# Quantitative Risk Management
# Assignment 3 Solutions

October 15, 2019

**Question 1:** The values of $VaR_{0.95}$ and $VaR_{0,99}$ are plotted in Figure 1 along with the losses that were realized on those particular days. The days on which $VaR_\alpha$ was breached are indicated with a marker.

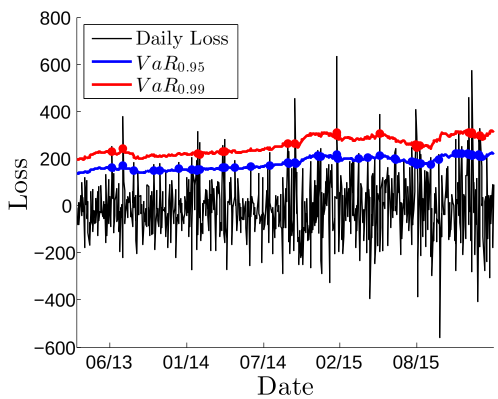


Figure 1: Daily $VaR_\alpha$ estimates and losses from March 11, 2013 to March 10, 2016.

The total number of days on which $VaR_\alpha$ is computed and then corresponding losses observed is 751. Of those days, $VaR_{0.95}$ was breached 48 times and $VaR_{0.99}$ was breached 18 times. If the methods are accurate, the probabilities of observing this many or more breaches in each case are given by:

$$\begin{aligned} 1 - Bino^{-1}(47, 751, 0.05) &= 0.05185 \\ &= 5.185\% \\ 1 - Bino^{-1}(17, 751, 0.01) &= 7.5 \cdot 10^{-4} \\ &= 0.075\% \end{aligned}$$

The probability corresponding to $\alpha = 0.95$ is not completely unreasonable, but the result for $\alpha = 0.99$ shows that we saw an extremely unlikely large number of breaches. It is likely that this method of computing $VaR$ should be supplemented with other methods. A simple visual analysis of the stock prices over the five year period of consideration shows much more volatility in the last two years than in the time leading up to it.

**Question 2:** Part 1) The geometric distribution with 0 included in the support has probability mass function $\mathbb{P}(L = k) = (1-p)^k p$. For $p = 0.5$, a table of values of the cumulative distribution function is given by:

| | |
|---|---|
| $\mathbb{P}(L \leq 0)$ | 0.5 |
| $\mathbb{P}(L \leq 1)$ | 0.75 |
| $\mathbb{P}(L \leq 2)$ | 0.875 |
| $\mathbb{P}(L \leq 3)$ | 0.935 |
| $\mathbb{P}(L \leq 4)$ | 0.96875 |

We see that the CDF changes from less than $\alpha = 0.95$ to greater than $\alpha = 0.95$ as the index $k$ changes from 3 to 4. This means that $VaR_{0.95} = 4$ as can be checked from the definition of $VaR_\alpha$:

$$VaR_{0.95} = \inf\{x \in \mathbb{R} : F_L(x) \geq 0.95\} = 4$$

Figure 2 plots $VaR_\alpha$ as a function of $\alpha$ for this particular distribution. Note that it has discontinuities due to the discreet nature of the random variable $L$.

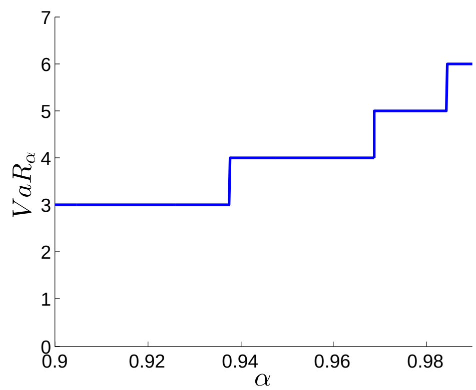


Figure 2: $VaR_\alpha$ as a function of $\alpha$ for the geometric distribution, $p = 0.5$.

Part 2) If $X$ and $Y$ are independent Poisson with parameters $\lambda_X$ and $\lambda_Y$, then $L = X + Y$ is Poisson with parameter $\lambda_Z = \lambda_X + \lambda_Y$. Values of $VaR_\alpha$ for each random variable can be found via the CDF as in the previous part of this question. The resulting values are shown in Figure 3.

