Anthony Vasquez
MATH 114
HW-7

3.1

10, 12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figure 3 and 9 and, if neccessary, the transformations of section 1.2.

(10) $y = e^{|x|}$

See attched sketch

(12) $y = 2(1 - e^{x})$

See attached sketch

16(b) Find the domain of the function. $g(t) = \sqrt{1 - 2^{t}}$

For the domain of $g(t)$ to be defined: $1 - 2^{t} > 0$

$$2^{t} < 1$$

Take the log of both sides:

$$\log(2^{t}) < \log(1) \Leftrightarrow t\log(2) < \log(1) \Leftrightarrow t < \frac{\log(1)}{\log(2)} \Leftrightarrow t < 0$$

31 If you graph the function:

$$f(x) = \frac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}$$

you'll see that it's an odd function. Prove it.

A function is odd if $f(-x) = -f(x)$

$$f(-x) = \frac{1 - e^{\frac{1}{-x}}}{1 + e^{\frac{1}{-x}}} = \frac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} = \frac{\frac{1}{e^{\frac{1}{x}}} - 1}{1 + \frac{1}{e^{\frac{1}{x}}}} = \frac{\frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}}}}{\frac{1 + e^{\frac{1}{x}}}{e^{\frac{1}{x}}}} = \frac{e^{\frac{1}{x}} - 1}{1 + e^{\frac{1}{x}}} = \frac{-(1 - e^{\frac{1}{x}})}{1 + e^{\frac{1}{x}}} \Leftrightarrow$$

$$\frac{-(1 - e^{\frac{1}{x}})}{1 + e^{\frac{1}{x}}} = -\left(\frac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}\right)$$

$$\boxed{\therefore f(x) \text{ is odd since, } f(-x) = -\left(\frac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}\right) = -f(x)}$$

3.2

16 If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$

For $x = 3$:

$$f^{-1}(3) = x \Leftrightarrow f(x) = 3 \Leftrightarrow x^5 + x^3 + x = 3 \Leftrightarrow x^5 + x^3 + x - 3 = 0$$

Based on the rational root theorem, the possible roots are: $\pm 1, \pm\frac{1}{3}$

However, only 1 satisfies for $f(x)$, so:

$$\boxed{f^{-1}(3) = 1}$$

$$\boxed{f(f^{-1}(2)) = 2 \text{ by the proprty of cancellation equations for } f \text{ and } f^{-1}}$$

18 The graph of f is given.

(a) Why is f one-to-one

$\boxed{f \text{ is one-to-one because for every } x \text{ there is a unique } f(x)}$

(b) What are the domain and range of f^{-1}?

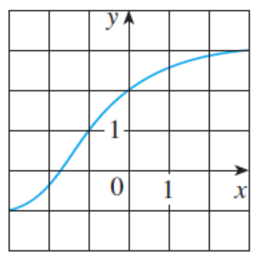
$\boxed{Dom(f^{-1}) = [-1, 3], Ran(f^{-1}) = [-2, 3]}$

(c) What is the value of $f^{-1}(2)$?

$\boxed{0}$

(d) Estimate the value of $f^{-1}(0)$?

$\boxed{1.90}$



32, 34 - (a) Show that f is one-to-one

(b) Use Theorem 7 to find $(f^{-1})'(a)$

(c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.

(d) Calculate $(f^{-1})'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).

(e) Sketech the graphs of f and f^{-1}

(32) $f(x) = \sqrt{x-2}, a = 2$

(a)

If f is one-to-one(injective), then suppose:

$$f(x_1) = f(x_2) \Leftrightarrow \sqrt{x_1 - 2} = \sqrt{x_2 - 2} \Leftrightarrow x_1 - 2 = x_2 - 2 \Leftrightarrow x_1 = x_2$$

Then f is injective since x_1 and x_2 are the same value.

(b)

$$\text{Inverse Function Theorem Formula: } (f^{-1})'(a) = \frac{1}{f'((f^{-1})(a))}$$

For $a = 2$:

$$(f^{-1})'(2) = x \Leftrightarrow f(x) = 2 \Leftrightarrow \sqrt{x-2} = 2 \Leftrightarrow x - 2 = 4 \Leftrightarrow x = 6 \Leftrightarrow (f^{-1})'(2) = 6$$

Then find $f'(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x-2}) = \frac{1}{2\sqrt{x-2}}$$

Thus,

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$$(f^{-1})'(2) = \frac{1}{f'(6)} \Leftrightarrow (f^{-1})'(2) = \frac{1}{\frac{1}{2\sqrt{6-2}}} = \frac{1}{\frac{1}{4}} = \boxed{(f^{-1})'(2) = 4}$$

