1. **Constructing entanglement witness from the partial transpose** (10 Points: 1+2+2+2+2+1)
In the lecture, we saw that every separable bi-partite quantum state has a positive partial transpose, which means that the positivity is an entanglement criterion. First, we show that this criterion is valid.

a) Show that for an arbitrary separable bi-partite quantum state $\rho = \sum_i p_j(\rho_{Ai} \otimes \rho_{Bi})$, all eigenvalues of ρ^{T_A} are greater than or equal to 0, i.e., $\rho^{T_A} \geq 0$.

- $\rho = \sum_i p_i \left(\rho_{Ai} \otimes \rho_{Bi}\right)$
- $\rho^{T_A} = \sum_i p_i \, \rho_{Ai}^T \otimes \rho_{Bi}$
- $\rho_{Ai} \geq 0 \Rightarrow \rho_{Ai}^T \geq 0$

$\rho_{Ai} = UDU^+ \Rightarrow \rho_{Ai}^T = \underbrace{U^* D^* U^t}_{} = U^* D U^{*+}$, $D^* = D$, ≥ 0

$\Rightarrow \rho_{Ai}^T \otimes \rho_{Bi} \geq 0$

The eigenvalues of $\rho_{Ai}^T \otimes \rho_{Bi}$ are the products of eigenvalues of ρ_{Ai}^T and ρ_{Bi}.

$\Rightarrow \rho^{T_A} = \sum_i \underset{\geq 0}{p_i} \underset{\geq 0}{\left(\rho_{Ai}^T \otimes \rho_{Bi}\right)} \Rightarrow \square$

$\langle v | \rho^{T_A} | v \rangle \geq 0 \quad \forall v.$

- ρ sep $\Rightarrow \rho^{T_A} \geq 0$
- $\rho^{T_A} \ngeq 0 \Rightarrow \rho$ entangled
- $\rho^{T_A} \geq 0 \overset{??}{\Rightarrow} \rho$ sep ? No, there are states with $\rho^{T_A} \geq 0$ but entangled.

In general, the opposite direction is not true. However, if we restrict a quantum state to a pure state, the opposite is also true as the following.

b) Show that a bi-partite pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is separable if it has a positive partial transpose.

Hint: Prove the contraposition: if $|\psi\rangle$ is entangled, $(|\psi\rangle\langle\psi|)^{T_A}$ has at least one negative eigenvalue. To this end, use Schmidt decomposition.

· If a state is pure:

$\exists$ $\{|u_i\rangle\}, \{|v_i\rangle\}$ orth. basis : $|\psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |u_i\rangle \otimes |v_i\rangle$

PROOF:

$$|\psi\rangle = \sum_{i,j} c_{i,j}\, |i\rangle \otimes |j\rangle = \sum_{i,j} (UDV^\dagger)_{i,j} |i\rangle \otimes |j\rangle = \sum_l D_{ll} \Big(\underbrace{\sum_i U_{il} |i\rangle}_{U|l\rangle}\Big) \otimes \Big(\underbrace{\sum_j V^\dagger_{lj} |j\rangle}_{V^*|l\rangle}\Big)$$

$C = UDV^\dagger$ with $D \geq 0$ ($\uparrow$ SVD)

$$= \sum_l \sqrt{\lambda_l}\, \underbrace{U|l\rangle}_{|u_l\rangle} \otimes \underbrace{V^*|l\rangle}_{|v_l\rangle}$$

$D_{ll} = \sqrt{\lambda_l}$

· $\sum_i \lambda_i = 1$

$$\sum_i \lambda_i = \sum_i D_{ii}^2 = tr(D^\dagger D) = tr(C^\dagger C) = \sum_j \langle j|C^\dagger C|j\rangle = \sum_{i,j} \langle j|C^\dagger|i\rangle\langle i|C|j\rangle$$

$\langle j|C^\dagger|i\rangle = \langle i|C^*|j\rangle = \langle i|C|j\rangle^*$

$$= \sum_{i,j} |C_{i,j}|^2 = \langle\psi|\psi\rangle = 1$$

$$|\psi\rangle\langle\psi| = \sum_{i,j} \sqrt{\lambda_i \lambda_j}\, |u_i\rangle\langle u_j| \otimes |v_i\rangle\langle v_j|$$

$$(|\psi\rangle\langle\psi|)^{T_A} = \sum_{i,j} \sqrt{\lambda_i \lambda_j}\, |u_j\rangle^* \langle u_i|^* \otimes |v_i\rangle\langle v_j|$$

· $(|u_i\rangle\langle u_j|)^t = (|u_i\rangle\langle u_j|)^{\dagger *} = (|u_j\rangle\langle u_i|)^*$

