HW - Oct 10

1. Proof: Secant method carries out the following iteration:

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Error Analysis: suppose $e_n = x_n - r$

therefore $e_{n+1} = x_n - r - f(x_n)\cdot\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

$$= e_n - f(x_n)\cdot\frac{e_n - e_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)\,e_{n-1} - f(x_{n-1})\cdot e_n}{f(x_n) - f(x_{n-1})} = \frac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{f(x_n) - f(x_{n-1})}e_n e_{n-1}$$

$$= \frac{\dfrac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{x_n - x_{n-1}}}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}\,e_n\cdot e_{n-1}$$

we expand f around its zero point $x = r$:

$$f(r+e) = \underbrace{f(r)}_{\overset{\shortparallel}{0}} + f'(r)\cdot e + \frac{f''(r)}{2}\cdot e^2 + O(e^3)$$

$f(x_n)$ $f(x_{n-1})$ $f(e_n+r)$ $f(e_{n-1}+r)$

therefore: $$\frac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{x_n - x_{n-1}} = \frac{\frac{f(e_n+r)}{e_n} - \frac{f(e_{n-1}+r)}{e_{n-1}}}{e_n - e_{n-1}}$$

$$= \frac{f'(r) + \frac{f''(r)}{2} \cdot e_n - \left(f'(r) + \frac{f''(r)}{2} \cdot e_{n-1}\right) + O(e_n^2)}{e_n - e_{n-1}}$$

$$= \frac{1}{2} f''(r) + O(e_n)$$

$$\Rightarrow \lim_{n \to \infty} \frac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{x_n - x_{n-1}} = \frac{1}{2} f''(r)$$

Apparently, we have: $$\lim_{n \to \infty} \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = f'(r)$$

So we have:

$$e_{n+1} = \lim_{n \to \infty} \frac{\dfrac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{x_n - x_{n-1}}}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} \cdot e_n \cdot e_{n-1}$$

$$\approx \frac{f''(r)}{2 f'(r)} e_n \cdot e_{n-1} = A \cdot e_n \cdot e_{n-1}$$

$$\Rightarrow \ln e_{n+1} = \ln A + \ln e_n + \ln e_{n-1}$$

we let $a_n = \ln e_n \Rightarrow a_{n+1} = A' + a_n + a_{n-1}$

corresponding equation is: $x^2 - x - 1 = 0$

the root of this function is: $\alpha = \frac{1 \pm \sqrt{5}}{2}$

therefore $a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n C_1(n) + \left(\frac{1-\sqrt{5}}{2}\right)^n \cdot C_2(n)$

$$\Rightarrow e_n = e^{a_n} = e^{\left(\frac{1+\sqrt{5}}{2}\right)^n C_1(n)} \cdot e^{\left(\frac{1-\sqrt{5}}{2}\right)^n \cdot C_2(n)}$$

$$\Rightarrow e_n = C^{\frac{\sqrt{5}-1}{2}} \cdot e_{n-1}^{\frac{\sqrt{5}+1}{2}}$$

2. Proof:

$$\lim_{n\to\infty} \left| \frac{p_{n+1} - p}{p_n - p} \right| = 0 \Longleftrightarrow \lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = 0$$

therefore $\lim_{n\to\infty} \left(\frac{p_{n+1} - p_n}{p_n - p}\right) = \lim_{n\to\infty} \left(\frac{p_{n+1} - p}{p_n - p} - 1\right)$

$$= 0 - 1 = -1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{p_{n+1} - p_n}{p_n - p} \right| = 1$$