ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Quantitative Risk Management - Problem Set 4
Group - G02

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Question 1

First, we have the following function for the CDF of X:

$$\begin{aligned}\mathbb{P}(X \leq x) &= \mathbb{P}(\mu + \sqrt{W}Z \leq x) \\ &= \sum_{i=1}^{k} \mathbb{P}\left((\mu + \sqrt{W}Z \leq x) \cap (W = k_i)\right) \\ &= \sum_{i=1}^{k} \mathbb{P}\left((\mu + \sqrt{k_i}Z \leq x) \cap (W = k_i)\right) \\ &= \sum_{i=1}^{k} \mathbb{P}(\mu + \sqrt{k_i}Z \leq x)\mathbb{P}(W = k_i) \\ &= \sum_{i=1}^{k} \mathbb{P}\left(Z \leq \frac{x-\mu}{\sqrt{k_i}}\right) p_i \\ &= \sum_{i=1}^{k} \Phi\left(\frac{x-\mu}{\sqrt{k_i}}\right) p_i\end{aligned}$$

Where Φ is the CDF function of the gaussian distribution.

If we want to find a function of a variable ν such that a root of the function ν_0 satisfies $\nu_0 = VaR_\alpha(X)$ mu must have $\mathbb{P}(X \leq \nu_0) = \alpha$. Therefore the function that we construct is :

$$f(v) = \sum_{i=1}^{k} \Phi\left(\frac{v-\mu}{\sqrt{k_i}}\right) p_i - \alpha$$

Since f is the sum of k CDF, which are increasing functions, f is an increasing continuous function. Therefore we know that is it has a root, it must be a unique root. Furthermore we have:

$$\lim_{v \to -\infty} f(v) = -\alpha$$
$$\lim_{v \to +\infty} f(v) = 1 - \alpha$$

Since $-\alpha < 0$ and $1 - \alpha > 0$. Therefore f has a unique root ν_0.

We constructed a function f which has a unique root ν_0 such that $\nu_0 = VaR_\alpha(X)$.

Question 2

Let's consider two random variables X which have a uniform distribution on $\{-1, 0, 1\}$ and $Y = ZX$ where Z is a uniform distribution on $\{-1, 1\}$ independent from X. Therefore we have:

$$\mathbb{P}\left((X = 0) \cap (Y = 0)\right) = \mathbb{P}(X = 0) = 1/3$$

$$\mathbb{P}(X = 0)\mathbb{P}(Y = 0) = 1/3 \times 1/3 = 1/9$$

Since $\mathbb{P}\left((X = 0) \cap (Y = 0)\right) \neq \mathbb{P}(X = 0)\mathbb{P}(Y = 0)$ the variables are not independent. Moreover we have:

$$\mathbb{E}(X) = \mathbb{E}(Y) = 0$$
$$\mathbb{E}(XY) = \mathbb{E}(X^2Z) = \mathbb{E}(X^2)\mathbb{E}(Z) = 0$$

Therefore the variables aren't correlated.

Question 3

3.1

We downloaded daily prices for IBM, McDonald's, 3M Company,Wal-Mart Stores and SP500 index for the period March 17^{th}, 2011 to March 16^{th}, 2016. Then, we computed the log-returns and ran the linear regression according to the following model:

$$X = a + \mathbf{BF} + \epsilon$$

with **a** the intercept, supposed to be the risk free rate according to the CAPM, **B** the comparative factor of return volatility between the stocks and SP500, and F the SP500 returns.
We obtained the following results for those regressions:

	IBM	MCD	MMM	WMT
a	-0.000350	0.000198	0.000122	0.000030
Beta	0.812348	0.583537	0.977800	0.522851

Figure 1: Linear Regression Coefficients

3.2

For this question, we used the estimators we found in the previous question to construc the matrix of residual errors such that : $\hat{\epsilon} = X - F\hat{B}$. We obtained the following correlation matrix for the residual errors:

	IBM	MCD	MMM	WMT
IBM	1.000000	0.063572	0.001717	-0.002494
MCD	0.063572	1.000000	0.023304	0.121076
MMM	0.001717	0.023304	1.000000	0.029484
WMT	-0.002494	0.121076	0.029484	1.000000

Figure 2: Correlation matrix of residual errors

Then, we computed the sample correlation matrix of the returns and obtained the following results:

	IBM	MCD	MMM	WMT
IBM	1.000000	0.426359	0.528267	0.320711
MCD	0.426359	1.000000	0.497808	0.381751
MMM	0.528267	0.497808	1.000000	0.419896
WMT	0.320711	0.381751	0.419896	1.000000

Figure 3: Correlation matrix of returns

First, we observe that the returns are more correlated than the residuals. In fact, the returns between stocks vary between 30% and 50% whereas the highest correlations between residuals are 12% between Wal-Mart and McDonald's and 6% between McDonald's and IBM.

The correlation between returns can be explained by the regression computed earlier: a non-negligible part of stock returns variance can be explained by market return variance.
Moreover, all our coefficients **B** are significant since their p-value is really close to zero.

On the other hand, residuals is the variance of stock returns that cannot be explained neither by movement from the market nor by constant return. We observe that the correlation between those residuals is really low. Some variability of stock returns could also be explained by factors such as specific industry chocs or just noise. For example, Wal-Mart and McDonald's are respectively in consumer staples and consumer discretionary industries whereas the 2 other are more B2B oriented. Then, change in buying power of consumer affect both McDonald's and Wal-Mart, it could explain why the correlation of their residuals is high compared to the others.

Question 4

1)

We compute matrix X using the matlab code then we compute the VaR_α for $\alpha = 0.95$ we obtain $VaR_\alpha = 10.42$.

2)

The third eigenvector correspond to the first principal component of X. We can find a link between the biggest component of the covariance matrix that correspond to the variance of the third component and the third coordinate of the first principal component. The first principal component is close to -1 so it explains a large part of the total variance. Then the third component of X contributes mostly to the first principal component.

3)

We approximate the matrix X using the two first principal components using the formula from the lecture note, we first compute $Y = (\Gamma^T(X-\mu)^T)^T$ then we use the formula :

$$X_{approx} = \mu + \Gamma_1 Y_1 + \Gamma_2 Y_2$$

We obtain an approximation of the matrix X then we compute the loss function and the VaR_α for $\alpha = 0.95$ we obtain $VaR_\alpha = 9.87$ that is close to the one obtain without approximation.