Rectifiable Graph Summer Research

Yutong Wu

Chap 1

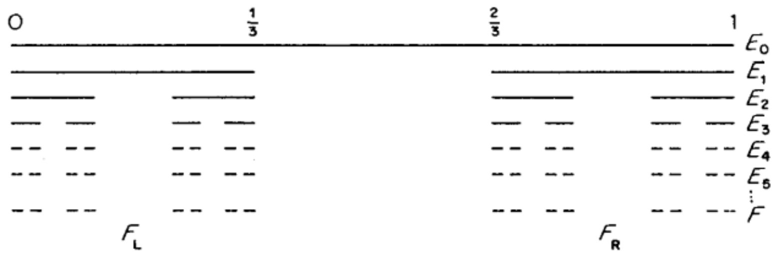


Figure 0.1 Construction of the middle third Cantor set F, by repeated removal of the middle third of intervals. Note that F_L and F_R, the left and right parts of F, are copies of F scaled by a factor $\frac{1}{3}$

1. The middle third Cantor Set : E_0, E_1, E_2,..., E_k, ... **F (E: operation of removing the middle third of the set of real numbers)**
   properties:
   a) Length = 0
   b) Generated by recursive process
   c) Awkward to describe the local geometry of it
   d) self -similar
   e) “Fine structure”,it contains details at arbitrarily small scale

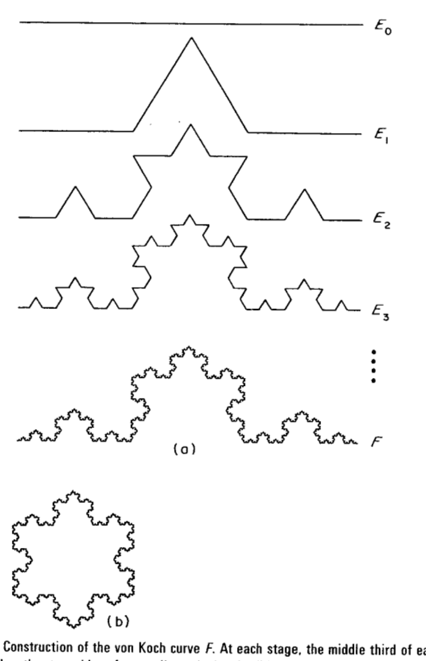


Figure 0.2 (a) Construction of the von Koch curve F. At each stage, the middle third of each interval is replaced by the other two sides of an equilateral triangle. (b) Three von Koch curves fitted together to form a snowflake curve

2. Von Koch Curves

It has similar properties as the middle third Cantor set. For defining middle third Cantor set as a curve, a simple calculation shows that it has length $(4/3)^k$ for E_k

Dimension of Cantor set and Koch curve:

D(Cantor set) = log2/log3 = 0.631, D(Koch Curve) = log4/log3 = 1.262

D(fractal object) = -log(number of fractals) / log(factor of its segment)

D(Cantor set) = -log 2/log 1/3 = log2/log3 = 0.631

D(Koch Curve) = log4/log3 = 1.262

Definition of a dimension: define how much space a set fills

It may vary depending on different mathematicians; the author prefers to define it as a list of characteristics that the fractal have.

(i) F has a fine structure, i.e. detail on arbitrarily small scales.

(ii) F is too irregular to be described in traditional geometrical language, both locally and globally.

(iii) Often F has some form of self-similarity, perhaps approximate or statistical.

(iv) Usually, the 'fractal dimension' of F (defined in some way) is greater than its topological dimension.

(v) In most cases of interest F is defined in a very simple way, perhaps recursively.

5.25 Chap 1.1

Basic Set Theory

1. Points in set x = (x1,x2,x3,...xn), y = (y1,y2,y3,...yn), operation: scalar addition/multiplication
2. Euclidean distance between two points $|x - y| = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$.
3. Three triangular inequality holds for all points

   $|x + y| \leqslant |x| + |y|$, the reverse triangle inequality $||x| - |y|| \leqslant |x - y|$ and the metric triangle inequality $|x - y| \leqslant |x - z| + |z - y|$ hold for all $x, y, z \in \mathbb{R}^n$.
4. Closed Ball: B(x,r) = { y : | y - x | <= r }
5. Open Ball: B^o(x,r) = { y : | y - x | < r }
6. Coordinate cube of side length = 2r : { y = {y1,y2,y3,...yn} : | yi - xi | <= r for all i = 0,1,2,...n}.
7. del-neighborhood and eps-parallel body: A_del = { x : | x - y | <= delta for some y in A}, like figure shown below:

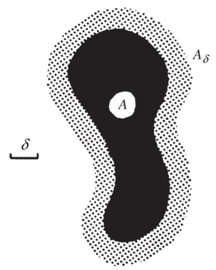


Figure 1.1 A set A and its δ-neighbourhood A_δ.

