

1. Examples of quantum channels (9 Points: 1+4+2+2)

Now we are ready to look at some examples of quantum channels acting on qubits, i.e., $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

$$F_\epsilon(A) := \epsilon XAX + (1 - \epsilon)A$$

$$D_\epsilon(A) := \epsilon \text{Tr}[A] \frac{1}{d} + (1 - \epsilon)A$$

$$A_\epsilon(A) := \epsilon \text{Tr}[A] |0\rangle\langle 0| + (1 - \epsilon)A,$$

Note first of all that they are all trace-preserving $\text{Tr}(\phi(p)) = \text{Tr}(p)$ and linear: $\phi(\alpha p_1 + \beta p_2) = \alpha \phi(p_1) + \beta \phi(p_2)$

where $\epsilon \in [0, 1]$.

- a) For each channel, show that it is CPT.

• CPT MAPS = COMPLETELY POSITIVE, TRACE-PRESERVING (LINEAR) MAPS



• LINEARITY: We need to show that:



$$\phi(\alpha p_1 + \beta p_2) = \alpha \phi(p_1) + \beta \phi(p_2)$$

for $\phi(\cdot) = F_\epsilon(\cdot), D_\epsilon(\cdot), A_\epsilon(\cdot)$.

OK, THIS IS EASY! 

• TRACE PRESERVING: We need to show that:



$$\text{Tr}(\phi(p)) = \text{Tr}(p) \quad \forall p \in S(H)$$

$$\text{Tr}(F_\epsilon(A)) = \epsilon \text{Tr}(XAX) + (1 - \epsilon) \text{Tr}(A) = \text{Tr}(A)$$

$$F_\epsilon(A) := \epsilon XAX + (1 - \epsilon)A$$

$X^2 = \text{Id}$

$$\text{Tr}(D_\epsilon(A)) = \epsilon \text{Tr}[A] \underbrace{\frac{\text{Tr}(A)}{d}}_{\text{Id}} + (1 - \epsilon) \text{Tr}[A] = \text{Tr}[A]$$

$$D_\epsilon(A) := \epsilon \text{Tr}[A] \frac{1}{d} + (1 - \epsilon)A$$

$$\cdot \text{Tr}[A_\epsilon(A)] = \underbrace{\epsilon \text{Tr}[A]}_{\text{①}} \underbrace{\text{Tr}[\mathbb{I}]}_{\text{②}} + (1-\epsilon) \text{Tr}[A] = \text{Tr}[A]$$

$$A_\epsilon(A) := \epsilon \text{Tr}[A] |0\rangle\langle 0| + (1-\epsilon)A,$$

• COMPLETELY POSITIVE: (i.e. Φ compl. positive $\stackrel{\text{DEF}}{\Leftrightarrow} \Phi \otimes \mathbb{I}(0) \geq 0 \forall \text{vec}(0)$)

To show it, we'll use one of the most important Q-CHANNEL theorems:

CHOI STATE THEOREM

$$P := \underbrace{\Phi \otimes \mathbb{I}}_{\substack{\Phi \\ \text{CHOI STATE}}} [|i\rangle\langle i|] \geq 0 \quad (\text{where } |i\rangle := \sum_{j=1}^d |ij\rangle \frac{|j\rangle}{\sqrt{d}}) \Rightarrow \Phi \text{ completely positive.}$$

• We'll use also a little LEMMA:

LEMMA: Φ_1, Φ_2 Q-CHANNEL $\Rightarrow \epsilon \Phi_1(\cdot) + (1-\epsilon) \Phi_2(\cdot)$ with $\epsilon \in [0,1]$ is Q-CHANNEL.
PROOF: EASY! \square

This LEMMA can be generalized: A CONVEX COMBINATION OF Q-CHANNELS IS A Q-CHANNEL.

$$F_\epsilon(A) := \underbrace{\epsilon XAX}_{\text{①}} + \underbrace{(1-\epsilon)A}_{\text{②}}$$

because of the LEMMA, we treat them separately.

$$\text{①} \Rightarrow X(\cdot) X \otimes \mathbb{I} [|i\rangle\langle i|] = \frac{1}{d} \sum_{j,s=1}^d X |ij\rangle\langle sj| X \otimes \mathbb{I} |sj\rangle\langle sj| \stackrel{?}{\geq} 0$$

$$\langle \psi | \frac{1}{d} \sum_{j,s=1}^d X |ij\rangle\langle sj| X \otimes \mathbb{I} |sj\rangle\langle sj| |\psi\rangle = \left(\langle \psi | X^\dagger \otimes \mathbb{I} \right) |i\rangle\langle i| \left(X \otimes \mathbb{I} |\psi\rangle \right) = \langle \psi | \omega \rangle \langle \omega | \psi \rangle$$

$X^\dagger = X$ $|\psi\rangle = X \otimes \mathbb{I} |\psi\rangle$

$$= |\langle \psi | \omega \rangle|^2 \geq 0$$

②

$$\cdot (\cdot) |i\rangle\langle i| = |i\rangle\langle i| \geq 0$$

$$D_\epsilon(A) := \epsilon \text{Tr}[A] \frac{1}{d} + (1 - \epsilon)A$$

(1)
CP.

$$\begin{aligned}
 \cdot \text{Tr}\left[\cdot \left(\frac{1}{d} \otimes \mathbb{1} \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)\right)\right] &= \frac{1}{d} \sum_{i,j} \text{Tr}\left[|i\rangle\langle j|\right] \frac{1}{d} \otimes |i\rangle\langle j| = \\
 &= \frac{1}{d} \sum_{i,j} S_{i,j} \frac{1}{d} \otimes |i\rangle\langle j| = \\
 &= \frac{1}{d} \otimes \frac{1}{d} \sum_i |i\rangle\langle i| = \frac{1}{d} \otimes \frac{\mathbb{1}}{d} \geq 0
 \end{aligned}$$

$$A_\epsilon(A) := \epsilon \text{Tr}[A] |0\rangle\langle 0| + (1 - \epsilon)A,$$

(2)
CP

$$\begin{aligned}
 \text{Tr}\left[\cdot \left(|0\rangle\langle 0| \otimes \frac{1}{d} \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)\right)\right] &= \frac{1}{d} \sum_{i,j} S_{i,j} |0\rangle\langle 0| \otimes |i\rangle\langle j| = \\
 &= |0\rangle\langle 0| \otimes \frac{1}{d} \geq 0
 \end{aligned}$$

$P_1 \otimes P_2$ state if P_1, P_2 state.

Subproof:

$$\cdot \text{Tr}[P_1 \otimes P_2] = \text{Tr}[P_1] \text{Tr}[P_2] = 1$$

$$\cdot P_1 \otimes P_2 = \sum_{i,j} S_{i,j} |x_i^{(1)} x_j^{(2)}\rangle \langle x_i^{(1)} x_j^{(2)}|$$

P_1 eigen.

P_2 eigen.

$$\Rightarrow \text{Eigenvalue} \geq 0 \Rightarrow P_1 \otimes P_2 \geq 0$$

Next, we represent each quantum channel in different three ways, discussed in the previous exercise.

- b) For each channel with fixed $\epsilon = 1$, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

Remember from my last tutorial that :

$$\text{CHOI STATE : } \rho_{\Phi} := \Phi \otimes \mathbb{1} [|\psi\rangle\langle\psi|]$$

KRAUS

$$\Phi(\rho) = \sum_{i=1}^d K_i \rho K_i^\dagger$$

with $\sum_{i=1}^d K_i^\dagger K_i = \mathbb{1}$

$$\text{where } K_i |j\rangle = \underbrace{\langle j|\Psi_i\rangle}_{\substack{\uparrow \\ \text{eigenstate}}} \underbrace{\sqrt{P_i}}_{\substack{\uparrow \\ \text{eigenvalue of } P_B}} \underbrace{|j\rangle}_{\substack{\uparrow \\ \text{AB}}}$$

(i.e.: $P_B = \sum_{i=1}^d P_i |\Psi_i\rangle\langle\Psi_i|_{AB}$)

$$\Rightarrow K_i = \sum_{j=1}^d K_{ij} |j\rangle\langle j| = \sqrt{P_i} \underbrace{\sqrt{d} \sum_{j=1}^d \langle j|\Psi_i\rangle_{AB} |j\rangle}_{\substack{\uparrow \\ \text{comp. bases}}}$$

$$\text{STINESPRINGS: } \Phi(\rho) = \text{Tr}_B (\mathbb{1}_{AB} \rho \otimes \mathbb{1}) \otimes U_{AB}^+ \quad \text{where } U_{AB} |j\rangle\otimes|i\rangle = \sum_{i=1}^d K_{ij} |j\rangle\otimes|i\rangle$$

$$\begin{aligned} F_\epsilon(A) &:= \epsilon X A X + (1 - \epsilon)A &= X A X \\ D_\epsilon(A) &:= \epsilon \text{Tr}[A] \frac{1}{d} + (1 - \epsilon)A &= \text{Tr}[A] \frac{1}{d} \\ A_\epsilon(A) &:= \epsilon \text{Tr}[A] |0\rangle\langle 0| + (1 - \epsilon)A, &= \text{Tr}[A] |0\rangle\langle 0| \\ &\uparrow \\ &\epsilon = 1 \end{aligned}$$

• CHOI STATE :

$$\begin{aligned} F_{\epsilon=1}(\cdot) \otimes \mathbb{1} [|\psi\rangle\langle\psi|] &= \underbrace{X(\cdot) X \otimes \mathbb{1} [|\psi\rangle\langle\psi|]}_{\substack{\uparrow \\ \text{BEFORE}}} \\ &= \frac{1}{d} \sum_{i,j=1}^d X |i\rangle\langle j| X \otimes |i\rangle\langle j| = \frac{1}{d} \sum_{i,j \in \{0,1\}} |i\rangle\langle j| \otimes |i\rangle\langle j| \\ &= \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}} \right) \left(\frac{\langle 01| + \langle 10|}{\sqrt{2}} \right) =: |\psi_1\rangle\langle\psi_1| \end{aligned}$$

$$\begin{aligned} D_{\epsilon=1}(\cdot) \otimes \mathbb{1} [|\psi\rangle\langle\psi|] &= \text{Tr}[\cdot] \frac{1}{d} \otimes \mathbb{1} [|\psi\rangle\langle\psi|] \\ &= \frac{1}{d} \otimes \frac{1}{d} \end{aligned}$$

$$\cdot A_{\epsilon=1}(p) \otimes \mathbb{I} [|\psi\rangle\langle\psi|] = \text{Tr}[\cdot] |\psi\rangle\langle\psi| \otimes \mathbb{I} [|\psi\rangle\langle\psi|]$$

$$= |\psi\rangle\langle\psi| \otimes \frac{\mathbb{I}}{d}$$

~~• KRAUS~~

$$\overline{P}(p) = \sum_{i=1}^d K_i p K_i^\dagger \quad \text{where} \quad K_i = \sum_{j=1}^d K_{ji} |\psi_j\rangle\langle\psi_j| = \sqrt{p_i} \sqrt{d} \sum_{j=1}^d |\psi_j\rangle_{AB} \langle\psi_j|_{AB}$$

with $\sum_i K_i^\dagger K_i = \mathbb{I}$

eigenstates of P_B
 i.e.: $P_B = \sum_{i=1}^d p_i |\psi_i\rangle_{AB} \langle\psi_i|_{AB}$

$$\cdot F_{\epsilon=1}(p) = X_p X \rightsquigarrow \text{KRAUS} \quad K_1 = X, K_{i>1} = 0$$

$$\cdot \sum_i K_i^\dagger K_i = X^\dagger X = XX = \mathbb{I}$$

$$\cdot D_{\epsilon=1}(p) = \text{Tr}[p] \frac{\mathbb{I}}{d}$$

$$P_{D_{\epsilon=1}} = \frac{1}{d^2} \mathbb{I} \otimes \mathbb{I} \rightsquigarrow K_{(i,s)} = \frac{\sqrt{2}}{\sqrt{d}} \sum_{l=1}^d \langle l | i, s \rangle_{AB} |l\rangle = \frac{1}{\sqrt{d}} |i\rangle \langle s|$$

