### 1. Constructing entanglement witness from the partial transpose (10 Points: 1+2+2+2+2+1

In the lecture, we saw that every separable bi-partite quantum state has a positive partial transpose, which means that the positivity is an entanglement criterion. First, we show that this criterion is valid.

a) Show that for an arbitrary separable bi-partite quantum state  $\rho = \sum_i p_i(\rho_{Ai} \otimes \rho_{Bi})$ , all eigenvalues of  $\rho^{T_A}$  are greater than or equal to 0, i.e.,  $\rho^{T_A} \geq 0$ .

$$P = \sum_{i} p_{i} \left( P_{Ai} \otimes P_{Bi} \right)$$

$$P_{A_{i}} \Rightarrow P_{A_{i}} \Rightarrow P_{A_{i}}$$

$$P_{A_{\lambda}} = 0 \Rightarrow P_{A_{\lambda}} = 0$$

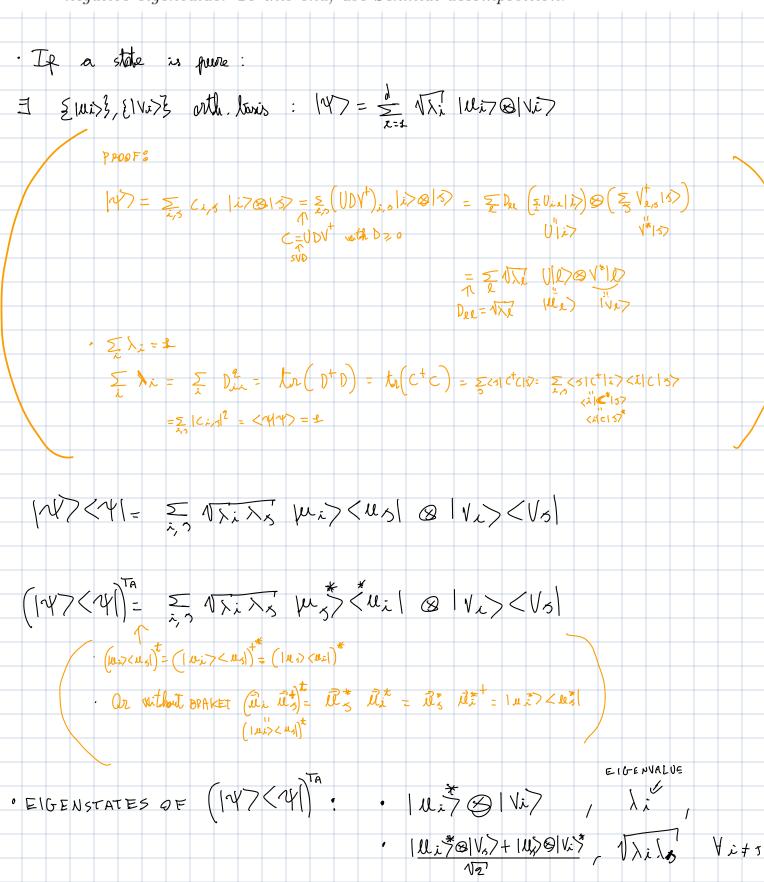
$$P_{A_{\lambda}} = 0$$

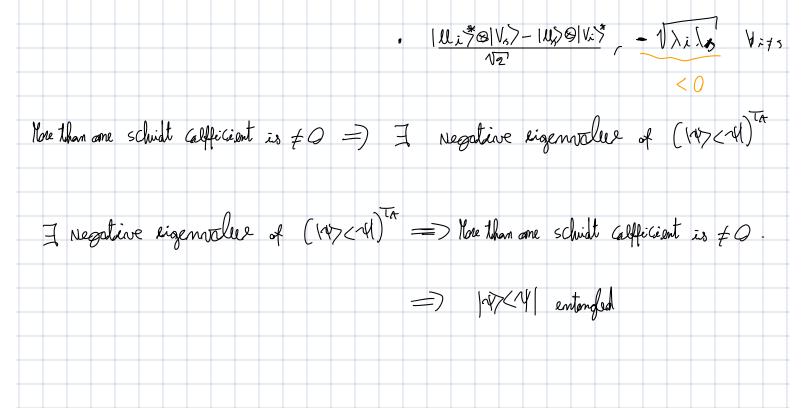
The eigenvalues of PAT and PBT are the paraducts

In general, the opposite direction is not true. However, if we restrict a quantum state to a pure state, the opposite is also true as the following.

b) Show that a bi-partite pure state  $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$  is separable if it has a positive partial transpose.

Hint: Prove the contraposition: if  $|\psi\rangle$  is entangled,  $(|\psi\rangle\langle\psi|)^{T_A}$  has at least one negative eigenvalue. To this end, use Schmidt decomposition.





