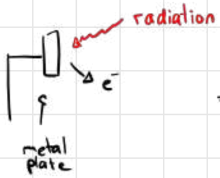
Lecture 1

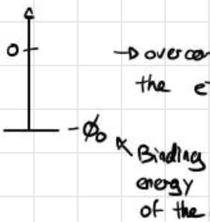
Early 1900 Three major unexplained phenomenon: (1) Photoelectric effect
(2) Blackbody radiation
(3) Heat capacity of gases

Photoelectric effect



Problem: The energy of the radiation can be high but no e^- get ripped off $\rightarrow$ only for high enough frequencies

$\Rightarrow$ until then no link between frequency and Energy



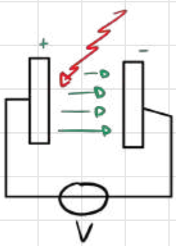
$\rightarrow$ overcome the Binding energy to get the e^- lose

introducing $\boxed{E = h\nu}$

Planchs constant
$h = 6.626 \cdot 10^{-34}$

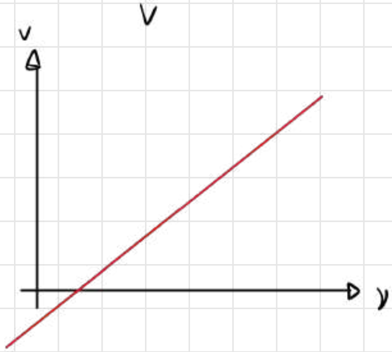
$$\boxed{E = h\nu \overset{!}{=} \phi_0 + \frac{1}{2}mv^2}$$

Experiment:



e^- goes away
$\rightarrow$ electric field
$\rightarrow$ searching for equilibrium

Result:

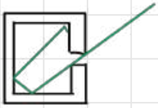


$$h\nu = \phi_0 + eV$$

$$V(\nu) = \frac{h\nu}{e} - \frac{\phi_0}{e}$$ $\Rightarrow$ now h can be found

Black Body radiation

↳ object in thermal equilibrium with environment but can absorb any radiation

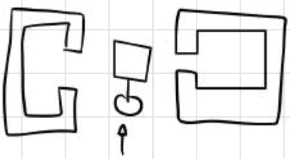


⇒ has to radiate: as in thermal equilibrium

Spektrum of radiation is universal

→ same for all black bodies

↳ Demonstration with counter example



filter (select frequencies)

→ for this frequency ν_1 Blackbody 1 emits more than Blackbody 2

⇒ not in thermal equilibrium anymore as it violates thermodynamics principles

→ Universal curves only depends on the internal temperature of the blackbody

↳ higher photon energy with higher temperature

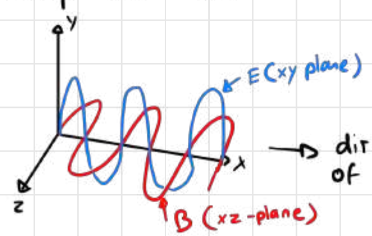
previous two models

$U(\omega, T) = \frac{\omega^2}{\pi^2 c^3} kT$ → diverges Rayleigh-Jeans

$U(\omega, T) = C\omega^3 e^{-\lambda\omega/T}$ Wien (empirical)

Lecture 2

Recap EM Waves

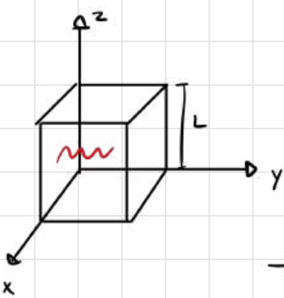


→ direction of the wave

→ E and B perpendicular

→ standing EM-waves possible

→ 2 polarization for Electric field (2 modes)



Blackbody with perfectly reflecting walls

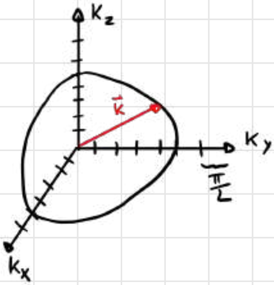
$E_x(x, y, z) = E_{0x} \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z)$ → cos and 2 sin because Maxwell's equation

$[2\pi\nu = \omega = ck = c \frac{2\pi}{\lambda}]$

→ boundary condition: direction of propagation has to be perpendicular to the walls (as perfectly reflecting

→ standing wave: $\boxed{k_x = l\frac{\pi}{L}\ ,\ k_y = m\frac{\pi}{L}\ ,\ k_z = n\frac{\pi}{L}}$ $\quad l,m,n \in \mathbb{N}$

$$\sqrt{k_x^2 + k_y^2 + k_z^2} = k \quad , \quad \omega = ck \quad , \quad \lambda\nu = c$$



→ count the amount of oscilators for a value of k have to conside 1/8 of a sphere

$$N(k) = \frac{1}{8}\frac{4}{3}\pi k^3 \frac{1}{\left(\frac{\pi}{L}\right)^3} \cdot 2$$

polarization of E

number of oscilators

$$= \frac{k^3 L^3}{3\pi^2}$$

$$\mathcal{D}(k) = \frac{dN(k)}{dk}\frac{1}{V} = \frac{k^2}{\pi^2}$$

density

L^3

$$\mathcal{D}(k)\,dk = \frac{k^2}{\pi^2}dk \rightarrow \boxed{\mathcal{D}(\omega)\,d\omega = \frac{\omega^3}{c^3\pi^2}d\omega}$$

↳ similar to Rayleigh-Jeans

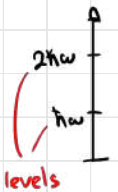
$$u(\omega)\,d\omega = \mathcal{D} \cdot kT\,d\omega$$

small part of the spectrum

thermal energy

Rethinking: like photoelectric effect

→ need an activation energy



$\boxed{kT < \hbar\omega}$

Boltzmann constant

$\boxed{\text{Boltzmann-factor}}$

$$N_1 = N_0 e^{-\frac{\Delta E}{kT}}$$

↳ probability of occupation of level 1 based on the difference of energy ΔE

↳ in this case $\Delta E = \hbar\omega$

$$N_1 = N_0 e^{-\hbar\omega/kT}$$
$$N_2 = N_0 e^{-2\hbar\omega/kT}$$
$$\vdots$$

average energy

$$\langle E \rangle = \frac{N_0 \times 0 + N_1\hbar\omega + N_2\, 2\hbar\omega + \dots}{N_0 + N_1 + N_2 + \dots}$$

prob. times energy

total prob.

$$= \frac{N_0\hbar\omega\left(e^{-\frac{\hbar\omega}{kT}} + 2e^{-\frac{2\hbar\omega}{kT}} + \dots\right)}{N_0\left(1 + e^{-\frac{\hbar\omega}{kT}} + e^{-\frac{2\hbar\omega}{kT}} + \dots\right)}$$

$x = e^{-\frac{\hbar\omega}{kT}}$

$$= \frac{\hbar\omega\left(x + 2x^2 + 3x^3 \dots\right)}{1 + x + x^2 + x^3 \dots}$$

$\rightarrow = \frac{1}{1-x}$

$$x + 2x^2 + 3x^3 + \dots = S$$

$$= 1 + x + x^2 + x^3 \dots - 1 + \underbrace{x^2 + 2x^3 + \dots}_{\substack{x(x+2x^2\dots) \\ = x \cdot S}}$$

$$\Rightarrow S = \frac{1}{1-x} - 1 + x \cdot S$$

$$S = \frac{1}{1-x}\left[\frac{1}{1-x} - \frac{1-x}{1-x}\right] = \frac{x}{(1-x)^2}$$

$$= \hbar\omega \frac{\frac{x}{(1-x)^2}}{\frac{1}{1-x}} = \hbar\omega \frac{x}{1-x} = \hbar\omega \frac{1}{\frac{1}{x} - 1}$$

$$\Rightarrow \boxed{\langle E \rangle = \hbar\omega \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}}$$

$$u(\omega)\, d\omega = \frac{\omega^2}{\pi^2 c^3} \langle E \rangle\, d\omega = \boxed{\frac{\omega^3 \hbar}{\pi^2 c^3} \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}}$$

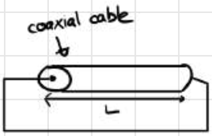
- Big Bang → Universe was a black body

Johnson-Nyquist noise

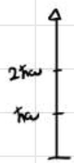
→ resistor acts like blackbody in 1-D ⇒ black body radiation read a voltage noise

cooling down restistance → less noise

Basic physical process:
- conductor has temperature T
- lattice of the metal is moved due to T
- electron move around and create fluctuation of voltage
  → creates noise



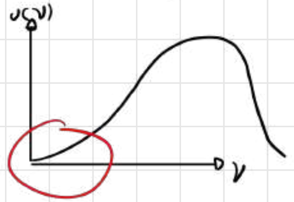
$$\boxed{c' = \frac{c}{n}}$$



but this time

$$\boxed{\hbar\omega \ll kT}$$

→ Energy levels have to be activated



looking at this part

→ count numbers of oscilators

$$N(k)\, dk = \frac{k}{\pi/L}\, dk = \frac{Lk}{\pi}\, dk$$

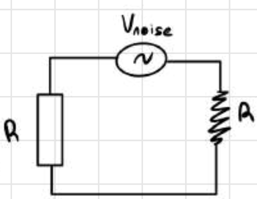
$$D(k)\, dk = \frac{dk}{\pi}$$

$$\boxed{D(\omega)\, d\omega = \frac{d\omega}{\pi c'}}$$

$$u(\nu)\, d\nu = \frac{2kT}{c'}\, d\nu$$

Lecture 3

Consider this circuit:



ideal resistor with extra noise source

radiated power $P_1 = u(\nu)\, d\nu \, \frac{c'}{2} = \frac{1}{2} \frac{c' kT}{c'} 2\, d\nu = kT\, d\nu$

consider just one side

power in the circuit $P_2 = \langle i^2 \rangle R = \frac{\langle v^2 \rangle}{R_{tot}^2} R = \frac{\langle v^2 \rangle}{4R} = P_1$

$R_{tot} = 2R$

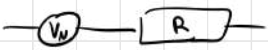
$$\Rightarrow \langle v^2 \rangle = 4 R k T \, d\nu$$

<u>Example</u>

$R = 50\,\Omega$, $T = 300K$, $\nu = 1\,GHz$

$$4 \cdot 50 \cdot 300 \cdot 1.38 \cdot 10^{-23} \cdot 10^9 = 8.3 \cdot 10^{-10}\, V^2 = \langle v^2 \rangle$$

$$\rightarrow V_{noise} \simeq 29\,\mu V$$



$\langle V_N^2 \rangle = 4R \cdot kT\, d\nu$

$\langle i_N^2 \rangle = \frac{4kT}{R} d\nu$

$\langle V_{N_{tot}}^2 \rangle = \langle V_{N_1}^2 \rangle + \langle V_{N_2}^2 \rangle + \dots$

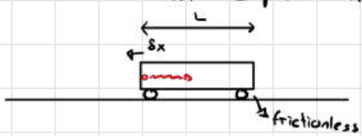
$R = R_1 + R_2 + \dots$

$\langle i_{N_{tot}}^2 \rangle = \langle i_{N_1}^2 \rangle + \langle i_{N_2}^2 \rangle + \dots$

$R^{-1} = R_1^{-1} + R_2^{-1} + \dots$

Momentum of a photon

Einstein's Gedankenexperiment



$\sum_i \vec{F}_i = 0 = \frac{d\vec{p}}{dt} \Rightarrow \vec{p}$ is conserved

$\vec{P}_{ph} + \vec{P}_{tube} = 0$

$\boxed{E = mc^2}$ photon doesn't mass

$\delta E = \delta m c^2 = h\nu$

$\delta m = \frac{h\nu}{c^2}$

recoil $\delta x\, M = \delta m \cdot L$

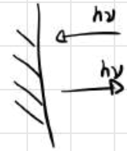
$\delta x = \frac{\delta m \cdot L}{M}$

$$M v_T = M \frac{\delta x}{\delta t} = M \frac{\delta m \cdot L}{M} \frac{1}{\delta t} = \delta m \cdot L \frac{1}{\delta t}$$

$$= \delta m\, L \frac{1}{L/c} = \delta m \cdot c = \frac{h\nu}{c^2} c = \frac{h\nu}{c}$$

$$\boxed{p_{photon} = \frac{h\nu}{c} = \frac{E}{c}} \rightarrow \boxed{E = cp}$$

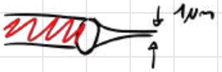
$$F = \frac{dp}{dt} = \Delta p \frac{1}{\Delta t} = 2\frac{h\nu}{c}\,\phi = 2\frac{h\nu}{c}\frac{P}{h\nu} = 2\frac{P}{c}$$



photon flux: $\phi = \frac{P}{h\nu}$

→ amount of photon/time or optical power/energy of 1 photon

$$\boxed{F = 2\frac{P}{c}}$$



optical lens can focus the beam of light

$$\Rightarrow P = \frac{F}{S} = \frac{10^{-8}\,N}{10^{-12}\,m^2} \approx 0.5 \cdot 10^{4}\,Pa$$

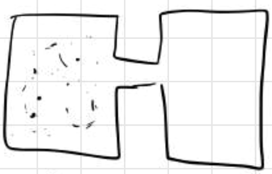
pressure

surface

↳ 1W, 1µm

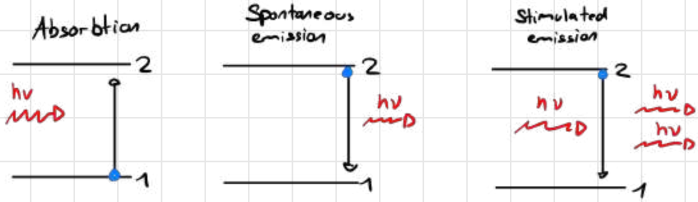
↳ not fare from atmospheric pressure

Absortion and Emission of photons



↑ vessel with gas

↑ black body source (create plancks distribution)



↳ kicked to new energy level by Temperature or photon

Rate equation analysis

Absorbtion — spont. em. — stim. emission

$$\frac{dn_2}{dt} = u(\nu)\cdot n_1 \cdot B_{12} - A_{21} n_2 - u(\nu)\cdot n_2 \cdot B_{21}$$

probability

n_i: population of e^- on level i

Steady state: (equilibrium) $\frac{dn_2}{dt} \overset{!}{=} 0 = \frac{B_{12}\,u(\nu)}{B_{21}\,u(\nu) + A_{21}}$

Boltzmann: $e^{-\frac{h\nu}{kT}} = \frac{B_{12}\,u(\nu)}{B_{21}\,u(\nu) + A_{21}}$

for $kT \to \infty$ $e^{-\frac{h\nu}{kT}} = 1 \Rightarrow B_{12} = B_{21} := B$

$$B\,u(\nu)\,e^{-\frac{h\nu}{kT}} + A_{21}\,e^{\frac{h\nu}{kT}} = B\,u(\nu)$$

$$A_{21} = B\,u(\nu)\left(1 - e^{-\frac{h\nu}{kT}}\right)e^{\frac{h\nu}{kT}}$$

$$= B\,u(\nu)\left(e^{\frac{h\nu}{kT}} - 1\right)$$

$$= B\,\frac{8\pi}{c^3}h\nu^3\,\frac{1}{e^{\frac{h\nu}{kT}} - 1}\left(e^{\frac{h\nu}{kT}} - 1\right)$$

conects absorbtion to spontaneous emission

$$\boxed{A_{21} = B\,\frac{8\pi}{c^3}h\nu^3}$$

→ always more on lower level

⇒ can produce a medium to amplify light (LASER
→ needs thermal energy to satisfy energy equilibrium

Lecture 4

Compton effect

$p = \frac{h\nu}{c}$

works well with e^- (not to much mass)

$$\begin{cases} \vec{p}_{\nu_1} = \vec{p}_{\nu_2} + \vec{p}_e \\ h\nu_1 + m_0 c^2 = h\nu_2 + \sqrt{m_e^2 c^4 + p_e^2 c^2} \end{cases}$$

4-Vector

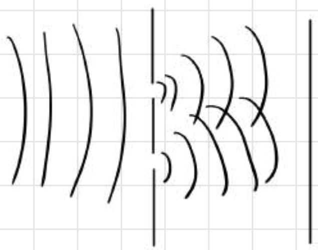
$E^2 = m^2c^4 + p^2c^2$

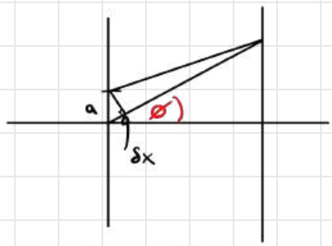
$\lambda_1 = \frac{c}{\nu_1}$, $\lambda_2 = \frac{c}{\nu_2}$

$$\rightarrow \boxed{\lambda_1 - \lambda_2 = \frac{h}{m_0 c}(1 - \cos\vartheta)}$$ → Compton effect formula

$\frac{h}{m_e c} \simeq 2.4 \cdot 10^{-12}\,m$ → need very short wave length (high energetic photon)

(Young's) Double slit experiment





if $\delta x = n\lambda$ → MAX

if $\delta x = (n+\frac{1}{2})\lambda$ → MIN

$\frac{\delta x}{a} = \sin\phi$

$a \sin\phi = n\lambda$ for MAX

Louis de Broglie (1924)

$$E = cp \quad \boxed{\frac{h\nu}{c} = p} = h\frac{1}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \boxed{\hbar k = p} = mv$$

$$\boxed{\lambda = \frac{h}{mv}}$$

for $m = 1kg$, $100\frac{m}{s} = v$ ⇒ $\lambda = 6.6 \cdot 10^{-36}\,m$

↳ no wave-like particle for very massive object

$\Rightarrow$ electron acts like a wave

Lecture 5

Recap:

① Photons have momentum!

