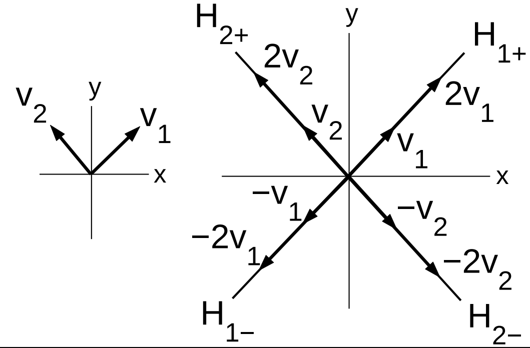
9.3-4: Phase Plane Portraits
Classification of 2d Systems:

$$\mathbf{x}' = A\mathbf{x},\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}:\ T = a + d,\ D = ad - bc,\ p(\lambda) = \lambda^2 - T\lambda + D$$

Case A: $T^2 - 4D > 0$

$\Rightarrow$ real distinct eigenvalues

$$\lambda_{1,2} = (T \pm \sqrt{T^2 - 4D})/2$$



General Solution:
($\mathbf{v}_1, \mathbf{v}_2$: eigenvectors)

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

$L_{1,2}$: Full lines generated by $\mathbf{v}_{1,2}$

Half line trajectories:

if $c_2 = 0 \Rightarrow \mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1$

$\Rightarrow$ trajectory is half line

$$H_{1+} = \{\mathbf{x} = \alpha\mathbf{v}_1 \,|\, \alpha > 0\} \text{ if } c_1 > 0$$
$$H_{1-} = \{\mathbf{x} = \alpha\mathbf{v}_1 \,|\, \alpha < 0\} \text{ if } c_1 < 0$$

Same for $H_{2\pm}$ if $c_1 = 0$, $c_2 > 0$ or < 0

- The 4 half line trajectories separate 4 regions of $\mathbf{R}^2$

Phase portrait:
Sketch trajectories. Indicate *direction of motion* by arrows pointing in the direction of increasing t

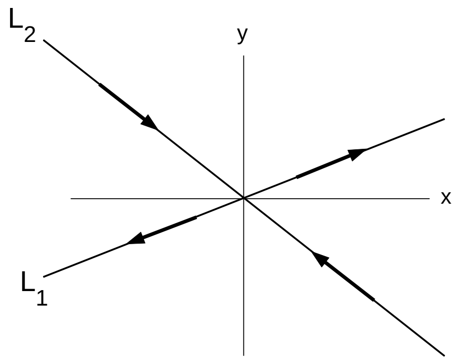
Direction of Motion on Half Line Trajectories:

- If $\lambda_1 > 0$ then $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1$
  - moves out to ∞ for $t \to \infty$ (outwards arrow on H_{1+})
  - approaches $\mathbf{0}$ for $t \to -\infty$
- If $\lambda_1 < 0$ then $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1$
  - approaches $\mathbf{0}$ for $t \to \infty$ (inwards arrow on H_{1+})
  - moves out to ∞ for $t \to -\infty$

