HW_4

1. Mandelstam Variables (10 points).

Consider the two-to-two scattering of massless particles (which may be the same or different), with incoming momenta $p^\mu_{1,2}$ and outgoing momenta $p^\mu_{1,2}$. Compute the Mandelstam variables in terms of the center-of-mass (CM) energies of the incoming particles, E, and CM scattering angle, θ:

It useful to first define the momenta,

$$\begin{aligned} p_1 &= E[1, \hat{p}] \\ p_2 &= E[1, -\hat{p}] \\ p_3 &= E[1, \hat{p}'] \\ p_4 &= E[1, -\hat{p}']. \end{aligned}$$

Remember that massless particles don't have a squared momentum,

$$p_i^2 = 0$$

so only the cross terms remain,

$$\begin{aligned} s &= \cancel{p_1^2} + 2p_1p_2 + \cancel{p_2^2} \\ t &= \cancel{p_1^2} - 2p_1p_3 + \cancel{p_3^2} \\ u &= \cancel{p_1^2} - 2p_1p_4 + \cancel{p_4^2}. \end{aligned}$$

So now all together,

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2[1, \hat{p}] \cdot [1, \hat{p}'] = -2E^2(1 - \cos\theta) \\ u &= -2E^2[1, \hat{p}] \cdot [1, -\hat{p}'] = -2E^2(1 + \cos\theta) \end{aligned}$$

Using momentum conservation show that, if the particles have masses, $p_i^2 = m_i^2$, then

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

The reason $s + t + u = 0$ in the first part of the problem is because of momentum conservation.

$$
\begin{aligned}
s+t+u &= 2p_1p_2 - 2p_1p_3 - 2p_1p_4 \\
&= 2p_1(p_2 - p_3 - p_4) \\
&= 2p_1p_1 \\
&= 0
\end{aligned}
$$

What this shows us is that when added together, the cross terms will ***always*** vanish when summing the Mandelstam variables.
Since the particles now have mass, the square terms in the Mandelstam sum no longer vanish, while the cross terms vanish no matter what.

2. Parton Model DIS Kinematics (10 points).

Consider the lab frame description of a generic DIS event: an incoming probe (massless electron, muon, or neutrino) of four-momentum $k = E_{\text{lab}}[1,0,0,1]$ **incident on a proton at rest,** $P = [m,0,0,0]$, which strikes a parton of momentum xP and scatters into a massless electron, muon, or neutrino with momentum

$$k' = [E'_{\text{lab}},\ 0,\ E'_{\text{lab}}\sin\theta_{\text{lab}},\ E'_{\text{lab}}\cos\theta_{\text{lab}}].$$

It is conventional to define the variables $Q^2 = -q^2 = -(k-k')^2$ and $v = E_{\text{lab}} - E'_{\text{lab}}$, and the partonic DIS observables

$$
\begin{aligned}
x &= \frac{Q^2}{2P\cdot q} \\
y &= \frac{P\cdot q}{P\cdot k}
\end{aligned}
$$

Note that as defined the quantities x and y are Lorentz-invariant.

a. Compute x and y in terms of the measured lab frame variables E_{lab}, E'_{lab} , and $\cos\theta_{\text{lab}}$.

$$
\begin{aligned}
P\cdot k &= mE \\
P\cdot q &= m(E-E') \\
Q^2 &= 2kk' = 2EE'(1-\cos\theta_{\text{lab}})
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{EE'(1-\cos\theta_{\text{lab}})}{m(E-E')} \\
y &= \frac{(E-E')}{E}
\end{aligned}
$$

b. Compute the partonic CM Mandelstam variables and scattering angle $\cos\theta$ in terms of x, y, and the lab (probe-proton) center-of-mass squared s – working in the limit where s, $\hat{s} \gg m^2$ and you may neglect the mass of the proton. Show that both x and y are positive numbers between zero and one. (Note that in the lab frame, the proton rest frame, you cannot set the proton mass to zero!)

Remember for this problem that $s = 4E^2$
The $\hat{s}$ variable is pretty easy

$$\hat{s} = (k + xP)^2 = 2xkP = xs.$$

Then, similarly for $\hat{t}$

$$\hat{t} = -Q^2 = -2xm(E - E') = -2xys.$$

For $\hat{u}$, we'll use the alternative definition

$$\hat{u} = (xP - k')^2 = -xk'P = -\frac{\hat{s}}{s}(mE') = -\hat{s}(1 - y)$$

Since $E > E'$ and both energies are positive, it is trivial to show that $0 < y < 1$.

To show that $0 < x < 1$, it is sufficient to show that,

$$EE'(1 - \cos\theta) < m(E - E')$$

After this I don't know what to do.

3.

I ran out of time.