$E = cp\,,\quad p = \hbar k \quad \Rightarrow$ radiation pressure

② Matter wavelength

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2mE}}$$

Heisenbergs Uncertainty Principle

$\boxed{\Delta x\, \Delta p \geq \hbar}$ $\rightarrow$ limit in precision
trade off

Math recap: Fourier transforms

$f(x) \overset{FT}{\Rightarrow} F(k) \qquad f(t) \overset{FT}{\Rightarrow} F(\omega)$

unit $[L] \qquad [L^{-1}]$

$\boxed{\Delta x \Delta k \geq 1}$

Example:

delta function

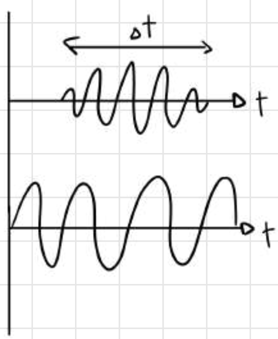
$\Delta x = 0 \rightarrow$ no uncertainty in position

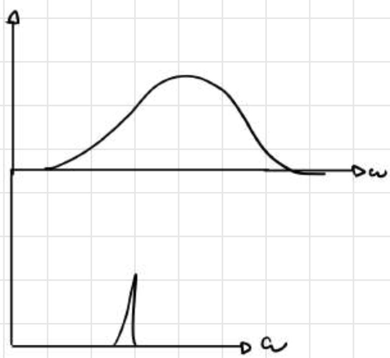
FT $\Rightarrow$

$\Delta k = \infty$

pulsed

continuous

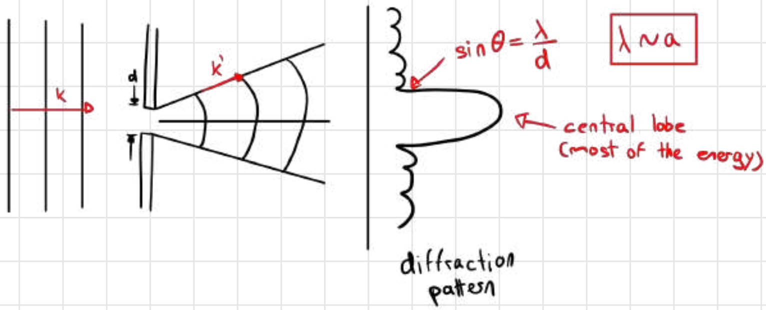




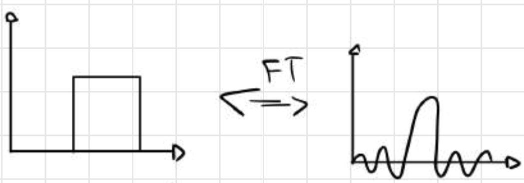
Diffraction

↳ property of waves

↳ interpherence as wave travels past aperture/object



Near field $\overset{FT}{\Longleftrightarrow}$ Far field



square pulse
from apparture

Diffraction for matter waves

$\lambda_{dB} = \frac{h}{p}$, $\Delta x = d \ (\approx \lambda)$

$\Delta p =$ Encoded uncertainty in momentum

$k \to k_x \to k\sin\theta$

$\Delta k = 2k\sin\theta$

$\Delta p = 2\hbar k \sin\theta$

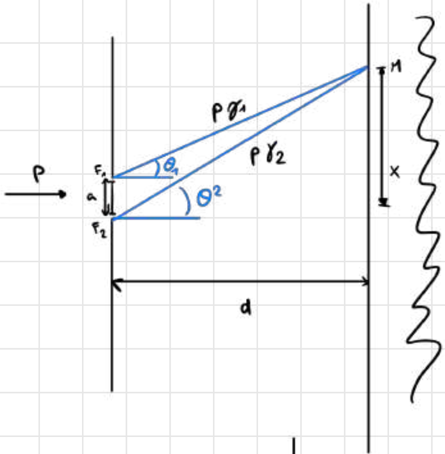
$= 2\hbar k \frac{\lambda}{d}$

for "most" e^-
between two first minima

$\Rightarrow \Delta x \Delta p = 2\hbar k \lambda = 2\frac{h}{2\pi}\frac{2\pi}{\lambda}\lambda = 2h$

$\to$ minimum $\Delta x \Delta p \gtrsim 2h$

or $E = \hbar\omega$ $\Rightarrow \Delta E \Delta t \geq \hbar$



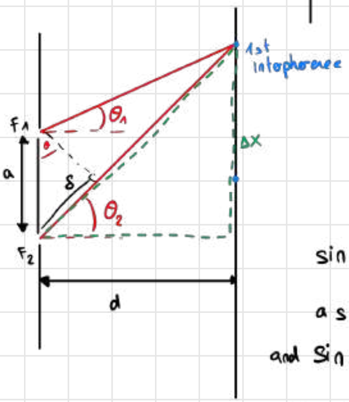
$d \gg a$

$$p_1 = -p\gamma_1 = -p\sin\theta_1$$
$$p_2 = -p\gamma_2 = -p\sin\theta_2$$
$$\Delta p \leqslant |p_2 - p_1|$$
$$\Delta p = p\sin\theta_2 - p\sin\theta_2$$
$$\approx p(\theta_2 - \theta_2)$$
$$\approx p\frac{a}{d} = \frac{h}{\lambda}\frac{a}{d}$$

$\sin\theta \sim \theta$
$d \gg a$

$$\theta_2 \approx \tan\theta_2 = \frac{x + a/2}{d}$$
$$\theta_1 \approx \tan\theta_1 = \frac{x - a/2}{d}$$

$$\Rightarrow \boxed{\Delta p \simeq \frac{h}{\lambda}\frac{a}{d}}$$



and $\Delta p \Delta x \geqslant h$

$$\Rightarrow \frac{h}{\lambda}\frac{a}{d}\Delta x \geqslant h$$

$$\Delta x \geqslant \frac{d\lambda}{a}$$ → distroys interference pattern

↘ fringe spacing

$$\sin\theta_2 = \frac{\delta}{a}$$
$$a\sin\theta_2 = n\lambda$$
and $\sin\theta_2 = \frac{\Delta x}{d}$
$$\Rightarrow a\frac{\Delta x}{d} = n\lambda$$
$$\Delta x = \frac{\lambda d}{a} \quad (\text{for } n=1)$$

Extra Info:

→ has been done with C_{60} (ball of 60 C atoms)
  ↳ smaller interference, hard to do

Lecture 6

Atomic Emission Spectrum
↳ Bohr – Sommerfeld quantisation

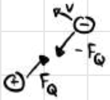
"If matter has wavelike properties, electron orbits should have integer values of λ"

orbit length $L = \lambda n$ $\lambda = \frac{h}{p}$

$$\rightarrow pL = nh$$
$$\rightarrow \boxed{\int_{traj} p\, dL = nh}$$

Calculating allowed energies of hydrogen atom:

Classical concepts:



$$|F_Q| = \frac{e^2}{4\pi\varepsilon_0 r^2} \qquad F_z = \frac{mv^2}{r} \qquad m \to \frac{m_e m_p}{m_p + m_e} \simeq m_e$$

$$\frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{m_e v^2}{r}$$

$$\rightarrow r = \frac{e^2}{4\pi\varepsilon_0 m_e v^2} \quad (1)$$

$$E = E_{kin} + E_{pot}$$

$$= \frac{1}{2} m_e v^2 - \frac{e^2}{4\pi\varepsilon_0 r} \quad V(r) = \int F(r)\,dr$$

$$= \frac{1}{2} m_e v^2 - m_e v^2$$

$$= -\frac{1}{2} m_e v^2 \quad (2)$$

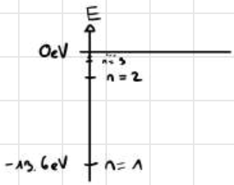
- $p \int_{traj} dL = mv \cdot 2\pi r = n h$

$$\rightarrow r = \frac{nh}{2\pi} \frac{1}{mv} = \frac{n\hbar}{m_e v} \rightarrow \frac{e^2}{4\pi\varepsilon_0 \cancel{m_e} \cancel{v^2}} = \frac{n\hbar}{\cancel{m_e} \cancel{v}} \quad (1) \rightarrow v = \frac{e^2}{4\pi\varepsilon_0 n\hbar}$$

$$\rightarrow E = \frac{e^4 m_e}{32\pi^2 \varepsilon_0^2 n^2 \hbar^2} = \boxed{\frac{-E_R}{n^2}} \qquad \boxed{E_R = \frac{1}{2} \frac{m_e e^4}{(4\pi\varepsilon_0)^2 \hbar^2}}$$

Quantised energy levels — Rydberg energy

"Balmer Series"



$n=1$ $\quad E = -E_R$ Ground state

$n=2$ $\quad E = -\frac{1}{4} E_R$

⋮

→ dropping a level

→ ΔE released as photon
$\quad = h\nu$

$$r = \boxed{\frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} = a_0} \quad \text{Bohr radius}$$

→ "old quantum theory"

Chapter 3 → Griphiths text book starts

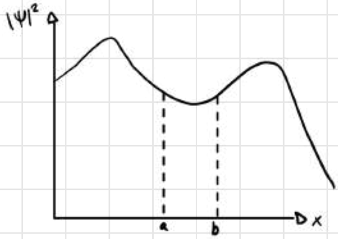
wavefunction $\psi(x,y,z,t)$ → spread out in space
→ complex

→ cannot directly measure

$|\psi|^2 \, dx\,dy\,dz$

↑ probability distribution in a small volume

$\psi\psi^*$



$$\int_a^b |\psi(x,t)|^2 \, dx$$ "probability of finding our particle between a and b"

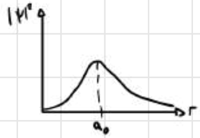
$$\iiint_{\mathbb{R}^3} |\psi|^2 \, dV = 1$$

↓

measurment at $t = t_1$

↓

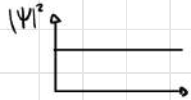
e.g $\psi(r,t) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} e^{-iE_n t/\hbar}$



e.g plane wave

$$\psi(x,t) = A e^{i(kx - \omega t)}$$

$\Delta x \to \infty \quad \Delta k \to 0$



Superposition principle

$\psi \Rightarrow$ Schrödinger equation

two functions $\phi_1(x,t)$ $\phi_2(x,t)$ are solutions to the SE

$$\psi = a_1 \phi_1 + a_2 \phi_2$$

$a_1, a_2 \in \mathbb{C}$

Schrödinger's equation (non relativistic)

$$\boxed{\Big[\underbrace{-\frac{\hbar^2}{2m}\nabla^2}_{E_{kin}} + \underbrace{V(x,y,z)}_{E_{pot}}\Big]\psi = i\hbar \frac{\partial}{\partial t}\psi}$$

potential (conservativ forces)

free particle : $V = 0$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\nabla^2 \psi = i\hbar \frac{\partial}{\partial t}\psi$$

$$\to 1D \qquad \frac{\hbar i}{2m}\psi_{xx} = \psi_t$$

Lecture 7

Wave function $\psi(\vec{x},t) \in \mathbb{C}$

$|\psi(\vec{x},t)|^2 \in \mathbb{R}$ probability density

$$\iiint |\psi(\vec{x},t)|^2 dx = 1 \qquad \forall t$$

For ψ_1 and ψ_1 (solutions of SE)

$\hookrightarrow \alpha \psi_1 + \beta \psi_2 = \psi_{tot} \qquad |\alpha^2 + \beta^2| = 1$

$$\Big[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x},t)\Big]\psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t)$$

→ "equivalent" to Newtons's law $\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$

for a conservative Force $\vec{F} = -\nabla V(x)$, $-\int_0^x F \, dx = V(x)$, $\oint \vec{F} d\vec{s} = 0$

Free particle $V(x)=0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}$$

Ansatz $\psi = Ae^{i(kx-\omega t)}$ plane wave

$$\rightarrow \frac{\hbar^2}{2m}Ak^2e^{i(kx-\omega t)} = \hbar A\omega e^{i(kx-\omega t)}$$

$$\frac{\hbar^2}{2m}k^2 = \omega\hbar = E = \frac{p^2}{2m}$$

kinetic energy

$$\boxed{k = \pm\sqrt{\frac{2m\omega}{\hbar}}}$$

$\rightarrow$ Mater waves are despersing

But $\int|\psi|^2dx = \int_{-\infty}^{\infty}A^2dx \rightarrow \infty$

$\rightarrow$ Wave packets with Fourier

Dispersion in waves

D'Alembert $\nabla^2 A(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A(\vec{r},t)$

$$\frac{\partial^2}{\partial x^2}A(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A(x,t)$$

Dispersion relation for : $\omega = ck$
- d'Alembert
- SE $\omega = \frac{\hbar k^2}{2m}$

$k^2 \sim E$

$\hookrightarrow$ different frequencies travel at different speeds

For $\omega = ck$

$$v_{phase} = \frac{\omega}{k} = c$$
$$\shortparallel$$
$$v_{group} = \frac{\partial\omega}{\partial k} = c$$

Matter waves

$$v_{phase} = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{p}{2m}$$

$$v_{group} = \frac{\hbar k}{m} = \frac{p}{m}$$

$$2v_{phase} = v_{group}$$

Time independent SE

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial t^2}\psi + V(x,t)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Separation of variables

$$\psi(x,t) = \varphi(x)\,\chi(t)$$

$$\underbrace{-\frac{\hbar^2}{2m}\varphi''(x)\frac{1}{\varphi(x)} + V(x)}_{\text{const.}} = \underbrace{i\hbar\frac{1}{\chi(t)}\chi'(t)}_{\text{const}} \quad \forall t,x$$

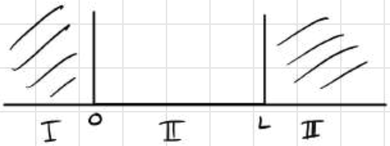
$$\rightarrow i\hbar\frac{1}{\chi(t)}\chi'(t) = E \rightarrow i\hbar\chi'(t) = E\chi(t) \rightarrow \boxed{\chi(t) = e^{-i\frac{E}{\hbar}t}} \qquad \frac{E}{\hbar} = \omega$$

$$\rightarrow -\frac{\hbar^2}{2m}\varphi''(x) + V(x)\varphi(x) = E\varphi(x) \rightarrow \boxed{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\varphi(x) = E\varphi(x)}$$

