

## Problem Set 3

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Before you begin this problem set, keep in mind the following:

- **Due date is Thursday, May 2, 5pm for the draft.**
  - **Due date is Tuesday, May 7, 5pm for the corrections.**
  - You should try each problem to the best of your ability. You can work with your class peers and consult internet resources in discussing a problem, but when writing/coding up your solution, you should not be consulting any other source specific to the problem.
  - Leave space for your corrections. Do not try to cram as many solutions into as small a space as possible.
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1. **QED**  $e^+e^- \rightarrow \mu^+\mu^-$  [**Bonus Problem**] (10 points).

Starting with the Feynman diagram for  $e^+e^- \rightarrow \mu^+\mu^-$  and the Feynman rules laid out in the lecture, calculate the spin-averaged differential cross section in the high-energy limit:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{cm}}} (1 + \cos^2 \theta). \quad (1)$$

Hint: Take a look at the useful trace and spinor completeness reference formulae in the appendix of Peskin& Schroeder at the end of this problem set (App. A). Also consider this helpful reference [1].

2. **MadGraph Bhabha scattering**  $e^+e^- \rightarrow e^+e^-$  (20 points).

The differential cross section for Bhabha scattering in QED in the high-energy limit can be written in terms of the Mandelstam variables  $s = (p_1 + p_2)^2$ ,  $t = (p_1 - p_3)^2$ , and  $u = (p_1 - p_4)^2$ ,

$$\frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{s} \left[ u^2 \left( \frac{1}{s} + \frac{1}{t} \right)^2 + \left( \frac{t}{s} \right)^2 + \left( \frac{s}{t} \right)^2 \right]. \quad (2)$$

Note that if we ignore the electron mass,  $s + t + u = 0$ .

- (5 points) Rewrite this formula in terms of  $s$  and  $\cos \theta$ .
- (5 points) What feature of the diagrams causes the differential cross section to diverge as  $\theta \rightarrow 0$ ? Why didn't we see this for  $e^+e^- \rightarrow \mu^+\mu^-$ ?

- (c) (10 points) Generate 10,000 events using MadGraph (excluding the Z boson exchange diagram) at  $\sqrt{s} = 1$  TeV. Plot the resulting distribution as a function of  $\cos \theta$  and compare to the theoretical expectation. What difference(s) do you observe?

### 3. MadGraph vs. ALEPH experimental results (20 points).

Using MadGraph, reproduce the experimental results from the ALEPH Collaboration shown in Fig. 1 [2], i.e. the total (inclusive) cross section  $\sigma$  and forward-backward asymmetry  $A_{\text{FB}}$  of the muons as a function of  $\sqrt{s}$  in the process  $e^+e^- \rightarrow \mu^+\mu^-$ . You will need to run MadGraph at a series of  $\sqrt{s}$  values, so you will need to edit the `runcard.dat` directly. Of course, both Z boson and  $\gamma$  exchange diagrams need to be included.

The forward-backward asymmetry is a measure of how many the imbalance between the forward and the backward directions

$$A_{\text{FB}} = \frac{\sigma(\cos \theta > 0) - \sigma(\cos \theta < 0)}{\sigma(\cos \theta > 0) + \sigma(\cos \theta < 0)} \quad (3)$$

For  $e^+e^- \rightarrow \mu^+\mu^-$ , this quantity is nonzero in the standard model because of the chiral couplings of the Z boson.

In particular, generate 1,000 events at  $\sqrt{s} = 60, 70, 80, 85, 88, 90, 91, 92, 95, 100, 110$ , and 120 GeV. You can manually run each individual generation by editing the run card or use the correlated scan syntax shown in [indico.jlab.org/event/413/contributions/7694](https://indico.jlab.org/event/413/contributions/7694) (page 18). Use [pylhe](#) (or another program of your choice) to plot  $\sigma$  and  $A_{\text{FB}}$  versus  $\sqrt{s}$  and compare to the data. Note the digitized data (from Tables 5 and 6) is available from HEPData [3].

### 4. $e^+e^- \rightarrow e^+e^-$ at NLO in QED (10 points).

- Draw all the LO and NLO in QED diagrams for  $e^+e^- \rightarrow e^+e^-$ . How many are there in total?
- Use MadGraph to generate the diagrams up to NLO order in QED. The MadGraph syntax for this is:

```
MG5_aMC> generate <process> / <excluded particles> [QED]
```

Note that you will have exclude all particles other than the electrons, positrons, and photons, including so-called “ghost” particles. You may also get a warning that you need to convert the model to python3 with:

```
convert model /usr/local/venv/MG5_aMC/models/loop_qcd_qed_sm
```

If they don’t match up to what you drew, which additional diagrams does MadGraph automatically include and why?