8. A \ B means points in A but not in B, R^n \ A means A's complement
9. Cartesian product:

   The set of all ordered pairs $\{(a, b) : a \in A \text{ and } b \in B\}$ is called the (*Cartesian*) *product* of A and B and is denoted by $A \times B$. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, then $A \times B \subset \mathbb{R}^{n+m}$.
10. Set operation:

    If A and B are subsets of $\mathbb{R}^n$ and λ is a real number, we define the *vector sum* of the sets as $A + B = \{x + y : x \in A \text{ and } y \in B\}$ and we define the *scalar multiple* $\lambda A = \{\lambda x : x \in A\}$.
11. sup A and inf A can be interpreted as the maximum and minimum of A but remember that sup A and inf A are not contained in A
12. Diameter |A| non-empty set of Rn has the greatest distance apart from points in A. Thus, |A| = sup { | x - y | < x,y in A }
13. Closed Vs Open sets: whether contains boundary or not. A set is open iff its complement is closed. U and intersections of any open set are open and vice versa.
14. Closure of A (cl(A)): intersection of all the closed sets containing a set A, the smallest closed set containing A.
    Interior of A (int(A)) : Union of all the open sets containing a set A, the largest open set contained in A.
    Boundary of A = cl(A) \ int(A)

Theorem 1.1.1 $x \in \partial A \Leftrightarrow \forall r > 0, B(x,r) \cap A \neq \emptyset, B(x,r) \cap A^C \neq \emptyset$

15. Set B is dense in A if A in cl(B), that is, there are arbitrarily. Set A is compact if any collection of open sets that covers A (i.e. with union containing A) has a finite subcollection which also covers A. The intersection of any compact set is compact.
16. Set A is connected if there do not exist open sets U and V such that U union V contains A with A union U and A union B disjoint and non-empty.
17. Convergence of sequence: a sequence $\{x_k\}$ in R^n converges to a point x of R^n as $k \to \infty$if, given $\epsilon > 0$, there exists a number k such that $\left|x_k - x\right| < \epsilon$ whenever k $>$ K, that is, if $\left|x_k - x\right|$converge to 0. The number x is called the limit of the sequence. Then we write $x_k \to x$ or $lim_{k\to\infty} x_k = x$.
18. Characteristics of open and closed set:
    A set is open iff its complement is a closed set. A subset A of R^n is open if for all points x in A, there is some ball B(x,r) centered at x and is of positive radius and contained in A, A

set is closed if whenever a sequence $\{x_k\}$ is a sequence of points converging to a point x of R^n, then x is in A. The empty set $\Phi$and R^n are regarded as both open and closed.

19. Borel Set:

    1. Every open set and every closed set is a Borel set.
    2. The union of every finite or countable collection of Borel sets is a Borel set, and the intersection of every finite or countable

Chap 1.2

1. Holder Function

   A function $f : X \to Y$ is called a *Hölder function of exponent* α if

   $$|f(x) - f(y)| \leqslant c|x - y|^{\alpha} \quad (x, y \in X)$$

   for some constant $c \geqslant 0$. The function f is called *Lipschitz* if α may be taken to be equal to 1, that is if

   $$|f(x) - f(y)| \leqslant c|x - y| \quad (x, y \in X)$$

   and *bi-Lipschitz* if

   $$c_1|x - y| \leqslant |f(x) - f(y)| \leqslant c_2|x - y| \quad (x, y \in X)$$

   for $0 < c_1 \leqslant c_2 < \infty$, in which case both f and $f^{-1} : f(X) \to X$ are Lipschitz functions. Lipschitz and Hölder functions play an important role in fractal geometry.

2. Lower and upper limit in terms of sup and inf

   $$\underline{\lim}_{x\to 0} f(x) \equiv \lim_{r\to 0}(\inf\{f(x) : 0 < x < r\}).$$

   $$\overline{\lim}_{x\to 0} f(x) \equiv \lim_{r\to 0}(\sup\{f(x) : 0 < x < r\}).$$

   if the lower one equals the upper one, the function value exists and equals this common value.

   value. Note that if $f(x) \leqslant g(x)$ for $x > 0$, then $\underline{\lim}_{x\to 0} f(x) \leqslant \underline{\lim}_{x\to 0} g(x)$ and $\overline{\lim}_{x\to 0} f(x) \leqslant \overline{\lim}_{x\to 0} g(x)$. In the same way, it is possible to define lower and upper limits as $x \to a$ for functions $f : X \to \mathbb{R}$ where X is a subset of $\mathbb{R}^n$ with a in $\overline{X}$.

3. Differentiable: if $f: R^n \to R^n$,f is differentiable at x and has derivative given by the linear mapping $f'(x): R^n \to R^n$ if $lim_{x\to 0} \frac{|f(x+h)-f(x)-hf'(x)|}{|h|} = 0.$

Chap 1.3

1. Measure: a measure is just a way of ascribing a numerical 'size' to sets
2. Theorem: $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, if A_i are disjoint Borel Sets.
3. The meaning of $\mu(A)$the measure of set A and think of $\mu(A)$as the size of A measured in some way. Conditions: a) Φhas zero measures c) the larger the set, the larger the measure. d) if a set is a union of a countable number of pieces (which may overlap), then the sum of the measure of the pieces is at least equal to the measure of the whole. If a set is decompose to a countable number of Borel sets, then the total measure of pieces equals the measure of the whole.b
4. From 1.4, if we have $A \supset B$, then we can rewrite A as $A = B \cup (A \setminus B)$, if A and B are Borel sets with $\mu(B)$finite, then we have $\mu(A \setminus B) = \mu(A) - \mu(B)$.
5. From last line, if $A_1 \subset A_2 \subset$...is an increasing sequence of Borel sets, then we have

   $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i)$.Since $\bigcup_{i=1}^{\infty} A_i = A_1 \cup (A_1 \setminus A_2) \cup (A_3 \setminus A_2)$..., we can get

   $\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A_1) + \sum_{i=1}^{\infty} (\mu(A_{i+1}) - \mu(A_i)) =$