(c)

Swap x & y and solve for y:

$$x = \sqrt{y-2} \Leftrightarrow x^2 = y-2 \Leftrightarrow x^2+2 = y \Leftrightarrow \boxed{f^{-1}(x) = x^2+2}$$

The domain and range of f^{-1} is the domain and range of $f(x)$ swapped:

$$\boxed{Dom(f^{-1}) = [0,\infty), Ran(f^{-1}) = [2,\infty)}$$

(d)

$$f^{-1}(x) = x^2+2$$

$$(f^{-1})'(x) = 2x$$

$$\boxed{(f^{-1})'(2) = 2*2 = 4}$$

(e)

See attached sketch

(34) $f(x) = \frac{1}{(x-1)}, x > 1, a = 2$
(a)

If f is one-to-one(injective), then suppose:

$$f(x_1) = f(x_2) \Leftrightarrow \frac{1}{x_1-1} = \frac{1}{x_2-1} \Leftrightarrow x_1 - 1 = x_2 - 1 \Leftrightarrow x_1 = x_2$$

Then f is injective since x_1 and x_2 are the same value.

(b)

Inverse Function Theorem Formula: $(f^{-1})'(a) = \dfrac{1}{f(f^{-1}(a))}$

For $a = 2$:

$$f^{-1}(2) = x \Leftrightarrow f(x) = 2 \Leftrightarrow \frac{1}{x-1} = 2 \Leftrightarrow x - 1 = \frac{1}{2} \Leftrightarrow x = \frac{3}{2} \Leftrightarrow f^{-1}(2) = \frac{3}{2}$$

Then find $f'(x)$:

$$f(x) = \frac{1}{x-1} \Leftrightarrow f'(x) = -\frac{1}{(x-1)^2}$$

Thus,

$$(f^{-1})'(a) = \frac{1}{f'(\frac{3}{2})} \Leftrightarrow (f^{-1})'(a) = \frac{1}{-\frac{1}{(\frac{3}{2}-1)^2}} \Leftrightarrow -\frac{1}{(x-1)^2} = -\frac{1}{\frac{1}{\frac{1}{4}}} = \Leftrightarrow -\frac{1}{(x-1)^2} = -\frac{1}{4}$$

(c)

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Swap x & y and solve for y:$x = \dfrac{1}{y-1} \Leftrightarrow \dfrac{1}{x} = y - 1 \Leftrightarrow \dfrac{1}{x} + 1 = y \Leftrightarrow \boxed{f^{-1} = \frac{1}{x} + 1}$

Domain and range of $f(x)$:

By the graph of $f(x)$ *No attached sketch yet*

$$Dom(f) = (1, \infty), Ran(f) = (0, \infty)$$

Swap the domain and range for $f(x)$ to get the domain and range for $f^{-1}(x)$:

$$\boxed{Dom(f^{-1}) = (0, \infty), Ran(f^{-1}) = (1, \infty)}$$

(d)

$$f^{-1}(x) = \frac{1}{x} + 1 \Leftrightarrow (f^{-1})'(x) = -\frac{1}{x^2} \Leftrightarrow (f^{-1})'(2) = -\frac{1}{2^2} \Leftrightarrow \boxed{(f^{-1})'(2) = -\frac{1}{4}}$$

(e)

See attached sketch

40 Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = \frac{1}{f^{-1}(x)}$. If $f(3) = 2$ and $f'(3) = \frac{1}{9,}$ find $G'(2)$.

Derivative of $G'(x)$ by the Chain rule :

$$G(x) = \frac{1}{f^{-1}(x)} \Leftrightarrow G(x) = [f^{-1}(x)]^{-1}$$

$$G'(x) = -[f^{-1}(x)]^{-2} * (f^{-1})'(x)$$

Since $f^{-1}(x)$ is the inverse function of a differentiable function f, then apply the Inverse Function Theorem Formula:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$G'(x) = -\frac{1}{[f^{-1}(x)]^2} * \frac{1}{f'(f^{-1}(x))}$$

$$f(3) = 2 \Leftrightarrow f^{-1}(2) = 3$$

$$G'(2) = -\frac{1}{[f^{-1}(2)]^2} * \frac{1}{f'(f^{-1}(2))} \Leftrightarrow G'(2) = -\frac{1}{3^2} * \frac{1}{f'(3)} \Leftrightarrow G'(2) = -\frac{1}{9} * \frac{1}{\frac{1}{9}} \Leftrightarrow G'(2) = -\frac{1}{9} * 9 \Leftrightarrow$$

$$\boxed{G'(2) = -1}$$

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50 Use the properties of logarithms to expand the quantity. $\ln\left(s^4\sqrt{t\sqrt{u}}\right)$

