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Group 2

QRM - Pb Set 7

Exercise 1

We have $\begin{cases} \log X_1 \sim \mathcal{N}(0,1) \\ \log X_2 \sim \mathcal{N}(0,\sigma^2) \end{cases}$

Therefore : $\begin{cases} X_1 = e^{Z_1} & \text{with } Z_1 \sim \mathcal{N}(0,1) \\ X_2 = e^{Z_2} & \text{with } Z_2 \sim \mathcal{N}(0,\sigma) \end{cases}$

$$\begin{aligned} \Rightarrow \quad \mathrm{Cov}(X_1, X_2) &= \mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\,\mathbb{E}(X_2) \\ &= \mathbb{E}\left(e^{Z_1+Z_2}\right) - \mathbb{E}(e^{Z_1})\,\mathbb{E}(e^{Z_2}) \\ &= \mathbb{E}\left(e^{(1,1)\binom{Z_1}{Z_2}}\right) - e^{\frac{1}{2}} \times e^{\frac{1}{2}\sigma^2} \\ &= e^{\frac{1}{2}(1,1)\begin{pmatrix} 1 & \sigma \\ \sigma & \sigma^2 \end{pmatrix}\binom{1}{1}} - e^{\frac{1}{2}(1+\sigma^2)} \\ &= e^{\frac{1}{2}(1+2\sigma+\sigma^2)} - e^{\frac{1}{2}(1+\sigma^2)} \\ &= e^{\frac{1}{2}(1+\sigma^2)}\left(e^{\sigma}-1\right) \end{aligned}$$

$$\begin{aligned} \mathbb{V}(X_1) &= \mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2 \\ &= \mathbb{E}(e^{2Z_1}) - \mathbb{E}(e^{Z_1})^2 \\ &= e^2 - e = e(e-1) \end{aligned}$$

$$\begin{aligned} \mathbb{V}(X_2) &= \mathbb{E}(X_2^2) - \mathbb{E}(X_2)^2 \\ &= \mathbb{E}(e^{2Z_2}) - \mathbb{E}(e^{Z_2})^2 \\ &= e^{2\sigma^2} - e^{\sigma^2} = e^{\sigma^2}\left(e^{\sigma^2}-1\right) \end{aligned}$$

$$\Rightarrow \quad \rho(X_1, X_2) = \frac{e^{\frac{1}{2}(1+\sigma^2)}\left(e^{\sigma}-1\right)}{\sqrt{(e^2-e)(e^{2\sigma^2}-e^{\sigma^2})}} = \frac{e^{\frac{1}{2}(1+\sigma^2)}\left(e^{\sigma}-1\right)}{\sqrt{e^{\sigma^2+1}(e-1)(e^{\sigma^2}-1)}} = \frac{e^{\sigma}-1}{\sqrt{(e-1)(e^{\sigma^2}-1)}}$$

$$\Rightarrow \quad \rho(X_1, X_2) = \frac{e^{\sigma} - 1}{\sqrt{(e-1)(e^{\sigma^2}-1)}}$$

Between 0 and 5, the maximum and minimum attainable of ρ is : $\rho_{max} = 1$ when $\sigma = 1$
and : $\rho_{min} = 0 \quad \left(= \lim_{\sigma \to +\infty} \rho(\sigma) \right)$

The graph is plot thank to our python code.

Exercise 2

We know that

$$\lambda_U(X_1, X_2) = \lim_{q \to 1^-} \mathbb{P}\left(X_2 > F_2^{\leftarrow}(q) \mid X_1 > F_1^{\leftarrow}(q)\right)$$