$\hookrightarrow$ Tim independent SE

$$\int |\psi|^2 dx = \int |\varphi(x)\chi(t)|^2 dx = \int |\varphi(x)|^2 dx = 1$$

The infinite quantum well



$I, III \to V(x) \to \infty$

$II \to V(x) = 0$

Boundary conditions $\varphi(0) = \varphi(L) = 0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi(x) = E_n \varphi(x)$$

Ansatz $A\sin(kx)$

$\sin(kL) = \sin(k0) = 0$ ✓

with $k_n = n\frac{\pi}{L}$

$$\to \frac{\hbar^2}{2m} A k^2 \sin(kx) = E_n A \sin(kx)$$

$$\to \boxed{\frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2 = E_n}$$

↳ Energy is quantised

$E_n \sim \left(\frac{n}{L}\right)^2$

$$\int_{-\infty}^{\infty} A^2 \sin^2(kx)dx = \int_0^L A^2 \sin^2(kx)dx$$

$$= \frac{A^2}{2}\int_0^L 1 - \cos(2x)dx = \frac{A^2}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^L$$

$$= \frac{A^2}{2}L = 1$$

$$A = \pm\sqrt{\frac{2}{L}}$$

$$\to \varphi_n(x) = \pm\sqrt{\frac{2}{L}}\sin(k_n x)$$

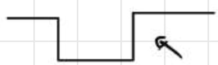
$$\to \psi_n(x,t) = \pm\sqrt{\frac{2}{L}}\, e^{-i\frac{E_n}{\hbar}t}\sin(k_n x)$$

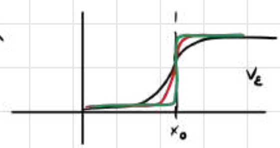
Lecture 8

Harmonics $\omega_0, 2\omega_0, \dots$ vibrating string

Q.U. $\omega_1 = \omega_0 \quad \omega_2 = 4\omega_0 \quad \omega_3 = 9\omega_0$

$\omega_n = n^2\omega_0$





$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi(x) + (V_e(x) - E)\varphi(x) = 0$$

$$\int_{x_0-\theta}^{x_0+\theta}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_e(x) - E\right)\varphi(x)\,dx = 0$$

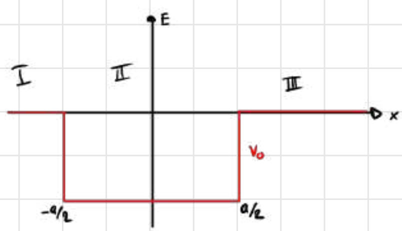
$$-\frac{\hbar^2}{2m}\frac{\partial\varphi}{\partial x}\Big|_{x_0-\theta}^{x_0+\theta} = \int_{x_0-\theta}^{x_0+\theta}(V_e - E)\varphi(x)\,dx$$

$\varepsilon \to 0$

↳ 0

$$\to \frac{\partial\varphi}{\partial x}\Big|_{x_0+\theta} = \frac{\partial\varphi}{\partial x}\Big|_{x_0-\theta}$$

One-dimensional potential of finite depth

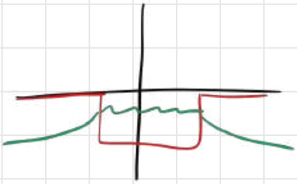


$$V = \begin{cases} 0 & x < -\frac{a}{2} \\ -V_0 & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ 0 & x > \frac{a}{2} \end{cases}$$

Bound states $-V_0 < E < 0$

(I) $\varphi_I(x) = B_1 e^{\varphi x} + \overset{0}{B_1'} e^{-\varphi x}$ $\quad \underset{\mathbb{R}}{\varphi} = \sqrt{\frac{-2mE}{\hbar^2}}$ $\quad \frac{\hbar^2 k^2}{2m} = E$

(III) $\varphi_{III}(x) = \underset{0}{B_3} e^{\varphi x} + B_3' e^{-\varphi x}$



(II) $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi(x) = (E+V_0)\varphi(x)$

Ansatz $A_2 e^{-ikx} + A_2' e^{ikx}$ $\quad k_n = n\frac{\pi}{a}$

$$\frac{\hbar^2}{2m} k_n^2 \left(A_2 e^{-ik_n x} + A_2' e^{ik_n x}\right) = (E_i + V_0)\left(A_2 e^{-ik_n x} + A_2' e^{ik_n x}\right)$$

$$k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$E_n =$

$\to$ continuity of function $\quad \varphi_I\left(-\frac{a}{2}\right) = B_1 e^{-\varphi\frac{a}{2}} = \varphi_{II}\left(-\frac{a}{2}\right) = A_2 e^{ik_n\frac{a}{2}} + A_2' e^{-ik_n\frac{a}{2}}$ (1)

$$\varphi_I'\left(-\frac{a}{2}\right) = -\varphi B_1 e^{-\varphi\frac{a}{2}} = \varphi_{II}'\left(-\frac{a}{2}\right) = ik_n\left(A_2 e^{ik_n\frac{a}{2}} - A_2' e^{-ik_n\frac{a}{2}}\right) \quad (2)$$

$\to$ $\frac{1}{ik_n}(2) + (1)$: $\left(1 + i\frac{\varphi}{k_n}\right) B_1 e^{-\varphi\frac{a}{2}} = 2A_2 e^{ik_n\frac{a}{2}}$

$$\to A_2 = \frac{1}{2}\left(1 + i\frac{\varphi}{k_n}\right) B_1 e^{(ik_n - \varphi)\frac{a}{2}}$$

$\frac{1}{ik_n}(2) - (1)$: $A_2' = \frac{1}{2}\left(1 - i\frac{\varphi}{k_n}\right) B_1 e^{-(ik_n + \varphi)\frac{a}{2}}$

$\Rightarrow$ Same with φ_{III}

$$\frac{B_3}{B_1} = \frac{e^{-\varphi a}}{4ik\varphi}\left[(\varphi + ik_n)^2 e^{ika} - (\varphi - ik_n)^2 e^{-ika}\right] \to 0$$

$$\frac{B_3'}{B_1'} = \frac{\varphi^2 + k_n^2}{2k_n\varphi} \sin(k_n a)$$

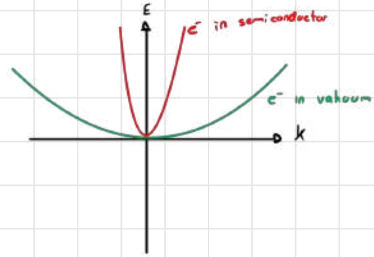
$$\frac{\varphi - ik_n}{\varphi + ik_n} = \pm e^{ik_n a}$$

even state $\quad \left|\cos\left(\frac{k_n a}{2}\right)\right| = \frac{k}{k_0}$ $\quad$ linear

$$k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$$

Lecture 9

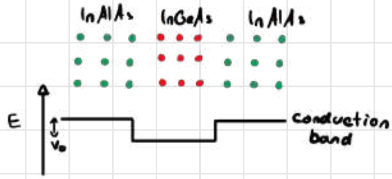
$E = \frac{\hbar k^2}{2m_i}$



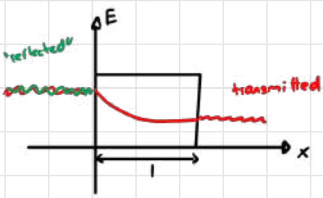
Semiconductor crystals:

- The potential $V(x)$ is periodic with the crystal
- The electrons in the crystal behave as if they were free
- But different mass $m^* \neq m_e$, often $m^* < m_e$
- e^- "feels" a potential V_0 dependent on the crystal

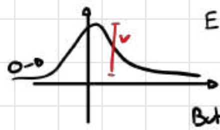
Heterojunctions:



Tunneling



Classical analogie



$E_{kin} < V \rightarrow P = 0$

But Q.M $\rightarrow P \neq 0$

Time independent

For $E < V_0$

$$\Psi(x,t) = \varphi(x)\,\chi(t), \quad \chi(t) = e^{-E/\hbar \cdot t}$$

I, III $V = 0$:

$$\varphi_I = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x}$$

$$\varphi_{III} = A_3 e^{ik_1 x} + A_3' e^{-ik_1 x}$$

$$\varphi_{II} = B_2 e^{i\varphi_2 x} + B_2' e^{-\varphi_2 x}$$

$$E = \frac{\hbar k_1^2}{2m} \rightarrow k_1 = \sqrt{\frac{2mE}{\hbar}}$$

$$\varphi_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$\in R$

transmission and reflexion coefficient

$$\boxed{T = \left|\frac{A_3}{A_1}\right|^2 \quad R = \left|\frac{A_1'}{A_1}\right|^2}$$

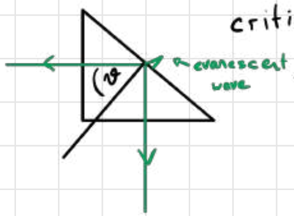
$$\boxed{T + R = 1}$$

↳ proof in Griffith's

$$\varphi_2 \cdot l \gg 1 \rightarrow T \simeq \frac{16E(V_0 - E)}{V_0^2} e^{-2\varphi_2 l}$$

$$T = \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \sinh^2\left[\sqrt{2m(V_0 - E)} \cdot l/\hbar\right]}$$

Lecture 10



critical ϑ → total reflection

⇒ small enough gap → passes through

Chapter 4

SE : $\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x) = E\,\Psi(x)$

observables:
x position
p momentum
E energy

↳ $p_1 = 0.2$ ↳ $p_2 = 0.8$

probability

weigthed average : $\langle x \rangle = x_1 p_1 + x_2 p_2$

for continuous values $\langle x \rangle = \int_{-\infty}^{\infty} x\,|\varphi(x,t)|^2\,dx$

$= \int_{-\infty}^{\infty} \varphi^* x\, \varphi\, dx$

position operator $\hat{x}$

$= \int$

Parenthesis on Hilbert space

$\mathbb{R}^n$ vectors:

$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $a_i \in \mathbb{R}$

$a \cdot b = \sum_n a_i b_i$ scalar prod.
$a \cdot b = 0$ ortho.
$|a|^2 = a \cdot a$ norm

Finite Hilbert space

$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $a_i \in \mathbb{C}$

$\langle a|b \rangle = \sum_n a_i^* b_i$ inner prod.
$a \cdot b = 0$ ortho.
$|a|^2 = \langle a|a \rangle$ norm

Hilbert space of wavefunctions

$\Psi(x) \in \mathbb{C}$

Operator $\hat{p}$: $\varphi(x,t) = \hat{p}\,\Psi(x,t)$

$\langle \Psi | \varphi \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,\varphi(x,t)\,dx$

$\langle \Psi | \varphi \rangle = 0$ ortho.

$\langle \Psi | \Psi \rangle$ 1 Norm of wavefunction momentum operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ $(= -i\hbar \nabla)$

Ex: $\psi = e^{i(kx - \omega t)}$ $\hat{p}\psi(x) = k\hbar e^{i(kx-\omega t)} = \hbar k \ \psi(x)$

$$\langle p \rangle_\psi = \int_{-\infty}^{\infty} p \, |\psi(p,t)|^2 dp = \int_{-\infty}^{\infty} \psi^* p \, \psi \, dp$$

with $\psi(p,t)$

related by Fourier Transform

$$\left[\begin{array}{l} \psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p,t) \, e^{i p/\hbar \, x} \, dp \\ \phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,t) \, e^{-i\frac{px}{\hbar}} \, dx \end{array}\right]$$

Parseval

$$\int_{-\infty}^{\infty} f_1^* f_2 \, dx = \int_{-\infty}^{\infty} g_1^* g_2 \, dp$$

if $g_1 = \mathcal{F}\{f_1\}$

$g_2 = \mathcal{F}\{f_2\}$

$$-i\hbar \frac{\partial}{\partial x} \psi(x,t) = \int_{-\infty}^{\infty} \left(\frac{-i\hbar}{\sqrt{2\pi\hbar}}\right) \frac{\partial}{\partial x}\left(e^{ip\frac{x}{\hbar}}\right) \phi(p,t)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} p \, \phi(p,t) \, e^{ip\frac{x}{\hbar}} \, dp$$

$$\hat{p} \begin{cases} p & \text{for } \phi(p,t) \\ -i\hbar \frac{\partial}{\partial x} & \text{for } \psi(x,t) \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \psi^* \left(i\hbar \frac{\partial \psi}{\partial x}\right) dx = \int_{-\infty}^{\infty} \phi^* p \, \phi \, dp$$

$$\hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$\frac{1}{2m} \hat{p}^2 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

SE $\underbrace{\left[\frac{1}{2m}\hat{p}^2 + V(x)\right]}_{\hat{H}} \psi(x) = E \, \psi(x)$

$\hat{H}$ → Hamiltonian

$$\Rightarrow \quad \hat{H}\psi(x) = E \, \psi(x)$$

Eigenvalues of the Hamiltonians are the energies of the system

$$\psi(x) \rightarrow \varphi_n \rightarrow \hat{H}\varphi_n = E_n \varphi_n$$

$$\langle E \rangle_\varphi = \int_{-\infty}^{\infty} \varphi_n^*(x)\, \hat{H} \varphi_n \, dx = E_n \int_{-\infty}^{\infty} \varphi_n^* \varphi_n \, dx = E_n$$

$\varphi_n \to$ one solution of the S.E

$$\hat{H}\varphi = \varphi E$$

$\hookrightarrow \varphi_n$ integer

$$\psi(x) = \sum_n c_n \varphi_n(x)$$

$$\langle E \rangle_\psi = \int_{-\infty}^{\infty} \sum_n c_n^* \varphi_n^* \cdot \hat{H} \underbrace{\sum_m c_m \varphi_m}_{\sum_m c_m E_m \varphi_m} dx = \sum_n \sum_m c_n^* c_m E_m \overbrace{\int_{-\infty}^{\infty} \varphi_n^* \varphi_m^* \, dx}^{\delta_{nm}}$$

$$= \sum_n |c_n|^2 E_n$$

$$\boxed{\langle E \rangle_\psi = \sum_n |c_n|^2 E_n}$$

$\hookrightarrow$ weighted sum

$\hat{a}, \hat{b}$ operators do not commute (in general)

$\hat{x}, \hat{p}$

$$\underbrace{(\hat{x}\hat{p} - \hat{p}\hat{x})}_{[\hat{x},\hat{p}] = i\hbar} \psi = x\left(-i\hbar \frac{\partial}{\partial x}\psi\right) + i\hbar \frac{\partial}{\partial x}(x\psi)$$

$$= i\hbar \psi \neq 0$$

commutator

Lecture 11

Hermitian $\to$ operator have real eigenvalues

$\hookrightarrow$ if not $\to$ not observable

$\to$ discrete spectrum infinite square well

$\to$ continuous spectrum (free particle)

For discrete spektrum: Eigenfunctions belonging to different eigenvalues are orthogonal $\perp$.

$\hat{Q}f = qf$, $\hat{Q}g = q'g$ $\quad \langle \hat{Q}f | g \rangle = \langle f | \hat{Q} g \rangle$ $\quad \to f \perp g$

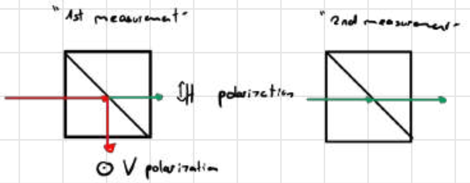
$q \neq q'$ $\quad \to q\langle f|g\rangle = q'\langle f|g\rangle \to \langle f|g\rangle = 0$

$$\hat{P} = |\alpha\rangle\langle\alpha| \rightarrow \hat{P}|\beta\rangle = |\alpha\rangle\langle\alpha|\beta\rangle$$

$$\langle e_m | e_n\rangle = \delta_{mn}$$

$$\sum_n |e_n\rangle\langle e_n| = \mathbb{1}$$

We consider a polarizing beam splitter



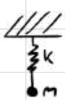
$\rightsquigarrow$ colapse of the wave function

45° polarization $\rightarrow$ $\Psi = \frac{1}{\sqrt{2}}\left(|V\rangle + |H\rangle\right)$

50/50 for photon

Harmonic Oscillator

classical



$\vec{F} = -k\vec{x} \Rightarrow V(x) = \frac{1}{2}kx^2$

$\omega = \sqrt{\frac{k}{m}}$ independent from the amplitude

$\ddot{x} = -\frac{k}{m}x$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 \qquad \boxed{\hat{p} = i\hbar\frac{\partial}{\partial x}}$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2\right)\Psi(x) = E\,\Psi(x)$$

subs $\quad y = \frac{x}{\sqrt{\hbar\sqrt{km}}} \quad a = \sqrt{\hbar\sqrt{km}} \quad \epsilon = \frac{E}{\hbar\sqrt{km}}$

...