Recall that an entanglement witness is an observable W with the following conditions: (i)  $\text{Tr}(W\boldsymbol{\sigma}) \geq 0$  for all separable states  $\sigma$  and (ii) there exists an entangled state  $\widetilde{\rho}$  satisfying  $\text{Tr}(W\widetilde{\rho}) < 0$ .

c) Consider an entangled state  $\rho$ . Let  $|\mu\rangle$  be an eigenvector of  $\rho^{T_A}$  whose eigenvalue is negative. Then show that  $W = (|\mu\rangle \langle \mu|)^{T_A}$  is an entanglement witness and  $|\mu\rangle$  is an entangled state.

$$P = \sum_{i} |\langle u_{i} \rangle \langle v_{i} | \rangle = \sum_{i} \sum_{i} |\langle v_{i} \rangle \langle v_{i} | \rangle^{T_{A}}$$

$$|| u_{i} \rangle || v_{i} \rangle || v_$$

As an application of this witness, we consider the following setting. In our (fictitious) lab, we are trying to prepare a two-qubit state  $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ . We use a simple model for what is actually happening in the lab, namely that we prepare a state with some noise

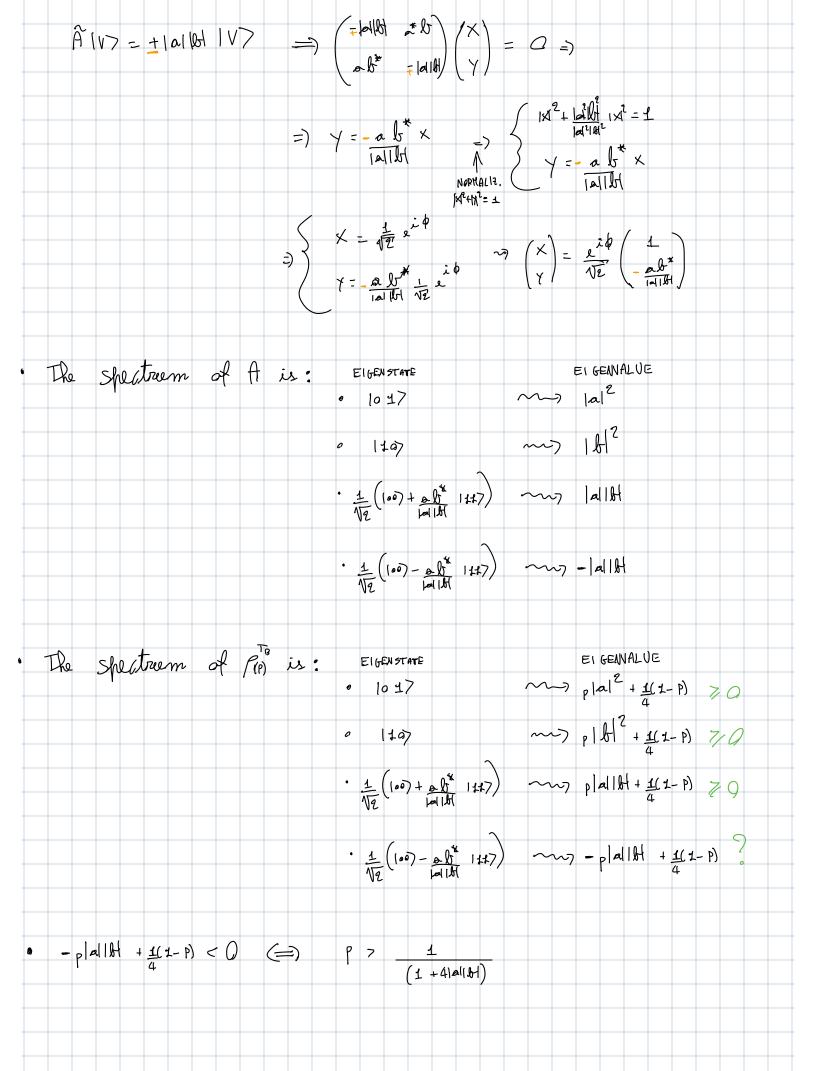
$$\rho(p) := p |\psi\rangle\langle\psi| + (1-p)\frac{1}{4}.$$

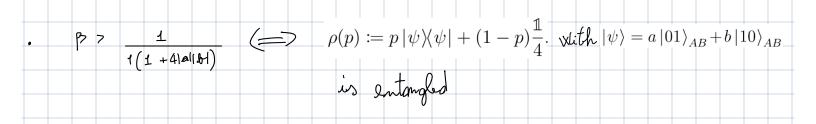
Our goal is to have an observable witness that decides whether  $\rho(p)$  is entangled or not. To this end, we will use the fact that for two-qubits system there exist no entangled. positive partial transpose (PPT) states. Therefore, the partial transpose  $T^A$  will always detect entanglement of  $\rho(p)$ .

d) Assume  $|\psi\rangle = a\,|01\rangle_{AB} + b\,|10\rangle_{AB}$ . Calculate eigenvalues of  $\rho(p)^{T_B}$ , and determine the values of p depending on a, b such that  $\rho(p)$  is entangled.

Hint: Use the fact that  $\rho(p)$  is entangled if and only if  $\rho(p)^{T_B} \ngeq 0$ .

