ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Quantitative Risk Management - Problem Set 2
Group - G02

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Question 1

1. We plot the VaR_α, VaR_α^{mean}, and ES_α for every distributions with the α in x-axis. (we used the same parameter for student approximation as we find in the problem set 1). We obtained:

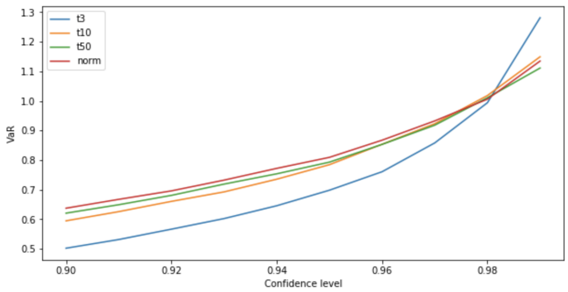

Figure 1: VaR_α in function of α

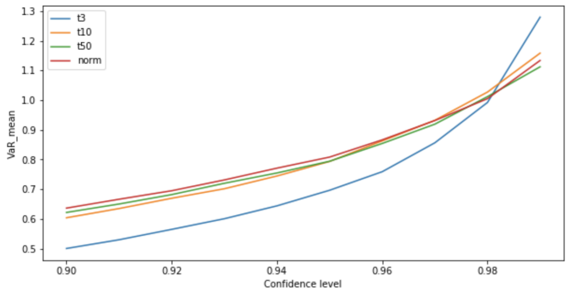

Figure 2: VaR_α^{mean} in function of α

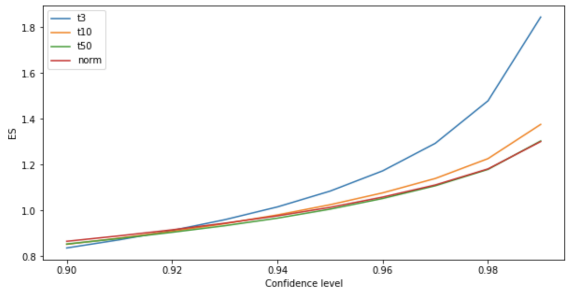

Figure 3: ES_α in function of α

2a. We plot the difference of the Var_α in compare of VaR_α^{mean}; we obtained:

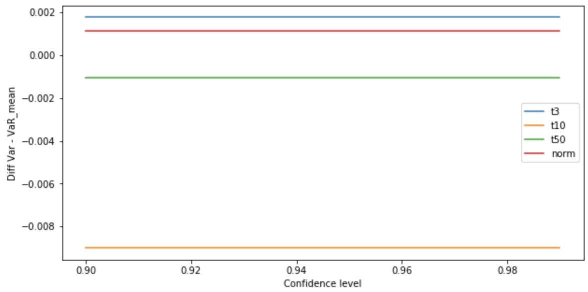

Figure 4: $VaR_\alpha - VaR_\alpha^{mean}$ in function of α

On the graph above, we observe that the difference between VaR_α and VaR_α^{mean} is independent of the confidence level. It can be explained by the fact that $VaR_\alpha^{mean} - VaR_\alpha = mean(Losses)$ which does not depend on the confidence level chosen. Theoretically, we should have a mean of zero for all distributions, but in practice we obtain mean losses in order 10^{-3}. It explains why there is no logic in the order between the different distribution.

2b. On all 3 graphs, the distributions show that when you take student distribution with low degree of freedom you obtain more extreme values than with high degree of freedom. Knowing that a normal distribution is a student distribution, we should obtain the following order (or the inverse) : t3 , t10, t50 , norm.

Moreover, on both the VaR_α^{mean} and VaR_α graphs, we observe that the student distribution with low degree of freedom tends to have lower VaR for low confidence level and higher VaR for high confidence level. For confidence level of 90% we observe the logic order: norm , t50, t10, t3. On the opposite, for a confidence level of 99% we observe the inverse order: t3 , t10, t50 , norm.