   $\mu(A_1) + \lim_{k \to \infty} \sum_{i=1}^{k} (\mu(A_{i+1}) - \mu(A_i)) = \lim_{k \to \infty} \mu(A_k)$
6. Extension: $\delta > 0$, $A_\delta$are Borel sets that are increasing as δdecreases, that is $A_{\delta'} \subset A_\delta$for

   $0 < \delta < \delta'$, then $\mu(\bigcup_{\delta > 0} A_\delta) = \lim_{\delta \to 0} \mu(A_\delta)$
7. The support of μ: written as spt $\mu$is the smallest closed set s.t.$\mu(R^n \setminus X) = 0$. x is in the support iff $\mu(B(x, r)) > 0$ for all positive radii r.we say that $\mu$is a measure on set A if A contains the support of μ.
8. Measure distribution: A measure of on a bound subset of $R^n$for which $0 < \mu(R^n) < \infty$ will be called a mass distribution.
9. Examples of measure distributions:
   a) Counting Measure: For each subset A of R^n, let $\mu(A)$be the number of points in A if A is finite and ∞otherwise, then $\mu$is a measure on R^n.
   b) Point Mass: Let a be a point in R^n and define μ(A) to be 1 if A contains a and 0 otherwise. Then $\mu$is a mass distribution, thought of as a unit point mass concentrated at a.
   c) Lebesgue Measure on R:

Lebesgue measure $\mathcal{L}^1$ extends the idea of 'length' to a large collection of subsets of $\mathbb{R}$ that includes the Borel sets. For open and closed intervals, we take $\mathcal{L}^1(a,b) = \mathcal{L}^1[a,b] = b - a$. If $A = \bigcup_i [a_i, b_i]$ is a finite or countable union of disjoint intervals,we let $\mathcal{L}^1(A) = \sum(b_i - a_i)$ be the length of A, thought of as the sum of the length of the intervals. This leads us to the definition of the *Lebesgue measure* $\mathcal{L}^1(A)$ of an arbitrary set A. We define

$$\mathcal{L}^1(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} [a_i, b_i] \right\},$$

that is, we look at all coverings of A by countable collections of intervals and take the smallest total interval length possible. It is not hard to see that (1.1)–(1.3) hold; it is rather harder to show that (1.4) holds for disjoint Borel sets A_i, and we avoid this question here. (In fact, (1.4) holds for a much larger class of sets than the Borel sets, 'the Lebesgue measurable sets', but not for all subsets of $\mathbb{R}$.) Lebesgue measure on $\mathbb{R}$ is generally thought of as 'length', and we often write $\text{length}(A)$ for $\mathcal{L}^1(A)$ when we wish to emphasise this intuitive meaning.

d) Lebesgue Measure on R^n:

We call a set of the form $A = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \leqslant x_i \leqslant b_i\}$ a *coordinate parallelepiped* in $\mathbb{R}^n$, its n-dimensional volume of A is given by

$$\text{vol}^n(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

(Of course, if $n = 1$, a coordinate parallelepiped is just an interval with vol^1 as length, as in Example 1.3; if $n = 2$, it is a rectangle with vol^2 as area, and if $n = 3$, it is a cuboid with vol^3 the usual 3-dimensional volume.) Then *n-dimensional Lebesgue measure* $\mathcal{L}^n$ may be thought of as the extension of n-dimensional volume to a large class of sets. Just as in Example 1.3, we obtain a measure on $\mathbb{R}^n$ by defining

$$\mathcal{L}^n(A) = \inf \left\{ \sum_{i=1}^{\infty} \text{vol}^n(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

where the infimum is taken over all coverings of A by coordinate parallelepipeds A_i. We get that $\mathcal{L}^n(A) = \text{vol}^n(A)$ if A is a coordinate parallelepiped or, indeed, any set for which the volume can be determined by the usual rules of mensuration. Again, to aid intuition, we sometimes write $\text{area}(A)$ in place of $\mathcal{L}^2(A)$, vol(A) for $\mathcal{L}^3(A)$ and $\text{vol}^n(A)$ for $\mathcal{L}^n(A)$.

Sometimes, we need to define 'k-dimensional' volume on a k-dimensional plane X in $\mathbb{R}^n$; this may be done by identifying X with $\mathbb{R}^k$ and using $\mathcal{L}^k$ on subsets of X in the obvious way.

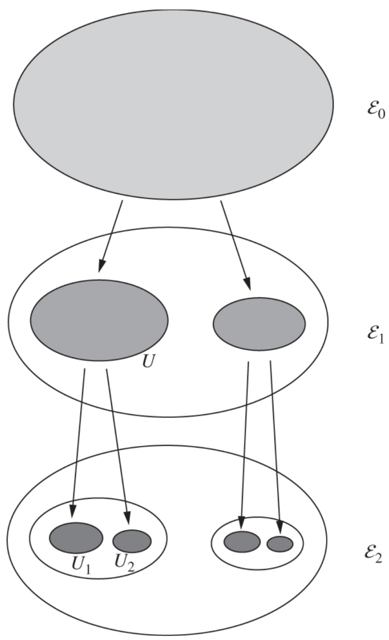
e) Uniform mass distribution on line segment:

f) Let L be a line segment of unit length in the plane. For $A \subset \mathbb{R}^2$ define $\mu(A) = \mathcal{L}^1(L \cap A)$, that is, the 'length' of intersection of A with L. Then μ is a mass distribution with support L, because $\mu(A) = 0$ if $A \cap L = \emptyset$. We may think of μ as unit mass spread evenly along the line segment L.