\star

$\left\{ |i\rangle \otimes |s\rangle, i, s = 0, \pm \right\}$
 eigenstates

CHECK: $D(p) = \sum_{(i,s) \in \{0, \pm\}^2} K_{(i,s)} P K_{(i,s)}^\dagger =$
 $= \sum_{(i,s) \in \{0, \pm\}^2} \frac{1}{2} |i\rangle \langle s| p |s\rangle \langle i| = \frac{\text{Tr}[p]}{2} = \frac{\text{Tr}[p]}{d}$

Other equiv. KRAUS OPERATORS are: $\frac{1}{2}\{I, X, Y, Z\}$.

$$\cdot A_{\epsilon=1}(p) = \text{Tr}[A] |\psi\rangle\langle\psi|$$

$$P_{A_{\epsilon=1}} = |\psi\rangle\langle\psi| \otimes \frac{\mathbb{I}}{d} \rightsquigarrow K_\psi = \sum_{l=1}^d \sqrt{p_l} \sqrt{d} \langle l | \psi \rangle_{AB} |l\rangle = \sqrt{\frac{1}{d}} \sqrt{d} \langle l | 0, \psi \rangle_{AB} |l\rangle = |\psi\rangle\langle\psi|$$

\star

eigenstates: $|0\rangle \otimes |i\rangle$
 with $i = 0, \pm$

CHECK:

$$A_{E=1}(p) = \sum_{i=1}^d K_i |p| K_i^+ = \sum_{i=1}^d |\psi(i)| p |\psi(i)\rangle\langle\psi(i)| = \text{Tr}[\tilde{P}] |\psi\rangle\langle\psi|$$

~~• STINESPRIN G:~~ $\tilde{\Phi}(p) = \text{tr}_B(U_{AB} p |\psi\rangle\langle\psi| U_{AB}^+)$ where $U_{AB} |\psi\rangle\langle\psi| = \sum_{i=1}^d K_i |\psi(i)\rangle\langle i|$

• $F_{E=1}(p) = X_p X \rightsquigarrow F_{E=1}(p) = \text{tr}_B(X \otimes \mathbb{1} p |\psi\rangle\langle\psi| X^+ \otimes \mathbb{1})$

$X = X^+$

$$U_{AB} = X \otimes \mathbb{1} = (-i) e^{i \frac{(X \otimes \mathbb{1}) \pi}{2}}$$

LEMMA : P such that $P^2 = \mathbb{1} \Rightarrow e^{iPt} = \cos(t) \mathbb{1} + i \sin(t) P$

PROOF:

$$e^{iPt} := \sum_{k=0}^{+\infty} \frac{(iPt)^k}{k!} = \sum_{k=0}^{\infty} \frac{(iPt)^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{(iPt)^{2k+1}}{(2k+1)!} =$$

even/odd splitting

$$= \sum_{k=0}^{+\infty} \frac{i^{2k} P^{2k} t^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{i^{2k+1} P^{2k+1} t^{2k+1}}{(2k+1)!} =$$

$$\begin{aligned} P^2 \mathbb{1} &= \left(\sum_{k=0}^{+\infty} \frac{(-i)^k t^{2k}}{(2k)!} \right) \mathbb{1} + i P \left(\frac{(-i)^k t^{2k+1}}{(2k+1)!} \right) \\ \cdot P^2 \mathbb{1} &= \mathbb{1} \\ \cdot i^2 &= -1 \end{aligned}$$

$\cos(t)$

$\sin(t)$

• $D_{E=1}(p) = \text{Tr}[\tilde{P}] \frac{\mathbb{1}}{\mathbb{1}} = \sum_{(E_1, E_2) \in \{0, 1\}^2} \left(\frac{|\psi(E_1)\rangle\langle\psi(E_2)|}{\sqrt{2}} \right) p \left(\frac{|\psi(E_2)\rangle\langle\psi(E_1)|}{\sqrt{2}} \right)$

$K(E_1, E_2)$

~~•~~ $\rightarrow \tilde{\Phi}(p) = \text{tr}_B(U_{AB} p |\psi\rangle\langle\psi| U_{AB}^+)$ where $U_{AB} |\psi\rangle\langle\psi| = \sum_{i=1}^d K_i |\psi(i)\rangle\langle i|$

$$U_{AB} |l\rangle\langle l| = \sum_{i=1}^d K_i |l\rangle\langle i| =$$

(RELABELLING) $= K_1 |l\rangle\langle 1| + K_2 |l\rangle\langle 2| + K_3 |l\rangle\langle 3| + K_4 |l\rangle\langle 4|$

$$= \frac{|\psi(l)\rangle\langle\psi(l)|}{\sqrt{2}} + \frac{|\psi(l+1)\rangle\langle\psi(l)|}{\sqrt{2}} + \frac{|\psi(l+2)\rangle\langle\psi(l)|}{\sqrt{2}} + \frac{|\psi(l+3)\rangle\langle\psi(l)|}{\sqrt{2}}$$

$$|U_{AB}|0\rangle \otimes |0\rangle = \frac{|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle}{\sqrt{2}} = :|L_0\rangle$$

$$|U_{AB}|1\rangle \otimes |0\rangle = \frac{|0\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle}{\sqrt{2}} = :|L_1\rangle$$

$$\Rightarrow |U_{AB}|1\rangle \otimes |0\rangle = \underset{\uparrow}{\langle 0|1\rangle} |L_0\rangle + \underset{\uparrow}{\langle 2|1\rangle} |L_1\rangle$$

$$|1\rangle = |0\rangle \langle 0|1\rangle + |1\rangle \langle 1|1\rangle$$

$$(|U_{AB}|1\rangle \otimes |0\rangle) (\langle 1|0\rangle \langle U_{AB}^+|) = (\langle 0|1\rangle |L_0\rangle + \langle 2|1\rangle |L_1\rangle) (\langle 1|0\rangle \langle L_0| + \langle 1|2\rangle \langle L_1|)$$

$$|U_{AB}|P \otimes |0\rangle \langle 0| |U_{AB}^+| = \underset{\uparrow}{\langle 0|P|0\rangle} |L_0\rangle \langle L_0| + \langle 0|P|2\rangle |L_0\rangle \langle L_1| + \langle 2|P|0\rangle |L_1\rangle \langle L_0| + \langle 2|P|2\rangle |L_1\rangle \langle L_1|$$

$P = \sum_i \lambda_i |i\rangle \langle i|$

CHECK: $\text{Tr}_B(|U_{AB}|P \otimes |0\rangle \langle 0| |U_{AB}^+|) = \langle 0|P|0\rangle \left(\frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2} \right) + 0 + 0 + \langle 2|P|2\rangle \left(\frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2} \right) = \frac{\text{Tr}[P]}{2}$

$$\bullet A_{c=1}(P) = \text{Tr}[P] |0\rangle \langle 0| = \sum_{i=1}^d (|0\rangle \langle i| P |i\rangle \langle 0|)$$

$\textcircled{*}$ $\rightarrow \text{Tr}(P) = \text{Tr}_B(|U_{AB}|P \otimes |0\rangle \langle 0| |U_{AB}^+|)$ where $|U_{AB}| \cdot |0\rangle \otimes |0\rangle = \sum_{i=1}^d |k_i\rangle \otimes |i\rangle$

$$|U_{AB}| \cdot |0\rangle \otimes |0\rangle = \sum_{i=1}^d |0\rangle \langle i| \otimes |i\rangle = |0\rangle \otimes |0\rangle$$

$$\Rightarrow |U_{AB}|(1\rangle \otimes |0\rangle) = \sum_{i=1}^d \langle 0|1\rangle \underset{\substack{\dots \\ |0\rangle \otimes |0\rangle}}{|U_{AB}| \cdot |0\rangle \otimes |0\rangle} = |0\rangle \otimes |1\rangle$$

$$\bullet \text{I can take } |U_{AB}| = F \quad (F |i\rangle \otimes |j\rangle := |i\rangle \otimes |j\rangle)$$

$$\bullet \text{CHECK: } \text{Tr}_B(F P \otimes |0\rangle \langle 0| F) =$$

$$= |0\rangle \underset{\substack{\dots \\ |0\rangle \otimes |0\rangle}}{\underline{|0\rangle \langle 0|}} \langle 0|$$

$$= \text{Tr}[P] |0\rangle \langle 0|$$

$$= A_{c=1}(P)$$

$$\bullet \text{CURIOSITY: } F = \frac{1}{d} \sum_{i,j=1}^d P_i \otimes P_j$$

(\therefore HILBERT-SCHMIDT SCALAR PRODUCT $\Rightarrow F = \sum_{i,j} c_{ij} P_i \otimes P_j \Rightarrow \text{Tr}(P_i \otimes P_j F) = c_{ii} d^2$)
 $\therefore P_i \otimes P_j$ is a basis
 δ_{ij}, d H.S. basis

$$\bullet |U_{AB}| = F = \begin{pmatrix} i F \frac{1}{2} \\ e^{-\frac{i}{4}(1+X \otimes X + Y \otimes Y + Z \otimes Z)} \end{pmatrix} (-i) = (-i) e^{\frac{i}{4}(1+X \otimes X + Y \otimes Y + Z \otimes Z)}$$

\uparrow
 $\therefore F^2 = \mathbb{1}$
 $\therefore \text{LENHA } \boxed{\text{smiley}}$

XXX - ISOTROPIC-HEISENBERG EVOLUTION!

- c) Generalise the previous results to an arbitrary $\epsilon \in [0, 1]$. (Hint: First compute the respective representations for $\epsilon = 0$ and then reason for the Choi state on the one hand, and for the Kraus and Stinespring representations on the other hand how to combine the $\epsilon = 0$ and $= 1$ cases into an arbitrary ϵ case.)

\approx First for $\epsilon = 0$:

- All of them are equal to $\Phi(p) = p$
- So their Choi-state is $\Phi(\cdot) \otimes \mathbb{I} (\mathbb{I} \otimes \mathbb{I}) = |\Omega\rangle\langle\Omega|$
- A KRAUS decomposition is: $K_i = \mathbb{I} \rightsquigarrow \Phi_p = \sum_i K_i p K_i^+ = p \mathbb{I}^+$
- A STINESPRING repn. is: $\Phi(p) = \text{tr}_B (U_{AB} p \otimes |\Omega\rangle\langle\Omega| U_{AB}^+)$
with $U_{AB} = \mathbb{I}$.