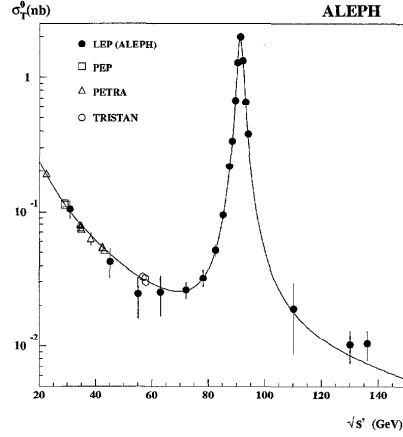


Fig. 2. Measured cross sections of muon-pair production compared with the fit results. The ALEPH measurements below 60 GeV correspond to the exclusive hard ISR selection that are not used in the fit. For comparison the measurements at lower energies from PEP, PETRA and TRISTAN are included.

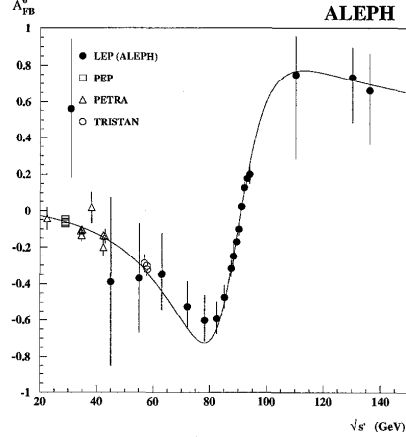


Fig. 3. Measured forward-backward asymmetries of muon-pair production compared with the fit results. The ALEPH measurements below 60 GeV correspond to the exclusive hard ISR selection that are not used in the fit. For comparison the measurements at lower energies from PEP, PETRA and TRISTAN are included.

Figure 1: Cross section (left) and forward-backward asymmetry of the muons (right) as a function of  $\sqrt{s}$  as measured by the ALEPH Collaboration [2].

## References

- [1] K. Kumericki, “Feynman diagrams for beginners”, 2016. [arXiv:1602.04182](#).
- [2] ALEPH Collaboration, “Study of the muon-pair production at centre-of-mass energies from 20 to 136 GeV with the ALEPH detector”, *Phys. Lett. B* **399** (1997) 329, [doi:10.1016/S0370-2693\(97\)00353-5](#).
- [3] HEPData record for [2], 1997. [doi:10.17182/hepdata.47490](#).

## A Reference Formulae

## Appendix

### Reference Formulae

This Appendix collects together some of the formulae that are most commonly used in Feynman diagram calculations.

#### A.1 Feynman Rules

In all theories it is understood that momentum is conserved at each vertex, and that undetermined loop momenta are integrated over:  $\int d^4p/(2\pi)^4$ . Fermion (including ghost) loops receive an additional factor of  $(-1)$ , as explained on page 120. Finally, each diagram can potentially have a symmetry factor, as explained on page 93.

$$\phi^4 \text{ theory: } \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

$$\text{Scalar propagator: } \begin{array}{c} \text{---} \leftarrow \\ p \end{array} = \frac{i}{p^2 - m^2 + i\epsilon} \quad (\text{A.1})$$

$$\phi^4 \text{ vertex: } \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = -i\lambda \quad (\text{A.2})$$

$$\text{External scalar: } \begin{array}{c} \text{---} \leftarrow \\ \text{---} \end{array} = 1 \quad (\text{A.3})$$

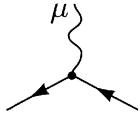
(Counterterm vertices for loop calculations are given on page 325.)

$$\text{Quantum Electrodynamics: } \mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2 - e\bar{\psi}\gamma^\mu\psi A_\mu$$

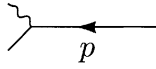
$$\text{Dirac propagator: } \begin{array}{c} \text{---} \leftarrow \\ p \end{array} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \quad (\text{A.4})$$

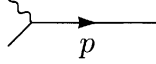
$$\text{Photon propagator: } \begin{array}{c} \text{~~~~~} \\ \leftarrow p \end{array} = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon} \quad (\text{A.5})$$

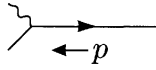
(Feynman gauge; see page 297 for generalized Lorentz gauge.)