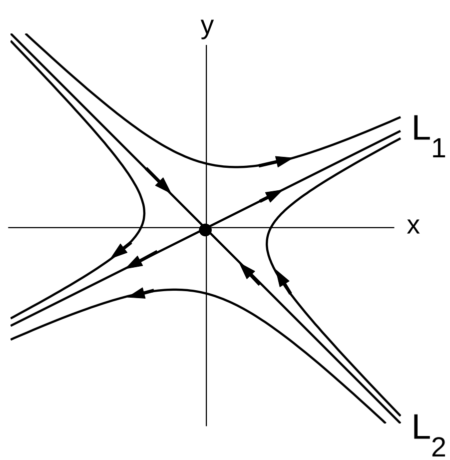
Subcases of Case A

Saddle
$\lambda_1 > 0 > \lambda_2$

Half line trajectories



Generic Trajectories

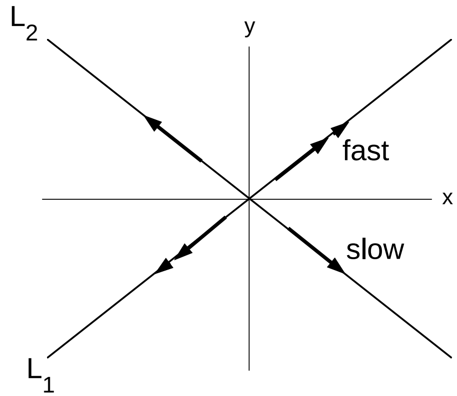


Generic trajectory in each region approaches

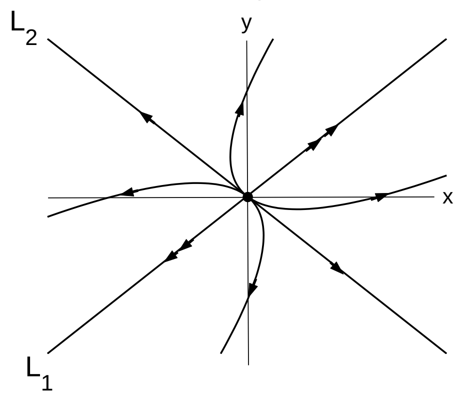
- L_1 for $t \to \infty$
- L_2 for $t \to -\infty$

Nodal source
$\lambda_1 > \lambda_2 > 0$

Half line trajectories



Generic Trajectories



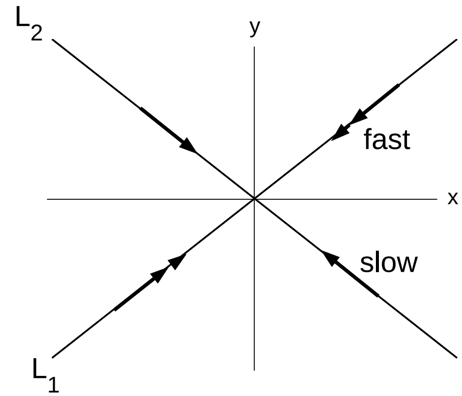
$\rightarrow\!\!\rightarrow$: fast escape to ∞

Generic trajectory is

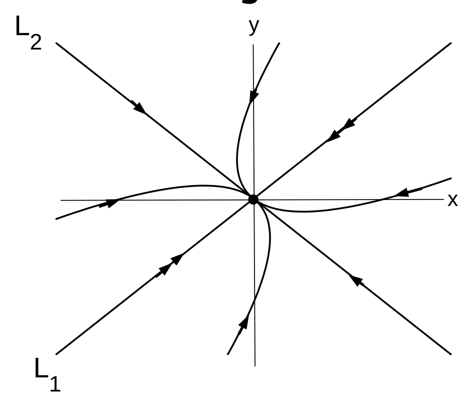
- parallel to L_1 for $t \to \infty$
- tangent to L_2 for $t \to -\infty$

Nodal sink
$\lambda_1 < \lambda_2 < 0$

Half line trajectories



Generic Trajectories



$\rightarrow\!\!\rightarrow$: fast approach to $\mathbf{0}$

Generic trajectory is

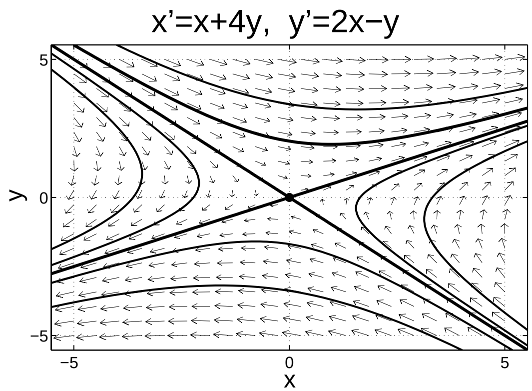
- parallel to L_1 for $t \to -\infty$
- tangent to L_2 for $t \to \infty$