Apply the Logarithm product rule: $\log(a * b) = \log(a) + \log(b)$

$$\ln\left(s^4\sqrt{t\sqrt{u}}\right) \Leftrightarrow \ln\left(s^4\right) + \ln\left(\sqrt{t\sqrt{u}}\right)$$

$$\ln\left(\sqrt{t\sqrt{u}}\right) = \ln\left((t * \sqrt{u})^{\frac{1}{2}}\right) = \frac{1}{2}\ln\left(t * \sqrt{u}\right)$$

$$\ln\left(s^4\right) + \frac{1}{2}\ln\left(t * \sqrt{u}\right) = 4\ln(s) + \frac{1}{2}\ln\left(t * \sqrt{u}\right) \Leftrightarrow \frac{1}{2}[8\ln(s) + \ln\left(t * (u)^{\frac{1}{2}}\right)] \Leftrightarrow$$

$$\frac{1}{2}[8\ln(s) + \ln(t) + \ln\left((u)^{\frac{1}{2}}\right)] \Leftrightarrow \frac{1}{2}[8\ln(s) + \ln(t) + \frac{1}{2}\ln(u)] =$$

$$\boxed{4\ln(s) + \tfrac{1}{2}\ln(t) + \tfrac{1}{4}\ln(u)}$$

52 Express the given quantity as a single logarithm. $\ln(a+b) + \ln(a-b) - 2\ln(c)$

$$\ln([(a+b) * (a-b)]) - \ln\left(c^2\right) \Leftrightarrow \ln\left(\frac{(a+b) * (a-b)}{c^2}\right) \Leftrightarrow \ln\left(\frac{a^2-b^2}{c^2}\right)$$

60 Make a rough sketch of the graph of each function. Just use the graphs given in Figures 14 and 15 and, if neccessary, the transformations of section 1.2.

(a) $y = \ln(-x)$
See attached sketch
(b) $y = ln(|x|)$
See attached sketch

64, 66 Solve each equation for x.

(64)
- (a) $\ln\left(x^2 - 1\right) = 3$

$$e^{\ln\left(x^2-1\right)} = e^3 \Leftrightarrow x^2 - 1 = e^3 \Leftrightarrow x^2 = e^3 + 1 \Leftrightarrow \boxed{x = \pm\sqrt{e^3+1}}$$

- (b) $e^{2x} - 3e^x + 2 = 0$

Factorize & Determine solutions:

$$(e^x - 1)(e^x - 2) = 0$$

$$e^x - 1 = 0 \Leftrightarrow e^x = 1 \Leftrightarrow \ln(e^x) = \ln(1) \Leftrightarrow x = \ln(1)$$

$$e^x - 2 = 0 \Leftrightarrow e^x = 2 \Leftrightarrow \ln(e^x) = \ln(2) \Leftrightarrow x = \ln(2)$$

$$\boxed{\therefore x = \{0, \ln(2)\}}$$

(66)
- (a) $\ln(\ln(x)) = 1$

Raise to the power of e:

$$e^{\ln(\ln(x))} = e^1 \Leftrightarrow \ln(x) = e$$

$$e^{\ln(x)} = e^e \Leftrightarrow \boxed{x = e^e}$$

- (b) $e^{ax} = Ce^{bx}$, where $a \neq b$

$$\ln(e^{ax}) = \ln\left(Ce^{bx}\right) \Leftrightarrow \ln(e^{ax}) = \ln(C) + \ln\left(e^{bx}\right) \Leftrightarrow$$

$$ax = \ln(C) + bx \Leftrightarrow ax - bx = \ln(C) \Leftrightarrow x(a-b) = \ln(C) \Leftrightarrow \boxed{x = \tfrac{\ln(C)}{a-b}}$$

68 Solve each inequality for x

(a) $1 < e^{3x-1} < 2$

Take the log for both sides:

$$\ln(1) < \ln\left(e^{3x-1}\right) < \ln(2) \Leftrightarrow \ln(1) < 3x - 1 < \ln(2) \Leftrightarrow \ln(1) + 1 < 3x < \ln(2) + 1 \Leftrightarrow \frac{\ln(1)+1}{3} < x < \frac{\ln(2)+1}{3} \Leftrightarrow$$

$$\boxed{\tfrac{1}{3} < x < \tfrac{\ln(2)+1}{3}}$$

(b) $1 - 2\ln(x) < 3$

$$-2\ln(x) < 2 \Leftrightarrow 2\ln(x) > -2 \Leftrightarrow \ln\left(x^2\right) > -2$$