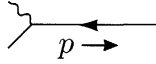
QED vertex:   $= iQe\gamma^\mu$  (A.6)

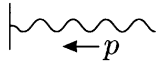
(Q = -1 for an electron)

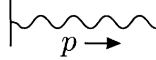
External fermions:   $= u^s(p)$  (initial) (A.7)

  $= \bar{u}^s(p)$  (final)

External antifermions:   $= \bar{v}^s(p)$  (initial) (A.8)

  $= v^s(p)$  (final)

External photons:   $= \epsilon_\mu(p)$  (initial) (A.9)

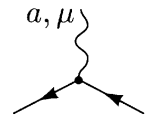
  $= \epsilon_\mu^*(p)$  (final)

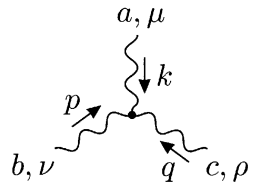
(Counterterm vertices for loop calculations are given on page 332.)

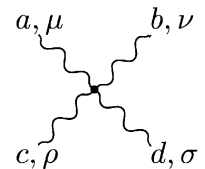
**Non-Abelian Gauge Theory:**

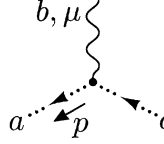
$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + gA_\mu^a \bar{\psi}\gamma^\mu t^a \psi - g f^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{\mu c}A^{\nu d})$$

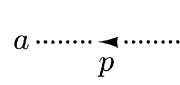
The fermion and gauge boson propagators are the same as in QED, times an identity matrix in the gauge group space. Similarly, the polarization of external particles is treated the same as in QED, but each external particle also has an orientation in the group space.

Fermion vertex:   $= i g \gamma^\mu t^a$  (A.10)

3-boson vertex:   $= g f^{abc} [g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$  (A.11)

4-boson vertex:   $= -i g^2 [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})]$  (A.12)

Ghost vertex:   $= -gf^{abc}p^\mu$  (A.13)

Ghost propagator:   $= \frac{i\delta^{ab}}{p^2 + i\epsilon}$  (A.14)

(Counterterm vertices for loop calculations are given on pages 528 and 532.)

**Other theories.** Feynman rules for other theories can be found on the following pages:

Yukawa theory	page 118
Scalar QED	page 312
Linear sigma model	page 353
Electroweak theory	pages 716, 743, 753

## A.2 Polarizations of External Particles

The spinors  $u^s(p)$  and  $v^s(p)$  obey the Dirac equation in the form

$$\begin{aligned} 0 &= (\not{p} - m)u^s(p) = \bar{u}^s(p)(\not{p} - m) \\ &= (\not{p} + m)v^s(p) = \bar{v}^s(p)(\not{p} + m), \end{aligned} \quad (\text{A.15})$$

where  $\not{p} = \gamma^\mu p_\mu$ . The Dirac matrices obey the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (\text{A.16})$$

We use a chiral basis,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A.17})$$

where

$$\sigma^\mu = (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}). \quad (\text{A.18})$$

In this basis the normalized Dirac spinors can be written

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix}, \quad (\text{A.19})$$

where  $\xi$  and  $\eta$  are two-component spinors normalized to unity. In the high-energy limit these expressions simplify to

$$u(p) \approx \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \boldsymbol{\sigma}) \xi^s \\ \frac{1}{2}(1 + \hat{p} \cdot \boldsymbol{\sigma}) \xi^s \end{pmatrix}, \quad v(p) \approx \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \boldsymbol{\sigma}) \eta^s \\ -\frac{1}{2}(1 + \hat{p} \cdot \boldsymbol{\sigma}) \eta^s \end{pmatrix}. \quad (\text{A.20})$$

Using the standard basis for the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.21})$$

we have, for example,  $\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  for spin up in the  $z$  direction, and  $\xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  for spin down in the  $z$  direction. For antifermions the physical spin is opposite to that of the spinor:  $\eta^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  corresponds to spin *down* in the  $z$  direction, and so on.

In computing unpolarized cross sections one encounters the polarization sums

$$\sum_s u^s(p) \bar{u}^s(p) = \not{p} + m, \quad \sum_s v^s(p) \bar{v}^s(p) = \not{p} - m. \quad (\text{A.22})$$

For polarized cross sections one can either resort to the explicit formulae (A.19) or insert the projection matrices

$$\left( \frac{1 + \gamma^5}{2} \right), \quad \left( \frac{1 - \gamma^5}{2} \right), \quad (\text{A.23})$$

which project onto right- and left-handed spinors, respectively. Again, for antifermions, the helicity of the spinor is opposite to the physical helicity of the particle.

