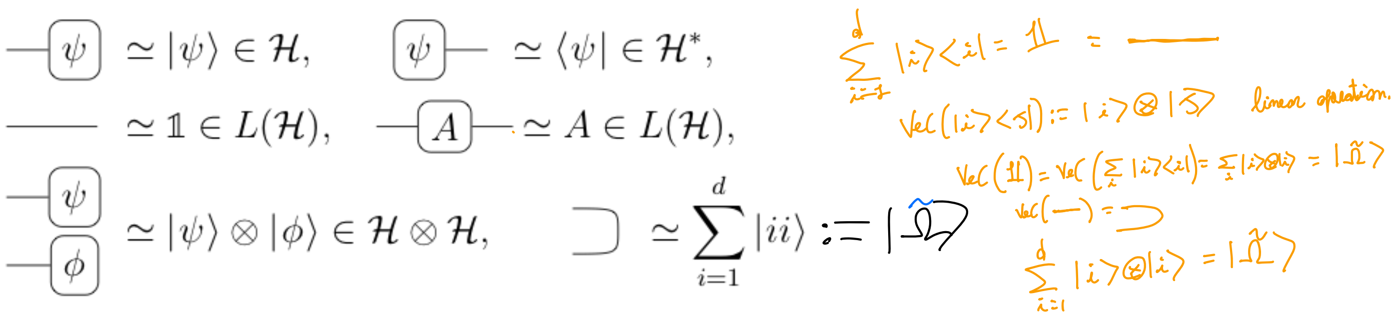
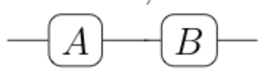
1. **Introduction to graphical calculus with tensor networks** (6 Points: 1+1+1+2+1)

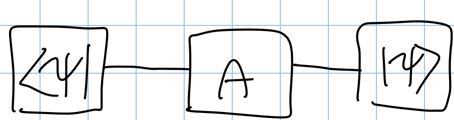
As you might have noticed, already for a little number of tensor factors even simple calculations can become hard to follow quite easily. Hence, an alternative approach to visualize such calculations was developed. Namely, graphical calculus with tensor networks, often atributed to Roger Penrose. We will give a short introduction into the basics of this calculation technique in this exercise. However, we encourage you to have a look into `https://arxiv.org/pdf/1603.03039.pdf`, which gives a nice and complete overview over tensor networks. For this course, you won't need most of the content but it constitutes a good reference where you can find any concept we will use (in particular, in chapter 1 and 2).

In tensor network notation, a tensor is simply an object that has indices, usually a set of complex numbers $A_{i_1,\dots,i_n}$. A tensor with one index is a vector, one with two indices is a matrix. A tensor with n indices is denoted as a box with n legs, hence we have the following correspondences

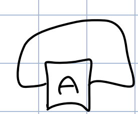


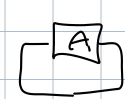
One can think of each unconnected leg carrying a (dual) Hilbert space. Connecting two legs denotes contraction of the indices, so that for example the matrix product $(AB)_{ij} = \sum_k A_{ik}B_{kj}$ is denoted by 

a) Draw the expectation value $\langle\psi| A |\psi\rangle$ as a tensor network.



b) What does the following tensor network represent? 

$\to tr(A) =$ 

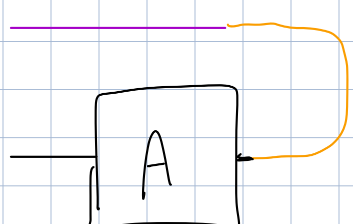
$=$ 

In fact:

- $A =$ 

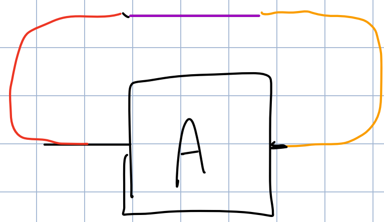
$= \mathbb{1} A \mathbb{1} = \sum_{i,j} \langle i|A|j\rangle\, |i\rangle\langle j|$

$\mathbb{1} = \sum |i\rangle\langle i|$, $\mathbb{1} = \sum |j\rangle\langle j|$

- 

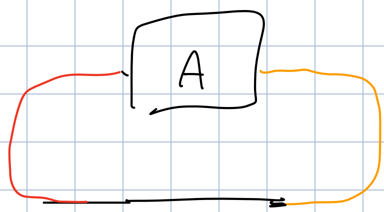
$= \mathbb{1} \otimes A\, |\tilde{\Omega}\rangle$ with $|\tilde{\Omega}\rangle := \sum_{i=1}^{d} |ii\rangle$

$|ii\rangle = |i\rangle\otimes|i\rangle$

•  $= \langle \tilde{\Omega} | \mathbb{1} \otimes A | \tilde{\Omega} \rangle = \sum_{i=1}^{d} \langle i i | \mathbb{1} \otimes A \sum_{j=1}^{d} | j j \rangle =$

$= \sum_{\substack{i=1 \\ j=1}}^{d} \langle i | A | j \rangle \underbrace{\langle i | j \rangle}_{\delta_{ij}} = \sum_{i=1}^{d} \langle i | A | i \rangle = tr(A)$

$\Rightarrow \square$

•  $= \langle \Omega | A \otimes \mathbb{1} | \Omega \rangle = tr(A) \Rightarrow \square$

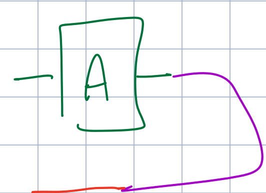
VECTORIZATION

• $Vec(\cdot)$ is a linear operator such that: $Vec\left(|i\rangle\langle j| \right) := |i\rangle \otimes |j\rangle$

• $A = \sum_{i,j} \langle i | A | j \rangle \, |i\rangle\langle j| \Rightarrow Vec(A) := \sum_{i,j} \langle i | A | j \rangle \, |i\rangle \otimes |j\rangle$

$A \underset{d=2}{\overset{e.g.}{=}} \begin{pmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{pmatrix} \rightsquigarrow Vec(A) = \begin{array}{cl} \begin{pmatrix} \langle 0|A|0\rangle \\ \langle 0|A|1\rangle \\ \langle 1|A|0\rangle \\ \langle 1|A|1\rangle \end{pmatrix} & \begin{array}{l} \leftarrow |0\rangle\otimes|0\rangle \\ \leftarrow |01\rangle \\ \leftarrow |10\rangle \\ \leftarrow |11\rangle \end{array} \end{array}$

Vectorize the matrix.

$Vec(A) = A \otimes \mathbb{1} \, |\tilde{\Omega}\rangle =$  with $|\tilde{\Omega}\rangle = \sum_{i=1}^{d} |i i\rangle$

PROOF:

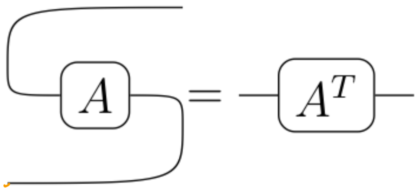
$A \otimes \mathbb{1} |\tilde{\Omega}\rangle = \sum_{i=1}^{d} A \otimes \mathbb{1} |i i\rangle = \sum_{i=1}^{d} \left(\sum_{j,k=1}^{d} \langle j | A | k \rangle \, |j\rangle\langle k| \right) \otimes \mathbb{1} |i i\rangle = \sum_{i,j} \langle j | A | i \rangle \, \underbrace{|j\rangle \otimes |i\rangle}_{Vec(|j\rangle\langle i|)}$

$= Vec\left(\sum_{i,j} \langle j | A | i \rangle \, |j\rangle\langle i| \right) \overset{\uparrow}{=} Vec(A)$

• $\sum_{i} |i\rangle\langle i| = \mathbb{1}$

• $\sum_{j} |j\rangle\langle j| = \mathbb{1}$

c) Prove

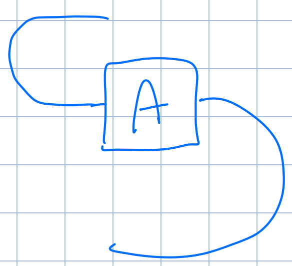


PROOF:

1.

2.

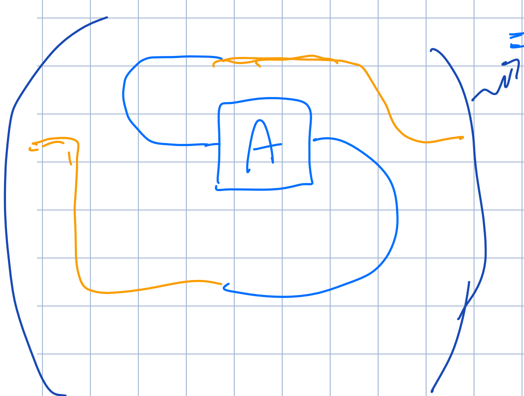
3.



$$= {}_{1,2}\langle\tilde{\Omega}|\, A\, |\tilde{\Omega}\rangle_{2,3} = \sum_{i=1}^{d} \langle i|_1 \otimes \langle i|_2 \; A \sum_{j=1}^{d} |j\rangle_2 \otimes |j\rangle_3 =$$

$$= \sum_{j=1}^{d} \sum_{i=1}^{d} \langle i|_2 A |j\rangle_2 \; |j\rangle_3 \langle i|_1$$

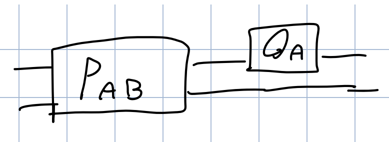
$$= \sum_{j=1}^{d} \sum_{i=1}^{d} \langle i|A|j\rangle \; |j\rangle\langle i| = \sum_{j=1}^{d} \sum_{i=1}^{d} \langle j|A^t|i\rangle \, |j\rangle\langle i| = A^t$$

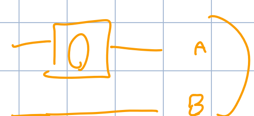
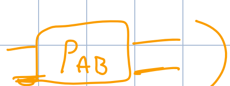


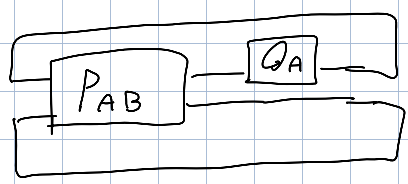
d) Using tensor networks, prove the following statement from Exercise sheet 1

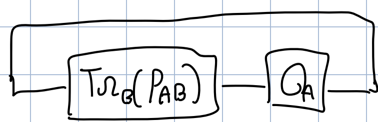
$$\mathrm{Tr}(\rho_{AB} O_A \otimes \mathbb{1}_B) = \mathrm{Tr}(\mathrm{Tr}_B(\rho_{AB}) O_A) \tag{1}$$

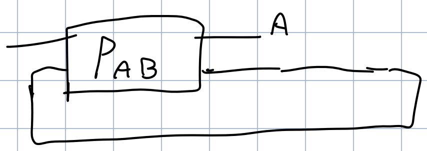
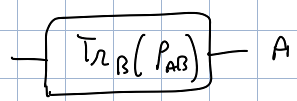
Hint: recall that any ρ_{AB} can be decomposed in bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ as $\rho_{AB} = \sum_{ijkl} \rho_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|$.

$\mathrm{Tr}(\rho_{AB}\; O_A \otimes \mathbb{1}_B) = \mathrm{Tr}($  $) =$

- $O_A \otimes \mathbb{1}_B =$ 
- $\rho_{AB} =$ 

$=$ 

$=$  $= \mathrm{Tr}\left(\mathrm{Tr}_B(\rho_{AB})\, O_A \right)$

 $= \mathrm{Tr}_B\left(\rho_{AB} \right) =$ 

e) Prove that $\mathrm{Tr}(A^2) = \mathrm{Tr}((A \otimes A)F)$ using tensor networks. F denotes the *flip operator* exchanging the two subsystems, i.e. $F : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, |i\rangle \otimes |j\rangle \mapsto |j\rangle \otimes |i\rangle$.

