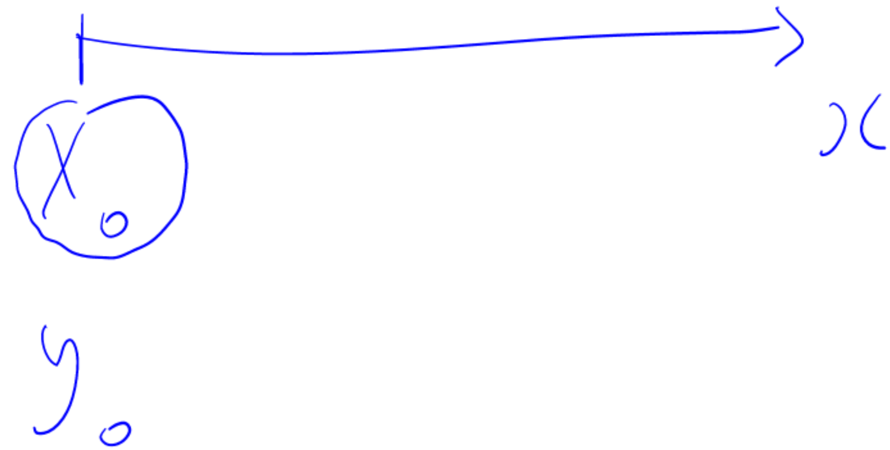
1

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

$$\int_{x_0}^{x} \frac{dy}{h(y)} = \int_{x_0}^{x} g(x)\,dx$$

$f(x,y) = h(y)\,g(x)$



$$\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi} \qquad (*)$$

(1) $$\int_{\mathbb{R}} e^{-ax^2}\,dx \qquad u = \sqrt{a}\,x$$

Using $(*)$ get $\sqrt{\frac{\pi}{a}}$

(2) $$\int_{\mathbb{R}} e^{-ax^2/2}\,dx \qquad u = \sqrt{\frac{a}{2}}\,x$$

$$\longrightarrow \sqrt{\frac{2\pi}{a}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u(x,0) = f(x)$$

$$t \geqslant 0$$

$u(x,t)$: Diff eq^{n} is invariant under

$$t' = \lambda t \; ; \quad x' = \sqrt{\lambda}\, x \qquad u' = \lambda^{-\frac{1}{2}} u$$

$$\xi = \frac{x}{\sqrt{t}} \; : \qquad \phi = u\sqrt{t}$$

$$\phi = \phi(\xi) \quad \underline{\text{OR}} \quad \underline{u\sqrt{t} = \phi\left(\frac{x}{\sqrt{t}}\right)}$$

Start with

$$u(x,t) = \frac{1}{\sqrt{t}}\,\phi\left(\frac{x}{\sqrt{t}}\right)$$

c.f $\quad t^{\alpha} f\left(y/t^{\beta}\right)$

① Invariant Scaling $\qquad \alpha = -\frac{1}{2} \qquad \beta = \frac{1}{2}$

② Similarity sol^{n}

$$\int_{\mathbb{R}} u(x,t)\,dx = \frac{1}{\sqrt{t}} \int \phi\left(\frac{x}{\sqrt{t}}\right) dx$$

$$= \int_{\mathbb{R}} \phi(\xi)\, d\xi$$

So that sol^{n} can be <u>normalised</u>

indep. of time t.

$$\Pi = V - \Delta S$$

$$V = \underbrace{\Pi}_{\substack{\text{cash} \\ \text{bonds}}} + \underbrace{\Delta S}_{\substack{\text{an amount} \\ \text{of asset}}}$$

$$dB = r\,\underline{\underline{B}}\,dt$$

$$\boxed{B_0 e^{rt}}$$

$$\int \frac{dB}{B} = r \int dt$$

$$B(t) = \underbrace{B_0}_{£1} e^{rt}$$

$$\boxed{e^{rt}} \longrightarrow \boxed{e^{r}}$$

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + \mu S \frac{\partial p}{\partial S}$$

$$p = \frac{1}{\sigma S' \sqrt{2\pi(t'-t)}} e^{-\left[\log \frac{S}{S'} + \left(\mu - \frac{1}{2}\sigma^2\right)(t'-t)\right]^2 / 2\sigma^2(t'-t)}$$

