HW Oct - 22

1. Solve: $x_0 = 0.0$ $x_1 = 0.4$ $x_2 = 0.7$

$f[x_2] = 6$

$$f[x_1, x_2] = 10 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 10$$

$$\Rightarrow f(x_2) - f(x_1) = 3$$

since $f[x_2] = 6$, we have $f(x_1) = 3$

And $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \dfrac{50}{7}$

$$\Rightarrow f[x_1, x_2] - f[x_0, x_1] = 5 \Rightarrow f[x_0, x_1] = 5$$

therefore $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = 5 \Rightarrow f(x_0) = 1$

so the missing values are: $f[x_0] = 1$, $f[x_1] = 3$

$f[x_0, x_1] = 5$

2. Proof: for function $f(x)$, suppose we're using Lagrange interpolation and Newton interpolation to approximate the function. we have:

$$f(x) = P_n(x) + R_n(x).$$

here, $P_n(x)$ is a polynomial of n degree.
$R_n(x)$ is the residual.

For lagrange interpolation, the residual equals to:

$$R_L(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Considering the uniqueness of the solution to the interpolation polynomial, we'll see that for newton interpolation:

$$R_N(x) = R_L(x) = f[x_0, x_1, \cdots, x_n, x] \cdot \prod_{i=0}^{n}(x-x_i)$$

There fore:

$$f[x_0, x_1, \cdots, x_n, x] \cdot \prod_{i=0}^{n}(x-x_i) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

it gives out:

$$f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

here $\xi \in (\min\{x_0, x_1, \cdots, x_n\}, \max\{x_0, x_1, \cdots, x_n\})$

The uniqueness of the interpolation polynomial can be proven by Cramer's rule.

Suppose the polynomial we use to approximate function f is expressed in the following form:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for $(n+1)$ different points and function values, it gives out a set of linear function consists of $(n+1)$ functions:

$$a_n x_i^n + a_{n-1} x_i^{n-1} + \cdots + a_1 x_i + a_0 = y_i$$

$$i = 0, 1, \cdots, n$$

and $y_i = f(x_i)$

the coefficients a_n consistitute a $(n+1)$ degree Vandermonde matrix, when these points were different to each other, the matrix is nonsingular. Therefore, according to Cramer's rule the solution $(a_n, a_{n-1}, \cdots, a_1, a_0)$ coefficients to the function set is unique, the same for the polynomial.