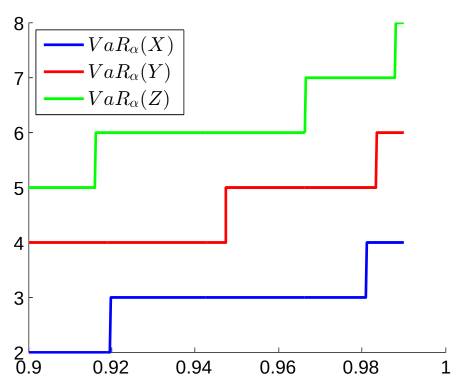


Figure 3: $VaR_\alpha$ as a function of $\alpha$ for the Poisson distributed random variables.

**Question 3:** There are of course an infinite number of counterexamples to the subadditivity of $VaR_\alpha$. The one given here is one of the most simplistic. Let $L_1$ and $L_2$ be independent Bernoulli random variables with success parameter $1-\alpha-\epsilon$ where $0<\epsilon<1-\alpha$. Then

$$\begin{aligned} \mathbb{P}(L_i = 0) &= \alpha + \epsilon \\ \mathbb{P}(L_i = 1) &= 1 - \alpha - \epsilon \end{aligned}$$

Clearly, $VaR_\alpha(L_i) = 0$. The distribution of $L = L_1 + L_2$ is Binomial with parameters $n = 2$ and $p = 1-\alpha-\epsilon$. This random variable takes on three possible values with the following probabilities:

$$\begin{aligned} \mathbb{P}(L = 0) &= (\alpha + \epsilon)^2 \\ \mathbb{P}(L = 1) &= 2(\alpha + \epsilon)(1 - \alpha - \epsilon) \\ \mathbb{P}(L = 2) &= (1 - \alpha - \epsilon)^2 \end{aligned}$$

If $\epsilon$ is chosen such that $(\alpha+\epsilon)^2 < \alpha$, then $VaR_\alpha(L) > 0 = VaR_\alpha(L_1) + VaR_\alpha(L_2)$, showing that $VaR_\alpha$ is not subadditive.

**Question 4:** Part 1) The current value bond $i$ is 1000CHF, meaning $V_{i,0} = 1000$CHF. If company $i$ does not default after one year, then the value of the bond will be $V_{i,1} = 1050$CHF. If company $i$ does default, then the bond will have a value of $V_{i,1} = 0$CHF. The loss $L_i$ as a function of $I_i$ is then:

$$\begin{aligned} L_i &= -(V_{i,1} - V_{i,0}) \\ &= -(1050(1 - I_i) - 1000) \\ &= 1050I_i - 50 \end{aligned}$$

Part 2) The probability distribution is given by the following:

$$\mathbb{P}(L_i = -50) = 0.98$$
$$\mathbb{P}(L_i = 1000) = 0.02$$

Part 3) The value of portfolio $V_a$ is a multiple of a single bond so that $L_a = 100(1050I_i - 50)$. Thus, the probability distribution of $L_a$ is:

$$\mathbb{P}(L_a = -5000) = 0.98$$
$$\mathbb{P}(L_a = 100000) = 0.02$$

The portfolio $V_b$ is the sum of 100 independent copies of a single bond, so the losses are given by:

$$\begin{aligned} L_b &= \sum_{i=1}^{100} 1050I_i - 100 \cdot 50 \\ &= 1050 \sum_{i=1}^{100} I_i - 100 \cdot 50 \\ &= 1050N - 100 \cdot 50 \end{aligned}$$

where $N$ counts the number of defaults. Since each $I_i$ is Bernoulli and they are independent, $N$ has Binomial distribution with parameters $n = 100$ and $p = 0.02$. For portfolio $V_a$, $VaR_\alpha$ takes the values:

$$VaR_{0.95}(L_a) = -5000$$
$$VaR_{0.99}(L_a) = 100000$$

For portfolio $V_b$, we can first compute quantiles of $N$:

$$Bino^{-1}(0.95, 100, 0.02) = 5$$
$$Bino^{-1}(0.99, 100, 0.02) = 6$$

Since $L_b$ is a monotone function with respect to $N$, these quantiles of $N$ give us the corresponding quantiles of $L_b$:

$$VaR_{0.95}(L_b) = 1050 \cdot 5 - 100 \cdot 50 = 250$$
$$VaR_{0.99}(L_b) = 1050 \cdot 6 - 100 \cdot 50 = 1300$$