Quantitative Risk Management
Assignment 8 Solutions

November 26, 2019

Question 1: The pseudo-sample is computed and plotted in Figure 1.

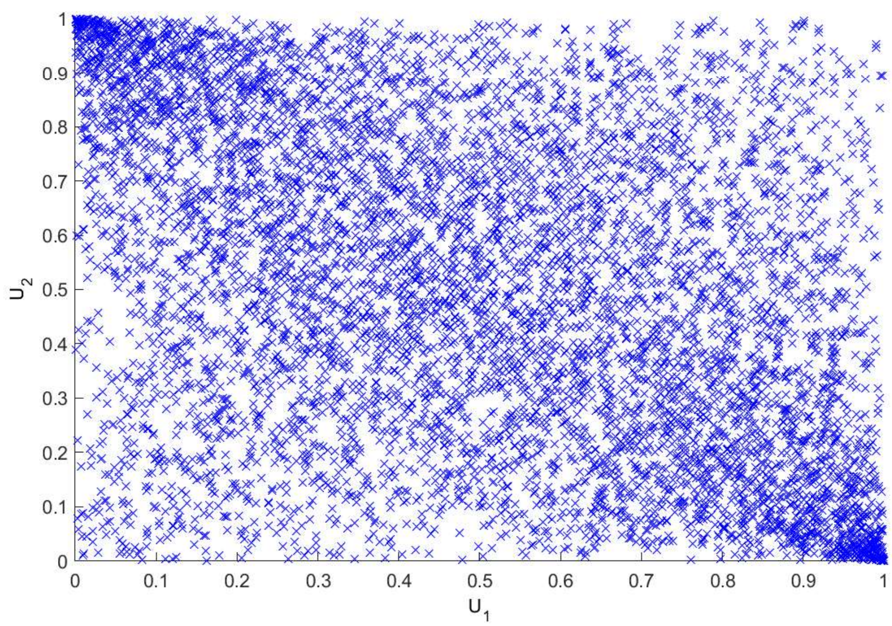

Figure 1: Pseudo-sample using empirical distribution.

Using this set of data, the value of the likelihood function can be computed given a copula family and a value of parameter θ using the Matlab function `copulapdf`. The value of θ which maximizes the likelihood is computed for the Gumbel, Clayton, and Frank copulas. The resulting parameters and log-likelihood are shown in Table 1.

Family	θ^*	$\log(L(\theta^*, \hat{\boldsymbol{U}}))$
Gumbel	1.00	-0.0059
Clayton	0.00	-0.0032
Frank	-3.13	825.73

Table 1: Maximum log-likelihood estimates of θ for each type of copula.

We see that the Frank coplua with $\theta = -3.13$ maximizes the likelihood over all possible copluas of interest, and so this is likely the coplua family from which the original data was generated.

Question 2: The marginal distribution of both X and Y is $\mathcal{N}(0,1)$, which is continuous, so we may write:

$$\begin{aligned}
C(u_1, u_2) &= \mathbb{P}(F(X) \leq u_1, F(Y) \leq u_2) \\
&= \mathbb{P}(F(X) \leq u_1, F(X) \leq u_2)\mathbb{P}(Z = 1) + \mathbb{P}(F(X) \leq u_1, F(-X) \leq u_2)\mathbb{P}(Z = -1) \\
&= p \min(u_1, u_2) + (1 - p) \max(u_1 + u_2 - 1, 0)
\end{aligned}$$

Note that this is simply a convex combination of the comonotonicity and countermonotonicity copulas.

Question 3: By first conditioning on the value of V then integrating over the distribution of V we have

$$\mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d) = \int_0^\infty \mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d | V = v) dG(v) \, .$$

Since $U_1, \ldots, U_d$ are conditionally independent given V by assumption, the integrand above factors into the given conditional distributions:

$$\begin{aligned}
\int_0^\infty \mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d | V = v) dG(v) &= \int_0^\infty \prod_{i=1}^d F_{U_i|V}(u_i; v) dG(v) \\
&= \int_0^\infty \exp\Big(-v(\hat{G}^{-1}(u_1) + \cdots + \hat{G}^{-1}(u_d)) \Big) dG(v) \\
&= \hat{G}(\hat{G}^{-1}(u_1) + \cdots + \hat{G}^{-1}(u_d)) \, ,
\end{aligned}$$

where the last equality is simply the definition of the Laplace-Stieltjes transform of G.