Analysis 3 (D-ITET) HS25

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Basics

in general: $u = u(x,y,z)$

Laplacian: $\Delta u = u_{xx} + u_{yy} + u_{zz}$

Gradient: $\nabla u = (u_x, u_y, u_z)^T$

Burger's eq.: $u_t - u u_x = 0$ Poisson eq.: $\Delta u = f(u)$

Wave eq.: $u_{tt} - c^2 \Delta u = 0$ Transport eq.: $u_y + c u_x = 0$

Heat equ.: $\Delta u - k u_t = 0$ Laplace eq.: $\Delta u = 0$

Well-posed problem: (otherwise ill-posed)

1) Existence: Problem has a solution
2) Uniqueness: Probem has only one solution
3) Stability: Small change in problem leads to small change in solution.

- Strong solution: All derivatives in the PDE exist and are continuous on the whole domain of the PDE
- Weak solution: ∃ points in the domain where the derivatives do not exist/are not continuous. (cannot be plugged into the PDE)

Classification properties of PDEs

- Order: Order of the highest derivative
- Linearity: If all coeffitients of u (and its derivatives) do not depend on u or its derivatives → allows superposition
- Quasi-linearity: Linearity in highest order derivative term
- Homogeneity: $f(x) = 0$

ODE solving methods:

linear ODE w. constant coeff. ($a_n x^{(n)} + \dots + a_0 x = f$)

→ $x = x_p + x_h \implies x_h = e^{\lambda t}$ (Ansatz), $x_h = A e^{\lambda_1 t} + \dots$

$\implies x_p \sim$ similar to f

separable ODE ($\frac{dy}{dx} = f(x)g(y)$)

$G(y) = \int \frac{1}{g(y)} dy = \int f(x) dx = F(x)$ (solve for y)

Integrating factor ($x'(t) + a(t)x(t) = b(t)$)

Let $A'(t) = a(t)$: and with $x(0) = x_0$

$$x(t) = e^{-A(t)} \int_0^t e^{A(\tau)} b(\tau) d\tau + e^{A(0) - A(t)} \cdot x_0$$

Classification of 2nd order PDE

$L[u] := a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$

$\delta = b^2 - ac$ Examples:

Hyperbolic:	$\delta > 0$	(Wave eq.)
Parabolic:	$\delta = 0$	(Heat eq.)
Elliptic:	$\delta < 0$	(Laplace eq.)

Method of Characteristics

↳ for 1st order (quasi)linear PDEs

$$\begin{cases} a(x,y) u_x(x,y) + b(x,y) u_y(x,y) = c(x,y,u) \\ u(x_0, y_0) = \tilde{u}_0 \end{cases}$$

1) Initial curve: $\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s))$
2) Solve ODE system: $\equiv (x(s,0), y_0(s,0), \tilde{u}(s,0))$

$$\begin{cases} x_t(t,s) = a(x,y) \\ y_t(t,s) = b(x,y) \\ u_t(t,s) = c(x,y,u) \end{cases}$$

3) solve for $t(x,y)$ and $s(x,y)$
4) $u(x,y) = \tilde{u}(t,s)$

Transversality condition:

$$\det \begin{bmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{bmatrix} = \det \begin{bmatrix} a_0(s) & b_0(s) \\ x_0'(s) & y_0'(s) \end{bmatrix} \neq 0$$

⇒ Gives local existence and uniqueness around $(x_0(s), y_0(s))$

Obstacles to global existence:

1) solutions can blow up in finite time
2) characteristics cross the initial curve $\Gamma(s)$ more than once
3) derivatives cross each other

Conservation laws:

Def.: PDEs describing the evolution of conserved quantities. x is the spacial variable, t/y the time variable.

$$\begin{cases} u_y + c(u) u_x = 0 \Leftrightarrow u_y + \partial_x f(u) = 0, \quad (x,y) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) \end{cases}$$

Flux function: $f(u): \mathbb{R} \to \mathbb{R}$, Velocity $c(u) = f'(u)$

- characteristics are straight lines
- Traveling waves: satisfy this equation:

$\forall y \in [0, y_c) \quad u(x,y) = u_0(x - c(u) \cdot y)$

Burger's eq.: $c(u) = u$, $f(u) = \frac{1}{2} u^2$

Transport eq.: $c \in \mathbb{R}$, $f(u) = c \cdot u$

Critical time

The transversality condition garantees local existence and uniqueness on $[0, y_c)$.