$\epsilon = \frac{1}{2}$ $\qquad k = \omega^2 m$

$\rightarrow \boxed{\Psi(x) = e^{-\frac{m}{2\hbar}\omega x^2}}$

$E = \frac{\hbar\omega}{2}$ ground state

Gaussian fn $\varphi(y) = e^{-y^2/2}$

$$\frac{\partial\varphi}{\partial y} = -y\,e^{-y^2/2} \qquad \frac{\partial^2\varphi}{\partial y^2} = (y^2-1)e^{-y^2/2}$$

Solution: $\boxed{\varphi_n = c_n\, e^{-y^2/2} H_n(y)}$

$$H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}\left(e^{-y^2}\right)$$

$$H_0(y) = 1 \quad H_1(y) = 2y \quad H_2 = 4y^2 - 2 \quad \dots$$

"Nice way"

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \qquad \text{Factor the Hamiltonian}$$

We define new operators

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{p} + m\omega\hat{x}\right)$$

$$\hat{a}_+ = a^\dagger \qquad \hat{a}_- = \hat{a}$$

$$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + m^2\omega^2\hat{x}^2 + m\omega i\underbrace{\left[-\hat{x}\hat{p} + \hat{p}\hat{x}\right]}_{-[\hat{x},\hat{p}] = i\hbar}\right]$$

$$= \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + m^2\omega^2\hat{x}^2 + \hbar m\omega\right]$$

$$= \frac{1}{\hbar\omega}\left[\underbrace{\frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2}_{\hat{H}} + \frac{1}{2}\hbar\omega\right] = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} =$$

$$\boxed{\hat{H} = \hbar\omega\left[\hat{a}_-\hat{a}_+ - \frac{1}{2}\right]}$$

Value of commutator $[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$

$$\boxed{\hat{H} = \hbar\omega\left[\hat{a}_+\hat{a}_- + \frac{1}{2}\right]}$$

$\hat{H}\Psi = E\Psi$ Claim $\hat{a}_+\Psi$ is also an eigenvector of $\hat{H}$ with eigenvalue $E + \hbar\omega \Rightarrow \hat{H}(\hat{a}_+\Psi) = (E + \hbar\omega)\hat{a}_+\Psi$

$$\hat{H}(\hat{a}_+\Psi) = \hbar\omega\left[\hat{a}_+\hat{a}_- + \frac{1}{2}\right]\hat{a}_+\Psi = \hbar\omega\left[\hat{a}_+\hat{a}_-\hat{a}_+ + \frac{\hat{a}_+}{2}\right]\Psi = \hbar\omega\,\hat{a}_+\left[\hat{a}_-\hat{a}_+ + \frac{1}{2}\right]\Psi$$

$$= \hbar\omega\,\hat{a}_+\left[\hat{a}_+\hat{a}_- + \frac{3}{2}\right]\Psi = \hat{a}_+\left[\hbar\omega + \underbrace{\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)}_{\hat{H}}\right]\Psi = \hat{a}_+\left[\hbar\omega\Psi + E\Psi\right]$$

$$= (E + \hbar\omega)\hat{a}_+\Psi$$

$$\Rightarrow \quad \boxed{\hat{H}(\hat{a}_\pm)^n\Psi = (E \pm n\hbar\omega)(\hat{a}_\pm)^n\Psi}$$

Lecture 12

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \sqrt{\frac{2}{3}}\,|\psi_2\rangle$$

$$\hat{H}|\psi\rangle = \frac{1}{\sqrt{3}}\hat{H}|\psi_1\rangle + \sqrt{\frac{2}{3}}\,\hat{H}|\psi_2\rangle = \frac{1}{\sqrt{3}}E_1|\psi_1\rangle + \sqrt{\frac{2}{3}}\,E_2|\psi_2\rangle$$

$$\underbrace{\langle\psi|}_{\langle \frac{1}{\sqrt{3}}\psi_1 + \sqrt{\frac{2}{3}}\psi_2|}\hat{H}|\psi\rangle = \frac{1}{\sqrt{3}}E_1\underbrace{\langle\psi|\psi_1\rangle}_{\frac{1}{\sqrt{3}}} + \sqrt{\frac{2}{3}}\,E_2\underbrace{\langle\psi|\psi_2\rangle}_{\sqrt{\frac{2}{3}}} = \frac{1}{3}E_1 + \frac{2}{3}E_2$$

$$\underbrace{\hat{a}_-\psi_0 = 0}_{\text{ground state}} \qquad \frac{1}{\sqrt{2m\omega\hbar}}\Big(\overbrace{\hbar\frac{\partial}{\partial x}}^{i\hat{p}} + m\omega x\Big)\psi_0 = 0$$

$$\frac{\partial}{\partial x}\psi_0 = -\frac{m\omega}{\hbar}x\,\psi_0$$

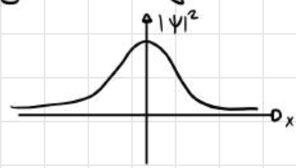
$$\rightarrow \quad \psi_0 = Ae^{-\frac{m\omega}{2\hbar}x^2} \xrightarrow{\text{normalize}} \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{\hbar}x^2}$$

1st state: $\hat{a}_+\psi_0 = \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)\psi_0$

$$\Rightarrow H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}\left(e^{-y^2}\right)$$

with $y = \sqrt{\frac{m\omega}{2\hbar}}\,x$

gaussian ground state and Heisenberg uncertainty principle



$\Delta x \Delta p$

$\Delta x \rightarrow$ HWHM "half width half maximum"

$$\left(e^{-x^2/2}\right)^2 = \frac{1}{2}$$

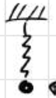
$$\Delta x = \sqrt{\ln(2)}\,a$$

$$\mathcal{F}\left\{e^{-\frac{1}{2}\frac{x^2}{a^2}}\right\} = e^{\frac{1}{2}k^2a^2}$$

$$\Delta p = \hbar\Delta k = \hbar\sqrt{\ln(2)}\cdot\frac{1}{a}$$

$$\Delta x\,\Delta p = \sqrt{\ln(2)}\,a\,\hbar\sqrt{\ln(2)}\,\frac{1}{a} = \ln(2)\hbar$$

$$\boxed{\Delta x\Delta p = \ln(2)\hbar}$$



$KT = 26\,meV \approx 10^{-21} J$

$\leftarrow \hbar\omega \approx 1 Hz \cdot 10^{-34} Js \approx 10^{-34} J$

$\frac{KT}{\hbar\omega} \approx 10^{12}$ # of modes excited

$KT \gg \hbar\omega \qquad \Psi(x,t) = \varphi(x)\,\chi(t)$

$\chi(t) = e^{-i\frac{E}{\hbar}t}$

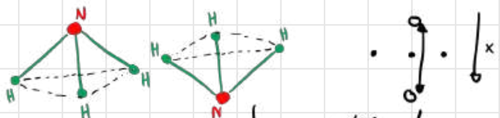
Coherent state

$$\Psi_\alpha'(x,t) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \Psi_n(x,t)$$

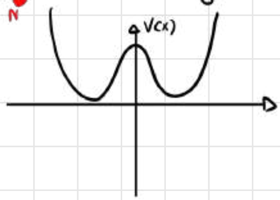
Ψ_n: eigenstates of H.O.

$$\Psi_{cat} = \frac{1}{\sqrt2}\Psi_\alpha(x,t) + \frac{1}{\sqrt2}\Psi_{\alpha e^{i\theta}}(x,t)$$

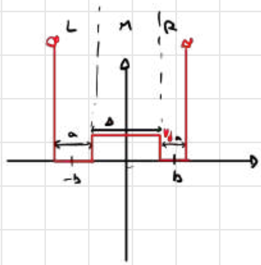
4.5 Coupled well $\rightarrow$ Ammonia Molecule (NH_3)



Potential



$\Rightarrow$



$\rightarrow$ tunnel barrier thickness

"guess" the solutions" + boundary condition

$L: \quad \Psi_L = \pm \lambda \sin\left(kx + k(b + a/2)\right)$

$\Psi_L(-b - \frac{a}{2}) = 0$ ✓

$R: \quad \Psi_R = \lambda \sin\left(kx - k(b + a/2)\right)$

$$M: \begin{cases} \nu \cosh(Kx) \\ \nu \sinh(Kx) \end{cases}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad K = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

Lecture 13

$\rightarrow$ solution for boundary conditions

$$\tan(ka) = \begin{cases} -\frac{k}{K}\coth\left(K(a - \frac{b}{2})\right) & \text{symetric} \\ -\frac{k}{K}\sinh\left(K(a - \frac{b}{2})\right) & \text{antysymetric} \end{cases}$$

$\rightarrow$ Tunneling is small $\quad K(\frac{b}{2} - a) \gg 1$

$$e^{-K(\frac{b}{2} - a)} \ll 1$$

$\coth(x) \cong 1 + 2e^{-2x}$ $\qquad \Delta = 2\left(\frac{b}{2} - a\right)$

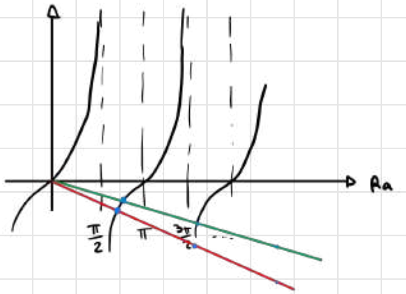
$$\tan(ka) = -\frac{k}{K}\left(1 \pm e^{-K\Delta}\right)$$

sym

antisym

$\longrightarrow E \ll V_0 \qquad K = \sqrt{\frac{2m(V_0-E)}{\hbar}} \sim$ constant

$$\tan(ka) = \overbrace{\left[-\frac{1}{aK}\left(1 \pm e^{-K\Delta}\right)\right]}^{e_s/e_a} ka$$



$$k_s = \frac{\pi}{a(1+e_s)} \qquad k_a = \frac{\pi}{a(1+e_a)}$$

$$\Rightarrow E_s = \frac{\hbar^2 k_s^2}{2m} \qquad E_A = \frac{\hbar^2 k_a^2}{2m}$$

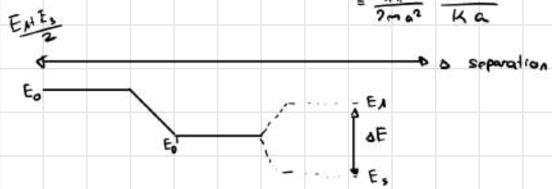
$$= \frac{\hbar^2 \pi^2}{\underbrace{2ma^2}(1+e_s)^2} \qquad = \frac{\hbar^2 \pi}{2ma(1+e_a)^2}$$

was solution for infinite quantum well

$$\Delta E = E_A - E_s = \frac{\hbar^2\pi^2}{2ma^2}\left(\frac{1}{(1+e_s)^2} + \frac{1}{(1+e_a)^2}\right)$$

$$\simeq \frac{\hbar^2\pi^2}{2ma^2}\left(2(e_s - e_a)\right) \qquad e_s, e_a \ll 1 \text{ Taylor expand}$$

$$= \frac{\hbar^2\pi^2}{2ma^2}\,\frac{8e^{-K\Delta}}{Ka}$$

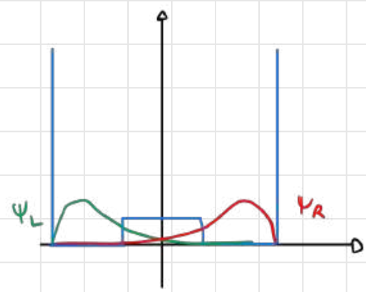


Inversion of NH_3

$$\psi(x,t) = \varphi(x)\,\chi(t), \qquad \chi(t) = e^{-i\frac{E}{\hbar}t}$$

We build new states ψ_A, ψ_s

$$\psi_L = \frac{1}{\sqrt{2}}\left(\psi_s - \psi_A\right) \qquad \psi_R = \frac{1}{\sqrt{2}}\left(\psi_s + \psi_A\right)$$



look at the time dependence

$$\psi_L(x,t) = \frac{1}{\sqrt{2}}\left(\psi_s(x,0)\,e^{-i\frac{E_s t}{\hbar}} - \psi_A(x,0)\,e^{-i\frac{E_a}{\hbar}t}\right)$$

$$= \frac{1}{\sqrt{2}}\,e^{-i\frac{E_a}{\hbar}t}\left(\psi_s(x,0)\,e^{-i\frac{\overbrace{(E_s - E_A)}^{-\Delta E}}{\hbar}t} - \psi_A(x,0)\right)$$

$|\psi_L|^2$ as a function of time

$$|\psi_L|^2 = \psi^*\psi = \frac{1}{2}\left(\psi_s\,e^{i\frac{\Delta E t}{\hbar}} + \psi_A\right)^2$$

$\omega = \frac{\Delta E}{\hbar}$ frequency $\rightarrow$ changing between left and right

$\frac{\omega}{2\pi} = \nu \simeq 25\ \text{GHz}$

with $kT - 26\,\text{meV}$

$\Downarrow$

$6.5\ \text{THz}$

or with AsH_3

$m(As) = 74u$ vs $m(N) = 14u$

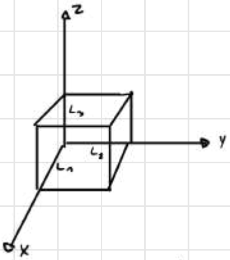
$\rightarrow \nu_{As} \simeq 1.6 \cdot 10^{-8}\ \text{Hz}$

~ 2 years

Quantum mechanics in 3D $\rightarrow$ Hydrogen Atom

Time independent SE

$$-\frac{\hbar^2}{2m}\left(\nabla^2 + V(x,y,z)\right)\psi(x,y,z) = E\,\psi(x,y,z)$$



L_1, L_2, L_3

$$V(x,y,z) = V(x) + V(y) + V(z)$$

$$V(x) = \begin{cases} 0 & 0 < x < L_1 \\ \infty & x > L_1 \end{cases}$$

...

