FIN-417, Quantitative Risk Management
Final Exam, 25 January 2019

NAME:____________________________________

- Duration: 3 hours.
- Numerical results without analytical derivations receive no grade.
- Writings with a pencil, or on a sheet not distributed during the exam session will not be graded.

Question 1: [10 marks] Consider an AR(1)-GARCH(1,1) model:

$$\begin{aligned} r_t &= \mu + \phi r_{t-1} + \epsilon_t \\ \epsilon_t &= \sigma_t z_t,\ z_t \stackrel{iid}{\sim} \mathcal{N}(0,1) \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned} \tag{1}$$

where r_t is the return of an equity index. The parameter estimates (MLE) of this model are shown as follows:
$\hat{\mu} = 0.1251 \times 10^{-2}$, $\hat{\phi} = -0.0794$, $\hat{\omega} = 0.0686 \times 10^{-4}$, $\hat{\alpha} = 0.0913$, $\hat{\beta} = 0.9004$.

Part a) [2 marks] Lets assume that the conditional mean model in (1) collapses to zero and we have $r_t = \epsilon_t$. Calculate the unconditional variance and check if the process is covariance stationary?

Part b) [3 marks] The last observations you have for this model correspond to $r_T = 1.085\%$, $\epsilon_T = 0.959\%$ and $\hat{\sigma}_T = 2.391\%$. Estimate the next period return and variance. Estimate the $VaR_{0.99}$ for a portfolio with \$1000 invested in the index.

Part c) [2 marks] Given the stylized facts about equity returns, explain in one sentence how you would improve the specification in (1) so that you have a more accurate VaR estimate?

Part d) [3 marks] With the same assumption as in part (a), show the dynamics of squared returns in the GARCH(1,1) process is an ARMA(1,1)?

Hint: Start with $\epsilon_t^2 = \mathbb{E}_{t-1}[\epsilon_t^2] + \nu_t$, where $\nu_t = \sigma_t^2(z_t^2 - 1)$ is the mean zero innovation in the ϵ_t^2 process.

Question 2: [10 marks]

Part a) [4 marks] Let $X_i \sim F$ be a sequence of independent and identically distributed random variables each having exponential distribution with parameter λ. Show that $F \in MDA(H_\xi)$ for some value of ξ.

Hint: recall that $F \in MDA(H_\xi)$ means there are sequences of constants a_n and b_n such that:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_n - a_n}{b_n} \le x\right) = H_\xi(x)$$

It may help to choose the sequence $a_n = \frac{\log(n)}{\lambda}$, and you should already know which value of ξ is valid.

Part b) [4 marks] Let X_t be an ARCH(1) process defined as:

$$\begin{aligned} X_t &= \sigma_t z_t,\ z_t \overset{iid}{\sim} \mathcal{N}(0,1) \\ \sigma_t^2 &= \omega + \alpha X_{t-1}^2 \end{aligned}$$

with $0 < \alpha < 1$. In assignment 5 you obtained that the unconditional fourth moment of X_t is positive and equal to:

$$\mathbb{E}[X_t^4] = \frac{\omega^2(1+\alpha)}{(1-\alpha)(1-3\alpha^2)}\mathbb{E}[z_t^4]$$

where $\mathbb{E}[z_t^4] = 3$.

(1) Calculate the kurtosis of X_t as a function of parameters using:

$$\kappa = \frac{\mathbb{E}[X_t^4]}{\sigma^4}$$

(2) Then compare κ with the kurtosis of z_t, $\kappa_z = 3$.

(3) Based on your comparison, explain in one sentence what is likely to be the domain of attraction (Fréchet, Weibull or Gumbel) for X_t?

Part c) [2 marks] Consider a sequence of independent and identically distributed daily returns R_t. Take a block of 5 days of returns $\{R_{t_k}, R_{t_k+1}, ..., R_{t_k+4}\}$ and consider their minimum as $m_{5,k}$. The probability that $m_{5,k}$ is below -2% is 1%. Calculate the probability that the return process R_t is below -2%.

Question 3: [10 marks] An Archimedean copula with generator ϕ takes the form:

$$C(u_1, \ldots, u_d) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)).$$

Let G be a distribution function on $\mathbb{R}^+$ with $G(0) = 0$ and let $\hat{G}$ be the Laplace-Stieltjes transform of G, extended such that $\hat{G}(\infty) = 0$. Let V have distribution G, and let $U_1, \ldots, U_d$ be conditionally independent given V with conditional distribution:

$$F_{U_i|V}(u_i; v) = e^{-v\hat{G}^{-1}(u_i)}, \quad u_i \in [0,1].$$

Show that:

$$\mathbb{P}(U_1 \le u_1, \ldots, U_d \le u_d) = \hat{G}(\hat{G}^{-1}(u_1) + \cdots + \hat{G}^{-1}(u_d)),$$

thus, $(U_1, \ldots, U_d)$ has distribution given by an Archimedean copula with generator $\phi = \hat{G}^{-1}$.

Question 4: [10 marks]

Part a) [2 marks] Suppose you have access to a computer software that can generate a standard uniform variable. How can you use this to generate a standard normal variable?

Part b) [2 marks] Suppose you have access to computer software that can generate independent standard normal variables. How can you use this to generate a multivariate normal random vector with covariance matrix $\mathbf{\Sigma}$ where $\mathbf{\Sigma}$ is a symmetric positive definite matrix?

Part c) [3 marks] Suppose you have access to computer software that can generate a multivariate normal vector with covariance matrix $\mathbf{\Sigma}$. How can you use this to generate a random vector with distribution $C_{\boldsymbol{P}}^{Ga}$, the Gaussian copula with correlation matrix $\boldsymbol{P}$?

Part d) [3 marks] Let $X \sim Exp(\lambda_X)$ and $Y \sim Exp(\lambda_Y)$. The copula of (X, Y) is the Farlie-Gumbel-Morgenstern copula:

$$C(u, v) = uv + \theta(1-u)(1-v)uv, \qquad -1 \leq \theta \leq 1\,.$$

Let $A = X^2$ and $B = Y^3$. What is the copula of (A, B)?

Question 5: [10 marks] Let $X \sim \mathcal{N}(0, \sigma^2)$ and:

Part a) [3 marks] Suppose you use a standard Monte Carlo simulation to estimate $\theta = \mathbb{E}[(X-K)_+]$. When the number of samples used is large, what fraction of the total samples will make the integrand evaluate to 0?

Part b) [4 marks] Suppose an importance sampling method is used with the likelihood ratio chosen to be:

$$r(x) = e^{\frac{-2xK+K^2}{2\sigma^2}}\,.$$

(1) Derive the importance sampling density. (2) what is the modified integrand? (3) what percentage of the samples will make the integrand evaluate to 0?

Part c) [3 marks] Compute the exact value of θ for $\sigma = 1$ and $K = 1$. You may find the following quantities useful:

$$\int_1^\infty x \exp\left\{-\frac{x^2}{2}\right\} dx = 0.6065 \qquad \Phi(1) = 0.8413$$