Raise to the power of e on both sides:

$$e^{\ln\left(x^2\right)} > e^{-2} \Leftrightarrow x^2 > e^{-2} \Leftrightarrow x^2 > \frac{1}{e^2} \Leftrightarrow x > \sqrt{\frac{1}{e^2}} \Leftrightarrow \boxed{x > \tfrac{1}{|e|}}$$

76 Find the limit. $\lim\limits_{x\to\infty} [\ln(2+x) - \ln(1+x)]$

Apply the Quotient property of logs: $\log(x) - \log(y) = \log\left(\dfrac{x}{y}\right)$

$$\lim_{x\to\infty} \ln\left(\frac{2+x}{1+x}\right)$$

Since the polynomials and $\ln(x)$ are continuous functions, then its limit can be interchanged:

$$\ln\left(\lim_{x\to\infty} \frac{2+x}{1+x}\right)$$

Factorize the greatest power of x on numerator & denominator:

$$\ln\left(\lim_{x\to\infty} \frac{x(\frac{2}{x}+1)}{x(\frac{1}{x}+1)}\right) \Leftrightarrow \ln\left(\lim_{x\to\infty} \frac{\frac{2}{x}+1}{\frac{1}{x}+1}\right) \Leftrightarrow \ln\left(\frac{1}{1}\right) = \ln(1) = \boxed{0}$$

3.3

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10, 16, 20, 30, 36 Differentiate the function.

(10) $f(u) = \frac{u}{1+\ln(u)}$

Apply the quotient rule and derivative of $\ln(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{x}$$

$$f'(u) = \frac{1 * (1 + \ln(u)) - u * \frac{1}{u}}{(1 + \ln(u))^2} = \frac{1 + \ln(u)) - 1}{(1 + \ln(u))^2} = \boxed{\frac{\ln(u)}{(1+\ln(u))^2}}$$

(16) $y = \ln(|cos(\ln(x))|)$

Apply the Chain rule

$$y' = \frac{1}{|\cos(\ln(x))|} * [|\cos(\ln(x))|]'$$

Derivative of $|\cos(\ln(x))|$ by the Chain rule:

$$|\cos(\ln(x))| = \sqrt{(|\cos(\ln(x))|)^2} = ((|\cos(\ln(x))|)^2)^{\frac{1}{2}}$$

$$[((|\cos(\ln(x))|)^2)^{\frac{1}{2}}]' = \frac{1}{2}((|\cos(\ln(x))|)^2)^{-\frac{1}{2}} * [-\sin(\ln(x)) * \frac{1}{x}] = \frac{-\sin(\ln(x))}{2x\sqrt{(|\cos(\ln(x))|)^2}} = \frac{-\sin(\ln(x))}{2x|\cos(\ln(x))|}$$

Thus,

$$y' = \frac{1}{|\cos(\ln(x))|} * \frac{-\sin(\ln(x))}{2x|\cos(\ln(x))|} = \boxed{\frac{-\sin(\ln(x))}{2x\cos^2(\ln(x))}}$$

(20) $g(x) = \sqrt{x}e^x$

Apply the product rule and derivate of e^x:

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$$

$$g'(x) = \frac{1}{2\sqrt{x}} * e^x + \sqrt{x} * (e^x) = \frac{e^x}{2\sqrt{x}} + \sqrt{x}e^x = \boxed{\frac{2xe^x+e^x}{2\sqrt{x}}}$$

(30) $y = \sqrt{1 + xe^{-2x}}$

Apply the Chain rule:

$$\frac{1}{2\sqrt{1 + xe^{-2x}}} * [1 + xe^{-2x}]'$$

Derivative of $[1 + xe^{-2x}]'$ by product rule:

$$1 * e^{-2x} + x * [e^{-2x}]'$$

Derivative of e^{-2x} by logarithmic differentiation:

$$f(x) = e^{-2x} \Leftrightarrow \ln(f(x)) = -2x\ln(e) \Leftrightarrow \ln(f(x)) = -2x \Leftrightarrow \frac{f'(x)}{f(x)} = -2 \Leftrightarrow f'(x) = -2f(x) \Leftrightarrow f'(x) = -2e^{-2x}$$

Thus, $[1+xe^{-2x}]'$ is:

$$1 * e^{-2x} + x * -2e^{-2x} = e^{-2x} - 2xe^{-2x}$$

Which completes the calculation for the Chain rule:

$$\frac{1}{2\sqrt{1+xe^{-2x}}} * e^{-2x} - 2xe^{-2x} = \boxed{\frac{e^{-2x}-2xe^{-2x}}{2\sqrt{1+xe^{-2x}}}}$$