$F(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle$

- Prove before without diagrams.

PROOF:

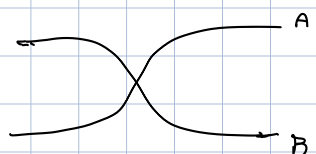
$$\mathrm{Tr}\left((A \otimes B)\, F \right) = \sum_{\substack{i=1\\ j=1}}^{d} \langle i,j|\, A \otimes B\, F\, |i,j\rangle \underset{F|i\rangle\otimes|j\rangle = |j\rangle \otimes |i\rangle}{=} \sum_{\substack{i=1\\ j=1}}^{d} \langle i,j|\, A \otimes B\, |j,i\rangle$$

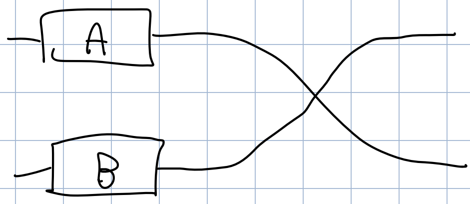
$$= \sum_{\substack{i=1\\ j=1}}^{d} \langle i|A|j\rangle \langle j|B|i\rangle \underset{\sum_j |j\rangle\langle j| = \mathbb{1}}{=} \sum_{\substack{i=1\\ j=1}}^{d} \langle i|AB|i\rangle$$

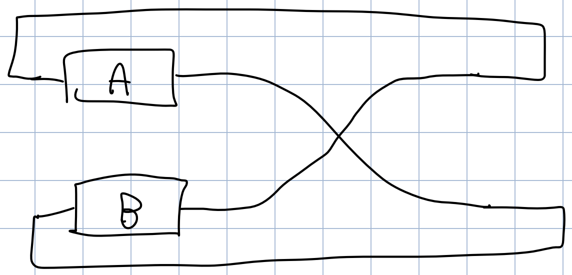
$$= \mathrm{tr}(AB)$$

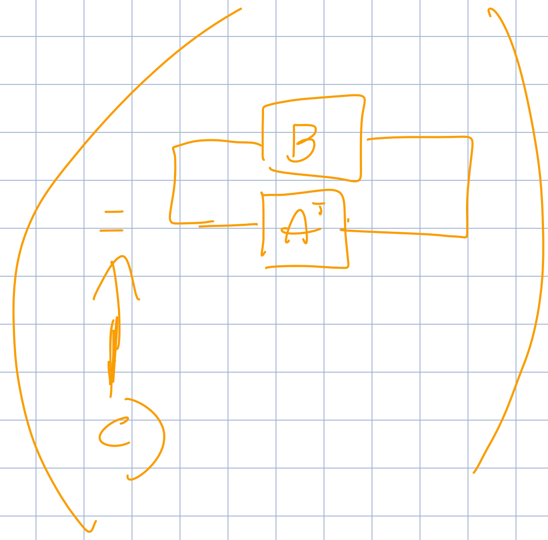
DIAGRAMS:

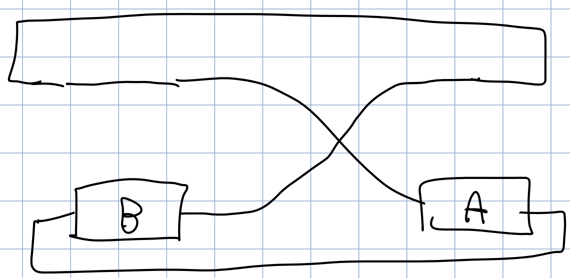
$$F(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle$$

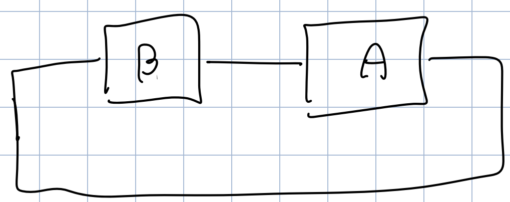
$F =$ 

$(A\otimes B)\mathbb{F} =$ 

$\mathrm{Tr}\big((A\otimes B)\mathbb{F}\big) =$ 



$=$ 

$=$  $= \mathrm{Tr}(BA) = \mathrm{Tr}(AB)$

$\Rightarrow \square$

OTHER EXAMPLES:

without diagrams:

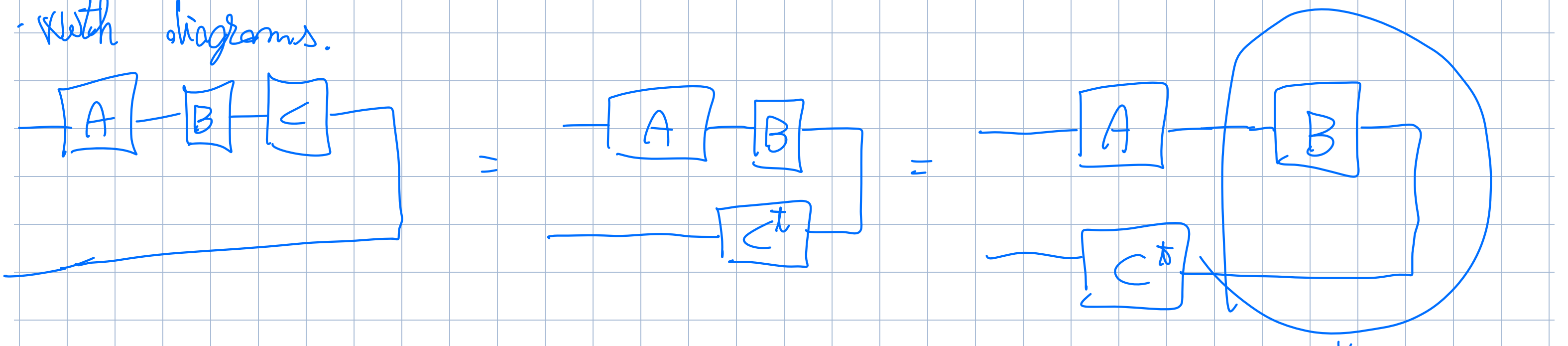
- $\mathrm{Vec}(ABC) = ABC\otimes \mathbb{1}|\Omega\rangle = (AB\otimes\mathbb{1})(C\otimes\mathbb{1})|\Omega\rangle = (AB\otimes\mathbb{1})\,\mathbb{1}\otimes C^t|\Omega\rangle$

$M\otimes\mathbb{1}|\Omega\rangle = \mathbb{1}\otimes M^T|\Omega\rangle$

PREVIOUS LECTURE
(TRANSPOSE-TRICK)

$= (A\otimes\mathbb{1})(\mathbb{1}\otimes C^t)\,B\otimes\mathbb{1}|\Omega\rangle = A\otimes C^t\,\mathrm{Vec}(B)$

· with diagrams.



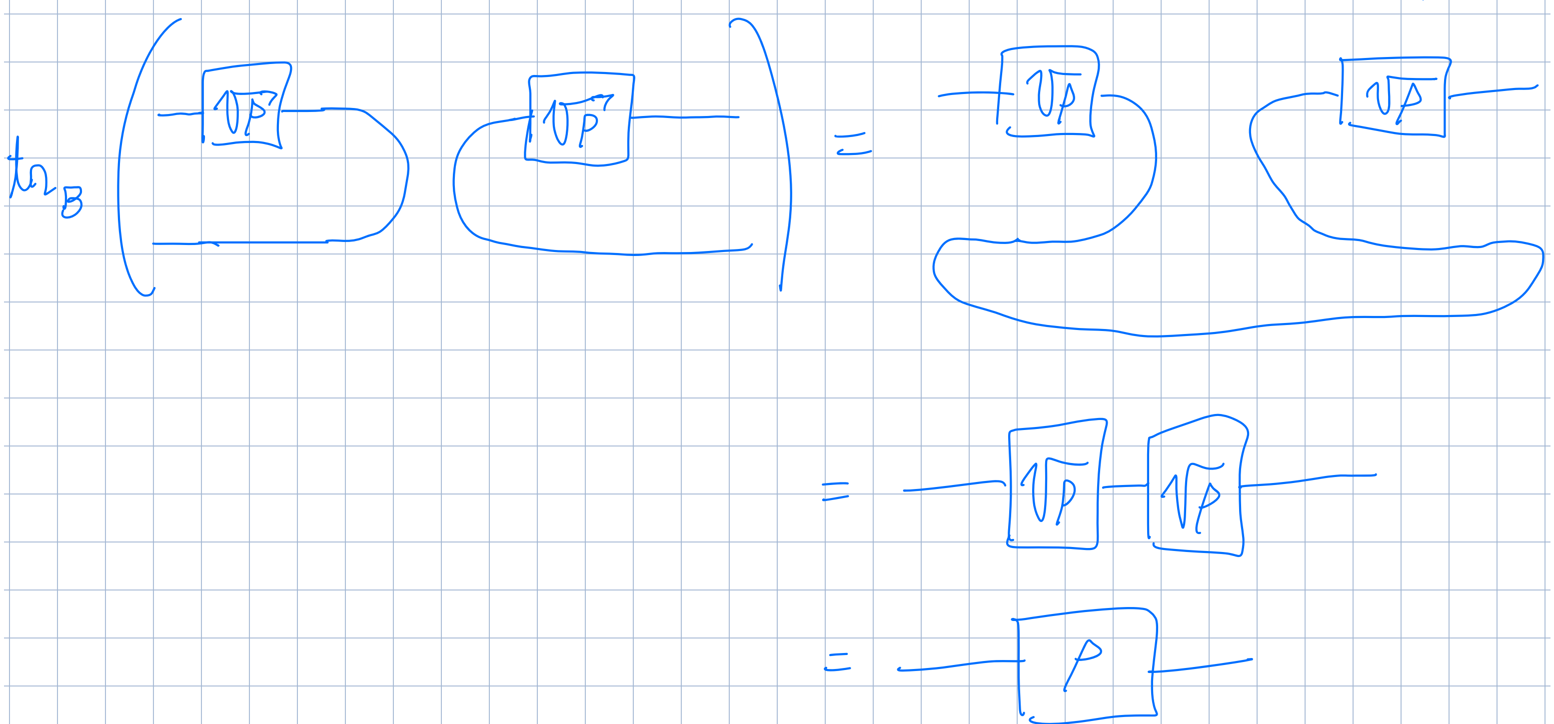
$= A \otimes C^t \; vec(B)$

$vec(B) = B \otimes \mathbb{1} |\Omega\rangle$

· $|\psi\rangle := \sqrt{\rho} \otimes \mathbb{1} \, |\tilde{\Omega}\rangle$. Prove that $tr_B(|\psi\rangle\langle\psi|) = \rho$

we call $|\psi\rangle$ "PURIFICATION" OF ρ.

