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THE DECAY OF A LIGHT HIGGS BOSON*

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We discuss the hadronic form factors which are required for the prediction of the decay rate of scalar Higgs bosons, including non-standard generalizations. We calculate the next-to-leading order corrections to the low-energy theorems of chiral symmetry, and use dispersion relations to determine the couplings at higher energy.

1. Introduction

In spontaneously broken gauge theories, it is the Higgs sector which drives the symmetry breaking. There always remain physical scalar Higgs bosons, whose masses are surprisingly unconstrained. The simplest case, the Minimal Standard Model, has only one such Higgs particle, but extensions of the Standard Model often have more. For a Higgs particle with a mass below 1 GeV, but above twice the pion mass, the prime decay channels are into $\pi\pi$ and $\mu^+\mu^-$. It would seem a simple exercise to calculate the rates into these channels. Nevertheless, the issue is remarkably subtle and there has been much confusion in the literature, both in the past and at present [1–7]. In this paper we attempt to resolve this issue by a careful treatment of the form factors which enter into the decay $H \to \pi\pi$.

The prime innovation of our work is that we simultaneously use a next-to-leading order chiral description of the decay amplitude and one based on dispersion theory, and we merge these two descriptions. The very low-energy behaviour of the $H \to \pi\pi$ amplitudes is governed by a set of chiral low-energy theorems discovered by Voloshin and collaborators [3]. We extend these low-energy theorems to the next-to-leading order in the energy expansion, thereby including loop diagrams, unitarity effects, and the contribution of the operator $m_s\bar{s}s$. As emphasized by Truong and collaborators [18], the higher-order terms in the chiral expansion of

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the $I=0$ S-wave turn out to be large already at a surprisingly low energy. The results can be extended to higher energy by use of dispersion relations, where the subtraction constants are fixed by the chiral low-energy theorems. We examine the uncertainty in this procedure due to different parametrizations of the pion scattering data, and find it to be relatively small. The result is a reasonably accurate determination of the scalar form factors relevant for Higgs decay. We present our results in a fashion where they can be used in alternate theories of the Higgs sector, into which different combinations of these form factors may enter.

In sect. 2 we set up the framework for the discussion of light Higgs boson decay. Sect. 3 is a detailed treatment of the form factors in chiral perturbation theory. In sects. 4–6 we explain the dispersion theory problem and its solution, and the matching of the results to chiral perturbation theory. Sect. 7 describes the final results and sect. 8 gives our conclusions.

2. Higgs boson coupling and decay

Within the Standard Model, the coupling of the Higgs boson to fermions is proportional to the fermion mass,

$$\mathscr{L} = -m_{\mathrm{f}}\left(1+\frac{H}{v}\right)\bar{\psi}_{\mathrm{f}}\psi_{\mathrm{f}}\,, \tag{1}$$

where $v = 246$ GeV and H is the Higgs field. For the leptons it is then a straightforward matter to calculate the decay rates,

$$\Gamma(\mathrm{H}\to\ell^+\ell^-) = \frac{\sqrt{2}\,G_{\mathrm{F}}}{8\pi}m_\ell^2 m_{\mathrm{H}}\left(1-\frac{4m_\ell^2}{m_{\mathrm{H}}^2}\right)^{3/2}. \tag{2}$$

For quarks, the QCD radiative corrections produce an important effect: heavy quarks can influence light hadronic couplings through the couplings to gluons (fig. 1). For large quark masses this can be treated perturbatively, and is independent of the mass of the quark. At low energy, this provides a local effective lagrangian,

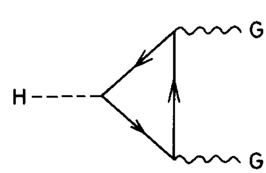

Fig. 1. Diagram for H → GG, in which heavy quarks may contribute.

which is

$$\mathscr{L}_{\text{eff}}=\frac{\alpha_s N_h}{12\pi v} H F_{\mu\nu}^A F^{A\mu\nu}-\sum_{i=u,d,s}\frac{m_i}{v} H\bar{\psi}_i\psi_i, \tag{3}$$

where N_h is the number of heavy flavours ($N_h = 3$ for c, t, b) and $F_{\mu\nu}^A$ is the field strength tensor for the gluons.

The gluonic matrix elements needed can be related to those of the energy–momentum tensor in QCD. This occurs because the energy–momentum tensor has the following trace:

$$\theta_\mu{}^\mu=-\frac{b\alpha_s}{8\pi}F_{\mu\nu}^A F^{A\mu\nu}+\sum_{i=u,d,s} m_i\bar{\psi}_i\psi_i, \tag{4}$$

where $b = 9$ is related to the first term in the beta function for three light quarks, and again the heavy quarks have been integrated out of the theory. This allows us to write the Higgs coupling in a useful form,

$$\mathscr{L}_{\text{eff}}=-\frac{H}{v}\left\{\tfrac{2}{9}\theta_\mu{}^\mu+\tfrac{7}{9}\sum_{i=u,d,s} m_i\bar{\psi}_i\psi_i\right\}, \tag{5}$$

valid for three heavy flavours.

The remaining issue is the calculation of hadronic matrix elements. Neglecting isospin breaking, the decay amplitude $H \to \pi\pi$ involves three kinematically independent form factors, viz.

$$\begin{aligned}\langle\pi^i(p)\pi^k(p')|\theta_\mu{}^\mu|0\rangle &\equiv \theta_\pi(s)\delta^{ik},\\ \langle\pi^i(p)\pi^k(p')|m_u\bar{u}u+m_d\bar{d}d|0\rangle &\equiv \Gamma_\pi(s)\delta^{ik},\\ \langle\pi^i(p)\pi^k(p')|m_s\bar{s}s|0\rangle &\equiv \Delta_\pi(s)\delta^{ik},\end{aligned} \tag{6}$$

where $s = (p + p')^2$. In terms of these, the transition amplitude is given by

$$\begin{aligned}\langle\pi^i(p)\pi^k(p')|\mathscr{L}_{\text{eff}}|H\rangle &= -\frac{1}{v}G(s)\delta^{ik},\\ G(s) &= \tfrac{2}{9}\theta_\pi(s)+\tfrac{7}{9}\{\Gamma_\pi(s)+\Delta_\pi(s)\}.\end{aligned} \tag{7}$$

Of more direct interest is the relative rate of the decays into $\pi\pi$ vs. $\mu^+\mu^-$.

Summing over the $\pi^+\pi^-$ and $\pi^0\pi^0$ possibilities, one finds

$$\frac{\Gamma(\mathrm{H}\to\pi\pi)}{\Gamma(\mathrm{H}\to\mu^+\mu^-)}=\frac{3}{4}\left(\frac{m_\mathrm{H}}{m_\mu}\right)^2\frac{\left(1-4m_\pi^2/m_\mathrm{H}^2\right)^{1/2}}{\left(1-4m_\mu^2/m_\mathrm{H}^2\right)^{3/2}}\left|\frac{G(m_\mathrm{H}^2)}{m_\mathrm{H}^2}\right|^2. \tag{8}$$

For a non-standard Higgs model, the couplings can be modified by ratios of vacuum expectation values, mixing angles, etc. This, however, only changes the weight of the operators $\theta_\mu{}^\mu$, $m_\mathrm{u}\bar{u}u$, etc. occurring in the effective lagrangian. For a general interaction of the type

$$\mathscr{L}_\mathrm{eff}=-\frac{H}{v}\left\{K_\theta\theta_\mu{}^\mu+K_\mathrm{u}m_\mathrm{u}\bar{u}u+K_\mathrm{d}m_\mathrm{d}\bar{d}d+K_\mathrm{s}m_\mathrm{s}\bar{s}s+K_\mu m_\mu\bar{\mu}\mu+\dots\right\}, \tag{9}$$

the branching ratio is of the form given in eq. (8), with $G(s)$ of the form

$$G(s)=\frac{1}{K_\mu}\left\{K_\theta\theta_\pi(s)+\frac{K_\mathrm{u}m_\mathrm{u}+K_\mathrm{d}m_\mathrm{d}}{m_\mathrm{u}+m_\mathrm{d}}\Gamma_\pi(s)+K_\mathrm{s}\Delta_\pi(s)\right\}. \tag{10}$$

3. Scalar form factors and Higgs decay in chiral perturbation theory

The interactions of pions are highly constrained by the approximate chiral symmetry of QCD. If the pions were massless, their interactions would be exactly specified at zero energy. For non-zero mass and energy, their couplings can be analyzed in terms of a series expansion in powers of the mass and energy. Chiral perturbation theory provides a systematic framework which allows one to determine the corresponding expansion of T-matrix elements or Green functions. In practice, the expansion parameter is m_π^2/Λ_χ^2 or E^2/Λ_χ^2, where the scale Λ_χ is of order 1 GeV. Note the qualitative difference between chiral perturbation theory and the standard perturbative expansion of QCD in powers of α_s. In the sense of the standard perturbative theory, the chiral energy expansion is a non-perturbative method. While the expansion in powers of α_s works best if all the energies involved approach infinity, the chiral expansion works best if they approach zero.

The expansion of the pion interaction in powers of the energy and mass starts with a term of order p^2 (E^2 or m_π^2). The present state of art is to work at order p^4 (E^4 or $m_\pi^2E^2$ or m_π^4). At leading order, chiral symmetry allows one to express all T-matrix elements, as well as the matrix elements of the quark currents and of the energy–momentum tensor, in terms of only two experimental inputs, the pion decay constant $F_\pi=93$ MeV and m_π. The leading terms occurring in the chiral expansion of matrix elements are therefore known. It is at this order that Voloshin's results for Higgs decay were obtained. At order p^4, new couplings appear – chiral symmetry only relates different processes, it does not provide the absolute normalization. However, most of the new coupling constants can be determined from low-energy phenomenology [8]. Furthermore, progress has been made in understanding their origin and their magnitude [9]. We are now able to estimate several of the coupling constants occurring at order p^4 on theoretical grounds, such that essentially parameter-free predictions can be made also at this order of the chiral expansion.