\approx General $\epsilon \in [0, 1]$

$$\text{We have } \Phi_\epsilon(p) = \epsilon \Phi_{\epsilon=0}(p) + (1-\epsilon) \underbrace{\Phi_{\epsilon=0}(p)}_{|\Omega\rangle\langle\Omega|}$$

- The Choi state is $\Phi_\epsilon \otimes \mathbb{I} (\mathbb{I} \otimes \mathbb{I}) = \epsilon \Phi_{\epsilon=0} \otimes \mathbb{I} (\mathbb{I} \otimes \mathbb{I}) + (1-\epsilon) \underbrace{\Phi_{\epsilon=0} \otimes \mathbb{I} (\mathbb{I} \otimes \mathbb{I})}_{|\Omega\rangle\langle\Omega|}$
- The KRAUS decomposition is $\Phi_\epsilon(p) = \sum_i \tilde{K}_i p \tilde{K}_i^+$

$$\text{where } \{\tilde{K}_i\} = \left\{ \underbrace{\sum_{\epsilon=0}^{1-\epsilon} K_i|}_{\sum_{\epsilon=0}^{1-\epsilon} \mathbb{I}} \right\}_{i=1, \dots} \cup \left\{ \underbrace{\sum_{\epsilon=0}^1 K_i|}_{\sum_{\epsilon=0}^1 \mathbb{I}} \right\}_{i=1, \dots}$$

- The STINES. repn. is:

$$\Phi(p) = \text{tr}_B (U_{AB} p \otimes |\Omega\rangle\langle\Omega| U_{AB}^+) \text{ where } U_{AB} |\Omega\rangle\langle\Omega| = \sum_i K_i |i\rangle\langle i|$$

- d) For arbitrary $\epsilon \in [0, 1]$ compute the action of each channel on the inputs $|0\rangle\langle 0|$ and $\rho = \frac{1}{2}I$. What is the physical interpretation of each channel?

$$F_\epsilon(A) := \epsilon XAX + (1 - \epsilon)A$$

$$D_\epsilon(A) := \epsilon \text{Tr}[A] \frac{1}{d} + (1 - \epsilon)A$$

$$A_\epsilon(A) := \epsilon \text{Tr}[A] |0\rangle\langle 0| + (1 - \epsilon)A,$$

• $F_\epsilon(|0\rangle\langle 0|) = \epsilon |+\rangle\langle +| + (1 - \epsilon) |0\rangle\langle 0|$

$\underbrace{\quad}_{\text{BIT-FLIP ERROR}}$

$$F_\epsilon\left(\frac{1}{2}\right) = \epsilon \frac{1}{2} + (1 - \epsilon) \frac{1}{2} = \frac{1}{2}$$

• $D_\epsilon(|0\rangle\langle 0|) = \epsilon \frac{1}{2} + (1 - \epsilon) |0\rangle\langle 0|$

$\underbrace{\quad}_{\text{depolarizing-channel}}.$
 $(\epsilon \rightarrow 1 \Rightarrow \text{maximally mixed state})$

$$D_\epsilon\left(\frac{1}{2}\right) = \frac{1}{2}$$

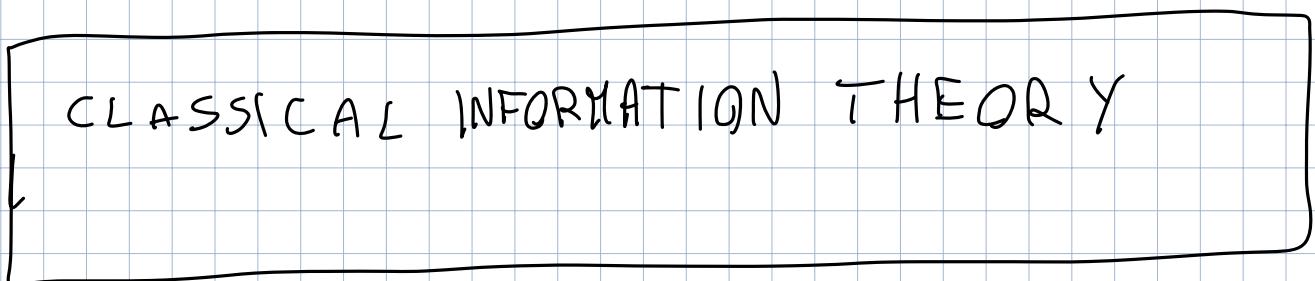
• $A_\epsilon(|0\rangle\langle 0|) = \epsilon |0\rangle\langle 0| + (1 - \epsilon) |0\rangle\langle 0| = |0\rangle\langle 0|$

$$A_\epsilon\left(\frac{1}{2}\right) = \epsilon |0\rangle\langle 0| + (1 - \epsilon) \frac{1}{2}$$

$\underbrace{\quad}_{\text{reset in the 0-state.}}$



INTERLUDE :



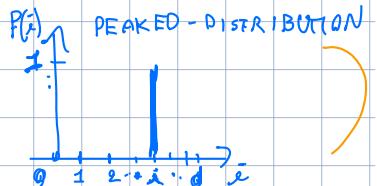
• $X = \{X_1, \dots, X_d\}$ Random Variable \Leftrightarrow

\downarrow \downarrow
 P_{x_1} P_{x_d}
 Probability to extract x_i

$\left\{ \begin{array}{l} \cdot P_i \geq 0 \\ \cdot \sum_{i=1}^d P_i = 1 \end{array} \right.$

• DEF: (SHANNON ENTROPY of X) $H(X) := - \sum_{i=1}^d P_i \log(P_i)$

• $H(X) \geq 0$ $(H(X) = 0 \Leftrightarrow \exists j \in \{1, \dots, d\} : P_i = \delta_{x_j})$



PROOF:

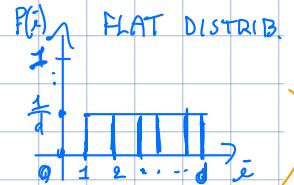
$$\bullet 0 \leq P_i \leq 1 \Rightarrow -P_i \log(P_i) \geq 0 \Rightarrow H(X) \geq 0$$

$$\bullet H(X) = 0 \Rightarrow P_i \log(P_i) = 0 \quad \forall i, \text{ with } P_i \geq 0 \text{ and } \sum_{i=1}^d P_i = 1$$

$$\Rightarrow P_i = 0 \text{ or } \log(P_i) = 0 \Rightarrow \exists j : P_i = \delta_{x_j} \quad \forall i.$$

\uparrow
 $P_i = 1$

• $H(X) \leq \log(d)$ $(H(X) = \log(d) \Leftrightarrow P_i = \frac{1}{d} \quad \forall i)$



PROOF:

• DEF: $f(x)$ is CONVEX $\Leftrightarrow f\left(\sum_{i=1}^d P_i x_i\right) \leq \sum_{i=1}^d P_i f(x_i)$ where $\begin{array}{l} x_i \in \mathbb{R} \\ P_i \in [0, 1] \\ \sum_{i=1}^d P_i = 1 \end{array}$

$$\bullet H(X) = \sum_{i=1}^d P_i \left(-\log(P_i) \right) = \sum_{i=1}^d P_i \left(\log\left(\frac{1}{P_i}\right) \right) =$$

$$\leq \log\left(\sum_{i=1}^d P_i \cdot \frac{1}{P_i}\right) = \log(d)$$

\uparrow
 • $f(x) = \log(x)$ is CONCAVE
 $\log(x)$

• " \Rightarrow ": If $P_i = \frac{1}{d} \forall i \Rightarrow H(X) = \log(d)$

• " \Leftarrow ": The inequality is saturated if $f\left(\sum_i p_i x_i\right) = \sum_i p_i f(x_i)$ with $x_i = \frac{1}{P_i}$.

- Now $f(x) = -\log(x)$ is strictly CONCAVE: $\sum_i p_i f(x_i) = f\left(\sum_i p_i x_i\right) \Leftrightarrow x_i = \bar{x} \forall i$.

$$\Rightarrow \frac{1}{P_i} = \bar{x} \stackrel{\sum_i P_i = 1}{\Rightarrow} P_i = \frac{1}{\bar{x}} \quad \forall i.$$

• DEF. (JOINT PROB. DISTRIBUTION $P(X, Y)$)

$$X = \{x_1, \dots, x_d\}, Y = \{y_1, \dots, y_d\}$$

$P(x, y)$ joint prob. distribution $\stackrel{\text{DEF.}}{\Leftrightarrow} \begin{cases} \cdot P(x, y) \geq 0 \quad \forall x \in X, y \in Y \\ \cdot \sum_{\substack{x \in X \\ y \in Y}} P(x, y) = 1 \end{cases}$

• DEF. (JOINT ENTROPY $H(X, Y)$)

$$H(X, Y) = - \sum_{\substack{x \in X \\ y \in Y}} P(x, y) \log(P(x, y))$$

DEF. (MARGINAL DISTRIBUTION $P(x)$ and $P(y)$ given $P(x, y)$).

Given JOINT PROB. DISTRIB. $P(x, y)$, we define:

$$P(x) := \sum_{y \in Y} P(x, y)$$

(marginal distrib.
of X)

$$P(y) := \sum_{x \in X} P(x, y)$$

(marginal distrib.
of Y)

$$\begin{aligned} & \cdot P(x) \geq 0, \sum_x P(x) = 1 \\ & \cdot H(X) = - \sum_{x \in X} P(x) \log(P(x)) \end{aligned}$$

$$\begin{aligned} & \cdot P(y) \geq 0, \sum_y P(y) = 1 \\ & \cdot H(Y) = - \sum_{y \in Y} P(y) \log(P(y)) \end{aligned}$$

• TH① (SUB-ADITIVE PROPERTY OF $H(X, Y)$)

$$H(X, Y) \leq H(X) + H(Y) \quad \left(H(X, Y) = H(X) + H(Y) \Leftrightarrow P(X, Y) = \underset{\text{INDP.}}{\underset{\text{RANDOM VARIABLES}}{\uparrow}} P(X) \cdot P(Y) \right)$$

PROOF:

$$H(X, Y) - H(X) - H(Y) = - \sum_{x,y} P(x, y) \log(P(x, y)) + \sum_x P(x) \log(P(x)) + \sum_y P(y) \log(P(y))$$

$$= - \sum_{x,y} P(x, y) \log(P(x, y)) + \sum_{x,y} P(x, y) \log(P(x)) + \sum_{x,y} P(x, y) \log(P(y))$$

• $P(X) = \sum_y P(x, y)$

• $P(Y) = \sum_x P(x, y)$

$$= - \sum_{x,y} P(x, y) \log \left(\frac{P(x, y)}{P(x) P(y)} \right) = \sum_{x,y} P(x, y) \log \left(\frac{P(x) P(y)}{P(x, y)} \right)$$

$$\leq \log \left(\sum_{x,y} P(x, y) \frac{P(x) P(y)}{P(x, y)} \right) = \log(1) = 0$$

(• $\log(\cdot)$ CONCAVE)

• $\sum_x P(x) = 1$

• $\sum_y P(y) = 1$

• Since $\log(\cdot)$ is strictly CONCAVE \Rightarrow

the inequality is saturated when $\frac{P(x) P(y)}{P(x, y)} = \text{CONST } \forall x, y$.

$\hookrightarrow P(x, y) = P(x) \cdot P(y)$

↑
Product distribution.