· DIAGRAMS:



· without diagrams:

$$tr_B(|\psi\rangle\langle\psi|) = tr_B\left(\sqrt{\rho} \otimes \mathbb{1} |\tilde{\Omega}\rangle\langle\tilde{\Omega}| \sqrt{\rho}^{\dagger} \otimes \mathbb{1}\right) \underset{\uparrow}{=} \sum_{i,j} tr_B\left(\sqrt{\rho} \otimes \mathbb{1} |i,i\rangle\langle j,j| \sqrt{\rho} \otimes \mathbb{1}\right)$$

· $|\Omega\rangle = \sum_{i=1}^{d} |i\,i\rangle$

· $\sqrt{\rho}^{\dagger} = \sqrt{\rho}$

$$= \sum_i \sqrt{\rho} \, |i\rangle\langle i| \sqrt{\rho} \underset{\uparrow}{=} \sqrt{\rho}\sqrt{\rho} = \rho \quad \rightarrow \blacksquare$$

$\sum |i\rangle\langle i| = \mathbb{1}$

2. **On the Kraus Representation of Quantum Channels**
We have seen in the lecture as well as in previous exercise sheets that many of the notions in quantum information theory can be understood by starting with pure-state quantum mechanics and demanding a description for subsystems of such quantum systems. Some examples of this are the following statements

- Given an arbitrary pure state $|\psi\rangle \in H_A \otimes H_B$ describing the joint state of two physical systems A and B, all measurement statistics of measurements on subsystem A (or B) are fully contained in the reduced density matrices $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ (or $\rho_B = \mathrm{Tr}_A |\psi\rangle\langle\psi|$). I.e. density matrices are required to describe the possible states of subsystems of larger systems whose states are pure.

$\rho_A = tr_B\left(|\psi\rangle_{AB} \langle\psi|_{AB}\right)$ is not pure $\left(\rho_A^2 = \rho_A\right)$ in general (when $|\psi\rangle_{AB}$ is entangled)

SCHMIDT DECOMPOSITION.

- Given an arbitrary mixed state $\rho \in D(H_A)$ there always exists a second Hilbert space H_B and a pure state $|\psi\rangle \in H_A \otimes H_B$ such that $\rho = tr_B |\psi\rangle\langle\psi|$(Such a $|\psi\rangle$ is called a purification of ρ). This means that all density matrices can be interpreted as states of a subsystem of a larger system which is in a pure state.

Now I'm going to explain a bit of theory.

DEF: $|\psi\rangle_{AB}$ purification of $\rho_A \overset{\text{DEF.}}{\Longleftrightarrow} tr_B\left(|\psi\rangle\langle\psi|\right) = \rho_A$ $\left(\rho_A \in D(H_A),\ |\psi\rangle_{AB} \in H_A \otimes H_B\right)$

FACT: Given $\underset{\text{eigendecomp.}}{\rho_A} = \sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i| \Rightarrow$ • $|\psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |v_i\rangle_A \otimes |v_i\rangle_B$ is a PURIFICATION of ρ_A

• More generally if $\{|i\rangle\}_{i=1}^{d_B}$ is orton. basis for H_B then $|\psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |v_i\rangle_A \otimes |i\rangle_B$ is a PURIFICATION of ρ_A

This is exactly what we saw in the previous exercise: $|\psi\rangle_{AB} = \mathrm{Vec}\left(\sqrt{\rho_A}\right)$

PROOF: Verify that $\rho_A = tr_B\left(|\psi\rangle_{AB} \langle\psi|_{AB}\right)$.

OBSERVATION: Each state $\rho \in D(H_A)$ admits a purification $|\psi\rangle_{AB} \in H_A \otimes H_B$ using an auxiliary H_B of the same dimension of the original system.

- POVMs, also called generalized measurements, can be understood as projective measurements on a larger system.

Now I'm going to explain a bit of theory and then prove such statement.

Projection-Valued Measure ← PROJECTIVE MEASUREMENTS

A PVM measurement is obtained through a projector system [1], which is defined as a set of operators $\{P_i, i \in M\}$ of Hilbert space H, where M is an alphabet set of all possible outcomes of the measurement, if these operators have properties: 1) P_i is Hermitian: $P_i = P_i^\dagger$; 2) P_i is positive semi-definite: $P_i \geq 0$; 3) P_i is idempotent: $P_i^2 = P_i$; 4) P_i is pairwise orthogonal: $P_i P_j = \delta_{ij} = 0$, for $i \neq j$; 5) $\{P_i, i \in M\}$ forms a resolution of the identity on H: $\sum_{i \in M} P_i = I_H$.

The probability of obtaining outcome i for a given state $s = |\psi\rangle$ is specified by

$$p_m(\mathrm{i}|\psi) = \mathrm{P}(\mathrm{m} = \mathrm{i}|s = |\psi\rangle) = \langle\psi|P_i|\psi\rangle \tag{1}$$

And the post-measurement state is given by

$$|\psi_{post}^{(i)}\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}} \tag{2}$$

For mixed state, specified by the density matrix ρ, the probability of obtaining outcome i is given by

$$p_m(\mathrm{i}|\rho) = \mathrm{tr}(P_i\rho) \tag{3}$$

where $\mathrm{tr}(\cdot)$ is the trace operation. And the post-measurement state is specified by the following density matrix:

$$\rho_{post}^{(i)} = \frac{P_i\rho P_i}{\mathrm{tr}(P_i\rho)} \tag{4}$$

← normalization.

- POVM: set $\{E_a\}_{a=1}^{N}$ such that $\begin{cases} \bullet\ E_a \geq 0 \ ; a=1,\dots,N \\ \bullet\ \sum_{a=1}^{N} E_a = \mathbb{1} \end{cases}$ $(\Rightarrow E_A^\dagger = E_A)$

- AXIOMS: Given a state $\rho \in D(\mathcal{H})$, If I perform a POVM on ρ, (generalized measurement)

  - the $\boxed{\text{probability}}$ of getting the outcome "a" is;

  $$\mathrm{Prob}(a) = \mathrm{tr}(E_a \rho)$$

- The state after getting the measurement outcome "a" is not uniquely determined if we only know the POVM.

  We need something more like "the physical implementation" of this POVM.

NAIMARK DILATATION THEOREM:

- POVMs, also called generalized measurements, can be understood as projective measurements on a larger system.

- $\boxed{\text{POVM}}$: • $\{E_a\}_{a=1}^{N}$ such that : $\begin{cases} \bullet\ E_a \geq 0 \\ \bullet\ \sum_{a=1}^{N} E_a = 1 \end{cases}$

  • $\text{Prob}(a) = tr(\rho E_a)$

- $\boxed{\text{PVM}}$: • $\{\Pi_a\}_{a=1}^{N'}$ such that : $\begin{cases} \bullet\ \Pi_a \geq 0 \\ \bullet\ \sum_{a=1}^{N} \Pi_a = 1 \\ \bullet\ \Pi_a \Pi_b = \delta_{ab} \Pi_a \end{cases}$

  • $\text{Prob}(a) = tr(\rho \Pi_a)$

  • $\rho_a \Big|_{\substack{\text{after} \\ \text{seen "a"}}} = \dfrac{\Pi_a \rho \Pi_a}{tr(\Pi_a \rho)}$

- We want to show that given a POVM $\{E_k\}_{k=1}^{N}$ acting on $\mathcal{H}_A$, $\exists$ PVM $\{\Pi_k\}_{k=1}^{N}$ and a unitary U_{AB} acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that:

$$\text{Prob}(k) = tr(E_k \rho) = tr\left(\Pi_k \left(U_{AB}\, \rho \otimes |0\rangle_B \langle 0|_B\, U_{AB}^{+}\right)\right)$$

which means that the probability of measuring "k" using $\{E_a\}_{a=1}^{N}$ POVM, can be understood as the probability of measuring "k" performing a PVM $\{\Pi_a\}_{a=1}^{N}$ after evolving the system $\rho \otimes |0\rangle_B \langle 0|_B$ using a unitary U_{AB}.

- In particular we can show that we can choose an $\mathcal{H}_B$ of dimension N = size of POVM set.

  It's a PVM! • $\Pi_k = \mathbb{1} \otimes |k\rangle_B \langle k|_B$ ← The PVM is performed only on $\mathcal{H}_B$.

  • $U_{AB}\left(|\cdot\rangle \otimes |0\rangle\right) = \sum_{i=1}^{N} \left(V_i \sqrt{E_i}\right) \otimes \mathbb{1} \left(|\cdot\rangle \otimes |i\rangle\right)$

  where V_i are unitaries that we can choose.

  • It is unitary since orthonormal vector are sent to orthonormal vector.
  • We're defining it only on the subspace $\mathcal{H}\otimes|0\rangle$ $\forall|\cdot\rangle$.

PROOF:

We want to show: $tr\left(\Pi_k \left(U_{AB}\, \rho \otimes |0\rangle_B \langle 0|_B\, U_{AB}^{+}\right)\right) = tr(E_k \rho)$

Let's write ρ in the eigendec. $\Rightarrow \rho = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$

$$\mathrm{tr}\left(\Pi_K\left(U_{AB}\,\rho\otimes|0\rangle_B\langle 0|_B\, U_{AB}^{\dagger}\right)\right) = \sum_{i=1}^{d}\lambda_i\,\mathrm{tr}\left(\Pi_K\, U_{AB}\,|\psi_i\rangle\langle\psi_i|\otimes|0\rangle\langle 0|\, U_{AB}^{\dagger}\right) =$$

$$= \sum_{i=1}^{d}\lambda_i\,\mathrm{tr}\left(\Pi_K\, U_{AB}\left(|\psi_i\rangle\otimes|0\rangle\right)\left(\langle\psi_i|\otimes\langle 0|\right)U_{AB}^{\dagger}\right) =$$

$$= \sum_{i=1}^{d}\lambda_i\,\mathrm{tr}\left(\Pi_K \sum_{j=1}^{N}\left(V_j\sqrt{E_j}\right)\otimes\mathbb{1}\left(|\psi_i\rangle\otimes|j\rangle\right)\left(\langle\psi_i|\otimes\langle 0|\right)U_{AB}^{\dagger}\right) =$$

$$U_{AB}\left(|\cdot\rangle\otimes|0\rangle\right) = \sum_{j=1}^{N}\left(V_j\sqrt{E_j}\right)\otimes\mathbb{1}\left(|\cdot\rangle\otimes|j\rangle\right)$$

$$= \sum_{i=1}^{d}\lambda_i\,\mathrm{tr}\left(\Pi_K \sum_{j=1}^{N}\left(V_j\sqrt{E_j}\right)\otimes\mathbb{1}\left(|\psi_i\rangle\otimes|j\rangle\right)\sum_{l=1}^{N}\left(\langle\psi_i|\otimes\langle l|\right)\left(V_l\sqrt{E_l}\right)^{\dagger}\otimes\mathbb{1}\right)$$

$$= \sum_{i=1}^{d}\lambda_i\,\mathrm{tr}\left(\mathbb{1}\otimes|k\rangle\langle k| \sum_{j=1}^{N}\left(V_j\sqrt{E_j}\right)\otimes\mathbb{1}\left(|\psi_i\rangle\otimes|j\rangle\right)\sum_{l=1}^{N}\left(\langle\psi_i|\otimes\langle l|\right)\left(V_l\sqrt{E_l}\right)^{\dagger}\otimes\mathbb{1}\right)$$

$$\Pi_K = \mathbb{1}\otimes|K\rangle\langle K|$$

$$= \sum_{i=1}^{d}\lambda_i\,\mathrm{tr}_{AB}\left(\mathbb{1}\otimes|k\rangle\langle k|\,\left(V_K\sqrt{E}\right)\otimes\mathbb{1}\left(|\psi_i\rangle\otimes|k\rangle\right)\left(\langle\psi_i|\otimes\langle k|\right)\left(V_K\sqrt{E_K}\right)^{\dagger}\otimes\mathbb{1}\right)$$