There are two levels of chiral perturbation theory which are used. One exploits the fact that in the limit $m_u = m_d = 0$ the lagrangian of QCD is symmetric under $SU(2)_L \times SU(2)_R$ and expands about this limit. In many applications, one also treats the mass of the strange quark as small and expands in powers of m_s. The corresponding symmetry group is $SU(3)_L \times SU(3)_R$. In the context of $H \to \pi\pi$, the main features are consequences of the $SU(2)_L \times SU(2)_R$ symmetry, and do not rely on expansion in powers of m_s. In most of the following we work in this limit, treating only m_u and m_d as small and keeping m_s fixed at its physical value. We will invoke the extension to SU(3) × SU(3) symmetry when analyzing the impact of $K\bar{K}$ intermediate states on the form factors. In the SU(2) × SU(2) limit these manifest themselves only indirectly through some of the low-energy constants.

The leading terms in the chiral expansion of the matrix elements $\theta_\pi(s)$, $\Gamma_\pi(s)$ and $\Delta_\pi(s)$ are obtained as follows. The start of the procedure is to write the most general effective lagrangian for pions, consistent with the chiral symmetry of QCD. The basic variable is an SU(2) matrix $U(x)$ which contains the pion field,

$$U = \exp\left(i \frac{\boldsymbol{\tau} \cdot \boldsymbol{\pi}}{F}\right), \tag{11}$$

and transforms according to the representation D(1/2, 1/2) of SU(2) × SU(2). To form an invariant, at least two derivatives of the field $U(x)$ are required. The leading term in the effective lagrangian is therefore of order p^2,

$$\mathscr{L}_2 = \tfrac{1}{4}F^2 \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \tfrac{1}{2}F^2 m^2 \operatorname{Tr} U, \tag{12}$$

where $F = 93$ MeV is the pion decay constant and where the second term describes the symmetry breaking generated by m_u and m_d,

$$m^2 = (m_u + m_d)B, \tag{13}$$

with B a constant. Note that the trace of an SU(2) matrix is real, $\operatorname{Tr} U = \operatorname{Tr} U^\dagger$. Expanding $\mathscr{L}_2$ in the pion field and retaining quadratic terms, one obtains

$$\mathscr{L}_2 = \tfrac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} - \tfrac{1}{2}m^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi}. \tag{14}$$

This shows that, up to corrections of higher order in m_u and m_d, the constant m is the pion mass. The energy–momentum tensor associated with $\mathscr{L}_2$ is

$$\theta_{\mu\nu}^{\text{eff}} = \tfrac{1}{2}F^2 \operatorname{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right) - g_{\mu\nu}\mathscr{L}_2$$

$$= \partial_\mu \boldsymbol{\pi} \cdot \partial_\nu \boldsymbol{\pi} - \tfrac{1}{2}g_{\mu\nu}\left(\partial_\lambda \boldsymbol{\pi} \cdot \partial^\lambda \boldsymbol{\pi} - m^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi}\right) + \dots \tag{15}$$

At leading order in the chiral expansion, the matrix element $\langle \pi\pi | \theta_{\mu\nu} | 0 \rangle$ therefore coincides with the canonical expression for a free scalar field. (Note that the term $(\partial_\mu \partial_\nu - g_{\mu\nu}\Box)\boldsymbol{\pi}^2$ breaks chiral symmetry – the canonical tensor must not be improved [10].) Accordingly, the form factor associated with the trace $\theta_\mu{}^\mu$ is given by

$$\theta_\pi(s) = s + 2m^2 + \mathrm{O}(p^4). \tag{16}$$

The operators $m_u \bar{u}u$, $m_d \bar{d}d$ and $m_s \bar{s}s$ are contained in the mass term of the QCD lagrangian, i.e.

$$m_u \bar{u}u = -m_u \frac{\partial}{\partial m_u}\mathscr{L}. \tag{17}$$

On the level of the effective theory, the quark masses m_u and m_d only enter the lagrangian through $m^2 = (m_u + m_d)B$, such that

$$\left(m_u \bar{u}u + m_d \bar{d}d\right)^{\text{eff}} = -\tfrac{1}{2}F^2 m^2 \operatorname{Tr} U$$

$$= \tfrac{1}{2}m^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi}. \tag{18}$$

To leading order in the chiral expansion, the matrix element

$$\Gamma_\pi(s) = m^2 + \mathrm{O}(p^4) \tag{19}$$

is therefore energy independent.

The mass of the strange quark does not explicitly occur in the lagrangian given in eq. (12). It does occur implicitly, however, through the constants F and B. Using the logarithmic derivatives of F and B with respect to m_s,

$$d_F \equiv \frac{m_s}{F}\frac{\partial F}{\partial m_s}, \qquad d_B \equiv \frac{m_s}{B}\frac{\partial B}{\partial m_s} = \frac{m_s}{m^2}\frac{\partial m^2}{\partial m_s}, \tag{20}$$

one finds at the level of the low-energy theory

$$(m_s \bar{s}s)^{\text{eff}} = -\tfrac{1}{2}F^2 \operatorname{Tr}\left\{d_F\, \partial_\mu U \partial^\mu U^\dagger + \left(d_F + \tfrac{1}{2}d_B\right)m^2(U + U^\dagger)\right\}. \tag{21}$$

Comparison of eqs. (15), (18) and (21) shows that at leading order in the energy expansion, all pion matrix elements of the operator $m_s \bar{s}s$ can be expressed through those of θ^μ_μ and $m_u \bar{u}u + m_d \bar{d}d$,

$$(m_s \bar{s}s)^{\mathrm{eff}} = d_F (\theta^\mu_\mu)^{\mathrm{eff}} + (d_B - 2d_F)(m_u \bar{u}u + m_d \bar{d}d)^{\mathrm{eff}} + \mathrm{O}(p^4). \qquad (22)$$

In particular, we have

$$\begin{aligned} \Delta_\pi(s) &= d_F \theta_\pi(s) + (d_B - 2d_F)\Gamma_\pi(s) + \mathrm{O}(p^4) \\ &= d_F s + d_B m^2 + \mathrm{O}(p^4). \end{aligned} \qquad (23)$$

This result has an unusual status in the power counting of the chiral expansion, because the constants d_F and d_B are proportional to m_s. Within chiral SU(2), m_s counts as a term of order one and the leading contribution to Δ_π is therefore of order p^2. (In much the same way, heavy quark effects also occur at order p^2. If the strange quark were very heavy, one would have $d_F = d_B = 2/29$; see the discussion below.) In the context of chiral SU(3), on the other hand, m_s counts as a term of order p^2 and the leading term in the expansion of Δ_π is therefore of order p^4. In this case, tree graphs of $\mathscr{L}_2$ do not contribute; the leading term is due to a tree graph from $\mathscr{L}_4$ and a one-loop graph from $\mathscr{L}_2$.

At leading order in the low-energy expansion, the hadronic matrix element relevant for the decay $\mathrm{H} \to \pi\pi$ is a linear function of $s = m_{\mathrm{H}}^2$. For the standard couplings, the factor $G(s)$ defined in eq. (7) becomes

$$G(s) = \tfrac{1}{9}\left\{2s\left(1 + \tfrac{7}{2} d_F\right) + 11 m_\pi^2 \left(1 + \tfrac{7}{11} d_B\right)\right\}. \qquad (24)$$

This result agrees with that of Voloshin [4],

$$G_{\mathrm{V}}(s) = \tfrac{1}{9}\left\{2s + 11 m_\pi^2\right\}, \qquad (25)$$

except for the contributions proportional to d_F and d_B which arise from the matrix element of the operator $m_s \bar{s}s$.

Let us turn to estimates of d_F and d_B. Since they are related to how F_π and m_π change, as m_s is varied, they represent effects which violate the Zweig rule. Equivalently, they are of order $1/N_c$ if the number of colours is sent to infinity. Note that $m_\pi^2 d_B$ is the pion analogue of the nucleon matrix element $\langle N | m_s \bar{s}s | N \rangle$, which has played a prominent role in the discussion of the σ-term problem.

There is an interesting hierarchy of the leading terms in θ_π, Γ_π and Δ_π. In the limit $m_u, m_d \to 0$, the matrix element Γ_π vanishes, while $\theta_\pi = s$ and $\Delta_\pi = s d_F \sim s m_s$ remain non-zero. We would therefore expect Δ_π to be smaller than θ_π, but nevertheless larger than Γ_π (except for small values of s of order m_π^2). This runs counter to the naive Zweig rule reasoning, but is indeed borne out by the detailed analysis described below.

First let us consider the extreme case of what would happen if the strange quark was very heavy. In this case, the quark could be integrated out from the theory using perturbation theory, as was done for the c-, b- and t-quarks. If that is done, the result already described in sect. 2 corresponds to

$$(m_s \bar{s}s)^{\text{eff}} \underset{m_s \to \infty}{\to} \tfrac{2}{29}\{\theta^{\mu}_{\mu} - m_u \bar{u}u - m_d \bar{d}d\}, \tag{26}$$

such that

$$\Delta_\pi(s) \underset{m_s \to \infty}{\to} \tfrac{2}{29}\{\theta_\pi(s) - \Gamma_\pi(s)\}. \tag{27}$$

By comparison with the result of eq. (23), one finds

$$d_F = d_B = \tfrac{2}{29} = 0.07. \tag{28}$$

This result occurs simply because the effect of a heavy quark corresponds to a modification of the beta function by one fermion flavour. This amounts to a change in the energy scale by a common factor. Both F and B are dimensionally of the order of energy to the first power, and they then change in the same way. The strange quark is far from being heavy enough for this estimate to be realistic. Note, however, that a Zweig-rule violating contribution of this order of magnitude is not insignificant because, in the Higgs decay matrix element, the coefficient of Δ_π is larger than the coefficient of θ_π by the factor $7/2$. Dropping the contribution proportional to m_π^2 in the lowest-order formula (24), a value like $d_F = 0.07$ corresponds to an increase in the decay rate of about 50%.

