Notes, Jan 19 '22

8.9 Improper Intgerals

There are typically two types of improper integrals:

1. The interval of integration is infinite
   $\int_1^\infty \frac{1}{x^2}\, dx$
2. The integrand is unbounded on the interval of integration
   $\int_{-1}^1 \frac{1}{x}\, dx$

How do you find the integral $\int_1^\infty \frac{1}{x^2}\, dx$?
Consider the integral $\int_1^b \frac{1}{x^2}\, dx$, where $b > 1$ is an arbitrary real number.
Then we can write $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx$
Changing the infinity to a variable, and then taking the limit of the integral to infinity allows for a possible solution.

Examples

- $\int_0^\infty \frac{sin(x)}{x}\, dx = \lim_{b\to\infty} \int_0^b \frac{sin(x)}{x}\, dx$
- $\int_{-\infty}^1 \frac{e^x}{x^2}\, dx = \lim a \to \infty \int_a^1 \frac{e^x}{x^2}\, dx$

Example

Find, if possible: $\int_1^\infty \frac{1}{x^2}\, dx$

1. Rewrite as a limit
   $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx$
2. Solve the integral first:
$$\int_1^b \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} + 1$$
3. Then use that to find the limit:
$$\lim_{b\to\infty} -\frac{1}{b} + 1 = -\frac{1}{\infty} + 1 = 0 + 1 = 1$$
4. Thus $\int_1^\infty \frac{1}{x^2}\, dx = 1$