MATH 471 - Quantitative Risk Management
2017 Final Exam

NAME:______________________________________

Question 1: [10 marks]

Let $X_1, \ldots, X_n$ be independent and identically distributed normal random variables.

Part a) [4 marks] Let $\hat{\mu}$ and $\hat{\sigma}^2$ be the sample mean and variance. Also, let μ^* and σ^* be the maximum likelihood estimates of the mean and standard deviation. Show that $\hat{\mu} = \mu^*$ and $\hat{\sigma} = \sigma^*$.

Part b) [4 marks] Suppose that the true mean and variance of the X_i's are μ and σ^2. Compute the expected value of $\hat{\mu}$ and $\hat{\sigma}^2$.

Part c) [2 marks] One of the results from part b) indicates that the corresponding estimator is biased. Which estimator is biased? What is the formula for the sample estimator which results in it being unbiased?

Question 2: [10 marks]

Part a) [5 marks] A portfolio consists of three stocks with values $A_0 = 20$ CHF, $B_0 = 10$ CHF, and $C_0 = 50$ CHF. The vector of one day returns denoted $(X_A, X_B, X_C)^T$ is multivariate normal with mean zero and the following covariance matrix:

$$\begin{bmatrix} 0.0020 & 0.0010 & 0.0010 \\ 0.0010 & 0.0030 & 0.0010 \\ 0.0010 & 0.0010 & 0.0040 \end{bmatrix}$$

i) Compute the expected value of the loss after one day.

ii) Write an expression for the one day linearized loss in terms of A_0, B_0, C_0, X_A, X_B, and X_C. For the linearized loss, compute VaR_α and ES_α with $\alpha = 0.95$ and the expected value of the linearized loss. The following may be useful:

$$\Phi^{-1}(0.95) = 1.64\,, \qquad \frac{1}{1-0.95}\int_{\Phi^{-1}(0.95)}^{\infty} x\phi(x)dx = 2.06\,.$$

Part b) [5 marks] A portfolio consists of a single stock with value $S_0 = 100$ USD. Denote the USD-CHF exchange rate by F, where $F_0 = 1$, so that the value of the stock to a Swiss institution is FS. As usual, the risk factor change corresponding to the stock is the return, denoted X_S. Similarly, the risk factor change corresponding to the exchange rate is $X_F = \log(F_1/F_0)$. The risk factor changes are jointly normal with mean zero and the following covariance matrix:

$$\begin{bmatrix} \sigma_S^2 & \rho\sigma_S\sigma_F \\ \rho\sigma_S\sigma_F & \sigma_F^2 \end{bmatrix} = \begin{bmatrix} 0.0020 & 0.0005 \\ 0.0005 & 0.0005 \end{bmatrix}$$

Write an expression for the one day linearized loss in terms of S_0, F_0, X_S, and X_F. Compute VaR_α and ES_α of the linearlized loss for $\alpha = 0.95$.

Question 3: [10 marks]

Let $\tau_1, \dots, \tau_n$ denote the default times of n reference entities. The default intensities are constant with values $\lambda_1, \dots, \lambda_n$ so that each default time has an exponential distribution. In all parts of this question recovery rates are zero and the interest rate is constant.

Part a) [3 marks] A 1^{st}-to-default swap is written on reference entities 1 and 2. Compute the fair spread of the swap assuming the default times are i) independent and ii) comonotonic.

Part b) [3 marks] A 1^{st}-to-default swap is written on all n reference entities. Compute the fair spread of the swap assuming the default times are independent.

Part c) [4 marks] A k^{th}-to-default swap is written on all n reference entities. Compute the fair spread of the swap assuming the default times are comonotonic.

Question 4: [10 marks] An Archimedean copula with generator ϕ takes the form:

$$C(u_1, \dots, u_d) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)) .$$

Let G be a distribution function on $\mathbb{R}^+$ with $G(0) = 0$ and let $\hat{G}$ be the Laplace-Stieltjes transform of G, extended such that $\hat{G}(\infty) = 0$. Let V have distribution G, and let $U_1, \dots, U_d$ be conditionally independent given V with conditional distribution:

$$F_{U_i|V}(u_i; v) = e^{-v\hat{G}^{-1}(u_i)}, \quad u_i \in [0,1] .$$

Show that:

$$\mathbb{P}(U_1 \le u_1, \dots, U_d \le u_d) = \hat{G}(\hat{G}^{-1}(u_1) + \cdots + \hat{G}^{-1}(u_d)) ,$$

thus, $(U_1, \dots, U_d)$ has distribution given by an Archimedean copula with generator $\phi = \hat{G}^{-1}$.

Question 5: [10 marks]

Part a) [2 marks] Suppose you have access to computer software that can generate a standard uniform variable. How can you use this to generate a standard normal variable?

Part b) [2 marks] Suppose you have access to computer software that can generate independent standard normal variables. How can you use this to generate a multivariate normal random vector with covariance matrix $\mathbf{\Sigma}$ where $\mathbf{\Sigma}$ is a symmetric positive definite matrix?

Part c) [3 marks] Suppose you have access to computer software that can generate a multivariate normal vector with covariance matrix $\mathbf{\Sigma}$. How can you use this to generate a random vector with distribution $C^{Ga}_{\boldsymbol{P}}$, the Gaussian copula with correlation matrix $\boldsymbol{P}$?

Part d) [3 marks] Let $X \sim Exp(\lambda_X)$ and $Y \sim Exp(\lambda_Y)$.) The copula of (X, Y) is the Farlie-Gumbel-Morgenstern copula:

$$C(u, v) = uv + \theta(1-u)(1-v)uv, \qquad -1 \leq \theta \leq 1\,.$$

Let $A = X^2$ and $B = Y^3$. What is the copula of (A, B)?

Question 6: [10 marks]

Part a) [2 marks] Let $L \sim Exp(\lambda)$. Compute $VaR_\alpha(L)$ and $ES_\alpha(L)$.

Part b) [4 marks] Let $X_i \sim F$ be a sequence of i.i.d. variables each having exponential distribution with parameter λ. Show that $F \in MDA(H_\xi)$ for some value of ξ.

Hint: recall that $F \in MDA(H_\xi)$ means there are sequences of constants a_n and b_n such that:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_n - a_n}{b_n} \leq x\right) = H_\xi(x)$$

It may help to choose the sequence $a_n = \frac{\log(n)}{\lambda}$, and you should already know which value of ξ is valid.

Part c) [4 marks] Let $X \sim F$ be exponentially distributed with parameter λ. Compute the excess distribution function over threshold u, $F_u(x)$.