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QRM - Pb Set 9

group 2

Question 1

$(X_i)_{i\in\mathbb{N}}$ be independent with distribution F

1. $F(x) = 1 - \exp(-\beta x)$, $\beta > 0$ and $x \geq 0$

Let setting $M_n = \max\{X_1, \dots, X_n\}$

$$c_n = \frac{\log(n)}{\beta} \quad , \quad d_n = \frac{1}{\beta}$$

Therefore we have.

$$\begin{aligned}
\mathbb{P}\left(\frac{M_n - c_n}{d_n} \leq x\right) &= \mathbb{P}\left(M_n \leq d_n x + c_n\right) \\
&= \mathbb{P}\left(\max\{X_1, \dots X_n\} \leq d_n x + c_n\right) \\
&= \mathbb{P}\left(\bigcap_{i=1}^{n} (X_i \leq d_n x + c_n)\right) \quad \text{independent} \\
&= \prod_{i=1}^{n} \mathbb{P}(X_i \leq d_n x + c_n) \\
&= F^n(d_n x + c_n) \\
&= \left(1 - e^{-\beta(d_n x + c_n)}\right)^n \\
&= \left(1 - e^{-\beta\left(\frac{1}{\beta}x + \frac{\log n}{\beta}\right)}\right)^n \\
&= \left(1 - \frac{1}{n} e^{-x}\right)^n \\
&= e^{n \ln\left(1 - \frac{1}{n} e^{-x}\right)} \\
&\underset{n\to+\infty}{\sim} e^{-e^{-x}}
\end{aligned}$$

Therefore when we take the limit of $\mathbb{P}\left(\frac{M_n - c_n}{d_n} \leq x\right)$ we have

$$H(x) = e^{-e^{-x}}$$

Which is a GEV distribution with parameter $\xi = 0$

$\Rightarrow F \in MDA(H_0)$

2. Now we set up: $M_n = \max\{X_1, \dots, X_n\}$

$$c_n = Kn^{1/\alpha} - K$$

$$d_n = \frac{Kn^{1/\alpha}}{\alpha}$$

And $F(x) = 1 - \left(\frac{K}{K+x}\right)^\alpha$

As before we have:

$$P\left(\frac{M_n - c_n}{d_n} \leq x\right) = F^n(d_n x + c_n)$$

$$= \left[1 - \left(\frac{K}{K + d_n x + c_n}\right)^\alpha\right]^n$$

$$= \left[1 - \left(\frac{K}{K + \frac{Kn^{1/\alpha}}{\alpha}x + Kn^{1/\alpha} - K}\right)^\alpha\right]^n$$

$$= \left[1 - \left(\frac{1}{\frac{n^{1/\alpha}x}{\alpha} + n^{1/\alpha}}\right)^\alpha\right]^n$$

$$= \left[1 - \frac{1}{n}\left(\frac{\alpha}{\alpha + x}\right)^\alpha\right]^n$$

$$= e^{n\ln\left(1 - \frac{1}{n}\left(\frac{\alpha}{\alpha+x}\right)^\alpha\right)}$$

$$\underset{n\to\infty}{\sim} e^{-\left(\frac{\alpha}{\alpha+x}\right)^\alpha}$$

Therefore by taking the limit in $P\left(\frac{M_n - c_n}{d_n} \leq x\right)$ we have:

$$H(x) = e^{-\left(\frac{\alpha}{\alpha+x}\right)^\alpha} = e^{-\left(1 + \frac{x}{\alpha}\right)^{-\alpha}}$$

$$H(x) = e^{-(1+\xi x)^{-1/\xi}} \quad \text{avec} \quad \xi = \frac{1}{\alpha}$$

Which is a GEV distribution with parameter $\xi = 1/\alpha$

$\Rightarrow F \in MDA(H_{1/\alpha})$

Question 2

X has excess distribution over the threshold u of $F_u = G_{\xi,\beta}$:

$$= \begin{cases} \mathbb{P}(X - u \leq x \mid X > u) = F_u(x) \\ G_{\xi,\beta} = \begin{cases} 1 - \left(1 + \frac{\xi x}{\beta}\right)^{-1/\xi} & , \ \xi \neq 0 \\ 1 - \exp\left(-\frac{x}{\beta}\right) & , \ \xi = 0 \end{cases} \end{cases}$$

Now we take $v \geq u$:

$$F_v(x) = \mathbb{P}(X - v \leq x \mid X > v)$$

$$= 1 - \mathbb{P}(X - v > x \mid X > v)$$

$$= 1 - \frac{\mathbb{P}((X > v + x) \cap (X > v))}{\mathbb{P}(X > v)} \qquad x \geq 0 \Rightarrow (X \geq v) \subset (X + v + x)$$

$$= 1 - \frac{\mathbb{P}(X > v + x)}{\mathbb{P}(X > v)}$$

$$= 1 - \frac{\mathbb{P}(X - u > v + x - u)}{\mathbb{P}(X - u > v - u)} \quad \frac{\mathbb{P}(X > u)}{\mathbb{P}(X > u)} \qquad v \geq u \Rightarrow (X > v) \subset (X > u)$$

$$= 1 - \frac{\mathbb{P}[(X - u > v + x - u) \cap (X > u)]}{\mathbb{P}(X > u)} \quad \frac{\mathbb{P}(X > u)}{\mathbb{P}[(X - u > v - u) \cap (X > u)]}$$

$$= 1 - \frac{\mathbb{P}(X - u > x + v - u \mid X > u)}{\mathbb{P}(X - u > v - u \mid X > u)}$$

$$= 1 - \frac{1 - \mathbb{P}(X - u \leq x + v - u \mid X > u)}{1 - \mathbb{P}(X - u \leq v - u \mid X > u)}$$

$$= 1 - \frac{1 - G_{\xi,\beta}(x + v - u)}{1 - G_{\xi,\beta}(v - u)}$$

$$= 1 - \left(\frac{1 + \xi/\beta\,(x + v - u)}{1 + \xi/\beta\,(v - u)}\right)^{-1/\xi}$$

$$= 1 - \left(\frac{\beta + \xi(v - u) + \xi x}{\beta + \xi(v - u)}\right)^{-1/\xi}$$

$$= 1 - \left(1 + \frac{\xi x}{\beta + \xi(v - u)}\right)^{-1/\xi}$$

$$= G_{\xi,\, \beta + \xi(v - u)}(x)$$

We find out what we wanted : $\forall x \geq 0$:

$$F_v(x) = G_{\xi, \beta + (u-v)\xi}(x)$$

Question 3

1. We computed the negative log-returns with our matlab code then we construct the sample mean excess plot :

$$\{(X_i , e_n(X_i))\}_{X_i > 0}$$

where $e_n(v) = \dfrac{\sum_{i=1}^{n} (X_i - v)\, \mathbb{1}_{X_i > v}}{\sum_{i=1}^{n} \mathbb{1}_{X_i > v}}$

The plot is on our python code

2. We selected a threshold of 0.01 for the negative log returns and fit a GPD model to the excess distribution over the threshold by maximum likelihood estimation.

The function that we maximized is :

$$\sum_{i=1}^{N_v} \log(g_{\xi,\beta}(\hat{X}_i))$$

where N_v is the positive $\hat{X}_i$

$$\hat{X}_i = X_i - v$$

$$g_{\xi,\beta}(\hat{X}_i) = \frac{1}{\beta}\left(1 + \frac{\xi \hat{X}_i}{\beta}\right)^{-(1+\frac{1}{\xi})}$$

We find out $\hat{\xi} = 0.2189$

$\hat{\beta} = 0.0074$