(36) $y = x^2 e^{-\frac{1}{x}}$

Apply the product rule:

$$2x * e^{-\frac{1}{x}} + x^2 * [e^{-\frac{1}{x}}]'$$

The derivative of $e^{-\frac{1}{x}}$:

$$f(x) = e^{-\frac{1}{x}} \Leftrightarrow \ln(f(x)) = \ln(e) \Leftrightarrow \ln(f(x)) = -\frac{1}{x}\ln(e) \Leftrightarrow \ln(f(x)) = -\frac{1}{x}$$

Take derivative of both sides and apply the derivate of $\ln(f(x))$: $\frac{f'(x)}{f(x)}$

$$\frac{f'(x)}{f(x)} = \frac{1}{x^2} \Leftrightarrow f'(x) = f(x) * \frac{1}{x^2} = \frac{e^{-\frac{1}{x}}}{x^2}$$

$$y' = 2x * e^{-\frac{1}{x}} + x^2 * \frac{e^{-\frac{1}{x}}}{x^2} = \boxed{2xe^{-\frac{1}{x}} + e^{-\frac{1}{x}}}$$

56, 58, 60 Use the logarithmic differentiation or an alternative method to find the derivative of the function.

(56) $y = x^{cos(x)}$

Let $y = f(x)$:

$$f(x) = x^{\cos(x)}$$

Take logs of both sides:

$$\ln(f(x)) = \ln\left(x^{\cos(x)}\right) \Leftrightarrow \ln(f(x)) = \cos(x)\ln(x)$$

Differentiate both sides:

$$\frac{f'(x)}{f(x)} = -\sin(x) * \ln(x) + cos(x) * \frac{1}{x} \Leftrightarrow$$

$$f'(x) = f(x) * [\frac{\cos(x)}{x} - \sin(x)\ln(x)] = \boxed{x^{\cos(x)}[\frac{\cos(x)}{x} - \sin(x)\ln(x)]}$$

(58) $y = \sqrt{x^x}$

Let $y = f(x)$:

$$f(x) = \sqrt{x^x} \Leftrightarrow f(x) = (x^x)^{\frac{1}{2}} \Leftrightarrow f(x) = (x)^{\frac{x}{2}}$$

Takes logs of both sides:

$$\ln(f(x)) = \ln\Big((x)^{\frac{x}{2}}\Big) \Leftrightarrow \ln(f(x)) = \frac{x}{2}\ln(x)$$

Differentiate both sides:

$$\frac{f'(x)}{f(x)} = \frac{1}{2} * \ln(x) + \frac{x}{2} * \frac{1}{x} \Leftrightarrow f'(x) = f(x) * [\frac{\ln(x)}{2} + \frac{1}{2}] = \boxed{\sqrt{x^x}[\frac{\ln(x)}{2} + \frac{1}{2}]}$$

(60) $y = (sin(x))^{\ln(x)}$

Let $y = f(x)$

$$f(x) = (sin(x))^{\ln(x)}$$

Take log of both sidse:

$$\ln(f(x)) = \ln\Big((sin(x))^{\ln(x)}\Big) \Leftrightarrow \ln(f(x)) = \ln(x)\ln(\sin(x))$$

Differentiate both sides:

$$\frac{f'(x)}{f(x)} = \frac{1}{x}*\ln(\sin(x))+\ln(x)*[\frac{1}{\sin(x)}*\cos(x)] \Leftrightarrow f'(x) = f(x)*[\frac{\ln(\sin(x))}{x}+\ln(x)\cot(x)] = \boxed{(sin(x))^{\ln(x)}[\frac{\ln(\sin(x))}{x} + \ln(x)\cot(x)]}$$

68 For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 5y' - 6y = 0$?

Find y'by logarithmic differentiation:

$$\text{Recall: } \frac{\mathrm{d}}{\mathrm{d}u}(e^u) = e^u * \frac{\mathrm{d}u}{\mathrm{d}x}$$

$$y' = r * e^{rx} = re^{rx}$$

Differentiate again for y":

$$y'' = r * re^{rx} = r^2e^{rx}$$

Subsitute y, y', y'' into equation:

$$r^2e^{rx} + 5re^{rx} - 6e^{rx} = 0$$

$$e^{rx}(r^2 + 5r - 6) = 0 \Leftrightarrow (r^2 + 5r - 6) = 0 \Leftrightarrow (r + 6)(r - 1) = 0$$

Thus, the values of r that satisfy the equation are:

$$\boxed{r = \{1, -6\}}$$

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8, 10 Simplify the expression.
(8) $\tan\left(\sin^{-1}(x)\right)$ *See attached work*
(10) $\cos\left(2\tan^{-1}(x)\right)$ *See attached work*

12 (a) Prove that $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$
(b) Use part (a) to prove formula 6.

