Quantitative Risk Management
Assignment 1 Solutions

Due: October 1, 2019

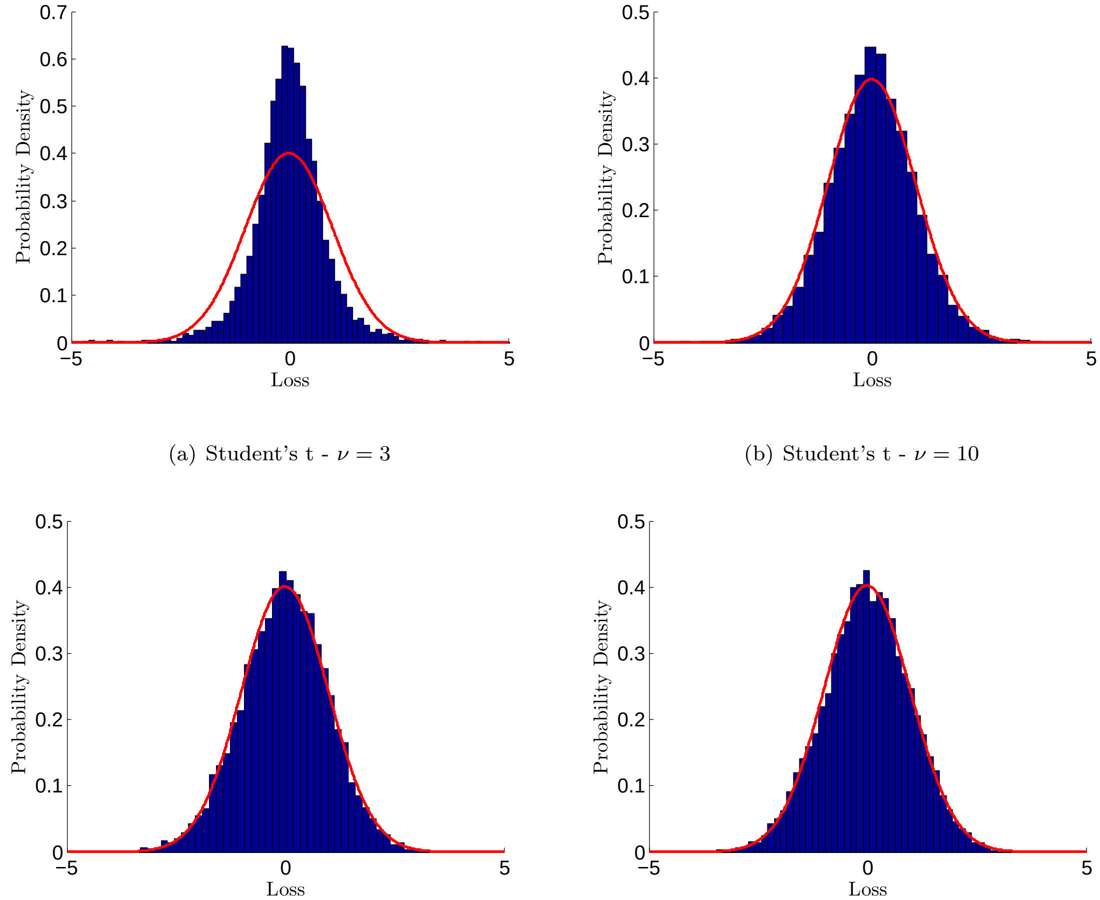
Question 1: In each case, the loss is equal to:

$$L(t, t + \Delta) = -\lambda S(e^X - 1) \tag{1}$$

Since Matlab can only simulate from a standard t-distribution with ν degrees of freedom, we must multiply the simulated variables by a constant to achieve the correct variance. Let T be a standard t-distributed variable with ν degrees of freedom, and let $X = mT$. Then:

$$\begin{aligned}
\mathbb{V}[X] &= m^2\mathbb{V}[T] \\
m &= \sqrt{\frac{\mathbb{V}[X]}{\mathbb{V}[T]}} \\
m &= \frac{0.01}{\sqrt{\frac{\nu}{\nu-2}}}
\end{aligned}$$

For each of the distributions of X of interest, the empirical distribution of losses is shown. The red curve represents the probability density function of a normal distribution with mean and variance equal to the empirical mean and variance of the loss.



(a) Student's t - $\nu = 3$

(b) Student's t - $\nu = 10$

(c) Student's t - $\nu = 50$

(d) Gaussian

Figure 1: Loss distribution for various distribution of risk factor change X.

For larger values of ν, the empirical distribution appears to be roughly normal. When ν is small, we see the normal distribution underestimates the weight in the center of the distribution. In none of these cases is the distribution of $L(t, t+\Delta)$ actually a normal distribution. From the expression (1) we see that $L(t, t+\Delta)$ is bounded above by λS. A normal random variable is unbounded, and so $L(t, t+\Delta)$ is not normal.

The linearized loss is equal to:

$$L^\delta(t, t+\Delta) = -\lambda S X$$

The mean and variance of $L^\delta(t, t+\Delta)$ are computed easily:

$$\begin{aligned}
\mathbb{E}[L^\delta(t, t+\Delta)] &= -\lambda S \mathbb{E}[X] \\
&= 0 \\
\mathbb{V}[L^\delta(t, t+\Delta)] &= \lambda^2 S^2 \mathbb{V}[X] \\
&= 1^2 \cdot 100^2 \cdot 0.01^2 \\
&= 1
\end{aligned}$$

The exact distributions of $L^\delta(t, t+\Delta)$ corresponding to each distribution of X are the following:

1. A scaled Student's t-distribution with 3 degrees of freedom, mean zero, and standard deviation 1.

2. A scaled Student's t-distribution with 10 degrees of freedom, mean zero, and standard deviation 1.

3. A scaled Student's t-distribution with 50 degrees of freedom, mean zero, and standard deviation 1.
4. A standard normal distribution.

