Quantitative Risk Management
Assignment 6 Solutions

November 12, 2019

Question 1: Part 1) We have $Y_t = \phi Y_{t-1} + \epsilon_t$ with $|\phi| < 1$ and $(\epsilon_t)_{t\in\mathbb{Z}}$ being white noise. The variance of Y_t is:

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \mathbb{E}[\phi^2 Y_{t-1}^2 + 2Y_{t-1}\epsilon_t + \epsilon_t^2] \\
\mathbb{E}[Y_t^2] &= \phi^2 \mathbb{E}[Y_t^2] + 1 \\
\mathbb{E}[Y_t^2] &= \frac{1}{1-\phi^2}
\end{aligned}$$

Next, we compute the covariance of Y_t and Y_{t+h}. Assume $h > 0$ so that:

$$\begin{aligned}
Y_{t+h} &= \phi^h Y_t + \sum_{i=1}^{h} \phi^{i-1}\epsilon_{t-i} \\
Y_{t+h}Y_t &= \phi^h Y_t^2 + \sum_{i=1}^{h} \phi^{i-1}\epsilon_{t-i}Y_t \\
\mathbb{E}[Y_{t+h}Y_t] &= \frac{\phi^h}{1-\phi^2} + \sum_{i=1}^{h} \phi^{i-1}\mathbb{E}[\epsilon_{t-i}Y_t] \\
\mathbb{E}[Y_{t+h}Y_t] &= \frac{\phi^h}{1-\phi^2}
\end{aligned}$$

and so the autocorrelation is ϕ^h. Repeating the calculation for $h < 0$ gives a similar result to show that we can write $\rho(h) = \phi^{|h|}$.

Part 2) We start with the identity $X_t^2 = \sigma_t^2 + \sigma_t^2(Z_t^2 - 1)$ and show that $\sigma_t^2(Z_t^2-1)$ is a martingale difference sequence. Since we are assuming that $\mathbb{E}[X_t^4] < \infty$, we also have $\mathbb{E}[X_t^2] < \infty$ which implies $\mathbb{E}[\sigma_t^2] < \infty$. Since Z_t is independent of σ_t, we have:

$$\begin{aligned}
\mathbb{E}\left[|\sigma_t^2(Z_t^2-1)|\right] &\leq \mathbb{E}[\sigma_t^2]\mathbb{E}[Z_t^2+1] \\
&= 2\mathbb{E}[\sigma_t^2] \\
&< \infty
\end{aligned}$$

Also:

$$\begin{aligned}
\mathbb{E}[\sigma_t^2(Z_t^2-1)|\mathcal{F}_{t-1}] &= \sigma_t^2\mathbb{E}[Z_t^2-1|\mathcal{F}_{t-1}] \\
&= \sigma_t^2\mathbb{E}[Z_t^2-1] \\
&= 0
\end{aligned}$$

And so $\sigma_t^2(Z_t^2-1)$ is a martingale difference sequence. Next, we verify that $\sigma_t^2(Z_t^2-1)$ has finite variance:

$$\begin{aligned}
\mathbb{E}[\sigma_t^4(Z_t^2-1)^2] &= \mathbb{E}[\sigma_t^4 Z_t^4 - 2\sigma_t^4 Z_t^2 + \sigma_t^4] \\
&= \mathbb{E}[X_t^4 - 2X_t^2\sigma_t^2 + \sigma_t^4] \\
&= \mathbb{E}[X_t^4 - 2X_t^2(\alpha_0 + \alpha_1 X_{t-1}^2) + \alpha_0^2 + 2\alpha_0\alpha_1 X_{t-1}^2 + \alpha_1^2 X_{t-1}^4] \\
&= \mathbb{E}[X_t^4 - 2\alpha_0 X_t^2 + 2\alpha_1 X_t^2 X_{t-1}^2 + \alpha_0^2 + 2\alpha_0\alpha_1 X_{t-1}^2 + \alpha_1^2 X_{t-1}^4]
\end{aligned}$$

This expression is finite because we assume $\mathbb{E}[X_t^4] < \infty$, and it is constant because $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary. Since $\sigma_t^2(Z_t^2 - 1)$ is a martingale difference sequence with finite constant variance, it is white noise which we will denote ϵ_t. The equation for X_t^2 is then:

$$
\begin{aligned}
X_t^2 &= \sigma_t^2 + \sigma_t^2(Z_t^2 - 1) \\
X_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \epsilon_t
\end{aligned}
$$

The desired expression is:

$$X_t^2 - c = \phi(X_{t-1}^2 - c) + \epsilon_t$$

Clearly we must choose $\phi = \alpha_1$ and $c = \frac{\alpha_0}{1-\alpha_1}$ for $X_t^2 - c$ to be $AR(1)$.