$\rightarrow$ separation of variables $\psi(x,y,z) = \psi_1(x)\,\psi_2(y)\,\psi_3(z)$

$$\hat{H}\psi = E\psi$$

$$\boxed{-\frac{\hbar^2}{2m}\left[\frac{\psi_1''}{\psi_1} + \frac{\psi_2''}{\psi_2} + \frac{\psi_3''}{\psi_3}\right] + V(x) + V(y) + V(z) = E}$$

$$\underbrace{-\frac{\hbar^2}{2m}\frac{\psi_1''}{\psi_1} + V(x)}_{\text{const.}} = E_1 - V(y) - V(z) + \frac{\hbar^2}{2m}\left[\frac{\psi_2''}{\psi_2} + \frac{\psi_3''}{\psi_3}\right]$$

$\Rightarrow$ like 1D: $-\frac{\hbar^2}{2m}\psi_1'' = (E - V(x))\psi_1$

$$E_{1n_1} = \frac{n_1^2 \hbar^2 \pi^2}{2mL_1^2}$$

$$E = E_{1n_1} + E_{2n_2} + E_{3n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right)$$

Wavefunction 3D Box

$$\psi(x,y,z) = \frac{\sqrt{8}}{\sqrt{L_1L_2L_3}}\sin\left(\frac{n_1\pi}{L_1}x\right)\sin\left(\frac{n_2\pi}{L_2}y\right)\sin\left(\frac{n_3\pi}{L_3}z\right)$$

Cube $L_1 = L_2 = L_3 = L$

$$E_{n_1 n_2 n_3} = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_1^2 + n_2^2 + n_3^2\right)$$

n_1	n_2	n_3	E	Degeneracy
1	1	1	$3E_1$	1
1	1	2	$6E_1$	3
1	2	1		
2	1	1		

Schrödinger equation in spherical coordinates

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r,\theta,\varphi)\right]\psi = E\psi$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right)$$

Separation of variables $\psi(r,\theta,\varphi) = R(r)\,Y(\theta,\varphi)$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2 Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{r^2 Y\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right] + V(r) = E$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left[\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right)\right] + V(r) = E + \frac{\hbar^2}{2m}\left[\frac{1}{r^2 Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{r^2 Y\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right]$$

$$\Rightarrow \quad \boxed{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left(V(r)-E\right) = l(l+1)} \qquad l \in \mathbb{Z}$$

$$\frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right) = -l(l+1)$$

with $Y(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi)$

$$\sin\theta\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + l(l+1)\sin^2\theta + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = 0 \qquad m \in \mathbb{Z}$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2 \;\Rightarrow\; \boxed{\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2\Phi}$$

$$\Rightarrow \boxed{\Phi = e^{im\varphi}}$$

$$\boxed{\sin\theta\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + l(l+1)\sin^2\theta = m^2}$$

$$P_l^m(x) = \left(1+x^2\right)^{\frac{|m|}{2}}\left(\frac{d}{dx}\right)^{|m|}P_l(x)$$

Solution $\Theta(\theta) = A\,P_l^m(\cos\theta)$

$$P_l(x) = \frac{1}{2^l\, l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l$$

Example $P_1^1 = \sin\theta$ $\qquad |m| \leqslant l$

$$P_1^0 = \cos\theta$$

$$P_2^2 = 3\sin^2\theta$$

$$P_2^1 = 3\sin\theta\cos\theta$$

$$P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$$

Normalization of wavefunction

$$\psi(r,\theta,\varphi) = R(r)\underbrace{Y_m^l(\theta,\varphi)}_{e^{im\varphi}\cdot P_l^m(\cos\theta)}$$ Spherical harmonic

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} |R\, Y_m^l|^2 \, r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$\int_0^\infty r^2 |R|^2 dr = 1 \qquad \int_0^\pi \int_0^{2\pi} |Y_m^l|^2 \sin\theta \, d\theta \, d\varphi = 1$$

$$\Rightarrow Y_l^m(\theta,\phi) = \varepsilon\sqrt{\frac{(2l+1)}{4\pi}\,\frac{(l-|m|)!}{(l+|m|)!}}\; e^{im\phi} P_l^m(\cos\theta)$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}(V(r)-E) = l(l+1)$$

$$\rightarrow u(r) = r\cdot R(r), \quad R = \frac{u}{r} \qquad \frac{dR}{dr} = \frac{1}{r^2}\left[r\frac{du}{dr} - u\right]$$

$$\rightarrow \left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \overbrace{\left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]}^{V_{ef}} u(r) = E\,u(r)\right.$$

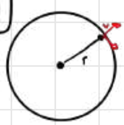
$$V' = \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$ "extra potential" → due to angular momentum

$$\Rightarrow \frac{1}{\cancel{2}}\frac{L^2}{\cancel{mr^2}} = \frac{\hbar^2}{\cancel{2m}}\frac{l(l+1)}{\cancel{r^2}}$$

$$L^2 = \hbar^2 l(l+1)$$

is quantised

$\boxed{\text{classical}}$



$$E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mv_t^2$$ v_t tangential $= \omega r$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2$$

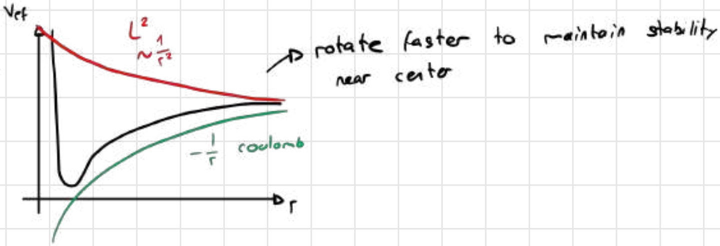
angular velocity

$$\vec{M} = \vec{r}\times\vec{F} \qquad \vec{M} = 0 \rightarrow \vec{r}\parallel\vec{F}$$

$$E_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{L^2}{I} = \frac{1}{2}\frac{L^2}{mr^2}$$

moment of Inertia

with $V = -\frac{e^2}{4\pi\varepsilon_0 r}$ (Coulomb potential)

$\Rightarrow$ 

Spherical well $V(r) = \begin{cases} \infty & r > a \\ 0 & r < a \end{cases}$

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - K^2\right]u \qquad K = \sqrt{\frac{2mE}{\hbar^2}}$$

$$l = 0: \quad \frac{d^2u}{dr^2} = -K^2 u$$

$$u(r) = A\sin(Kr) + B\cos(Kr)$$

$$R(r) = A\frac{\sin(Kr)}{r} + B\frac{\cos(Kr)}{r}$$

$$\lim_{r\to 0} R(r) \text{ diverges} \to B = 0$$

$$R(r) = A\frac{\sin(Kr)}{r}$$

Lecture 14

$K = \frac{\sqrt{-2mE}}{\hbar} \qquad \rho = Kr \qquad \rho_0 = \frac{me^2}{2\pi\varepsilon_0\hbar^2 K}$

$$\Rightarrow \frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

$$\rho \to \infty \quad \frac{d^2u}{d\rho^2} = u \qquad u(\rho) = Ae^{-\rho} + \overset{0}{B}e^{\rho}$$

$$\rho \to 0 \quad \frac{d^2u}{d\rho^2} = \frac{l(l+1)}{\rho^2}u \qquad u(\rho) = C\rho^{l+1} + \overset{0}{D}\rho^{-l}$$

$$\Rightarrow u(\rho) = \rho^{l+1}e^{-\rho}v(\rho)$$

$$\frac{du}{d\rho} = \rho^l e^{-\rho}\left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} = \rho^l e^{-\rho}\left[\left[-2l-2+\rho+\frac{l(l+1)}{\rho}\right]v + 2(l+1-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$\Rightarrow \rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(l+1)]v = 0$$

Ansatz $v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$ determine c_j

$$\Rightarrow \sum_{j=2}^{\infty} j(j-1) c_j \rho^{j-1} + 2(l+1) \sum_{j=1}^{\infty} j c_j \rho^{j-1} - \sum_{j=1}^{\infty} j c_j \rho^j + [\rho_0 - 2(l+1)] \sum_{j=0}^{\infty} c_j \rho^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) c_{j+2} \rho^{j+1} + 2(l+1) \sum_{j=0}^{\infty} (j+1) c_{j+1} \rho^j - \sum_{j=0}^{\infty} (j+1) c_{j+1} \rho^{j+1} + [\rho_0 - 2(l+1)] \sum_{j=0}^{\infty} c_j \rho^j = 0$$

$$\sum_{j=0}^{\infty} [(j+2)(j+1) c_{j+2} - (j+1) c_{j+1}] \rho^{j+1} + [2(l+1)(j+1) c_{j+1} + [\rho_0 - 2(l+1)] c_j] \rho^j] = 0$$

$$\boxed{c_{j+1} = \left[\frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)} \right] c_j}$$

for large j $\quad c_{j+1} \approx \frac{2j}{j(j+1)} c_j = \frac{2}{j+1} c_j \rightarrow c_j = \frac{2^j}{j!} c_0$

$$v(\rho) = c_0 \sum_{j=0}^{\infty} \frac{2^j}{j!} \rho^j = c_0 e^{2\rho} \quad \times$$

$\hookrightarrow$ avoid exponential, must be a finite series

$\rightarrow$ find $c(j_{max}+1) = 0$

$$c_{j+1} = \left[\frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)} \right] c_j$$

$\rightarrow \quad 2(j_{max} + l + 1) - \rho_0 = 0$

$$\Rightarrow \rho_0 = 2 \underbrace{(j_{max} + l + 1)}_{\text{integer } n} = 2n \qquad \rho_0 = \frac{m_e e^2}{2\pi \varepsilon_0 K} \qquad K = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$\rightarrow \frac{m_e e^2}{4\pi n \varepsilon_0} = K = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$-\frac{\hbar^2}{8 m_e \hbar} \left(\frac{m e^2}{2\pi \varepsilon_0 \hbar^2} \right)^2 = \boxed{-\underbrace{\left[\frac{m}{2\hbar} \left(\frac{e^2}{4\pi \varepsilon_0} \right)^2 \right]}_{\text{Rydberg}} \frac{1}{n^2} = E}$$

$$K = \underbrace{\left(\frac{m_e e^2}{4\pi \varepsilon_0 \hbar^2} \right)}_{a_0} \frac{1}{n}$$

$$\boxed{E_n = -E_{Ry} \frac{1}{n^2}} \qquad E_{Ry} = 13.6\,eV$$

Principal quantum number

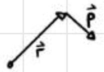
$\rightarrow a_0 \cong 0.52 \cdot 10^{-10} m$
(Bohr radius)

Quantum numbers n, $l = 0, 1, 2, \ldots$, $|m| \leq l$

$$\Rightarrow \boxed{n-1 \geq l \geq |m|}$$

Angular Momentum

Orbital angular momentum



$\vec{L} = \vec{r} \times \vec{p}$

In central potential, it's a constant of motion

$$\vec{M} = \dot{\vec{L}} \qquad \vec{F} = \frac{d\vec{p}}{dt}$$

$$V(r) \qquad \vec{M} = 0 \qquad \vec{L} = \text{const.}$$

Quantum Mechanics

$$\hat{L} = \hat{r} \times \hat{p}$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}p_y$$
$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$
$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}p_x$$

Lecture 15

Degeneracy $\rightarrow$ commuting observables

if two observables are commuting they have a common spektrum of eigenstate

$$\hat{A}|u_n\rangle = \overset{\text{eigenvalue}}{\underset{\text{eigenvalue}}{a_n}}|u_n\rangle \qquad [\hat{H}, \hat{A}] = 0 \qquad |u_n'\rangle = \hat{A}|u_n\rangle$$

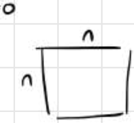
$$\rightarrow \quad \hat{H}(\hat{A}|u_n\rangle) = \hat{A}(\hat{H}|u_n\rangle) = \hat{A}(E_n|u_n\rangle) = \overset{\text{Eigenvalue}}{E_n}\underbrace{\hat{A}|u_n\rangle}_{\text{Eigenvector of Hamiltonian}}$$

Complete Set of commuting observables (CSCO)

n, l, m $\rightarrow E_n$

$\quad\Downarrow$ Angular momentum

$$d(n) = \sum_{l=0}^{n-1}(2l+1) = n^2$$

$E_n \rightarrow$ 

from last time: L_x, L_y, L_z

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] - [\overset{=0}{zp_y, zp_x}] - [\overset{=0}{zp_y, zp_x}] + [zp_y, xp_z]$$

$$= yp_z zp_x - zp_x yp_z + zp_y xp_z - xp_z zp_y$$

$$= yp_x p_z z - yp_x zp_z + xp_y zp_z - xp_y p_z z$$

$$= yp_x\underset{-i\hbar}{[p_z, z]} + xp_y\underset{i\hbar}{[zp_z]} = i\hbar[-yp_x + xp_y] = i\hbar L_z$$

if position and momentum vectors in different direction, they commute

$L^2 = L_x^2 + L_y^2 + L_z^2$

$\hat{L_x} = \hat{y}\hat{p_z} - \hat{z}p_y$

$\hat{L_y} = \hat{z}\hat{p_x} - \hat{x}\hat{p_z}$

$\hat{L_z} = \hat{x}\hat{p_y} - \hat{y}p_x$

Find $[L^2, L_x]$

$[L_x^2, L_x] = 0$

$[A^2, B] = A[A,B] + [A,B]A$

$\left(A^2B - BA^2 = A \cdot AB - A \cdot BA + AB \cdot A - BA \cdot A\right)$

(1) $[L_y^2, L_x] = L_y[L_y, L_x] + [L_y, L_x]L_y$
$= -i\hbar\left(L_yL_z + L_zL_y\right)$

(2) $[L_z^2, L_x] = i\hbar\,[L_zL_y + L_yL_z]$

(1)+(2) $= 0$

$\Rightarrow \boxed{[L^2, L_x] = 0}$

$\underline{L_+ = L_x + iL_y}$ $\quad$ $\underline{L_- = L_x - iL_y}$

$[L_+, L_-] = (L_x + iL_y)(L_x - iL_y) - (L_x - iL_y)(L_x + iL_y)$

$= L_x^2 - iL_xL_y + iL_yL_x + L_y^2 - L_x^2 - iL_xL_y + iL_yL_x - L_y^2$

$= i[L_yL_x - L_xL_y] = i\,[L_y, L_x] = \hbar L_z$

$[L_z, L_\pm] = [L_z, L_x] + [L_z, \pm iL_y]$

$= i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar L_\pm$

$[L^2, L_\pm] = [L^2, L_x] + [L^2, \pm iL_y] = 0$

ex: $\quad f \quad L^2 f = \lambda f$

$L^2(L_\pm f) = L_\pm L^2 f = \underline{\lambda L_\pm f}$

ex: $\quad L_z f = \mu f$

$\underline{L_z(L_\pm f)} = (L_zL_\pm - L_\pm L_z + L_\pm L_z)f$

$= (\pm\hbar L_\pm + L_\pm L_z)f$

$= \pm\hbar L_\pm f + L_\pm \underbrace{L_z f}_{\mu f} = \underline{(\mu \pm \hbar)L_\pm f}$

$\Rightarrow$ Ladder of values $\quad f_+$ "top" $\quad L_+ f_+ = 0$

$\rightarrow$ has to satisfy $L_x^2 + L_y^2 + L_z^2 = L^2$

$\underline{L_z f_+ = l\hbar f_+}$

Express L^2 in terms of $L_\pm$

$L_\pm \times L_\mp = (L_x \pm iL_y)(L_x \mp iL_y)$

$= L_x^2 + L_y^2 \mp iL_xL_y \pm iL_yL_x$

$= \underbrace{L_x^2 + L_y^2}_{L^2 - L_z^2} \mp i\underbrace{[L_x, L_y]}_{i\hbar L_z} = L^2 - L_z^2 \pm \hbar L_z$

$\rightarrow \underline{L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z}$

$$\underline{L^2 f_+} = \left(\underset{\text{choos } \ominus}{L_\pm} L_\mp + L_z^2 \mp \hbar L_z \right) f_+$$

$$= \underset{\overset{\shortparallel}{0}}{L_- L_+ f_+} + L_z^2 f_+ + \hbar L_z f_+ = l^2 \hbar^2 f_+ + l\hbar^2 f_+ = \underline{l(l+1)\hbar^2 f_+}$$

"bottom" f_b

$$\underline{L^2 f_b} = (L_+ L_- + L_z^2 - \hbar L_z) f_b \qquad L_z f_b = \hbar \bar{l} f_b$$

$$= \underset{\overset{\shortparallel}{0}}{L_+ L_- f_b} + L_z^2 f_b - \hbar L_z f_b$$

$$= \bar{l}^2 \hbar^2 f_b - \bar{l}\hbar^2 f_b = \underline{\bar{l}(\bar{l}-1)\hbar^2 f_b}$$

$$\Rightarrow l(l+1) = \bar{l}(\bar{l}-1) \rightarrow \bar{l} = -l$$

goes from $-l \rightarrow l$

$-l + n = l$

$l = n/2 \quad n \in \mathbb{N}_0$

$$\rightarrow -l \leq m \leq l$$

$\underset{\text{EW of } L^2}{l(l+1)\hbar^2}$ and $\underset{\text{EW of } L_z}{m\hbar}$

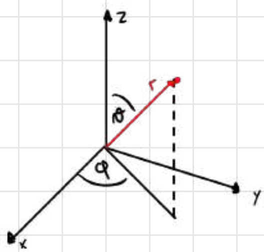
$$\sqrt{L^2} = \sqrt{l(l+1)}\,\hbar$$

$\rightarrow$ you can never have $L = L_z$ due to quantization

$L \geq L_z$

Lecture 16

$$\hat{L} = \hat{r} \times \hat{p} = -i\hbar \hat{r} \times \nabla \qquad \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + \vec{e}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}$$



$$\hat{L} = -i\hbar\, r\hat{e}_r \times \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + \vec{e}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi} \right)$$

$$= -i\hbar \left[\vec{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \vartheta} + \vec{e}_\vartheta \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi} \right]$$

In cartesian coordinates:

$$\hat{L}_x = -i\hbar \left(-\sin\phi \frac{\partial}{\partial \theta} - \cos\theta\cos\phi \frac{1}{\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_y = -i\hbar \left(\cos\phi \frac{\partial}{\partial \theta} - \frac{\cos\theta}{\sin\theta}\sin\phi \frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \right]$$

$\Rightarrow$ same as the angular equation we had before

$$\underline{\hat{L}^2 Y = \hbar^2 l(l+1) Y}$$

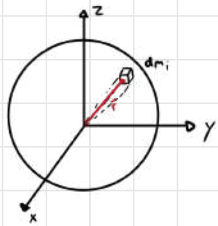
$$\hat{L}_z \Psi = b\Psi \rightarrow \frac{\partial}{\partial \phi} = \frac{ib}{\hbar}\Psi \rightarrow \Psi = Ae^{\frac{ib}{\hbar}\phi} = Ae^{im\phi} \qquad b = \hbar m$$

Angular momentum is quatised!