g) Restriction of a measure:

Let μ be a measure on $\mathbb{R}^n$ and E a Borel subset of $\mathbb{R}^n$. We may define a measure ν on $\mathbb{R}^n$, called the *restriction of μ to E*, by $\nu(A) = \mu(E \cap A)$ for every set A. Then ν is a measure on $\mathbb{R}^n$ with support contained in $\overline{E}$.

h) Mass distribution of subsets of R^n: $\sum_i \mu(U_i) = \mu(U)$



Chap 2.1

1. δ- cover: We say that $\{U_i\}$is δ- cover of set F, if $\{U_i\}$is a finite or countable collection of sets with $|U_i| < \delta$and covers F ($F \subset \bigcup_{i=0}^{\infty} U_i$).
2. $N_\delta(F)$: the least number of set $U_i$that covers F
3. Upper and lower box-counting dimension of F:
   a) Lower: $\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$

b) Upper: $\overline{\dim}_B F = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log \delta}$

4. Common value of box counting dimension: $dim_B F = \lim_{\delta \to 0} \frac{log N_\delta(F)}{-log(\delta)}$
5. We assume that box-dimension is used only for non-empty bounded sets.
6. Precise relationship: $N_\delta \delta^s \to \infty$, if $s < dim_B F$,$N_\delta \delta^s \to 0$, if $s > dim_B F$
7. Equivalent definition of $N_\delta(F)$: a) the smallest number of sets with diameter at most δ

that covers F. b) the smallest number of closed balls of radius at most δcovers F c) the smallest number of closed cube with side at most δcovers F. d) the number of δ-mesh cubes that intersects F. e) the largest number of disjoint balls of radius δwith centres in F.

8. Box dimension has been variously termed *Kolmogorov entropy*, *entropy dimension*, *capacity dimension* (a term best avoided in view of potential theoretic associations), *metric dimension*, *logarithmic density* and *information dimension*.

9. More generally a set F made up of m similar disjoint copies of itself at scale r has $dim_B F = log\, m / - log\, r$

10. If F is a subset of $\mathbb{R}^n$, and $\lim_{\delta \to 0} \frac{L^n(F_\delta)}{\delta^{n-s}} = c$ for some s > 0 and $0 < c < \infty$. The number

c is called the Minkowski content of F and it's a quantity that is useful in some concepts but has the disadvantages that it does not exist for many standard fractals and that it is not necessarily additive on disjoint subsets, that is, is not a measure.

11. The box dimension of a compact subset depends only on the lengths of its complementary intervals and not on their relative positions.

Chap 2.2

1. Monotonicity: if $E \subset F, N_\delta(E) < N_\delta(F)$for all δ.
2. Range of values: $0 < \dim_{B_{low}} F < \dim_{B_{up}} F < 0$
3. Finite stability: $\dim_{B_{up}} (E \cup F) = \max \{\dim_{B_{up}} E, \dim_{B_{up}} F\}$
4. Open Sets: If $F \subset R^n$is open, then $\dim_B F = n$since F contains a cube C,

   $N_\delta(F) \geq N_\delta(C) \geq c\delta^{-n}$where c is independent of δ.
5. Finite Sets: If F is non-empty and finite, then $\dim_B F = 0$, since if F comprises m distinct points, then $N_\delta(F) = m$ for all sufficiently small δ.
6. Smooth sets: If F is a smooth (i.e. continuously differentiable) bounded m-dimensional submanifold (i.e. m-dimensional surface) of R^n, then $\dim_B F = m$.
7. Proposition 2.5:

a) If $F \subset R^n$ and $f: F \rightarrow R^m$ is a Lipschitz transformation, that is,
$|f(x) - f(y)| \leq c|x - y|$, $(x, y \in F)$., then
$\underline{\dim}_B f(F) \leq \underline{\dim}_B F$ *and* $\overline{\dim}_B f(F) \leq \overline{\dim}_B F$

b) If $F \subset R^n$ and $f: F \rightarrow R^m$ is a bi-:Lipschitz transformation, that is,
$c_1|x - y| \leq |f(x) - f(y)| \leq c|x - y|$ $(x, y \in F)$, where
$0 < c_1 \leq c < \infty$, then $\underline{\dim}_B f(F) = \underline{\dim}_B F$ *and* $\overline{\dim}_B f(F) = \overline{\dim}_B F$

8. Add from Chap 3, Hausdorff Measure: Suppose that F is a subset of R^n and $s \geq 0$, we define $\mathcal{H}^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta - \text{cover of } F\}$.

Chap 3.1 Hausdorff Measure

Hausdorff Measure: Suppose that F is a subset of R^n and $s \geq 0$, we define
$H^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta - \text{cover of } F\}$.

1. $H^s(F) = \lim_{\delta \to 0} H^s_\delta(F)$, and we call $H^s(F)$ the s-dimensional Hausdorff measure of F.

Figure 3.1 A set F and two possible δ-covers for F. The infimum of $\Sigma|U_i|^s$ over all such δ-covers $\{U_i\}$ gives $\mathcal{H}^s_\delta(F)$.