14 Prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1}(x)) = \dfrac{1}{x\sqrt{x^2-1}}$

See attached work

26, 28 Find the derivative of the function. Simplify where possible.
(26) $f(x) = x\ln(\arctan(x))$

Apply the product rule:

$$f'(x) = 1 * \ln(\arctan(x)) + x * [\ln(\arctan(x))]'$$

The derivative of $[\ln(\arctan(x))]'$ by the Chain rule

$$[\ln(\arctan(x))]' = \frac{1}{\arctan(x)} * \frac{1}{x^2+1}$$

Thus,

$$f'(x) = 1 * \ln(\arctan(x)) + x * [\frac{1}{\arctan(x)} * \frac{1}{x^2+1}] = \boxed{\ln(\arctan(x)) + \frac{x}{(x^2+1)\arctan(x)}}$$

(28) $y = \arctan\left(\sqrt{\frac{1-x}{1+x}}\right)$

Apply the Chain rule:

$$y' = \frac{1}{(\sqrt{\frac{1-x}{1+x}})^2+1} * \frac{1}{2\sqrt{\frac{1-x}{1+x}}} * \frac{-1*(x+1)-(1-x)*1}{(x+1)^2} \Leftrightarrow$$

$$y' = \frac{1}{\frac{1-x}{1+x}+1} * \frac{1}{2\sqrt{\frac{1-x}{1+x}}} * \frac{-x-1-1+x}{(x+1)^2} \Leftrightarrow$$

$$y' = \frac{1}{\frac{1-x}{1+x}+\frac{1+x}{1+x}} * \frac{1}{2\sqrt{\frac{1-x}{1+x}}} * \frac{-2}{(1+x)^2} \Leftrightarrow$$

$$y' = \frac{1}{\frac{2}{1+x}} * \frac{1}{2\sqrt{\frac{1-x}{1+x}}} * \frac{-2}{(1+x)^2} \Leftrightarrow y' = \frac{1+x}{2} * \frac{1}{2\sqrt{\frac{1-x}{1+x}}} * \frac{-2}{(1+x)^2} \Leftrightarrow y' = \frac{-1}{(1+x)} * \frac{1}{\frac{2\sqrt{1-x}}{\sqrt{1+x}}} \Leftrightarrow$$

$$y' = \frac{-1}{(1+x)} * \frac{\sqrt{1+x}}{2\sqrt{1-x}} \Leftrightarrow y' = \frac{-1}{\sqrt{1+x}} * \frac{1}{2\sqrt{1-x}} \Leftrightarrow y' = \frac{-1}{2\sqrt{(1-x)(1+x)}} = \boxed{\frac{-1}{2\sqrt{1-x^2}}}$$

3.6

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4 Find the numerical value of the experssion.
(a) cosh(3)

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh(3) = \frac{e^3 + e^{-3}}{2} \approx \boxed{10.067}$$

(b) cosh(ln(3))

$$\cosh(\ln(3)) = \frac{e^{\ln(3)} + e^{-\ln(3)}}{2} = \cosh(\ln(3)) = \frac{e^{\ln(3)} + e^{\ln\left(3^{-1}\right)}}{2}$$

$$\text{Note: } e^{\ln(x)} = x$$

$$\frac{3 + \frac{1}{3}}{2} = \frac{\frac{10}{3}}{2} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

12 Prove the identity. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

16 If $\tanh(x) = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

$$\text{Use the trig identity: } \tanh(x) = \sqrt{1 - sech^2(x)}$$

$$\sqrt{1 - sech^2(x)} = \frac{12}{13} \Leftrightarrow 1 - \operatorname{sech}^2(x) = \frac{144}{169} \Leftrightarrow \operatorname{sech}(x) = \sqrt{1 - \frac{144}{169}} \Leftrightarrow \operatorname{sech}(x) = \sqrt{\frac{25}{169}} \Leftrightarrow \operatorname{sech}(x) = \frac{5}{13} \Leftrightarrow$$

$$\frac{1}{\cosh(x)} = \frac{5}{13} \Leftrightarrow \boxed{\cosh(x) = \frac{13}{5}}$$

$$\tanh(x) = \frac{12}{13} \Leftrightarrow \frac{\sinh(x)}{\cosh(x)} = \frac{12}{13} \Leftrightarrow \frac{\sinh(x)}{\frac{13}{5}} = \frac{12}{13} \Leftrightarrow 5\sinh(x) = 12 \Leftrightarrow \boxed{\sinh(x) = \frac{12}{5}}$$