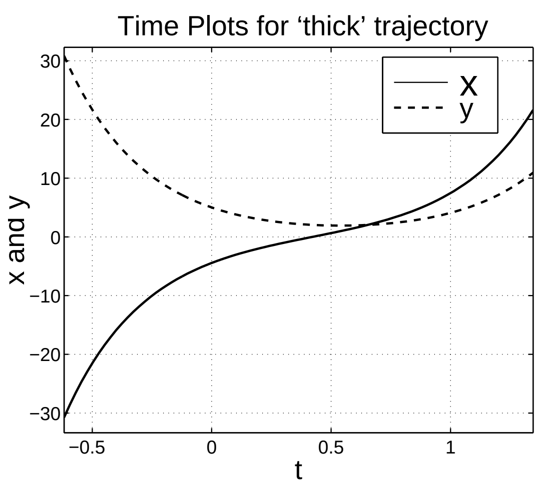
Phase Portraits and Time Plots for Cases A (*pplane6*)

Saddle

Ex.: $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$

$\lambda_1 = 3 \leftrightarrow \mathbf{v}_1 = [2,1]^T$
$\lambda_2 = -3 \leftrightarrow \mathbf{v}_2 = [-1,1]^T$

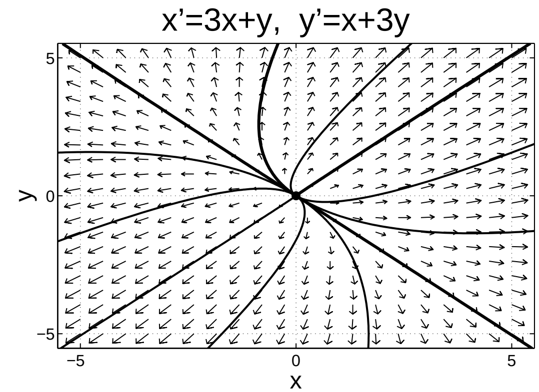


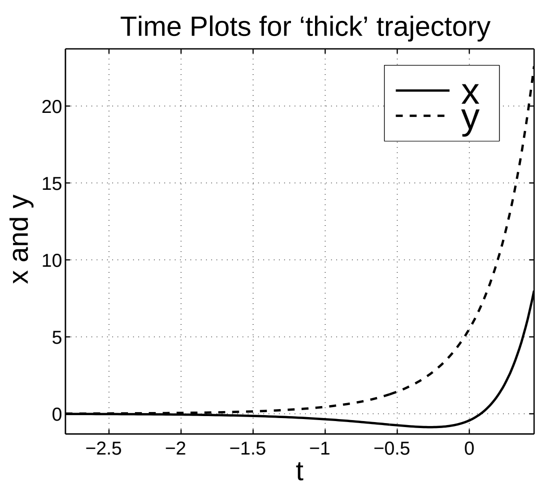


Nodal Source

Ex.: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\lambda_1 = 4 \leftrightarrow \mathbf{v}_1 = [1,1]^T$
$\lambda_2 = 2 \leftrightarrow \mathbf{v}_2 = [-1,1]^T$