· Or without BRAKET $\big(\underbrace{\vec{u}_i\, \vec{u}_j^{\,\dagger}}_{(|u_i\rangle\langle u_j|)^t}\big)^t = \vec{u}_j^{\,*}\, \vec{u}_i^{\,t} = \vec{u}_j^{\,*}\, \vec{u}_i^{\,*\dagger} = |u_j^*\rangle\langle u_i^*|$

· EIGENSTATES OF $(|\psi\rangle\langle\psi|)^{T_A}$:

· $|u_i^*\rangle \otimes |v_i\rangle$, λ_i (EIGENVALUE) ,

· $\dfrac{|u_i^*\rangle \otimes |v_j\rangle + |u_j^*\rangle \otimes |v_i\rangle}{\sqrt{2}}$, $\sqrt{\lambda_i \lambda_j}$ $\quad \forall\, i \neq j$

- $\frac{|u_i\rangle^* \otimes |v_j\rangle - |u_j\rangle \otimes |v_i\rangle^*}{\sqrt{2}}$, $\underbrace{-\sqrt{\lambda_i \lambda_j}}_{<0}$ $\forall i \neq j$

More than one schmidt coefficient is $\neq 0 \Rightarrow \exists$ Negative eigenvalue of $(|\Psi\rangle\langle\Psi|)^{T_A}$

$\exists$ Negative eigenvalue of $(|\Psi\rangle\langle\Psi|)^{T_A} \Longrightarrow$ More than one schmidt coefficient is $\neq 0$.

$\Longrightarrow |\Psi\rangle\langle\Psi|$ entangled

Recall that an entanglement witness is an observable W with the following conditions: (i) $\mathrm{Tr}(W\sigma) \geq 0$ for all separable states σ and (ii) there exists an entangled state $\tilde{\rho}$ satisfying $\mathrm{Tr}(W\tilde{\rho}) < 0$.

c) Consider an entangled state ρ. Let $|\mu\rangle$ be an eigenvector of ρ^{T_A} whose eigenvalue is negative. Then show that $W = (|\mu\rangle\langle\mu|)^{T_A}$ is an entanglement witness and $|\mu\rangle$ is an entangled state.

- $\underset{\substack{\uparrow \\ \text{eigendecomp.}}}{\rho} = \sum \lambda_i |\Psi_i\rangle\langle\Psi_i| \Rightarrow \rho^{T_A} = \sum_i \lambda_i \left(\underbrace{|\Psi_i\rangle\langle\Psi_i|}\right)^{T_A}$

  PURE STATE $\Rightarrow$ Before we saw that $(|\Psi_i\rangle\langle\Psi_i|)^{T_A}$ is diagonalizable.

- $\rho : \rho^{T_A} \not\geq 0 \Rightarrow \exists |\mu\rangle : \rho^{T_A}|\mu\rangle = \lambda|\mu\rangle$ with $\lambda < 0$

- $W := (|\mu\rangle\langle\mu|)^{T_A}$

- σ sep. $\Rightarrow \mathrm{Tr}[W\sigma] = \mathrm{Tr}[(|\mu\rangle\langle\mu|)^{T_A}\sigma] = \mathrm{Tr}[|\mu\rangle\langle\mu|\sigma^{T_A}]$

  $= \langle\mu|\sigma^{T_A}|\mu\rangle \geq 0$

  σ SEP $\Rightarrow$ PPT. $(\sigma^{T_A} \geq 0)$

- $\tilde{\rho} := \rho$ is entangled and $\mathrm{Tr}[W\tilde{\rho}] = \mathrm{Tr}[|\mu\rangle\langle\mu|\tilde{\rho}^{T_A}] = \langle\mu|\tilde{\rho}^{T_A}|\mu\rangle$

  $= \lambda\langle\mu|\mu\rangle = \lambda < 0$

As an application of this witness, we consider the following setting. In our (fictitious) lab, we are trying to prepare a two-qubit state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We use a simple model[1] for what is actually happening in the lab, namely that we prepare a state with some noise

$$\rho(p) := p\,|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{4}.$$

Our goal is to have an observable witness that decides whether $\rho(p)$ is entangled or not. To this end, we will use the fact that for two-qubits system there exist no entangled. positive partial transpose (PPT) states. Therefore, the partial transpose T^A will always detect entanglement of $\rho(p)$.

d) Assume $|\psi\rangle = a\,|01\rangle_{AB} + b\,|10\rangle_{AB}$. Calculate eigenvalues of $\rho(p)^{T_B}$, and determine the values of p depending on a, b such that $\rho(p)$ is entangled.