• 
$$|V| < tH = |o|^2 |o| + |c| + |c|$$

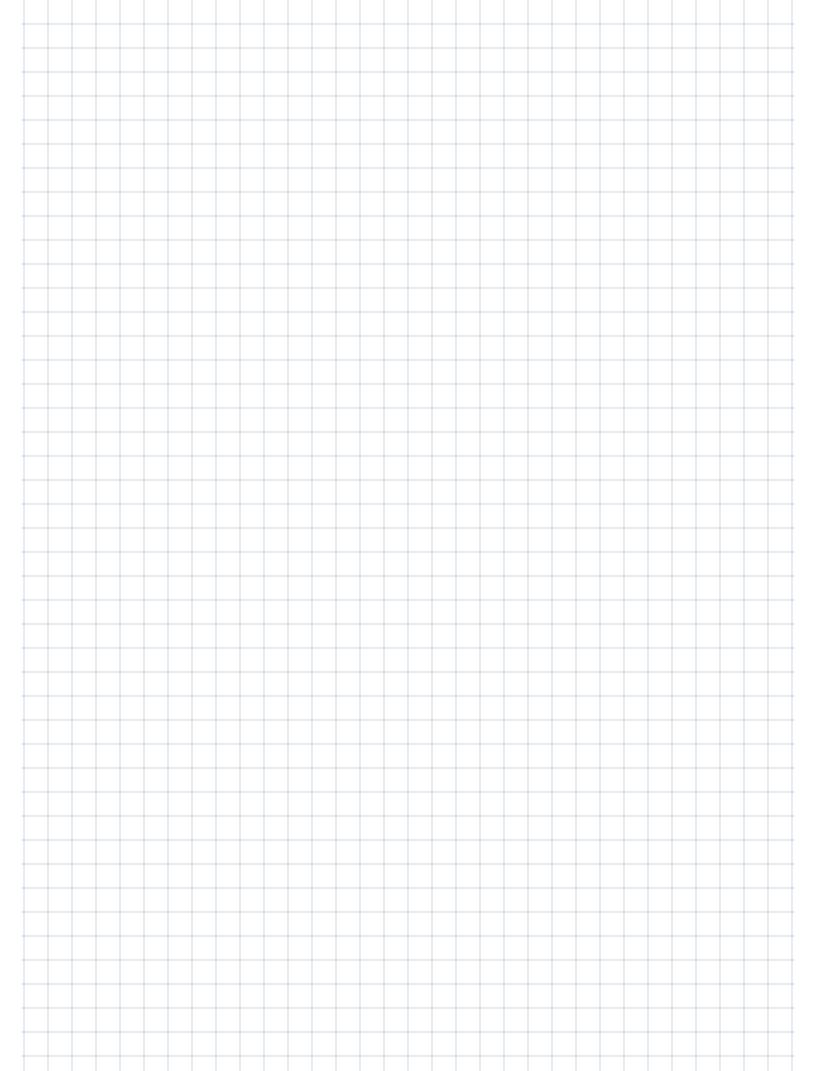




e) Use the eigenvector corresponding to a negative eigenvalue of  $(\rho(p))^{T_B}$  in order to derive an entanglement witness  $\mathcal{W}$  for  $\rho(p)$ .

$$P > \frac{1}{(1+4)al(bt)} = \frac{1}{\sqrt{2}} \left( \frac{147}{4} \right)^{TB}$$
 with 
$$[ij] := \frac{1}{\sqrt{2}} \left( \frac{100}{4} \right)^{-\frac{1}{2}} \left( \frac{1}{100} \right)^$$

f) Show that, in fact, the witness W detects all entangled states of the form  $\rho(p)$ .



#### Freie Universität Berlin

### Tutorials on Quantum Information Theory

Winter term 2022/23

# Problem Sheet 8 Entanglement Witnesses and Cryptography

J. Eisert, A. Townsend-Teague, A. Mele, A. Burchards, J. Denzler

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Recall that an entanglement witness is an observable W with the following conditions: (i)  $\text{Tr}(W\rho) \geq 0$  for all separable states  $\sigma$  and (ii) there exists an entangled state  $\rho$  satisfying  $\text{Tr}(W\rho) < 0$ .

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$$\rho(p) := p |\psi\rangle\langle\psi| + (1-p)\frac{1}{4}.$$

Our goal is to have an observable witness that decides whether  $\rho(p)$  is entangled or not. To this end, we will use the fact that for two-qubits system there exist no entangled positive partial transpose (PPT) states. Therefore, the partial transpose  $T^A$  will always detect entanglement of  $\rho(p)$ .

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Hint: Use the fact that  $\rho(p)$  is entangled if and only if  $\rho(p)^{T_B} \ngeq 0$ .

- e) Use the eigenvector corresponding to a negative eigenvalue of  $(\rho(p))^{T_B}$  in order to derive an entanglement witness W for  $\rho(p)$ .
- f) Show that, in fact, the witness W detects all entangled states of the form  $\rho(p)$ .

<sup>&</sup>lt;sup>1</sup>What is the corresponding noise channel for this model?