2. With a certain amount of effort, Hausdorff may be shown to be a measure. Hausdorff measure have following characteristics:

a) $H^s(\emptyset) = 0$

b) $H^s(E) \le H^s(F)\ if\ E \subset H$

c) If $\{F_i\}$ is any countable collection of sets, then $H^s(\bigcup_{i=1}^{\infty} F_i) \le \sum_{i=1}^{\infty} H^s(F_i)$

3. If F is a Borel subset of R^n, then $H^s(F) = c_n^{-1}\ vol^n(F)$, where c_n is the volume of an n-dimensional ball of diameter 1, so that $c_n = \pi^{\pi/2}/2^n(n/2)!$ if n is even and $c_n = \pi^{(n-1)/2}((n-1)/2)!\,/n!$if n is odd. (For lower dimensional subsets of R^n, we have that $H^0(F)$is the number of points in F; $H^1(F)$gives the length of a smooth curve F; $H^2(F) = (4/\pi) \times area(F)$if F is a smooth surface; $H^3(F) = (6/\pi) \times vol(F)$and $H^m(F) = c_m^{-1} \times vol^m(F)$ if F is a smooth m-dimensional submanifold of R^n(i.e. an m-dimensional surface is the classical sense))

3. Proposition 3.1: Let $F \subset R^n$and $f: F \to R^m$ be a mapping such that $|f(x) - f(y)| \le c\,|x - y|^{\alpha},\ (x, y, \in F)$. for constants $\alpha > 0$ and $c > 0$. Then for each s we have $H^{s/\alpha}(f(F)) \le c^{s/\alpha} H^s(F)$. In particular, if f is a Lipschitz mapping, that is, if $\alpha = 1$, then $H^s(f(F)) \le c^s H^s(F)$.

4. Scaling Property 3.2: Let $f\colon R^n \to R^n$be a similarity transformation of scale factor$\lambda > 0$. If $F \subset R^n$, then $H^s(f(F)) = \lambda^s H^s(F)$

5. If f is a congruence or isometry, that is, $|f(x) - f(y)| = |x - y|$ for all $x, y \in R^n$, then $H^s(f(F)) = H^s(F)$. Thus, Hausdorff measures are translation invariant (i.e. $H^s(f(F)) = H^s(F)$)Thus, Hausdorff measures are translation invariant (i.e. $H^s(F + z) = H^s(F)$), where $F + z = \{x + z : x \in F\}$, and rotation invariant, as would certainly be expected.

Chap 3.2 Hausdorff Dimension

1. Hausdorff Dimension: $\dim_H f$, the critical value of s when $H^s(F)$ | jumps from ∞| to 0. Formal definition: $\dim_H F = \inf\{s \ge 0 : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}$(taking the supremum of the empty set to be 0), so that we have $H^s(F) = \infty\ if\ 0 \le s < \dim_H F,\ 0\ if\ s > \dim_H F$. If $s = \dim_H F$, then $H^s(F)$ may be zero or infinite or may satisfy $0 < H^s(F) < \infty$. A Borel set satisfying this last condition is called an s-set. Mathematically, s-sets are by far the most convenient sets to study, and fortunately, they occur surprisingly often.
2. Monotonically, if $E \subset F$, then $\dim_H E \le \dim_H F$.This is immediate from the measure property that $H^s(F)$ for each s.
3. Range of values: If $F \subset R^n$, then $0 \le \dim_H F \le n$. Clearly, Hausdorff dimensions are non-negative.

4. Countable stability: If F_1, F_2, . . . is a countable sequence of sets, then $\dim_H \bigcup_{i=1}^{\infty} F_i = \sup_{1 \le i < \infty}\{\dim_H F_i\}$. Certainly, $\dim_H \bigcup_{i=1}^{\infty} F_i \ge \dim_H F_j$ for each j by monotonically. On the other hand, if $s > \dim_H F_i$, $\forall i$, then $H^s(F_i) = 0$, so that $H^s(\bigcup_{i=1}^{\infty} F_i) \le \sum_{i=1}^{\infty} H^s(F_i) = 0$, give the opposite inequality
5. Countable sets. If F is countable, then $\dim_H F = 0$. For if F_i is a single point, $H^0(F_i) = 1$ and $\dim_H F_i = 0$, so by countable stability, $\dim_H \bigcup_{i=1}^{\infty} F_i = 0$.
6. Open sets: If $F \subset R^n$ is open, then $\dim_H F = n$. Since F contains a ball of positive n-dimensional volume, $\dim_H F \ge n$.
7. Proposition 3.3:
    a. Let $F \subset R^n$ and suppose that $f\colon F \to R^m$ satisfies the *Hölder Condition*.
    $|f(x) - f(y)| \le c|x - y|^{\alpha}$, $(x, y \in F)$.
    Then $\dim_H f(F) \le (1/\alpha)\dim_H F$. In particular, if f is a Lipschitz mapping, that is, if $\alpha = 1$, then $\dim_H f(F) \le \dim_H F$.
    b. If $f\colon F \to R^m$ is a bi-Lipschitz transformation, that is,
    $c_1|x - y| \le |f(x) - f(y)| \le c|x - y|$ $(x, y \in F)$. *where* $0 < c_1 \le c < \infty$, *then* $\dim_H f(F) = \dim_H F$.

8. Geometric Invariance. Let f be a congruence, similarity or affine transformation on R^n, Then $\dim_H f(F) = \dim_H F$

9. Smooth sets: If F is a smooth m-dimensional manifold (curve, surface,etc.),then $\dim_H F = m$.

10. Projections: $\dim_H proj\ F \le \min\{1, \dim_H F\}$. proj denotes the orthogonal project from R^2 onto some given line through the origin.