Hint: Use the fact that $\rho(p)$ is entangled if and only if $\rho(p)^{T_B} \ngeq 0$.

- $|\psi\rangle\langle\psi| = |a|^2\,|01\rangle\langle 01| + a b^*\,|01\rangle\langle 10| + |b|^2\,|10\rangle\langle 10| + a^* b\,|10\rangle\langle 01|$

- $\rho(p) = p\left(|a|^2\,|01\rangle\langle 01| + a b^*\,|01\rangle\langle 10| + |b|^2\,|10\rangle\langle 10| + a^* b\,|10\rangle\langle 01|\right) + (1-p)\frac{\mathbb{1}}{4}$

- $\rho(p)^{T_B} = p\left(|a|^2\,|01\rangle\langle 01| + a b^*\,|00\rangle\langle 11| + |b|^2\,|10\rangle\langle 10| + a^* b\,|11\rangle\langle 00|\right) + (1-p)\frac{\mathbb{1}}{4}$

$$= p\,\underbrace{\begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & a^* b \\ 0 & |a|^2 & 0 & 0 \\ 0 & 0 & |b|^2 & 0 \\ a b^* & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array}}_{=: A} + (1-p)\frac{\mathbb{1}}{4}$$

- The spectrum of A is:

EIGENSTATE	EIGENVALUE
$\lvert 01\rangle$	$\leadsto \lvert a\rvert^2$
$\lvert 10\rangle$	$\leadsto \lvert b\rvert^2$

$\tilde{A} := A\big|_{\{|00\rangle, |11\rangle\}} = \begin{pmatrix} 0 & a^* b \\ a b^* & 0 \end{pmatrix} \Rightarrow \det(\tilde{A} - \lambda\mathbb{1}) = \det\left[\begin{pmatrix} -\lambda & a^* b \\ a b^* & -\lambda \end{pmatrix}\right] = 0 \Rightarrow \lambda^2 - |a|^2|b|^2$

$\Rightarrow \lambda = \pm |a||b|$

$$\hat{A}|v\rangle = \pm|a||b|\,|v\rangle \implies \begin{pmatrix} \mp|a||b| & a^* b \\ a b^* & \mp|a||b| \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies$$

$$\implies y = \pm\frac{a\,b^*}{|a||b|}\,x \quad \underset{\substack{\text{NORMALIZ.}\\ |x|^2+|y|^2=1}}{\implies} \quad \begin{cases} |x|^2 + \frac{|a|^2|b|^2}{|a|^2|b|^2}|x|^2 = 1 \\ y = \pm\frac{a\,b^*}{|a||b|}\,x \end{cases}$$

$$\implies \begin{cases} x = \frac{1}{\sqrt{2}}\,e^{i\phi} \\ y = \pm\frac{a\,b^*}{|a||b|}\frac{1}{\sqrt{2}}\,e^{i\phi} \end{cases} \rightsquigarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{e^{i\phi}}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm\frac{a b^*}{|a||b|} \end{pmatrix}$$

- The spectrum of A is:

EIGENSTATE		EIGENVALUE
$\|01\rangle$	$\rightsquigarrow$	$\|a\|^2$
$\|10\rangle$	$\rightsquigarrow$	$\|b\|^2$
$\frac{1}{\sqrt{2}}\left(\|00\rangle + \frac{a b^*}{\|a\|\|b\|}\|11\rangle\right)$	$\rightsquigarrow$	$\|a\|\|b\|$
$\frac{1}{\sqrt{2}}\left(\|00\rangle - \frac{a b^*}{\|a\|\|b\|}\|11\rangle\right)$	$\rightsquigarrow$	$-\|a\|\|b\|$

- The spectrum of $\rho_{(p)}^{T_B}$ is:

EIGENSTATE		EIGENVALUE	
$\|01\rangle$	$\rightsquigarrow$	$p\|a\|^2 + \frac{1}{4}(1-p)$	≥ 0
$\|10\rangle$	$\rightsquigarrow$	$p\|b\|^2 + \frac{1}{4}(1-p)$	≥ 0
$\frac{1}{\sqrt{2}}\left(\|00\rangle + \frac{a b^*}{\|a\|\|b\|}\|11\rangle\right)$	$\rightsquigarrow$	$p\|a\|\|b\| + \frac{1}{4}(1-p)$	≥ 0
$\frac{1}{\sqrt{2}}\left(\|00\rangle - \frac{a b^*}{\|a\|\|b\|}\|11\rangle\right)$	$\rightsquigarrow$	$-p\|a\|\|b\| + \frac{1}{4}(1-p)$	?

