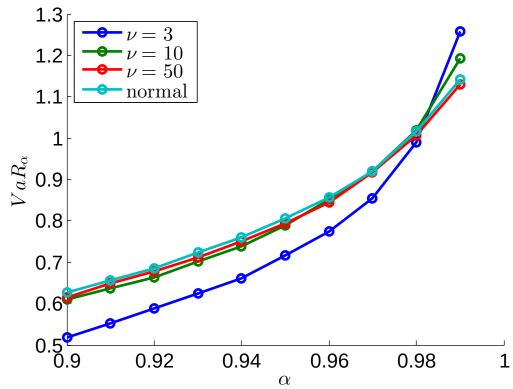
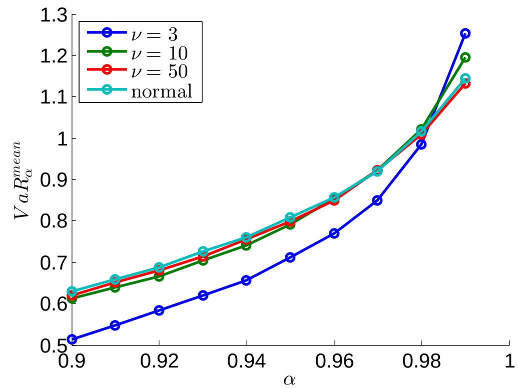
Quantitative Risk Management
Assignment 2 Solutions

Due: October 8, 2019

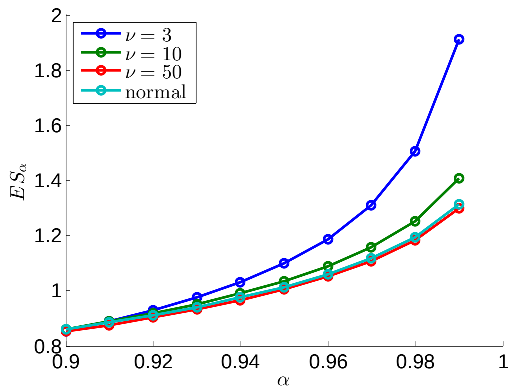
Question 1: The values of VaR_α, VaR_α^{mean} and ES_α are plotted with respect to α below for each distribution of interest.



(a) VaR_α



(b) VaR_α^{mean}



(c) ES_α

Figure 1: Risk versus confidence level for various distributions.

To compare the resulting values of VaR_α and VaR_α^{mean}, we plot the difference below:

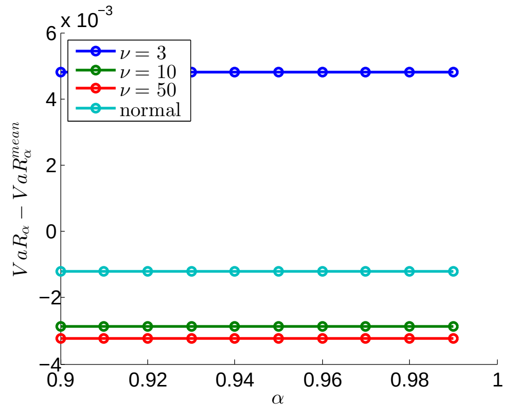


Figure 2: Difference between VaR_α and VaR_α^{mean}.

The difference is constant with respect to α for each distribution because the same random variates were used across all values of α. Note that the difference in each case is close to zero. This is because all of our risk factors have a mean of zero and a small variance, so the expected losses in each case are very close to zero (but not identically zero because the loss is not linear with respect to risk factor changes).

In Figure 1a) above, the value of VaR_α is shown for each distribution. The results may be considered surprising because the t-distributions with 3 degrees of freedom yields the lowest value of VaR_α for most confidence levels, whereas the normal yields the highest. Not until $\alpha = 0.99$ does the typical "fat tail" behaviour of the t-distribution take over and the expected ordering is seen.

In Figure 1c) above, we see what is to be more expected behaviour of ES_α between each distribution: the t-distribution with $\nu = 3$ has the largest value of ES_α for each confidence level, as expected from its "fat tail" characteristic.

Question 2: The value of the portfolio at time 0 is equal to:

$$V_0 = C^{BS}(0, T, S_0, K, r, \sigma_0) + \lambda S_0$$

where the position taken in the stock is equal to:

$$\lambda = -\frac{\partial C^{BS}}{\partial S}(0, T, S_0, K, r, \sigma_0)$$

The position in the stock is held constant over the time period Δ, independent of changes in S. With this in mind note that a derivative of V_0 with respect to S_0 is equal to zero, hence the portfolio is delta neutral. The value of the portfolio at time Δ is:

$$V_\Delta = C^{BS}(\Delta, T, S_0 e^{X_{1,\Delta}}, K, r, \sigma_0 + X_{2,\Delta}) + \lambda S_0 e^{X_{1,\Delta}}$$

where the vector $(X_{1,\Delta}, X_{2,\Delta})$ has the distribution indicated in the question. Thus the loss is equal to:

$$L_\Delta = -C^{BS}(\Delta, T, S_0 e^{X_{1,\Delta}}, K, r, \sigma_0 + X_{2,\Delta}) + C^{BS}(0, T, S_0, K, r, \sigma_0) - \lambda S_0 (e^{X_{1,\Delta}} - 1)$$

