2016 12th

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Questions

1 Vectors

1. If vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are such that $\left|\overrightarrow{a}\right| = \frac{1}{2}, \left|\overrightarrow{b}\right| = \frac{4}{\sqrt{3}}$ and $\left|\overrightarrow{a} \times \overrightarrow{b}\right| = \frac{1}{\sqrt{3}}$, then find $\left|\overrightarrow{a}.\overrightarrow{b}\right|$.

2. If $\overrightarrow{a}$ and $\overrightarrow{b}$ are unit vectors, then what is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ for $\overrightarrow{a} - \sqrt{2}\overrightarrow{b}$ to be an unit vector ?

3. Find the distance between the planes

$$\overrightarrow{r}.\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right) - 4 = 0$$

and

$$\overrightarrow{r}.\left(6\hat{i} - 9\hat{j} + 18\hat{k}\right) + 30 = 0$$

4. Given that vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ form a triangle such that $\overrightarrow{a} = \overrightarrow{b} + \overrightarrow{c}$. Find p, q, r, s such that area of triangle is $5\sqrt{6}$ where $\overrightarrow{a} = p\hat{i} + q\hat{j} + r\hat{k}$, $\overrightarrow{b} = s\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overrightarrow{c} = 3\hat{i} + \hat{j} - 2\hat{k}$.

5. Find the co-ordinates of the point where the line $\overrightarrow{r} = (-\hat{i} - 2\hat{j} - 3\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 3\hat{k})$ meets the plane which is perpendicular to the vector $\overrightarrow{n} = \hat{i} + \hat{j} + 3\hat{k}$ and at a distance of $\frac{4}{\sqrt{11}}$ from origin.

6. Write the sum of intercepts cut off by the plane $\overrightarrow{r}.\left(2\hat{i} + \hat{j} - \hat{k}\right) - 5 = 0$ on the three axes.

7. Find λ and μ if

$$\left(\hat{i} + 3\hat{j} + 9\hat{k}\right) \times \left(3\hat{i} - \lambda\hat{j} + \mu\hat{k}\right) = \overrightarrow{0}.$$

8. If $\overrightarrow{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector parallel to the vector $\overrightarrow{a} + \overrightarrow{b}$.

9. Find the equation of the plane which contains the line of intersection of the planes

$$\overrightarrow{r}.\left(\hat{i} - 2\hat{j} + 3\hat{k}\right) - 4 = 0 \text{ and}$$
$$\overrightarrow{r}.\left(-2\hat{i} + \hat{j} + \hat{k}\right) + 5 = 0$$

and whose intercept on x-axis is equal to that of on y-axis.

2 Linear Forms

10. Find the equation of plane passing through the points $A\left(3, 2, 1\right)$, $B\left(4, 2, -2\right)$ and $C\left(6, 5, -1\right)$ and hence find the value of λ for which $A\left(3, 2, 1\right)$, $B\left(4, 2, -2\right)$, $C\left(6, 5, -1\right)$ and $D\left(\lambda, 5, 5\right)$ are coplanar.

11. Find the equation of the plane containing two parallel lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x}{4} = \frac{y-2}{-2} = \frac{z+1}{6}$ Also, find if the plane thus obtained contains the line $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-2}{5}$ or not.

12. For what value of k, the system of linear equations

$$\begin{aligned} x + y + z &= 2 \\ 2x + y - z &= 3 \\ 3x + 2y + kz &= 4 \end{aligned}$$

has a unique solution?

13. Show that the four points $A(4, 5, 1)$, $B(0, -1, -1)$, $C(3, 9, 4)$ and $D(-4, 4, 4)$ are coplanar.

14. Find the coordinates of the foot of perpendicular drawn from the point $A(-1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$. Hence find the image of the point A in the line BC.

3 Differentiation

15. Differentiate $(\sin 2x)^x + \sin^{-1} \sqrt{3x}$ with respect to x.

16. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1} x^2$.

17. Determine the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is strictly increasing or strictly decreasing.

18. Find the maximum and minimum values of $f(x) = \sec x + \log \cos^2 x$, $0 < x < 2\pi$.

19. Find the eqaution of normal to the curve $ay^2 = x^3$ at the point whose x coordinate is am^2

20. If $\cos(a + y) = \cos y$ then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a + y)\frac{dy}{dx} = 0$.

21. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left[\frac{6x-4\sqrt{1-4x^2}}{5}\right]$

22. Find the equation of the tangents to the curve $y = x^3 + 2x - 4$ which are perpendicular to line $x + 14y + 3 = 0$

23. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

24. Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ ia an increasing function of θ on $\left[0, \frac{\pi}{2}\right]$.

4 Martrices

25. If A is a square matrix such that $|A| = 5$, write the value of $\left|AA^{\mathrm{T}}\right|$

26. $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -4 \\ 3 & -2 \end{pmatrix}$, find $|AB|$.

27. If $A = \begin{pmatrix} 0 & 3 \\ 2 & -5 \end{pmatrix}$ and $KA = \begin{pmatrix} 0 & 4a \\ -8 & 5b \end{pmatrix}$ find the values of k and a.

28. Ishan wants to donate a rectangular plot of land for a school in his village. When he was asked to give dimensions of the plot, he told that if its length is decreased by $50m$ and breadth is increased by $50m$, then its area will remain same, but if length is decreased by $10m$ and breadth is decreased by $20m$, then its area will decrease by $5300m^2$. Using matrices, find the dimensions of the plot. Also give reason why he wants to donate the plot for a school.