$$y_c = \begin{cases} -\inf\limits_{s \in \mathbb{R}} \frac{1}{c'(u_0(s)) \cdot u_0'(s)} & \partial_s c(u_0(s)) < 0 \\ \infty & \partial_s c(u_0(s)) \geq 0 \; \forall s \in \mathbb{R} \end{cases}$$

- If $u(x,0)$ never decreases → no critical time
- If $y_c < 0$, no critical time

↳ after y_c ⇒ weak solution

The notion of weak solutions

$$\int_a^b [u(x,y)]_{y=y_1}^{y=y_2} dx = -\int_{y_1}^{y_2} [f(u(x,y))]_{x=a}^{x=b} dy$$

→ holds even if u isn't smooth on the whole domain D

- weak solutions satisfy the PDE in each subdomain D_i of D ($D = \bigcup_{i=1} D_i$).
- boundaries between D_i are called shocks

Rankine-Hugoniot condition

$x = \gamma(y)$ a smooth curve across which $u(x,y)$ is discontinuous. γ is called shock wave.

$$\gamma'(y) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}, \quad u^\pm = \lim_{x \to \gamma(y)^\pm} u(x,y)$$

(= speed of shock wave)

- u^+/u^- solution on the right/left side

General approach

1) Find y_c 2) use MoC to find strong solution
3) for $y \geq y_c$: Find characteric crossing (x_0, y_c)

→ solve: $x_0 = \gamma(y) = \gamma'(y_c) y_c + b$ (Find b)

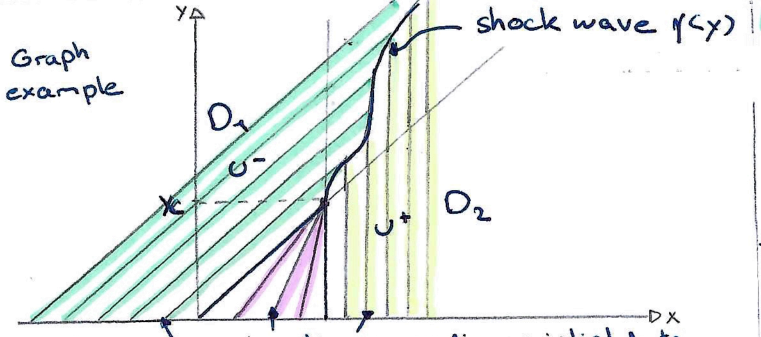
→ $u(x,y) = \begin{cases} u^-, & x < \gamma(y) \\ u^+, & x > \gamma(y) \end{cases}$

4) use entropy condition if there is no unique weak solution:

$c(u^+) < \gamma'(y) < c(u^-) \Leftrightarrow f'(u^+) < \gamma'(y) < f'(u^-)$

(→ characteristics must enter the shock)

Graph example



different characteristics depending on intial data

Hyperbolic PDEs: The Wave eq. in $\mathbb{R}$

General solution (even if F/G are not smooth):

$$u(x,t) = F(x+ct) + G(x-ct)$$

- G is called forward wave (if $c>0$)
- F is called backward wave

$\rightarrow$ are constant along the characteristics $x+ct = \alpha \in \mathbb{R}$ and $x-ct = \beta \in \mathbb{R}$

Let $\xi(x,t) = x+ct$, $\eta(x,t) = x-ct$ and $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$. Recall: $w_t = w_\xi \frac{\partial \xi}{\partial t} + w_\eta \frac{\partial \eta}{\partial t} \Rightarrow u_{tt} = c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta})$

and $u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \Rightarrow u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta} = 0$

$w_{\xi\eta} = 0$ is the canonical form of the wave eq.

