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PHYS 239: Modern Collider Physics
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Problem Set 4

Before you begin this problem set, keep in mind the following:

- **Due date is Monday, May 8, 5pm for the draft.**
- **Due date is Thursday, May 11, 5pm for the corrections.**
- You should try each problem to the best of your ability. You can work with your class peers and consult internet resources in discussing a problem, but when writing/coding up your solution, you should not be consulting any other source specific to the problem.
- Leave space for your corrections. Do not try to cram as many solutions into as small a space as possible.

1. **Mandelstam Variables** [10 points]. Consider the two-to-two scattering of massless particles (which may be the same or different), with incoming momenta $p^\mu_{1,2}$ and outgoing momenta $p^\mu_{3,4}$. Compute the Mandelstam variables in terms of the center-of-mass (CM) energies of the incoming particles E and CM scattering angle θ:

$$s = (p_1 + p_2)^2 = 4E^2 \tag{1}$$

$$t = (p_1 - p_3)^2 = -2E^2(1 - \cos\theta) \tag{2}$$

$$u = (p_1 - p_4)^2 = -2E^2(1 + \cos\theta)\ . \tag{3}$$

Note that $s + t + u = 0$. Using momentum conservation show that, if the particles have masses, $p_i^2 = m_i^2$, then

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2\ . \tag{4}$$

2. **Parton Model DIS Kinematics** [10 points]. Consider the lab frame description of a generic DIS event: an incoming probe (massless electron, muon, or neutrino) of four-momentum $k = (E_{\text{lab}}, 0, 0, E_{\text{lab}})$ incident on a proton at rest $P = (m, 0, 0, 0)$ which strikes a parton of momentum xP and scatters into a massless electron, muon, or neutrino with momentum

$$k' = (E'_{\text{lab}}, 0, E'_{\text{lab}}\sin\theta_{\text{lab}}, E'_{\text{lab}}\cos\theta_{\text{lab}}))\ . \tag{5}$$

It is conventional to define the variables $Q^2 = -q^2 = -(k - k')^2$ and $\nu = E_{\text{lab}} - E'_{\text{lab}}$, and the partonic DIS observables

$$x = \frac{Q^2}{2P \cdot q} \tag{6}$$

$$y = \frac{2P \cdot q}{2P \cdot k}\ . \tag{7}$$

Note that as defined the quantities x and y are Lorentz-invariant.

(a) Compute x and y in terms of the measured lab frame variables $E_{\rm lab}$, $E'_{\rm lab}$, and $\cos\theta_{\rm lab}$.

(b) Compute the partonic CM Mandelstam variables and scattering angle $\cos\theta$ in terms of x, y, and the lab (probe-proton) center-of-mass squared s – working in the limit where $s, \hat{s} \gg m^2$ and you may neglect the mass of the proton. Show that both x and y are positive numbers between zero and one. (Note that in the lab frame, the proton rest frame, you cannot set the proton mass to zero!)

3. **Charged-Current DIS in the Parton Model** [20 points]. In terms of the helicity cross-sections discussed in class (see Peskin, "Concepts in Particle Physics", Chs. 8–9) the center-of-mass cross-section for the annihilation of electrons into muons for a fixed helicity can be written

$$\frac{d\sigma(e^-_L e^+_R \to \mu^-_L \mu^+_R)}{d\cos\theta} = \frac{1}{32\pi\hat{s}}|\mathcal{M}(LR \to LR)|^2 \,, \tag{8}$$

where the factor of in front of the matrix element comes from two-body scattering kinematics, and the matrix-element squared in QED is equal to

$$|\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R)|^2 = e^4(1+\cos\theta)^2 \,. \tag{9}$$

We will use this expression to compute the deep inelastic charged-current scattering cross section of muon neutrinos on protons.