11. Proposition 3.4: For every non-empty bounded $F \subset R^n$, $\dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F$

12. Fundamental property of dimension: Hausdorff dimension, lower box dimension and upper box dimensions are all invariant under bi-Lipschitz transformation. Thus, if two sets have different dimensions, there cannot be a bi-Lipschitz mapping one onto the other. This is reminiscent of the situation in topology where various 'invariants' (such as homotopy or homology groups) are set up to distinguish between sets that are not homeomorphic: if the topological invariants of two sets differ, then there cannot be a homeomorphism (continuous one-to-one mapping with continuous inverse) between the two sets.

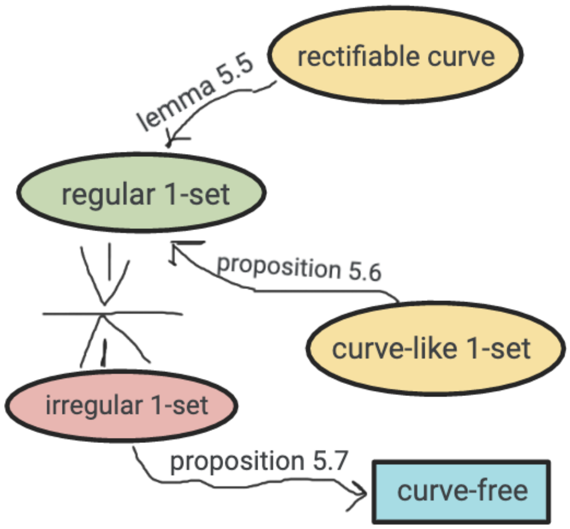
13. One approach to fractal geometry is to regard two sets as equivalent if there is a bi-Lipschitz mapping between them.

14. Proposition 3.5: every set $F \subset R^n$ with $\dim_H F < 1$ is totally disconnected.

15. RadeMacher's Theorem: If U is an open subset of R^n and $f: U \to R^m$ is Lipschitz continuous, then f is differentiable almost everywhere in U; that is, the points in U at which f is not differentiable form a set of Lebesgue measure zero.

Chap 5.1 Densities

1. The Lebesgue density theorem: Let F be a subset of the plane. The density of F at x is $\lim_{r\to 0} \frac{area(F \cap B(x,r)))}{area(B(x,r))} = \lim_{r\to 0} \frac{area(F \cap B(x,r))}{\pi r^2}$, For a Borel set F, this limit exists and equals 1 when $x \in F$ and 0 when $x \notin F$, except for a set of x of area 0. Similarly, if F is a smooth curve in the plane and x is a point of F (other than the end point), then $F \cap B(x, r)$ is close to a diameter of $B(x, r)$ for small r and $\lim_{r\to 0} \frac{length(F \cap B(x,r))}{2r} = 1$
2. The lower and upper densities of an s-set F at a point $x \in R^n$ as
$$\underline{D}^s(F, x) = \lim_{r\to 0} \frac{H^s(F \cap B(x,r))}{(2r)^s},\ \overline{D}^s(F, x) = \lim_{r\to 0} \frac{H^s(F \cap B(x,r))}{(2r)^s}$$
3. If a point at which $\underline{D}^s(F, x) = \overline{D}^s(F, x) = 1$, we say that point a is called a regular point of F, otherwise x is an irregular point. An s -set is termed regular if H^s- almost all of its points are regular and irregular [vice versa]. A fundamental result is that an s-set F must be irregular unless s is an integer.
4. A regular 1-set consists of portions of rectifiable curves of finite length, whereas an irregular 1-set is totally disconnected and dust-like and typically of fractal form.
5. Proposition 5.1: Let F be an s-set in R^n.Then: a. $\underline{D}^s(F, x) = \overline{D}^s(F, x) = 0$ for H^s -almost all $x \notin F$. b. $2^{-s} \le \overline{D}^s(F, x) \le 1$ for almost all $x \in F$
6. If set E is a subset of an s-set F with $H^s(E) > 0$, then E is regular if f is regular and E is irregular if F is irregular. In particular, the intersection of a regular and an irregular set, being a subset of both, has H^s − measure zero.
7. Theorem 5.2: Let F be an s-set in R^2. Then F is irregular unless s is an integer.
8. Curves are not self-intersecting, have two ends and are compact connected subsets of the plane. The length L(C) of the curve C is given by polygonal approximation $L(C) = \sup \sum_{i=1}^{m} |x_i - x_{i-1}|$. If the length $L(C)$ is positive and finite, we call C a rectifiable curve.
9. Lemma 5.4: If C is a rectifiable curve, then we have $H^1(C) = L(C)$.
10. Lemma 5.5: A rectifiable curve is a regular 1-set
11. Proposition 5.6: A curve-like 1-set is a regular 1-set
12. Proposition 5.7: An irregular 1-set is curve-free



13.
14. Proposition 5.8: Let F be a curve-free 1-set in R^2.Then $\underline{D}^1(F, x) \leq \frac{3}{4}$at almost all $x \in F$.
15. Theorem 5.9: a. A 1-set in $R^2$is irregular iff it is curve-free. B. A 1-set in $R^2$is regular iff it is the union of a curve-like set and a set of H^1-measure zero.
16. In any 1-set, the set of points for which $\frac{3}{4} < \underline{D}^1(F, x) < 1$has H^1 −measure zero.
17. Regular 1-set may be connected but, similar to sets of dimension less than 1, irregular 1-set must be totally disconnected

