Quantitative Risk Management
Assignment 9 Solutions

December 3, 2019

Question 1: Part 1) Selecting the sequences $c_n = \frac{\log(n)}{\beta}$ and $d_n = \frac{1}{\beta}$ and setting $M_n = \max\{X_1, \dots, X_n\}$, we compute:

$$\begin{aligned}
\mathbb{P}\left(\frac{M_n - c_n}{d_n} \leq x\right) &= \mathbb{P}\left(M_n \leq d_n x + c_n\right) \\
&= F^n(d_n x + c_n) \\
&= \left(1 - e^{-\beta(d_n x + c_n)}\right)^n \\
&= \left(1 - e^{-\beta(\frac{x}{\beta} + \frac{\log(n)}{\beta})}\right)^n \\
&= \left(1 - \frac{1}{n}e^{-x}\right)^n
\end{aligned}$$

Taking the limit as $n \to \infty$ gives us the limiting distribution:

$$H(x) = \exp\{-e^{-x}\}$$

which is a GEV distribution with parameter $\xi = 0$.

Part 2) We now select the sequences $c_n = \kappa n^{1/\alpha} - \kappa$ and $d_n = \frac{\kappa n^{1/\alpha}}{\alpha}$ and again compute:

$$\begin{aligned}
\mathbb{P}\left(\frac{M_n - c_n}{d_n} \leq x\right) &= \mathbb{P}\left(M_n \leq d_n x + c_n\right) \\
&= F^n(d_n x + c_n) \\
&= \left(1 - \left(\frac{\kappa}{\kappa + d_n x + c_n}\right)^\alpha\right)^n \\
&= \left(1 - \left(\frac{\kappa}{\kappa + \frac{\kappa n^{1/\alpha}}{\alpha}x + \kappa n^{1/\alpha} - \kappa}\right)^\alpha\right)^n \\
&= \left(1 - \frac{1}{n}\left(1 + \frac{x}{\alpha}\right)^{-\alpha}\right)^n
\end{aligned}$$

Taking the limit $n \to \infty$ gives us the limiting distribution:

$$H(x) = \exp\{-(1 + \frac{x}{\alpha})^{-\alpha}\}$$

which is a GEV distribution with $\xi = \frac{1}{\alpha}$.

Question 2: If X has excess distribution over the threshold u of $F_u = G_{\xi,\beta}$, this means:

$$\mathbb{P}(X - u \leq x | X > u) = F_u(x) = G_{\xi,\beta}(x)$$

We now compute:

$$\begin{aligned}
\mathbb{P}(X - v \leq x | X > v) &= 1 - \mathbb{P}(X - v > x | X > v) \\
&= 1 - \frac{\mathbb{P}(X > x + v)}{\mathbb{P}(X > v)} \\
&= 1 - \frac{\mathbb{P}(X > x + v)}{\mathbb{P}(X > u)} \frac{\mathbb{P}(X > u)}{\mathbb{P}(X > v)} \\
&= 1 - \frac{\mathbb{P}(X - u > x + v - u)}{\mathbb{P}(X > u)} \frac{\mathbb{P}(X > u)}{\mathbb{P}(X - u > v - u)} \\
&= 1 - \mathbb{P}(X - u > x + v - u | X > u) \frac{1}{\mathbb{P}(X - u > v - u | X > u)} \\
&= 1 - \frac{1 - \mathbb{P}(X - u \leq x + v - u | X > u)}{1 - \mathbb{P}(X - u \leq v - u | X > u)} \\
&= 1 - \frac{1 - G_{\xi,\beta}(x + v - u)}{1 - G_{\xi,\beta}(v - u)} \\
&= 1 - \left(\frac{1 + \frac{\xi(x+v-u)}{\beta}}{1 + \frac{\xi(v-u)}{\beta}} \right)^{-\frac{1}{\xi}} \\
&= 1 - \left(\frac{\beta + \xi(x + v - u)}{\beta + \xi(v - u)} \right)^{-\frac{1}{\xi}} \\
&= 1 - \left(1 + \frac{\xi x}{\beta + \xi(v - u)} \right)^{-\frac{1}{\xi}} \\
&= G_{\xi,\beta+(v-u)\xi}(x)
\end{aligned}$$

Question 3: Part 1) The negative log returns are computed, and the sample mean excess function defined by:

$$e_n(v) = \frac{\sum_{i=1}^{n}(X_i - v)\mathbb{1}_{X_i > v}}{\sum_{i=1}^{n}\mathbb{1}_{X_i > v}}$$

is plotted at the points $X_1, \ldots, X_n$ to give the result in Figure 1.

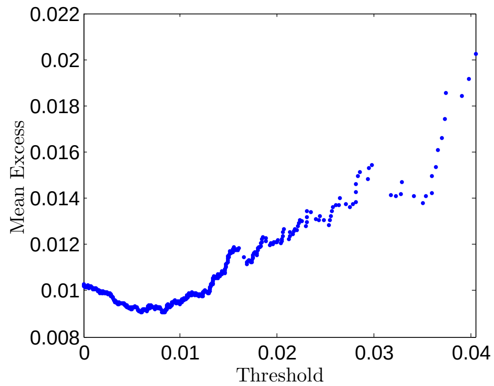

Figure 1: Sample mean excess plot for MSFT data.

Part 2) In performing a GPD fit to the data, a threshold of $u = 0.01$ is selected. The data is transformed into $\hat{X}_i = X_i - u$. This leaves N_u points that are positive, which are the only ones we consider, and we relabel the data as $\hat{X}_1, \ldots, \hat{X}_{N_u}$. We assume that $\hat{X}_1, \ldots, \hat{X}_{N_u}$ come from a GPD distribution and estimate the parameters ξ and β by maximizing the log likelihood function:

$$\log(L(\xi, \beta; \hat{X}_1, \ldots, \hat{X}_{N_u})) = \sum_{i=1}^{N_u} \log(g_{\xi,\beta}(\hat{X}_i))$$

The resulting estimates are:

$$\hat{\xi} = 0.2171$$
$$\hat{\beta} = 0.0075$$