Quantitative Risk Management
Assignment 6

Due: November 12, 2019

Question 1: Recall that $(X_t)_{t\in\mathbb{Z}}$ is an $ARCH(1)$ process if it is strictly stationary and satisfies:

$$\begin{aligned} X_t &= \sigma_t Z_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 \end{aligned}$$

for some $(Z_t)_{t\in\mathbb{Z}}$ that is $SWN(0,1)$, $\alpha_0 > 0$ and $\alpha_1 \geq 0$. The process $(X_t)_{t\in\mathbb{Z}}$ is white noise if and only if $\alpha_1 < 1$. In this case it has no serial correlation. This question will show that $(X_t^2)_{t\in\mathbb{Z}}$ does have serial correlation.

Part 1) Consider an $AR(1)$ process $(Y_t)_{t\in\mathbb{Z}}$ of the form $Y_t = \phi Y_{t-1} + \epsilon_t$ where $|\phi| < 1$ and $(\epsilon_t)_{t\in\mathbb{Z}}$ is white noise. Compute the autocorrelation function $\rho(h)$ of $(Y_t)_{t\in\mathbb{Z}}$.

Part 2) Assume that the $ARCH(1)$ process $(X_t)_{t\in\mathbb{Z}}$ satisfies $\mathbb{E}[X_t^4] < \infty$. Show that for some $c \in \mathbb{R}$ the process $(X_t^2 - c)_{t\in\mathbb{Z}}$ is an $AR(1)$ process with zero mean. Why is the condition $\mathbb{E}[X_t^4] < \infty$ needed?
(**Hint**: Start with $X_t^2 = \sigma_t^2 + \sigma_t^2(Z_t^2 - 1)$ and prove that $\sigma_t^2(Z_t^2 - 1)$ is a martingale difference sequence).

Part 3) Find the autocorrelation function $\rho(h)$ of $(X_t^2)_{t\in\mathbb{Z}}$.

Question 2: Download 5 years of Microsoft stock prices starting on April 7, 2011. For each day in the period from April 8, 2013 to April 7, 2016:

1. Fit a $GARCH(1,1)$ model to the previous two years of returns:
   (a) Compute the log returns, R_t, and set $X_t = R_t - \hat{\mu}$ where $\hat{\mu}$ is the sample mean of R_t.
   (b) Assume $\sigma_0 = \hat{S}$, where $\hat{S}$ is the sample standard deviation of X_t.
   (c) Perform an estimation of α_0, α_1, and β_1 using maximum conditional likelihood estimation as discussed in lecture assuming the innovations Z_t are standard normal.
2. Estimate VaR_α for $\alpha = 0.95$ and $\alpha = 0.99$
3. Observe if the return on the next day breaches your estimate of VaR_α.
4. Repeat the previous process except assume that Z_t has a scaled t-distribution, and also include the degrees of freedom ν as one of the parameters to be estimated.
5. In each of the estimated models, do the parameters result in a covariance stationary process? If yes, compute the theoretical variance of X_t and compare it with the sample variance of the data.

Note: Matlab only has a function for the standard t-distribution probability density function. You will need to find out how to write the PDF of a scaled t-distributed variable in terms of a standard t-distributed variable so that the scaled version has a variance of 1. Also, the Matlab function `fmincon` will probably be useful. A sample of how to use this function will be posted on the course website.

Question 3: Read the article "Is GARCH(1,1) as good a model as the Nobel prize accolades would imply?" posted on the course website.