HW Oct-24

1. Solve: for an integration problem: $\int_{x_0}^{x_2} f(x)\,dx$

Suppose we know the function of x_0, x_1, x_2 where $x_0 < x_1 < x_2$

Then we can use polynomial interpolation to approximate $f(x)$:

$f(x) = P_2(x) + R_2(x)$ where

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$R_2(x) = \frac{f^{(3)}(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$$

Therefore: $\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} P_2(x)\,dx$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} \int_{x_0}^{x_2} (x-x_1)(x-x_2)\,dx$$

$$+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} \int_{x_0}^{x_2} (x-x_0)(x-x_2)\,dx$$

$$+ \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} \int_{x_0}^{x_2} (x-x_0)(x-x_1)\,dx$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} \left[\frac{x^3}{3} - (x_1+x_2)\frac{x^2}{2} + x_1x_2 \cdot x\right]_{x_0}^{x_2}$$

$$+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} \left[\frac{x^3}{3} - \frac{x_0+x_2}{2}\cdot x^2 + x_0x_2 \cdot x\right]_{x_0}^{x_2}$$

$$+ \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}\left[\frac{x^3}{3} - \frac{x_0+x_1}{2}x^2 + x_0x_1\cdot x\right]_{x_0}^{x_2}$$

$$= \frac{1}{(x_0-x_1)(x_0-x_2)}\left[\frac{1}{3}(x_2^3 - x_0^3) - \frac{(x_1+x_2)}{2}(x_2^2 - x_0^2) + x_1x_2(x_2-x_0)\right]f(x_0)$$

$$+ \frac{1}{(x_1-x_0)(x_1-x_2)}\left[\frac{1}{3}(x_2^3 - x_0^3) - \frac{(x_0+x_2)}{2}(x_2^2 - x_0^2) + x_0x_2(x_2-x_0)\right]f(x_1)$$

$$+ \frac{1}{(x_2-x_0)(x_2-x_1)}\left[\frac{1}{3}(x_2^3 - x_0^3) - \frac{x_0+x_1}{2}(x_2^2 - x_0^2) + x_0x_1(x_2-x_0)\right]f(x_2)$$

$$= \frac{1}{x_1-x_0}\left[\frac{1}{3}(x_2^2 + x_0x_2 + x_0^2) - \frac{(x_1+x_2)(x_0+x_2)}{2} + x_1x_2\right]f(x_0)$$

$$+ \frac{x_2-x_0}{(x_1-x_0)(x_1-x_2)}\left[\frac{1}{3}(x_2^2 + x_0x_2 + x_0^2) - \frac{(x_0+x_2)^2}{2} + x_0x_2\right]f(x_1)$$

$$+ \frac{1}{x_2-x_1}\left[\frac{1}{3}(x_2^2 + x_0x_2 + x_0^2) - \frac{(x_0+x_1)(x_0+x_2)}{2} + x_0x_1\right]f(x_2)$$

$$= \frac{1}{x_1-x_0}\left[-\frac{1}{6}x_2^2 + \frac{1}{3}x_0^2 - \frac{1}{6}x_0x_2 + \frac{x_1}{2}(x_2-x_0)\right]f(x_0)$$

$$+ \frac{x_2-x_0}{(x_1-x_0)(x_1-x_2)}\left[-\frac{1}{6}x_2^2 + \frac{1}{3}x_0x_2 - \frac{1}{6}x_0^2\right]f(x_1)$$

$$+ \frac{1}{x_2-x_1}\left[\frac{1}{3}x_2^2 - \frac{1}{6}x_0^2 - \frac{1}{6}x_0x_2 - \frac{x_1}{2}(x_2-x_0)\right]f(x_2)$$

$$= \frac{1}{6(x_0-x_1)}\left[x_2^2 + x_0x_2 - 2x_0^2 - 3x_1(x_2-x_0)\right] f(x_0)$$

$$- \frac{(x_2-x_0)^3}{6(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{1}{6(x_2-x_1)}\left[2x_2^2 - x_0x_2 - x_0^2 - 3x_1(x_2-x_0)\right] f(x_2)$$