TH ②

$$H(X) \leq H(X, Y) \quad ("=" \Leftrightarrow P(X, Y) = P(X) \delta_{Y, \bar{Y}})$$

$$H(Y) \leq H(X, Y) \quad ("=" \Leftrightarrow P(X, Y) = P(Y) \delta_{X, \bar{X}})$$

$$\left(\text{or } \max(H(X), H(Y)) \leq H(X, Y) \right)$$

PROOF:

$$H(X) - H(X, Y) = - \sum_x P(x) \log(P(x)) + \sum_{x,y} P(x,y) \log(P(x,y))$$

$\sum_y P(x,y)$

$$= - \sum_{x,y} P(x,y) \log\left(\frac{P(x)}{P(x,y)}\right)$$

$$= \sum_{x,y} P(x,y) \log\left(\frac{P(x,y)}{P(x)}\right) \stackrel{\log(\cdot) \text{ CONCAVE}}{\leq} \log\left(\sum_{x,y} \frac{P(x,y)}{P(x)}\right)$$

$$\stackrel{\uparrow}{\leq} \log\left(\sum_{x,y} \frac{P(x,y)}{P(x)}\right) \leq \log\left(\sum_x \frac{P(x)}{B(x)}\right) = \log(1) = 0 \Rightarrow \square$$

- $\frac{P(x,y)}{P(x)} = \frac{P(x,y)}{P(x)} \cdot P(x,y) \leq 1 \cdot P(x,y)$

- $P(x) = \sum_{y'} P(x, y') \Rightarrow \frac{P(x,y)}{P(x)} \leq 1$

- SATURATED ($\Leftrightarrow \frac{P(x,y)}{P(x)} = 1 \Leftrightarrow P(x,y) = P(x) \delta_{Y, \bar{Y}}$)

- $\log(\cdot)$ monotone

- We'll see that $\max(H(X), H(Y)) \leq H(X, Y)$ is violated in the "QUANTUM VERSION" due to ENTANGLEMENT.

DEF: $(\text{CONDITIONAL PROBABILITY } P(X|Y))$

$$P(X|Y) := \frac{P(X,Y)}{P(Y)}$$

(Fixed y , $P(X|y)$ is a prob. dist.: $\begin{cases} \cdot P(X|y) \geq 0 \\ \sum_x P(x|y) = 1 \end{cases}$)

↓
This defines a random variable $X|_y$
distributed according to $P(X|y)$.

$$P(Y|X) := \frac{P(X,Y)}{P(X)}$$

(Fixed x , $P(Y|x)$ is a prob. dist.: $\begin{cases} \cdot P(Y|x) \geq 0 \\ \sum_y P(y|x) = 1 \end{cases}$)

↓
This defines a random variable $Y|_x$

OBS! We have $\sum_X P(X|Y) P(Y) = P(X)$

DEF: $(\text{CONDITIONAL ENTROPY } H(X|Y))$

$$H(X|Y) := \mathbb{E}_Y H(X|_Y) = \mathbb{E}_Y \left(- \sum_X P(X|Y) \log(P(X|Y)) \right)$$

COR:

$$\begin{aligned} H(X|Y) &= - \sum_{x,y} P(x,y) \log(P(x|y)) \\ &= H(X,Y) - H(Y) \end{aligned}$$

PROOF:

$$\bullet H(X|Y) = \mathbb{E}_Y \left(- \sum_X P(X|Y) \log(P(X|Y)) \right)$$

$$= - \sum_{x,y} P(x|y) P(y) \log(P(x|y)) =$$

$$= - \sum_{x,y} P(x,y) \log(P(x|y))$$

$$P(x|y) := \frac{P(x,y)}{P(y)}$$

$$\bullet H(X|Y) = - \sum_{x,y} P(x,y) \log(P(x|y)) = - \sum_{x,y} P(x,y) \log \left(\frac{P(x,y)}{P(y)} \right) =$$

$$= - \sum_{x,y} P(x,y) \log(P(x,y)) + \underbrace{\sum_{y,x} P(x,y) \log(P(y))}_{P(y)} = H(X) - H(Y)$$

OBS.

$$H(X|Y) \geq 0$$

PROOF:

$$H(X|Y) = H(X, Y) - H(Y) \geq 0$$

↑
THEOREM ②

(This inequality does not hold for the Quantum version of $H(X|Y)$ because TH.② is violated.)

DEF. (MUTUAL INFORMATION)

$$I(X:Y) := H(X) + H(Y) - H(X, Y)$$

OBS.

$$\bullet I(X:Y) \geq 0$$

↑
(SUB-ADD.)

$$\bullet I(X:Y) = I(Y:X)$$

↑
 $H(X,Y) = H(Y,X)$

$$\bullet I(X:Y) = H(X, Y) - H(X|Y) - H(Y|X)$$

$$I(X:Y) = \overbrace{(H(X, Y) - H(X|Y))}^{\bullet H(X|Y) = H(X, Y) - H(Y)} + \overbrace{(H(X, Y) - H(Y|X))}^{\bullet H(Y|X) = H(X, Y) - H(X)} - H(X, Y)$$

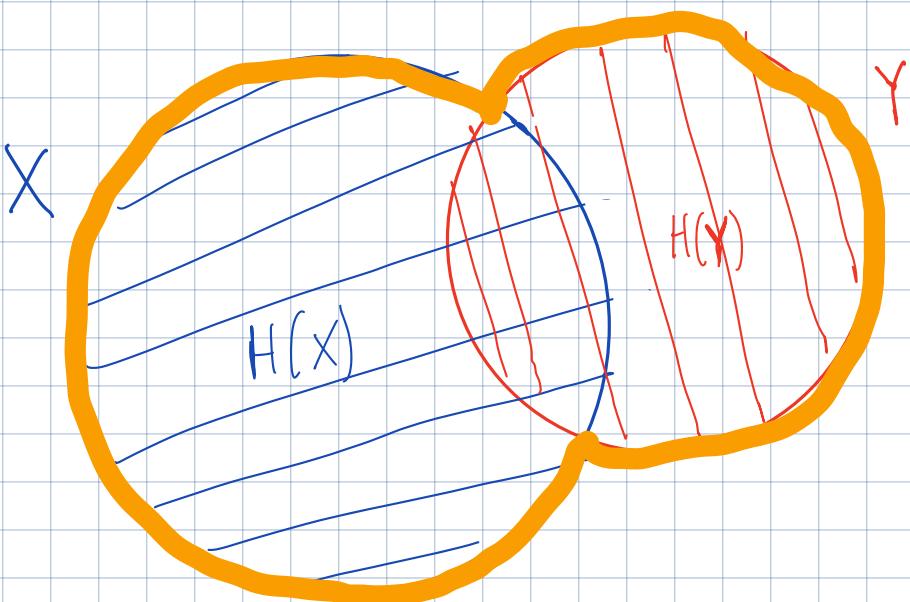
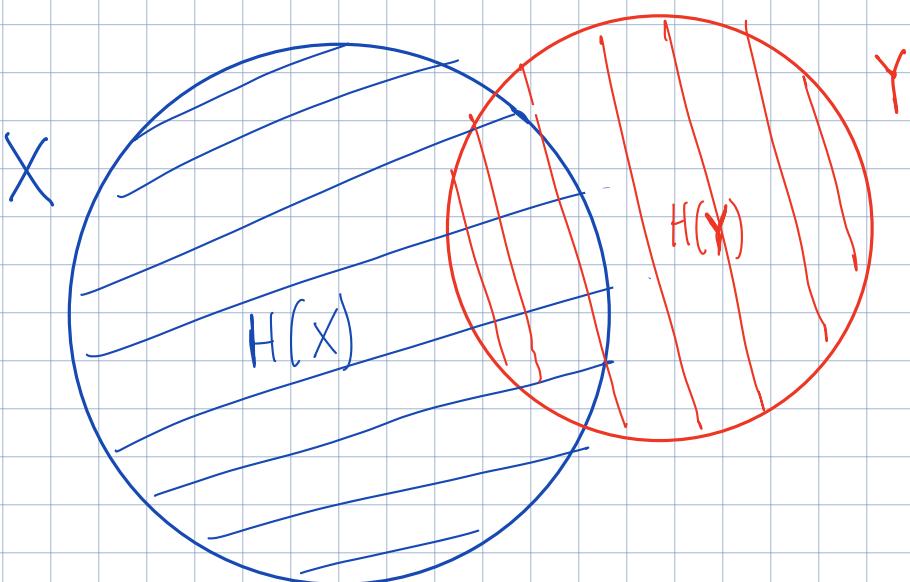
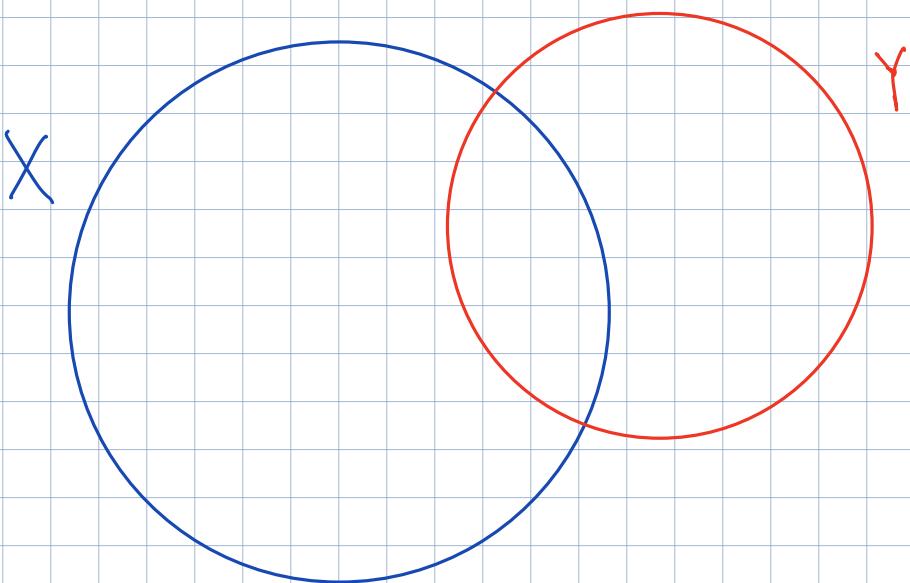
• $H(X|Y) = H(X, Y) - H(Y)$

• $H(Y|X) = H(X, Y) - H(X)$

$$\bullet I(X:Y) \leq \min(H(X), H(Y))$$

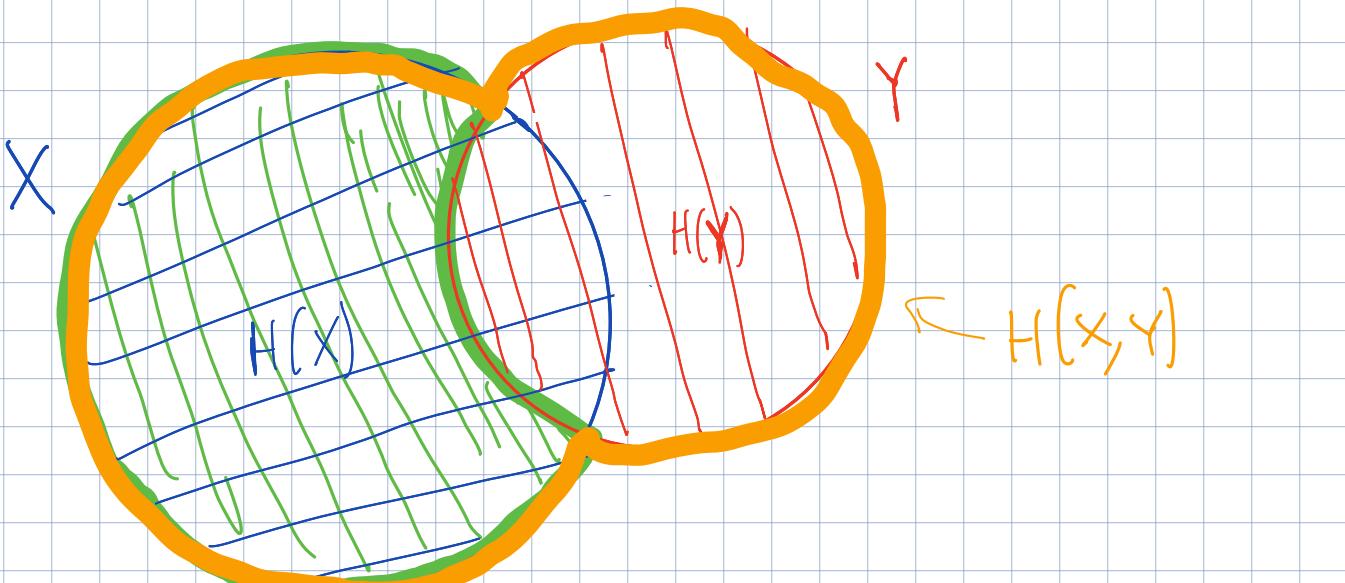
↑
 $I(X:Y) = H(X) + H(Y) - H(X, Y) \leq H(X) + H(Y) - \max(H(X), H(Y)) = \min(H(X), H(Y))$