D'Alembert's formula

The solution to:
$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

is given by:

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau$$

Simple inhomogenities ($F(x)$ or $F(t) = F(x,t)$)

$\rightarrow$ find $v(x)/v(t)$ with $F(x) = -c^2 v_{xx}$ / $F(t) = v_{tt}$

$\rightarrow$ use superposition: $u(x,t) = v(x,t) + w(x,t)$

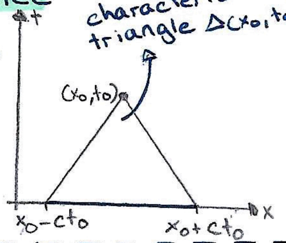
Theorems:

- Uniqueness: The 1D-waveeq. has a unique sol.
- Singularities propagate along characteristics
- If g and f are even/odd/periodic $\Rightarrow$ u is also even/odd/periodic (possibility to extend f(x) and g(x) oddly or evenly)

The domain of dependence

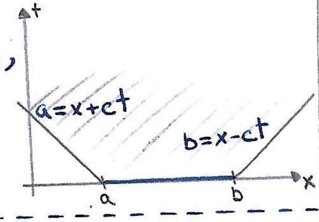
Given by $[x_0 - ct_0, x_0 + ct_0]$ for a point (x_0, t_0)

$\rightarrow$ value changes if the values in the interval change



The region of influence

All points satisfying $x-ct \leq b$, $x+ct \geq a$ are dependent on the intial condition in $[a,b]$



Method of Separation of Variables

Ansatz always the same: Insert $u(x,t) = X(x)T(t)$ into the PDE and decouple it into two ODEs.

$X''(x) = -\lambda X(x)$ is reoccuring. Solution are:

$$X(x) = \begin{cases} A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x), & \lambda > 0 \\ C + Dx, & \lambda = 0 \\ A\sinh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x), & \lambda < 0 \end{cases}$$

Boundary conditions used to find the non-trivial solutions:

Dirichlet: $u(0,t) = u(L,t) = 0$ $\quad X_n(x) = \sin(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

Neumann: $u_x(0,t) = u_x(L,t) = 0$ $\quad X_n(x) = \cos(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

Mixed 1: $u(0,t) = u_x(L,t) = 0$ $\quad X_n(x) = \sin(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{\frac{n+1}{2}\pi}{L}\right)^2$

Mixed 2: $u_x(0,t) = u(L,t) = 0$ $\quad X_n(x) = \cos(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{\frac{n+1}{2}\pi}{L}\right)^2$

Homogeneous 1D heat equation

Let $k \in \mathbb{R}^+$ be the constant of diffusivity:

$$\begin{cases} u_t - k u_{xx} = 0, & (x,t) \in (0,L)\times(0,\infty) \\ \text{Boundary condition}, & t \in \mathbb{R}^+ \\ u(x,0) = f(x), & x \in (0,L) \end{cases}$$

Separation of variable gives:

$$\begin{cases} X'' = -\lambda X, & x \in (0,L) \\ T' = -\lambda k T, & t > 0 \end{cases} \Rightarrow T_n(t) = A_n e^{-k\lambda_n t}$$

1) General solution:

- Dirichlet & Mixed 1: $u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$
- Neumann & Mixed 2: $u(x,t) = A_0 + \sum_{n=1}^{\infty} X_n(x)T_n(t)$

2) Apply initial condion $u(x,0)$:

$A_n = \frac{2}{L}\int_0^L f(x)\dots dx$

Homogeneous 1D wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in [0,L]\times[0,\infty) \\ \text{Boundary condition}, & t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$

Separation of variable gives:

$$\begin{cases} X'' = -\lambda X, & x \in (0,L) \\ T'' = -\lambda c^2 T, & t > 0 \end{cases}$$

$\Rightarrow T_n = A_n\cos(ct\sqrt{\lambda_n}) + B_n\sin(ct\sqrt{\lambda_n})$

1) general solution:

- Dirichlet & Mixed 1: $u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$
- Neumann & Mixed 2: $u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} X_n T_n$

2) Apply initial condition $u(x,0)$ and $u_t(x,0)$

$A_n = \frac{2}{L}\int_0^L f(x)\dots dx$ $\quad B_n = \frac{2}{cn\pi}\int_0^L g(x)\dots dx$

$\left(B_0 = \frac{2}{L}\int_0^L g(x)dx\right.$

Inhomogeneous boundary condition

- Find a $w(x,t)$ that satisfies the boundary condition.
- Define $u = w + v$
- Solve for v, combine solutions

Inhomogeneous heat/wave eq.

