CSCI 570 - Spring 2019 - HW 11 Solutions

1 Practice Problems

1. State True/False. Let A be *NP-complete*, and B be *NP-hard*. Then, $A \leq_p B$.

   True. A is *NP-complete* and thus, $A \in NP$. Then, since B is *NP-hard*, we have $A \leq_p B$ by definition.

2. State True/False. If $P = NP$, then every NP-hard problem can be solved in polynomial time.

   False. If $P = NP$, only the *NP-complete* problems will be surely poly-time solvable (because they are in NP), but not necessarily all the *NP-hard* problems.

3. Given an undirected graph $G = (V, E)$, a clique is a subset $A \subseteq V$ such that For every pair of vertices $u, v \in A$, if $u = v$, then $(u, v) \in E$. Given a graph and an integer m, the *CLIQUE* problem is to decide if the graph has a clique of size m. The *HALF-CLIQUE* problem is to decide if a given graph $G = (V, E)$ has a clique of size at least $\frac{|V|}{2}$.

   First, show that *CLIQUE* is *NP-complete* by showing a reduction from the *INDEPENDENT-SET* problem which is known to be *NP-complete*. Further, show that *HALF-CLIQUE* is *NP-complete* by showing a reduction from *CLIQUE*.

   Solution. Given a set of vertices A as the certificate, it is easy to verify (by iterating through pairs of vertices in A) that it is a clique, and that $|A| = k$ or similarly $|A| = \frac{|V|}{2}$. Hence, *CLIQUE* and *HALF-CLIQUE* are both in NP.

   First, we prove that *CLIQUE* is *NP-hard*. For a graph $G = (V, E)$, its complement $\bar{G} = (V, \bar{E})$ is defined so that an edge $e \in \bar{E}$ if and only if $e \notin E$. The key observation is that a set of vertices B is an independent set in G if and only if it is a clique for its complement $\bar{G}$. Thus an *INDEPENDENT-SET* instance $\langle G, k\rangle$ can be reduced to the *CLIQUE* problem by mapping it to the *CLIQUE* instance $\left\langle \bar{G}, m = k\right\rangle$, and the reduction is clearly poly-time and poly-size. Thus, *INDEPENDENT-SET* $\leq_p$ *CLIQUE*.

Next, we show that *HALF-CLIQUE* is *NP-hard*. Given a *CLIQUE* instance $\langle G = (V, E), m\rangle$ we reduce it to a *HALF-CLIQUE* instance. If $m = \frac{|V|}{2}$, then we already have a *HALF-CLIQUE* instance.

If $m < \frac{|V|}{2}$, we add $|V| - 2m$ new vertices to G. Then, we add an edge between every distinct pair of new vertices and also an edge between every new vertex and every existing vertex as well. Call this graph $G' = (V', E')$. G' has $2m$ vertices. Now, if a set of vertices B is a clique in G, then $B \cup (V' \setminus V)$ is a clique in G' and vice versa. Hence, G has a clique of size at least m if and only if G' has a clique of size at least $m + (|V|2m) = |V|m = \frac{|V'|}{2}$.

On the other hand, if $m > \frac{|V|}{2}$, we add $2m - |V|$ new vertices to G and do not introduce any new edges. Call this graph $G' = (V', E')$. G' has $2m$ vertices. Now, G has a clique of size at least m if and only if G' had a clique of size $m = \frac{|V'|}{2}$.

It's easy to see that the reductions are poly-time computable and of poly-size. This shows *CLIQUE* $\leq_p$ *HALF-CLIQUE*.

4. Given an undirected graph with positive edge weights, the *BIG-HAM-CYCLE* problem is to decide if it contains a Hamiltonian cycle C such that the sum of weights of edges in C is at least half of the total sum of weights of edges in the graph. Show that *BIG-HAM-CYCLE* is *NP-complete*. You are allowed to use the fact that deciding if an undirected graph has a Hamiltonian cycle is *NP-complete*.

   Solution. The certifier takes as input an undirected graph (the *BIG-HAM-CYCLE* instance) and a sequence of edges (certificate). It verifies that the sequence of edges is a Hamiltonian cycle and that the total weight of the cycle is at least half the total weight of the edges in the graph. Thus *BIG-HAM-CYCLE* is in *NP-complete*.

   We claim that *HAM-CYCLE* is polynomial time reducible to *BIG-HAM-CYCLE*. To see this, given an undirected graph $G = (V, E)$ (instance of *HAM-CYCLE*), fix a node u. For a neighbor v of u, set the weight of edge uv to $|E|$ and assign the rest of the edges a weight of 1. This gives a weighted graph G' as an instance of *BIG-HAM-CYCLE*. G' is a yes instance if and only if G has a Hamiltonian cycle containing the edge uv. Now, G has a hamiltonian cycle if and only if it has a hamiltonian cycle containing uv for some 2 neighbors v of the fixed node u. Hence, by repeating the above procedure for every neighbor v of the fixed vertex u, we can decide if G has a Hamiltonian cycle, (if and only if we find a neighbor v for which the resultant graph G' is a YES instance of *BIG-HAM-CYCLE*. Since these are only $|V|$ calls to the *BIG-HAM-CYCLE* black box and the other steps are also poly-time, *BIG-HAM-CYCLE* must also be NP-hard.