At the opposite extreme, that of a light strange quark, the constants d_F and d_B are related to the coupling constants L_4 and L_6 occurring in the effective $SU(3) \times SU(3)$ lagrangian at order p^4 [8]. Unfortunately, direct experimental information on the magnitude of these particular couplings is not available. The theoretical estimates given in two papers by Ecker et al. [9] are based on the assumption that the couplings of the pseudoscalars to the scalar mesons obey the Zweig rule. Even if this assumption is taken for granted, the prediction for d_F and d_B involves a logarithmic scale μ, whose value strongly affects the numerical result (μ is related to the energy at which the continuum underneath the resonance is cut off). With $\mu \simeq 1$ GeV, one finds $d_F \simeq 0.04$, $d_B \simeq 0.01$. A change in scale by a factor of two modifies this estimate by ± 0.12 in the case of d_F and by ± 0.05 in the case of d_B.

We conclude that, within the estimated range, the constant d_B has very little effect on our results and we therefore set $d_B = 0$. For d_F, on the other hand, the available information is too crude to permit a significant discussion of the role of

Zweig-rule violating contributions to the decay matrix element. We clearly need additional information here.

As will be shown in the following sections, a considerable amount of information can be gathered from a dispersive analysis of the form factors, and we will see that this analysis in particular also provides us with a reliable determination of the constant d_F. The dispersive method does not rely on an expansion of the form factors in powers of the momentum transfer, and it therefore allows us to study the behaviour of the decay matrix element beyond the range of the chiral expansion and to consider Higgs masses which are not small compared to the scale of QCD. The disadvantage of the method is that it involves non-trivial numerical work and that the connection between the information used as an input and the outcome of the calculation is not very transparent. For this reason, we consider it useful to discuss briefly the chiral expansion at next-to-leading order, where this problem does not occur.

At the next order in the chiral expansion, the form factors develop an imaginary part due to $\pi\pi$ intermediate states. In the language of the effective lagrangian, the imaginary part stems from a one-loop diagram generated by $\mathscr{L}_2$. To order p^4, this diagram accounts for all of the contributions from the low-energy region of the dispersion integral and in particular for the so-called unitarity corrections. At this order of the expansion, the high-energy region generates a purely real contribution to the form factors, given by a polynomial of the form $b_1 s^2 + b_2 s m_\pi^2 + b_3 m_\pi^4$. This contribution is associated with the effective p^4 lagrangian $\mathscr{L}_4$, and the constants b_i are related to the coupling constants occurring in $\mathscr{L}_4$. For the quark density $\Gamma_\pi(s)$, the chiral expansion to order p^4 is analyzed in detail in ref. [8]. For the energy–momentum tensor, new terms in the lagrangian, not present in previous work, must be included. We have classified these and will present them in a future publication [11]. In the present context we are only interested in the pion matrix elements of the trace $\theta_\mu{}^\mu$ and we do not need the explicit expression for the effective lagrangian.

The loop calculation is straightforward. Unitarity implies that the phase of the form factors is given by the $I = J = 0$ $\pi\pi$ scattering phase shift, for which current algebra implies

$$\delta_\pi(s) = \frac{(2s - m_\pi^2)}{32\pi F^2}\sqrt{1 - \frac{4m_\pi^2}{s}} + \mathrm{O}(p^4). \tag{29}$$

We denote the corresponding twice-subtracted loop integral by $\phi(s)$,

$$\begin{aligned}\phi(s) &= \frac{s^2}{32\pi^2 F^2}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'^2}\,\frac{1}{s' - s - i\varepsilon}\left(2s' - m_\pi^2\right)\sqrt{1 - \frac{4m_\pi^2}{s'}} \\ &= \frac{2s - m_\pi^2}{32\pi^2 F^2}\left\{\sigma \ln\left(\frac{1-\sigma}{1+\sigma}\right) + 2 + i\pi\sigma\right\} + \frac{s}{192\pi^2 F^2},\end{aligned} \tag{30}$$

where $\sigma = (1 - 4m_\pi^2/s)^{1/2}$. In this notation, the first two terms in the SU(2) × SU(2) expansion of the form factors are given by

$$\theta_\pi(s) = \left(s + 2m_\pi^2\right)\{1 + \phi(s)\} + b_\theta s^2 + \mathrm{O}\left(m_\pi^2 s, p^6\right),$$

$$\Gamma_\pi(s) = m_\pi^2\{1 + \phi(s) + b_\Gamma s\} + \mathrm{O}\left(m_\pi^4, p^6\right),$$

$$\Delta_\pi(s) = d_F s\{1 + \phi(s) + b_\Delta s\} + \mathrm{O}\left(m_\pi^2, m_\pi^2 s, p^6\right). \tag{31}$$

In these formulae, we have replaced the polynomial $b_1 s^2 + b_2 s m_\pi^2 + b_3 m_\pi^4$ by a single term, dropping contributions which merely renormalize the coefficients of the chiral representation at order p^2. Since these renormalizations are proportional to m_π^2, they are expected to be very small. (In the case of Γ_π, this expectation is confirmed by the estimate of the relevant coupling constant given in ref. [8], which implies that the term $b_3 m_\pi^4$ renormalizes the leading term m_π^2 only at the level of one or two per cent. In the expression for Δ_π, we have also omitted the contribution $d_B m_\pi^2$, for the reasons given above.)

The constants b_θ, b_Γ and b_Δ are of dimension (mass)$^{-2}$. The quantity b_Γ represents the slope of the form factor $\Gamma_\pi(s)$ at $s = 0$. The constants b_θ and b_Δ are related to the second derivatives,

$$\frac{1}{2}\frac{\mathrm{d}^2\theta_\pi}{\mathrm{d}s^2}\bigg|_{s=0} = b_\theta + \frac{19}{60}\frac{1}{(4\pi F)^2}, \tag{32}$$

$$\frac{1}{2}\frac{\mathrm{d}^2\Delta_\pi}{\mathrm{d}s^2}\bigg|_{s=0} = b_\Delta d_F. \tag{33}$$

The scale of these constants is set by the mass of the scalar intermediate states. Saturating the form factors by a single resonance of mass M, one expects $b_\theta \simeq b_\Gamma \simeq b_\Delta \simeq M^{-2}$. In the scalar channel, saturation by a single narrow resonance is, however, a questionable hypothesis, because the continuum starting at $s = 4m_\pi^2$ generates a sizeable contribution to the dispersion integral. This contribution manifests itself by the fact that the constants b_θ, b_Γ and b_Δ contain a chiral logarithm, $b_\alpha \sim (\ln \mu^2/m_\pi^2)/(4\pi F)^2$ (the function $\phi(s)$ diverges if m_π tends to zero; in order for the form factors to remain finite, the low-energy constants must compensate for this divergence). The point here is that the coefficient of the chiral logarithm is large. In the vector channel, the coefficient of the corresponding chiral logarithm is smaller by a factor of 6 – the vector meson dominance formula for the slope of the electromagnetic form factor works perfectly well. For the scalar form factors, saturation by a narrow resonance only provides a rough estimate. In fact, the slope of the form factor $\Gamma_\pi(s)$ can be determined without invoking the saturation hypothesis. The value $\langle r^2 \rangle_s^\pi = 0.55 \pm 0.15$ fm^2 given in ref. [12] corresponds to $b_\Gamma = 2.4 \pm 0.7$ GeV^{-2}. This illustrates the importance of the continuum contribution below the scalar bound states which occur at and above $M \simeq 1$ GeV. The dispersive analysis to be described below not only provides us with an independent determination of the constant b_Γ, but it also allows us to determine the constants b_θ and b_Δ. Together with the value of d_F which follows from the same analysis, we then have a parameter-free chiral representation of all three form factors to order p^4. We will confront this representation with the numerical results of the dispersive analysis in sect. 7.

4. Analyticity and unitarity

The chiral expansion provides significant information only at low energy. We use dispersion relations to extend the analysis to higher energies and we now turn to this topic.

One of the standard problems with dispersion relations is the issue of subtraction constants, which are not controlled by unitarity and analyticity. In the present context the problem can be solved, because the subtraction constants are related to the values and to the slopes of the form factors at zero energy which, as discussed in sect. 3, are very strongly constrained by chiral symmetry. Since chiral perturbation theory satisfies the general properties of field theory which are at the origin of the dispersion relations, it is of course compatible with these, order-by-order in the energy expansion. In fact, we will see that the dispersion method feeds back into chiral perturbation theory as it allows one to determine some of the coupling constants occurring in the effective lagrangian. The two methods are complementary and, taken together, they provide a rather satisfactory description of the hadronic matrix element involved in the decay $\mathrm{H} \to \pi\pi$, for Higgs masses in the range $2m_\pi \leqslant M_\mathrm{H} \leqslant 1$ GeV.