29. Using the properties of determinants, prove that:

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

30. Using elementary row operations, find the inverse of the following matrix :

$$A = \begin{pmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{pmatrix}$$

31. If $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, find α satisfying $0 < \alpha < \frac{1}{2}$ when $A + A^{\mathrm{T}} = \sqrt{2}I_2$, where A^{T} is transpose of A

32. If A is a 3×3 matrix and $|3A| = k|A|$ then write the value of k

33. Using properties of determinants, prove that

$$\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix} = 2xyz(x+y+z)^3$$

34. If

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$$

and $A^3 - 6A^2 + 7A + kI_3 = 0$ find k.

5 Integration

35. Find: $\int \frac{1-\sin x}{\sin x(1+\sin x)} dx$

36. Find: $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

37. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

38. Evaluate : $\int_0^1 \cot^{-1}\left(1 - x + x^2\right) dx$

39. Solve the differential equation: $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^3$

40. Solve the differential equation : $2ye^{\frac{x}{y}} dx + \left(y - 2xe^{\frac{x}{y}}\right) dy = 0$

41. Using integration find the area of the region $(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2$.

42. Find : $\int \frac{(2x-5)e^{2x}}{(2x-3)^3} dx$

43. Find : $\int \frac{x^2+x+1}{(x^2+1)(x+2)} dx$

44. Evaluate : $\int_{-2}^{2} \frac{x^2}{1+5^x} dx$

45. Find : $\int (x+3)\sqrt{3-4x-x^2}dx$

46. Find the particular solution of difference equation :

$$\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x} \tag{1}$$

given that $y = 1$ when $x = 0$.

47. Find the particualr solution of the differential equation

$$2ye^{\frac{x}{y}}dx + (y - 2xe^{\frac{x}{y}})dy = 0$$

given that $x = 0$ when $y = 1$.

48. Using the method of integration, find the area of the triangular region whose vertices are $(2, 2)$, $(4, 3)$ and $(1, 2)$.

6 Function

49. Find k, if

$$f(x) = \begin{cases} k\sin\frac{\pi}{2}(x+1) & , x \leq 0 \\ \frac{\tan x - \sin x}{x^3} & , x > 0 \end{cases}$$

is continous at $x = 0$

50. Let $f : \mathrm{N} \to \mathrm{N}$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : \mathrm{N} \to \mathrm{S}$ is invertible (where S is range of f). Find the inverse of f and hence find $f^{-1}(31)$ and $f^{-1}(87)$.

51. If

$$f(x) = \begin{cases} \frac{\sin(a+1)x + 2\sin}{x} & , x < 0 \\ 2 & , x = 0 \\ \frac{\sqrt{1+bx}-1}{x} & , x > 0 \end{cases}$$

is continuous at $x = 0$, then find the values of a and b.

52. Let $A = R \times R$ and $*$ be a binary operation on A defined by

$$(a, b) * (c, d) = (a + c, b + d)$$

Show that $*$ is commutative and associative. Find the identity element for $*$ on A. Also find the inverse of every element $(a, b) \in A$.

7 Probability

53. There are two bags A and B. Bag A contains 3 white and 4 red balls whereas bag B contains 4 white and 3 red balls. Three balls are drawn at random (without replacement) from one of the bags and are found to be two white and one red. Find the probability that these were drawn from bag B.

54. Three numbers are selected at random (without replacement) from first six positive integers. If X denotes the smallest of the three numbers obtained, find the probability distribution of X. Also find the mean and variance of the distribution.

55. A bag X contains 4 white balls and 2 black balls, while another bag Y contains 3 white balls and 3 black balls. Two balls are drawn (without replacement) at random from one of the bags and were found to be one white and one black. Find the probability that the balls were drawn from bag Y.

56. A and B throw a pair of dice alternately, till one of them gets a total of 10 and wins the game. Find their respective probabilities of winning, if A starts first.

57. Three numbers are selected at random (without replacement) from first six positive integers. Let X denote the largest of the three numbers obtained. Find the probability distribution of X. Also, find the mean and variance of the distribution.

8 Optimization

58. A diet is to contain at least 80 units of Vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available costing 5 rupees per unit and 6 rupees per unit respectively. One unit of food F_1 contains 4 units of vitamin A and 3 units of minerals whereas one unit of food F_2 contains 3 units of vitamin A and 6 units of minerals. Formulate this as a linear programming problem. Find the minimum cost of diet that consists of mixture of these two foods and also meets minimum nutritional requirement.

59. A retired person wants to invest an amount of $50,000$ rupees. His broker recommends investing in two type of bonds A and B yielding 10% and 9% return respectively on the invested amount. He decides to invest at least $20,000$ rupees in bond A and at least $10,000$ rupees in bond B. He also wants to invest at least as much in bond A as in bond B. Solve this linear programming problem graphically to maximise his returns.

60. A typist charges 145 rupees for typing 10 English and 3 Hindi pages, while charges for typing 3 English and 10 Hindi pages are 180 rupees. Using matrices, find the charges of typing one English and one Hindi page separately. However typist charged only 2 rupees per page from a poor student Shayam for 5 Hindi pages. How mcuh less was charged from ths poor boy? which values are reflected in this problem?

9 Algebra

61. Prove that $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

62. Solve the equation for x: $\cos(\tan^{-1}) = \sin\left(\cot^{-1}\frac{3}{4}\right) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

63. Solve for

$$x : \tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$$

64. Prove that

$$\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right) - \tan^{-1}\left(\frac{4x}{1-4x^2}\right) = \tan^{-1}2x;$$

$$|2x| < \frac{1}{\sqrt{3}}$$