1) Use solution for the homogeneous case without solving for T

2) Express the inhomogenity in the corresponding basis:

ex: $\sum_n T_n'' X_n - c^2 \sum_n T_n X_n'' = h(x,t) = \sum_n c_n(t) X_n$

$\sum_n T_n(0) X_n = f(x) = \sum_n a_n X_n$

$\sum_n T_n'(0) X_n = g(x) = \sum_n b_n X_n$

$$\Rightarrow \begin{cases} T_n'' + c^2\left(\frac{n\pi}{L}\right)^2 T_n = c_n \\ T_n(0) = \frac{2}{L}\int_0^L X_n f(x)\,dx \\ T_n'(0) = \frac{2}{L}\int_0^L X_n g(x)\,dx \end{cases}$$

Uniqueness with energy method

- Let u_1 and u_2 be solution to the PDE
- Energy function: $E(t) = \int_0^L w_t^2 + c^2 w_x^2\,dx$ (for wave eq.)
- Show that $\frac{d}{dt}E(t) = 0$ and $E(0) = 0$

$\Rightarrow w = 0 \Rightarrow u_1 = u_2$ (solution is unique)

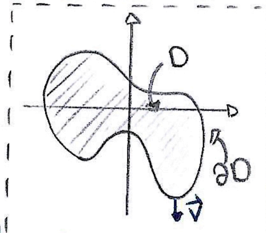
Elliptic PDEs - Laplace & Poisson eq.

$u(x,y)$ is called harmonic if it solves the Laplace eq.

Dirichlet: $\begin{cases} \Delta u = p(x,y) & (x,y)\in D \\ u = g(x,y) & (x,y)\in\partial D \end{cases}$

Neumann: $\begin{cases} \Delta u = p(x,y), & (x,y)\in D \\ \partial_v u = \vec{v}\cdot\nabla u = g(x,y), & (x,y)\in\partial D \end{cases}$

Mixed: $\begin{cases} \Delta u = p(x,y), & (x,y)\in D \\ u + \partial_v u = g(x,y), & (x,y)\in\partial D \end{cases}$



$\bar{D} = D \cup \partial D$

$\vec{v}$: normal vector

∂_v: directional derivative with respect to v

Compability condition for Neumann:

necessary for existence of a solution:

$$\int_D \Delta u\, dS = \int_{\partial D} \partial_v u$$

for circles:

$$\int_{\partial D} \partial_v u = \int_0^{2\pi} \partial_v u(\gamma(s))\,\|\dot\gamma(s)\|_2\,ds \quad \gamma(s) = \begin{pmatrix} r\cos s \\ r\sin s \end{pmatrix}$$

Maximum Principles

Let $x_0 \in D$ be a maximum point, then:

$Du(x_0) = 0$ (gradient) and $D^2 u(x_0) \leq 0$ (Hessian)

Weak maximum principle

Let D be a bounded domain and let $u(x,y) \in C^2(D) \cap C(\bar{D})$ be a harmonic function in D. The maximum (and minimum) of u in $\bar{D}$ is achieved on ∂D:

$$\max_{\bar{D}} u = \max_{\partial D} u \qquad \min_{\bar{D}} u = \min_{\partial D} u$$

Mean value principle

Let u be a harmonic function on D and let $B_R(x_0,y_0) \subset D$ be a ball of radius R $\Leftrightarrow$

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,d\theta$$

The strong maximum principle

Let u be a harmonic in D and D a connected subset of $\mathbb{R}^2$, the Dirichlet problem

$\begin{cases} \Delta u = f & \text{, in } D \\ u = g & \text{, on } \partial D \end{cases}$ has at most one solution $u \in C^2(D) \cap C(\bar{D})$

Theorem: Let $u_1, u_2 \in C^2(D) \cap C(\bar{D})$ solve

$\begin{cases} \Delta u_1 = 0 & \text{in } D \\ u_1 = g_1 & \text{on } \partial D \end{cases}$ $\begin{cases} \Delta u_2 = 0 & \text{in } D \\ u_2 = g_2 & \text{on } \partial D \end{cases}$

$$\Rightarrow \max_{\bar{D}} |u_1 - u_2| = \max_{\partial D} |g_1 - g_2|$$

Maximum principle for parabolic equations

$0 = u_t - k u_{xx} \Leftrightarrow u_t = k\Delta u$

problem lives in space time $Q_T = [0,T]\times D$, $x\in D$, $t\in[0,T]$. Define the parabolic boundary

$$\partial_p Q_T = \{\{0\}\times D\} \cup \{[0,T]\times\partial D\}$$

$\Rightarrow$ u achieves its maximum and minimum on $\partial_p Q_T$

Laplace eq. in rectangular and circular domains

Combinations of Dirichlet and Neumann also possible. $\rightarrow$ choose the right bases!