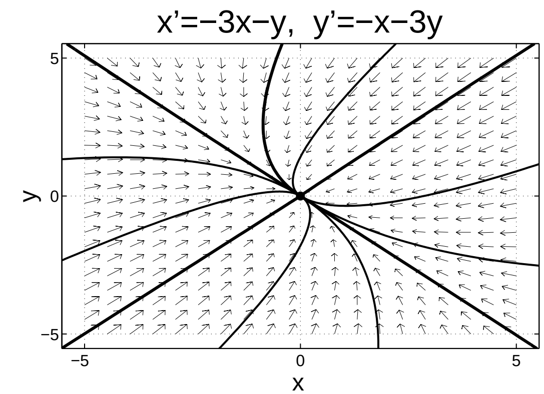


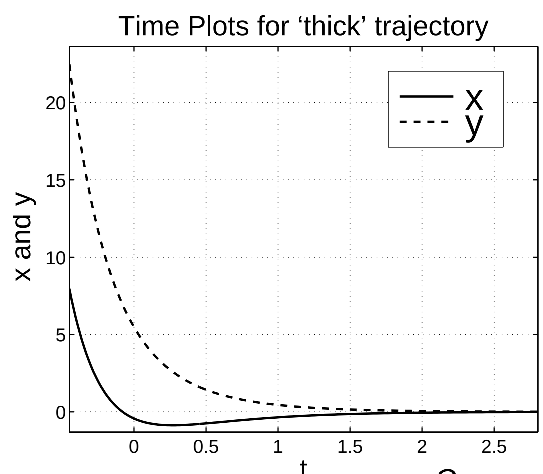


Nodal Sink

Ex.: $A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}$

$\lambda_1 = -4 \leftrightarrow \mathbf{v}_1 = [1,1]^T$
$\lambda_2 = -2 \leftrightarrow \mathbf{v}_2 = [-1,1]^T$





Case B: $T^2 - 4D < 0 \Rightarrow \lambda = \alpha + i\beta$; $\alpha = T/2$, $\beta = \sqrt{4D - T^2}/2$

λ complex $\Rightarrow$ eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ complex $\Rightarrow$ no half line solutions

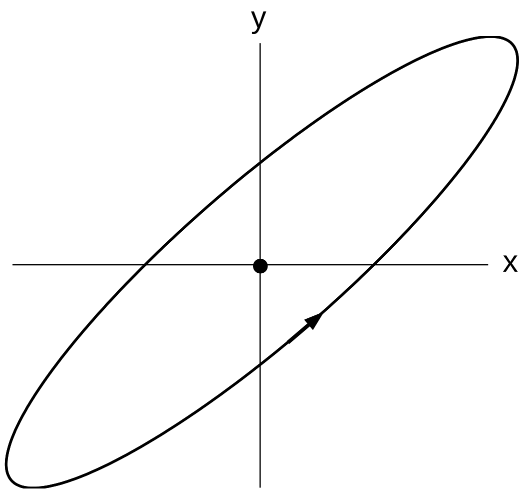
General Solution: $\mathbf{x}(t) = e^{\alpha t}[c_1(\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t) + c_2(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t)]$

Subcases of Case B

Center: $\alpha = 0$

$\Rightarrow$ $\mathbf{x}(t)$ periodic

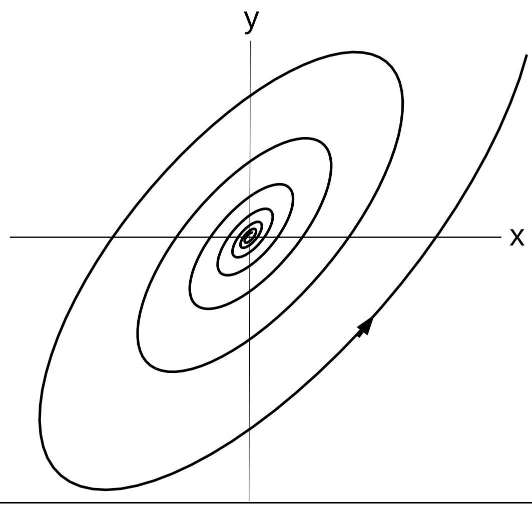
$\Rightarrow$ trajectories are closed curves



Spiral Source: $\alpha > 0$

$\Rightarrow$ growing oscillations

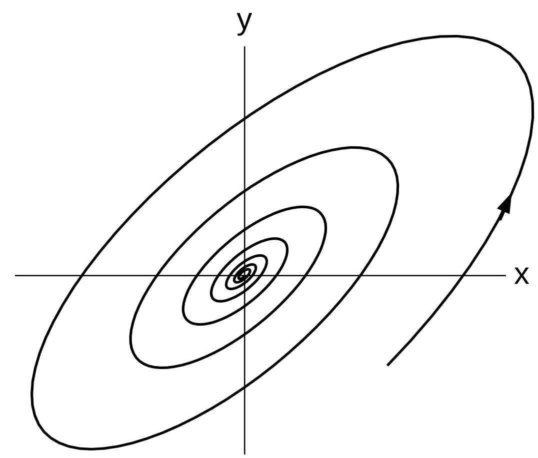
$\Rightarrow$ trajectories are outgoing spirals



