Quantitative Risk Management
Assignment 5 Solutions

November 5, 2019

Question 1: First compute the variance of the process. Since $(\epsilon)_{t\in\mathbb{Z}}$ is white noise $WN(0, \sigma_\epsilon^2)$, then $\mathbb{E}[\epsilon_i \epsilon_j] = 0$ for $i \neq j$, and

$$\begin{aligned}
\mathbb{V}[X_t] &= \mathbb{E}[X_t^2] \\
&= \mathbb{E}\Big[\Big(\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}\Big)^2\Big] \\
&= \sigma_\epsilon^2 \sum_{i=0}^{\infty} \psi_i^2
\end{aligned}$$

Now we compute the covariance between X_t and X_{t+h}. Assume $h > 0$:

$$\begin{aligned}
\mathbb{E}[X_t X_{t+h}] &= \mathbb{E}\Big[\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} \sum_{j=0}^{\infty} \psi_j \epsilon_{t+h-j}\Big] \\
&= \mathbb{E}\Big[\sum_{j=0}^{\infty} \psi_j \psi_{j+h} \epsilon_{t-j}^2\Big] \\
&= \sigma_\epsilon^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}
\end{aligned}$$

If $h < 0$:

$$\begin{aligned}
\mathbb{E}[X_t X_{t+h}] &= \mathbb{E}\Big[\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} \sum_{j=0}^{\infty} \psi_j \epsilon_{t+h-j}\Big] \\
&= \mathbb{E}\Big[\sum_{i=0}^{\infty} \psi_i \psi_{i-h} \epsilon_{t-i}^2\Big] \\
&= \sigma_\epsilon^2 \sum_{i=0}^{\infty} \psi_i \psi_{i-h}
\end{aligned}$$

In either case this can be written as:

$$\mathbb{E}[X_t X_{t+h}] = \sigma_\epsilon^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+|h|}$$

Thus, the autocorrelation function is:

$$\rho(h) = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+|h|}}{\sum_{i=0}^{\infty} \psi_i^2}$$

Question 2: We must find ψ_i such that:

$$\sum_{i=0}^{\infty} \psi_i z^i = \frac{1+\theta z}{1-\phi z}$$

Let $h(z) = \sum_{i=0}^{\infty} \psi_i z^i$, $f(z) = 1 + \theta z$, and $g(z) = (1-\phi z)^{-1}$. Then $h(z) = f(z)g(z)$. We also have:

$$h^{(n)}(z) = \sum_{i=n}^{\infty} \psi_i \frac{i!}{(i-n)!} z^{i-n}$$

which gives:

$$h^{(n)}(0) = n!\psi_n$$

If we compute $h^{(n)}(z)$ in terms of f and g we have:

$$h^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z)$$

Explicitly computing these derivatives gives:

$$\begin{aligned}
f^{(0)}(z) &= 1 + \theta z \\
f^{(1)}(z) &= \theta \\
f^{(k)}(z) &= 0, \quad k \geq 2 \\
g^{(k)}(z) &= k!\phi^k (1-\phi z)^{-1-k}
\end{aligned}$$

Substituting these into the sum above gives:

$$h^{(n)}(z) = \begin{cases} (1+\theta z) n! \phi^n (1-\phi z)^{-1-n} + n\theta(n-1)!\phi^{n-1}(1-\phi z)^{-n} & n > 0 \\ (1+\theta z)(1-\phi z)^{-1} & n = 0 \end{cases}$$

Finally, we have:

$$h^{(n)}(0) = n!\psi_n = \begin{cases} n!\phi^n + n!\theta\phi^{n-1} & n > 0 \\ 1 & n = 0 \end{cases}$$

giving:

$$\psi_n = \begin{cases} \phi^{n-1}(\theta + \phi) & n > 0 \\ 1 & n = 0 \end{cases}$$