Dirichlet problem with rectangular domain

$$\begin{cases} \Delta u = 0 & , (x,y)\in(a,b)\times(c,d) \\ u(a,y) = f(y) & , (x,y)\in\{a\}\times[c,d] \\ u(b,y) = g(y) & , (x,y)\in\{b\}\times[c,d] \\ u(x,d) = h(x) & , (x,y)\in[a,b]\times\{d\} \\ u(x,c) = k(x) & , (x,y)\in[a,b]\times\{c\} \end{cases}$$

$\rightarrow$ split into two subproblems $u = u_1 + u_2$

$\begin{cases} \Delta u_1 = 0 \\ u_1(a,y) = f(y) \\ u_1(b,y) = g(y) \\ u_1(x,d) = 0 \\ u_1(x,c) = 0 \end{cases}$ $\quad$ $\begin{matrix} \Delta u_2 = 0 \\ u_2(a,y) = 0 \\ u_2(b,y) = 0 \\ u_2(x,d) = h(x) \\ u_2(x,c) = k(x) \end{matrix}$

Separation of variable (switch x & y for u_2)

$X'' - \lambda X = 0$ $\quad$ $\begin{cases} Y'' + \lambda Y = 0 \\ Y(c) = Y(d) = 0 \end{cases}$

1) General solutions:

$$u_1 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{d-c}(y-c)\right)\left[A_n \sinh\left(\frac{n\pi}{d-c}(x-a)\right) + B_n \sinh\left(\frac{n\pi}{d-c}(x-b)\right)\right]$$

$$u_2 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b-a}(x-a)\right)\left[C_n \sinh\left(\frac{n\pi}{b-a}(y-c)\right) + D_n \sinh\left(\frac{n\pi}{b-a}(y-d)\right)\right]$$

2) Use initial conditions to find coefficients.

Issue for numerical approach

By splitting the problem the boundary data may not be continuous anymore. The trick to avoid this issue is defining

$\bar{u} = u - P$ where $P(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$ is harmonic. $\rightarrow$ choose a_i so that at the corners $\bar{u} = 0$.

Neumann problem with rectangular domain

$u = C \in \mathbb{R}$ is always a solution $\rightarrow \infty$ solutions due to superposition.

$$\begin{cases} \Delta u = 0 & , (x,y)\in(a,b)\times(c,d) \\ u_x(a,y) = f(y) & , (x,y)\in\{a\}\times[c,d] \\ u_x(b,y) = g(y) & , (x,y)\in\{b\}\times[c,d] \\ u_y(x,c) = h(x) & , (x,y)\in[a,b]\times\{d\} \\ u_y(x,d) = k(x) & , (x,y)\in[a,b]\times\{c\} \end{cases}$$

Compability condition:

$$\int_c^d [u_x(x,y)]_{x=a}^{x=b}\,dy + \int_a^b [u_y(x,y)]_{y=c}^{y=d}\,dx = 0$$

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Once again we want to split the problem into subproblems. But this can lead to problems in the compability condition. Therefore, we add a harmonic polynomial:

$\rightarrow v = u + \alpha(x^2 - y^2) \rightarrow v = v_1 + v_2$

$$\begin{cases} \Delta v_1 = 0 \\ (v_1)_x(a,y) = f(y) + 2\alpha a \\ (v_1)_x(b,y) = g(y) + 2\alpha b \\ (v_1)_y(x,c) = 0 \\ (v_1)_y(x,d) = 0 \end{cases} \qquad \begin{matrix} \Delta v_2 = 0 \\ (v_2)_x(a,y) = 0 \\ (v_2)_x(b,y) = 0 \\ (v_2)_y(x,c) = k(x) - 2\alpha c \\ (v_2)_y(x,d) = h(x) - 2\alpha d \end{matrix}$$

using the compatibility condition to find α:

$$\int_c^d g + 2\alpha b - \int_c^d f + 2\alpha a = 0, \quad \alpha = \frac{1}{2(b-a)(d-c)} \int_c^d f - g$$

1) general solution given by:

$$v_1 = A_0 x + B_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{d-c}(y-c)\right) \cdot \left[A_n \cosh\left(\frac{n\pi}{d-c}(x-a)\right) + B_n \cosh\left(\frac{n\pi}{d-c}(x-b)\right)\right]$$

$$v_2 = C_0 y + D_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{b-a}(x-a)\right) \cdot \left[C_n \cosh\left(\frac{n\pi}{b-a}(x-c)\right) + D_n \cosh\left(\frac{n\pi}{b-a}(x-d)\right)\right]$$

2) use initial conditions to find coeff.