We have :

$$\mathbb{P}(X_2 > F_2^{\leftarrow}(q) \mid X_1 > F_1^{\leftarrow}(q)) = \frac{\mathbb{P}\left((X_2 > F_2^{\leftarrow}(q)) \cap (X_1 > F_1^{\leftarrow}(q))\right)}{\mathbb{P}(X_1 > F_1^{\leftarrow}(q))}$$

$$= \frac{1 - \mathbb{P}\left((X_2 \leq F_2^{\leftarrow}(q)) \cup (X_1 \leq F_1^{\leftarrow}(q))\right)}{1 - \mathbb{P}(X_1 \leq F_1^{\leftarrow}(q))}$$

$$= \frac{1 - \mathbb{P}(X_2 \leq F_2^{\leftarrow}(q)) - \mathbb{P}(X_1 \leq F_1^{\leftarrow}(q)) + \mathbb{P}((X_2 \leq F_2^{\leftarrow}(q)) \cap (X_1 \leq F_1^{\leftarrow}(q)))}{1 - q}$$

$$= \frac{1 - q - q + C(q,q)}{1 - q}$$

$$= \frac{1 - 2q + C(q,q)}{1 - q}$$

$$\Rightarrow \quad \lambda_U(X_1, X_2) = \lim_{q \to 1^-} \frac{1 - 2q + C(q,q)}{1 - q}$$

Exercise 3:

We know that:

$$C_\theta^{Gu}(u_1,u_2) = \exp\left(-\left((-\log(u_1))^\theta + (-\log(u_2))^\theta\right)^{1/\theta}\right)$$

$$\Rightarrow C_\theta^{Gu}(q,q) = \exp\left[-\left(2(-\log(q))^\theta\right)^{1/\theta}\right] \quad \Big\} \; 0<q<1$$

$$= \exp\left[2^{1/\theta}\log(q)\right]$$

$$= q^{2^{1/\theta}}$$

$$\Rightarrow \lambda_L^{Gu} = \lim_{q\to0^+} \frac{C_\theta^{Gu}(q,q)}{q}$$

$$= \lim_{q\to0^+} q^{2^{1/\theta}-1}$$

$\Rightarrow \theta \in [1;+\infty[$

$\frac{1}{\theta} \in]0;1]$

$2^{1/\theta} \in]1;2]$

$\Rightarrow 2^{1/\theta}-1 > 0$

$$\Rightarrow \underline{\underline{\lambda_L^{Gu} = 0}}$$

$$\Rightarrow \lambda_U^{Gu} = \lim_{q\to1^-} \frac{1-2q+C_\theta^{Gu}(q,q)}{1-q}$$

$$= \lim_{q\to1^-} \frac{1-2q+q^{2^{1/\theta}}}{1-q}$$

$$q^{2^{1/\theta}} \underset{\substack{q\to1^- \\ u\to0^- \\ q=1+u}}{=} (1+u)^{2^{1/\theta}} \underset{u\to0^-}{=} 1+2^{1/\theta}u \underset{q\to1^-}{=} 1+2^{1/\theta}(q-1)$$

$$\Rightarrow \frac{1-2q+q^{2^{1/\theta}}}{1-q} \underset{q\to1^-}{=} \frac{2-2q+2^{1/\theta}(q-1)}{(1-q)} \underset{q\to1^-}{=} \frac{(1-q)(2-2^{1/\theta})}{(1-q)} \underset{q\to1^-}{=} 2-2^{1/\theta}$$

$$\Rightarrow \underline{\underline{\lambda_U^{Gu} = 2-2^{1/\theta}}}$$

We know that :

$$C_\theta^{cl}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$$

Therefore :

$$C_\theta^{cl}(q, q) = (2q^{-\theta} - 1)^{-1/\theta} \quad ; \quad \theta \geq 1$$

$$\underset{q \to 0^+}{=} (2q^{-\theta})^{-1/\theta} \underset{q \to 0^+}{=} 2^{-1/\theta} q$$

Therefore :

$$\lambda_\ell^{cl} = \lim_{q \to 0^+} \frac{C_\theta^{cl}(q, q)}{q} \underset{q \to 0^+}{=} \frac{2^{-1/\theta} q}{q}$$