26, 30, 34, 36, 40 Find the derivative. Simplify where possible.
(26) $f(x) = \tanh\left(1 + e^{2x}\right)$

Apply the chain rule:

$$[\tanh(x)]' * [1 + e^{2x}]$$

$$\text{Recall the derivative of } \tanh(x)\text{: } \operatorname{sech}^2(x)$$

$$\operatorname{sech}^2(1 + e^{2x}) * 2e^{2x} = \boxed{2e^{2x}\operatorname{sech}^2(1 + e^{2x})}$$

(30) $y = x\coth\left(1 + x^2\right)$

Apply the product rule:

$$y' = 1 * \coth\left(1 + x^2\right) + x * [\coth\left(1 + x^2\right)]'$$

$$\text{Recall the derivative of } \coth(x)\text{: } \frac{\mathrm{d}}{\mathrm{d}x}(\coth(x)) = -\operatorname{csch}^2(x)$$

Then apply the Chain rule for $\coth(1+x^2)$:

$$-\operatorname{csch}(1+x^2) * 2x = -2x\operatorname{csch}(1+x^2)$$

So,

$$y' = \coth(1+x^2) + x * [-2x\operatorname{csch}(1+x^2)] = \boxed{\coth(1+x^2) - 2x^2\operatorname{csch}(1+x^2)}$$

(34) $y = \sinh(\cosh(x))$

Apply the Chain rule:

Recall:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh(x)) = \cosh(x), \frac{\mathrm{d}}{\mathrm{d}x}(\cosh(x)) = \sinh(x)$$

$$y' = \cosh(\cosh(x)) * \sinh(x) = \boxed{\sinh(x)\cosh(\cosh(x))}$$

(36) $y = \sinh^{-1}(\tan(x))$

$$\text{Recall: } \frac{\mathrm{d}}{\mathrm{d}x}(\sinh^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

So apply the Chain rule:

$$y' = \frac{1}{\sqrt{1+\tan^2(x)}} * \sec^(x) = \frac{sec^2(x)}{\sqrt{1+\tan^2(x)}}$$

Use trig identity $\sec^2(x) = 1 + tan^2(x)$

$$y' = \frac{sec^2(x)}{\sqrt{sec^2(x)}} = \boxed{\frac{sec^2(x)}{|sec(x)|}}$$

(40) $y = \operatorname{sech}^{-1}(e^{-x})$

$$\text{Recall: } \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech}^{-1}(x)) = -\frac{1}{x\sqrt{1-x^2}}$$

So apply the Chain rule:

$$y' = \frac{1}{e^{-x}\sqrt{1-(e^{-x})^2}} * -e^{-x} = \frac{e^{-x}}{e^{-x}\sqrt{1-(e^{-x})^2}} = \boxed{\frac{1}{\sqrt{1-e^{-2x}}}}$$

47 A telephone line hangs between two poles 14m apart in the shape of the catenary $y = 20\cosh(\frac{x}{20})$ - 15, where x and y are measured in meters.

(a) Find the slope of the curve where it meets the right pole.

(b) Find the angle θ between the line and the pole.

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50 Evaluate $\lim\limits_{x\to\infty} \frac{\sinh(x)}{e^x}$

$$\text{Recall: } \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\lim_{x\to\infty} \frac{\frac{e^x - e^{-x}}{2}}{e^x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2e^x}$$

Factor out e^x:

$$\lim_{x\to\infty} \frac{e^x(1 - e^{-2x})}{e^x(2)} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{2} = \frac{1 - e^{-\infty}}{2} = \frac{1-0}{2} = \boxed{\frac{1}{2}}$$

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4, 8, 10, 12, 16, 30, 32, 36, 38 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

(4) $\lim\limits_{x\to 1} \frac{x^2-1}{x^2-x}$

Factorize the numerator and denominator:

$$\lim_{x\to 1} \frac{(x+1)(x-1)}{x(x-1)} \Leftrightarrow x \neq 1, \lim_{x\to 1} \frac{x+1}{x} = \frac{2}{1} = \boxed{2}$$

(8) $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{1-\sin(\theta)}{\csc(\theta)}$

Direct substitution:

$$\frac{1 - \sin\left(\frac{\pi}{2}\right)}{\csc\left(\frac{\pi}{2}\right)} = \frac{1-1}{1} = \boxed{0}$$