TAKING TRACE RESPECT TO B

$$= \sum_{i=1}^{d}\lambda_i\,\mathrm{tr}_A\left(\left(V_K\sqrt{E_k}\right)|\psi_i\rangle\langle\psi_i|\left(V_K\sqrt{E_K}\right)^{\dagger}\right) =$$

$$= \mathrm{tr}\left(\left(V_K\sqrt{E_K}\right)^{\dagger}V_K\sqrt{E_K}\,\rho\right) = \mathrm{tr}\left(\sqrt{E_K}\sqrt{E_K}\,\rho\right) = \mathrm{tr}\left(E_K\,\rho\right)$$

- $\rho = \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$

- $V_K V_K^{\dagger} = \mathbb{1}$
- $\left(\sqrt{E_K}\right)^{\dagger} = \sqrt{E_K}$

$E_K \geq 0 \Rightarrow \sqrt{E_K}\geq 0 \Rightarrow \left(\sqrt{E_K}\right)^{\dagger} = \sqrt{E_K}$

$\Rightarrow \square$

• If the POVM is actually implemented physically by a PVM in a larger system, then the post measurement state would be:

AXIOMS PVM.

$$\rho_{\substack{\text{AFTER}\\ \text{"k" OUTCOME}}} = \frac{\Pi_k \, \rho' \, \Pi_k}{tr(\rho' \Pi_k)} \qquad \text{where } \rho' = U_{AB}\, \rho \otimes |0\rangle_B\langle 0|_B \, U_{AB}^\dagger \, .$$

• Using our construction with $\Pi_k = \mathbb{1} \otimes |k\rangle\langle k|$ and U_{AB} we have:

$$\rho^{(AB)}_{\substack{\text{AFTER}\\ \text{"k" OUTCOME}}} = \frac{\Pi_k \, \rho' \, \Pi_k}{tr(\rho' \Pi_k)} = \frac{\Pi_k \, \rho' \, \Pi_k}{tr(\rho E_k)} = \left(V_k \frac{\sqrt{E_k}\, \rho \sqrt{E_k}}{tr(\rho E_k)} V_k^\dagger \right) \otimes |k\rangle\langle k|$$

BEFORE

• $U_{AB} (|\cdot\rangle \otimes |0\rangle) = \sum_{j=1}^{N} (V_j \sqrt{E_j}) \otimes \mathbb{1} \, (|\cdot\rangle \otimes |j\rangle)$
• $\rho' = U_{AB}\, \rho \otimes |0\rangle_B\langle 0|_B \, U_{AB}^\dagger$
• $\Pi_k = \mathbb{1} \otimes |k\rangle\langle k|$
• SIMILARLY AS BEFORE

tracing out the system "B".

$$\Rightarrow \quad \boxed{\rho_{\substack{\text{AFTER}\\ \text{"k" OUTCOME}\\ \text{POVM}}} = V_k \frac{\sqrt{E_k}\, \rho \sqrt{E_k}}{tr(\rho E_k)} V_k^\dagger}$$

$\Leftarrow$ POST POVM OUTCOME "k" STATE

"we need to know V_k which depend by the "physical" implementation of the PVM."

2. **On the Kraus Representation of Quantum Channels**

We have seen in the lecture as well as in previous exercise sheets that many of the notions in quantum information theory can be understood by starting with pure-state quantum mechanics and demanding a description for subsystems of such quantum systems. Some examples of this are the following statements

- Given an arbitrary pure state $|\psi\rangle \in H_A \otimes H_B$ describing the joint state of two physical systems A and B, all measurement statistics of measurements on subsystem A (or B) are fully contained in the reduced density matrices $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ (or $\rho_B = \mathrm{Tr}_A |\psi\rangle\langle\psi|$). I.e. density matrices are required to describe the possible states of subsystems of larger systems whose states are pure.
- Given an arbitrary mixed state $\rho \in D(H_A)$ there always exists a second Hilbert space H_B and a pure state $|\psi\rangle \in H_A \otimes H_B$ such that $\rho = tr_B |\psi\rangle\langle\psi|$(Such a $|\psi\rangle$ is called a purification of ρ). This means that all density matrices can be interpreted as states of a subsystem of a larger system which is in a pure state.
- POVMs, also called generalized measurements, can be understood as projective measurements on a larger system.

In this exercise we want to develop a similar picture for quantum channels by exploring the fact that quantum channels are exactly set of operations one can implement on a quantum system H_A by implementing a unitary operation on a joint system $H_A \otimes H_B$ and then looking at how the state of the subsystem A has transformed.

Recall that a map $\mathcal{C} : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$ is a proper quantum channel if and only if it is completely positive and trace preserving, which is equivalent to

$$\mathcal{C} : \rho \mapsto \sum_k E_k \rho E_k^\dagger \tag{2}$$

for some *Kraus operators* E_k such that $\sum_k E_k^\dagger E_k = \mathbb{I}$. In the following, we investigate the operational meaning of Kraus operators. For simplicity, we restrict ourselves to quantum channels with the same input and output space $L(\mathcal{X})$. Suppose we apply a unitary U to the joint system and environment in the state $\rho \otimes |0\rangle\langle 0| \in L(\mathcal{X} \otimes \mathcal{Z})$, where $|0\rangle \in \mathcal{Z}$ is some reference state, and then we measure system $\mathcal{Z}$ in the computational basis.

Now I'm going to explain a bit of theory.

- $\Phi : L(\mathcal{H}_1) \longrightarrow L(\mathcal{H}_2)$ QUANTUM CHANNEL $\overset{DEF}{\iff}$
  - LINEAR
    $\Phi(\alpha \rho_1 + \beta \rho_2) = \alpha \Phi(\rho_1) + \beta \Phi(\rho_2)$
  - COMPLETELY POSITIVE
    - $A \in \mathcal{H}_1 \otimes \mathcal{H}_E$
    - $A \geq 0$ $\Rightarrow (\Phi \otimes \mathbb{1})(A) \geq 0$
  - TRACE PRESERVING.
    $tr(\Phi(X)) = tr(X)$

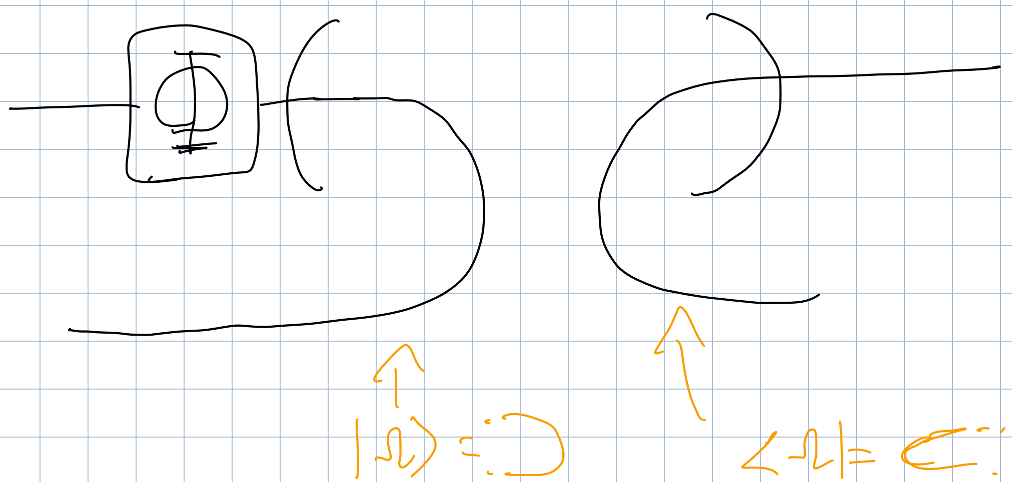
• DEF: CHOI - STATE of $\Phi(\cdot) : L(\mathcal{H}_A) \to L(\mathcal{H}_A)$

$$\rho_\Phi := (\Phi \otimes \underbrace{\mathbb{1}}_{\mathcal{H}_B \text{ space of same dimension } d_B})\left(|\Omega\rangle\langle\Omega|\right) \quad \text{with} \quad |\Omega\rangle := \sum_{i=1}^{d} \frac{1}{d} \underbrace{|i\rangle \otimes |i\rangle}_{\text{ortonormal basis states}}$$

We have $\rho_\Phi = \frac{1}{d}(\Phi \otimes \mathbb{1})\left(\sum_{i,j}^{d} |i\,i\rangle\langle j\,j|\right) = \frac{1}{d}\sum_{i,j=1}^{d} \Phi\left(|i\rangle\langle j|\right) \otimes |i\rangle\langle j|$

· Using diagrams a Choi state is:

$\rho_\Phi = \frac{1}{d}$:



We have $\boxed{\rho_\Phi \geq 0 \iff \Phi(\cdot) \text{ is completely positive.}}$

PROOF :

"$\Leftarrow$" Easy : $\rho_\Phi := \Phi \otimes \mathbb{1}\left(|\Omega\rangle\langle\Omega|\right) \underset{\Phi \text{ compl. positive}}{\geq} 0$

"$\Rightarrow$" : I have to show : $\rho_\Phi \geq 0 \Rightarrow \left(\sigma \geq 0 \Rightarrow (\Phi \otimes \mathbb{1})(\sigma) \geq 0\right)$

$$\underset{\dim(H_B) \geq \dim(H_A)}{\Phi \otimes \mathbb{1}}(\sigma) \underset{\sigma = \sum_{i=1}^{d_A} \lambda_i |V_i\rangle\langle V_i|,\ \lambda_i \geq 0}{=} \sum_{i}^{d_A} \lambda_i (\Phi \otimes \mathbb{1})\left(|V_i\rangle\langle V_i|\right)$$

Let's focus on $(\Phi \otimes \mathbb{1})(|V_i\rangle\langle V_i|)$.

$d_B \geq d_A \Rightarrow \min(d_A, d_B) = d_A$

- $|V_i\rangle = \sum_{j=1}^{\min(d_A,d_B)} \underset{\geq 0}{s_j} |j\rangle \otimes |j\rangle = \sum_{j=1}^{\min(d_A,d_B)} X \otimes \mathbb{1} |j\rangle \otimes |j\rangle$

  - $|V_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
  - SCHMIDT DECOMP.

$(\exists\, X$ such that $X|j\rangle = s_j|j\rangle)$

$= (X \otimes \mathbb{1}) W_A \otimes W_B |\Omega\rangle$

- Where W_A and W_B are unitary such that: $\sum_j (W_A^\dagger|j\rangle) \otimes (W_B^\dagger|j\rangle) = |\Omega\rangle$

$= X \otimes \mathbb{1} (\mathbb{1} \otimes W_B W_A^t)|\Omega\rangle =$

TRANSPOSE-TRICK

$= \mathbb{1} \otimes W_B W_A^t X^t |\Omega\rangle$

- $|V_i\rangle\langle V_i| = (\mathbb{1} \otimes W_B W_A^T X^T)\, |\Omega\rangle\langle\Omega|\, (\mathbb{1} \otimes (W_B W_A^t X^t)^\dagger)$

- $(\Phi \otimes \mathbb{1})\left[|V_i\rangle\langle V_i|\right] = \underbrace{(\mathbb{1} \otimes W_B W_A^T X^T)}_{= M} (\Phi \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|) (\mathbb{1} \otimes (W_B W_A^t X^t)^\dagger)$

$= M P_\Phi M^\dagger$

Now $\forall\, |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ $\quad \langle\psi| M P_\Phi M^\dagger |\psi\rangle = \langle M^\dagger|\psi\rangle|\, P_\Phi \,|\underset{= v}{M^\dagger|\psi\rangle}\rangle$

$= \langle v | P_\Phi | v\rangle \geq 0$

$P_\Phi \geq 0$

ASSUMPTION.