The numerical accuracy of a dispersion relation is only as good as the experimental data which are the input. In the present case the data are that of $I = 0$, $J = 0$ $\pi\pi$ scattering. This system has some notorious experimental disagreements, and we have to explore the effects of these uncertainties on the Higgs decay amplitude. At very low energies we normalize to the chiral predictions [8] which are in agreement with the scattering lengths and slopes of a full analysis of the data obtained by Donoghue et al. [9]. Around 700 or 800 MeV, all experiments agree that the phase shift is close to $\pi/2$. It turns out that the decay amplitude is rather insensitive to the interpolation used between these two regimes. In contrast, the behaviour of the scattering amplitude above 800 MeV plays a significant role in our analysis. This is because the dispersion relations involve integrals which in principle extend to infinity; a dispersive evaluation of the form factors up to a given energy therefore requires information on the phase shifts also above this energy. In particular, the behaviour of the scattering amplitude in the inelastic region above the threshold of the reaction $\pi\pi \to K\bar{K}$ constitutes an important part of the input. We illustrate the sensitivity of our results to the phenomenological uncertainties associated with this region by evaluating the decay matrix element for two different parametrizations of the data, taken from the literature. Note, however, that this variation in the input does not account for the possibility [13] of a glueball state occurring in the vicinity of the established scalar resonance $f_0(975)$. It is evident that an additional narrow resonance strongly enhances the rate of the decay $H \to \pi\pi$ if the Higgs mass happens to be right there. However, a resonance significantly affects the decay matrix element even if the Higgs mass is outside the resonance peak (recall that the presence of a resonance can be seen in the electromagnetic form factor even near $s = 0$: the slope is almost entirely generated by the ρ). If the lightest scalar glueball indeed occurs in the 1 GeV region, then our numerical predictions presumably underestimate the decay rate.

In the remainder of this section we briefly review the constraints imposed by analyticity and unitarity. The method used to solve these constraints is outlined in sect. 5. A more detailed account, which in particular also contains a discussion of the available information on the phase shifts, will be given in ref. [14].

The form factors $\theta_\pi(s)$, $\Gamma_\pi(s)$ and $\Delta_\pi(s)$ are analytic in the complex s-plane, except for a cut along the positive real axis, starting at $s = 4m_\pi^2$. For definiteness, we consider $\theta_\pi(s)$ – the analysis applies without change to the two-pion matrix element of any scalar, isospin-invariant operator. If the variable s approaches the cut from above, $\theta_\pi(s)$ represents the matrix element $\langle \pi\pi \text{ out} | \theta_\mu{}^\mu | 0 \rangle$, while the corresponding in-matrix element is reached by approaching the cut from below. For real and negative values of s, the form factor coincides with the matrix element $\langle \pi | \theta_\mu{}^\mu | \pi \rangle$ and is therefore real. This implies that the values above and below the cut are complex conjugates of one another. In the elastic region, $4m_\pi^2 < s < 16m_\pi^2$, the in- and out-states only differ in phase. Denoting the value of the form factor on the upper side of the cut by $\theta_\pi(s)$, we therefore have

$$\theta_\pi(s) = e^{2i\delta_\pi(s)}\theta_\pi^*(s), \tag{34}$$

where $\delta_\pi(s) = \delta_0^0(s)$ is the $I = J = 0$ $\pi\pi$ scattering phase shift – the familiar final-state interaction theorem of Watson.

If s exceeds $16m_\pi^2 \simeq 0.3\ \text{GeV}^2$, $\pi\pi$ collisions may give rise to inelastic reactions involving four or more pions in the final state. The probability of this occurring is, however, strongly suppressed, both by phase space and by chiral symmetry. If the $I = 0$, S-wave projection of the matrix element $\langle \pi\pi \text{ out} | \pi\pi \text{ in} \rangle$ is written in the form $(1 - \varepsilon)\exp(2i\delta_\pi)$, then chiral perturbation theory predicts that ε is of order p^8, while the phase δ_π is of order p^2. Indeed, the phenomenological analysis of the $\pi\pi$ reactions [13, 15] shows that final states containing more than two particles start playing a significant role only well above $s = 4m_K^2 \simeq 1\ \text{GeV}^2$, where the inelastic two-body channel $\pi\pi \to K\bar{K}$ opens. [In the chiral counting of powers, this channel generates an inelasticity ε of order p^4 and it is therefore expected to be more important than states consisting of four pions. Note, however, that in the immediate vicinity of the $K\bar{K}$ threshold, the spectrum contains a scalar resonance which strongly influences the scattering amplitudes in this region – chiral perturbation theory cannot be trusted all the way up to $s = 1$ GeV^2.] To the extent that reactions other than $\pi\pi \to \pi\pi$, $\pi\pi \leftrightarrow K\bar{K}$ and $K\bar{K} \to K\bar{K}$ are insignificant, the $I = 0$, S-wave projection of the S-matrix reduces to a unitary 2×2 matrix S_{mn}. The indices m, n label the two channels in action; S_{11}, for example, corresponds to $\pi\pi \to \pi\pi$, while S_{21} is the amplitude for $\pi\pi \to K\bar{K}$. A unitary 2×2 matrix contains four real parameters. One of these is, however, of no interest, as it can be adjusted at will by changing the relative phase of the two states. In the standard phase convention, S takes the form

$$S = \begin{pmatrix} \cos\gamma\, e^{2i\delta_\pi} & i\sin\gamma\, e^{i(\delta_\pi + \delta_K)} \\ i\sin\gamma\, e^{i(\delta_\pi + \delta_K)} & \cos\gamma\, e^{2i\delta_K} \end{pmatrix}. \tag{35}$$

The generalization of the one-channel S-matrix $\exp(2i\delta_\pi)$ to two channels thus involves three real functions of the energy, $\delta_\pi(s)$, $\delta_K(s)$ and $\gamma(s)$. The probability for a pion collision generating a $K\bar{K}$ pair is given by $\sin^2\gamma$. Accordingly, $|\cos\gamma|$ is referred to as the elasticity parameter. The corresponding T-matrix is defined by

$$S_{mn} = \delta_{mn} + 2i\sqrt{\sigma_m \sigma_n}\, T_{mn}, \tag{36}$$

where the kinematical factor σ_m stands for the velocity of the two particles in the centre-of-mass frame,

$$\begin{aligned} \sigma_1(s) &= \sqrt{1 - 4m_\pi^2/s}\, \theta(s - 4m_\pi^2), \\ \sigma_2(s) &= \sqrt{1 - 4m_K^2/s}\, \theta(s - 4m_K^2). \end{aligned} \tag{37}$$

At leading order in the chiral expansion, T is given by

$$\begin{aligned} T_{11} &= \frac{1}{32\pi F^2}(2s - m_\pi^2) + O(p^4), \\ T_{12} = T_{21} &= \frac{\sqrt{3}}{64\pi F^2}s + O(p^4), \\ T_{22} &= \frac{3}{64\pi F^2}s + O(p^4). \end{aligned} \tag{38}$$

An imaginary part only develops at order p^4. The leading contribution to $\operatorname{Im} T$ is readily worked out with the relation

$$\operatorname{Im} T = T^* \sigma T\,, \tag{39}$$

which immediately follows from the unitarity of S.

Since the process $\pi\pi \to K\bar{K}$ relates the pion matrix element $\theta_\pi \sim \langle \pi\pi | \theta_\mu{}^\mu | 0 \rangle$ to the corresponding $K\bar{K}$ matrix element,

$$\langle K^\alpha \bar{K}^\beta | \theta_\mu{}^\mu | 0 \rangle = \delta^{\alpha\beta} \theta_K(s)\,, \tag{40}$$

the unitarity condition obeyed by the pion form factor involves the kaon form factor and vice versa,

$$\theta_m(s) = \sum_n \{ \delta_{mn} + 2iT_{mn}(s) \sigma_n(s) \} \theta_n^*(s)\,. \tag{41}$$

Here θ_1 and θ_2 stand for

$$\theta_1(s) = \theta_\pi(s)\,, \qquad \theta_2(s) = \frac{2}{\sqrt{3}} \theta_K(s)\,. \tag{42}$$

[Since the relation (41) is linear, only the relative normalization of θ_1 and θ_2 matters. In the convention adopted here, the Clebsch–Gordan coefficient occurring in the projection of the $\pi\pi$ state onto $I = 0$ is shifted into θ_2.]

Below the $K\bar{K}$ threshold, the relation (41) reduces to the one-channel final state interaction formula discussed above, if we set $m = 1$. For $m = 2$, it takes the form ($s < 4m_K^2$)

$$\operatorname{Im} \theta_2(s) = \sigma_1(s) T_{21}(s) \theta_1^*(s)\,, \tag{43}$$

showing that the form factor $\langle K\bar{K} | \theta_\mu{}^\mu | 0 \rangle$ contains a $\pi\pi$ cut with a discontinuity determined by the product of the matrix element $\langle \pi\pi | \theta_\mu{}^\mu | 0 \rangle$ with the element T_{21} of the T-matrix. The phenomenon is referred to as an anomalous threshold. The presence of a $\pi\pi$ discontinuity is readily seen in chiral perturbation theory, where (at leading order in the expansion) it arises from the one-loop graph shown in fig. 2. Note that the relation (43) involves an analytic continuation of the T-matrix element $T_{21}(s)$ outside the physical region of the process $\pi\pi \to K\bar{K}$.

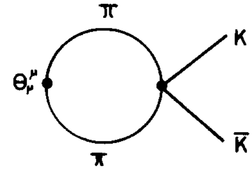

Fig. 2. Anomalous threshold in the form factor $\theta_K(s)$. In chiral perturbation theory, the corresponding cut at $s \geqslant 4m_\pi^2$ starts contributing at order p^4, through the one-loop graph shown here.

5. The Muskhelishvili–Omnès problem

The unitarity condition obeyed by the form factors gives rise to a well-known mathematical problem [16]. Let us first ignore inelastic reactions and assume that the one-channel unitarity condition (34) holds in the entire region $4m_\pi^2 < s < \infty$. The form factor then has the following properties: (i) $\theta_\pi(s)$ is analytic in the complex s-plane, except for the cut $4m_\pi^2 < s < \infty$. (ii) If s is real and less than $4m_\pi^2$, then $\theta_\pi(s)$ is real. (iii) If s approaches the cut from above, the function $\theta_\pi(s)\exp[-\mathrm{i}\delta_\pi(s)]$ is real.