GRAPHICAL INTERPRETATION

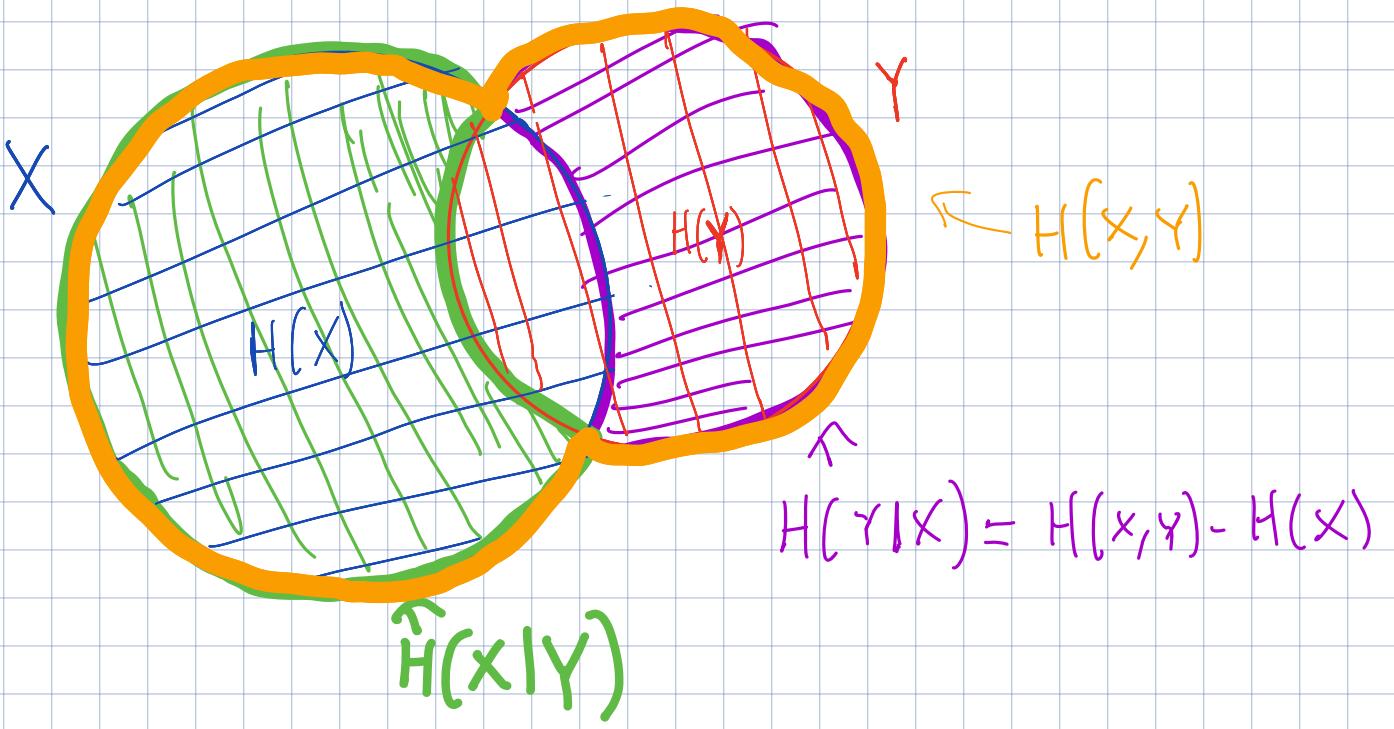


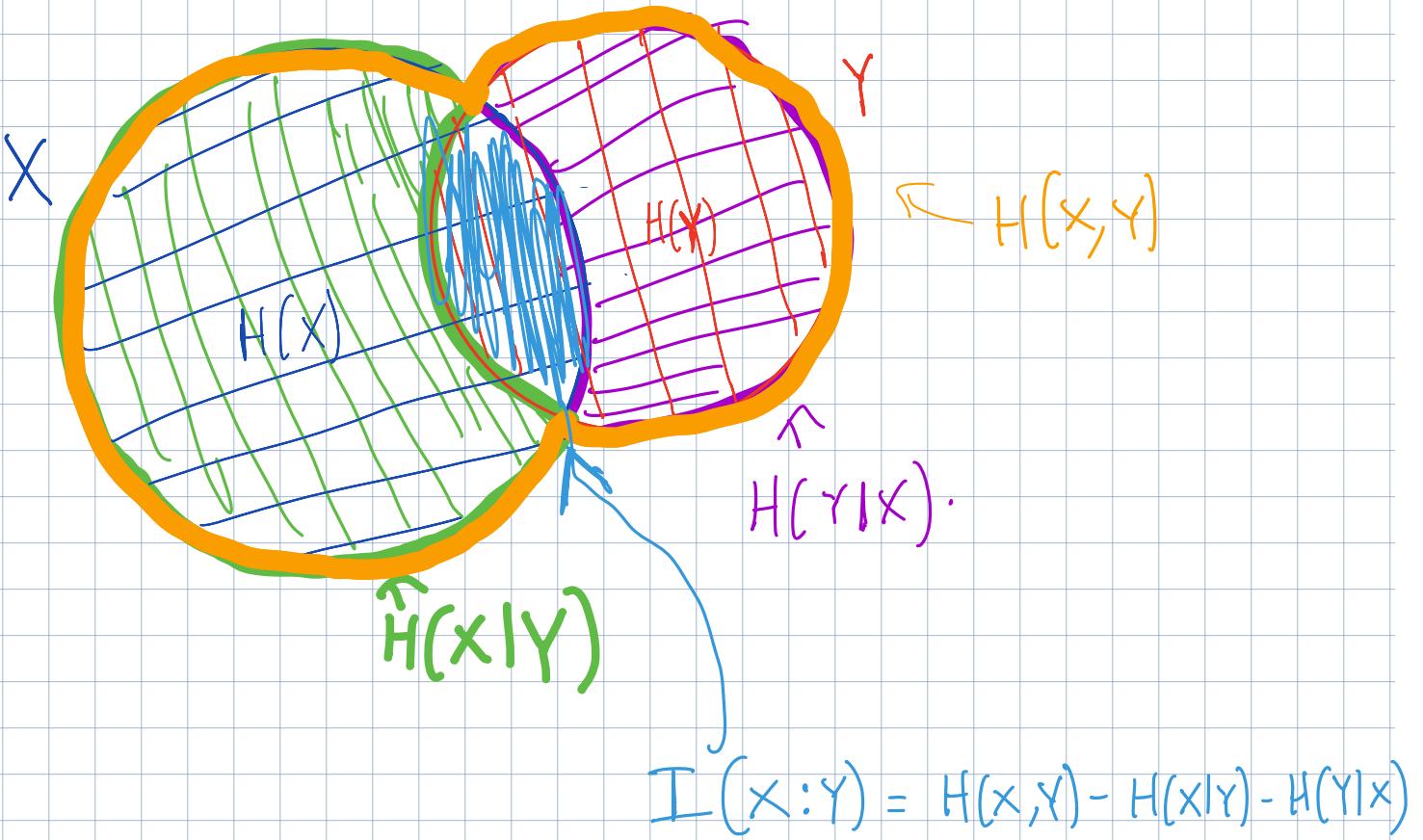
$H(X, Y)$

- $H(X, Y) \leq H(X) + H(Y)$
- $H(X, Y) \geq \max(H(X), H(Y))$
- They are obvious here!



$$\hat{H}(x|y) = H(x,y) - H(y)$$





- DEF: (RELATIVE ENTROPY)

- $P(x) \geq 0$; $\sum_x P(x) = 1$. $q(x) \geq 0$; $\sum_x q(x) = 1$.

$$\begin{aligned}
 H(P(x) \parallel q(x)) &:= \sum_x P(x) \log_2 \left(\frac{P(x)}{q(x)} \right) = \\
 &= -H(P) - \sum_x P(x) \log_2(q(x))
 \end{aligned}$$

• OBS.

$$H(P(x) \parallel q(x)) \geq 0 \quad \left(H(P(x) \parallel q(x)) = 0 \iff P(x) = q(x) \forall x \right)$$

PROOF:

$$H(P(x) \parallel q(x)) := \sum_x P(x) \log_2 \left(\frac{P(x)}{q(x)} \right) \geq \sum_x \frac{P(x)}{\ln(2)} \left(1 - \frac{q(x)}{P(x)} \right)$$

$$\begin{aligned} & \bullet e^x \geq 1+x \\ \Rightarrow & x \geq \ln(1+x) \\ \Rightarrow & 1-x \geq \ln(1-x) = \log_2(1-x) \ln(2) \\ \Leftrightarrow & -\log_2(1-x) \geq \frac{1-x}{\ln(2)} \\ & \bullet \log_2 \left(\frac{p}{q} \right) = -\log_2 \left(\frac{q}{p} \right) \end{aligned}$$

$$= \frac{1}{\ln(2)} \left(1 - \frac{1}{d} \right) = 0$$

• The inequality $\log_2(y) \geq \frac{1}{\ln(2)}(1-y)$ is saturated $\iff y=1 \iff P(x)=q(x)$

OBS:

$$H(P(X,Y) \parallel P(X) \cdot P(Y)) = -H(X,Y) + H(X) + H(Y) \geq 0$$

\uparrow
DEF
 \uparrow
OBS.

$$\Rightarrow H(X,Y) \leq H(X) + H(Y)$$

$$H(P(x) \parallel q(x)) = -H(x) - \underbrace{\sum_x P(x) \log_2 \left(\frac{1}{d} \right)}_{\text{if } q(x) = \frac{1}{d} \forall x} \geq 0 \quad \Rightarrow H(x) \leq \log(d)$$

\uparrow
 \uparrow
 \uparrow
OBS

2. On Shannon entropy...

To begin with let us first show some simple properties of entropies, in particular, of the mutual information.

Recall the definition of the Shannon entropies for random variables X, Y which take values in \mathcal{X}, \mathcal{Y} and are distributed according to probability distributions p, q over \mathcal{X} and \mathcal{Y} , respectively.

$$H(X) = - \sum_{x \in \mathcal{X}} p(x) \log p(x) \text{ (Shannon entropy)} \quad (1)$$

$$H(X|Y) = H(X, Y) - H(Y) = \sum_{x \in \mathcal{X}} p(x) H(Y|X = x) \text{ (Conditional entropy)} \quad (2)$$

- Show that $0 \leq H(X) \leq \log |\mathcal{X}|$, where the first equality holds iff there is an $x \in \mathcal{X}$ for which $p(x) = 1$ and the second inequality holds iff $p(x) = 1/|\mathcal{X}|$ for all x .
- Show that the Shannon entropy is *subadditive*, i.e., that $H(X, Y) \leq H(X) + H(Y)$ with equality if X and Y are independent..