Laplace eq. in circular domains

$\rightarrow$ for D with radial symmetry

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad A_D = \int_0^r \int_0^{2\pi} w(r,\theta) \cdot r \, d\theta \, dr$$

where $r^2 = x^2 + y^2$ and $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \Delta u = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}$$

Approach for disk and annulus

Let the domain be $D := \{r \in [a,b], \theta \in [0,2\pi)\}$

With separation of variable $w(r,\theta) = R(r)\Theta(\theta)$

$$\begin{cases} r^2 R'' + rR' = \lambda R \\ \Theta'' = -\lambda\Theta \end{cases} \qquad \begin{matrix} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{matrix}$$

The solutions are given by:

$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$, $n \in \mathbb{N}$

$$R_n(r) = \begin{cases} c_0 + D_0 \log(r) & n = 0 \\ c_n r^n + D_n r^{-n} & n \neq 0 \end{cases}$$

1) general solution is:

$$w = A_0 + B_0 \log(r) + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) + \sum_{n=1}^{\infty} r^{-n} (C_n \cos(n\theta) + D_n \sin(n\theta))$$

$\Rightarrow$ Δ diregard r^{-n} and $\log(r)$ for disk, as they have a singularity at $r=0$

2) use boundary condition to find coeffitients

Approach for disk sectors or annulus sector

Let the domain $D = \{r \in [a,b], \theta \in [0,\gamma]\}$ with $\gamma \in (0, 2\pi)$. We assume $w(r,0) = w(r,\gamma) = 0$ $\forall r \in (a,b)$

Same operation but $\Theta(0) = \Theta(\gamma) = 0$:

$\Theta_n(\theta) = A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) \qquad \lambda_n = \left(\frac{n\pi}{\gamma}\right)^2$

$R_n(r) = C_n r^{n\pi/\gamma} + D_n r^{-n\pi/\gamma}$

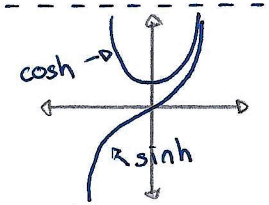
1) General solution becomes

$$w = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{n\pi/\gamma} + B_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{-n\pi/\gamma}$$

2) use boundary conditions for coeff.

Appendix

Trigonometry



sinh is odd, cosh is even

$\sinh(0) = 0$, $\cosh(0) = 1$

$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

$\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x)$

$\cos(x + \frac{\pi}{2}) = -\sin(x) \qquad \sin(x + \frac{\pi}{2}) = \cos(x)$

$\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

$\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$

$\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y) + \sin(x+y))$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$

$\cos^3(x) = \frac{1}{4}(3\cos(x) + \cos(3x))$

$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}$

$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}$

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
sin	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	-	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0

Analysis 2 stuff:

- Min/Max von f auf dem Rand mit Parametrisierung $\gamma: [a,b] \rightarrow \mathbb{R}^n$ erfüllen $\frac{d}{dt}(f \circ \gamma)(t) = 0$
- Richtungsableitung:

$$\partial_v f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h}$$

- Wegintegral über Skalarfeld:

$$\int_\gamma f(x)dx = \int_a^b f(\gamma(t)) \|\dot{\gamma}(t)\| dt$$

- Koordinatentransformation:

$$\int_{\Phi(\Omega)} f(y)dy = \int_\Omega f(\Phi(x)) |\det d\Phi| dx$$

Polar: $r \, dr d\theta$ Kugel: $r^2 \sin\theta \, dr d\theta d\phi$

- Kettenregel: 1D: $(f(u(s)))_s = \partial_u f(u) \partial_s u$

2D: $\frac{\partial}{\partial s} f(x(s,t), y(s,t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$

$\frac{\partial^2}{\partial s^2} f = \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial s^2}$ (only for smooth functions)