Chap 5.3-5.4

1. An s-set F in R^nhas a tangent at x in direction θ, where θis a unit vector, if $\overline{D}^s(F, x) > 0$ and for every angle $\phi > 0$, $\lim_{r \to 0} r^{-s} H^s(F \cap B(x, r) \setminus S(x, \theta, \phi)) = 0$.
   Where $S(x, \theta, \phi)$ is a double sector with vertex x.
2. Proposition 5.10: A rectifiable curve C has a tangent at almost all of its points.
3. Proposition 5.11: A regular 1-set F in R^2has a tangent at almost all of its points
4. Proposition 5.12: At almost all points of an irregular 1-set, no tangent exists
5. Proposition 5.13: If F is an s-set in $R^2$with $1 < s < 2$then at almost all points of F no tangent exists. $H^1(f(F) \cap B(f(x), r)) = H^1(F \cap B(x, r))$

June 14th
Prof. Lisa Naples
Research Paper

1. Definition of a Good Cone: $C_G(x, V, \alpha) = \{y \in H: dist(y - x, V) \leq \alpha |x - y|\}$(H denotes a finite or infinite dimensional Hilbert Space. This is a definition of a good cone at x with respect to V and alpha)
2. Hilbert space: A Hilbert space is a vector space equipped with an inner product, an operation that allows lengths and angles to be defined. Furthermore, Hilbert spaces are

complete, which means that there are enough limits in the space to allow the techniques of calculus to be used.

3. Definition of a Bad Cone: $C_B(x,\ V,\ \alpha)\ =\ H \setminus C_G(x\ ,\ V,\ \alpha)$
4. Theorem 7.1 (Geometric Lemma): Let $F\ \subset\ H$, let V be an m-dimensional linear plane in H, and let $\alpha\ \in\ (0,1)$. If $F \setminus C_G(x, V, \alpha)\ =\ \emptyset$for all $x\ \in\ F$Then F is contained in an m-Lipschitz graph. In particular, $F\ \subset\ \Gamma$where $\Gamma$is a Lipschitz graph with respect to V and Lipschitz constant corresponding to $\Gamma$is at most $1\ +\ 1/(1\ -\ \alpha^2)^{1/2}$.

   Proof. Let $x\ \in\ F$. Let $P_v: H\ \rightarrow V$denotes standard projection onto the m-plane V. Suppose that $|\ P_Vx\ -\ P_Vy\ |\ <\ (1\ -\ \alpha^2)^{1/2}|\ x\ -\ y\ |$.Then $y\ \in C_B(, V, \alpha)$, and by assumption of F this means that $y\ \notin\ F$. Thus we may assume that if $x, y\ \in\ F$then $|\ P_Vx\ -\ P_Vy\ |\ \geq\ (1\ -\ \alpha^2)^{1/2}|\ x\ -\ y\ |$. From this inequality we see that $P_V|F$is one-to-one with Lipschitz inverse $f\ =\ (P_V|F)^{-1}$and $Lip(f)\ \leq\ (1\ -\ \alpha^2)^{1/2}$. Note that $F\ =\ f(P_V|F)$. Then there exists a Lipschitz extension $\widehat{f}\colon\ V\ \rightarrow H$so that $F\ \subset \widehat{f}(V)$. Thus the desired result holds.

5. Corollary 7.1 Let μbe a locally finite Borel measure on H, V be an m-dimensional linear plane in H, $\alpha\ \in\ (0,1)$, and $0\ <\ r\ <\ \infty$. If for μ-a.e. $x\ \in\ H$, $\mu(C_B(x, r, V, \alpha))\ =\ 0$, then $\mu$is carried by m-Lipschitz graphs.

   Proof. Let F denote the set of $x\ \in\ H$that satisfy $\mu(C_B(x, r, V, \alpha))\ =\ 0$ (18). We may assume that $F\ \subset\ B(0,\ r/2)$, otherwise we may write F as a union of countably many sufficiently small sets and show that each one is an m-Lipschitz graph. Let $\{x_i\}$be a countable dense subset of F, It follows from (18) and the containment $F\ \subset\ B(0,\ r/2)$that for each $x_i$there exists $F_i\ \subset\ F$such that :

$$F_i \cap C_B(x,\ r,\ V,\ \alpha)\ =\ F_i \cap C_B(x, V, \alpha)\ =\ \emptyset$$

   and $\mu(F \setminus \bigcup_{i=1}^{\infty} F_i)\ =\ 0$, Define $F' :=\ \bigcap_{i=1}^{\infty} F_i$. Then

$$\mu(F \setminus F')\ =\ \mu(F \setminus \bigcap_{i=1}^{\infty} F_i)\ =\ \mu(\bigcup_{i=1}^{\infty} F \setminus F_i)\ \leq\ \sum_{i=1}^{\infty} \mu(F \setminus F_i)\ =\ 0$$

   We claim that $F'\ \cap C_B(x,\ V,\ \alpha)\ =\ \emptyset$for every $x\ \in\ F'$. Fix $x\ \in\ F'$, and let $y\ \in C_B(x,\ V,\ \alpha)$. By definition of bad cone we have that $dist(y\ -\ x,\ V)\ >\ \alpha|y\ -\ x|$. Now let $\epsilon\ >\ 0$such that $dist(y\ -\ x,\ V)\ \geq\ \alpha(|y\ -\ x|\ +\ \epsilon)$. Recalling that $0\ <\ \alpha\ <\ 1$, choose x, such that $|x_i\ -\ x|\ <\ \alpha/2\ <\ \epsilon/2$ Then

$$dist\ (y\ -\ x_i,\ V)\ \geq\ dist(y\ -\ x,\ V)\ -\ |\ x\ -\ x_i\ |$$

$$\geq \alpha(|y - x| + \epsilon) - \alpha(\epsilon/2)$$
$$= \alpha(|y - x| + \epsilon/2)$$
$$> \alpha(|y - x| + |x_i - x|)$$
$$\geq \alpha(|y - x_i|)$$

In particular, we conclude that $y \in C_B(x_i, V, \alpha) = \emptyset$Since

$F_i \cap C_B(x_i, V, \alpha) = \emptyset$ for all $x \in F'$. By an application of Theorem 7.1we conclude that there exists an m-Lipschitz graph Γsuch that $F' \subset \Gamma$. So we have $\mu(F \setminus \Gamma) = 0$.