Hint: Show that $H(X, Y) - H(X) - H(Y) \leq 0$ using that $\log_2 x \ln 2 = \ln x \leq x - 1$.

- Show that $H(Y|X) \geq 0$ equality if and only if Y is a (deterministic) function of X .

Hint: Use Bayes' rule: $p(x, y) = p(y|x)p(x)$

• We showed all these points explaining the theory.

DEF. VON NEUMANN ENTROPY

$$S(p) := -\text{Tr}(p \log p) \quad \text{where } p \text{ is a density matrix}$$

$$P = \sum_{i=1}^d p_i |V_i\rangle \langle V_i| \implies S(p) = -\sum_{i=1}^d p_i \log(p_i) = H(\{p_i\})$$

↑
SHANNON
ENTROPY

Eigendec.

eigenvalues
(with $\sum_{i=1}^d p_i = 1$)

PROOF:

$$\text{In general we have } f(p) = \sum_{i=1}^d f(p_i) |V_i\rangle \langle V_i|.$$

FACT \oplus :

$$\text{This because every } f(\cdot) \text{ can be written as } f(p) = \sum_{s=0}^{\infty} c_s p^s$$

$$\begin{aligned} \Rightarrow f(p) &= \sum_{s=0}^{\infty} c_s p^s \sum_{i=1}^d |V_i\rangle \langle V_i| = \sum_i \sum_s c_s \underbrace{(p^s |V_i\rangle)}_{p_i^s |V_i\rangle} \langle V_i| = \\ &= \sum_i \left(\sum_s c_s p_i^s \right) |V_i\rangle \langle V_i| = \sum_{i=1}^d \underbrace{f(p_i)}_{\sum_s c_s p_i^s} |V_i\rangle \langle V_i| \end{aligned}$$

$$\therefore \text{ given } f(p) := p \log(p) \Rightarrow p \log(p) = \sum_{i=1}^d (p_i \log(p_i)) |V_i\rangle \langle V_i|$$

$$\begin{aligned} \Rightarrow S(p) &:= -\text{Tr}(p \log p) = -\sum_{i=1}^d (p_i \log(p_i)) \underbrace{\text{Tr}(|V_i\rangle \langle V_i|)}_{1} \\ &= -\sum_{i=1}^d p_i \log p_i \end{aligned}$$

QBS.

$$0 \leq S(p) \leq \log(d)$$

PROOF:

$$S(p) = H(\{p_i\}) \quad \text{and we showed that } 0 \leq H(x) \leq \log(d).$$

OBS)

$$\cdot S(p) = 0 \iff p = |\psi\rangle\langle\psi| \text{ (PURE)}$$

$$\cdot S(p) = \log(d) \iff p = \frac{1}{d} \text{ (MAXIMALLY MIXED STATE)}$$

PROOF:

$$\cdot S(p) = H(\sum p_i |v_i\rangle\langle v_i|) = 0 \iff \sum p_i |v_i\rangle\langle v_i| \text{ is a peaked distribution i.e. } p_i = \delta_{i,a} \forall i.$$

PREVIOUS LECTURE

$$\cdot S(p) = H(\sum p_i |v_i\rangle\langle v_i|) = \log(d) \iff \sum p_i |v_i\rangle\langle v_i| \text{ is a flat distribution i.e. } p_i = \frac{1}{d} \forall i.$$
$$\iff p = \sum_{i=1}^d p_i |v_i\rangle\langle v_i| = \frac{1}{d} \underbrace{\sum_{i=1}^d |v_i\rangle\langle v_i|}_{\mathbb{I}} \Rightarrow \mathbb{I}$$

$$\cdot S(p) = S(U_p U^\dagger) \quad \forall U \text{ unitary.}$$

PROOF:

$$P = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{and} \quad (U_p U^\dagger) = \sum_i p_i (U |v_i\rangle\langle v_i| U^\dagger)$$

have the same eigenvalues.

and $S(\cdot)$ depends only by the eigenvalues.

• DEF: (RELATIVE ENTROPY)

$$S(P_1 \| P_2) := \text{tr} [P_1 \log(P_2)] - \text{tr} [P_1 \log(P_1)]$$

$$= -S(P_1) - \text{tr} [P_1 \log(P_2)]$$

THEOREM: (KLEIN INEQUALITY)

$$S(P_1 \parallel P_2) \geq 0 \quad \left(S(P_1 \parallel P_2) = 0 \iff P_1 = P_2 \right)$$

PROOF:

$$P_1 = \sum_{i=1}^d p_i |V_i\rangle\langle V_i|, \quad P_2 = \sum_{j=1}^d q_j |W_j\rangle\langle W_j|$$

$$\begin{aligned} S(P_1 \parallel P_2) &= -S(P_1) - \text{tr}[P_1 \log(P_2)] \\ &= -S(P_1) - \text{tr}\left[\sum_{i=1}^d p_i |V_i\rangle\langle V_i| \log(P_2)\right] \end{aligned}$$

$$= -S(P_1) - \sum_{i=1}^d p_i \langle V_i | \log(P_2) | V_i \rangle$$

$$\begin{aligned} &= -S(P_1) - \sum_{i=1}^d \sum_{j=1}^d p_i \log(q_j) \underbrace{\langle V_i | W_j \rangle \langle W_j | V_i \rangle}_{|\langle V_i | W_j \rangle|^2} = \\ &\log(P_2) = \sum_{j=1}^d \log(q_j) |W_j\rangle\langle W_j| \quad \text{↑} \\ &\text{★} \end{aligned}$$

$$= -S(P_1) - \sum_{i=1}^d \sum_{j=1}^d p_i \log(q_j) |\langle V_i | W_j \rangle|^2$$

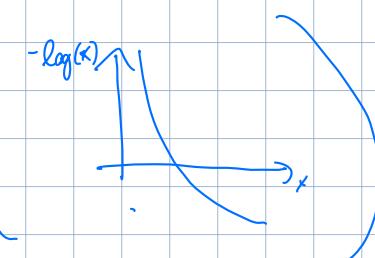
$$= + \sum_{i=1}^d p_i \log(p_i) - \sum_{i=1}^d \sum_{j=1}^d p_i \log(q_j) |\langle V_i | W_j \rangle|^2$$

$$= + \sum_i \sum_j |\langle V_i | W_j \rangle|^2 p_i \log(p_i) - \sum_{i=1}^d \sum_j |\langle V_i | W_j \rangle|^2 p_i \log(q_j)$$

$$\sum_j |\langle V_i | W_j \rangle|^2 = 1$$

$$= + \sum_i \sum_j |\langle V_i | W_j \rangle|^2 p_i \left(\log\left(\frac{p_i}{q_j}\right) \right) = \sum_i \sum_j \underbrace{|\langle V_i | W_j \rangle|^2}_{P(i,j)} p_i \left(-\log\left(\frac{q_j}{p_i}\right) \right)$$

$$\geq \sum_i p_i \left[-\log \left(\sum_j P(i,j) \frac{q_j}{p_i} \right) \right] = \sum_i p_i \left[-\log \left(\frac{k_i}{p_i} \right) \right]$$

• $\sum_j P(i,j) = 1, P(i,j) \geq 0$
 • $-\log(x)$ CONVEX

 • $k_i := \sum_j P(i,j) q_j$
 { Prob. dist.
 ($\sum_i k_i = 1$)
 ($k_i \geq 0$)

$$= \sum_i p_i \log \left(\frac{p_i}{k_i} \right) \geq 0$$

Bozzi's lecture

- The first inequality is saturated when $p_i = q_j \forall i, j$
 $\stackrel{\text{def}}{=} q_j = \frac{1}{d} \forall i, j$

or when $P(i,j) = \delta_{j,i_0}, \forall j = 1, \dots, d$

- But $P(i,j) = |\langle v_i | w_j \rangle|^2 = \delta_{j,i_0} \forall j = 1, \dots, d \Leftrightarrow |w_i\rangle = |w_{i_0}\rangle \forall i$

$$|v_i\rangle = \sum_j \langle w_j | v_i \rangle |w_j\rangle$$

δ_{j,i_0} $\stackrel{\text{def}}{=} |\langle w_j | v_{i_0} \rangle|^2$
 $= e^{\frac{i\pi}{d} j(i_0)}$

- The second inequality is saturated when $p_i = k_i \forall i$ (previous lecture)

$$\Leftrightarrow p_i = \sum_j P(i,j) q_j = \sum_j \delta_{j,i_0} q_j = q_{i_0} \forall i$$

$$\Leftrightarrow p_1 = \sum_i p_i |v_i\rangle \langle v_i| = \sum_i q_{i_0} |w_{i_0}\rangle \langle w_{i_0}| = p_2$$

• OBS.

$$\bullet S(P_1 \parallel P_2) \neq S(P_2 \parallel P_1) \quad (\text{NOT SYMMETRIC})$$

$$\bullet \text{Supp}(P_1) \cap \text{Ker}(P_2) \neq \{\vec{0}\} \Rightarrow S(P_1 \parallel P_2) = +\infty$$

PROOF:

$$S(P_1 \parallel P_2) = -S(P_1) - \sum_{i=1}^d \sum_{j=1}^d p_i \log(p_j) |v_i(w_j)|^2$$

$$\text{If } p_j = 0 \text{ then } p_i |v_i(w_j)| \neq 0 \forall i \Rightarrow S(P_1 \parallel P_2) = +\infty$$

\Updownarrow

$$w_j \notin \text{Ker}(P_2) \quad w_j \in \text{Supp}(P_2) = \text{Supp}(P_1)$$

$$\bullet \text{So } \text{Ker}(P_2) \cap \text{Supp}(P_1) \neq \{\vec{0}\} \Rightarrow S(P_1 \parallel P_2) = +\infty \Rightarrow \square$$

$$\bullet O = O_1 \otimes O_2 \quad \text{with} \quad O_1 = O_1^+, O_2 = O_2^+$$

$$\log(O) = \log(O_1) \otimes 1L + 1L \otimes \log(O_2) \quad \star$$

PROOF:

$$\log(O) = \log(O_1 \otimes O_2) = \log \left[\sum_{i,s} \lambda_i^{(1)} \lambda_s^{(2)} (|v_i^{(1)}\rangle \langle v_i^{(1)}| \otimes |v_s^{(2)}\rangle \langle v_s^{(2)}|) \right] =$$

$O_1 = \sum_i \lambda_i^{(1)} |v_i^{(1)}\rangle \langle v_i^{(1)}|$
 $O_2 = \sum_s \lambda_s^{(2)} |v_s^{(2)}\rangle \langle v_s^{(2)}|$