Assume that the phase shift $\delta_\pi(s)$ is known. The problem is to find all functions $\theta_\pi(s)$ which obey the above three conditions. In the single-channel case, the problem has a simple, explicit solution, referred to as the Omnès solution. If the phase shift $\delta_\pi(s)$ tends to a finite value as $s \to \infty$, and if the form factor does not grow faster than a power of s as $|s| \to \infty$, then the solution is given by

$$\theta_\pi(s) = P(s)\exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}t}{t}\frac{\delta_\pi(t)}{t-s}\right\}, \tag{44}$$

where $P(s)$ is a polynomial.

In the two-channel case, the unitarity relation (41) intertwines two form factors, $\theta_1(s) = \theta_\pi(s)$ and $\theta_2(s) = 2\theta_{\mathrm{K}}(s)/\sqrt{3}$. Except for this complication, the mathematical problem is the same: assume that the T-matrix which occurs in the unitarity condition is given, and determine the general solution $\{\theta_1(s), \theta_2(s)\}$ of this condition. To our knowledge, an explicit solution which generalizes the formula (44) however, only exists for a restricted class of T-matrices [17]. We make use of an iterative method to solve the problem.

According to Muskhelishvili, the problem always admits two independent "canonical" solutions $\{C_1(s), C_2(s)\}$ and $\{D_1(s), D_2(s)\}$. Furthermore, the general solution which at infinity does not grow faster than a power of s can be expressed in terms of these as

$$\theta_n(s) = P(s)C_n(s) + Q(s)D_n(s), \qquad (n = 1,2), \tag{45}$$

where $P(s)$ and $Q(s)$ are polynomials. Clearly, the canonical solutions represent the generalization of the exponential factor in eq. (44). To determine the form factors $\theta_\pi(s)$, $\Gamma_\pi(s)$ and $\Delta_\pi(s)$ from the experimental information about the scattering matrix, we thus have to take three steps: (i) specify the T-matrix, (ii) find the two corresponding canonical solutions and (iii) determine the two polynomials $P(s)$ and $Q(s)$ for the three cases of interest.

In the remainder of this section we briefly discuss the first two steps, referring for details to ref. [14]. The third step is discussed in detail in sect. 6.

(i) In the interval from 0.8 to 1.6 GeV, we take the T-matrix from one of the available phase shift analyses. Below 0.8 GeV, we interpolate between the experimental phase shift and threshold in the manner discussed above. Since the uncertainties associated with the interpolation do not affect the decay rate significantly, we do not discuss the details here, but refer to ref. [14]. Above 1.6 GeV, we essentially set the T-matrix equal to zero. More precisely, to avoid the fictitious singularities generated by a jump in T, we add a tail and guide the T-matrix to zero smoothly, in accordance with the unitarity condition. The behaviour of the T-matrix above 1.4 GeV does not significantly affect our results either.

(ii) Using the criteria in Muskhelishvili's book, we first show that for the T-matrix under consideration, the two canonical solutions tend to zero as $|s| \to \infty$, with the power $1/s$ (the crucial property here is the number of times the determinant of the S-matrix winds around the unit circle). Next we generate a family of solutions $\{X_1(s), X_2(s)\}$ of the unitarity condition by means of the following iteration★. Start with $X_1^{(1)}(s) = 1$, $X_2^{(2)}(s) = \lambda$, where λ is a real parameter. Define the imaginary part of $\{X_1^{(N+1)}, X_2^{(N+1)}\}$ by

$$\operatorname{Im} X_n^{(N+1)}(s) = \sum_m \operatorname{Re}\{T_{nm}^*(s)\sigma_m(s) X_n^{(N)}(s)\} \tag{46}$$

and set the real part equal to

$$\operatorname{Re} X_n^{(N+1)}(s) = \frac{1}{\pi}\int \frac{dt}{t-s} \operatorname{Im} X_n^{(N+1)}(t). \tag{47}$$

The iteration converges after about 20 steps. By construction, all of these solutions, which are labelled by the parameter λ, tend to zero as $|s| \to \infty$. Comparison with eq. (45) shows that the corresponding polynomials P and Q must therefore be constants. Indeed, the iterative process is linear and the result of the iteration is therefore a linear function of λ; the family therefore contains only two linearly independent members. It is convenient to choose the two linearly independent solutions $\{C_1(s), C_2(s)\}$ and $\{D_1(s), D_2(s)\}$ such that

$$C_n(s)|_{s=0} = \delta_{n1}, \qquad D_n(s)|_{s=0} = \delta_{n2}. \tag{48}$$

6. Fixing the subtraction constants with chiral symmetry

We now turn to the problem of fixing the polynomials which occur in the general solution of the unitarity condition. Consider first the form factors associated with

★ There are more efficient ways to iterate these integral equations [14]. The one outlined here merely illustrates the principle.

the operator $m_u\bar{u}u + m_d\bar{d}d$,

$$\Gamma_1(s) = \Gamma_\pi(s), \qquad \Gamma_2(s) = \frac{2}{\sqrt{3}}\Gamma_K(s), \tag{49}$$

and represent this solution of the unitarity condition in the standard form

$$\Gamma_n(s) = P_\Gamma(s)C_n(s) + Q_\Gamma(s)D_n(s). \tag{50}$$

Since π and K are composite objects, the form factors tend to zero as $s \to \infty$. This implies that the polynomials P_Γ and Q_Γ are constants. Their value is fixed by the value of the form factors at $s = 0$,

$$P_\Gamma = \Gamma_1(0) = \Gamma_\pi(0), \qquad Q_\Gamma = \Gamma_2(0) = \frac{2}{\sqrt{3}}\Gamma_K(0). \tag{51}$$

According to the Feynman–Hellman theorem, these quantities represent derivatives of the meson masses with respect to the quark masses,

$$\Gamma_\pi(0) = \left(m_u\frac{\partial}{\partial m_u} + m_d\frac{\partial}{\partial m_d}\right)m_\pi^2,$$

$$\Gamma_K(0) = \left(m_u\frac{\partial}{\partial m_u} + m_d\frac{\partial}{\partial m_d}\right)m_K^2. \tag{52}$$

At leading order in the expansion in powers of m_u, m_d and m_s, the mass of the K-meson, for example, is given by $m_{K^+}^2 = (m_u + m_s)B_0$. Neglecting the higher-order terms, this implies

$$\Gamma_\pi(0) = m_\pi^2, \quad \Gamma_{K^+}(0) = \frac{m_u}{m_u + m_d}m_\pi^2, \quad \Gamma_{K^0}(0) = \frac{m_d}{m_u + m_d}m_\pi^2. \tag{53}$$

In the case of $\Gamma_\pi(0)$, the neglected terms are of order m_π^4 and are tiny. In $\Gamma_K(0)$, the corrections are, however, not a priori negligible as they are of order $m_\pi^2 m_s$. They can be expressed in terms of the coupling constants occurring in the effective SU(3) $\times$ SU(3) lagrangian and can be evaluated, for example, with the estimates of these couplings obtained in refs. [8, 9]. Note that in the kaon matrix elements, isospin symmetry is strongly broken.

In the context of the Higgs decay matrix element, these details are of no relevance, as this matrix element does not contain the form factor $\Gamma_K(s)$ directly, and the indirect effect of the constant $\Gamma_K(0)$ on the shape of $\Gamma_\pi(s)$ is small. The result for the decay rate remains practically the same, whether we set $\Gamma_K(0) = m_\pi^2$

or $\Gamma_K(0)=0$. In the numerical analysis we use the approximation (53) with $m_u=m_d$. The matrix elements of the operator $m_u\bar{u}u+m_d\bar{d}d$ then become

$$\Gamma_\pi(s)=m_\pi^2\left\{C_1(s)+\frac{1}{\sqrt{3}}D_1(s)\right\},$$

$$\Gamma_K(s)=m_\pi^2\left\{\tfrac{1}{2}\sqrt{3}\,C_2(s)+\tfrac{1}{2}D_2(s)\right\}. \tag{54}$$

In particular, this result provides us with a parameter-free representation of the form factor $\Gamma_\pi(s)$ in terms of the canonical solutions which in turn are fixed by the T-matrix.

The analysis of the matrix elements $\Delta_\pi(s)$, $\Delta_K(s)$ associated with the operator $m_s\bar{s}s$ is entirely analogous. The values at $s=0$ are related to the response of the meson masses to a change in m_s,

$$\Delta_\pi(0)=m_s\frac{\partial}{\partial m_s}m_\pi^2,\qquad \Delta_K(0)=m_s\frac{\partial}{\partial m_s}m_K^2. \tag{55}$$

At leading order in the SU(3) × SU(3) expansion, we therefore obtain

$$\Delta_\pi(0)=0,\qquad \Delta_K(0)=m_K^2-\tfrac{1}{2}m_\pi^2. \tag{56}$$

The higher-order terms can again be expressed through the coupling constants of the effective SU(3) × SU(3) lagrangian. As discussed in sect. 3, the corrections in $\Delta_\pi(0)$ are tiny. Dropping them, the form factors of the operator $m_s\bar{s}s$ become

$$\Delta_\pi(s)=\frac{2}{\sqrt{3}}\Delta_K(0)D_1(s),\qquad \Delta_K(s)=\Delta_K(0)D_2(s). \tag{57}$$

The higher-order contributions to the quantity $\Delta_K(0)$ are more significant. Taking the values of the coupling constants either from ref. [8] or from the two papers of Ecker et al. [9], one finds that the corrections amount to an increase of about 10%. In the numerical analysis, we drop these corrections and use eq. (56).

