CSCI 570 - HW 12

1.

A variation of the satisfiability problem is the MIN 2-SAT problem. The goal in the MIN 2-SAT problem is to find a truth assignment that minimizes the number of satisfied clauses. Give the best approximation algorithm that you can find for the problem.

We assume that no clause contains a variable as well as the complement of the variable. (Such clauses are satisfied regardless of the truth assignment used.)

We give a 2-approximation algorithm as follows:

Let I be an instance of MINSAT consisting of the clause set C_I and variable set X_I. Construct an auxiliary graph $G_I(V_I, E_I)$ corresponding to I as follows. The node set V_I is in one-to-one correspondence with the clause set C_I. For any two nodes v_i and v_j in V_I, the edge (v_i, v_j) is in E_I if and only if the corresponding clauses c_i and c_j are such that there is a variable x belongs to X_I that appears in un-complemented form in c_i and complemented form in c_j, or vice versa.

To construct a truth assignment, we construct an approximate vertex cover V' for G_I such that $|V'|$ is at most twice that of a minimum vertex cover for G_I. Then construct a truth assignment that causes all clauses in $V_I - V'$ to be false. (For a method to find an approximate vertex cover, please refer to section 11.4 in the textbook.)

2.

Consider the following vertex cover algorithm for an undirected graph $G = (V, E)$.

0. Initialize C=Null set.
1. Pick any edge $e = (u, v) \in E$, add u and v to C. Delete all edges incident on either u or v.
2. If E is empty, output C and exit. Otherwise, go to step 1.

Show that C is a vertex cover and that the size of C is at most twice as big as the minimum vertex cover. (Thus we have a 2-approximation).

Every edge $e \in E$ was either picked in step 1 or deleted in step 1. If it was picked, then both its edges were added to C and is hence covered. If it was deleted, then it already had at least one incident vertex contained in C and is hence covered. Thus C is indeed a vertex cover.

Let M be the set of edges picked in step 1. let C_{opt} be an minimal vertex cover. Observe that no two edges in M share a vertex (When we pick an e, the deletion step ensures that no other edge that shares an edge with e is ever picked). Thus for each edge $e \in M$, C_{opt} has to contain at least one of its incident vertex (Moreover these vertex are distinct as reasoned in the previous line). Hence

$$|C_{opt}| \geq |M|.$$

From the algorithm, we have that $|C| = 2|M|$. Thus

$$|C| \leq 2|C_{opt}|$$

3.

720 students have pre-enrolled for the "Analysis of Algorithms" class in Fall. Each student must attend one of the 16 discussion sections, and each discussion section i has capacity for D_i students. The happiness level of a student assigned to a discussion section i is proportionaate to $\alpha_i(D_i - S_i)$, where α_i is a parameter reflecting how well the air-conditioning sysem works for the room used for section i (the higher the better), and S_i is the actual number of students assigned to that section. We want to find out how many students to assign to each section in order to maximize total student happiness.

Express the problem as a linear programming problem.

Our variables will be the S_i. Our objective function is:

$$\text{maximize} \sum_{i=1}^{16} \alpha_i(D_i - S_i)$$

$$\text{subject to: } D_i - S_i \geq 0 \text{ for } 0 < i \leq 16$$
$$S_i \geq 0 \text{ for } 0 < i \leq 16$$
$$\sum_{i=1}^{16} S_i = 720$$

Grading:

- Objective function (4 points)
- Each constraint (2 points)

4. Show that vertex cover remains **NP**-Complete even if the instances are restricted to graphs with only even degree vertices.

Let $< G; K >$ be an input instance of VERTEX-COVER, where $G = (V; E)$ is the input graph.

Because each edge in E contributes a count of 1 to the degree of each of the vertices with which it is incident, the sum of the degrees of the vertices is exactly $2|E|$, an even number. Hence, there is an even number of vertices in G that have odd degrees.

Let U be the subset of vertices with odd degrees in G.

Construct a new instance $< \bar{G}; k + 2 >$of VERTEX-COVER, where $\bar{G} = (V0; E0)$ with $V0 = V \cup \{x, y, z\}$ and $E0 = E \cup \{(x, y), (y, z), (z, x)\} \cup \{(x, v)|v \in U\}$. In words, we make a triangle with the three new vertices, and then connect one of them (say x) to all the vertices in U.

The degree of every vertex in $V0$ is even. Since a vertex cover for a triangle is of (minimum) size 2, it is clear that $\bar{G}$ has a vertex cover of size $k + 2$ if and only if G has a vertex cover of size k.

5.

A set of n space stations need your help in building a radar system to track spaceships traveling between them. The i^{th} space station is located in 3D space at coordinates (x_i, y_i, z_i). The space stations never move. Each space station i will have a radar with power r_i, where r_i is to be determined. You want to figure how powerful to make each space station's radar transmitter, so that whenever any spaceship travels in a straight line from one station to another, it will always be in radar range of either the first space station (its origin) or the second space station (its destination). A radar with power r is capable of tracking space ships anywhere in the sphere with radius r centered at itself. Thus, a space ship is within radar range through its strip from space station i to space station j if every point along the line from (x_i, y_i, z_i) to (x_j, y_j, z_j) falls within either the sphere of radius r_i centered at (x_i, y_i, z_i) or the sphere of radius r_j centered at (x_j, y_j, z_j). The cost of each radar transmitter is proportional to its power, and you want to minimize the total cost of all of the radar transmitters. You are given all of the $(x_1, y_1, z_1), ..., (x_n, y_n, z_n)$ values, and your job is to choose values for $r_1, ..., r_n$. Express this problem as a linear program.

(a) Describe your variables for the linear program (3 pts).

Solution: r_i=the power of the i^{th} radar transmitter., i=1,2,...n (3 pts)

(b) Write out the objective function (5 pts).

Solution: Minimize $r_1 + r_2 + ... + r_n$.

Defining the objective function without mentioning r_i: -3 pts

(c) Describe the set of constraints for LP. You need to specify the number of constraints needed and describe what each constraint represents (8 pts).

Solution: $r_i + r_j \geq \sqrt{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)}$. Or, $r_i + r_j \geq d_{i,j}$ for each pair of stations i and j, where $d_{i,j}$ is the distance from station i to station j (6 pts).

We need $\sum_{i=1}^{n-1} i = (n^2 - n)/2$ constraints of inequality (2pts).