↑ eigenstates for $O_1 \otimes O_2$

$$= \sum_{i,s} \log(\lambda_i^{(1)} \lambda_s^{(2)}) (|v_i^{(1)}\rangle \langle v_i^{(1)}| \otimes |v_s^{(2)}\rangle \langle v_s^{(2)}|)$$

$$f(A) = \sum_i f(\lambda_i) |v_i\rangle \langle v_i|$$

$\sum_i |v_i\rangle \langle v_i|$

$$\log(ab) = \log(a) + \log(b)$$

$$= \sum_{i,s} \log(\lambda_i^{(1)}) (|v_i^{(1)}\rangle \langle v_i^{(1)}| \otimes |v_s^{(2)}\rangle \langle v_s^{(2)}|) + \sum_{i,s} \log(\lambda_s^{(2)}) (|v_i^{(1)}\rangle \langle v_i^{(1)}| \otimes |v_i^{(2)}\rangle \langle v_i^{(2)}|)$$

$$= \sum_i \log(\lambda_i^{(1)}) (|v_i^{(1)}\rangle \langle v_i^{(1)}| \otimes 1L) + \sum_s \log(\lambda_s^{(2)}) (1L \otimes |v_s^{(2)}\rangle \langle v_s^{(2)}|) =$$

$$= \log(O_1) \otimes \mathbb{1} + \log(O_2) \otimes \mathbb{1}$$

COR.

$$\cdot O = O_1 \otimes O_2 \otimes \dots \otimes O_n, O_i = O_i^+ \text{ } H_i$$

$$\cdot \log(O) = \sum_{i=1}^N \mathbb{1} \otimes \log(O_i) \otimes \mathbb{1}_{i+1, \dots, n}$$

COR:

$$S(P_1 \otimes P_2 \otimes \dots \otimes P_N) = \sum_{i=1}^N S(P_i)$$

PROOF:

$$\log(P) = \sum_{i=1}^N \mathbb{1} \otimes \log(p_i) \otimes \mathbb{1}_{i+1, \dots, n}$$

$$\begin{aligned} S(P) &= -\text{tr}_B(P \log(P)) = -\sum_{i=1}^N \text{tr}\left(P \left(\mathbb{1} \otimes \log(p_i) \otimes \mathbb{1}_{i+1, \dots, n}\right)\right) = \\ &= -\sum_{i=1}^N \text{tr}_{\frac{\mathbb{1}}{P_i}}\left(\text{tr}_{\frac{\mathbb{1}}{P_i}}(P) \log(p_i)\right) = \sum_{i=1}^N S(P_i) \end{aligned}$$

THEOR.] ②

$$\cdot P_A = \text{tr}_B(P_{AB}), P_B = \text{tr}_A(P_{AB})$$

$$\cdot S(P_{AB}) \leq S(P_A) + S(P_B) \quad (S(P_{AB}) = S(P_A) + S(P_B) \Leftrightarrow P_{AB} = P_A \otimes P_B)$$

PROOF:

$$\cdot S(P_{AB} \parallel P_A \otimes P_B) \geq 0$$

↑
KLEIN INEQUALITY

$$\cdot S(P_{AB} \parallel P_A \otimes P_B) = -S(P_{AB}) - \text{tr}\left(P_{AB} \frac{\log(P_A \otimes P_B)}{\mathbb{1} \otimes \mathbb{1}}\right)$$

$\underbrace{\mathbb{1} \otimes \mathbb{1}}_Q$ $\log(P_A) \otimes \mathbb{1} + \mathbb{1} \otimes \log(P_B)$

$$= -S(P_{AB}) - \underbrace{\text{tr}(P_{AB} \log(P_A \otimes P_B))}_{\text{tr}(P_A \log(P_A))} - \text{tr}(P_{AB} \# \otimes \log(P_B))$$

$$= -S(P_{AB}) + S(P_A) + S(P_B) \geq 0 \Rightarrow \underline{\Sigma}$$

• " $=$ " $\Leftrightarrow S(P_{AB} \parallel P_A \otimes P_B) = 0 \Leftrightarrow P_{AB} = P_A \otimes P_B$

FACT :

• $P_A := \text{tr}_B(P)$, $P_B := \text{tr}_A(P)$

• $P = |\psi\rangle\langle\psi|$ pure $\Rightarrow S(P_A) = S(P_B)$

(This is not true in general if P is not pure)
Find a counter-example (5).

PROOF:

SCHRIOT.

$$|\psi\rangle = \sum_{i=1}^{\text{dim}(A \otimes B)} \sqrt{\lambda_i} |V_i^A\rangle \otimes |V_i^B\rangle$$

With $\langle V_i^A, V_j^A \rangle = \delta_{i,j}$, $\langle V_i^B, V_j^B \rangle = \delta_{i,j}$.

SUBPROOF:

$$\cdot |\psi\rangle = \sum_{i,s} \langle i, s | \psi \rangle |i, s\rangle = \sum_{i,s} (UDV^+)_{i,s} |i, s\rangle =$$

\uparrow
 $(C)_{i,s}$
SVD
 $\bullet C = UDV^+$

$$= \sum_e \sum_{i,s} U_{i,e} D_{e,s} (V^+)^*_{e,s} |i\rangle \otimes |s\rangle = \sum_e \lambda_e \left(\sum_i U_{i,e} |i\rangle \right) \otimes \left[\sum_s (V^+)^*_{e,s} |s\rangle \right]$$

$\uparrow \lambda_e := D_{e,e} \geq 0$
 $|V_i^A\rangle$
 $|V_i^B\rangle$

$$\text{tr}_B(|\psi\rangle\langle\psi|) = \sum_i \lambda_i |V_i^A\rangle\langle V_i^A|$$

\Rightarrow Some eigenvalues \Rightarrow same entropies.

$$\text{tr}_A(|\psi\rangle\langle\psi|) = \sum_i \lambda_i |V_i^B\rangle\langle V_i^B|$$

THEOR | ④

$$S(P_{AB}) \geq |S(P_A) - S(P_B)|$$

PROOF?

- Given a mixed state P_{AB} , then $\exists R$ such that $P_{AB} = \text{tr}_R [|\psi\rangle\langle\psi|]$

with $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$

SUBPROOF (ALREADY SEEN)

E.g. take $|\psi\rangle := \sqrt{P_{AB}} \otimes \frac{1}{\sqrt{d}} \left(\sum_{i=1}^d |i\rangle_{AB} \otimes |i\rangle_R \right)$ and verify.

$$\left(-\sqrt{P_{AB}} \otimes \sqrt{P_{AB}} \right) = -P_{AB}$$

We have:

$$S(P_B) = S(P_{AR}) \stackrel{\substack{\uparrow \\ (\text{TH ②})}}{\leq} S(P_A) + S(P_R) = S(P_A) + S(P_{AB})$$

$$\begin{aligned} \bullet S(P_B) &= S(P_{AR}) \\ (\bullet \text{FACT ③}) \\ (\bullet P_{ABR} \text{ pure}) \end{aligned}$$

$$\begin{aligned} \bullet S(P_{AB}) &= S(P_R) \\ (\bullet \text{FACT ③}) \\ (\bullet P_{ABR} \text{ pure}) \end{aligned}$$

$$\Rightarrow \boxed{S(P_{AB}) \geq S(P_B) - S(P_A)} = *_1$$

$$S(P_A) = S(P_{BR}) \stackrel{\substack{\uparrow \\ \text{FACT ③}}}{\leq} S(P_B) + S(P_R) = S(P_B) + S(P_{AB})$$

$$\Rightarrow \boxed{S(P_{AB}) \geq S(P_A) - S(P_B)} = *_2$$

\Rightarrow



*₁, *₂

OBS] Also in the classical case I have $H(X, Y) \geq |H(X) - H(Y)|$

But we did not mention it because in the classical case

We have an even stronger inequality: $H(X, Y) \geq \max(H(X), H(Y))$

Which can be violated quantumly.

$$S(P_{AB}) \geq \max(S(P_A), S(P_B))$$

No!

No! No!

FALSE!

" " " "

PROOF.

COUNTER-EXAMPLE: $P_{AB} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = |1\rangle$ max. entangled state.

$$S(P_{AB}) = 0$$

↑
state pure.

$$\max(S(P_A), S(P_B)) = \max\left(S\left(\frac{1}{d_A}\right), S\left(\frac{1}{d_B}\right)\right) \neq 0$$

$\frac{1}{d_A}$ $\frac{1}{d_B}$

DEF.

$$P \text{ Separable} \iff P = \sum_{i=1}^N p_i^{(i)} p_i^{(i)} \otimes p_i^{(i)} \quad \text{with} \quad \begin{cases} p_i \geq 0 \\ \sum_{i=1}^N p_i = 1 \end{cases}$$

$p_i^{(i)}$ and $p_i^{(i)}$ states.

DEF

P entangled $\iff P$ not separable.

FACT 1:

$$P \text{ Separable} \Rightarrow S(P_{AB}) \geq S(P_A)$$

PROOF: NOT SHOWN HERE.

THEOR. (CONCAVITY)

$$S\left(\sum_{i=1}^N p_i p_i\right) \geq \sum_{i=1}^N p_i S(p_i)$$

$$\left(S\left(\sum_{i=1}^N p_i p_i\right) = \sum_{i=1}^N p_i S(p_i) \Leftrightarrow p_i = p \forall i \right)$$

PROOF:

$$P_{AB} := \sum_{i=1}^N p_i p_i \otimes |i\rangle\langle i| = \sum_{i=1}^N \sum_{j=1}^d p_i \lambda_j^{(i)} |V_j^{(i)}\rangle\langle V_j^{(i)}| \otimes |i\rangle\langle i|$$

↑
EIGEN.

$$S(P_{AB}) = H\left(\{p_i \lambda_j^{(i)}\}\right) = -\sum_{i,j} p_i \lambda_j^{(i)} \log(p_i \lambda_j^{(i)}) = -\sum_{i,j} p_i \lambda_j^{(i)} \log(p_i) - \sum_{i,j} p_i \lambda_j^{(i)} \log(\lambda_j^{(i)})$$

$$= H\left(\{p_i\}\right) + \sum_{i=1}^N p_i S(p_i)$$

↑
($\sum_j \lambda_j^{(i)} = 1$)

$$P_A = \text{tr}_B(P_{AB}) = \sum_{i=1}^N p_i p_i$$

$$S(P_A) = S\left(\sum_{i=1}^N p_i p_i\right)$$

$$P_B = \text{tr}_A(P_{AB}) = \sum_{i=1}^N p_i |i\rangle\langle i|$$

$$S(P_B) = H\left(\{p_i\}\right)$$

$$S(P_{AB}) \leq S(P_A) + S(P_B)$$

↑
SUB-ADD.

$$H\left(\{p_i\}\right) + \sum_{i=1}^N p_i S(p_i) \leq S\left(\sum_{i=1}^N p_i p_i\right) + H\left(\{p_i\}\right) \Rightarrow \square$$

THEOR.

$$\sum_{i=1}^N p_i S(p_i) \leq S\left(\sum_{i=1}^N p_i p_i\right) \leq H\left(\{p_i\}\right) + \sum_{i=1}^N p_i S(p_i)$$

PROOF:

$$\cdot S\left(\sum_{i=1}^N p_i |i\rangle\langle i|\right) \stackrel{\uparrow}{=} S\left(\sum_{i=1}^N \sum_{j=1}^d p_i |\lambda_j^{(i)}\rangle\langle V_j^{(i)}| V_j^{(i)}\langle V_j^{(i)}|\right)$$