Spiral Sink: $\alpha < 0$

$\Rightarrow$ decaying oscillations

$\Rightarrow$ trajectories are ingoing spirals



Direction of Rotation: At $\mathbf{x} = [1, 0]^T$: $y' = c$. If $\begin{cases} c > 0 & \Rightarrow \text{counterclockwise} \\ c < 0 & \Rightarrow \text{clockwise} \end{cases}$

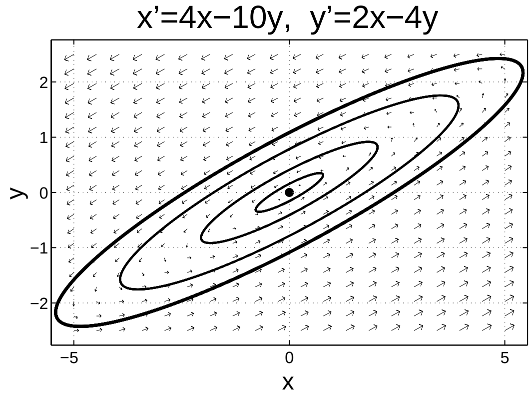
Borderline Case:
Center ($\alpha = 0$) is border between spiral source ($\alpha > 0$) and spiral sink ($\alpha < 0$).

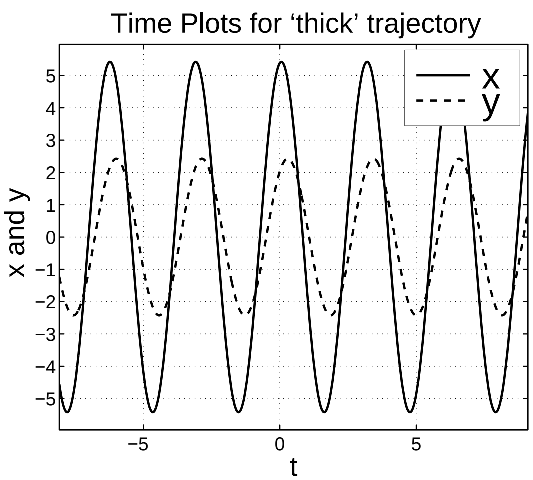
Phase Portraits and Time Plots for Cases B (*pplane6*)

Center

Ex.: $A = \begin{bmatrix} 4 & -10 \\ 2 & -4 \end{bmatrix}$

$$\lambda = 2i \leftrightarrow \mathbf{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