- Now we have proven that $P_\Phi \geq 0 \Rightarrow \Phi \otimes \mathbb{1} \geq 0$

$\mathcal{H}_B$ with $\dim(\mathcal{H}_B) = \dim(\mathcal{H}_A)$

• If $\dim(H_B) < \dim(H_A)$:

$$\phi \otimes \mathbb{1}_B(\sigma) = tr_{B'}\Big(\phi \otimes \underbrace{\mathbb{1}_B \otimes \mathbb{1}_{B'}}_{\text{Same dimension of A in total}}(\sigma \otimes |0\rangle_{B'}\langle 0|_{B'})\Big) \geq 0 \quad \Rightarrow \square$$

• ρ_Φ uniquely define Φ using :

$$\Phi(\sigma) = d\, tr_B\left(\mathbb{1} \otimes \sigma^t \; \rho_\Phi\right)$$

PROOF :

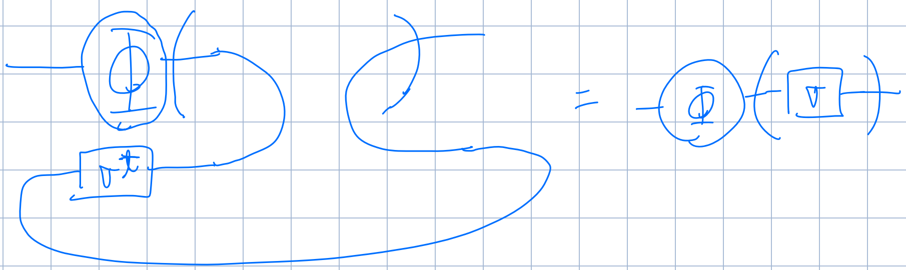
$$d\, tr_B\left(\mathbb{1} \otimes \sigma^t \; \rho_\Phi\right) = \frac{d}{d} tr_B\left(\mathbb{1} \otimes \sigma^t \sum_{i,j} \phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|\right) =$$

$$= \sum_{i,j} tr_B\left(\phi(|i\rangle\langle j|) \otimes \sigma^t |i\rangle\langle j|\right) =$$

$$= \sum_{i,j} \phi(|i\rangle\langle j|)\langle j|\sigma^t|i\rangle =$$

$$= \sum_{i,j} \phi(|i\rangle\langle j|)\langle i|\sigma|j\rangle \underset{\sum |i\rangle\langle i| = \mathbb{1}}{=} \phi(\sigma)$$

Or Using diagrams:



• $tr_A(\rho_\phi) = \frac{\mathbb{1}_B}{d}$ ⟵ Only the state ρ such that $tr_A(\rho) = \frac{1}{d}$ can be Choi state.

PROOF :

$$tr_A\left(\Phi \otimes \mathbb{1}(|\Omega\rangle\langle\Omega|)\right) = \frac{1}{d} tr_A\left(\sum_{i,j} \phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|\right) = \frac{1}{d}\sum_{i,j} tr\left(\phi(|i\rangle\langle j|)\right)|i\rangle\langle j| =$$

$$\underset{\phi \text{ trace preserving}}{=} \frac{1}{d}\sum_{i,j} tr(|i\rangle\langle j|)\,|i\rangle\langle j| = \frac{1}{d}\sum_i |i\rangle\langle i| = \frac{\mathbb{1}}{d}$$

THEOREM KRAUS

- Φ quantum channel $\Longleftrightarrow$ $\exists \{K_i\}_{i=1}^{d^2} \in \mathcal{L}(H)$ such that :

KRAUS OPERATORS

$$\begin{cases} \cdot\ \Phi(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^\dagger \\ \cdot\ \sum_{i=1}^{d^2} K_i^\dagger K_i = \mathbb{1} \end{cases}$$

PROOF:

" $\Longleftarrow$ "

- Linear : easy

- Trace preserving : $\mathrm{Tr}(\phi(\rho)) = \sum_{i=1}^{d^2} \mathrm{Tr}(K_i \rho K_i^\dagger) = \sum_i \mathrm{Tr}(K_i^\dagger K_i \rho) =$

$$= \mathrm{Tr}\Big(\underbrace{\sum_i K_i^\dagger K_i}_{\mathbb{1}} \rho\Big) = \mathrm{Tr}(\rho)$$

- Completely positive : $\sigma \geq 0 \Rightarrow \forall\, |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \quad \langle\psi| \Phi \otimes \mathbb{1}(\sigma)|\psi\rangle \geq 0$

SUBPROOF:

$$\langle\psi|(\phi\otimes\mathbb{1})(\sigma)|\psi\rangle = \langle\psi|\sum_i K_i\otimes\mathbb{1}\ \sigma\ K_i^\dagger\otimes\mathbb{1}|\psi\rangle =$$

$$= \sum_i \underbrace{\langle\psi|(K_i\otimes\mathbb{1})}_{\langle v_i|}\ \sigma\ \underbrace{(K_i^\dagger\otimes\mathbb{1})|\psi\rangle}_{|v_i\rangle}$$

$$= \sum_i \langle v_i|\sigma|v_i\rangle \underset{\sigma\geq 0}{\geq} 0$$

" $\Longrightarrow$ "

- $\Phi(\rho) \underset{\rho=\rho^\dagger \Rightarrow \text{eigendec.}}{=} \sum_{i=1}^{d} \lambda_i\, \Phi(|v_i\rangle\langle v_i|)$

- $\Phi(|\psi\rangle\langle\psi|) = d\ {}_B\langle\psi^*|\ \rho_\Phi\ |\psi^*\rangle_B = ⭐$

SUB PROOF:

$${}_B\langle\psi^*|\ \rho_\Phi\ |\psi^*\rangle_B = {}_B\langle\psi^*|\,(\Phi\otimes\mathbb{1})(|\Omega\rangle\langle\Omega|)\,|\psi^*\rangle_B =$$

$$= \frac{1}{d}\sum_{i,j} \Phi(|i\rangle_A\langle j|)\ \langle\psi^*|i\rangle\ \langle j|\psi^*\rangle$$

$$\left(\phi\otimes\mathbb{1}_B(|\Omega\rangle\langle\Omega|) = \frac{1}{d}\sum_{i,j}\Phi(|i\rangle_A\langle j|_A)\otimes|i\rangle_B\langle j|_B\right)$$

$$\underset{\uparrow}{=} \frac{1}{d} \sum_{i,j} \Phi\left(\langle i|\psi\rangle |i\rangle\langle j| \langle\psi|j\rangle\right) = \frac{1}{d}\Phi\left(|\psi\rangle\langle\psi|\right)$$

$$\left(\begin{aligned} \langle\psi^*|i\rangle &= (\langle i|\psi^*\rangle)^* \\ &= \langle i|^*|\psi\rangle \\ &\underset{\uparrow}{=} \langle i|\psi\rangle \\ \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\end{pmatrix} &= |i\rangle \text{ comp. basis.}\end{aligned}\right)$$

$\Rightarrow \square$ SUBPROOF.

- $$\Phi(\rho) = \sum_{i=1}^{d} \lambda_i \Phi(|v_i\rangle\langle v_i|) = \sum_{i=1}^{d} \lambda_i \, {}_B\langle v_i^*| \rho_\Phi |v_i^*\rangle_B \, d$$

$$\underset{\uparrow}{=} \sum_{i=1}^{d} \lambda_i \sum_{j=1}^{d^2} p_j \, {}_B\langle v_i^*|\psi_j\rangle_{AB} \, {}_{AB}\langle\psi_j|v_i^*\rangle_B \, d =$$

$$\left(\begin{aligned} &\rho_\Phi \geq 0 \Rightarrow \rho_\Phi^\dagger = \rho_\Phi \Rightarrow \text{diagonaliz.} \\ &\Rightarrow \rho_\Phi = \sum_{j=1}^{d^2} p_j |\psi_j\rangle_{AB} \, {}_{AB}\langle\psi_j| \end{aligned}\right)$$

$$= \sum_{j=1}^{d^2} p_j \, d \sum_{i=1}^{d} \lambda_i \, {}_B\langle v_i^*|\psi_j\rangle_{AB} \, {}_{AB}\langle\psi_j|v_i^*\rangle_B =$$

$$\underset{\uparrow}{=} \sum_{j=1}^{d^2} \sum_{i=1}^{d} \lambda_i K_j |v_i\rangle\langle v_i| K_j^\dagger =$$

$$\left(\begin{aligned} &K_j|\cdot\rangle_A := {}_B\langle\cdot|^* |\psi_j\rangle_{AB} \sqrt{p_j}\sqrt{d} \\ &\text{linear operators} \end{aligned}\right)$$

$$\underset{\uparrow}{=} \sum_{j=1}^{d^2} K_j \rho K_j^\dagger$$

$$\sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i| = \rho$$

- $$\mathrm{tr}(\Phi(\rho)) = 1 \Rightarrow \mathrm{tr}\left(\sum_j K_j^\dagger K_j \rho\right) = 1 \quad \forall \rho \Rightarrow \sum_{j=1}^{d^2} K_j^\dagger K_j = \mathbb{1}$$

STINESPRING REPRESENTATION.

$\Phi(\cdot)$ quantum channel $\Longleftrightarrow$ $\exists\; U_{AB} \in \mathcal{H}_A \otimes \underset{\text{auxiliary system}}{\mathcal{H}_B}$ unitary such that

$$\Phi(\rho) = tr_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right)$$

PROOF:

"$\Longrightarrow$" Φ q. channel $\Leftrightarrow$ $\exists\; \{K_i\}_{i=1}^{d^2}$ such that $\Phi(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^{\dagger}$ and $\sum_i K_i^{\dagger} K_i = \mathbb{1}$

CLAIM: $U_{AB} |\cdot\rangle \otimes |0\rangle = \sum_{i=1}^{d^2} K_i |\cdot\rangle \otimes |i\rangle$

It's unitary

SUBPROOF:

$$U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \underset{\rho = \sum_i \lambda_i |v_i\rangle\langle v_i|}{=} \sum_i \lambda_i\, U_{AB} \left(|v_i\rangle \otimes |0\rangle\right) \left(\langle v_i| \otimes \langle 0| \right) U_{AB}^{\dagger} =$$

$$= \sum_i \lambda_i \sum_{j=1}^{d^2} \left(K_j |v_j\rangle \otimes |j\rangle \right) \sum_{l=1}^{d^2} \left(\langle v_l| K_l^{\dagger} \otimes \langle l| \right)$$

$$tr_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right) = \sum_i \lambda_i \sum_{j=1}^{d^2} \sum_{l=1}^{d^2} \left(K_j |v_j\rangle\langle v_l| K_l^{\dagger} \right) \underbrace{tr_B\left(|j\rangle\langle l| \right)}_{= \delta_{j,l}}$$

$$\underset{\sum_i \lambda_i |v_i\rangle\langle v_i| = \rho}{=} \sum_{j=1}^{d^2} K_j \rho K_j^{\dagger}$$

"$\Longleftarrow$" $\Phi(\rho) = tr_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right)$ is:

- linear : Yes!
- trace preserving : Yes! $\leftarrow$ $tr\left(\Phi(\rho) \right) = \ldots = 1$
- Complet. positive :

$$(\phi \otimes \underbrace{\mathbb{1}}_{H_2})(\underset{\uparrow\; H_A \otimes H_2}{\sigma}) =: \mathrm{tr}_B\left(U_{AB}\, \underbrace{\sigma}_{H_A \otimes H_2} \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right) \geq 0 .$$

Now let's restart with the exercises:

Recall that a map $\mathcal{C} : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$ is a proper quantum channel if and only if it is completely positive and trace preserving, which is equivalent to

$$\mathcal{C} : \rho \mapsto \sum_k E_k \rho E_k^\dagger \tag{2}$$

for some *Kraus operators* E_k such that $\sum_k E_k^\dagger E_k = \mathbb{I}$. In the following, we investigate the operational meaning of Kraus operators. For simplicity, we restrict ourselves to quantum channels with the same input and output space $L(\mathcal{X})$. Suppose we apply a unitary U to the joint system and environment in the state $\rho \otimes |0\rangle\langle 0| \in L(\mathcal{X} \otimes \mathcal{Z})$, where $|0\rangle \in \mathcal{Z}$ is some reference state, and then we measure system $\mathcal{Z}$ in the computational basis.

a) Show that the action of any unitary on the joint system can be written as

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle\langle l| \,,$$

with respect to the basis $\{|i\rangle\}_i$ on the second system for a set of operators $\{E_k\}$. In particular show how these operators are related to the unitary U.

$$\cdot\ \rho' = U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \Big(\underbrace{\sum_i \mathbb{1} \otimes |i\rangle\langle i|}_{\mathbb{1}}\Big) U \Big(\underbrace{\sum_j \mathbb{1} \otimes |j\rangle\langle j|}_{\mathbb{1}}\Big)(\rho \otimes |0\rangle\langle 0|)\, U^\dagger =$$

$$= \sum_{i,j} {}_z\langle i| U_{x,z} |j\rangle_z \otimes |i\rangle\langle j| \,(\rho \otimes |0\rangle\langle 0|)\, U^\dagger =$$

$$= \sum_i^d \left({}_z\langle i| U_{x,z} |0\rangle_z \rho\right) \otimes (|i\rangle\langle 0|)\; U^\dagger =$$

$$= \sum_i^d \left({}_z\langle i| U_{x,z} |0\rangle_z \rho\right) \otimes |i\rangle\langle 0| \sum_{l,m} {}_z\langle l| U_{x,z}^\dagger |m\rangle_z \otimes |l\rangle\langle m|$$

$$U^\dagger = \sum_{l,m} {}_z\langle l| U^\dagger_{x,z} |m\rangle_z \otimes |l\rangle\langle m|$$

AS BEFORE

$$= \sum_i^d \left({}_z\langle i| U_{x,z} |0\rangle_z \rho\right) \otimes |i\rangle\langle 0| \left(\sum_{m=1}^d {}_z\langle 0| U^\dagger_{x,z} |m\rangle_z \otimes |0\rangle\langle m|\right)$$

$$= \sum_{i=1}^{d} \sum_{m=1}^{d} {}_z\langle i | U_{x,z} | 0 \rangle_z \, \rho \, {}_z\langle 0 | U^{\dagger}_{x,z} | m \rangle_z \otimes |i\rangle \underbrace{\langle 0|0\rangle}_{=1} \langle m|$$

$$= \sum_{i,m}^{d} \underbrace{{}_z\langle i | U_{x,z} | 0 \rangle_z}_{= E_i} \, \rho \, \underbrace{{}_z\langle 0 | U^{\dagger}_{x,z} | m \rangle_z}_{= \left({}_z\langle m | U_{x,z} | 0 \rangle_z\right)^{\dagger} = E_m^{\dagger}} \otimes |i\rangle\langle m|$$

$$= \sum_{i,m}^{d} E_i \, \rho \, E_m^{\dagger} \otimes |i\rangle\langle m| \qquad \text{with } E_i := {}_z\langle i | U_{x,z} | 0 \rangle_z$$

- $\rho_x' = tr_z(\rho_x') = \sum_{i=1}^{d} E_i \, \rho \, E_i^{\dagger}$

b) Now, we perform a von-Neumann (that is, projective) measurement on $\mathcal{Z}$ in the same basis. Determine the post-measurement state conditioned on outcome i.

$$\rho_{\text{post "}i\text{"}} = \boxed{\frac{\Pi_i \, \rho' \, \Pi_i}{tr(\Pi_i \rho')}} = \frac{E_i \rho E_i^{\dagger}}{tr(E_i^{\dagger} E_i \rho)} \otimes |i\rangle_z \langle i|_z$$

- $\Pi_i = \mathbb{1}_A \otimes |i\rangle\langle i|_z$
- $\rho' = \sum_{l,m}^{d} E_l \, \rho \, E_m^{\dagger} \otimes |l\rangle\langle m|$
- $\Pi_i \rho' = \sum_m E_m \rho E_m^{\dagger} \otimes |i\rangle\langle m|$
- $tr(\Pi_i \rho') \overset{\Downarrow}{=} tr_x(E_i \rho E_i^{\dagger}) = tr_x(E_i^{\dagger} E_i \rho)$

c) What is the probability of obtaining outcome i? What does this entail for the operators E_k?

$$\text{Prob}(\text{"}i\text{"}) = tr(\Pi_i \rho') \underset{\substack{\uparrow \\ \text{computed before}}}{=} tr(E_i^{\dagger} E_i \rho)$$

- This implies that $E_k^{\dagger} E_k$ is a POVM element of the set $\{E_k^{\dagger} E_k\}_{k=1}^{d}$

since
- $\text{Prob}(\text{"}k\text{"}) = tr((E_k^{\dagger} E_k)\rho)$
- $E_k^{\dagger} E_k \geq 0$

  $(\langle\psi| E_k^{\dagger})(E_k|\psi\rangle) = \| E_k |\psi\rangle \|^2 \geq 0$
- $\sum_{k=1}^{d} E_k^{\dagger} E_k = \sum_{k=1}^{d} {}_z\langle 0 | U^{\dagger}_{x,z} | k \rangle_z \, {}_z\langle k | U_{x,z} | 0 \rangle_z \underset{\sum_k |k\rangle\langle k| = \mathbb{1}}{=} {}_z\langle 0 | \underbrace{U^{\dagger}_{x,z} U_{x,z}}_{=\mathbb{1}} | 0 \rangle_z = \mathbb{1}$

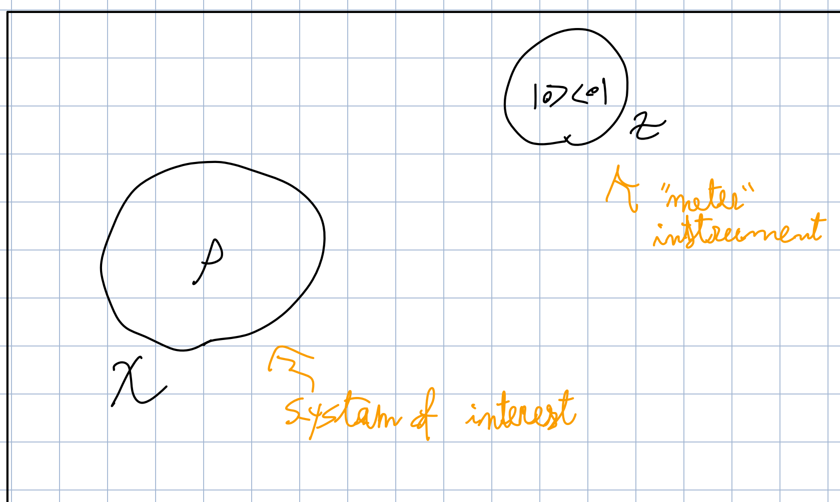
  $E_i := {}_z\langle i | U_{x,z} | 0 \rangle_z$

Note that we are able to determine the post-measurement state here

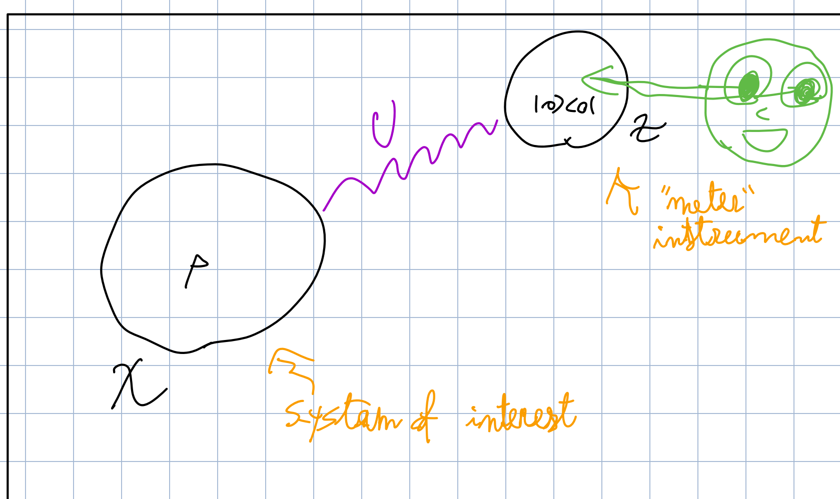
d) Give the corresponding operational interpretation of the Kraus operators E_k and the unitary U.

- The operational interpretation is the following:

- I want to "measure" the system "X", but in reality I can do that making it interacting with an instrument ("like a meter") that I call Z:



I make them interacting via unitary U and then I measure my "meter" system Z.



The KRAUS OPERATORS E_K in this case represent the effect of the interaction "U" on the subsystem $\mathcal{X}$.

$$\cdot\ \rho' = U\left(\rho \otimes |0\rangle\langle 0|\right)U^{+} \underset{\text{tracing out } \mathcal{Z}}{\Rightarrow} \quad \rho'_{\mathcal{X}} = tr_{\mathcal{Z}}\left(U\left(\rho \otimes |0\rangle\langle 0|\right)U^{+}\right) =$$

$$\underset{\text{before}}{=} \sum_{i=1}^{d} E_i \rho E_i^{+}$$

$$= \sum_{i=1}^{d} \text{Prob}(\text{"}i\text{"}) \frac{E_i \rho E_i^{+}}{tr\left(E_i^{+} E_i \rho\right)}$$

$$tr\left(E_i^{+} E_i \rho\right) = \text{Prob}(\text{"}i\text{"})$$

e) Now, suppose we want to implement a projective measurement on $\mathcal{X}$ via a global unitary and a projective measurement on $\mathcal{Z}$. Consider the unitaries $U \in U(\mathcal{X} \otimes \mathcal{Z})$ on the joint system that give rise to this situation. What conditions do they have to satisfy?

Hint: the measurement needs to collapse the state of the first system as well.

- Before $\{E_K^{+} E_K\}_{K=1}^{d}$ were shown to be POVM elements for $\mathcal{X}$: $\begin{cases} \cdot\ E_K^{+} E_K \geq 0 \\ \cdot\ \sum_{K=1}^{d} E_K^{+} E_K = \mathbb{1} \end{cases}$

- Now we want to see when they are PVM.

$$\begin{cases} \cdot\ E_K^{+} E_K \geq 0 \\ \cdot\ \sum_{K=1}^{d} E_K^{+} E_K = \mathbb{1} \\ \circ\ \left(E_i^{+} E_i\right)\left(E_j^{+} E_j\right) = \delta_{i,j}\left(E_i^{+} E_i\right) \end{cases}$$

PVM condition.

with $E_i := {}_{\mathcal{Z}}\langle i | U_{\mathcal{X},\mathcal{Z}} | 0 \rangle_{\mathcal{Z}}$

PVM condition:

$$(E_i^\dagger E_i)(E_j^\dagger E_j) = \delta_{i,j}(E_i^\dagger E_i)$$

$$\left({}_z\langle 0|U^\dagger_{x,z}|i\rangle_z \langle i|U|0\rangle\right)\left(\langle 0|U^\dagger|j\rangle\langle j|U|0\rangle\right) = \delta_{i,j}\left(\langle 0|U^\dagger|i\rangle\langle i|U|0\rangle\right)$$

Alternative, the state on $\mathcal{X}$ after outcome "i" is: $\rho_{\mathcal{X}}^{1,(i)} = \dfrac{E_i \rho E_i^\dagger}{tr(E_i^\dagger E_i \rho)}$

- If on $\mathcal{X}$ we would have had a projective measurement $\Rightarrow \rho_{\mathcal{X}}^{1,(i)} = \dfrac{\Pi_i \rho \Pi_i^\dagger}{tr(\Pi_i \rho)}$ ⟵ projector.

and so a sufficient condition is $E_i E_j = \delta_{i,j} E_i$ and $E_i = E_j^\dagger$.