Question 2: For the given parameters, the empirical distribution of $L(t, t+\Delta)$ is shown:

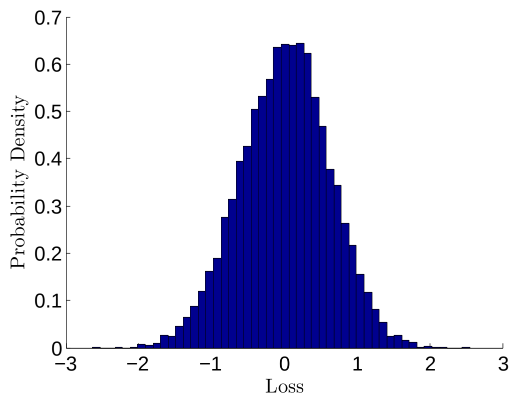


Figure 2: Loss distribution for European call option.

When simulating values of $r_{t+\Delta}$ and $\sigma_{t+\Delta}$, we require that they remain positive. Since they are declared to be normally distributed, there is a small chance that they take on negative values. In this case, they are set equal to the absolute value of the simulated quantity. This should happen in a very small number of cases and so will not have a significant effect on the overall distribution.

The linearized loss is equal to the following:

$$L^{\delta}(t, t+\Delta) = -\frac{\partial C^{BS}}{\partial t}\Delta - \frac{\partial C^{BS}}{\partial S} S X_1 - \frac{\partial C^{BS}}{\partial r} X_2 - \frac{\partial C^{BS}}{\partial \sigma} X_3 \tag{2}$$

These partial derivatives have closed form expressions equal to the following:

$$\begin{aligned}
\frac{\partial C^{BS}}{\partial t} &= -\frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{rT}\Phi(d_2) \\
\frac{\partial C^{BS}}{\partial S} &= \Phi(d_1) \\
\frac{\partial C^{BS}}{\partial r} &= KTe^{-rT}\Phi(d_2) \\
\frac{\partial C^{BS}}{\partial \sigma} &= S\phi(d_1)\sqrt{T}
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

Using the same simulated values of $\boldsymbol{X}$ and substituting into (2) gives the following empirical distribution:

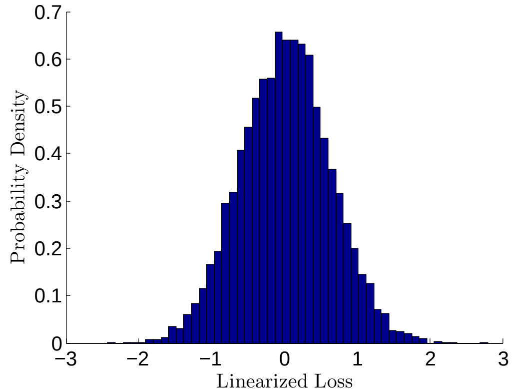


Figure 3: Linearized loss distribution for European call option.

There are three risk factor changes that contribute to the linearized loss, each of them multiplied by a scale factor. To asses which of these contributes the most to the linearized loss, we compute the variance of each of the three terms:

$$
\begin{aligned}
\mathbb{V}\left[\frac{\partial C^{BS}}{\partial S} S X_1\right] &= \left(\frac{\partial C^{BS}}{\partial S}\right)^2 S^2 \mathbb{V}[X_1] \\
&= 0.64^2 \cdot 100^2 \cdot 0.01^2 \\
&= 0.41 \\
\mathbb{V}\left[\frac{\partial C^{BS}}{\partial r} X_2\right] &= \left(\frac{\partial C^{BS}}{\partial r}\right)^2 \mathbb{V}[X_2] \\
&= 53.23^2 \cdot 10^{-8} \\
&= 2.8 \cdot 10^{-5} \\
\mathbb{V}\left[\frac{\partial C^{BS}}{\partial \sigma} X_3\right] &= \left(\frac{\partial C^{BS}}{\partial \sigma}\right)^2 \mathbb{V}[X_3] \\
&= 37.52^2 \cdot 10^{-6} \\
&= 1.4 \cdot 10^{-3}
\end{aligned}
$$

These results show that the most of the randomness present in $L^{\delta}(t, t+\Delta)$ is caused by randomness in underlying stock price.

Question 3: $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$
\begin{aligned}
\mathbb{E}[e^X] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^x dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2\mu x+\mu^2)}{2\sigma^2}} e^x dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2(\mu+\sigma^2)x+\mu^2)}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2(\mu+\sigma^2)x)}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2-2(\mu+\sigma^2)x+(\mu+\sigma^2)^2-(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu-\sigma^2)^2}{2\sigma^2}} e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} dx \\
&= \frac{e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x'-\mu)^2}{2\sigma^2}} dx' \qquad x' = x - \sigma^2 \\
&= e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x'-\mu)^2}{2\sigma^2}} dx' \\
&= e^{\frac{\mu^2+2\sigma^2\mu+\sigma^4}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} \\
&= e^{\mu+\frac{1}{2}\sigma^2}
\end{aligned}
$$