

SCHMIDT DECOMPOSITION

• $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

CLAIM: $\exists |u_i\rangle \in \mathbb{C}^{d_A}, |v_i\rangle \in \mathbb{C}^{d_B} \forall i \in \{1, \dots, \min(d_A, d_B)\} : |\psi\rangle = \sum_{i=1}^{\min(d_A, d_B)} \lambda_i |u_i\rangle \otimes |v_i\rangle$

PROOF:

$$|\psi\rangle = \text{vec}(A) \xrightarrow{\text{SVD}} \text{vec}\left(\sum_{i=1}^{\min(d_A, d_B)} \lambda_i |u_i\rangle \langle v_i|\right) = \sum_{i=1}^{\min(d_A, d_B)} \lambda_i |u_i\rangle \otimes |v_i\rangle$$

\uparrow
 $V \mapsto \exists A \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$

$$\left(\begin{aligned} |\psi\rangle &= \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i, j | \psi \rangle |i\rangle \otimes |j\rangle \\ A = \text{vec}^{-1}(|\psi\rangle) &= \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i, j | \psi \rangle |i\rangle \langle j| \end{aligned} \right)$$

SUPEROPERATOR DECOMPOSITION

• $\Phi : \mathcal{L}(\mathbb{C}^d) \rightarrow \mathcal{L}(\mathbb{C}^d)$ linear map.

CLAIM: $\exists A_i, B_i \in \mathcal{L}(\mathbb{C}^d) \forall i \in \{1, \dots, d^2\} : \Phi(X) = \sum_{i=1}^{d^2} A_i X B_i^\dagger$

PROOF:

• $\rho_\Phi := \Phi \otimes \mathbb{I} \left(|\Omega\rangle \langle \Omega| \right)$ CHOI-STATE

\uparrow
 $|\Omega\rangle = \sum_{i=1}^d |i, i\rangle$
 \uparrow
(Id. channel)

$$\rho_\Phi \xrightarrow{\text{SVD}} \sum_{i=1}^{d^2} \lambda_i |u_i\rangle \langle v_i| = \sum_{i=1}^{d^2} \lambda_i \text{vec}(\tilde{A}_i) (\text{vec}(\tilde{B}_i))^\dagger = \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i \otimes |\Omega\rangle \langle \Omega| \tilde{B}_i^\dagger \otimes \mathbb{I}$$

\uparrow $\text{vec}(\tilde{A}_i) = \tilde{A}_i \otimes |\Omega\rangle$

\uparrow $\begin{aligned} &\text{vec}(\tilde{A}_i) \in \mathcal{L}(\mathbb{C}^d) : |u_i\rangle = \text{vec}(\tilde{A}_i) \\ &\exists \tilde{B}_i \in \mathcal{L}(\mathbb{C}^d) : |v_i\rangle = \text{vec}(\tilde{B}_i) \end{aligned}$

$$\Phi(X) = \text{tr}_B \left(\mathbb{I} \otimes X^T \rho_\Phi \right) = \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i X \tilde{B}_i^\dagger$$

\uparrow
 $\rho_\Phi = \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i \otimes |\Omega\rangle \langle \Omega| \tilde{B}_i^\dagger \otimes \mathbb{I}$

$A_i := \sqrt{\lambda_i} \tilde{A}_i \Rightarrow \Pi$
 $B_i := \sqrt{\lambda_i} \tilde{B}_i$

CHOI ISOMORPH.

TENSOR NETWORK PROOF:

$$\Phi(X) = \text{Diagram 1} = \text{Diagram 2} =: P_\Phi$$

Diagram 1: A box labeled Φ with two inputs and two outputs. The top input is connected to a box labeled X , which then connects to the top output. The bottom input and output are connected directly.

Diagram 2: A box labeled Φ with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to a box labeled X^T , which then connects to the bottom output.

$$= \text{Diagram 3} \leftarrow \text{CHOI-ISOMORPHISM}$$

Diagram 3: A box labeled P_Φ with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to a box labeled X^T , which then connects to the bottom output.

$$P_\Phi \xrightarrow{\text{SVD}} \sum_{i=1}^d \lambda_i |\mu_i\rangle \langle \nu_i| = \sum_{i=1}^d \lambda_i \tilde{A}_i \tilde{B}_i^\dagger$$

Diagram 1: A box labeled $|\mu_i\rangle$ with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to the bottom output.

Diagram 2: A box labeled \tilde{A}_i with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to the bottom output.

Diagram 3: A box labeled \tilde{B}_i^\dagger with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to the bottom output.

$$\Phi(X) = \text{Diagram 4} = \sum_{i=1}^d \lambda_i \text{Diagram 5} = \sum_{i=1}^d \lambda_i \tilde{A}_i X \tilde{B}_i^\dagger$$

Diagram 4: A box labeled P_Φ with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to a box labeled X^T , which then connects to the bottom output.

Diagram 5: A box labeled \tilde{A}_i with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to the bottom output.

Diagram 6: A box labeled \tilde{B}_i^\dagger with two inputs and two outputs. The top input is connected to the top output. The bottom input is connected to the bottom output.

KRAUS DECOMPOSITION

• $\Phi : \mathcal{L}(\mathcal{H}^d) \rightarrow \mathcal{L}(\mathcal{H}^d)$ linear map.

• Φ COMPLETELY POSITIVE. (C.P.)

CLAIM: $\exists K_i \in \mathcal{L}(\mathcal{H}^d) \forall i \in \{1, \dots, d^2\} : \Phi(X) = \sum_{i=1}^{d^2} K_i X K_i^\dagger$

PROOF:

• Same as before but with EIG. DEC. on the Choi state.
(instead of SVD)

• In fact $P_\Phi := \Phi \otimes \mathbb{I}(|\psi\rangle\langle\psi|) \geq 0$
 \uparrow
C.P.

$$\Rightarrow P_\Phi = P_\Phi^\dagger$$

$$\Rightarrow P_\Phi = \sum_{i=1}^{d^2} \lambda_i |\mu_i\rangle\langle\mu_i|. \text{ (conclude as before)}$$


• $\Phi : \mathcal{L}(\mathcal{H}^d) \rightarrow \mathcal{L}(\mathcal{H}^d)$ linear map.

CLAIM: $P_\Phi := \Phi \otimes \mathbb{I}(|\psi\rangle\langle\psi|) \geq 0 \Rightarrow \Phi$ Completely Positive (C.P.)
 \uparrow
CHOI STATE

PROOF:

$$(\Rightarrow) : \Phi \text{ C.P.} \Rightarrow \left(\forall \sigma \geq 0 \Rightarrow \Phi \otimes \mathbb{I}(\sigma) \geq 0 \right) \Rightarrow P_\Phi \geq 0.$$

$$(\Leftarrow) : \text{Let } \sigma \in \mathcal{L}(\mathcal{H}^d \otimes \mathcal{H}^d) : \sigma \geq 0.$$

$$\Phi \otimes \mathbb{I}(\sigma) =$$


$$= \sum_i \lambda_i \quad \uparrow \quad (\Gamma \geq 0 \Rightarrow \Gamma^+ = \Gamma)$$

$$= \sum_i \quad \nearrow \quad A_{i,:}^T := \sqrt{\lambda_i} \cdot V_{\Phi}^{-1}(|v_i\rangle)$$

$$= \sum_i$$

• Given $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d_B}$,

$$\langle \psi | \Phi \otimes \mathbb{I}(\Gamma) | \psi \rangle = \sum_i \langle \psi | \quad \downarrow \quad 0$$