Fixed Income Analysis
Exercise Sheet 7

Teacher: Prof. Damir Filipović
Assistant: Lotfi Boudabsa

Please hand in your solutions on Wednesday 06.11.2019 at the beginning of the lecture.

Exercise 1[Jamshidian Decomposition] Consider an affine diffusion short-rate model $r(t)$ with zero-coupon bond prices

$$P(t,T) = e^{-A(t,T)-B(t,T)r(t)}$$

for some deterministic functions $A, B > 0$.

a) Show that in this model, the price of a put option on a coupon bond is identical to the price of a portfolio of put options on zero-coupon bonds with appropriate strike prices. Hint: the function

$$r \mapsto p(r) = \sum_{i=1}^{n} c_i e^{-A(T_0,T_i)-B(T_0,T_i)r}$$

is strictly monotone in r. Show that $p(r) > p(r^*)$ if and only if $e^{-A(T_0,T_i)-B(T_0,T_i)r} > e^{-A(T_0,T_i)-B(T_0,T_i)r^*}$, for all i, for $r, r^* \in \mathbb{R}$.

b) Now consider the Vasiček short-rate model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW^*(t), \quad r(0) = 0.08,$$

with $\beta = -0.86$, $b/|\beta| = 0.09$, and $\sigma = 0.0148$. Use a) to price a put option on the coupon bond with: coupon dates $T_i = (1+i)/4$, $i = 0, \ldots, 3$, maturity of the bond $T_3 = 1$, coupons $c_i = 4$, nominal

$N = 100$, the maturity of the option is $T_0 = 1/4$ (the option payoff does not include the coupon payment c_0), and the strike price of the option is $K =$ price of the underlying coupon bond at $t = 0$ divided by $P(0, T_0)$ ("at-the-money").

3 points

Exercise 2

a) Prove Lemma 28.2.

b) Use Lemma 28.2 to derive the relation between instantaneous forward and futures rates in the Ho-Lee model

$$dr(t) = b(t)dt + \sigma dW^*(t).$$

c) Compute the Eurodollar futures rate convexity adjustment

$$\mathbb{E}_{\mathbb{Q}}[F(T, S)|\mathcal{F}_t] - F(t; T, S)$$

for the Hull-White extended Vasiček short rate model

$$dr(t) = (b(t) + \beta r(t))dt + \sigma dW^*(t).$$

3 points

Exercise 3 In this exercise we will investigate an extension of the Black-Scholes setup where both the volatility of the stock returns and the short rate are stochastic with Gaussian dynamics and we allow all of these stochastic processes to be correlated. More specifically we consider the following system of differential equations:

$$\begin{cases} dS_t &= S_t r_t dt + S_t \sigma_t dW_t^s \\ dr_t &= \lambda(\bar{r} - r_t)dt + \eta dW_t^r \\ d\sigma_t &= \kappa(\bar{\sigma} - \sigma_t)dt + \gamma dW_t^\sigma \end{cases} \tag{0.1}$$

where W^s, W^r and W^σ are Brownian motions under $\mathbb{Q}$ with the following instantaneous correlations:

$$dW_t^s dW_t^r = \rho_{s,r}dt, \quad dW_t^s dW_t^\sigma = \rho_{s,\sigma}dt \quad \text{and} \quad dW_t^r dW_t^\sigma = \rho_{r,\sigma}dt.$$

a) Show that $X_t = [\log(S_t),\, r_t,\, \sigma_t, \sigma_t^2]^\top$ is an affine process.

b) Write down the system of Ricatti equations for ϕ and ψ that have to be solved (you don't have to solve it) in order to compute the characteristic function of X_T:

$$\mathbb{E}_t[e^{u^\top X_T}] = e^{\phi(T-t,u)+\psi(T-t,u)^\top X_t}, \quad u \in i\mathbb{R}^4.$$

4 points