The autocorrelation function can be computed from the formula in the solution to question 1:

$$\begin{aligned}
\rho(h) &= \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+|h|}}{\sum_{i=0}^{\infty} \psi_i^2} \\
&= \frac{\psi_{|h|} + \sum_{i=1}^{\infty} \psi_i \psi_{i+|h|}}{1 + \sum_{i=1}^{\infty} \psi_i^2} \\
&= \frac{\phi^{|h|-1}(\phi+\theta) + \sum_{i=1}^{\infty} \phi^{i-1}(\theta+\phi)\phi^{i+|h|-1}(\theta+\phi)}{1 + \sum_{i=1}^{\infty}(\phi^{i-1}(\theta+\phi))^2} \\
&= \frac{\phi^{|h|-1}(\phi+\theta) + (\theta+\phi)^2\phi^{|h|-2}\sum_{i=1}^{\infty}\phi^{2i}}{1 + (\theta+\phi)^2\phi^{-2}\sum_{i=1}^{\infty}\phi^{2i}} \\
&= \frac{\phi^{|h|-1}(\phi+\theta) + (\theta+\phi)^2\phi^{|h|-2}\phi^2(1-\phi^2)^{-1}}{1 + (\theta+\phi)^2\phi^{-2}\phi^2(1-\phi^2)^{-1}} \\
&= \frac{\phi^{|h|-1}(\phi+\theta)(1-\phi^2) + (\theta+\phi)^2\phi^{|h|}}{1 - \phi^2 + (\theta+\phi)^2} \\
&= \frac{\phi^{|h|-1}(\phi+\theta)\Big((1-\phi^2) + (\theta+\phi)\phi\Big)}{1 + \theta^2 + 2\theta\phi} \\
&= \frac{\phi^{|h|-1}(\phi+\theta)(1+\theta\phi)}{1 + \theta^2 + 2\theta\phi}
\end{aligned}$$

Question 3: We begin by writing the equations satisfied by X_t:

$$\begin{aligned} X_t &= \sigma_t Z_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 \end{aligned}$$

Squaring the first equation and substituting the second gives:

$$\begin{aligned} X_t^2 &= \sigma_t^2 Z_t^2 \\ &= (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2 \end{aligned}$$

Squaring again gives:

$$X_t^4 = (\alpha_0^2 + 2\alpha_0\alpha_1 X_{t-1}^2 + \alpha_1^2 X_{t-1}^4) Z_t^4$$

Now we take expectations of both sides and note that because $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary, $\mathbb{E}[X^4] = \mathbb{E}[X_{t-1}^4]$:

$$\begin{aligned} \mathbb{E}[X_t^4] &= \alpha_0^2\mathbb{E}[Z_t^4] + 2\alpha_0\alpha_1\mathbb{E}[X_{t-1}^2]\mathbb{E}[Z_t^4] + \alpha_1^2\mathbb{E}[X_t^4]\mathbb{E}[Z_t^4] \\ \mathbb{E}[X_t^4] &= \alpha_0^2\mathbb{E}[Z_t^4] + 2\alpha_0\alpha_1\frac{\alpha_0}{1-\alpha_1}\mathbb{E}[Z_t^4] + \alpha_1^2\mathbb{E}[X_t^4]\mathbb{E}[Z_t^4] \\ \mathbb{E}[X_t^4] - \alpha_1^2\mathbb{E}[X_t^4]\mathbb{E}[Z_t^4] &= \alpha_0^2\mathbb{E}[Z_t^4] + 2\alpha_0\alpha_1\frac{\alpha_0}{1-\alpha_1}\mathbb{E}[Z_t^4] \\ \mathbb{E}[X_t^4](1 - \alpha_1^2\mathbb{E}[Z_t^4]) &= \frac{\alpha_0^2\mathbb{E}[Z_t^4](1-\alpha_1) + 2\alpha_0^2\alpha_1\mathbb{E}[Z_t^4]}{1-\alpha_1} \\ \mathbb{E}[X_t^4] &= \frac{\alpha_0^2\mathbb{E}[Z_t^4](1+\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2\mathbb{E}[Z_t^4])} \end{aligned}$$