(10) $\lim\limits_{x\to\infty} \frac{\ln\left(\sqrt{x}\right)}{x^2}$

Apply l'Hopital's Rule:

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}} * \frac{1}{2\sqrt{x}}}{2x} = \lim_{x\to\infty} \frac{\frac{1}{2x}}{2x} = \lim_{x\to\infty} \frac{1}{4x^2} = \frac{1}{\infty} = \boxed{0}$$

(12) $\lim\limits_{t\to 0} \frac{8^t - 5^t}{t}$

$$\text{Indeterminate form of } \frac{0}{0}, \text{ so apply L'Hoptial's rule:}$$

$$\text{Recall the derivative of } b^x\text{: } \frac{\mathrm{d}}{\mathrm{d}x}(b^x) = b^x \ln(b)$$

$$\lim_{t\to 0} \frac{8^t \ln(8) - 5^t \ln(5)}{1} = \frac{8^0 \ln(8) - 5^0 \ln(5)}{1} = \frac{\ln(8) - \ln(5)}{1}$$

Quotient property of logs:

$$\ln(8) - \ln(5) = \boxed{\ln\left(\frac{8}{5}\right)}$$

(16) $\lim\limits_{x\to 0} \frac{\cos mx - \cos mn}{x^2}$

$$\text{Indeterminate form of } \frac{0}{0}, \text{ so apply L'Hopital's rule:}$$

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$$\lim_{x\to 0}\frac{m*-\sin(mx)+n*\sin(nx)}{2x}$$

Still indeterminate form $\dfrac{0}{0}$, so apply L'Hopital's rule again:

$$\lim_{x\to 0}\frac{m^2*-\cos(mx)+n^2*\cos(nx)}{2}=\frac{m^2*-1+n^2+1}{2}=\boxed{\frac{n^2-m^2}{2}}$$

(30) $\lim\limits_{x\to 0}(\csc x-\cot x)$

$$\csc(x)=\frac{1}{\sin(x)},\cot(x)=\frac{\cos(x)}{\sin(x)}$$

Substitute $\csc(x)\&\cot(x)$:

$$\lim_{x\to 0}\frac{1}{\sin(x)}-\frac{\cos(x)}{\sin(x)}=\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)}(\text{Indeterminate form }\frac{0}{0})$$

Apply L'Hopital's rule:

$$\lim_{x\to 0}\frac{\sin(x)}{\cos(x)}=\frac{0}{1}=\boxed{0}$$

(32) $\lim\limits_{x\to 1^+}[\ln\big(x^7-1\big)-\ln\big(x^5-1\big)]$

Quotient property of logs:

$$\lim_{x\to 1^+}\ln\left(\frac{x^7-1}{x^5-1}\right)$$

Since log and polynomials are cts functions, then the limit can be interchanged in the composition of it:

$$\ln\left(\lim_{x\to 1^+}\frac{x^7-1}{x^5-1}\right)$$

Apply L'Hopital's rule:

$$\ln\left(\lim_{x\to 1^+}\frac{7x^6}{5x^4}\right)=\ln\left(\frac{7(1)^6}{5(1)^4}\right)=\boxed{\ln\left(\tfrac{7}{5}\right)}$$

(36) $\lim\limits_{x\to\infty}(1+\frac{a}{x})^{bx}$

(38) $\lim\limits_{x\to\infty}(e^x+x)^{\frac{1}{x}}$

Let $L=\lim\limits_{x\to\infty}(e^x+x)^{\frac{1}{x}}$:

Take log of both sides:

$$\ln(L)=\ln\Big(\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}}\Big)\Leftrightarrow\ln(L)=\lim_{x\to\infty}\frac{1}{x}\ln(e^x+x)\Leftrightarrow\ln(L)=\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}$$

Apply L'Hopital's rule on the RHS:

$$\lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1}=\lim_{x\to\infty}\frac{e^x+1}{e^x+x}$$

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L'Hopital's rule again:

$$\lim_{x\to\infty} \frac{e^x}{e^x+1}$$

L'Hoptial's rule again:

$$\lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1$$

$$\ln(L) = 1 \Leftrightarrow e^{\ln(L)} = e^1 \Leftrightarrow L = \boxed{e}$$

52 Suppose f is a positive function. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, show that

$$\lim_{x\to a} [f(x)]^{g(x)} = 0$$

This shows that 0^∞ is not an indeterminate form.

57a Let

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use the definition of the derivative to compute $f'(0)$.

Apply the Prinicple definition of a limit: $\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}$

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x-0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x} =$$

Apply L'Hopital's rule:

$$\lim_{x\to 0}$$