Many other identities involving Dirac spinors and matrices can be found in Chapter 3.

Polarization vectors for photons and other gauge bosons are conventionally normalized to unity. For massless bosons the polarization must be transverse:

$$\epsilon^\mu = (0, \boldsymbol{\epsilon}), \quad \text{where } \mathbf{p} \cdot \boldsymbol{\epsilon} = 0. \quad (\text{A.24})$$

If  $\mathbf{p}$  is in the  $+z$  direction, the polarization vectors are

$$\epsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0), \quad \epsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0), \quad (\text{A.25})$$

for right- and left-handed helicities, respectively.

In computing unpolarized cross sections involving photons, one can replace

$$\sum_{\text{polarizations}} \epsilon_\mu^* \epsilon_\nu \longrightarrow -g_{\mu\nu}, \quad (\text{A.26})$$

by virtue of the Ward identity. In the case of massless non-Abelian gauge bosons, one must also sum over the emission of ghosts, as discussed in Section 16.3. In the massive case, one must in addition include the emission of Goldstone bosons, as discussed in Section 21.1.

### A.3 Numerator Algebra

Traces of  $\gamma$  matrices can be evaluated as follows:

$$\begin{aligned}
\text{tr}(\mathbf{1}) &= 4 \\
\text{tr}(\text{any odd } \# \text{ of } \gamma\text{'s}) &= 0 \\
\text{tr}(\gamma^\mu \gamma^\nu) &= 4g^{\mu\nu} \\
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) &= 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
\text{tr}(\gamma^5) &= 0 \\
\text{tr}(\gamma^\mu \gamma^\nu \gamma^5) &= 0 \\
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma}
\end{aligned} \tag{A.27}$$

Another identity allows one to reverse the order of  $\gamma$  matrices inside a trace:

$$\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \dots) = \text{tr}(\dots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu). \tag{A.28}$$

Contractions of  $\gamma$  matrices with each other simplify to:

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= 4 \\
\gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma \gamma^\rho \gamma^\nu
\end{aligned} \tag{A.29}$$

(These identities apply in four dimensions only; see the following section.) Contractions of the  $\epsilon$  symbol can also be simplified:

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} &= -24 \\
\epsilon^{\alpha\beta\gamma\mu} \epsilon_{\alpha\beta\gamma\nu} &= -6\delta^\mu_\nu \\
\epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\rho\sigma} &= -2(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)
\end{aligned} \tag{A.30}$$

In some calculations, it is useful to rearrange products of fermion bilinears by means of *Fierz identities*. Let  $u_1, \dots, u_4$  be Dirac spinors, and let  $u_{iL} = \frac{1}{2}(1 - \gamma^5)u_i$  be the left-handed projection. Then the most important Fierz rearrangement formula is

$$(\bar{u}_{1L} \gamma^\mu u_{2L})(\bar{u}_{3L} \gamma_\mu u_{4L}) = -(\bar{u}_{1L} \gamma^\mu u_{4L})(\bar{u}_{3L} \gamma_\mu u_{2L}). \tag{A.31}$$

Additional formulae can be generated by the use of the following identities for the  $2 \times 2$  blocks of Dirac matrices:

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}; \quad (\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{A.32}$$

In non-Abelian gauge theories, the Feynman rules involve gauge group matrices  $t^a$  that form a representation  $r$  of a Lie algebra  $G$ . The symbol  $G$  also denotes the adjoint representation of the algebra. The matrices  $t^a$  obey

$$[t^a, t^b] = if^{abc}t^c, \tag{A.33}$$



where the structure constants  $f^{abc}$  are totally antisymmetric. The invariants  $C(r)$  and  $C_2(r)$  of the representation  $r$  are defined by

$$\text{tr}[t^a t^b] = C(r) \delta^{ab}, \quad t^a t^a = C_2(r) \cdot \mathbf{1}. \quad (\text{A.34})$$

These are related by

$$C(r) = \frac{d(r)}{d(G)} C_2(r), \quad (\text{A.35})$$

where  $d(r)$  is the dimension of the representation. Traces and contractions of the  $t^a$  can be evaluated using the above identities and their consequences:

$$\begin{aligned} t^a t^b t^a &= [C_2(r) - \tfrac{1}{2} C_2(G)] t^b \\ f^{acd} f^{bcd} &= C_2(G) \delta^{ab} \\ f^{abc} t^b t^c &= \tfrac{1}{2} i C_2(G) t^a \end{aligned} \quad (\text{A.36})$$