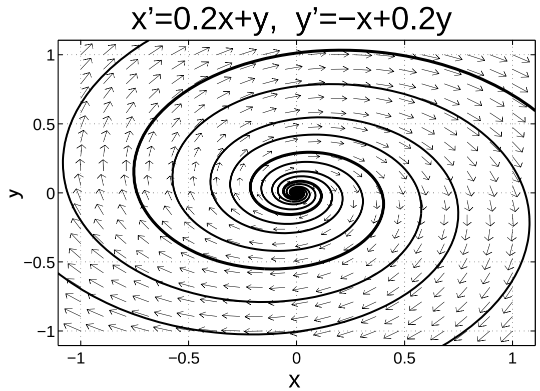


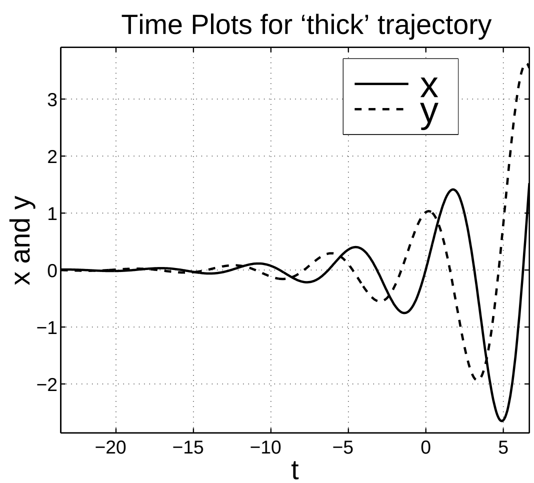


Spiral Source

Ex.: $A = \begin{bmatrix} 0.2 & 1 \\ -1 & 0.2 \end{bmatrix}$

$$\lambda = 0.2 + i \leftrightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

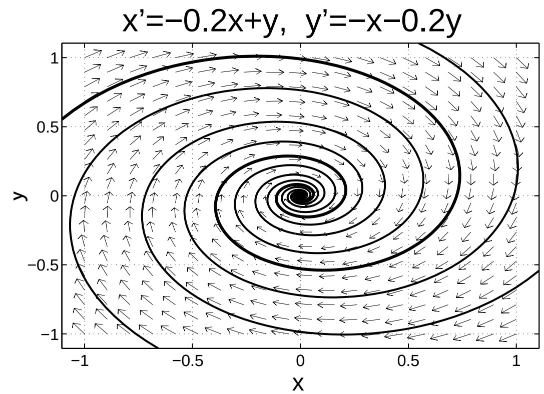


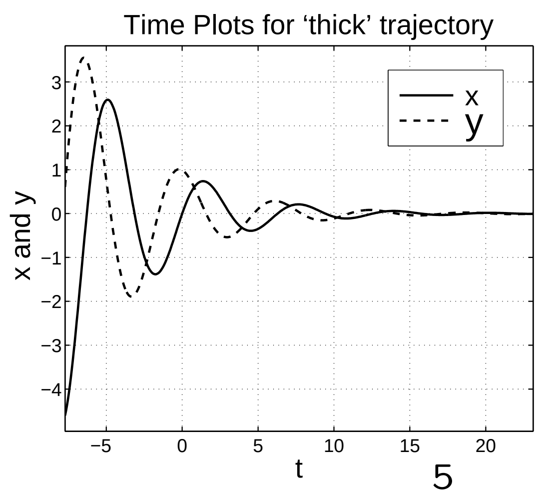


Spiral Sink

Ex.: $A = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix}$

$$\lambda = -0.2 + i \leftrightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$





Degenerate Node: Borderline Case Spiral/Node

- Assume $T^2 - 4D = 0 \Rightarrow$ single eigenvalue $\lambda = T/2$
- Assume generic case: $(A - \lambda I) \neq 0 \Rightarrow$ single eigenvector $\mathbf{v}$
- Let $(A - \lambda I)\mathbf{w} = \mathbf{v} \Rightarrow$ General solution:

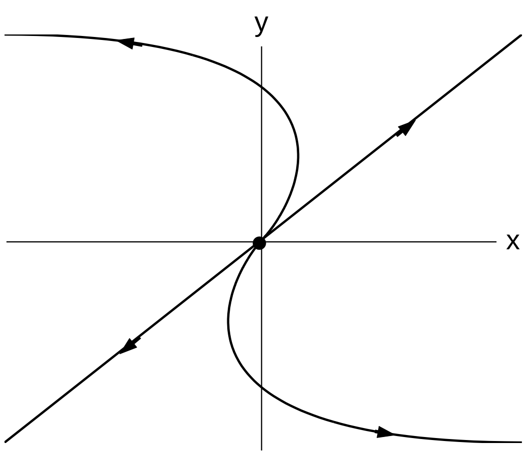
$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(\mathbf{w} + t\mathbf{v})$$

$\Rightarrow$ only two half line solutions on straight line generated by $\mathbf{v}$