Finally we turn to the matrix elements $\theta_\pi(s)$, $\theta_K(s)$. In this case there is a problem: although these form factors presumably also tend to zero for $|s|\to\infty$, our machinery does not produce meaningful results if we impose this condition. The origin of the problem is best seen in the limit $m_u=m_d=m_s=0$, where $\theta_\pi(s)=\theta_K(s)$ and where the problem involves a single channel, governed by the phase of the S-matrix in the SU(3)-singlet configuration. In this case, the general solution of the unitarity condition is given by the Omnès formula (44). Since $\theta_\pi(0)$ vanishes if the quarks are massless, the polynomial occurring in this formula cannot be a constant. The situation with the phenomenological parametrization of the T-matrix used in our input is the same. The assumption that the polynomials $P(s)$ and $Q(s)$ occurring in the representation (45) are constants is not consistent with the low-energy constraints imposed by chiral symmetry. On the other hand, if $P(s)$ and $Q(s)$ are not constant, then $\theta_\pi(s)$ and $\theta_K(s)$ cannot both tend to zero for large values of s. We conclude that unsubtracted dispersion relations for the matrix elements of the operator $\theta_\mu{}^\mu$ are not saturated by the low-energy intermediate states considered here. One possible solution is the occurrence of a glueball at relatively low energies – the corresponding modification of the T-matrix may provide the missing imaginary parts. Note that the problem only concerns the SU(3) singlet component of the matrix element. We expect the difference $\theta_K(s) - \theta_\pi(s)$, which is proportional to $m_s - m_u$, to decrease more rapidly with s than the individual terms. Indeed, as we will briefly discuss below, the phenomenological information on the T-matrix is consistent with the hypothesis that an unsubtracted dispersion relation for this difference is saturated by low-energy intermediate states.

We now relax the requirements on the behaviour of the form factors at large values of s, allowing the polynomials which occur in the representation (45) to contain terms linear in s. Using the kinematic constraints

$$\theta_\pi(0) = 2m_\pi^2 , \qquad \theta_K(0) = 2m_K^2 , \tag{58}$$

this representation takes the form

$$\begin{aligned} \theta_\pi(s) &= \left(2m_\pi^2 + ps\right)C_1(s) + \frac{2}{\sqrt{3}}\left(2m_K^2 + qs\right)D_1(s) , \\ \theta_K(s) &= \tfrac{1}{2}\sqrt{3}\left(2m_\pi^2 + ps\right)C_2(s) + \left(2m_K^2 + qs\right)D_2(s) . \end{aligned} \tag{59}$$

The parameters p, q are related to the slopes at $s = 0$,

$$\begin{aligned} p &= \dot{\theta}_\pi - 2m_\pi^2 \dot{C}_1 - \frac{4m_K^2}{\sqrt{3}}\dot{D}_1 , \\ q &= \dot{\theta}_K - \sqrt{3}\, m_\pi^2 \dot{C}_2 - 2m_K^2 \dot{D}_2 , \end{aligned} \tag{60}$$

where $\dot{F}$ is the derivative $\mathrm{d}F(s)/\mathrm{d}s$ at $s = 0$. As discussed in sect. 3, $\dot{\theta}_\pi$ tends to one if the quark masses m_u, m_d are sent to zero,

$$\dot{\theta}_\pi = 1 + \mathrm{O}\left(m_\pi^2\right) . \tag{61}$$

The same is true of $\dot{\theta}_K$, provided m_s is also set equal to zero. The terms of order m_u, m_d are expected to be small, but how large are the terms of order m_s? As the difference $\dot{\theta}_K - \dot{\theta}_\pi$ is an SU(3) breaking effect, the popular general rule of thumb suggests a value of order 20 or 30% of $\dot{\theta}_\pi = 1$. A quantitative estimate is obtained by saturating the unsubtracted dispersion relation for the difference $\theta_K(s) - \theta_\pi(s)$ with the low-energy intermediate states occurring in our analysis. In view of eq. (59), the difference tends to zero for $s \to \infty$ only if the parameters p and q are related by

$$q = -\tfrac{1}{2}\sqrt{3}\, p\left(\bar{C}_1 - \tfrac{1}{2}\sqrt{3}\,\bar{C}_2\right)/\left(\bar{D}_1 - \tfrac{1}{2}\sqrt{3}\,\bar{D}_2\right), \tag{62}$$

where $\bar{C}_n$ and $\bar{D}_n$ are integrals over the imaginary part of the canonical solutions,

$$\bar{C}_n = \frac{1}{\pi}\int \mathrm{d}s\, \mathrm{Im}\, C_n(s)\,; \qquad \bar{D}_n = \frac{1}{\pi}\int \mathrm{d}s\, \mathrm{Im}\, D_n(s)\,. \tag{63}$$

Using eq. (60), this condition determines the slope $\dot{\theta}_K$ in terms of $\dot{\theta}_\pi$. Together with eq. (61), this then provides us with a parameter-free representation of the form factors $\theta_\pi(s)$, $\theta_K(s)$. We have checked that the resulting value for $\dot{\theta}_K$ is not sensitive to the behaviour of the phase shifts above $\sqrt{s} = 1.2$ GeV or to the interpolation used at low energies. The analysis of Au et al. gives $\dot{\theta}_K = 0.9$, while the CERN–Munich phases lead to $\dot{\theta}_K = 1.1$. In either case, the deviation from unity is consistent with what was to be expected with the rule of thumb.

The higher-order terms in $\dot{\theta}_\pi$ are smaller than those in $\dot{\theta}_K$ by the factor m_π^2/m_K^2. In addition, $\dot{\theta}_\pi$ contains a chiral logarithm $\sim m_\pi^2 \log m_\pi^2$, however with a small coefficient. All in all, the higher-order terms reduce the value of $\dot{\theta}_\pi$ by less than 5%. We ignore these corrections and use $\dot{\theta}_\pi = 1$; the T-matrix input then fixes the value of $\dot{\theta}_K$ according to eqs. (60) and (62).

7. Numerical results and discussion

(i) Given a suitable T-matrix input, the iterative procedure specified in the preceding sections generates a set of form factors $\theta_\pi(s)$, $\Gamma_\pi(s)$, $\Delta_\pi(s)$. We have tested the functioning of this apparatus by using a T-matrix input for which the form factors can be given in closed analytic form. The ansatz proposed by Truong and Willey [7], which contains several free parameters, is suitable for this purpose. With this parametrization, the function $T_{1k}(s)/\sqrt{s}$ and $T_{2k}(s)/\sqrt{s}$ represent two linearly independent solutions of the unitarity condition which tend to zero as $s \to \infty$ and which therefore represent two canonical solutions of the Muskhelishvili–Omnès problem. We have verified that the canonical solutions generated by the iterative procedure indeed represent linear combinations of the two analytic solutions. This example also allows us to investigate the sensitivity of our results to the cut-off introduced at the upper end of the integration region. For Higgs masses smaller than 1.2 GeV, the behaviour of the T-matrix above 1.6 GeV indeed turns out to be insignificant. We conclude that our machine functions properly.

(ii) Our numerical results are based on two different phase shift analyses, due to the CERN–Munich group [15] and to Au et al. [13] respectively. Neither of these is claimed to be valid below 500 MeV. We exploit the fact that chiral symmetry very strongly constrains the behaviour of the phase shift in the low-energy region. To analyze the sensitivity of our results to the behaviour of the phase shift below 800 MeV, we consider two different interpolations A_1 and A_2 between the chiral expansion at threshold and the CERN–Munich phase shifts. Fig. 3 depicts the real and imaginary parts of the matrix element $\langle \pi\pi | \theta_\mu{}^\mu | 0 \rangle$ which result from these two T-matrix inputs. The difference between the curves A_1 and A_2 is the uncertainty in the form factor caused by the noise in the low-energy region.

(iii) Concerning the behaviour of the T-matrix above 800 MeV, we emphasize that the occurrence of an additional narrow (glueball) state in the 1 GeV region would affect the form factors significantly, and also below this energy. We do not discuss this possibility here and only consider phase shift solutions for which such a state does not occur.

(iv) A comprehensive analysis of the various data which bear on the T-matrix was recently performed by Au et al. [13]. Their phase shift solutions contain a narrow additional state. A close look at their solution K_1, however, reveals that the occurrence of the additional state hinges on small details of the K-matrix representation. In fact, a minuscule modification of one of the K-matrix coefficients ($C^0_{12} = -3.29$ instead of $C^0_{12} = -3.28$) suffices to eliminate the additional state*. Except for the immediate vicinity of the potential additional resonance, this modification does not have any noticeable consequences for the T-matrix. Curve B in fig. 3 shows the form factor $\theta_\pi(s)$ which results if this modified $T-$ matrix is used as an input. The difference between the curves A_1 and B shows how the form factor responds to a variation of the T-matrix input above 800 MeV. We conclude that the sensitivity of our results to the uncertainties in the input is modest.

(v) Next, we compare the numerical results for the form factors with the chiral expansion to order p^4 given in sect. 3. For this purpose, we need to determine the low-energy constants d_F, b_θ, b_Γ, b_Δ occurring in this expansion. Evaluating the derivatives of the form factors at $s = 0$, we find (for definiteness we use input B; the corresponding numbers for A_1 or A_2 are very similar)

$$d_F = 0.09, \qquad b_\theta = 2.7\ \mathrm{GeV}^{-2},$$

$$b_\Gamma = 2.6\ \mathrm{GeV}^{-2}, \qquad b_\Delta = 3.3\ \mathrm{GeV}^{-2}. \tag{64}$$

The value of b_Γ neatly confirms the estimate $2.4 \pm 0.7\ \mathrm{GeV}^{-2}$ quoted in sect. 3 and the values of the remaining constants are consistent with the crude theoretical estimates discussed there. The chiral representation of the form factor $\theta_\pi(s)$ to

* We neglect isospin breaking effects and used $m_{\pi^0} = m_{\pi^+} = 139.6$ MeV, $M_{K^0} = M_{K^+} = 493.7$ MeV.