6. Rectifiable sets: A set $A \subseteq R^n$ is m-rectifiable if $\exists f_i : R^m \to R^n$Lipschitz s.t.:

   $H^m(A \setminus \bigcup_{i=1}^{\infty} f_i(R^m)) = 0$

7. Lemma 7.1 Let $x \in H$, $\alpha \in (0, 1)$, and V be an m-dimensional linear plane. If $y \in C_B(x, V, \alpha + \frac{1-\alpha}{2})$ then there exists some constant $\eta_\alpha$depending on at most $\alpha$and the dimension of the space, n, such that $B(y, \eta_\alpha d) \subset C_B(x, V, \alpha)$where $d := |x - y|$

8. Lemma 7.2 Let μbe a locally finite Borel measure on H. For $x_o \in H$, V an m-dimensional linear plane, $\alpha \in (0, 1)$, and parameter $K > 0$, let E denote the set of points $x \in H$such that
   (i) The sequence of functions:

   $$f_r(x) := \frac{\mu(C_B(x,r,V,\alpha))}{\mu(B(x,r))}$$

   Converges to 0 uniformly on E, and
   (ii) there exists $r_1 > 0$such that at every E,

   $$\mu(B(x, 2r)) \leq K\mu(B(x, r)), \forall r \in (0, r_1]$$

   Then E is μ-carried by m-Lipschitz graphs with Lipschitz constants depending on at most K and α.

   Proof. Fix $\delta > 0$. By uniform convergence, choose $r_\delta \leq r_1$such that for all $r < r_\delta$and $\forall x \in E$,

   $$\frac{\mu(C_B(x,2r,V,\alpha))}{\mu(B(x,2r))} < \delta \qquad (19)$$

   Fix $x \in E$, and define $S := E \cap C_B(x, r, V, \alpha)$. Assuming the set is non-empty,, fix $y_0 \in S$ such that $|x - y_0| = max_{y \in S}|a - y| =: \lambda r$ for some $0 < \lambda \leq 1$. As an application of

Lemma 7.1 choose η_α such that $B(y_0, \eta_\alpha \lambda r) \subset C_B(x, 2r, V, \alpha)$. Let $d = \log_2(\frac{\lambda+2}{\eta_\alpha \lambda})$.

Then

$$2^d \eta_\alpha \lambda r = \frac{\lambda+2}{\eta_\alpha \lambda}\eta_\alpha \lambda r = (\lambda + 2)r = |x - y_0|r + 2r$$

In particular, for the specified value of d, $B(x, 2r) \subset B(y_0, 2^d \eta_\alpha \lambda r)$. Applying condition (ii) of the set E at the point y_0 we see that:

$$\mu(C_B(x, 2r, V, \alpha)) \geq \mu(B(x, \eta_\alpha \lambda r)) \geq K^{-d}\mu(B(y_0, 2^d \eta_\alpha \lambda r)) \geq K^{-d}\mu(B(x, 2r)) \tag{20}$$

Combining inequalities(19) and (20), we get the density ratio bounds

$$\delta > z\frac{\mu(C_B(x,2r,V,\alpha))}{\mu(B(x,2r))} \geq K^{-d}$$

$\forall r < r_\delta$, In particular, this implies that $d > \frac{-\log(\delta)}{\log K}$. Equivalently,

$$\log \frac{\lambda+2}{\eta_\alpha \lambda} > \frac{-\log\delta}{\log K}, \qquad 0$$

so that if δ is chosen to be less than $2^{-\log K \log(\frac{5}{\log \eta_\alpha})}$ then $\lambda < \frac{1}{2}$. From this result we conclude that for $r < r_\delta$ and for all $y \in S$, $|x - y| < \frac{1}{2}r$. Letting $r \to 0$ we conclude that $\mu(E \cap C_B(x, r_\delta, V, 2\alpha)) = 0$. Thus we can apply Corollary 7.1, and we obtain the desired conclusion.

9. Definition of bad cone using dyadic cubes; $y \in \partial C_B(x, V, \alpha + \frac{1-\alpha}{2})$

a) $C_B^{\ 1}(Q, V, \alpha) = \bigcup_{x \in R} C_B(x, V, \alpha)$

b) $C_B^{\ 1,1}(Q, V, \alpha) = \{R \mid R \cap (\bigcup_{x \in Q} C_B(x, V, \alpha))\}$

c) $C_B^{\ 1,2}(Q, V, \alpha) = \{R \mid R \subset (\bigcup_{x \in Q} C_B(x, V, \alpha))\}$,

R := dyadic cubes of same side length of Q

d) $C_B^{\ 2}(Q, V, \alpha) = \bigcup_{x \in R} C_B(x, V, \alpha)$