The general formula for the linearized loss is:

$$L^\delta_\Delta = -\left(\partial_t f(t, \mathbf{Z}_t)\Delta + \sum_{i=1}^{d} \partial_{Z_i} f(t, \mathbf{Z}_t) X_{i,t+\Delta}\right)$$

which in this specific case becomes:

$$L^\delta_\Delta = -\left(\frac{\partial C^{BS}}{\partial t}(0, T, S_0, K, r, \sigma_0)\Delta + \frac{\partial C^{BS}}{\partial S}(0, T, S_0, K, r, \sigma_0) S_0 X_{1,\Delta} + \lambda S_0 X_{1,\Delta} + \frac{\partial C^{BS}}{\partial \sigma}(0, T, S_0, K, r, \sigma_0) X_{2,\Delta}\right)$$

Since $\lambda = -\frac{\partial C^{BS}}{\partial S}(0, T, S_0, K, r, \sigma_0)$, the dependence on $X_{1,\Delta}$ vanishes:

$$L^\delta_\Delta = -\frac{\partial C^{BS}}{\partial t}(0, T, S_0, K, r, \sigma_0)\Delta - \frac{\partial C^{BS}}{\partial \sigma}(0, T, S_0, K, r, \sigma_0) X_{2,\Delta}$$

To use the variance-covariace method to compute VaR_α, VaR^{mean}_α, and ES_α, we require the mean and variance of L^δ_Δ. These are clearly given by:

$$\mathbb{E}[L^\delta_\Delta] = -\frac{\partial C^{BS}}{\partial t}(0, T, S_0, K, r, \sigma_0)\Delta$$
$$\mathbb{V}[L^\delta_\Delta] = \left(\frac{\partial C^{BS}}{\partial \sigma}(0, T, S_0, K, r, \sigma_0)\right)^2 \mathbb{V}[X_{2,\Delta}]$$

Thus, the following expressions hold for the variance-covariance method:

$$VaR_\alpha = \mathbb{E}[L^\delta_\Delta] + \sqrt{\mathbb{V}[L^\delta_\Delta]}\Phi^{-1}(\alpha)$$
$$VaR^{mean}_\alpha = \sqrt{\mathbb{V}[L^\delta_\Delta]}\Phi^{-1}(\alpha)$$
$$ES_\alpha = \mathbb{E}[L^\delta_\Delta] + \sqrt{\mathbb{V}[L^\delta_\Delta]}\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$$

The table below summarizes the numerical results for each method.

	$\alpha = 0.95$			$\alpha = 0.99$		
	VaR_α	VaR^{mean}_α	ES_α	VaR_α	VaR^{mean}_α	ES_α
Monte-Carlo	0.0364	0.0044	0.0376	0.0383	0.0063	0.0392
Linearized Monte-Carlo	0.0366	0.0044	0.0377	0.0385	0.0063	0.0395
Variance-Covariace	0.0367	0.0045	0.0378	0.0386	0.0064	0.0395

Question 3: Let L have the Student t distribution with ν degrees of freedom. The probability density function of L is:

$$g_\nu(x) = K\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where K is a constant which normalizes the integral of $g_\nu(x)$. Let $t_\nu(x)$ be the cumulative distribution function of L, which is continuous and strictly increasing. Then:

$$\begin{aligned}
ES_\alpha(L) &= \mathbb{E}[L|L > VaR_\alpha] \\
&= \frac{1}{1-\alpha}\int_{VaR_\alpha}^{\infty} x g_\nu(x)dx \\
&= \frac{K}{1-\alpha}\int_{t_\nu^{-1}(\alpha)}^{\infty} x\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\
&= \frac{K}{2(1-\alpha)}\int_{t_\nu^{-1}(\alpha)^2}^{\infty} \left(1+\frac{u}{\nu}\right)^{-\frac{\nu+1}{2}} du \\
&= \frac{K}{2(1-\alpha)}\frac{2\nu}{\nu-1}\left(1+\frac{u}{\nu}\right)^{-\frac{\nu-1}{2}}\Bigg|_{u=t_\nu^{-1}(\alpha)^2} \\
&= \frac{K}{1-\alpha}\frac{\nu}{\nu-1}\left(1+\frac{t_\nu^{-1}(\alpha)^2}{\nu}\right)^{-\frac{\nu-1}{2}} \\
&= \frac{K}{1-\alpha}\frac{\nu}{\nu-1}\left(1+\frac{t_\nu^{-1}(\alpha)^2}{\nu}\right)^{-\frac{\nu+1}{2}}\left(1+\frac{t_\nu^{-1}(\alpha)^2}{\nu}\right) \\
&= \left(\frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\right)\left(\frac{\nu+t_\nu^{-1}(\alpha)^2}{\nu-1}\right)
\end{aligned}$$