(a) First consider the matrix element squared for $\nu_{\mu L}\mu^+_R \to u_L \bar{d}_R$. This process occurs through the s-channel exchange of a W^+ boson, which couples to left-handed particles, here $\nu_{\mu L}$ and u_L, and right-handed antiparticles, in this case μ^+_R and $\bar{d}_R$. Write down the corresponding Feynman diagram: the external fermion states follow exactly the same pattern as the QED process given above (left-handed particle and right-handed anti-particle annihilate into left-handed particle and right-handed antiparticle)—so the factors associated with the external fermions (which we are treating as massless) are exactly the same! The only difference is that the factors of the coupling e at each vertex and the photon propagator change to the W boson coupling and propagator. In the squared matrix element this amounts to the substitution

$$\frac{e^4}{\hat{s}^2} \to \frac{g^4}{4M_W^4} = 8G_F^2 \,, \tag{10}$$

where G_F is the "Fermi constant" which is determined by measuring the muon lifetime. Making this replacement, and re-writing things in terms of the Mandelstam variables, show that

$$|\mathcal{M}(\nu_{\mu L}\mu^+_R \to u_L \bar{d}_R)|^2 = 32 G_F^2 \hat{u}^2 \,. \tag{11}$$

Are there any other helicity contributions to this (admittedly theoretical) scattering process $\nu_\mu \mu \to u\bar{d}$? Why or why not?

(b) Use crossing to relate the process in the previous step to the scattering process $\nu_{\mu L}\overline{u}_R \rightarrow \mu_L^- \overline{d}_R$. In terms of the Mandelstam variables, show that this corresponds to the "crossing" $\hat{s} \rightarrow \hat{t}$, $\hat{t} \rightarrow \hat{s}$, and $\hat{u} \rightarrow \hat{u}$, resulting in the matrix-element squared

$$|\mathcal{M}(\nu_{\mu L}\overline{u}_R \rightarrow \mu_L^- \overline{d}_R)|^2 = 32 G_F^2 \hat{u}^2 \,. \tag{12}$$

(c) Next, use crossing to relate the process in part (a) to $\nu_{\mu L} d_L \rightarrow \mu_L^- u_L$. Show that this corresponds to $\hat{s} \rightarrow \hat{t}$, $\hat{t} \rightarrow \hat{u}$, and $\hat{u} \rightarrow \hat{s}$, and hence the matrix-element squared

$$|\mathcal{M}(\nu_{\mu L} d_L \rightarrow \mu_L^- u_L)|^2 = 32 G_F^2 \hat{s}^2 \,. \tag{13}$$

(d) Use the results of parts (b) and (c) above to compute the partonic differential cross sections for the deep inelastic scattering process $\nu_\mu p \rightarrow \mu^- X$ by following these steps: (i) re-write the dependence of both squared matrix-elements in terms of $\hat{s}$, x, and y, using the kinematic formulae you derived in problem 2, (ii) use the two-to-two master formula for the partonic scattering cross section in Eq. (8), (iii) remember to divide by a factor of two to average over incoming quark spins (only the appropriate chirality contributes, but the incoming parton from the proton is *unpolarized*), (iv) multiply these cross sections by the appropriate parton distribution functions and convert $\cos\theta$ to y to compute the differential cross section in the form

$$\frac{d^2\sigma(\nu_\mu p \rightarrow \mu^- X)}{dxdy} = A(x,y,s)\overline{u}(x) + B(x,y,s)d(x) \,, \tag{14}$$

where $\overline{u}(x)$ and $\overline{d}(x)$ are the anti-up and down quark distribution functions, and A and B are functions you determine. Note that, having come to the physical neutrino-proton scattering process, we no longer can or should specify the helicities of the particles: ν_μ is only left-handed (in the standard model) and only the appropriate chirality components of the quarks inside the proton contribute to the scattering process.

(e) How would you do the analogous computation for the process $\overline{\nu}_\mu p \rightarrow \mu^+ X$? Show that the final answer is

$$\frac{d^2\sigma(\overline{\nu}_\mu p \rightarrow \mu^+ X)}{dxdy} = \frac{G_F^2 xs}{\pi}\left[\overline{u}(x) + (1-y)^2 d(x)\right] \,. \tag{15}$$

Use these to determine the cross-sections from an isoscalar target N, $\nu_{\mu L} N \rightarrow \mu^- X$ and $\overline{\nu}_{\mu R} N \rightarrow \mu^+ X$.

