HW 07 - Nov 05

1. $f(x) = e^{x}$ $\quad x \in [-1,1]$

Maclaurin polynomial : degree $=6$

$$P_6(x) = 1 + e^{x} \cdot x + \frac{e^{x}}{2} x^2 + \frac{e^{x}}{6} x^3 + \frac{e^{x}}{24} x^4 + \frac{e^{x}}{120} x^5 + \frac{e^{x}}{720} x^6$$

$$R_6(x) = \left| \frac{e^{x} \cdot x^7}{5040} \right| \leq \frac{e}{5040} = 5.4 \times 10^{-4}$$

$$P_5(x) = P_6(x) - a_6 \tilde{T}_6(x)$$

$$= \left(1 + e^{x} \cdot x + \frac{e^{x}}{2} x^2 + \frac{e^{x}}{6} x^3 + \frac{e^{x}}{24} x^4 + \frac{e^{x}}{120} x^5 + \frac{e^{x}}{720} x^6\right)$$

$$- \frac{1}{720}\left(64x^6 - 80x^4 + 24x^2 - 1\right)$$

$$\left| P_5(x) - P_6(x) \right| = \left| a_6 \tilde{T}_6(x) \right| = \frac{1}{720} \left| 64x^6 - 80x^4 + 24x^2 - 1 \right|$$

$$\leq \frac{7}{720} < 0.01$$

2. $w(x) = e^{-x}$, $L_0(x) = 1$, $x \in (0, +\infty)$

$$B_1 = \frac{\int_0^{+\infty} x e^{-x} \, dx}{\int_0^{+\infty} e^{-x} \, dx} = \frac{1}{1} = 1 \quad \Rightarrow L_1(x) = x - 1$$

$$B_2 = \frac{\int_0^{+\infty} x e^{-x} (x-1)^2 \, dx}{\int_0^{+\infty} e^{-x} (x-1)^2 \, dx} = \frac{3}{1} = 3$$

$$C_2 = \frac{\int_0^{+\infty} x e^{-x} (x-1)\, dx}{\int_0^{+\infty} e^{-x}\, dx} = \frac{1}{1} = 1$$

$$\therefore L_2(x) = (x - B_2) L_1(x) - C_2 L_0(x)$$

$$= (x-3)(x-1) - 1 = x^2 - 4x + 2$$

$$B_3 = \frac{\int_0^{+\infty} x e^{-x} (x^2-4x+2)^2\, dx}{\int_0^{+\infty} e^{-x} (x^2-4x+2)^2\, dx} = \frac{20}{4} = 5$$

$$C_3 = \frac{\int_0^{+\infty} x e^{-x} (x^2-4x+2)(x-1)\, dx}{\int_0^{+\infty} e^{-x} (x-1)^2\, dx} = \frac{4}{1} = 4$$

$$\therefore L_3(x) = (x - B_3) L_2(x) - C_3 L_1(x)$$

$$= (x-5)(x^2-4x+2) - 4(x-1)$$

$$= x^3 - 9x^2 + 22x - 10 - 4x + 4 = x^3 - 9x^2 + 18x - 6$$

$$\Rightarrow \begin{cases} L_0(x) = 1 \\ L_1(x) = x - 1 \\ L_2(x) = x^2 - 4x + 2 \\ L_3(x) = x^3 - 9x^2 + 18x - 6 \end{cases}$$

3. $x_j = -\pi + \frac{j}{m}\pi \quad j \in [0, 2m-1]$

$\therefore \cos^2 m x_j = \cos^2 (j-m)\pi = \frac{1}{2} + \frac{1}{2}\cos (j-m)\cdot 2\pi$

$\therefore \sum_{j=0}^{2m-1} \cos^2 m x_j = \sum_{j=0}^{2m-1}\left[\frac{1}{2} + \frac{1}{2}\cos (j-m)\cdot 2\pi\right]$

$$= m + \frac{1}{2}\sum_{j=0}^{2m-1} \cos 2m x_j$$

$\sum_{j=0}^{2m-1} \cos 2m x_j$:

$$\sum_{j=0}^{2m-1} e^{i\cdot 2m x_j} = \sum_{j=0}^{2m-1} \cos 2m x_j + i \sum_{j=0}^{2m-1} \sin 2m x_j$$

$$= 2m + 0 = 2m$$

$\therefore \sum_{j=0}^{2m-1} \cos^2 m x_j = m + \frac{1}{2}\cdot 2m = 2m$