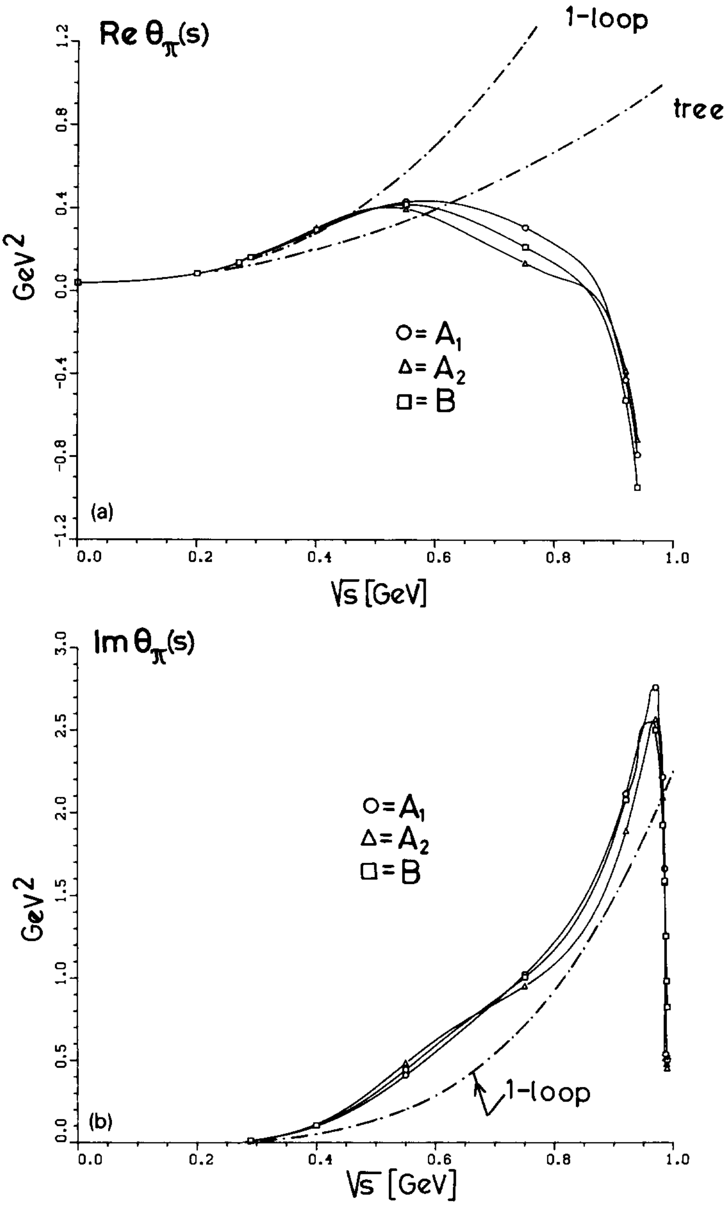

Fig. 3. Real and imaginary parts of the matrix element $\langle \pi\pi | \theta_\mu{}^\mu | 0 \rangle$. The three curves show the sensitivity to the input used in our calculation. Curves A_1 and A_2 are based on the CERN–Munich phase shifts, supplemented by two different interpolations between these data and the threshold behaviour required by chiral symmetry. Curve B results if one instead uses the phase shift of Au et al. Also shown are the lowest-order and the next-to-leading order chiral predictions.

order p^4, given in eq. (31), is shown as a dash–dotted line in fig. 3. At $s = 0$, the chiral representation for $\operatorname{Re}\theta_\pi(s)$ is of course tangent to curve B. Comparison with the lowest-order prediction, $\theta_\pi = 2m_\pi^2 + s$, shows that the correction of order p^4 shifts the result essentially into place up to $\sqrt{s} \simeq 500$ MeV. Beyond this point, the "correction" exceeds half of the leading term, indicating that it is not justified to truncate the series after the first two terms. In the imaginary part the situation is worse, because here the chiral representation to order p^4 only provides for the leading contribution. Quite generally, the imaginary part, calculated order-by-order in the chiral expansion, only reflects the real part at one order less. Unitarity is then only satisfied approximately, as one does not compare equivalent quantities. Since the chiral representations of both the scattering amplitude [8] and of $\operatorname{Re}\theta_\pi$ are available to order p^4, we could readily calculate $\operatorname{Im}\theta_\pi$ to order p^6 and convince ourselves that the corrections indeed also shift $\operatorname{Im}\theta_\pi$ into place at low energies. The exercise would, however, merely confirm that the chiral expansion is consistent with unitarity and that the first two terms in the expansion of the scattering matrix and of $\operatorname{Re}\theta_\pi$ provide a decent representation of these quantities.

(vi) It is instructive to compare the chiral perturbation theory predictions for the Higgs decay matrix element with the results of the dispersive analysis. Consider the Standard Model Higgs at a mass $m_H = 0.5$ GeV. The leading-order predictions of Voloshin and the next-to-leading order results are, respectively (in GeV units),

$$\begin{array}{ll} \text{Leading order} & \text{Next-to-leading order} \\ \theta_\pi = 0.29 & \theta_\pi = 0.461 + 0.132i \\ \Gamma_\pi = 0.019 & \Gamma_\pi = 0.032 + 0.009i \\ \Delta_\pi = 0 & \Delta_\pi = 0.043 + 0.011i \\ G = 0.079 & G = 0.161 + 0.044i. \end{array} \tag{65}$$

The corrections increase each one of the ingredients and lead to a surprisingly large net enhancement in the decay rate,

$$|G^{\text{next-to-leading}}/G^{\text{leading}}|^2 = 4.4. \tag{66}$$

The corresponding numbers which result from the dispersive analysis are

$$\begin{aligned} \theta_\pi &= 0.404 + 0.306i, & \Gamma_\pi &= 0.029 + 0.021i, \\ \Delta_\pi &= 0.041 + 0.031i, & G &= 0.144 + 0.109i, \end{aligned} \tag{67}$$

and the enhancement in the rate is

$$|G/G^{\text{leading}}|^2 = 5.2. \tag{68}$$

The enhancement originates in two distinct effects: the final-state interaction between the pions generates a substantial increase in the decay matrix element and the Zweig-rule violating term Δ_π provides for an additional contribution. Both of these features clearly manifest themselves in the chiral expansion. (If the Zweig-rule violating contribution Δ_π is disregarded, the chiral prediction 4.4 is reduced to 2.8 and the dispersive result drops from 5.2 to 3.1.)

(vii) The relevance of resonance phenomena in Higgs decay was first pointed out by Raby and West [5]. More recently, Truong and Willey [7] performed a two-channel analysis of the problem and Narison [6] investigated the effects generated by glueball states. Our results disagree with the conclusions of Truong and Willey, and we briefly comment on the origin of this discrepancy. The form factors obtained by Truong and Willey are different from ours because their T-matrix input is different in several respects. The main difference stems from the sign of the parameter λ occurring in their representation. This sign is irrelevant as far as the position of the resonance is concerned, but it is crucial for the decay matrix element. At low energies, the sign chosen by Truong and Willey gives rise to negative values of T_{12}, in conflict with current algebra [see eq. (38)] and in disagreement with the two phase-shift representations used in our analysis. If T_{12} is negative, the contribution to the form factors generated by the virtual transition $\pi\pi \leftrightarrow \mathrm{K\bar{K}}$ interferes destructively with elastic scattering, whereas in reality the interference is constructive. Note also that the TW ansatz corresponds to a negative value of the constant d_F, i.e. to destructive interference of the Zweig-rule

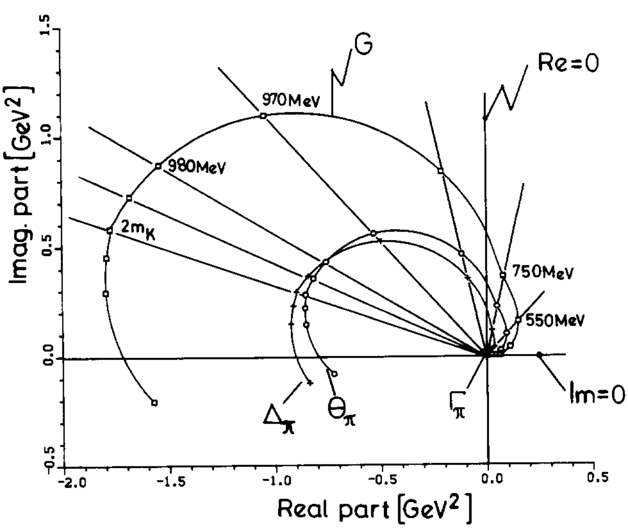

Fig. 4. Real and imaginary parts of the matrix elements occurring in the amplitudes of the decay $H \to \pi\pi$. In the region $2m_\pi < m_H < 2m_K$ the Higgs mass fixes the ratio Im/Re in terms of the $\pi\pi$ phase shift, as indicated by the radial lines.

violating contribution. If the sign of λ is flipped, the strong suppression of the branching ratio which results from their T-matrix input for a Higgs mass of 0.9 GeV is converted into an enhancement by a factor of about four. On a quantitative level, the fact that their phase $\delta_\pi(s)$ is too large near threshold [$a_0^0 = 0.29$ instead of $a_0^0 = 0.20$ as required by chiral symmetry] produces a significant distortion: the corresponding final-state interaction generates strong curvature in the mass dependence of the decay matrix element $|G(m_{\mathrm{H}}^2)|$, enhancing it near threshold and suppressing it higher up. We conclude that their T-matrix representation is very useful as a mathematical tool, but it does not provide an adequate quantitative parametrization.