(Note that these two conditions imply that $E_i \geq 0$ and $\sum_{i=1}^{d} E_i = \sum_{i=1}^{d} E_i^\dagger E_i = \underset{\text{before}}{\mathbb{1}}$)

$$(\star) = \begin{cases} E_i^\dagger = E_i \Rightarrow {}_z\langle 0|U^\dagger_{x,z}|i\rangle_z = {}_z\langle i|U_{x,z}|0\rangle_z & \equiv (1) \\ E_i E_j = \delta_{i,j} E_i \Rightarrow {}_z\langle i|U_{x,z}|0\rangle_z\, {}_z\langle j|U_{x,z}|0\rangle_z = \delta_{i,j}\, {}_z\langle i|U_{x,z}|0\rangle_z & \equiv (2) \end{cases}$$

f) Can you think of an example for the case of $\mathcal{X}$ and $\mathcal{Z}$ being each a qubit?

Let's consider condition $(\star)$:

$$(2) \Rightarrow \left((\mathbb{1}\otimes\langle i|)U(\mathbb{1}\otimes|0\rangle)\right)^2 = (\mathbb{1}\otimes\langle i|)\,U\,(\mathbb{1}\otimes|0\rangle)$$

This is satisfied if I choose U of the form: $U = |0\rangle\langle 0|_x \otimes \mathbb{1}_z + |1\rangle\langle 1|_x \otimes X_z$

$(1) \Rightarrow$ This is also satisfied

This is the so called C-NOT:

$U|0\rangle\otimes|0\rangle = |0\rangle\otimes|0\rangle$

$U|01\rangle = |01\rangle$

$U|10\rangle = |11\rangle$

$U|11\rangle = |10\rangle$

We will show one last property of the Kraus representation

g) Let $\{K_i\}_{i=1}^N$ and $\{\tilde{K}_j\}_{i=1}^N$ be two sets of linear operators in $L(\mathcal{X}, \mathcal{Z})$ fulfilling the completeness relation of Kraus operators. Show that if the two sets are related by a unitary transformations $U \in U(N)$ such that $\tilde{K}_i = \sum_j U_{ij} K_j$, the channels represented by the sets coincide.

- We have $\sum_{i=1}^N K_i^\dagger K_i = \mathbb{1}$, $\sum_{i=1}^N \tilde{K}_i^\dagger \tilde{K}_i = \mathbb{1}$.

and $\tilde{K}_i = \sum_{j=1}^N U_{ij} K_j$ and $\tilde{K}_i^\dagger = \sum_{j=1}^N U_{ij}^* K_j$

$$\Rightarrow \quad \Phi(\rho) = \sum_{i=1}^N \tilde{K}_i \rho \tilde{K}_i^\dagger = \sum_{i=1}^N K_i \rho K_i^\dagger$$

PROOF:

$$\sum_{i=1}^N \tilde{K}_i \rho \tilde{K}_i^\dagger = \sum_{i=1}^N \sum_{j,l} U_{ij} K_j \rho U_{il}^* K_l^\dagger = \sum_{\substack{l=1 \\ j=1}}^N K_j \rho K_l^\dagger \delta_{l,j} =$$

- $U_{il}^* = U_{li}^{t*} = U^\dagger_{li}$
- $\sum_i U^\dagger_{li} U_{ij} = (U^\dagger U)_{l,j} = \delta_{l,j}$

$$= \sum_{j=1}^N K_j \rho K_j^\dagger$$

3. **Equivalence between representations of quantum channels** (11 Points: 1+1+2+1+2+2+1+1)
The aim of this exercise is to establish a duality between quantum channels and quantum states. To this end, let

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i, i\rangle \tag{3}$$

be the maximally mixed state on a bipartite system $\mathcal{H}_d \otimes \mathcal{H}_d$ and denote by $\Omega = |\Omega\rangle\langle\Omega|$ its corresponding density matrix. Then define the Choi-Jamiołkowski map as

$$J : L(L(\mathcal{X}), L(\mathcal{Y})) \to L(\mathcal{X} \otimes \mathcal{Y}) :: T \mapsto (T \otimes \mathbb{1})(\Omega) \tag{4}$$

with Ω now the maximally entangled state in $\mathcal{X} \otimes \mathcal{X}$ ~~and $\mathbb{1}$ the identity on $\mathcal{X} \otimes \mathcal{X}^*$~~.
Throughout let d be the dimension of $\mathcal{X}$.

We will show that J as a map from the completely positive tracepreserving (CPTP) maps to the set of quantum states on a bipartite system $\mathcal{X} \otimes \mathcal{Y}$ with the restriction $\mathrm{Tr}_{\mathcal{Y}}\, \rho = \mathbb{1}/d$ is a bijection.

a) Use the criterion for positivity from the lecture and show that for a CPTP map T from operators on $\mathcal{X}$ to operators on $\mathcal{Y}$, $J(T) \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Y})$ is indeed a density matrix on the joined system.

$$J(T) = T \otimes \mathbb{1}\left(|\Omega\rangle\langle\Omega|\right) = \frac{1}{d} \sum_{i,j} T\left(|i\rangle\langle j|\right) \otimes |i\rangle\langle j|$$

$$|\Omega\rangle = \sum_{i=1}^{d} \frac{|i\rangle \otimes |i\rangle}{\sqrt{d}}$$

- $\mathrm{Tr}\left(J(T)\right) = 1$

PROOF:

$$\mathrm{Tr}\left(J(T)\right) = \frac{1}{d} \sum_{i,j} \mathrm{Tr}\left(T\left(|i\rangle\langle j|\right) \otimes |i\rangle\langle j|\right) =$$

$$= \frac{1}{d} \sum_{i,j} \mathrm{Tr}_A\left(T\left(|i\rangle\langle j|\right)\right) \mathrm{Tr}_B\left(|i\rangle\langle j|\right) =$$

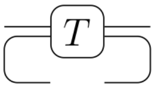
$$\underset{\uparrow}{=} \frac{1}{d} \sum_{i,j} \mathrm{Tr}\left(|i\rangle\langle j|\right) \delta_{i,j} = \frac{1}{d} \mathrm{Tr}\Big(\underbrace{\sum_i |i\rangle\langle i|}_{\mathbb{1}}\Big) = 1$$

$T(\cdot)$ trace preserving

- $J(T) \geq 0$

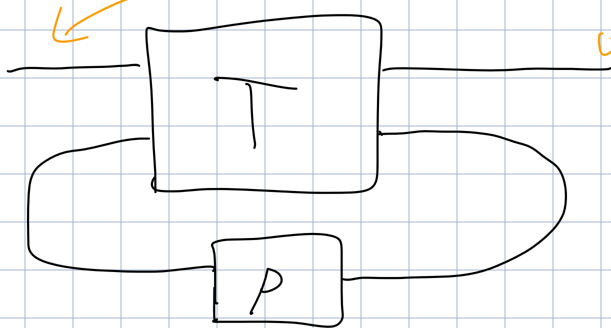
PROOF: $J(T) = T \otimes \mathbb{1}\left(|\Omega\rangle\langle\Omega|\right) \geq 0$ because of compl. positive.

b) Use the diagrammatic notation to first draw the action of T on a density matrix $\rho \in L(\mathcal{X})$. Then use that intuition to draw the Choi-state $J(T)$ in diagrammatic notation. (Hint: you can represent T diagramatically as

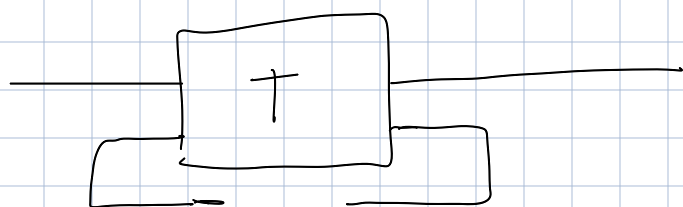
$T =$ 

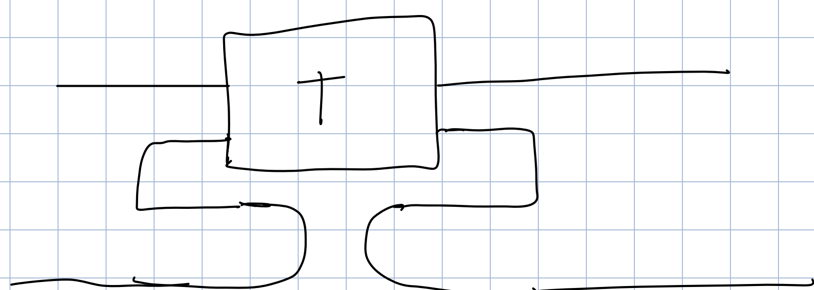
(5)

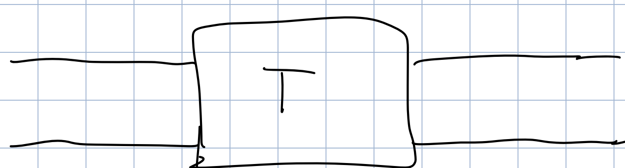
where the two bottom legs can be thought as corresponding to the "input" space $L(\mathcal{X}) \simeq \mathcal{X} \otimes \mathcal{X}^*$ and similarly for the two top legs. It may be convenient to think about how the density matrix Ω is expressed graphically.)

$T(\rho) =$ 

This because $T(\rho)$ is a matrix

$T(\cdot) =$ 

$J(T) = T \otimes \mathbb{1}(|\Omega\rangle\langle\Omega|) = \frac{1}{d}$ 

$= \frac{1}{d}$ 

c) Show that J is injective. (Hint: Do so by showing that for any $J(T)$ in the image of J you can define a $\tilde{T}$ that maps $X \in L(\mathcal{X})$ to $\tilde{T}(X) = d\,\mathrm{Tr}_{\mathcal{Y}}\left[J(T)(\mathbb{1}_{\mathcal{X}} \otimes X^T)\right]$. If you use this hint, explain what this implies?).