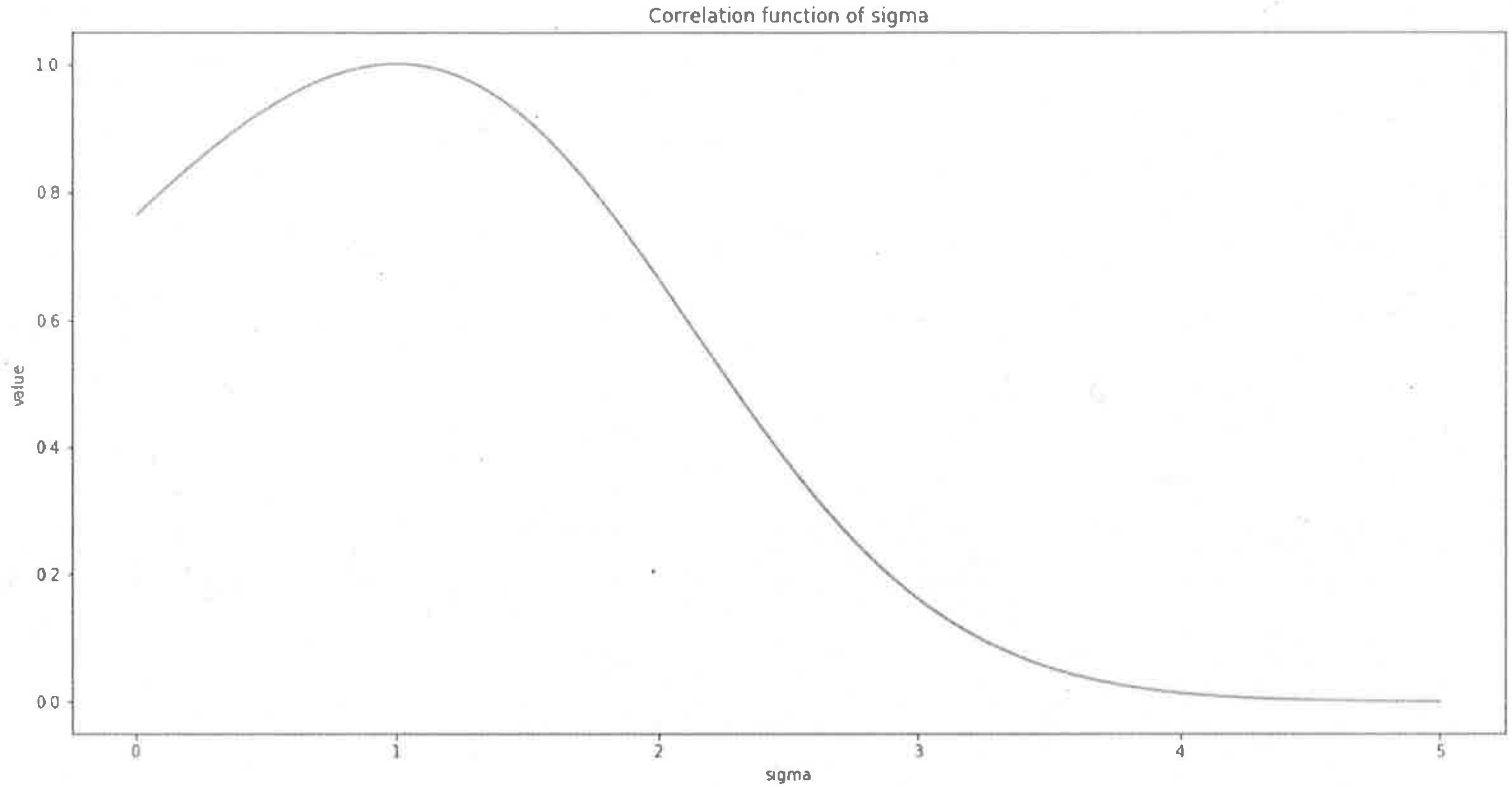
$$\underline{\underline{\lambda_\ell^{cl} = 2^{-1/\theta}}}$$

$$\lambda_u^{cl} = \lim_{q \to 1^-} \frac{1 - 2q + (2q^{-\theta} - 1)^{-1/\theta}}{1 - q}$$

$$1 - 2q + (2q^{-\theta} - 1)^{-1/\theta} \underset{q \to 1^-}{\longrightarrow} 0$$

$$\frac{1}{1-q} \underset{q \to 1^-}{\longrightarrow} 0$$

Therefore : $\underline{\underline{\lambda_u^{cl} = 0}}$


Correlation function of sigma
value
sigma
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5