

SCHMIDT DECOMPOSITION

• $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

CLAIM: $\exists |u_i\rangle \in \mathbb{C}^{d_A}, |v_i\rangle \in \mathbb{C}^{d_B} \forall i \in \{1, \dots, \min(d_A, d_B)\} : |\psi\rangle = \sum_{i=1}^{\min(d_A, d_B)} \lambda_i |u_i\rangle \otimes |v_i\rangle$

PROOF:

$$|\psi\rangle = \text{vec}(A) \xrightarrow{\text{SVD}} \text{vec}\left(\sum_{i=1}^{\min(d_A, d_B)} \lambda_i |u_i\rangle \langle v_i|\right) = \sum_{i=1}^{\min(d_A, d_B)} \lambda_i |u_i\rangle \otimes |v_i\rangle$$

\uparrow
 $V \mapsto \exists A \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$

$$\left(\begin{aligned} |\psi\rangle &= \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i, j | \psi \rangle |i\rangle \otimes |j\rangle \\ A = \text{vec}^{-1}(|\psi\rangle) &= \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i, j | \psi \rangle |i\rangle \langle j| \end{aligned} \right)$$

SUPEROPERATOR DECOMPOSITION

• $\Phi : \mathcal{L}(\mathbb{C}^d) \rightarrow \mathcal{L}(\mathbb{C}^d)$ linear map.

CLAIM: $\exists A_i, B_i \in \mathcal{L}(\mathbb{C}^d) \forall i \in \{1, \dots, d^2\} : \Phi(X) = \sum_{i=1}^{d^2} A_i X B_i^\dagger$

PROOF:

• $\rho_\Phi := \Phi \otimes \mathbb{I} \left(|\Omega\rangle \langle \Omega| \right)$ CHOI-STATE

\uparrow
 $|\Omega\rangle = \sum_{i=1}^d |i, i\rangle$
 \uparrow
(Id. channel)

$$\rho_\Phi \xrightarrow{\text{SVD}} \sum_{i=1}^{d^2} \lambda_i |u_i\rangle \langle v_i| = \sum_{i=1}^{d^2} \lambda_i \text{vec}(\tilde{A}_i) (\text{vec}(\tilde{B}_i))^\dagger = \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i \otimes |\Omega\rangle \langle \Omega| \tilde{B}_i^\dagger \otimes \mathbb{I}$$

\uparrow $\text{vec}(\tilde{A}_i) = \tilde{A}_i \otimes |\Omega\rangle$

\uparrow $\begin{aligned} &\text{vec}(\tilde{A}_i) \in \mathcal{L}(\mathbb{C}^d) : |u_i\rangle = \text{vec}(\tilde{A}_i) \\ &\exists \tilde{B}_i \in \mathcal{L}(\mathbb{C}^d) : |v_i\rangle = \text{vec}(\tilde{B}_i) \end{aligned}$

$$\Phi(X) = \text{tr}_B \left(\mathbb{I} \otimes X^T \rho_\Phi \right) = \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i X \tilde{B}_i^\dagger$$

\uparrow
 $\rho_\Phi = \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i \otimes |\Omega\rangle \langle \Omega| \tilde{B}_i^\dagger \otimes \mathbb{I}$

$\begin{aligned} A_i &:= \sqrt{\lambda_i} \tilde{A}_i \\ B_i &:= \sqrt{\lambda_i} \tilde{B}_i \end{aligned} \Rightarrow \Pi$

CHOI ISOMORP.

TENSOR NETWORK PROOF:

$$\Phi(X) = \text{[Diagram: } \Phi \text{ box followed by } X \text{ box]} \\ = \text{[Diagram: } \Phi \text{ box and } X \text{ box connected by a line, with a green box around them]} =: P_\Phi$$

$$= \text{[Diagram: } X^T \text{ box followed by } P_\Phi \text{ box]} \leftarrow \text{CHOI-ISOMORPHISM}$$

$$P_\Phi = \sum_{i=1}^d \lambda_i |u_i\rangle \langle v_i| \\ \uparrow \text{SVD} \\ = \sum_{i=1}^d \lambda_i \text{[Diagram: } \tilde{A}_i \text{ box and } \tilde{B}_i^+ \text{ box connected by a line]}$$

$$\Phi(X) = \text{[Diagram: } X^T \text{ box followed by } P_\Phi \text{ box]} \\ = \sum_{i=1}^d \lambda_i \text{[Diagram: } X^T \text{ box, } \tilde{A}_i \text{ box, and } \tilde{B}_i^+ \text{ box connected by lines]} = \sum_{i=1}^d \lambda_i \tilde{A}_i X \tilde{B}_i^+$$

KRAUS DECOMPOSITION

- $\Phi : \mathcal{L}(\mathcal{H}^d) \rightarrow \mathcal{L}(\mathcal{H}^d)$ linear map.
- Φ COMPLETELY POSITIVE. (CP)

CLAIM: $\exists K_i \in \mathcal{L}(\mathcal{H}^d) \forall i \in \{1, \dots, d^2\} : \Phi(X) = \sum_{i=1}^{d^2} K_i X K_i^\dagger$

PROOF:

- Same as before but with EIG. DEC. on the Choi state.
(instead of SVD)

- In fact $P_\Phi := \Phi \otimes I_{\mathcal{H}^d} (|\Omega\rangle\langle\Omega|) \geq 0$
 \uparrow
CP

$$\Rightarrow P_\Phi = P_\Phi^\dagger$$

$$\Rightarrow P_\Phi = \sum_{i=1}^{d^2} \lambda_i |u_i\rangle\langle u_i|. \quad (\text{conclude as before})$$