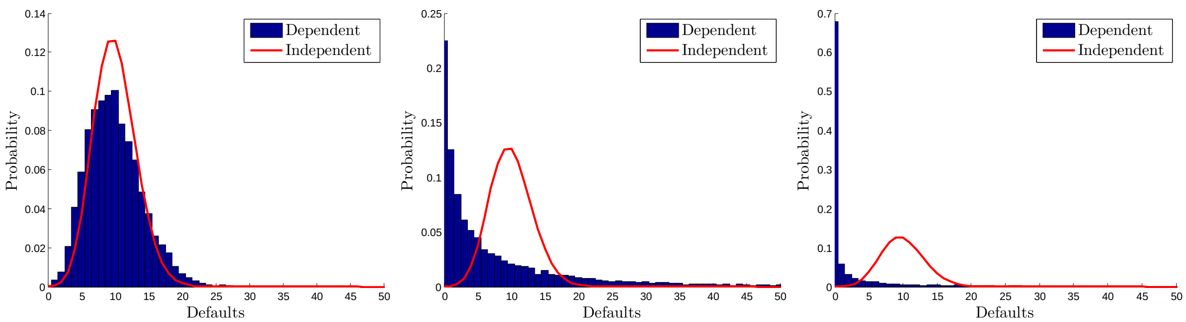
Quantitative Risk Management
Assignment 10 Solutions

December 10, 2019

Question 1: In each case, simulating from the appropriate copula is done by setting:

$$(X_1, \dots, X_N) = (F(Y_1), \dots, F(Y_N))$$

where Y_N has multivariate distribution associated with the corresponding copula (Gaussian or t with the given correlation matrix), and F is the CDF of the marginal distribution, either Gaussian or Student-t. For each simulation of a vector $\boldsymbol{X}$, the number of defaults can be counted, thus giving us a distribution of L by performing several simulations. The empirical distributions are shown in Figure 1.



(a) Gaussian (b) Student-t, $\nu = 10$ (c) Student-t, $\nu = 3$

Figure 1: Empirical distribution of losses for various copulas. The red curve represents the distribution given independent losses.

The quantiles of interest are given in Table 1.

Copula	95^{th} percentile	99^{th} percentile	99.9^{th} percentile
Gaussian	17	22	27
Student-t, $\nu = 10$	42	89	147
Student-t, $\nu = 3$	53	182	339

Table 1: Quantiles of number of defaults for each copula.

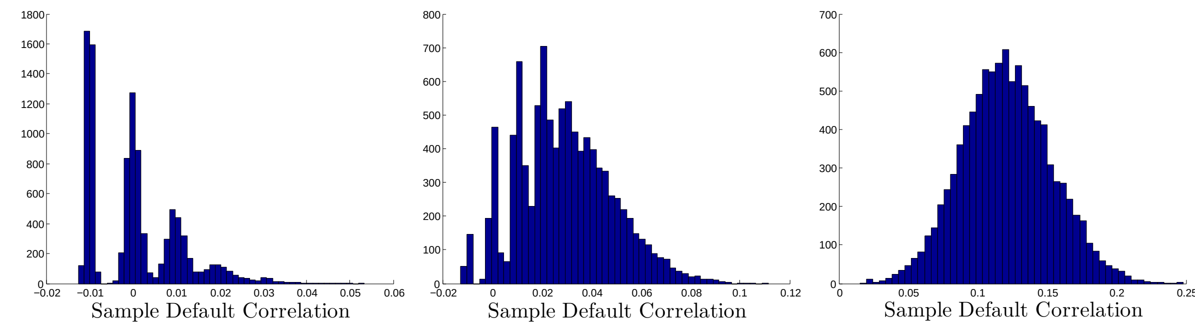
The behaviour of the quantiles is clearly very different for each copula. For the Gaussian copula, the 99.9^{th} percentile is approximately a factor of 2 larger than the 95^{th} percentile. However for the Student-t copula with $\nu = 3$, the different is between a factor of 6 and 7. Even more significant are the differences between a single quantile depending on which copula is used. For the 95^{th} percentile, the difference between the Gaussian and the t copula with $\nu = 3$ is a factor of approximately 3. For the 99.9^{th} percentile, this factor is approximately 13. This drastic difference shows that the default dependence structure is very sensitive to the exact nature of dependence, which can have significant consequences for the apparent riskiness of a credit portfolio, particularly credit derivatives which only take losses after a large number of defaults.

The sample correlations measured between $\mathbb{1}_{X_1 \leq \pi}$ and $\mathbb{1}_{X_2 \leq \pi}$ are given in Table 2.

Copula	Sample Default Correlation
Gaussian	0.0014
Student-t, $\nu = 10$	0.0476
Student-t, $\nu = 3$	0.1294

Table 2: Sample correlations of default pairs.

The confidence attributed to these sample values should be quite small, as repeating the simulation does not appear to reproduce similar results. To demonstrate the lack of accuracy of these estimates, the simulation is repeated 10,000 times, and a distribution of sample correlations is achieved.



(a) Gaussian (b) Student-t, $\nu = 10$ (c) Student-t, $\nu = 3$

Figure 2: Distribution of sample correlations for each copula. The wide distribution indicates little confidence in any single sample correlation estimate.

Question 2: Each standard Monte Carlo simulation uses the following algorithm:

1. Simulate $L_1, \dots, L_n$ independently from distribution F_L.
2. Discard all results which do not satisfy $L_k > VaR_\alpha$.
3. Take the average of all remaining results.

The algorithm for the importance sampling method is derived by first considering the following:

$$\begin{aligned} ES_\alpha &= \mathbb{E}[L|L > VaR_\alpha] \\ &= \frac{1}{1-\alpha}\mathbb{E}[L\mathbb{1}_{L>VaR_\alpha}] \\ &= \frac{1}{1-\alpha}\int_{VaR_\alpha}^{\infty} x f_L(x)dx \\ &= \frac{1}{1-\alpha}\int_{VaR_\alpha}^{\infty} x\frac{f_L(x)}{g_L(x)}g_L(x)dx \\ &= \frac{1}{1-\alpha}\int_{VaR_\alpha}^{\infty} xr(x)g_L(x)dx \\ &= \frac{1}{1-\alpha}\mathbb{E}_G[Lr(L)] \end{aligned}$$

Thus, in order to perform the importance sampling method, we perform a standard Monte Carlo simulation to estimate the expectation of $Lr(L)$ where L has distribution G_L, and then divide by $1-\alpha$. The algorithm is then:

1. Simulate independent exp(1) distributed random variables $X_1, \dots, X_n$ and set $L_k = X_k + VaR_\alpha$.
2. Compute $Z_k = L_k r(L_k)$ and take the average of all the Z_k's.
3. Divide the result by $1-\alpha$.

Using $\alpha = 0.99$ and performing the standard Monte Carlo algorithm 1000 times gives 1000 estimates of ES_α. The average of these estimates is 2.801 and the standard deviation is 0.168. Similarly, performing the importance sampling algorithm 1000 times gives an average result of 2.800 with a standard deviation of 0.039. The mean values are essentially equivalent, but the standard deviation is reduced by a factor of more than 4.