-v(s,T))^2\mathrm{d}s.$$

Hence, if we denote $\mu_S = \mu_S(T), \sigma_S = \sigma_S(T)$ and we denote $X \sim \mathcal{N}\left(0, \sigma_S^2\right)$, we have

$$\begin{aligned}
\pi &= P(t,S)\mathbb{E}_{\mathbb{Q}^S}\left[(\mu_S + X - K)^+\right] \\
&= P(t,S)\left(\frac{\sigma_S}{\sqrt{2\pi}}e^{-(K-\mu_S)^2/2\sigma_S^2} + (\mu_S - K)\Phi\left((\mu_S - K)/\sigma_S\right)\right).
\end{aligned}$$

Exercise 3

a) Set $u = S$. Under $\mathbb{Q}^S$ the S-bond discounted T-bond price is a martingale, which immediately implies that $F(t,T,S)$ is a $\mathbb{Q}^S$-martingale.

b) Note that

$$\begin{aligned}
\frac{B(t)}{P(t,T)} &= \frac{B(t)}{\mathbb{E}_{\mathbb{Q}}\left(\frac{B(t)}{B(T)} \mid \mathcal{F}_t\right)} \\
&= \frac{\mathbb{E}_{\mathbb{Q}}\left(\frac{B(T)}{P(0,T)B(T)} \mid \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\frac{1}{P(0,T)B(T)} \mid \mathcal{F}_t\right)} \\
&= \mathbb{E}_{\mathbb{Q}^T}\left(B(T) \mid \mathcal{F}_t\right),
\end{aligned}$$

where we have used the definition of the density process and the Bayes' rule in the last equality.

Exercise 4 A solution approach was presented during the exercise session.