Question 2

We defined $L(t, t+\Delta)$ by :

$$L(t,t+\Delta) = -\left(C^{BS}(S_{t+\Delta}, r_{t+\Delta}, \sigma_{t+\Delta}, t+\Delta t) + \lambda S_{t+\Delta} - \left(C^{BS}(S, r, \sigma, t) + \lambda S_t\right)\right)$$

Where $S_{t+\Delta} = S_t e^{X_{1,t+\Delta}}$, $\sigma_{t+\Delta} = \sigma_t + X_{2,t+\Delta}$. C^{BS} is the Black Scholes formula for a call option and $\lambda = -\frac{\partial C^{BS}}{\partial S}(0, T, S_0, K, r, \sigma_0)$.
Therefore we have:

$$L(0,\Delta) = -C^{BS}\left(S_\Delta, \sigma_\Delta, \Delta\right) + C^{BS}\left(S_0, \sigma_0, 0\right) - \lambda S_0\left(e^{X_{1,\Delta}} - 1\right)$$

For the linearized loss, we have:

$$L^\delta(0,\Delta) = -\left(\partial_t C^{BS}\left(S_0, \sigma_0, 0\right)\Delta + \partial_\sigma C^{BS}\left(S_0, \sigma_0, 0\right) X_{2,\Delta}\right)$$

Now that we have the formula for the loss and the linearized loss, we can compute the Monte Carlo method.
To compute the VaR_α, VaR_α^{mean}, and ES_α with the variance-covariance method, we have to ocmpute the mean and the variance of $L^\delta(0,\Delta)$:

$$\mathbb{E}\left[L^\delta(0,\Delta)\right] = -\partial C_t^{BS}\left(S_0, \sigma_0, 0\right)\Delta$$

$$\mathrm{V}\left[L^\delta(0,\Delta)\right] = \left(\partial_\sigma C^{BS}\left(S_0, \sigma_0, 0\right)\right)^2 \mathrm{V}\left[X_{2,\Delta}\right]$$

Now we can use the formula from the lecture for the variance-covariance method:

$$VaR_a = \mathbb{E}\left[L^\delta(0,\Delta)\right] + \sqrt{\mathbb{V}\left[L^\delta(0,\Delta)\right]}\Phi^{-1}(\alpha)$$

$$VaR_\alpha^{mean} = \sqrt{\mathbb{V}\left[L^\delta(0,\Delta)\right]}\Phi^{-1}(\alpha)$$

$$ES_\alpha = \mathbb{E}\left[L^\delta(0,\Delta)\right] + \sqrt{\mathbb{V}\left[L^\delta(0,\Delta)\right]}\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}$$

We can sum-up the numerical values in the following table:

	$VaR_{0.05}$	$VaR_{0.05}^{mean}$	$ES_{0.05}$	$VaR_{0.01}$	$VaR_{0.01}^{mean}$	$ES_{0.01}$
Loss - M-C	-0.0684737	-0.0684405	-0.0864117	-0.0975878	-0.0975545	-0.11065
Lin Loss - M-C	-0.226462	-0.258285	-0.291916	-0.335745	-0.367568	-0.385898
Variance Covariance	-0.228377	-0.228377	0.0494053	-0.336341	-0.336341	0.0364712

Tableau 1: VaR_α, VaR_α^{mean} and ES_α with the three methods and $\alpha = 0.05$ and $\alpha = 0.01$

Question 3

Let L have the Student t distribution with ν degrees of freedom, the density function of L is:

$$g_\nu = K \times \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where K is a constant to normalize the distribution. We also introduce t_ν the cumulative distribution of L. Therefore we have:

$$\begin{aligned}
ES_\alpha &= \mathbb{E}\left[L|L > VaR_\alpha\right] \\
&= \frac{1}{1-\alpha} \int_{VaR_\alpha}^{\infty} x g_\nu(x) dx \\
&= \frac{K}{1-\alpha} \int_{VaR_\alpha}^{\infty} x \times \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\
&= \frac{K}{2(1-\alpha)} \int_{VaR_\alpha^2}^{\infty} \left(1 + \frac{u}{\nu}\right)^{-\frac{\nu+1}{2}} du \\
&= \frac{K}{1-\alpha} \times \frac{\nu}{\nu-1} \left[\left(1 + \frac{u}{\nu}\right)^{-\frac{\nu-1}{2}}\right]_{VaR_\alpha^2}^{\infty}
\end{aligned}$$

Since $VaR_\alpha = t_\nu^{-1}(\alpha)$ we have:

$$\begin{aligned}
ES_\alpha &= \frac{K}{1-\alpha} \times \frac{\nu}{\nu-1} \times \left(1 + \frac{t_\nu^{-1}(\alpha)^2}{\nu}\right)^{-\frac{\nu-1}{2}} \\
&= \frac{K}{1-\alpha} \times \frac{\nu}{\nu-1} \times \left(1 + \frac{t_\nu^{-1}(\alpha)^2}{\nu}\right)^{-\frac{\nu+1}{2}} \times \left(1 + \frac{t_\nu^{-1}(\alpha)^2}{\nu}\right) \\
&= \left(\frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\right)\left(\frac{\nu + t_\nu^{-1}(\alpha)^2}{\nu-1}\right)
\end{aligned}$$