Degenerate Nodal Source:

$$T > 0$$

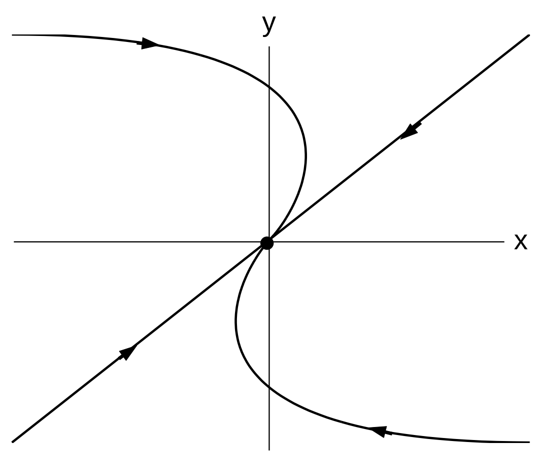
$$\text{borderline case} \begin{cases} \text{nodal source} \\ \text{spiral source} \end{cases}$$



Degenerate Nodal Sink:

$$T < 0$$

$$\text{borderline case} \begin{cases} \text{nodal sink} \\ \text{spiral sink} \end{cases}$$



Saddle–Node: Borderline Case Node/Saddle

- Assume $D = 0$, $T \neq 0 \Rightarrow$ eigenvalues $\lambda_1 = 0$, $\lambda_2 = T$
- Let $\mathbf{v}_1, \mathbf{v}_2$ be the eigenvectors $\Rightarrow$ General solution:

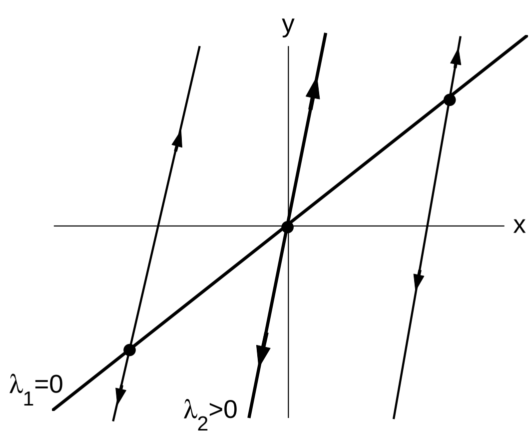
$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

$\Rightarrow$
- line of equilibrium points generated by $\mathbf{v}_1$
- infinitely many half line solutions on straight lines parallel to line generated by $\mathbf{v}_2$

Unstable Saddle–Node:

$$T > 0$$

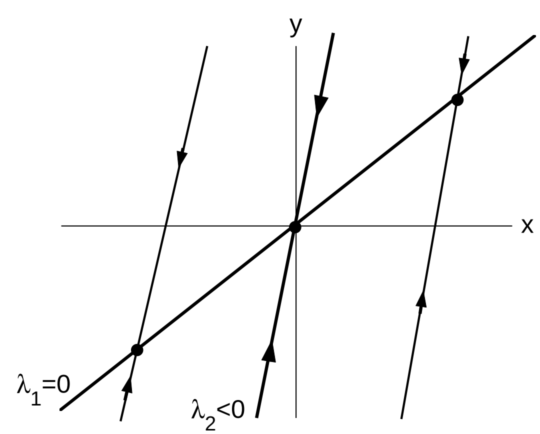
$$\text{borderline case}\begin{cases}\text{nodal source}\\ \text{saddle}\end{cases}$$



Stable Saddle–Node:

$$T < 0$$

$$\text{borderline case}\begin{cases}\text{nodal sink}\\ \text{saddle}\end{cases}$$



9.4: The (T,D)–Plane: $\lambda = \mathbf{T/2} \pm \sqrt{T^2 - 4D}/\mathbf{2}$

Five Generic Cases:

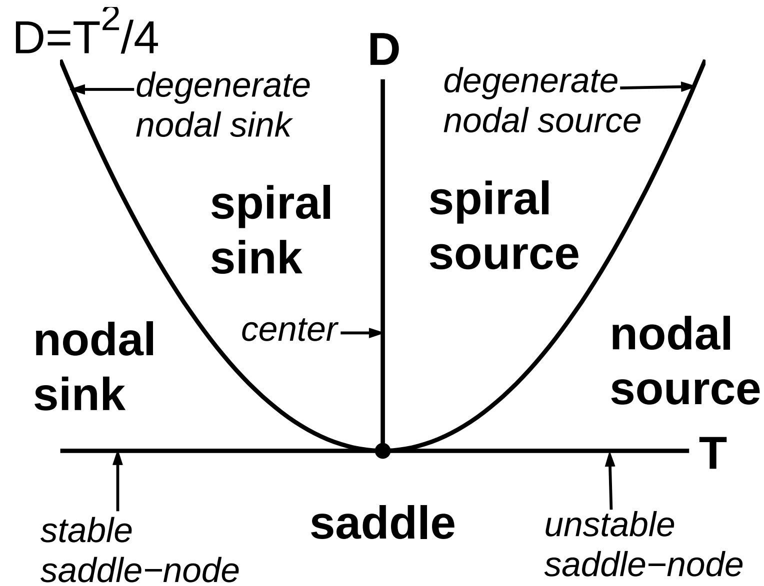
- if $D < 0 \quad \Rightarrow$ saddle
- if $D > 0$ and
  - $T > 0 \quad \Rightarrow$ source
  - $T < 0 \quad \Rightarrow$ sink
  - $T^2 > 4D \Rightarrow$ node
  - $T^2 < 4D \Rightarrow$ spiral

Borderline Cases:

- if $T = 0$ and $D > 0 \Rightarrow$ center
- if $D = 0$, $T \neq 0 \Rightarrow$ saddle-node
  - if $T > 0 \Rightarrow$ unstable
  - if $T < 0 \Rightarrow$ stable
- if $T^2 = 4D$, $A \neq (T/2)I$, and
  - $T > 0 \Rightarrow$ d. nodal source
  - $T < 0 \Rightarrow$ d. nodal sink

Other Special Case: $A = \lambda I$, $\lambda \neq 0$

- only half line solutions from origin
- Name: $\left\{\begin{matrix}\text{unstable}\\ \text{stable}\end{matrix}\right\}$ star if $\left\{\begin{matrix}\lambda > 0\\ \lambda < 0\end{matrix}\right\}$



Ex.: $A = \begin{bmatrix} 8 & 5 \\ -10 & -7 \end{bmatrix} \; \{ \; D = -6 \; \}$

$\Rightarrow$ saddle

Ex.: $A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \; \left\{ \begin{matrix} D = 2,\, T = -3 \\ T^2 - 4D = 1 \end{matrix} \right\}$

$\Rightarrow$ nodal sink

Ex.: $A = \begin{bmatrix} -10 & -25 \\ 5 & 10 \end{bmatrix} \; \left\{ \begin{matrix} D = 25 \\ T = 0 \end{matrix} \right\}$

$\Rightarrow$ center

$c = 5 > 0 \Rightarrow$ counterclockwise direction of rotation

Typical Homework and Exam Problems

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1. Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *classify* the type of phase portrait.

In the case of centers and spirals you may also be asked to determine the direction of rotation.

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2. Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *sketch* the phase portrait.

The sketch should show all special trajectories and a few generic trajectories. At each trajectory the direction of motion should be indicated by an arrow.

- In the case of centers, sketch a few closed trajectories with the right direction of rotation. For spirals, one generic trajectory is sufficient.

- In the case of saddles or nodes, the sketch should include all half line trajectories and a generic trajectory in each of the four regions separated by the half line trajectories. The half line trajectories should be sketched correctly, that is, you have to compute eigenvalues as well as eigenvectors.

- In the case of nodes you should also distinguish between fast (double arrow) and slow (single arrow) motions (see p.2).

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3. Given A, find the general solution (or a solution to an IVP), classify the phase portrait, and sketch the phase portrait.