H-H $\quad K_B T \rightarrow$ rotates $\qquad K_B T < l(l+1)\hbar^2 \quad l=1$

Classical: Earth-Sun

$\vec{L} = \vec{r} \times m_0 \vec{v}$



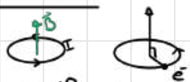
Sum the individual dm_i

$$\vec{L}_S = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times d\vec{p}_i = \sum_i dm_S\, \vec{r}_i \times \vec{v}_i = \sum_i dm_i\, r\vec{\omega}\, \underbrace{\vec{r}_i}_{r\vec{e}_r} \times \vec{e}_\varphi$$

$$= \sum_i r_i^2 \vec{\omega}\, \vec{e}_\varphi\, dm_i$$

$$|\vec{L}_S| = \frac{2}{5} m_0 R^2$$

Magnetism



$I = \frac{\Delta Q}{\Delta t} = e\frac{\omega}{2\pi}$

$\omega = 2\pi\nu = \frac{2\pi}{T}$

Magnetic moment

$\vec{M} = I \cdot A \cdot \hat{n} = I\vec{A}$

$= \frac{e\omega}{2\pi}\pi r^2 = \frac{e\omega r^2}{2}$

$\vec{L} = \vec{r} \times \vec{p} = |\vec{L}| = r m_e v = m_e \omega r^2$

$M = L$

$\frac{e\omega r^2}{2} = m_e \omega r^2$

$\left|\frac{\vec{M}}{\vec{L}}\right| = \frac{e}{2m_e}$ Classical gyromagnetic ratio

Energy associated $\vec{M}$ in a magnetic field $\vec{B}$: $W = -\vec{M}\cdot\vec{B}$

Spacial dependat $\vec{B} = \vec{B}(\vec{r})$

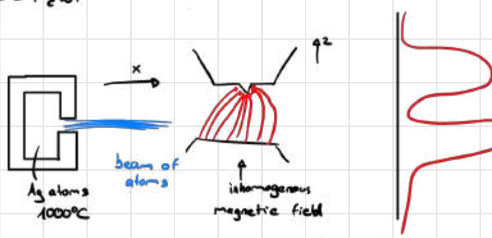
$$\vec{F} = -\nabla(W) = -\nabla(\vec{M}\cdot\vec{B}) = |\vec{M}|\,\nabla B$$

For simplicity $B = B(z)$

$F_z = M_z \nabla_z B \qquad M_z > 0 \rightarrow F_z > 0$

$\qquad\qquad\qquad\quad M_z < 0 \rightarrow F_z < 0$

should be random $\rightarrow$ but spliting "up" or "down"



$\boxed{\text{Stern-Gerlach experiment}}$

$\hat{S}$ spin operator $\hat{S}^2 |\pm\rangle = \hbar^2 (l+1)l|\pm\rangle = \hbar^2 \frac{3}{4}|\pm\rangle \quad l=\frac{1}{2}$ $\Rightarrow$ connection to spin was made later

$|+\rangle, |-\rangle = |\uparrow\rangle, |\downarrow\rangle$

up $\quad$ down

Ag behave like an e^-

Spin is an intrinsic property of Q.M object

2D Vectors $\vec{A} = A_x\hat{x} + A_y\hat{y}$

$\vec{B} = B_x\hat{x} + B_y\hat{y}$

$\vec{A}' = A'_x\hat{x}' + A'_y\hat{y}'$

$\vec{B}' = B'_x\hat{x}' + B'_y\hat{y}'$

$\vec{A}\cdot\vec{B} = \vec{A}'\cdot\vec{B}'$ $\rightarrow$ independent for coordinate system

$$\langle\varphi|\psi\rangle = \int_{-\infty}^{\infty} \varphi^* \psi\, dx$$

We can expand the $|\psi\rangle = \sum_n c_n |\psi_n\rangle$ span the space

$|\varphi\rangle = \sum_n b_n |\psi_n\rangle$

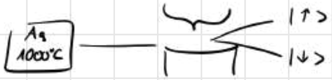
$$\langle \psi | \hat{A} | \psi \rangle = \langle \psi | \hat{A}\psi \rangle = \langle \hat{A}^{\dagger}\psi | \psi \rangle$$

$$\Rightarrow \langle \varphi | \psi \rangle = \sum_{n'} \sum_{n} b_n^* c_{n'} \int_{-\infty}^{\infty} \psi_n^* \psi_{n'}^* dx$$

$$= \sum_n b_n^* c_n$$

if $\hat{A}$ is such that $\hat{A} = \hat{A}^{\dagger}$ ($\hat{A}$ is observable)

$\rightarrow$ the spectrum of $\hat{A}$ has real eigenvalue



$$|\beta\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$

$$\hat{P}_{\uparrow} |\beta\rangle = |\uparrow\rangle\langle\uparrow| (a|\uparrow\rangle + b|\downarrow\rangle)$$
$$= |\uparrow\rangle [a\langle\uparrow|\uparrow\rangle + b\langle\uparrow|\downarrow\rangle]$$
$$= a|\uparrow\rangle$$
$$\hat{P}_{\downarrow} |\beta\rangle = b|\downarrow\rangle$$

Lecture 17

Spin 1/2

$l = \frac{1}{2} \rightarrow m = \pm\frac{1}{2}$ Understand the action of the operators

$|\frac{1}{2}\,\frac{1}{2}\rangle$, $|\frac{1}{2}\,-\frac{1}{2}\rangle$

$\approx |\uparrow\rangle$, $|\downarrow\rangle$

Prototypical 2-level system

$|\psi\rangle = \varphi|\uparrow\rangle + b|\downarrow\rangle \quad a, b \in \mathbb{C}$

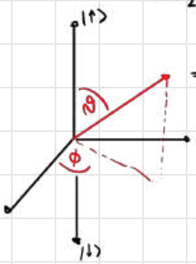
$a^*a + b^*b = 1$

$$|\psi\rangle = |a| e^{i\vartheta_a} |\uparrow\rangle + |b| e^{i\vartheta_b} |\downarrow\rangle$$
$$= e^{i\vartheta_a} \left(|a| |\uparrow\rangle + |b| e^{i(\vartheta_b - \vartheta_a)} |\downarrow\rangle \right)$$
$$= \cos\frac{\vartheta}{2} |\uparrow\rangle \quad \sin\frac{\vartheta}{2} e^{i\phi} |\downarrow\rangle$$

choose $e^{i\vartheta_a} = 1$

$\hookrightarrow |a| = \cos\vartheta/2 \qquad \phi = \vartheta_b - \vartheta_a$

$|b| = \sin\vartheta/2$



$\Rightarrow$ block sphere representation

$$\hat{S}^2 |\uparrow\rangle = \frac{3}{4}\hbar^2 |\uparrow\rangle \qquad \hat{S}^2 |\downarrow\rangle = \frac{3}{4}\hbar^2 |\downarrow\rangle$$

magnitude of the spin

$l = \frac{1}{2}$

$$L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$$

$$|\uparrow\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{S}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{3}{4}\hbar^2$$

$$|\downarrow\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\hat{S}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\hbar}{2} \qquad S_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hbar$$

$$\hat{S}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hbar/2 \qquad S_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \hbar$$

$$\hat{S}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{\hbar}{2}$$

$$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$$

$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

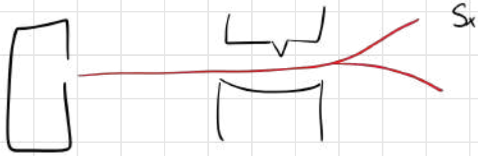
$\hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ Pauli matrices

$\hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

eigenstates of $\hat{S}_x$

$$\det \begin{bmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{bmatrix} = \lambda^2 - \frac{\hbar^2}{4} \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

EV $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Lecture 19

Lamor Precession

Spin + Magnetic field

Nuclear magnetic resonance

Classical case

$\vec{B} \parallel \vec{z}$ $\quad \vec{L}$ angular momentum $\Rightarrow$ electron (charged particle)

$\vec{M} = \mu \vec{L}$

gyromagnetic ratio

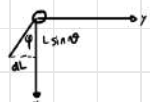
$\vec{\tau} = \vec{M} \times \vec{B} \qquad |\tau| = |M||B| \sin\vartheta$

torque

$$\frac{dL}{dt} = \tau = \mu L B \sin\vartheta$$

$$dL = \mu L B \sin\vartheta \, dt = L \sin\vartheta \, d\varphi$$

$$\frac{d\varphi}{dt} = \mu B = \omega$$



precession

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

$E = -\vec{M} \cdot \vec{B} \qquad |B| \ggg |M|$

$\vec{B} \parallel \vec{z} \qquad \hat{H} = -\mu \vec{S} \cdot \vec{B} = -\mu \hat{S}_z \cdot B = -\mu B \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

E

$E_+ = \mu B \frac{\hbar}{2}$

$E_- = -\mu B \frac{\hbar}{2}$

Time evolution for time-independent S.E.

$$\chi(t) = \cos\left(\frac{\vartheta}{2}\right) e^{-i\frac{E_+}{\hbar}t} \chi_+ + \sin\left(\frac{\vartheta}{2}\right) e^{-i\frac{E_-}{\hbar}t} \chi_-$$

$$\chi(0) = \cos\left(\frac{\vartheta}{2}\right) \chi_+ + \sin\left(\frac{\vartheta}{2}\right) \chi_-$$

$$\langle \hat{S}_x \rangle_\chi = \langle \chi(t) | \hat{S}_x | \chi(t) \rangle$$

$$= \frac{\hbar}{2} \begin{bmatrix} \cos\frac{\vartheta}{2} e^{\frac{i\mu B}{2}t} \\ \sin\frac{\vartheta}{2} e^{-\frac{i\mu B}{2}t} \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\vartheta}{2} e^{-\frac{i\mu B}{2}t} \\ \sin\frac{\vartheta}{2} e^{\frac{i\mu B}{2}t} \end{bmatrix} = \frac{\hbar}{2} \cos\frac{\vartheta}{2} \sin\frac{\vartheta}{2} \left(e^{-i\mu B t} + e^{i\mu B t} \right) = \frac{\hbar}{2} \sin\vartheta \cos(\mu B t)$$

frequency

$\langle \hat{S}_y \rangle_\chi = \frac{\hbar}{2} \sin\vartheta \sin\mu B t$

$\langle \hat{S}_z \rangle_\chi = \frac{\hbar}{2} \cos(\vartheta) \quad \Rightarrow$ time independent

Experiment

$B + B_1(t)$

??

$\mu B \hbar = \Delta E$

Sum of angular momenta

2 spins $\rightarrow$ 4 possibilities

Eigenfunctions $\chi_+^{(1)}\chi_-^{(1)}$, $\chi_+^{(1)}\chi_-^{(2)}$

$$\hat{S} = \hat{S}^{(1)} + \hat{S}^{(2)}$$

no interaction (assumption)

$$\hat{S}_z |\uparrow\downarrow\rangle = (\hat{S}_z^{(1)} + \hat{S}_z^{(2)}) \underbrace{\chi_+^{(1)}\chi_-^{(2)}}_{\text{tensor product}}$$

$$= \chi_-^{(2)} \underbrace{\hat{S}_z^{(1)}\chi_+^{(1)}}_{\hbar/2} + \chi_+^{(1)} \underbrace{S_z^{(2)}\chi_-^{(2)}}_{-\hbar/2} = 0$$

In general: $S_z \chi^{(1)}\chi^{(2)} = (m_1 + m_2)\hbar \chi^{(1)}\chi^{(2)}$

$m = \pm\frac{1}{2} \rightarrow l = 1$

$l = 1$:

triplet $\begin{cases} m \;\; 1 - |\uparrow\uparrow\rangle \\ \;\;\;\; 0 - ? \\ \;\; -1 - |\downarrow\downarrow\rangle \end{cases}$ $\downarrow S_-$

$$S_-|\uparrow\uparrow\rangle = (S_-^{(1)} + S_-^{(2)})|\uparrow\uparrow\rangle = S_-^{(1)}|\uparrow\uparrow\rangle + S_-^{(2)}|\uparrow\uparrow\rangle$$
$$= \hbar(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$

$\rightarrow$ normalize $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$l = 0$: $m = 0$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \rightarrow \text{Singlet}$$

Lecture 20

$$\hat{L}^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle$$
$$\hat{S}^2 |1,0\rangle = \hbar^2 2 |l,m\rangle$$

$$\rightarrow \hat{S}^2 = (\hat{S}^{(1)} + \hat{S}^{(2)})(\hat{S}^{(1)} + \hat{S}^{(2)})$$
$$= \hat{S}^{(1)2} + 2\hat{S}^{(1)}\hat{S}^{(2)} + \hat{S}^{(2)2}$$

$$\hat{S}^{(1)}\hat{S}^{(2)}|\uparrow\downarrow\rangle = |\hat{S}_x^{(1)}\uparrow \; S_x^{(2)}\downarrow\rangle + |\hat{S}_y^{(1)}\uparrow \; \hat{S}_y^{(2)}\downarrow\rangle + |S_z^{(1)}\uparrow \; S_z^{(2)}\downarrow\rangle$$

$\hat S_x|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}|\downarrow\rangle$

$\hat S_x|\downarrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$

$S_y|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{i\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{i\hbar}{2}|\downarrow\rangle$

$S_y|\downarrow\rangle = -\frac{i\hbar}{2}|\uparrow\rangle$

$S_z|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}|\uparrow\rangle$

$S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$

$= |\frac{\hbar}{2}\downarrow \ \ \frac{\hbar}{2}\uparrow\rangle + |\frac{i\hbar}{2}\downarrow \ \ -\frac{i\hbar}{2}\uparrow\rangle + |\frac{\hbar}{2}\uparrow \ \ -\frac{\hbar}{2}\downarrow\rangle$

$= \frac{\hbar^2}{4}\left[2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right]$

$\hat S^{(1)}\hat S^{(2)}|\downarrow\uparrow\rangle = \frac{\hbar^2}{4}\left[2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$

$\hat S^{(1)}\hat S^{(2)}|1\ 0\rangle = \hat S^{(1)}\hat S^{(2)}\left(\frac{1}{\sqrt2}|\uparrow\downarrow\rangle + \frac{1}{\sqrt2}|\downarrow\uparrow\rangle\right)$

$= \frac{\hbar^2}{4\sqrt2}\left(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle + 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

$= \frac{\hbar^2}{4\sqrt2}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) = \frac{\hbar^2}{4}|1\ 0\rangle$

$\hat S^2|1,0\rangle = \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 2\cdot\frac{\hbar^2}{4}\right)|1,0\rangle$