For  $SU(N)$  groups, the fundamental representation is denoted by  $N$ , and we have

$$C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}, \quad C(G) = C_2(G) = N. \quad (\text{A.37})$$

The following relation, satisfied by the matrices of the fundamental representation of  $SU(N)$ , is also very helpful:

$$(t^a)_{ij} (t^a)_{k\ell} = \frac{1}{2} \left( \delta_{i\ell} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{k\ell} \right). \quad (\text{A.38})$$

## A.4 Loop Integrals and Dimensional Regularization

To combine propagator denominators, introduce integrals over Feynman parameters:

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \delta(\sum x_i - 1) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots x_n A_n]^n} \quad (\text{A.39})$$

In the case of only two denominator factors, this formula reduces to

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}. \quad (\text{A.40})$$

A more general formula in which the  $A_i$  are raised to arbitrary powers is given in Eq. (6.42).

Once this is done, the bracketed quantity in the denominator will be a quadratic function of the integration variables  $p_i^\mu$ . Next, complete the square and shift the integration variables to absorb the terms linear in  $p_i^\mu$ . For a one-loop integral, there is a single integration momentum  $p^\mu$ , which is shifted to a momentum variable  $\ell^\mu$ . After this shift, the denominator takes the form

$(\ell^2 - \Delta)^n$ . In the numerator, terms with an odd number of powers of  $\ell$  vanish by symmetric integration. Symmetry also allows one to replace

$$\ell^\mu \ell^\nu \rightarrow \frac{1}{d} \ell^2 g^{\mu\nu}, \quad (\text{A.41})$$

$$\ell^\mu \ell^\nu \ell^\rho \ell^\sigma \rightarrow \frac{1}{d(d+2)} (\ell^2)^2 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}). \quad (\text{A.42})$$

(Here  $d$  is the spacetime dimension.) The integral is most conveniently evaluated after Wick-rotating to Euclidean space, with the substitution

$$\ell^0 = i\ell_E^0, \quad \ell^2 = -\ell_E^2. \quad (\text{A.43})$$

Alternatively, one can use the following table of  $d$ -dimensional integrals in Minkowski space:

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \quad (\text{A.44})$$

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \quad (\text{A.45})$$

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \quad (\text{A.46})$$

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{(\ell^2)^2}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \quad (\text{A.47})$$

$$\begin{aligned} \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu \ell^\rho \ell^\sigma}{(\ell^2 - \Delta)^n} &= \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \\ &\times \frac{1}{4} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \end{aligned} \quad (\text{A.48})$$

If the integral converges, one can set  $d = 4$  from the start. If the integral diverges, the behavior near  $d = 4$  can be extracted by expanding

$$\left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} = 1 - (2 - \frac{d}{2}) \log \Delta + \dots \quad (\text{A.49})$$

One also needs the expansion of  $\Gamma(x)$  near its poles:

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x) \quad (\text{A.50})$$

near  $x = 0$ , and

$$\Gamma(x) = \frac{(-1)^n}{n!} \left( \frac{1}{x+n} - \gamma + 1 + \dots + \frac{1}{n} + \mathcal{O}(x+n) \right) \quad (\text{A.51})$$

near  $x = -n$ . Here  $\gamma$  is the Euler-Mascheroni constant,  $\gamma \approx 0.5772$ . The following combination of terms often appears in calculations:

$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log \Delta - \gamma + \log(4\pi) + \mathcal{O}(\epsilon)\right), \quad (\text{A.52})$$

with  $\epsilon = 4 - d$ .

Notice that  $\Delta$  is positive if it is a combination of masses and *spacelike* momentum invariants. If  $\Delta$  contains timelike momenta, it may become negative. Then these integrals acquire imaginary parts, which give the discontinuities of  $S$ -matrix elements. To compute the  $S$ -matrix in a physical region, choose the correct branch of the function by the prescription

$$\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \rightarrow \left(\frac{1}{\Delta - i\epsilon}\right)^{n-\frac{d}{2}}, \quad (\text{A.53})$$

where  $-i\epsilon$  (not to be confused with  $\epsilon$  in the previous paragraph!) gives a tiny negative imaginary part.