8. Conclusions

In fig. 4 we give an overview, showing the results of the analysis described in the preceding sections for the mass range $2m_\pi < m_{\mathrm{H}} < 1$ GeV. In the region $m_{\mathrm{H}} < 2m_{\mathrm{K}}$ unitarity requires each of the three form factors to obey the relation

$$\frac{\operatorname{Im} T(s)}{\operatorname{Re} T(s)} = \tan \delta_\pi(s) \tag{69}$$

where $\delta_\pi(s)$ is the phase shift of elastic $\pi\pi$ scattering in the $I = J = 0$ state, contributions from intermediate states with four or more pions being neglected. At a given value of the Higgs mass, $m_{\mathrm{H}} = \sqrt{s}$, the ratio of the imaginary part to the real part is therefore fixed by the phase shift at this energy, as indicated by the radial lines in fig. 4. The main features emerging from this analysis are as follows:

(i) The contribution generated by the matrix element Γ_π of the operator $m_{\mathrm{u}}\bar{u}u + m_{\mathrm{d}}\bar{d}d$ is small.

(ii) Up to $m_{\mathrm{H}} \approx 900$ MeV, the main contribution comes from the form factor θ_π associated with the trace of the energy–momentum tensor.

(iii) The Zweig-rule violating term $\Delta_\pi = \langle\pi\pi|m_{\mathrm{s}}\bar{s}s|0\rangle$, however, generates a remarkably large contribution. Above 980 MeV it even dominates the decay matrix elements.

It is clear why the contribution from Γ_π is small – it is proportional to the masses of the u- and d-quarks. In contrast, the matrix elements of the operators

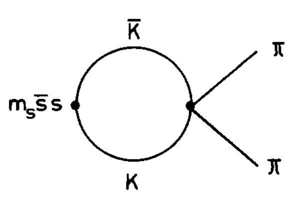

Fig. 5. Leading contributions to the Zweig-rule violating matrix element $\Delta_\pi = \langle\pi\pi|m_{\mathrm{s}}\bar{s}s|0\rangle$ in chiral perturbation theory.

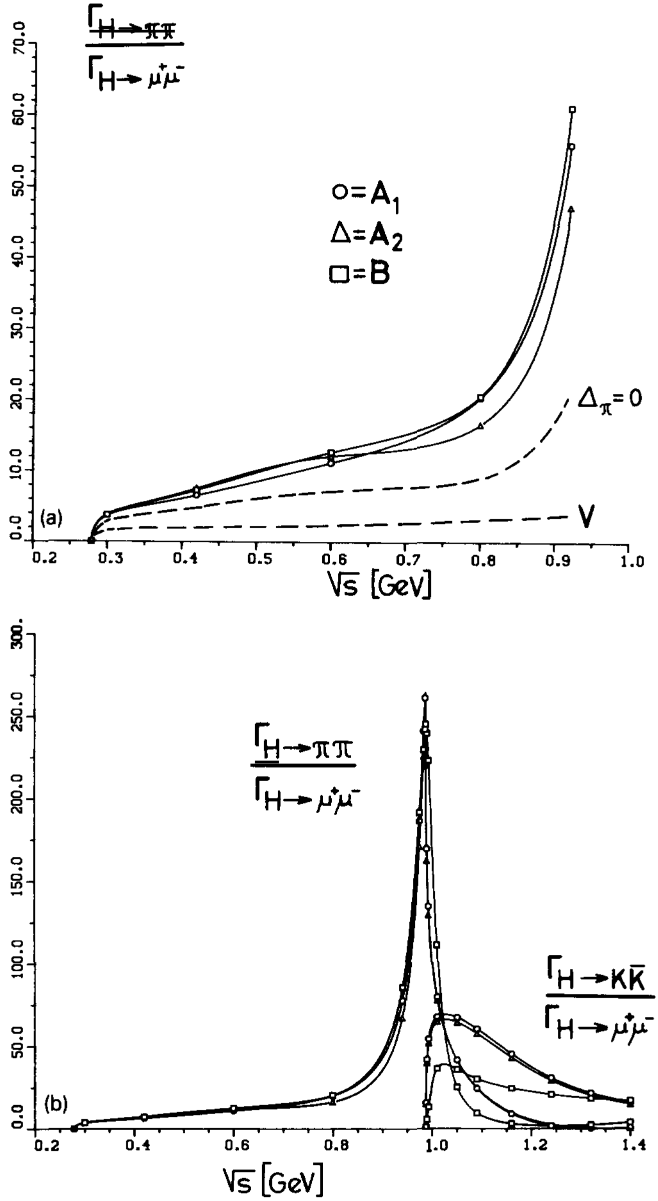

Fig. 6. (a) Branching ratios as a function of Higgs masses. The full curves correspond to different T-matrix inputs, specified in the caption to fig. 3. In addition, to exhibit the Zweig-rule violating contributions, we also show the branching ratio which results if the term Δ_π is dropped. V corresponds to the lowest-order prediction of Voloshin, $G = (2s + 11m_\pi^2)/9$. (b) The results for an extended range of Higgs mass, which includes the results of our calculation of the decay $H \to K\bar{K}$. The noise in the calculation visibly increases with the mass of the Higgs.

$\theta_\mu{}^\mu$ and $m_s\bar{s}s$ would not disappear if m_u and m_d were set equal to zero, except at threshold, $s=0$. The origin of the Zweig-rule violating term can be understood on the basis of the chiral expansion. Treating all the light quark masses as small quantities of order p^2, the leading contribution to Δ_π is of order p^4 and arises from the one-loop graph shown in fig. 5. Note that the matrix element $\langle\pi\pi|m_s\bar{s}s|0\rangle$ itself never exceeds 40% of the matrix element $\langle\pi\pi|\theta_\mu{}^\mu|0\rangle$, even at 1 GeV. However, its contribution to $\mathrm{H}\to\pi\pi$ is weighted with a factor 7/2.

In fig. 6a we plot the branching ratio of the decays $\mathrm{H}\to\pi\pi$ and $\mathrm{H}\to\mu^+\mu^-$ for the couplings of the Standard Model Higgs. Fig. 6b shows the results obtained for the branching ratio $\mathrm{H}\to\mathrm{K}\bar{\mathrm{K}}/\mathrm{H}\to\mu^+\mu^-$ for a somewhat broader range of mass. Clearly the sensitivity to the input used is more significant for Higgs masses above 1 GeV. However, in the low-mass region there remains very little uncertainty in these form factors. The triple constraints of chiral symmetry, analyticity and unitarity, plus the gross feature of $\pi\pi$ scattering, unambiguously determine the form factors up to 1 GeV. Even if a light Higgs meson turns out not to exist in nature, this is a theoretically significant result.

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References

[1] J.F. Gunion, H.E. Haber, G.L. Kane and S. Dawson, The Higgs hunter's guide (Addison Wesley, Menlo Park, 1990)

[2] J. Ellis, M.K. Gaillard and D.V. Nanopoulos, Nucl. Phys. B106 (1976) 292

[3] M.B. Voloshin and V.I. Zakharov, Phys. Rev. Lett. 45 (1980) 688;
M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Phys. Lett. B78 (1978) 443

[4] M.B. Voloshin, Sov. J. Nucl. Phys. 44 (1986) 478;
M.B. Voloshin and A.D. Dolgov, Sov. J. Nucl. Phys. 35 (1982) 213;
V.A. Novikov and M.A. Shifman, Z. Phys. C8 (1981) 43

[5] S. Raby and G.B. West, Phys. Rev. D38 (1988) 3488

[6] S. Narison, Phys. Lett. B228 (1989) 513; Montpellier preprint PM/89-35 (1989)

[7] T.N. Truong and R.S. Willey, University of Pittsburgh preprint PITT-89-05 (1989)

[8] J. Gasser and H. Leutwyler, Phys. Lett. B125 (1983) 321, 325; Ann. Phys. (N.Y.) 158 (1984) 142; Nucl. Phys. B250 (1985) 465

[9] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B321 (1989) 311;
G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, Phys. Lett. B223 (1989) 425;
J.F. Donoghue, C. Ramirez and G. Valencia, Phys. Rev. D39 (1989) 1947; D38 (1988) 2195;
J.F. Donoghue and B.R. Holstein, Phys. Rev. D40 (1990) 2378;
M. Praszalowicz and G. Valencia, Brookhaven preprint BNL43148-1989;
D. Espriu, E. de Rafael and J. Taron, CERN preprint TH.5599/89 (1989)

[10] M.B. Voloshin and A.D. Dolgov, Yad. Fiz. 35 (1982) 213 [Sov. J. Nucl. Phys. 35 (1982) 120];
H. Leutwyler and M.A. Shifman, Phys. Lett. B221 (1989) 384

[11] J.F. Donoghue and H. Leutwyler, Energy and momentum in chiral theories, in preparation

[12] J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 517
[13] K.L. Au, D. Morgan and M.R. Pennington, Phys. Rev. D35 (1987) 1633
[14] J. Gasser, P. Gerber, F. Hansen and H. Leutwyler, Scalar form factors, in preparation
[15] B. Hyams et al., Nucl. Phys. B64 (1974) 134;
W. Ochs, Thesis, University of Munich (1974)
[16] R. Omnès, Nuovo Cimento 8 (1958) 316;
N.I. Muskhelishvili, Singuläre Integralgleichungen, ed. L. Berg and H. Schubert (Akademie Verlag, Berlin 1965)
[17] J.D. Bjorken, Phys. Rev. Lett. 4 (1960) 473;
S.W. MacDowell, Phys. Rev. Lett. 6 (1961) 385;
T.N. Pham and T.N. Truong, Phys. Rev. D16 (1977) 896
[18] T.N. Truong, Phys. Rev. Lett. 61 (1988) 2526;
A. Dobado, M.J. Herrero and T.N. Truong, Phys. Lett. B235 (1990) 134