8.9 Improper Intgerals

There are typically two types of improper integrals:

1. The interval of integration is infinite
   $\int_1^\infty \frac{1}{x^2}\, dx$
2. The integrand is unbounded on the interval of integration
   $\int_{-1}^1 \frac{1}{x}\, dx$

How do you find the integral $\int_1^\infty \frac{1}{x^2}\, dx$?
Consider the integral $\int_1^b \frac{1}{x^2}\, dx$, where $b > 1$ is an arbitrary real number.
Then we can write $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx$
Changing the infinity to a variable, and then taking the limit of the integral to infinity allows for a possible solution.

Examples

- $\int_0^\infty \frac{sin(x)}{x}\, dx = \lim_{b\to\infty} \int_0^b \frac{sin(x)}{x}\, dx$
- $\int_{-\infty}^1 \frac{e^x}{x^2}\, dx = \lim a \to \infty \int_a^1 \frac{e^x}{x^2}\, dx$

Example

Find, if possible: $\int_1^\infty \frac{1}{x^2}\, dx$

1. Rewrite as a limit
   $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx$
2. Solve the integral first:
   $$\int_1^b \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} + 1$$
3. Then use that to find the limit:
   $$\lim_{b\to\infty} -\frac{1}{b} + 1 = -\frac{1}{\infty} + 1 = 0 + 1 = 1$$
4. Thus $\int_1^\infty \frac{1}{x^2}\, dx = 1$

- An interval is said to **converge** when it goes to a finite value, and **diverge** when it evaluates to an infinite value.

Examples, Infinite Integrals

Evaluate the following integrals, or state if they if diverge.

1.

$$\int_{2}^{\infty} \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \int_{2}^{b} \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} (\ln(x)\Big|_{2}^{b})$$
$$= \lim_{b\to\infty} (\ln(b) - \ln(2))$$
$$= \ln(\infty) - \ln(2) = \infty$$

Therefore, the integral diverges.

2.

$$\int_{-\infty} 0e^{2x}\,dx$$
$$= \lim a \to -\infty \int_{-a} 0e^{2x}\,dx$$
$$= \lim a \to -\infty(\frac{1}{2}e^{2x}\Big|_{a}^{0})$$
$$= \lim a \to -\infty(\frac{1}{2} - \frac{1}{2e^{2a}})$$
$$= \frac{1}{2} - \frac{1}{2e^{2\infty}}$$
$$= \frac{1}{2}$$

Therefore, the integral converges.

3.

$$\int_{-\infty}^{\infty} e^{4x}\,dx$$
$$= \int_{-\infty}^{0} e^{4x}\,dx + \int_{0}^{\infty} e^{4x}\,dx$$
$$\int_{-\infty}^{0} e^{4x}\,dx = \lim_{a\to\infty}\left(\frac{1}{4}e^{4x}\Big|_{a}^{0}\right)$$
$$= \lim_{a\to-\infty}\left(\frac{1}{4} - \frac{1}{4}e^{4a}\right) = \frac{1}{4}$$
$$\int_{0}^{\infty} e^{4x}\,dx = \lim_{b\to\infty}\left(\frac{1}{4}e^{4x}\Big|_{0}^{b}\right)$$
$$= \lim_{b\to\infty}\left(\frac{1}{4}e^{4b} - \frac{1}{4}\right) = \infty$$
$$= \int_{-\infty}^{0} e^{4x}\,dx + \int_{0}^{\infty} e^{4x}\,dx = \frac{1}{4} + \infty = \infty$$

Therefore, the integral diverges.