SCHMIDT DECOMPOSITION

- $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

CLAIM: $\exists\, |u_i\rangle \in \mathbb{C}^{d_A},\ |v_i\rangle \in \mathbb{C}^{d_B}\ \forall i \in \{1, \dots, \min(d_A, d_B)\} : \ |\psi\rangle = \sum_{i=1}^{\min(d_A,d_B)} \lambda_i\, |u_i\rangle \otimes |v_i\rangle$

PROOF:

$$|\psi\rangle = \underset{\substack{\uparrow\\ \forall|\psi\rangle\ :\ \exists A \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)}}{\mathrm{vec}}(A) \overset{\text{SVD}}{=} \mathrm{vec}\left(\sum_{i=1}^{\min(d_A,d_B)} \lambda_i\, |u_i\rangle\langle v_i| \right) = \sum_{i=1}^{\min(d_A,d_B)} \lambda_i\, |u_i\rangle \otimes |v_i\rangle$$

$$\left(\begin{array}{l} |\psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i, j|\psi\rangle\, |i\rangle \otimes |j\rangle \\ A = \mathrm{vec}^{-1}(|\psi\rangle) = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i, j|\psi\rangle\, |i\rangle\langle j| \end{array} \right)$$

SUPEROPERATOR DECOMPOSITION

- $\Phi : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ linear map.

CLAIM: $\exists\, A_i, B_i \in \mathcal{L}(\mathbb{C}^d)\ \forall i \in \{1, \dots, d^2\} : \ \Phi(X) = \sum_{i=1}^{d^2} A_i X B_i^\dagger$

PROOF:

- $\rho_\Phi := \Phi \otimes \underset{\substack{\uparrow\\ (\text{Id. channel})}}{\mathcal{I}} \big(\underset{\substack{\uparrow\\ |\Omega\rangle := \sum_{i=1}^{d} |i,i\rangle}}{|\Omega\rangle\langle\Omega|} \big)$ <u>CHOI-STATE</u>

$$\rho_\Phi \underset{\substack{\uparrow\\ \underline{\text{SVD.}}}}{=} \sum_{i=1}^{d^2} \lambda_i\, |u_i\rangle\langle v_i| \underset{\uparrow}{=} \sum_{i=1}^{d^2} \lambda_i\, \mathrm{vec}(\tilde{A}_i)\, (\mathrm{vec}(\tilde{B}_i))^\dagger \underset{\uparrow}{=} \sum_{i=1}^{d^2} \lambda_i\, \tilde{A}_i \otimes \mathbb{1}\, |\Omega\rangle\langle\Omega|\, \tilde{B}_i^\dagger \otimes \mathbb{1}$$

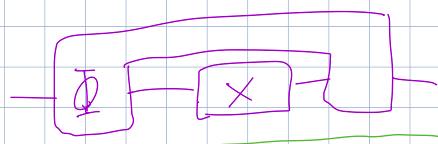
- VEC. $\left(\begin{array}{l} \exists\, \tilde{A}_i \in \mathcal{L}(\mathbb{C}^d) : |u_i\rangle = \mathrm{vec}(\tilde{A}_i) \\ \exists\, \tilde{B}_i \in \mathcal{L}(\mathbb{C}^d) : |v_i\rangle = \mathrm{vec}(\tilde{B}_i) \end{array} \right)$

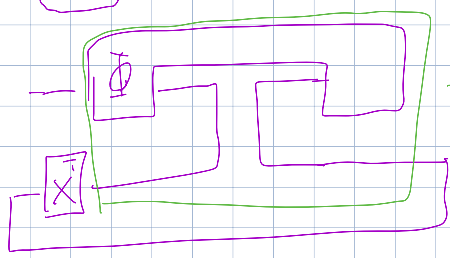
$\mathrm{vec}(\tilde{A}_i) = \tilde{A}_i \otimes \mathbb{1}\, |\Omega\rangle$

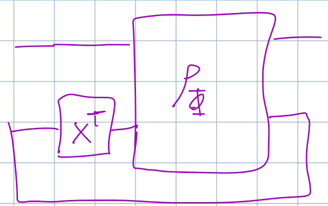
$$\Phi(X) \underset{\substack{\uparrow\\ \text{CHOI ISOMORP.}}}{=} \mathrm{tr}_B\left(\mathbb{1} \otimes X^T \rho_\Phi \right) \underset{\substack{\uparrow\\ \rho_\Phi = \sum_{i=1}^{d^2} \lambda_i\, \tilde{A}_i \otimes \mathbb{1}\, |\Omega\rangle\langle\Omega|\, \tilde{B}_i^\dagger \otimes \mathbb{1}}}{=} \sum_{i=1}^{d^2} \lambda_i\, \tilde{A}_i X \tilde{B}_i^\dagger .$$

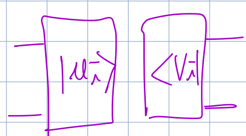
$A_i := \sqrt{\lambda_i}\, \tilde{A}_i$
$B_i := \sqrt{\lambda_i}\, \tilde{B}_i$ $\Rightarrow$ <u>II</u>

TENSOR NETWORK PROOF:

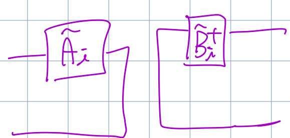
- $\Phi(X) =$ 

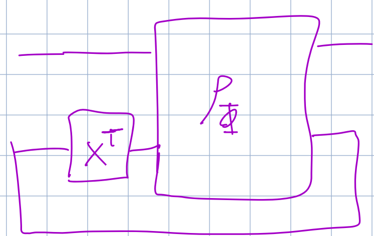
$=$  $=: \rho_\Phi$

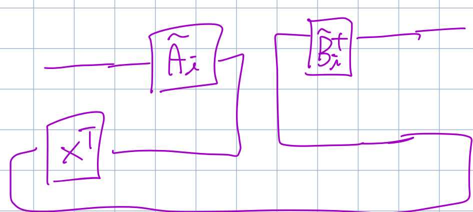
$=$  $\leftarrow$ CHOI - ISOMORPHISM

- $\rho_\Phi \overset{\uparrow}{=} \sum_{i=1}^{d^2} \lambda_i$ 

SVD

$= \sum_{i=1}^{d^2} \lambda_i$ 

- $\Phi(X) =$ 

$= \sum_{i=1}^{d^2} \lambda_i$  $= \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i X \tilde{B}_i^{\dagger}$

KRAUS DECOMPOSITION

- $\Phi : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ linear map.
- Φ COMPLETELY POSITIVE (CP)

CLAIM: $\exists K_i \in \mathcal{L}(\mathbb{C}^d)$ $\forall i \in \{1, \ldots, d^2\}$: $\Phi(X) = \sum_{i=1}^{d^2} K_i X K_i^\dagger$

PROOF:

- Same as before but with EIG.DEC. on the Choi state. (instead of SVD).

- In fact $\rho_\Phi := \Phi \otimes I_d (|\Omega\rangle\langle\Omega|) \underset{\substack{\uparrow \\ CP}}{\geq} 0$

$$\Rightarrow \rho_\Phi = \rho_\Phi^\dagger$$

$$\Rightarrow \rho_\Phi = \sum_{i=1}^{d^2} \lambda_i |u_i\rangle\langle u_i| .$$ (Conclude as before)