Quantitative Risk Management
Assignment 4 Solutions

October 21, 2019

Question 1: If we write the CDF of X we get:

$$\begin{aligned}
\mathbb{P}(X \leq v) &= \mathbb{P}(\mu + \sqrt{W}Z \leq v) \\
&= \sum_{i=1}^{k} \mathbb{P}(\mu + \sqrt{W}Z \leq v \cap W = k_i) \\
&= \sum_{i=1}^{k} \mathbb{P}(\mu + \sqrt{k_i}Z \leq v \cap W = k_i) \\
&= \sum_{i=1}^{k} \mathbb{P}(\mu + \sqrt{k_i}Z \leq v)\mathbb{P}(W = k_i) \\
&= \sum_{i=1}^{k} \mathbb{P}\left(Z \leq \frac{v-\mu}{\sqrt{k_i}}\right)p_i \\
&= \sum_{i=1}^{k} \Phi\left(\frac{v-\mu}{\sqrt{k_i}}\right)p_i
\end{aligned}$$

If $v = VaR_\alpha(X)$, then we need $\mathbb{P}(X \leq v) = \alpha$. Thus, the function we construct is:

$$f(v) = \sum_{i=1}^{k} \Phi\left(\frac{v-\mu}{\sqrt{k_i}}\right)p_i - \alpha$$

Note that f is strictly increasing because it is the sum of k strictly increasing functions. Also:

$$\begin{aligned}
\lim_{v\to-\infty} f(v) &= -\alpha \\
\lim_{v\to\infty} f(v) &= 1-\alpha
\end{aligned}$$

and so f has a unique root.

Question 2: There are many examples of pairs of random variables that are uncorrelated but not independent. One such example is given by:

$$\begin{aligned} X &\sim \mathcal{N}(0,1) \\ Y &= AX \end{aligned}$$

where A is independent of X and takes values as follows:

$$A = \begin{cases} 1 & p \\ -1 & 1-p \end{cases}$$

Both X and Y have mean zero, so the covariance between X and Y is:

$$\begin{aligned} \mathbb{E}[XY] &= \mathbb{E}[AX^2] \\ &= \mathbb{E}[A]\mathbb{E}[X^2] \\ &= 2p - 1 \end{aligned}$$

If we choose $p = \frac{1}{2}$, then X and Y are uncorrelated. However, for a fixed value of h, compute:

$$\begin{aligned} \mathbb{P}(X > h \cap Y > h) &= \mathbb{P}(X > h \cap AX > h) \\ &= \frac{1}{2}\mathbb{P}(X > h) + \frac{1}{2}\mathbb{P}(X > h \cap X < -h) \end{aligned}$$

If $h > 0$, then the second term above is zero, and we have:

$$\mathbb{P}(X > h \cap Y > h) = \frac{1}{2}(1 - \Phi(h))$$

On the other hand:

$$\mathbb{P}(X > h)\mathbb{P}(Y > h) = (1 - \Phi(h))^2$$

These two expressions are clearly not equal for all values of h showing that X and Y are not independent.

Question 3: Part 1) The expanded matrices $\mathcal{X}$ and $\mathcal{F}$ are constructed from the data as demonstrated in the lecture notes. The regression is performed to compute $\hat{\mathcal{B}}$ as:

$$\hat{\mathcal{B}} = (\mathcal{F}^T \mathcal{F})^{-1} \mathcal{F}^T \mathcal{X}$$

This results in the following numerical values for the vector $\boldsymbol{a}$ and factor loadings $\boldsymbol{B}$:

$$\boldsymbol{a} = \begin{bmatrix} -0.342 \times 10^{-3} \\ 0.121 \times 10^{-3} \\ 0.193 \times 10^{-3} \\ 0.021 \times 10^{-3} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0.813 \\ 0.978 \\ 0.583 \\ 0.522 \end{bmatrix}$$

The components of $\boldsymbol{a}$ and $\boldsymbol{B}$ correspond to the ordering of ticker symbols: IBM, MMM, MCD, WMT.

Part 2) The residual errors are computed as:

$$\hat{\mathcal{E}} = \mathcal{X} - \mathcal{F}\hat{\mathcal{B}}$$

We compute the sample correlation matrix of both the error sequence $\hat{\boldsymbol{\epsilon}}_1, \ldots, \hat{\boldsymbol{\epsilon}}_n$ and the original returns $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. These are summarized in the following two matrices:

$$\boldsymbol{\rho}_\epsilon = \begin{bmatrix} 1.000 & 0.002 & 0.063 & -0.004 \\ 0.002 & 1.000 & 0.023 & 0.030 \\ 0.063 & 0.023 & 1.000 & 0.122 \\ -0.004 & 0.030 & 0.122 & 1.000 \end{bmatrix}, \qquad \boldsymbol{\rho}_X = \begin{bmatrix} 1.000 & 0.528 & 0.426 & 0.320 \\ 0.528 & 1.000 & 0.498 & 0.420 \\ 0.426 & 0.498 & 1.000 & 0.382 \\ 0.320 & 0.420 & 0.382 & 1.000 \end{bmatrix}$$

Note that the correlations between the residual errors are significantly lower than the correlations between the original returns. This is to be expected since all of the equities at hand are part of the SNP500 index.

Question 4: Part 1) Based off of the given realizations of the loss, $VaR_{0.95}$ is found to be 10.41.

Part 2) The matrix $\bar{\boldsymbol{\Sigma}}$ is computed to be equal to:

$$\bar{\boldsymbol{\Sigma}} = \begin{bmatrix} 1.73 & 1.74 & -1.74 & 1.73 \\ 1.74 & 3.44 & 1.67 & 0.09 \\ -1.74 & 1.67 & 23.62 & -0.13 \\ 1.73 & 0.09 & -0.13 & 6.65 \end{bmatrix}$$

The matrix of eigenvectors is given by:

$$\bar{\boldsymbol{\Gamma}} = \begin{bmatrix} 0.07 & 0.34 & 0.41 & -0.84 \\ -0.07 & 0.18 & 0.85 & 0.48 \\ -0.99 & 0.02 & -0.04 & -0.10 \\ 0.01 & 0.92 & -0.32 & 0.21 \end{bmatrix}$$

The first column of $\bar{\boldsymbol{\Gamma}}$ is the eigenvector corresponding to the first principle component. Note that the magnitude of the third component of this vector is very close to 1. This indicates that most of the variability in the first principle component (and original data) is solely due to the third component of the data. This is further exemplified by the fact that the variance of the third component is 23.62, significantly larger than the variance of any other component.

Part 3) In this part we approximate $\boldsymbol{X}$ with a factor model where the factors are the first two principle components of $\boldsymbol{X}$. Performing the principle component transform on $\boldsymbol{X}$ gives us the 10,000 realizations of the principle components $\boldsymbol{Y}$:

$$\boldsymbol{Y}_k = \bar{\boldsymbol{\Gamma}}^T(\boldsymbol{X}_k - \bar{\boldsymbol{\mu}})$$

By using the notation:

$$\bar{\boldsymbol{\Gamma}} = \begin{bmatrix} \boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 & \boldsymbol{\gamma}_3 & \boldsymbol{\gamma}_4 \end{bmatrix}$$

and setting the error terms to zero, the factor model becomes:

$$\boldsymbol{X} = \bar{\boldsymbol{\mu}} + \begin{bmatrix} \boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

We can recompute $VaR_{0.95}$ to be equal to 9.87. This is close to, but lower than the original value of 10.41, as should be expected because the first two principle components account for 88% of the variability in this data.