

8.9 Improper Integrals

There are typically two types of improper integrals:

1. The interval of integration is infinite

$$\int_1^{\infty} \frac{1}{x^2} dx$$

2. The integrand is unbounded on the interval of integration

$$\int_{-1}^1 \frac{1}{x} dx$$

How do you find the integral $\int_1^{\infty} \frac{1}{x^2} dx$?

Consider the integral $\int_1^b \frac{1}{x^2} dx$, where $b > 1$ is an arbitrary real number.

Then we can write $\lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx$

Changing the infinity to a variable, and then taking the limit of the integral to infinity allows for a possible solution.

Examples

- $\int_0^{\infty} \frac{\sin(x)}{x} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{\sin(x)}{x} dx$

- $\int_{-\infty}^1 \frac{e^x}{x^2} dx = \lim a \rightarrow \infty \int_a^1 \frac{e^x}{x^2} dx$

Example

Find, if possible: $\int_1^{\infty} \frac{1}{x^2} dx$

1. Rewrite as a limit

$$\lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx$$

2. Solve the integral first:

$$\int_1^b \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^b = -\frac{1}{b} + 1$$

3. Then use that to find the limit:

$$\lim_{b \rightarrow \infty} -\frac{1}{b} + 1 = -\frac{1}{\infty} + 1 = 0 + 1 = 1$$

4. Thus $\int_1^{\infty} \frac{1}{x^2} dx = 1$

- An interval is said to **converge** when it goes to a finite value, and **diverge** when it evaluates to an infinite value.

Examples, Infinite Integrals

Evaluate the following integrals, or state if they if diverge.

1.

$$\begin{aligned} & \int_2^{\infty} \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} (\ln(x) \Big|_2^b) \\ &= \lim_{b \rightarrow \infty} (\ln(b) - \ln(2)) \\ &= \ln(\infty) - \ln(2) = \infty \end{aligned}$$

Therefore, the integral diverges.

2.

$$\begin{aligned} & \int_{-\infty}^0 e^{2x} dx \\ &= \lim_{a \rightarrow -\infty} \int_{-a}^0 e^{2x} dx \\ &= \lim_{a \rightarrow -\infty} \left(\frac{1}{2} e^{2x} \Big|_a^0 \right) \\ &= \lim_{a \rightarrow -\infty} \left(\frac{1}{2} - \frac{1}{2e^{2a}} \right) \\ &= \frac{1}{2} - \frac{1}{2e^{2\infty}} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the integral converges.