- $-p|a||b| + \frac{1}{4}(1-p) < 0 \iff p > \dfrac{1}{(1+4|a||b|)}$

- $p > \frac{1}{1(1+4|a||b|)} \Longleftrightarrow$ $\rho(p) := p\,|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{4}$. with $|\psi\rangle = a\,|01\rangle_{AB} + b\,|10\rangle_{AB}$ is entangled

e) Use the eigenvector corresponding to a negative eigenvalue of $(\rho(p))^{T_B}$ in order to derive an entanglement witness $\mathcal{W}$ for $\rho(p)$.

$$p > \frac{1}{(1+4|a||b|)} \Rightarrow W := \left(|u\rangle\langle u|\right)^{T_B} \quad \text{with} \quad |u\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle - \underbrace{\frac{a\,b^*}{|a||b|}}_{=\,1,\ \text{if } a,b\in\mathbb{R}^+.}|11\rangle\Big)$$

is entangl. witness for $\rho(p)$.

(shown before)

f) Show that, in fact, the witness $\mathcal{W}$ detects *all* entangled states of the form $\rho(p)$.

$\forall\ p > \frac{1}{(1+4|a||b|)}$, $\rho(p)$ is entangled and

$$\mathrm{Tr}\left[W\,\rho(p)\right] = \mathrm{Tr}\left[|u\rangle\langle u|\,\rho^{T_B}(p)\right] \Rightarrow$$

$$= \langle u|\,\rho^{T_B}(p)\,|u\rangle < 0$$

$|u\rangle$ is eigenstate of $\rho^{T_B}(p)$ with eigenvalue < 0.

Freie Universität Berlin
Tutorials on Quantum Information Theory
Winter term 2022/23

Problem Sheet 8
Entanglement Witnesses and Cryptography

J. Eisert, A. Townsend-Teague, A. Mele, A. Burchards, J. Denzler

1. **Constructing entanglement witness from the partial transpose** (10 Points: 1+2+2+2+2+1)
In the lecture, we saw that every separable bi-partite quantum state has a positive partial transpose, which means that the positivity is an entanglement criterion. First, we show that this criterion is valid.

   a) Show that for an arbitrary separable bi-partite quantum state $\rho = \sum_i p_j(\rho_{Ai} \otimes \rho_{Bi})$, all eigenvalues of ρ^{T_A} are greater than or equal to 0, i.e., $\rho^{T_A} \geq 0$.

   In general, the opposite direction is not true. However, if we restrict a quantum state to a pure state, the opposite is also true as the following.

   b) Show that a bi-partite pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is separable if it has a positive partial transpose.

   Hint: Prove the contraposition: if $|\psi\rangle$ is entangled, $(|\psi\rangle\langle\psi|)^{T_A}$ has at least one negative eigenvalue. To this end, use Schmidt decomposition.

   Recall that an entanglement witness is an observable W with the following conditions: (i) $\mathrm{Tr}(W\rho) \geq 0$ for all separable states σ and (ii) there exists an entangled state ρ satisfying $\mathrm{Tr}(W\rho) < 0$.

   c) Consider an entangled state ρ. Let $|\mu\rangle$ be an eigenvector of ρ^{T_A} whose eigenvalue is negative. Then show that $W = (|\mu\rangle\langle\mu|)^{T_A}$ is an entanglement witness and $|\mu\rangle$ is an entangled state.

   As an application of this witness, we consider the following setting. In our (fictitious) lab, we are trying to prepare a two-qubit state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We use a simple model[1] for what is actually happening in the lab, namely that we prepare a state with some noise

$$\rho(p) := p\,|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{4}.$$

   Our goal is to have an observable witness that decides whether $\rho(p)$ is entangled or not. To this end, we will use the fact that for two-qubits system there exist no entangled. positive partial transpose (PPT) states. Therefore, the partial transpose T^A will always detect entanglement of $\rho(p)$.

   d) Assume $|\psi\rangle = a\,|01\rangle_{AB} + b\,|10\rangle_{AB}$. Calculate eigenvalues of $\rho(p)^{T_B}$, and determine the values of p depending on a, b such that $\rho(p)$ is entangled.

   Hint: Use the fact that $\rho(p)$ is entangled if and only if $\rho(p)^{T_B} \not\geq 0$.

   e) Use the eigenvector corresponding to a negative eigenvalue of $(\rho(p))^{T_B}$ in order to derive an entanglement witness $\mathcal{W}$ for $\rho(p)$.

   f) Show that, in fact, the witness $\mathcal{W}$ detects *all* entangled states of the form $\rho(p)$.

[1] What is the corresponding noise channel for this model?