$P_i = \sum_{j=1}^d \lambda_j^{(i)} |V_j^{(i)}\rangle\langle V_j^{(i)}|$

orthogonal basis

$$\cdot |\psi\rangle_{AB} := \sum_{i=1}^N \sum_{j=1}^d \sqrt{p_i \lambda_j^{(i)}} |\lambda_j^{(i)}\rangle \otimes |(i,j)\rangle$$

$$\cdot S(P_A) = S(\text{tr}_{B_0} [|\psi\rangle\langle\psi|]) = S(\text{tr}_{A_0} [|\psi\rangle\langle\psi|]) = S(P_B)$$

|| || pure

$$S\left(\sum_{i,j} p_i \lambda_j^{(i)} |V_j^{(i)}\rangle\langle V_j^{(i)}|\right)$$

||

$$S\left(\sum_{i=1}^N p_i |i\rangle\langle i|\right)$$

$$\Rightarrow S(P_B) = S\left(\sum_{i=1}^N p_i |i\rangle\langle i|\right)$$

$$\begin{aligned} \cdot P_B &= \text{tr}_A [|\psi\rangle\langle\psi|] = \text{tr}_A \left(\sum_{i=1}^N \sum_{j=1}^d \sum_{k=1}^N \sum_{l=1}^d \sqrt{p_i \lambda_j^{(i)}} \sqrt{p_k \lambda_l^{(k)}} |V_j^{(i)}\rangle\langle V_k^{(k)}| \otimes |(i,j)\rangle\langle(k,l)| \right) = \\ &= \sum_{m=1}^d \langle m | \otimes \text{id} \left(\sum_{i=1}^N \sum_{j=1}^d \sum_{k=1}^N \sum_{l=1}^d \sqrt{p_i \lambda_j^{(i)}} \sqrt{p_k \lambda_l^{(k)}} |V_j^{(i)}\rangle\langle V_k^{(k)}| \otimes |(i,j)\rangle\langle(k,l)| \right) |m \rangle \otimes \text{id} \\ &= \sum_{m=1}^d \sum_{i=1}^N \sum_{j=1}^d \sum_{k=1}^N \sum_{l=1}^d \sqrt{p_i \lambda_j^{(i)}} \sqrt{p_k \lambda_l^{(k)}} \langle m | V_j^{(i)} \langle V_k^{(k)} | |m\rangle |(i,j)\rangle\langle(k,l)| \\ &= \sum_{i=1}^N \sum_{j=1}^d \sum_{k=1}^N \sum_{l=1}^d \sqrt{p_i \lambda_j^{(i)}} \sqrt{p_k \lambda_l^{(k)}} \langle V_k^{(k)} | V_j^{(i)} |(i,j)\rangle\langle(k,l)| \end{aligned}$$

\cdot If I perform a measurement on B (with $\sum_{i,j} |\langle(i,j)|\langle(i,j)|\}_{i,j}$), then the post measurement state will be:

$$P_B' = \sum_{i=1}^d \sum_{j=1}^N \text{tr}(P_B P_{i,j}) \left(\frac{P_{i,j} P_B P_{i,j}}{\text{tr}(P_B P_{i,j})} \right)$$

$$= \sum_{i=1}^d \sum_{j=1}^N P_{i,j} P_B P_{i,j} = \sum_{i,j} P_{i,j} \lambda_j^{(i)} |(i,j)\rangle\langle(i,j)|$$

$$P_B' = \sum_{i,s} P_i \lambda_s^{(i)} |(i,s)\rangle\langle(i,s)|$$

$$S(P_B') = H\left(\{P_i \lambda_s^{(i)}\}\right) = H\left(\{P_i\}\right) + \sum_i P_i \underbrace{\sum_{s=1}^d \lambda_s^{(i)} \log(\lambda_s^{(i)})}_{S(P_i)}$$

- LEMMA • $\sum P_k$ PVH set $\Rightarrow S\left(\underbrace{\sum_k P_k p_k}_{P'}\right) \geq S(p)$
- (PROOF LATER)

\uparrow
state after measurement.

$$\bullet S(P_B') \geq S(P_B) = S\left(\sum_i P_i p_i\right)$$

||

$$H\left(\{P_i\}\right) + \sum_i P_i S(P_i)$$

- LEMMA • $\sum P_k$ PVH set $\Rightarrow S\left(\underbrace{\sum_k P_k p_k}_{P'}\right) \geq S(p)$
- (PROOF LATER)

\uparrow
state after measurement.

SUB PROOF:

$$\bullet \underbrace{0 \leq S(p || p')}_{\substack{\uparrow \\ \text{KLEIN INEQ}}} = -S(p) - \text{tr}(p \log(p'))$$

KLEIN INEQ

$$\bullet -\text{tr}(p \log(p')) \geq S(p)$$

$$-\text{tr}(P \log(P)) = -\text{tr}(P' \log(P')) = S(P') \Rightarrow \square$$

SUB SUB PROOF:

$$\text{tr}(P \log(P)) = \text{tr}\left(\sum_i P_i p \log(P')\right) = \sum_i \text{tr}\left(P_i^T p \log(P')\right) = \sum_i \text{tr}\left(P_i p \log(P') P_i\right) =$$

$\stackrel{\text{P.M.} \Rightarrow P_i P_i^T = \delta_{ij}, P_i^T = P_i}{\cdot P_i P_i^T = \mathbb{1}}$

$\text{tr}(AB) = \text{tr}(BA)$

$$\log(P) P_i = P_i \log(P)$$

$$P' = \sum_j P_j P_j^T \Rightarrow P' P_i = P_i P_j P_i = P_i P' \Rightarrow \log(P) P_i = P_i \log(P)$$

$\log(P) = \sum_{k=0}^{\infty} c_k P^k$

$$= \sum_i \text{tr}(P_i P_i^T \log(P')) = \sum_i \text{tr}(P_i \log(P')) \Rightarrow \square$$

THEOR. (STRONG SUB-ADDITIONALITY)
(SSA)

$$S(P_{ABC}) + S(P_C) \leq S(P_{AC}) + S(P_{BC})$$

with $P_{AB} = \text{tr}_C(P_{ABC})$, $P_{BC} = \text{tr}_A(P_{ABC})$, $P_A = \text{tr}_{BC}(P_{ABC})$, ...

OBJS STRONG SUB-ADDITIONALITY \Rightarrow SUB-ADDITIONALITY

PROOF:

$$P_{ABC} := P_{AB} \otimes |0\rangle\langle 0|$$

$$\underbrace{S(P_{ABC})}_{\substack{\text{PRODUCT STATE} \\ \rightarrow |11\rangle}} + \underbrace{S(P_C)}_{\substack{|1\rangle \\ S(|0\rangle\langle 0|)}} \leq \underbrace{S(P_{AC})}_{\substack{|1\rangle \\ S(P_A \otimes |0\rangle\langle 0|)}} + \underbrace{S(P_{BC})}_{\substack{|1\rangle \\ S(P_B \otimes |0\rangle\langle 0|)}} \\ \underbrace{S(P_A)}_{\substack{|1\rangle \\ S(P_A + S(|0\rangle\langle 0|))}} + \underbrace{S(P_B)}_{\substack{|1\rangle \\ S(P_B + S(|0\rangle\langle 0|))}}$$

$$\Rightarrow S(P_{AB}) \leq S(P_A) + S(P_B) \Rightarrow \square$$

PROP.

$$\cdot P^{\otimes h} := \overbrace{P \otimes \dots \otimes P}^{n\text{-times}}$$

$$\cdot S(P^{\otimes h}) = h S(P)$$

SUB-ADD.
(PRODUCT)

$$\cdot S(P^{\otimes h} \| \tau^{\otimes h}) = h S(P \| \tau)$$

PROOF:

$$\begin{aligned} S(P^{\otimes h} \| \tau^{\otimes h}) &= -S(P^{\otimes h}) + \text{tr}(P^{\otimes h} \log(\tau^{\otimes h})) = \\ &= -h S(P) + h \text{tr}(P \log(\tau)) = h S(P \| \tau) . \end{aligned}$$

$\cdot S(P^{\otimes h}) = h S(P)$
 $\cdot \log(A \otimes B) = \log A \otimes 1 + 1 \otimes \log(B)$

3. ... and the von-Neumann entropy

For any state $\rho \in \mathcal{D}(\mathcal{H})$ with $\dim \mathcal{H} = d$ the von-Neumann entropy is defined as $S(\rho) = -\text{Tr}(\rho \log \rho)$. Throughout this problem, if the global state being referred to is clear, we will denote entropies of the reduced states using the corresponding Hilbert space as an argument, e.g. the entropy of a state ρ_{AB} reduced on subsystem A is denoted $S(A)$.

- Show that $0 \leq S(\rho)$ with equality if and only if ρ is pure. (One can also show the upper bound $S(\rho) \leq \log d$.)
 - Show that the von-Neumann entropy is *subadditive* in the sense that if two distinct systems A and B have a joint quantum state ρ^{AB} then $S(A, B) \leq S(A) + S(B)$, with equality if $\rho_{AB} = \rho_A \otimes \rho_B$.
- Hint: You may use the inequality $S(\rho) \leq -\text{Tr}[\rho \log \sigma]$ for an arbitrary quantum state σ and that for two matrices A and B , $\log(A \otimes B) = \log(A) \otimes \mathbb{1} + \mathbb{1} \otimes \log(B)$.*
- Suppose that $p = (p_i)_i$ is a probability vector and the states ρ_i are mutually orthogonal. Show that

$$S\left(\sum_i p_i \rho_i\right) = H(p) + \sum_i p_i S(\rho_i).$$

and use this result to infer that

$$S\left(\sum_i p_i \rho_i \otimes |i\rangle\langle i|\right) = H(p) + \sum_i p_i S(\rho_i),$$

where $\langle i|j\rangle = \delta_{ij}$ and the ρ_i are arbitrary quantum states.

- Use the results from b) and c) to infer that the von-Neumann entropy S is concave, that is, $S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$ for a probability distribution $\{p_i\}$.
- let Ω_{AB} be the maximally entangled state on two Hilbert spaces of equal dimension d , i.e. $\Omega = |\Omega\rangle\langle\Omega|$ with

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle. \quad (3)$$

Compute $S(A|B)$. What do you conclude?

A) PROVED BEFORE

B) PROVED BEFORE

$$\text{C)} \cdot S\left(\sum_i p_i \rho_i\right) = S\left(\sum_i p_i \lambda_i^{(1)} |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}|\right) =$$

other. eigenvectors. because ρ_i orthogonal.

$$= H\left(\sum_i p_i \lambda_i^{(s)}\right) = - \sum_{i,s} p_i \lambda_i^{(s)} \log(p_i \lambda_i^{(s)}) =$$

$$= - \sum_i p_i \underbrace{\left(\sum_s \lambda_i^{(s)} \right)}_{\frac{1}{1}} \log(p_i) - \sum_i p_i \sum_s \lambda_i^{(s)} \log(\lambda_i^{(s)}) =$$

$$= - H(\{p_i\}) + \sum_i p_i S(p_i)$$

• Take $\sum_i p_i p_i \otimes \underbrace{|i\rangle\langle i|}_{\text{ort. basis.}}$ $\Rightarrow \{p_i \otimes |i\rangle\langle i|\}_i$ is orthonorm. set.

$$\Rightarrow S\left(\sum_i p_i p_i\right) = - H(\{p_i\}) + \underbrace{\sum_i p_i S(p_i \otimes |i\rangle\langle i|)}_{S(p_i) + \underbrace{S(|i\rangle\langle i|)}_0}$$

D) Seem! $S(A|B) := S(A, B) - S(B)$

$$= \underbrace{S(|B\rangle\langle B|)}_0 - S(B) \stackrel{\uparrow}{=} - \log(d_B) < 0$$

$$S(B) = S(P_B) = S\left(\frac{1}{d_B}\right)$$

Not like in the classical case!