$= 2\hbar^2|1,0\rangle$ ✓

Now generalized to any spin and angular momenta

$|j_1 m_1\ j_2 m_2\rangle \to |J, M\rangle$

what is going to be the value for the projection

$J_z \to M$

$m_J \to m_1+m_2$, $m_J \to |m_1-m_2|$

projection along z of J

$|m_1-m_2| \leq m_J \leq m_1+m_2$

$|J, m_j\rangle = \sum_{m_j=m_l+m_s} C^{l_s}_{m_j m_l m_s}|l, m_l\rangle|s, m_s\rangle$

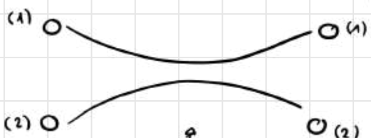
total angular momentum

$\hat J = \hat L + \hat S$

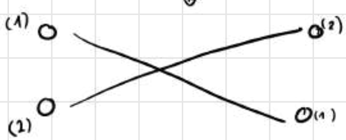
$\hat S = \hat S^{(1)} + \hat S^{(2)}$

Identical Particles

collision



classicaly, those are distinguishable



no collision

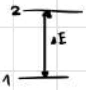
2-particle wavefunction

$\psi(r_1, r_2, t)\chi_1\chi_2$

Lecture 20

→ First part see book
(periodic Table)

Quantum Statistics



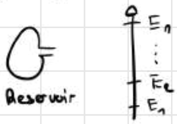
Boltzmann statistics $\frac{\langle \# e^- \text{ st } 2 \rangle}{\langle \# e^- \text{ state } 1 \rangle} = e^{-\frac{\Delta E}{kT}}$

Chemical potential : accounts for #particles → μ

Thermodynamics : $F = \underbrace{U}_{\text{inner energy}} - \underbrace{T}_{\text{temp}}\underbrace{S}_{\text{entropy}}$

$$\mu = \frac{dF}{dN}\bigg|_{T,V=\text{const.}}$$

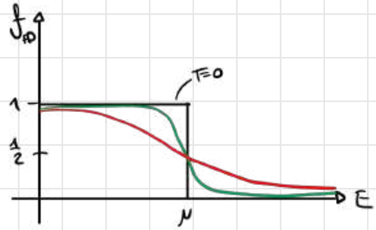
Expressing the occupancy if particles are fermions or bosons



Q : What is the probabity p to find a partical at E_i, T

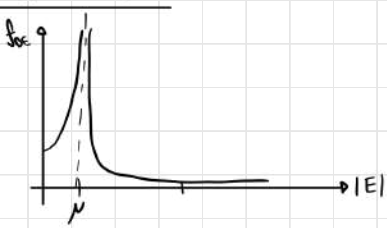
Boson $f_{BE}(\mu, E_i, T) = \frac{1}{e^{\frac{E_i - \mu}{kT}} - 1}$

Fermions $f_{FD}(\mu, E_i, T) = \frac{1}{e^{\frac{E_i - \mu}{kT}} + 1}$



$kT \gg E_f \Rightarrow f_{FD} \simeq f_{BE} \simeq e^{-\frac{E_i - \mu}{k_B T}}$

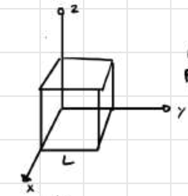
Lecture 21



for photon (massless) $\mu = 0$

Free electron gas

Model a piece of metal (Na)

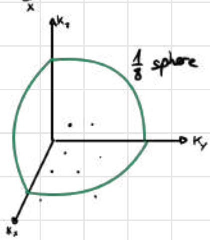


box with infinite potential

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$$

a possible solution: $\psi(x,y,z) = \frac{\sqrt{8}}{\sqrt{L^3}} \sin(k_x x)\sin(k_y y)\sin(k_z z)$

$k_x = l\frac{\pi}{L}$ $\quad l, n, m \in \mathbb{N}$
$k_y = n\frac{\pi}{L}$
$k_z = m\frac{\pi}{L}$

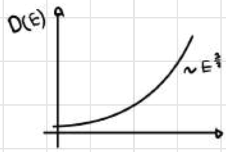


$$N(E) = \frac{1}{8}\frac{4}{3}\pi k^3 \frac{1}{\left(\frac{\pi}{L}\right)^3} \cdot 2 \quad \text{spin}$$

$$N(E) = \frac{1}{3}\frac{k^3}{\pi^2} L^3 \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$N(E) = \frac{1}{3\pi^2} L^3 \left(\frac{2mE}{\hbar^2}\right)^{\frac{3}{2}}$$

$$\frac{N(E)}{V} = D(E) = \frac{1}{3\pi^2}\left(\frac{2mE}{\hbar}\right)^{\frac{3}{2}}$$



in a metal you have a lot of electrons:

$T = 0 \quad E = \mu \to E_F$

$$D_{el} = \frac{1}{3\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{\frac{3}{2}}$$

Invert Formula $\quad E_F = \frac{\hbar^2}{2m}(3\pi^2 \cdot D_{el})^{\frac{2}{3}}$

Na : $\rho \approx 0.97 \cdot 10^{3} \frac{kg}{m^3}$, $m_n = 22.9 \cdot 10^{-3} \frac{kg}{mol}$

assume 1e per atom

$\to D_e = 2{,}5 \cdot 10^{28}\ m^{-3}$

$E_F = 3.14\,eV$ $\quad$ we are not considering the electron interaction and we are neglecting the crystal potential

$k_B T \to 26\,meV$ (at room temp)

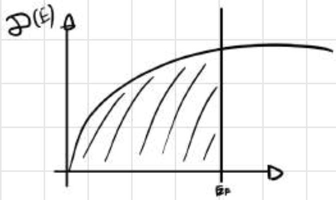
$E_F \gg k_B T$

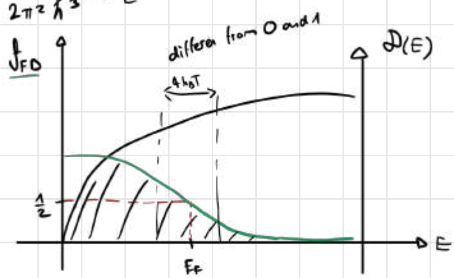
$T \neq 0$

I need to express the density of states

$$\mathcal{D}(E) = \frac{1}{V}\frac{\partial N(E)}{\partial E} = \frac{d}{dE}\left(\frac{1}{3\pi^2}\left(\frac{2mE}{\hbar^2}\right)^{\frac{3}{2}}\right)$$

$$= \frac{\sqrt{8m^3}}{2\pi^2\hbar^3} E^{\frac{1}{2}}$$





$$\Rightarrow n_e = \int_0^\infty \frac{1}{e^{\frac{E-\mu}{kT}}+1}\mathcal{D}(E)\,dE$$

$$= \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty \frac{1}{e^{\frac{E-\mu}{kT}}+1}\sqrt{E}\ dE =$$

Lecture 21

$|\psi\rangle = \frac{1}{\sqrt{3}} |1\ 0\rangle |\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}} |1\ 1\rangle |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$$\left.\begin{aligned} |1\,0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2}\ \tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle \\ |1\,1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}\,|\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle \end{aligned}\right\} \text{tabelle}$$

$|\psi\rangle = \frac{\sqrt{2}}{3}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle - \frac{1}{3}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \frac{\sqrt{2}}{3}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \frac{2}{3}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$= \frac{2\sqrt{2}}{3}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \frac{1}{3}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

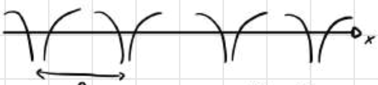
$J^2 |J\ m_J\rangle = J(J+1)\hbar^2 |J\ m_J\rangle$

$\frac{J^2|\frac{3}{2}\ \frac{1}{2}\rangle}{|\frac{3}{2}\ \frac{1}{2}\rangle} = \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2 = \frac{15}{4}\hbar^2 \quad \rightarrow P = \frac{8}{9}$

$\frac{J^2|\frac{1}{2}\ \frac{1}{2}\rangle}{|\frac{1}{2}\ \frac{1}{2}\rangle} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2 \quad \rightarrow P = \frac{1}{9}$

e^- in a periodic potential

→ chain of atoms



$\hat{H} = \frac{p_x^2}{2m} + V(x)$ $p_x^2 \rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$\hat{H}(x+a) = \hat{H}(x)$

$\hat{T}_a \psi(x) = \psi(x+a)$ → commutes with hamiltonian (share eigenvalues)

$\hat{T}_a(\hat{H}\psi(x)) = \hat{H}(x+a)\psi(x+a)$
$= \hat{H}(x)\hat{T}_a\psi(x)$

$T_a \varphi(x) = \lambda\varphi(x) \rightarrow \varphi(x+a) = \lambda\varphi(x)$

$\varphi(x) = e^{ikx} \quad T_a(e^{ikx}) = e^{ik(x+a)} = \underbrace{e^{ika}}_{\lambda} e^{ikx}$

$T_a\varphi = e^{ika}\varphi$

Larger class of solution $e^{ikx}u_{nk}(x)$ with $u_{nk}(x+a) = u_{nk}(x)$ → bloch function

$T_a(e^{ikx}u_{nk}(x)) = e^{ika}e^{ikx}u_{nk}(x+a)$
$= e^{ika}\left[e^{ikx}u_{nk}(x)\right]$

$\langle \nu_p | \hat{H} | \nu_p\rangle = E_0$

take $|\nu_p\rangle$ as a basis   index p is an integer and indicates the position in the 1D chain

$\langle \nu_p | \hat{H} | \nu_{p+1}\rangle = -A$

$\langle \nu_p | \hat{H} | \nu_{p+2}\rangle = 0$

$|\psi\rangle = \sum_p c_p |\nu_p\rangle$

$\hat{H}|\psi\rangle = E|\psi\rangle$

$$\hat{H}|\psi\rangle = \begin{pmatrix} E_0 & -A & & 0 \\ -A & E_0 & -A & \\ & -A & \ddots & -A \\ 0 & & -A & E_0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \end{pmatrix} E$$

$\rightarrow -Ac_{p-1} + E_0 c_p - A c_{p+1} = E c_p$

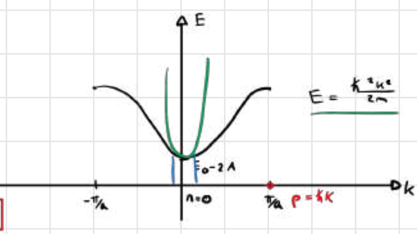
Bloch theorem $\psi(x \pm a) = e^{\pm ika}\psi(x)$

$c_{p+1} = e^{ika}c_p \quad c_{p-1} = e^{-ika}c_p$

$\rightarrow -Ae^{-ika}c_p + E_0 c_p - Ae^{ika}c_p = Ec_p$

$\rightarrow c_p\left(E_0 - E - A(e^{-ika} + e^{ika})\right) = 0$

$E_0 - E - 2A\cos(ka) \rightarrow \boxed{E = E_0 - 2A\cos(ka)}$



For values in $|\ |$ → assumption: like free particle → parabola → approximation to fit → adapt $m \to m'$ effective mass

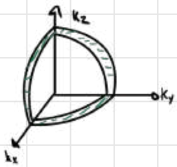
Lecture 22

Free electron model (Recap)

Fermi surface → Fermi energy

$$E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{\frac{2}{3}}$$

electron gas properties → we calculate the total energy



$\frac{1}{8}(4\pi k^2 dK) = \frac{1}{2}\pi k^2 dK$

* e^- in the shell

$$\frac{\frac{1}{2}\pi k^2\, dk}{\pi^3/V}\cdot \underset{\text{spin}}{2} = \frac{V}{\pi^2}k^2 dK$$

each state has an energy: $\frac{\hbar^2 k^2}{2m}$

$$dE = \frac{\hbar^2 k^2}{2m}\frac{V}{\pi^2}k^2\, dk$$

$$E_{tot} = \frac{\hbar^2 V}{2m\pi^2}\int_0^{K_F} k^4\, dK = \frac{\hbar^2 V}{10m\pi^2}K_F^5 = \frac{\hbar^2(3\pi^2 N_d)^{\frac{5}{3}}}{10\pi^2 m}V^{-\frac{2}{3}}$$

Plays a role analogous to the energy U of a gas

$PV = nRT \qquad dW = PdV$

$$dE_{tot} = -\frac{\frac{2}{3}\hbar^2(3\pi N d)^{\frac{5}{3}}}{10\pi^2 m}V^{-\frac{5}{3}}dV = -\frac{2}{3}E_{tot}\frac{dV}{V}$$

$$\underset{\text{pressure}}{P} = \frac{2}{3}\frac{E_{tot}}{V} = \frac{\frac{2}{3}\hbar^2 K_F^5}{10\pi^2 m} = \frac{(3\pi^2)^{\frac{2}{3}}\hbar^2}{5m}\underset{\text{density}}{\rho^{\frac{5}{3}}}$$

quantum pressure of the fermions

$E(K) = E_0 - 2A\cos(Ka)$

$E(K) \cong E_0 - 2A\left(1 - \frac{Ka^2}{2}\right)$

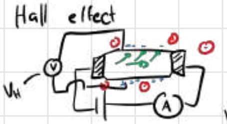
$\frac{\hbar^2 k^2}{2m} = A a^2 K^2$

$\frac{\hbar^2}{2m} = A\underset{\text{lattice constant}}{a^2}$ semiconductor $m_c' < m_e$

$m_{GaAs} \approx 0.067\, m_e$

→ behave like free particle

Hall effect



Magnetic field

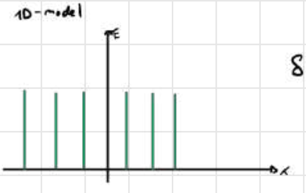
$\vec{F} = q\vec{v}\times\vec{B}$

V_H Hall Voltage

$R_H = \frac{1}{n|q|}$ Hall resistance

"Dirac comb"

1D-model



$\delta(x - Na)$

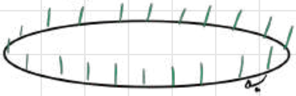
as QW

$\Psi(0) = \Psi(L) = 0$ standing waves

A solid → electron are moving
travelling wave

bloch's theorem

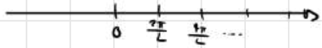
$\Psi_{nk}(x) = e^{ikx} u_{nk}(x)$ (cristall infinite)



Periodic boundary condition

N finite

$\Psi_k = Ae^{ikx}$ after L $e^{ikL} = 1 \to K = 2\pi \frac{n}{L}$ $n \in N$



$$N(k) = \frac{2K}{2\frac{\pi}{L}} = \frac{kL}{\pi}$$

$$V(x) = \alpha \sum_{j=0}^{N-1} \delta(x - ja) \to \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\varphi(x) = E\varphi(x)$$

We will consider boundary conditions $\Rightarrow$ periodic

$]0,a[$ $V(x) = 0 \Rightarrow \varphi(x) = A\sin(kx) + B\cos(kx)$

$\Psi_{nk} = e^{ikx} u_{nk}(x)$

$\to u_{nk}(x+a) = u_{nk}(x)$

$\Psi_{nk}(x+a) = e^{ika} e^{ikx} u_{nk}(x) = e^{ika}\Psi_{nk}(x)$

$\Psi_{nk}(x) = e^{-ika}\Psi_{nk}(x+a)$

Continuity at $x=0$

$A\sin(0) + B\cos(0) = B = e^{-ika}[A\sin(ka) + B\cos(ka)]$

$$\lim_{\varepsilon\to 0}\left[\int_{-\varepsilon}^{\varepsilon}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x)}{\partial x^2} + \underbrace{\alpha\sum_{j=0}^{N-1}\delta(x-ja)\Psi(x)}_{=\alpha\Psi(0)}\right)dx = \underbrace{\int_{-\varepsilon}^{\varepsilon} E\,\Psi(x)dx}_{\to 0}\right]$$

$$= \frac{\partial\Psi}{\partial x}\Big|_{\varepsilon} - \frac{\partial\Psi}{\partial x}\Big|_{-\varepsilon} = \frac{2m\alpha}{\hbar^2}\Psi(0)$$

$$\frac{\partial\Psi'}{\partial x}\Big|_{0^+} = kA\cos(0^+) - kB\sin(0^+) = kA$$

free particle $\to \alpha = 0$

$$\frac{\partial\Psi}{\partial x}\Big|_{0^-} = e^{ika}(kA\cos(ka) - kB\sin(ka))$$