There fore :

$$\begin{cases} a = \dfrac{x_2^2 + x_0x_2 - 2x_0^2 - 3x_1(x_2-x_0)}{6(x_0-x_1)} \\ b = -\dfrac{(x_2-x_0)^3}{6(x_1-x_0)(x_1-x_2)} \\ c = \dfrac{2x_2^2 - x_0x_2 - x_0^2 - 3x_1(x_2-x_0)}{6(x_2-x_1)} \end{cases}$$

if $x_2 - x_1 = x_1 - x_0 = h \Rightarrow$

$$\begin{cases} a = \dfrac{1}{12h}(x_2-x_0)^2 = \dfrac{h}{3} \\ b = \dfrac{8h^3}{6h^2} = \dfrac{4}{3}h \\ c = \dfrac{1}{12h}(x_2-x_0)^2 = \dfrac{h}{3} \end{cases}$$

$$\Rightarrow \int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

But: $f(x) = P_n(x) + R_n(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} w_{n+1}(x)$

where $W_{n+1}(x) = (x - x_0) \cdots (x - x_n)$

So we have: $\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} P_n(x)\,dx + \int_{x_0}^{x_n} \frac{f^{(n+1)}(\xi)}{(n+1)!} W_{n+1}(x)$

now we look at:

$$R[f] = \int_{x_0}^{x_n} \frac{f^{(n+1)}(\xi)}{(n+1)!} w_{n+1}(x)\,dx \leq \frac{M_{n+1}}{(n+1)!} \int_{x_0}^{x_n} |w_{n+1}(x)|\,dx$$

suppose $x_i = x_0 + i \cdot h$. then we have:

$$R[f] = \int_0^n \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \prod_{i=0}^{n} (t - i) \cdot h^{n+1} \cdot h\,dt$$

$$= \frac{h^{n+2}}{(n+1)!} \int_0^n f^{(n+1)}(\xi) \left[\prod_{i=0}^{n} (t - i) \right] dt$$

for Simpson's rule: $n = 2$

$$R[f] = \frac{h^4}{3!} \int_0^2 f^{(3)}(\xi) \cdot t(t-1)(t-2)\,dt$$

Noticing that "$t(t-1)(t-2)$"'s value varys from negative to positive, There fore we cannot use mean value theorem to calculate this integral.

We use Hermite interpolation with degree of 3 based on the following requirements:

$P_3(x_0) = f(x_0)$, $P_3(x_2) = f(x_2)$, $P_3(x_1) = f(x_1)$

$P_3'(x_1) = f'(x_1)$, and we have:

$$f(x) = P_3(x) + \frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)^2(x-x_2) \qquad \xi \in [x_0, x_2]$$

so:

$$\Rightarrow R[f] = \int_{x_0}^{x_2} \frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)^2(x-x_2)\,dx$$

when $x \in [x_0, x_2]$, $(x-x_0)(x-x_1)^2(x-x_2) \le 0$

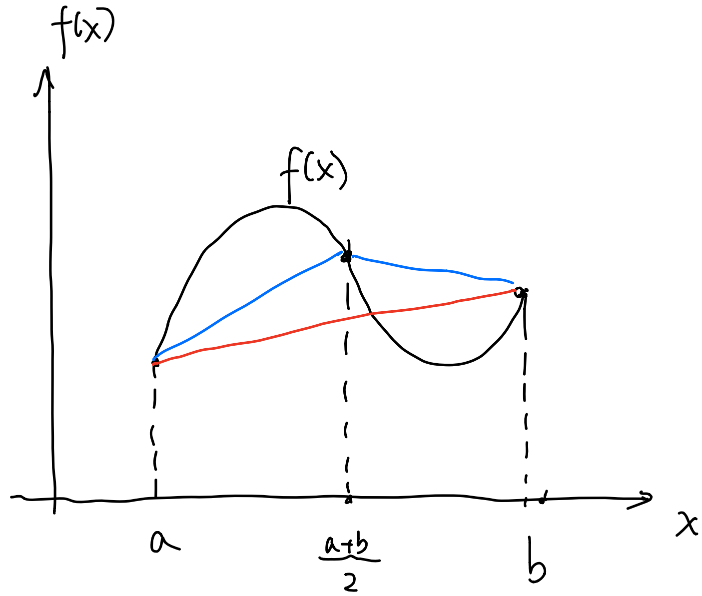
so we can use mean value theorem:

$$R[f] = \frac{f^{(4)}(\eta)}{4!} \int_{x_0}^{x_2} (x-x_0)(x-x_1)^2(x-x_2)\,dx$$

$$= -\frac{(x_2-x_0)^5}{2880} f^{(4)}(\eta) = -\frac{h^5}{90} f^{(4)}(\eta)$$

$$\Rightarrow k = -\frac{h^5}{90}$$

2. Solve:



single application:

$$T(a,b) = \frac{f(a)+f(b)}{2}(b-a) \qquad e_1 = \int_a^b f(x)dx - T(a,b)$$

$$= -\frac{(b-a)^3}{12} f''(\xi)$$

$\xi \in (a,b)$

double application:

$$T(a, \tfrac{a+b}{2}) = \frac{1}{2}\left[f(a) + f(\tfrac{a+b}{2})\right] \cdot \frac{b-a}{2}$$

$$T(\tfrac{a+b}{2}, b) = \frac{1}{2}\left[f(b) + f(\tfrac{a+b}{2})\right] \cdot \frac{b-a}{2}$$

$$e_2 = \int_a^b f(x)dx - \left[T(a, \tfrac{a+b}{2}) + T(\tfrac{a+b}{2}, b)\right]$$

$$= \left[\int_a^{\frac{a+b}{2}} f(x)\,dx - T(a, \tfrac{a+b}{2}) \right] +$$

$$\left[\int_{\frac{a+b}{2}}^{b} f(x)\,dx - T(\tfrac{a+b}{2}, b) \right]$$

$$= - \frac{(\frac{b-a}{2})^3}{12} f''(\xi_1) - \frac{(\frac{b-a}{2})^3}{12} f''(\xi_2)$$

$$= - \frac{(b-a)^3}{96} \left[f''(\xi_1) + f''(\xi_2) \right]$$

where
$\xi_1 \in (a, \frac{a+b}{2})$
$\xi_2 \in (\frac{a+b}{2}, b)$

$$\Rightarrow \left| \int_a^b f(x)\,dx - T(a, \tfrac{a+b}{2}) - T(\tfrac{a+b}{2}, b) \right|$$

$$= \frac{(b-a)^3}{96} \left| f''(\xi_1) + f''(\xi_2) \right|$$

And: $\left| T(a,b) - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b) \right|$

$$= \left| \frac{b-a}{2} [f(a) + f(b)] - \frac{b-a}{4} \left[f(a) + 2f(\tfrac{a+b}{2}) + f(b) \right] \right|$$

$$= \frac{b-a}{} \left| f(a) - 2f(\tfrac{a+b}{2}) + f(b) \right|$$

$$\frac{b-a}{4}\left| f(a) - 2f\left(\frac{a+b}{2}\right) + f(b) \right|$$

$$\le \frac{b-a}{4}\left(\left| f\left(\frac{a+b}{2}\right) - f(a) \right| + \left| f(b) - f\left(\frac{a+b}{2}\right) \right| \right)$$

$$= \frac{b-a}{4}\left(f'(\xi)\,\frac{b-a}{2} + f'(\eta)\,\frac{b-a}{2} \right)$$

$$\le \frac{(b-a)^2}{8}\left| f'(\xi) + f'(\eta) \right| \qquad \text{where } \xi \in \left(a, \frac{a+b}{2}\right),\ \eta \in \left(\frac{a+b}{2}, b\right)$$

Since $\frac{(b-a)^2}{8}\left| f'(\xi) + f'(\eta) \right| \approx \frac{(b-a)^3}{96}\left| f''(\xi_1) + f''(\xi_2) \right|$

So: $\left| T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right) \right| \approx \left| \int_a^b f(x)\,dx - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right) \right|$