4. **Kinematics of** $2 \rightarrow 2$ **Scattering at a Hadron Collider** [10 points]. Finish the discussion of pp collider kinematics for hard $2 \rightarrow 2$ processes begun in lecture.

(a) In the partonic CM frame, using the notation of problem 1, $E = \sqrt{\hat{s}}/2$, and in this frame the total CM momentum is

$$q = p_1 + p_2 = (\sqrt{\hat{s}}, 0, 0, 0) . \tag{16}$$

The collider (proton-proton CM or lab) frame is related to partonic CM frame via a boost along the beam axis by rapidity y. Compute the total momentum in the collider frame in two ways: by boosting the expression in the CM frame, and by considering that the momentum comes from partons with momentum fraction $x_{1,2}$. Show that

$$\cosh y = \frac{x_1 + x_2}{2} \sqrt{\frac{s}{\hat{s}}} , \tag{17}$$

where $\sqrt{s}$ is the collider energy. Recalling that (ignoring the mass of the proton) $\hat{s} = x_1 x_2 \sqrt{s}$, show that

$$x_{1,2} = \sqrt{\tau} e^{\pm y} , \tag{18}$$

where $\tau = x_1 x_2 = \hat{s}/s$.

(b) Compute the Jacobian to allow for the transformation between the variables

$$d\hat{s}\, dy = \frac{\hat{s}}{x_1 x_2} dx_1 dx_2 . \tag{19}$$

(c) In the partonic CM frame, the momenta $p_{3,4}$ may be written

$$p_{3,4} = (p_\mathrm{T} \cosh y^*, 0, \pm p_\mathrm{T}, \pm p_\mathrm{T} \sinh y^*) . \tag{20}$$

Show that the Mandelstam variables may be written in terms of p_T and y^* as

$$\hat{s} = 4 p_\mathrm{T}^2 \cosh^2 y^* , \tag{21}$$

$$\hat{t} = -2 p_\mathrm{T}^2 \cosh y^* e^{-y^*} , \tag{22}$$

$$\hat{u} = -2 p_\mathrm{T}^2 \cosh y^* e^{+y^*} , \tag{23}$$

and therefore that

$$x_{1,2} = \frac{2 p_\mathrm{T}}{\sqrt{s}} \cosh y^* e^{\pm y} . \tag{24}$$

(d) In the collider frame, the momenta $p_{3,4}$ may be written in terms of the p_T of the event and the rapidities of the outgoing states (remember, we are considering massless states)

$$p_{3,4} = (p_\mathrm{T} \cosh y_{3,4}, 0, \pm p_\mathrm{T}, p_\mathrm{T} \sinh y_{3,4}) . \tag{25}$$

Show that the partonic CM rapidity y^* and the boost rapidity relating the partonic CM frame and the collider frame are given by

$$y^* = \frac{y_3 - y_4}{2} \tag{26}$$

$$y = \frac{y_3 + y_4}{2} . \tag{27}$$

5. **Drell–Yan Prodution Cross Section at LHC** [10 points]. Use the machinery introduced above to compute the contribution to the Drell–Yan cross section from photons at a hadron collider, $pp \to \mu^+\mu^-$. Recall from our discussion in class that the total cross section from electron-positron annihilation is

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3E_{CM}^2} \,. \tag{28}$$

   (a) Show that the partonic level total cross section for $q\bar{q} \to \mu^+\mu^-$ is

$$\hat{\sigma}(q\bar{q} \to \mu^+\mu^-) = \frac{e_q^2}{3} \cdot \frac{4\pi\alpha^2}{3\hat{s}} \,, \tag{29}$$

   and explain the factor $e_q^2/3$. What is the significance of $\hat{s}$ in terms of the observed muons?

   (b) Multiply by the appropriate structure functions and make the necessary variable transformations to compute the differential hadronic-level cross section

$$\frac{d^2\sigma(pp \to \mu^+\mu^-)}{d\hat{s}\, dy} \,. \tag{30}$$

   for the production of a $\mu^+\mu^-$ pair with total rapidity y.

   (c) How would this computation change if you wanted to predict the angular distribution of the *individual* muon and antimuon?