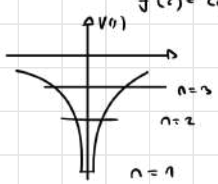
$$\to kA - e^{ika}(kA\cos(ka) - kB\sin(ka)) = \frac{2m\alpha}{\hbar^2}B$$

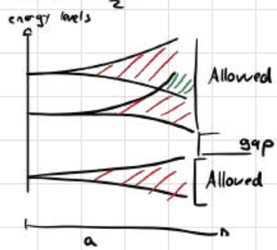
$$\to \underset{\text{free}}{\cos(Ka)} = \cos(ka) + \frac{m\alpha}{k\hbar}\sin(ka)$$

no solutions for $\left|\cos(ka) + \frac{m\alpha}{k\hbar^2}\sin(ka)\right| > 1$

$z = ka$ $\beta = \frac{m\alpha a}{\hbar^2}$ $\to$ energy gap

$$f(z) = \cos(z) + \beta\frac{\sin(z)}{z}$$





Pertubation theory

Non degenerate perturbation + time independent

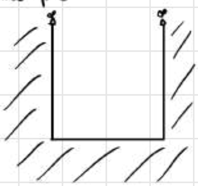
$H^0 \psi_n^0 = E_n^0 \psi_n^0$ subscript 0 $\to$ unperturbed Hamiltonian

$\langle \psi_n^0 | \psi_m^0 \rangle = \delta_{nm}$

$H = H^0 + \lambda H'$ $\to$ peturbation

Lecture 23

example



$K = k^0 + \lambda k'$

$\lambda \ll 1$

$k \psi_n = E_n \psi_n$

$\psi_n = \psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots$

$\Rightarrow E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$

$H_n \psi_n = E_n \psi_n$

$(H^0 + \lambda H^1)(\psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2) = (E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2)(\psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2)$

$\Rightarrow \lambda^3 H^1 \psi_n^2 + \lambda^2 (H^0 \psi_n^2 + H^1 \psi_n^1) + \lambda (H^0 \psi_n^1 + H^1 \psi_n^0) + H^0 \psi_n^0$

$= \lambda^4 (E_n^2 \psi_n^2) + \lambda^3 (E_n^1 \psi_n^2 + E_n^2 \psi_n^1) + \lambda^2 (E_n^0 \psi_n^2 + E_n^1 \psi_n^1 + E_n^2 \psi_n^0) +$

$\lambda (E_n^0 \psi_n^1 + E_n^1 \psi_n^0) + E_n^0 \psi_n^0$

$H^0 \psi_n^2 + H^1 \psi_n^1 = E_n^0 \psi_n^2 + E_n^1 \psi_n^1 + E_n^2 \psi_n^0$

$\left.\begin{aligned} \Rightarrow\ & E_n^2 \psi_n^2 = 0 \\ & H^1 \psi_n^2 = E_n^1 \psi_n^2 + E_n^2 \psi_n^1 \end{aligned}\right\}$ not power of 2

$\to$ What is E_n^1 $E \approx E_n^0 + E_n^1$

apply $\langle \psi_n^0 |$

$H^0 \psi_n^1 + H^1 \psi_n^2 = E_n^0 \psi_n^1 + E_n^1 \psi_n^0$

$H^0 \psi_n^0 = E_n^0 \psi_n^0$

$\Rightarrow \langle \psi_n^0 | H^1 | \psi_n^0 \rangle + \langle \psi_n^0 | H^0 | \psi_n^1 \rangle$

$= \langle \psi_n^0 | E_n^0 | \psi_n^1 \rangle + \langle \psi_n^0 | E_n^1 | \psi_n^0 \rangle$

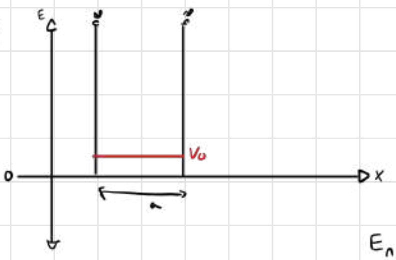
$= E_n^0 \langle \psi_n^0 | \psi_n^1 \rangle + E_n^1$

$\Rightarrow \langle \psi_n^0 | H^1 | \psi_n^0 \rangle + E_n^0 \langle \psi_n^0 | \psi_n^1 \rangle = E_n^0 \langle \psi_n^0 | \psi_n^1 \rangle + E_n^1$

$\boxed{E_n^1 = \langle \psi_n^0 | H^1 | \psi_n^0 \rangle}$ $E_n' \ll E_n^0$

1st order energy correction

example 1)



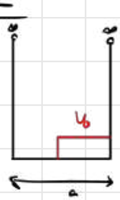
$E_n = \frac{n^2 \pi \hbar^2}{2ma}$

$\psi_n^0 = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$

Perturbation $H' = V_0$

$E_n = E_n^0 + \langle \psi_n^0 | V_0 | \psi_n^0 \rangle = E_n^0 + V_0$

example 2



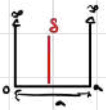
$E = E_n^0 + \langle \psi_n^0 | H' | \psi_n^0 \rangle$

$= E_n^0 + \int_{-\infty}^{\infty} \psi_n^{0*} H' \psi_n^0 \rangle = E_n^0 + V_0 \int_0^{a/2} \psi_n^{0*} \psi_n^0 \, dx = E_n^0 + \frac{2}{a} V_0 \int_0^{a/2} \sin^2\left(\frac{n\pi}{a} x\right) dx = E_n^0 + \frac{V_0}{2}$

if $V_0 \to \infty$ E_n more depended on second term

$E_n = 4 E_n^0$

example 3



$$H' = \alpha\,\delta(x-\tfrac{a}{2})$$

$$E_n = E_n^0 + \langle \psi_n^0 | \alpha\,\delta(x-\tfrac{a}{2}) | \psi_n^0 \rangle$$

$$= E_n^0 + \frac{2}{a}\alpha \int_0^a \sin^2\left(\frac{n\pi}{a}x\right)\delta(x-\tfrac{a}{2})\,dx$$

$$= E_n^0 + \frac{2}{a}\alpha \sin^2\left(\frac{n\pi}{2}x\right)$$

$$= \begin{cases} E_n^0 & n \text{ even} \\ E_n^0 + \frac{2}{a}\alpha & n \text{ odd} \end{cases}$$

What are our ψ_n^1? $\quad \psi_n^1 = \sum_{m\neq n} c_m^n \psi_m^0$

$$H^0\psi_n^1 + H^1\psi_n^2 = E_n^0\psi_n^1 + E_n^1\psi_n^0$$

$$(H^0 - E_n^0)\psi_n^1 = -(H^1 - E_n^1)\,\psi_n^0$$

$$\sum_{m\neq n} c_m^n (H^0 - E_n^0)\,\psi_m^0 = -(H^1 - E_n^1)\,\psi_n^0$$

$$\sum_{m\neq n} c_m^n \left(\langle \psi_L^0 | H^0 | \psi_m^0\rangle - E_n^0 \langle \psi_L^0 | \psi_m^0 \rangle\right) = -\langle \psi_L^0 | H^1 | \psi_n^0 \rangle + \underbrace{E_n \langle \psi_L^0 | \psi_n^0 \rangle}_{0}$$

$$\sum_{m\neq n} c_m^n (E_m^0 - E_n^0)\langle \psi_L^0 | \psi_m^0 \rangle = -\langle \psi_L^0 | H^1 | \psi_n^0 \rangle$$

$$c_L^1 = \frac{\langle \psi_L^0 | H^1 | \psi_n^0 \rangle}{E_n^0 - E_L^0}$$

$$\Rightarrow \boxed{|\psi_n^1\rangle = \sum_{m\neq n} \frac{\langle \psi_m^0 | H^1 | \psi_n^0 \rangle}{E_n^0 - E_m^0} |\psi_m^0\rangle}$$

λ^2

$$H^0\psi_n^2 + H^1\psi_n^1 = E_n^0\psi_n^2 + E_n^1\psi_n^1 + E_n^2\psi_n^0$$

...

$$E_n^2 = \langle \psi_n^0 | H^1 | \psi_n^1 \rangle - E_n^1 \langle \psi_n^0 | \psi_n^1 \rangle$$

$$= \sum_{m\neq n} \frac{|\langle \psi_n^0 | H^1 | \psi_m^0 \rangle|^2}{E_n^0 - E_m^0}$$

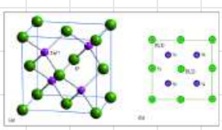
Lecture 23

Application of QM to semiconductor physics

semiconductor
- control conductivity
- bloch theorem
- electron velocity

semiconductor crystals
→ Periodic arrangement of atoms

Zincblende ($\to$ 2 types of atoms) and Diamond ($\to$ 1 type of atom)



→ potential is periodic

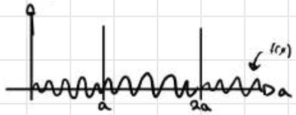
⇒ The implication is Bloch's theorem

Two quantum numbers

n is an integer and labels the band

$\vec{k}$ crystal momentum. Belongs to the "first Brillouin zone"

1D analogy $\Rightarrow$ periodic function over an interval $[0, a]$



$f(x) = \sum_{n=0}^{\infty} A_n e^{i k_n x}$ $\quad k_n = n\frac{\pi}{a}$

We can label the energies $E = E(n, \vec{k})$ $\Rightarrow$ called **band structure**

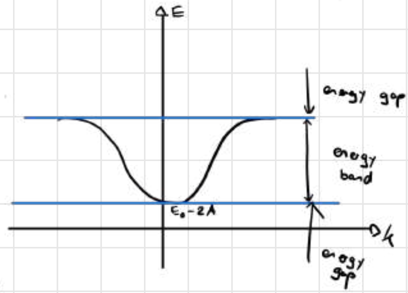
$$N = \int_0^{\infty} \frac{1}{\exp\left(\frac{E-\mu}{kT}\right)+1} D(E)\, dE$$

electron velocity $v_g = \frac{1}{\hbar} \frac{\partial E}{\partial k}$ minimum at k minima

it is maximum where $\left|\frac{\partial E}{\partial k}\right|$ is maximum

(in Si $\sim 10^5 \frac{m}{s}$)

$$E(k) = E_0 - 2A \cos(ka)$$



The "semiclassical" approximation: use Newton

$\vec{p} = \hbar \vec{k}$ $\quad F = \frac{dp}{dt} = \hbar \frac{dk}{dt}$

Motion of electron near a band extremum

$E(k) = E_0 + \frac{1}{2} \frac{\partial^2 E}{\partial k^2} \cdot k^2$ $\quad$ 1D $\quad$ $k=0$

look at $E(k) - E = \frac{1}{2} \left.\frac{\partial^2 E}{\partial k^2}\right|_{k_0} k^2$

free space $V(x) = 0$

$E(k) = \frac{\hbar^2 k^2}{2m} = \frac{1}{2} \frac{\hbar^2}{m} k^2$

$$\frac{1}{m^*} = \frac{1}{\hbar^2} \left(\frac{\partial^2 E}{\partial k^2}\right)$$

$\hookrightarrow$ low effective mass

easier to observe quantum nature

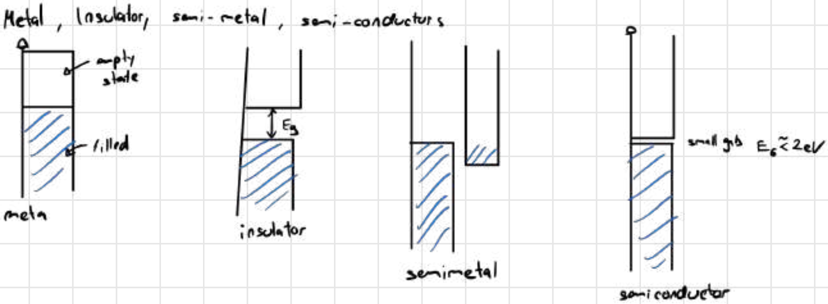
(longer wavelength)

electron statistics

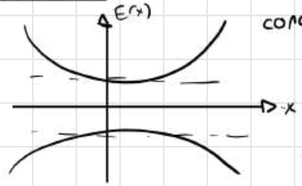
chemical potential μ, P(finding e^-):

$$f(E, \mu, T) = \frac{1}{\exp\left(\frac{E-\mu}{kT}\right)+1}$$

Metal, Insulator, semi-metal, semi-conductors



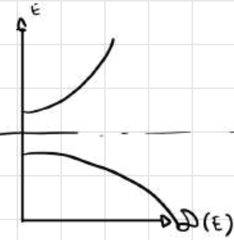
Lecture 24

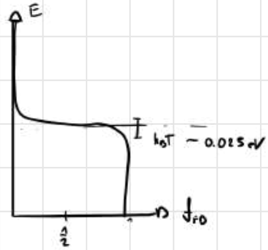


conduction band

gap — E_g

valence band





How to pass current in a semi conductor

$\psi_{nk}(x)$ solution

→ Resistance

What gives finite resistance → Impurity

Semiclassical approximation

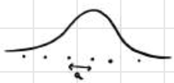
$\vec{k}$ crystal momentum

$$\vec{F} = \hbar \frac{d\vec{k}}{dt}$$

↳ if k changes

: there is already another e^- in the next state

→ cannot → no current



Filled bands are inert

ex: solar cell: photon absorbtion

$\hbar\omega > E_g$

↳ jump of the electron creates a hole

hole: 1) $K_h = -K_e$ $\quad \sum_k \vec{k} = 0$

2) $E(K_h) = -E(K_e)$

3) $m_h = -m_e$

4) $\vec{v}_h = \vec{v}_e$

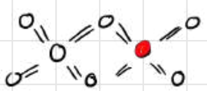
$$\hbar \frac{dK_h}{dt} = e\,(\vec{E} + v_h \times \vec{B})$$

positiv e^- = hole

Impurities

Silicon (4 valence e^-)

n-type



Arsenic: 5 valence e^-

→ but 4 bonds

→ 1 free e^-

$$E_d = \frac{e^4 m_e}{2\hbar^2(4\pi\varepsilon_0\varepsilon)^2} \overset{\sim 0.2\, m_e}{=} 20\,eV \Big\}^{Si} \ll 13.6\,eV$$

p-type

Ga

EPR - Paradox

Einstein - Podolski - Rosen 1964

2-particle

$H = H_1 \otimes H_2$

H_1 bases: $\{|0\rangle_1, |1\rangle_1\}$

H_2 bases: $\{|0\rangle_2, |1\rangle_2\}$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\hat{\sigma}_z|0\rangle = |0\rangle$

$\hat{\sigma}_z|1\rangle = -|1\rangle$

$\psi = \alpha|0\rangle + \beta|1\rangle$

basis for H $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$|\psi\rangle = \sum_{i,j=0} c_{ij}|i\,j\rangle \quad c_{ij} \in \mathbb{C}$

$\sum |c_{ij}|^2 = 1$

1) Reality Principle

2) Locality principle

$\Big\}$ contradict?

2 spin $\frac{1}{2}$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$ entangled state

Alice measures $\sigma_z^{(A)} = +1$

$\rightarrow |01\rangle = |\psi\rangle$

$\sigma_z^{(B)} = +1$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|-+\rangle - |+-\rangle\right)$

$|+\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$|-\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

$\hat{\sigma}_x |+\rangle = |+\rangle$

$\hat{\sigma}_x |-\rangle = -|-\rangle$

$\hat{\sigma}_x^{(A)} = +1$

$\hat{\sigma}_x^{(B)} = -1 \quad \hat{\sigma}_z = \begin{cases} +1 & 50\% \\ -1 & 50\% \end{cases}$

1964 → Bell's inequalities   1982 Alain aspect

(CHSH)

⟹ QM is not locally real

Qubit

Bit : 0,1

transistor 1948

Qubit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

$\alpha, \beta \in \mathbb{C}$

$0 \leq \theta \leq \pi$

$0 \leq \varphi < 2\pi$

- prepare qubit in a state
- manipulation
- measur it

entanglement