- $J(\cdot)$ is injective $\overset{\text{DEF}}{\Longleftrightarrow}$ $\left(J(T) = J(T') \Leftrightarrow T = T'\right)$

"$\Leftarrow$": easy! (DEFINITION)

"$\Rightarrow$": Using the HINT:

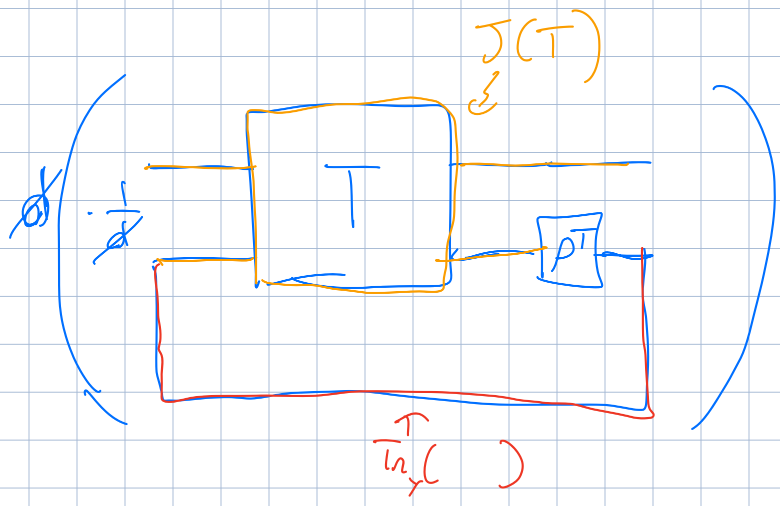
$$T(\rho) = d\,\mathrm{Tr}_y\left[J(T)\left(\mathbb{1} \otimes \rho^T\right)\right]$$

PROOF:

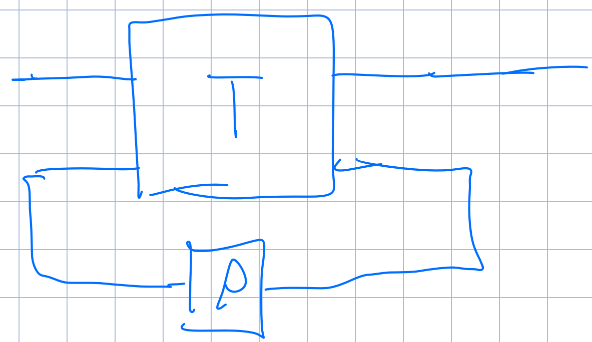
- $$d\,\mathrm{Tr}_y\left[J(T)\left(\mathbb{1}\otimes\rho^T\right)\right] = d\,\mathrm{Tr}_y\left[\left(T\otimes\mathbb{1}\left(|\Omega\rangle\langle\Omega|\right)\right)\left(\mathbb{1}\otimes\rho^T\right)\right]$$
$$= \frac{d}{d}\,\mathrm{Tr}_y\left[\left(T\otimes\mathbb{1}\sum_{i,j}|i\rangle\langle j|\otimes|i\rangle\langle j|\right)\left(\mathbb{1}\otimes\rho^T\right)\right]$$
$$= \mathrm{Tr}_y\left[\left(\sum_{i,j}T\left(|i\rangle\langle j|\right)\otimes|i\rangle\langle j|\right)\left(\mathbb{1}\otimes\rho^T\right)\right]$$
$$= \sum_{i,j}\mathrm{Tr}_y\left[T\left[|i\rangle\langle j|\right]\otimes|i\rangle\langle j|\rho^T\right]$$
$$= \sum_{i,j}T\left[|i\rangle\langle j|\right]\langle j|\rho^T|i\rangle =$$
$$= \sum_{i,j}T\left[|i\rangle\langle j|\right]\langle i|\rho|j\rangle =$$
$$= T\left[\sum_{i,j}\langle i|\rho|j\rangle\,|i\rangle\langle j|\right] =$$
$$\underset{\substack{\sum_i |i\rangle\langle i| = \mathbb{1} \\ \sum_j |j\rangle\langle j| = \mathbb{1}}}{=} T[\rho]$$

- Via diagrams:

$$d\left(\mathrm{Tr}_y\left[J(T)\left(\mathbb{1}\otimes\rho^T\right)\right]\right) =$$



$=$



$= T(\rho)$

- We have shown that given $J(T)$, I can determine $T(\cdot)$ via the above formula.

d) Before we show surjectivity of J we want to get used to some concepts from the lecture: determine a set of Kraus operators representing T (Hint: use the matrix representation of pure states on a bipartite system from two weeks ago together with the eigendecomposition of ρ_T.).

- Look at the $\boxed{\text{THEOREM KRAUS}}$ at beginning of this PDF.

We showed that: $K_\ell|\cdot\rangle_A := \langle\cdot|^*_B\,|\psi_\ell\rangle_{AB}\sqrt{p_\ell}\,\sqrt{d}$ for $\ell = 1, \dots, d^2$

eigenvalue of Choi state

eigenvector Choi state.

$\Rightarrow \quad K_\ell |\cdot\rangle$ is linear operator since $K_\ell |\alpha|v\rangle + \beta|w\rangle\rangle$
$= \alpha K_\ell |v\rangle + \beta K_\ell |w\rangle$

$K_\ell |\cdot\rangle = \langle\cdot|^* |\psi\rangle \sqrt{p_\ell}$

- $K_\ell = \sum_i K_\ell |i\rangle_A \langle i|_A = \sum_i \left({}_B\langle i|^* |\psi_\ell\rangle_{AB} \sqrt{p_\ell} \sqrt{d} \right) {}_A\langle i|$

- $\{|i\rangle\}_{i=1}^d$ orton. basis

$$K_\ell = \sum_{i=1}^{d} \sqrt{p_\ell} \sqrt{d}\, {}_B\langle i|^* |\psi_\ell\rangle_{AB} \langle i|$$

e) Assuming $\dim(\mathcal{X}) = \dim(\mathcal{Y})$, show that J is surjective. (Hint: Assume a given ρ with the restriction mentioned above and use the previous exercise to construct a CPTP map T such that $J(T) = \rho$.).

- Given ρ such that $tr_Y(\rho) = \frac{\mathbb{1}}{d}$, we need to show that $\exists\, \tilde{T}$ quantum channel such that $J(\tilde{T}) = \rho$

- Before we showed that $T(v) = d\, Tr_Y \left[J(T) \left(\mathbb{1}_X \otimes v^T \right) \right]$.

Now I define that $\tilde{T}(v) := d\, Tr_Y \left[\rho \left(\mathbb{1} \otimes v^T \right) \right]$

and I need to show: (1) $\tilde{T}$ is CPTP.

(2) $J(\tilde{T}) = \rho$

- (2) : Let's start by (2):

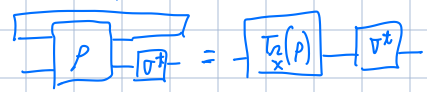
$$J(\tilde{T}) = \tilde{T} \otimes \mathbb{1} \left(|\Omega\rangle\langle\Omega| \right) = \frac{1}{d} \tilde{T}\left(|i\rangle\langle j| \right) \otimes |i\rangle\langle j| = \frac{d}{d} Tr_Y \left[\rho \left(\mathbb{1} \otimes |j\rangle\langle i| \right) \right] \otimes |i\rangle\langle j| =$$

$$= \sum_{i,j} {}_Y\langle i|\rho|j\rangle_Y \otimes |i\rangle_Y \langle j|_Y = \sum_{\substack{i,j \\ \ell, m}} \langle \ell, i|\rho|m, j\rangle\, |\ell, i\rangle\langle m, j| \underset{\uparrow}{=} \rho$$

$\mathbb{1} = \sum |m,j\rangle\langle m,j|$

• (1) • $\tilde{T}(\cdot)$ is linear

• $\tilde{T}(\cdot)$ is trace preserving: $\mathrm{Tr}(\tilde{T}(\sigma)) = d \, \mathrm{Tr}_x \mathrm{Tr}_y [\rho(\mathbb{1} \otimes \sigma^T)] = d \, \mathrm{Tr}_y [\mathrm{Tr}_x(\rho) \sigma^t]$



$\mathrm{Tr}_x(\rho) = \frac{\mathbb{1}}{d}$ $\Rightarrow$ $= d \, \mathrm{Tr}_y [\frac{\mathbb{1}}{d} \sigma^t] \underset{\mathrm{Tr}(A) = \mathrm{Tr}(A^t)}{=} \mathrm{Tr}[\sigma]$

• $\tilde{T}(\cdot)$ is CP:

I will use the fact that $J(\tilde{T}) \geq 0 \Rightarrow \tilde{T}(\cdot)$ is CP.

This is true since $J(\tilde{T}) = \rho \geq 0$ for assumption.

Let $\rho_T \in \mathcal{X} \otimes \mathcal{Y}$ be the Choi-Jamiołkowski state corresponding to the quantum channel T.

f) Determine a unitary U_T representing T via the Stinespring representation.

• $\Phi(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^+$ and $\sum_i K_i^+ K_i = \mathbb{1}$ with $K_i |\cdot\rangle := \langle \cdot |^* |\psi_i\rangle \sqrt{p_i} \sqrt{d}$

eigenvector of ρ_T — eigenvalue of ρ_T.

• We have to find U_{AB} such that $\Phi(\rho) = \mathrm{tr}_y (U_{xy} \, \rho \otimes |0\rangle\langle 0|_y \, U_{xy}^+)$

CLAIM: $U_{xy} \underbrace{|\cdot\rangle \otimes |0\rangle}_{\text{It's unitary}} = \sum_{i=1}^{d^2} K_i |\cdot\rangle \otimes |i\rangle$

SUBPROOF:

$$U_{xy} \, \rho \otimes |0\rangle\langle 0| \, U_{xy}^+ \underset{\rho = \sum_i \lambda_i |v_i\rangle\langle v_i|}{=} \sum_i \lambda_i \, U_{xy} (|v_i\rangle \otimes |0\rangle)(\langle v_i| \otimes \langle 0|) U_{xy}^+ =$$

$$= \sum_i \lambda_i \sum_{j=1}^{d^2} (K_j |v_j\rangle \otimes |j\rangle) \sum_{l=1}^{d^2} (\langle v_l| K_l^+ \otimes \langle l|)$$

$$\mathrm{tr}_B (U_{xy} \, \rho \otimes |0\rangle\langle 0| \, U_{xy}^+) = \sum_i \lambda_i \sum_{j=1}^{d^2} \sum_{l=1}^{d^2} (K_j |v_j\rangle\langle v_l| K_l^+) \underbrace{\mathrm{tr}_B(|j\rangle\langle l|)}_{= \delta_{j,l}}$$

$$\underset{\sum \lambda_i |v_i\rangle\langle v_i| = \rho}{=} \sum_{j=1}^{d^2} K_j \rho K_j^+ = T(\rho)$$

Now, let U_T be a unitary representing T in the Stinespring representation.

g) Determine the Choi-Jamiołkowski state representing T from U_T.

$$J(T) = \phi \otimes \mathbb{1}(|\Omega\rangle\langle\Omega|) = \frac{1}{d} \phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

where $\phi(\cdot) = tr_Y\left(U_T\left((\cdot) \otimes |0\rangle\langle 0|\right) U_T^{+}\right)$

The rank of a quantum channel is defined as the rank of its Choi matrix.

h) Show that a quantum channel with rank r can be represented as a Stinespring dilation using an auxiliary system of dimension r.

- $$J(T) \underset{\substack{\uparrow \\ \text{eigendec.}}}{=} \sum_{i=1}^{d^2} p_i |\psi_i\rangle\langle\psi_i| \underset{\substack{\uparrow \\ \text{I take only the eigenvalues} \neq 0.}}{=} \sum_{i=1}^{r} p_i |\psi_i\rangle\langle\psi_i|$$

- From this, we observed we can define $K_i|j\rangle := \langle \overset{*}{j}|\psi_i\rangle \sqrt{p_i} \sqrt{d}$ $\quad i = 1, \dots, r$

We observed in f) that $U_{XY} |\cdot\rangle \otimes |0\rangle = \sum_{i=1}^{d^2} K_i |\cdot\rangle \otimes |i\rangle$ is the unitary of Stinespring representation.

Taking only $K_i \neq 0$, we have $U_{XY} |\cdot\rangle \otimes |0\rangle = \sum_{i=1}^{r} K_i |\cdot\rangle \otimes |i\rangle$

$\Rightarrow$ we need only an ancilla of dimension "r".