Traces in Eq. (A.27) that do not involve  $\gamma^5$  are independent of dimensionality. However, since

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = d \quad (\text{A.54})$$

in  $d$  dimensions, the contraction identities (A.29) are modified:

$$\begin{aligned} \gamma^\mu \gamma_\mu &= d \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -(d-2)\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} - (4-d)\gamma^\nu \gamma^\rho \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma \end{aligned} \quad (\text{A.55})$$

## A.5 Cross Sections and Decay Rates

Once the squared matrix element for a scattering process is known, the differential cross section is given by

$$\begin{aligned} d\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \left( \prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\times |\mathcal{M}(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum p_f). \end{aligned} \quad (\text{A.56})$$

The differential decay rate of an unstable particle to a given final state is

$$d\Gamma = \frac{1}{2m_A} \left( \prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(m_A \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_A - \sum p_f). \quad (\text{A.57})$$

For the special case of a two-particle final state, the Lorentz-invariant phase space takes the simple form

$$\left( \prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(\sum p_i - \sum p_f) = \int \frac{d\Omega_{\text{cm}}}{4\pi} \frac{1}{8\pi} \left( \frac{2|\mathbf{p}|}{E_{\text{cm}}} \right), \quad (\text{A.58})$$

where  $|\mathbf{p}|$  is the magnitude of the 3-momentum of either particle in the center-of-mass frame.

## A.6 Physical Constants and Conversion Factors

Precisely known physical constants:

$$\begin{aligned} c &= 2.998 \times 10^{10} \text{ cm/s} \\ \hbar &= 6.582 \times 10^{-22} \text{ MeV s} \\ e &= -1.602 \times 10^{-19} \text{ C} \\ \alpha &= \frac{e^2}{4\pi\hbar c} = \frac{1}{137.04} = 0.00730 \\ \frac{G_F}{(\hbar c)^3} &= 1.166 \times 10^{-5} \text{ GeV}^{-2} \end{aligned}$$

The values of the strong and weak interaction coupling constants depend on the conventions used to define them, as explained in Sections 17.6 and 21.3. For the purpose of estimation, however, one can use the following values:

$$\begin{aligned} \alpha_s(10 \text{ GeV}) &= 0.18 \\ \alpha_s(m_Z) &= 0.12 \\ \sin^2 \theta_w &= 0.23 \end{aligned}$$

Particle masses (times  $c^2$ ):

$e$ :	0.5110 MeV	$p$ :	938.3 MeV
$\mu$ :	105.6 MeV	$n$ :	939.6 MeV
$\tau$ :	1777 MeV	$\pi^\pm$ :	139.6 MeV
$W^\pm$ :	80.2 GeV	$\pi^0$ :	135.0 MeV
$Z^0$ :	91.19 GeV		

Useful combinations:

Bohr radius:	$a_0 = \frac{\hbar}{\alpha m_e c} = 5.292 \times 10^{-9} \text{ cm}$
electron Compton wavelength:	$\lambda = \frac{\hbar}{m_e c} = 3.862 \times 10^{-11} \text{ cm}$
classical electron radius:	$r_e = \frac{\alpha \hbar}{m_e c} = 2.818 \times 10^{-13} \text{ cm}$
Thomson cross section:	$\sigma_T = \frac{8\pi r_e^2}{3} = 0.6652 \text{ barn}$
annihilation cross section:	$1 \text{ R} = \frac{4\pi\alpha^2}{3E_{\text{cm}}^2} = \frac{86.8 \text{ nbarn}}{(E_{\text{cm}} \text{ in GeV})^2}$

Conversion factors:

$$\begin{aligned}
 (1 \text{ GeV})/c^2 &= 1.783 \times 10^{-24} \text{ g} \\
 (1 \text{ GeV})^{-1}(\hbar c) &= 0.1973 \times 10^{-13} \text{ cm} = 0.1973 \text{ fm}; \\
 (1 \text{ GeV})^{-2}(\hbar c)^2 &= 0.3894 \times 10^{-27} \text{ cm}^2 = 0.3894 \text{ mbarn} \\
 1 \text{ barn} &= 10^{-24} \text{ cm}^2 \\
 (1 \text{ volt/meter})(e\hbar c) &= 1.973 \times 10^{-25} \text{ GeV}^2 \\
 (1 \text{ tesla})(e\hbar c^2) &= 5.916 \times 10^{-17} \text{ GeV}^2
 \end{aligned}$$

A complete, up-to-date tabulation of the fundamental constants and the properties of elementary particles is given in the *Review of Particle Properties*, which can be found in a recent issue of either *Physical Review D* or *Physics Letters B*. The most recent *Review* as of this writing is published in *Physical Review D* **50**, 1173 (1994).