Quantitative Risk Management
Assignment 7 Solutions

November 19, 2019

Question 1: If $\log(X_1) \sim \mathcal{N}(0,1)$, then $X_1 = e^{Z_1}$ for $Z_1 \sim \mathcal{N}(0,1)$. Similarly, $X_2 = e^{\sigma Z_2}$ for $Z_2 \sim \mathcal{N}(0,1)$. If we want X_1 and X_2 to have maximal correlation, we require them to be comonotonic which can be achieved by $Z_1 = Z_2 = Z$. In this case we have:

$$
\begin{aligned}
\mathbb{C}[X_1, X_2] &= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\
&= \mathbb{E}[e^{(1+\sigma)Z}] - \mathbb{E}[e^{Z}]\mathbb{E}[e^{\sigma Z}] \\
&= e^{\frac{1}{2}(1+2\sigma+\sigma^2)} - e^{\frac{1}{2}(1+\sigma^2)} \\
&= e^{\frac{1}{2}(1+\sigma^2)}(e^{\sigma} - 1)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V}[X_1] &= \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 \\
&= \mathbb{E}[e^{2Z}] - \mathbb{E}[e^{Z}]^2 \\
&= e^2 - e \\
\mathbb{V}[X_2] &= \mathbb{E}[X_2^2] - \mathbb{E}[X_2]^2 \\
&= \mathbb{E}[e^{2\sigma Z}] - \mathbb{E}[e^{\sigma Z}]^2 \\
&= e^{2\sigma^2} - e^{\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
\rho_{\max} = \rho(X_1, X_2) &= \frac{\mathbb{C}[X_1, X_2]}{\sqrt{\mathbb{V}[X_1]\mathbb{V}[X_2]}} \\
&= \frac{e^{\frac{1}{2}(1+\sigma^2)}(e^{\sigma} - 1)}{\sqrt{(e^2 - e)(e^{2\sigma^2} - e^{\sigma^2})}} \\
&= \frac{e^{\sigma} - 1}{\sqrt{(e-1)(e^{\sigma^2} - 1)}}
\end{aligned}
$$

For X_1 and X_2 to achieve minimal correlation, we require them to be countermonotonic which can be achieved by $Z_1 = -Z_2 = Z$. Alternatively, we can replace σ with $-\sigma$ in all expressions above. This gives:

$$
\rho_{\min} = \rho(X_1, X_2) = \frac{e^{-\sigma} - 1}{\sqrt{(e-1)(e^{\sigma^2} - 1)}}
$$

Figure 1 shows the attainable correlations as a function of σ.

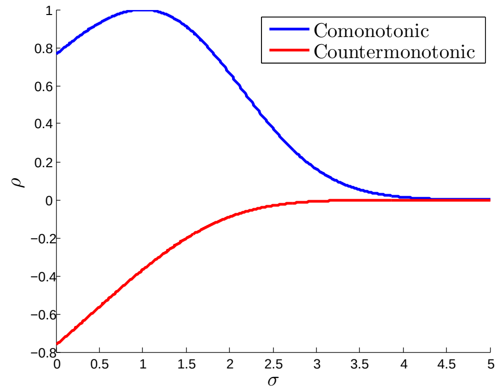

Figure 1: Attainable correlations for a pair of lognormal variables.

Question 2: By definition we have:

$$\begin{aligned}
\lambda_u(X_1, X_2) &= \lim_{q \to 1^-} \mathbb{P}\left(X_2 > F_2^{\leftarrow}(q) \middle| X_1 > F_1^{\leftarrow}(q) \right) \\
&= \lim_{q \to 1^-} \frac{\mathbb{P}(X_2 > F_2^{\leftarrow}(q), X_1 > F_1^{\leftarrow}(q))}{\mathbb{P}(X_1 > F_1^{\leftarrow}(q))} \\
&= \lim_{q \to 1^-} \frac{1 - \mathbb{P}(X_2 \leq F_2^{\leftarrow}(q)) - \mathbb{P}(X_1 \leq F_1^{\leftarrow}(q)) + \mathbb{P}(X_1 \leq F_1^{\leftarrow}(q), X_2 \leq F_2^{\leftarrow}(q))}{1 - q} \\
&= \lim_{q \to 1^-} \frac{1 - 2q + C(q, q)}{1 - q}
\end{aligned}$$

Question 3: The Gumbel copula is given by:

$$C_\theta^{Gu}(u_1, u_2) = \exp\{-((-\log u_1)^\theta + (-\log u_2)^\theta)^{1/\theta}\}$$

The lower and upper tail dependence coefficients are then given by:

$$\begin{aligned}
\lambda_l &= \lim_{q\to 0^+} \frac{C_\theta^{Gu}(q,q)}{q} \\
&= \lim_{q\to 0^+} \frac{\exp\{-(2(-\log q)^\theta)^{1/\theta}\}}{q} \\
&= \lim_{q\to 0^+} \frac{\exp\{2^{1/\theta}(\log q)\}}{q} \\
&= \lim_{q\to 0^+} \frac{q^{(2^{1/\theta})}}{q} \\
&= 0 \\
\lambda_u &= \lim_{q\to 1^-} \frac{1-2q+C_\theta^{Gu}(q,q)}{1-q} \\
&= \lim_{q\to 1^-} \frac{1-2q+q^{(2^{1/\theta})}}{1-q} \\
&= \lim_{q\to 1^-} \frac{-2+2^{1/\theta}q^{(2^{1/\theta}-1)}}{-1} \\
&= 2-2^{1/\theta}
\end{aligned}$$

The Clayton copula is given by:

$$C_\theta^{Cl}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$$

The lower and upper coefficients of tail dependence are given by:

$$\begin{aligned}
\lambda_l &= \lim_{q\to 0^+} \frac{C_\theta^{Cl}(q,q)}{q} \\
&= \lim_{q\to 0^+} \frac{(2q^{-\theta}-1)^{-1/\theta}}{q} \\
&= \lim_{q\to 0^+} (2q^{-\theta}-1)^{-(1+\theta)/\theta} 2q^{-\theta-1} \\
&= \lim_{q\to 0^+} \left((2q^{-\theta}-1)(2q^{-\theta-1})^{\frac{-\theta}{1+\theta}}\right)^{-(1+\theta)/\theta} \\
&= \lim_{q\to 0^+} \left(2^{1-\frac{\theta}{1+\theta}} - 2^{\frac{-\theta}{1+\theta}} q^\theta\right)^{-(1+\theta)/\theta} \\
&= 2^{-\frac{1}{\theta}} \\
\lambda_u &= \lim_{q\to 1^-} \frac{1-2q+C_\theta^{Cl}(q,q)}{1-q} \\
&= \lim_{q\to 1^-} \frac{1-2q+(2q^{-\theta}-1)^{-1/\theta}}{1-q} \\
&= \lim_{q\to 1^-} \frac{-2+(2q^{-\theta}-1)^{-(1+\theta)/\theta}2q^{-\theta-1}}{-1} \\
&= 0
\end{aligned}$$