Part 3) We have immediately from Part 1) that the autocorrelation function for $(X_t^2 - c)_{t\in\mathbb{Z}}$ is $\rho(h) = \alpha_1^{|h|}$.

Question 2: The conditional maximum likelihood estimation is performed as outlined in lecture. A two year moving window is used to estimate model parameters each day for three years. In the case of normally distributed innovations, the function which is maximized with respect to $(\alpha_0, \alpha_1, \beta_1)$ is:

$$\sum_{k=1}^{n} \log\left(\frac{1}{\sigma_k}\phi(\frac{x_k}{\sigma_k})\right)$$

where ϕ is the pdf of a standard normal and:

$$\sigma_k^2 = \alpha_0 + \alpha_1 x_{k-1}^2 + \beta_1 \sigma_{k-1}^2 \,, \tag{1}$$

$$\sigma_0 = \hat{S}_X \,. \tag{2}$$

In the case of the t-distributed innovations, the function to be maximized with respect to $(\alpha_0, \alpha_1, \beta_1, \nu)$ is:

$$\sum_{k=1}^{n} \log\left(\frac{c}{\sigma_k} g_\nu(\frac{c x_k}{\sigma_k})\right)$$

where $c = \sqrt{\frac{\nu}{\nu-2}}$ and g_ν is the pdf of the standard t variable with ν degrees of freedom.
Suppose at time t the parameters have been estimated using the previous two years of returns. We then compute σ_{t+1} using equations (1) and (2), and we make the VaR_α estimate as:

$$VaR_\alpha = \sigma_{t+1} q_\alpha(Z) - \hat{\mu} \,. \tag{3}$$

where $\hat{\mu}$ is the sample mean of the two year window of returns. Figure 1 shows the daily VaR_α estimates for both estimated models, as well as the observed return on that day. Breaches are indicated with coloured dots.

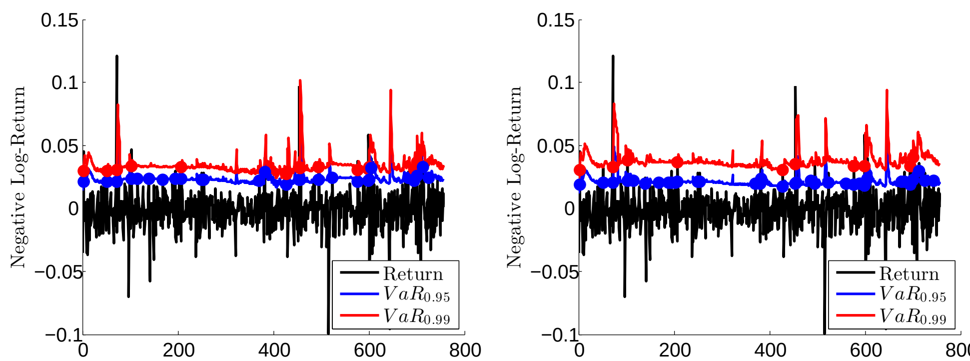

Figure 1: VaR_α estimates for normal and student-t innovations. When using normal innovations, the number of VaR_α breaches is equal to 27 for $\alpha = 0.95$ and 14 for $\alpha = 0.99$. With student-t innovations, the number of breaches is equal to 38 for $\alpha = 0.95$ and 10 for $\alpha = 0.99$.

In all estimated models, the conditions of covariance stationarity are satisfied by the parameters: $\alpha_1 + \beta_1 < 1$. This allows us to compute the variance of X_t implied by the parameters in each case. Figure 2 shows for each day the empirical variance of the two year moving window of returns, as well as the theoretical variance given by the estimated model in both cases.

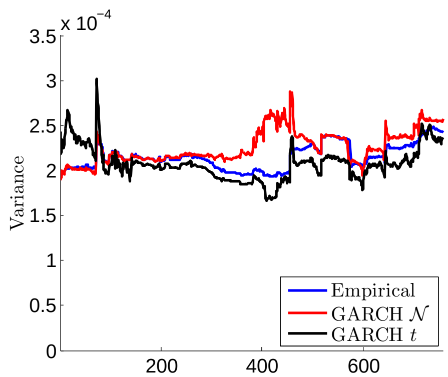

Figure 2: Daily two year window empirical variance and theoretical variance given by estimated model.