(1) Write $p = p(s, t)$ as

$p(\xi, t)$ $\xi = \log S$ $\frac{\partial \xi}{\partial s} = \frac{1}{S}$

$$\frac{\partial p}{\partial s} = \frac{1}{S} \frac{\partial p}{\partial \xi}$$ (chain rule)

$$\frac{\partial^2 \xi}{\partial s^2} = -\frac{1}{S^2}$$

$$\frac{\partial^2 p}{\partial s^2} = \frac{1}{S^2} \left(\frac{\partial^2 p}{\partial \xi^2} - \frac{\partial p}{\partial \xi} \right)$$

chain rule + product rule

$$\frac{\partial p}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial \xi^2} + \left(\mu - \frac{1}{2} \sigma^2 \right) \frac{\partial p}{\partial \xi}$$

2) $x = \xi + kt$ $\leftarrow$ TBD $\qquad \tau = t$

Now use Chain Rule II

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + k\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} \longrightarrow \frac{\partial^2}{\partial \xi^2} = \frac{\partial^2}{\partial x^2}$$

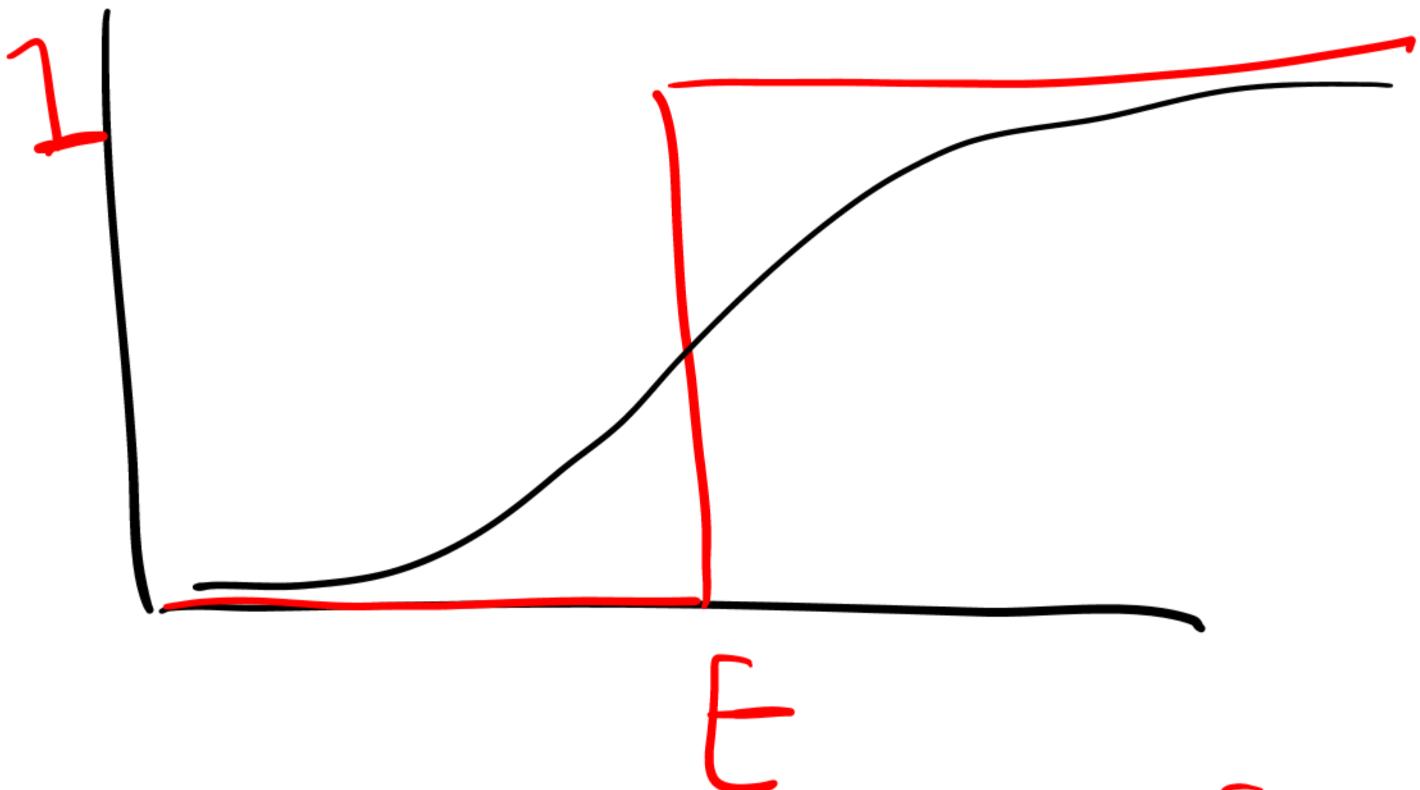
$$\frac{\partial p}{\partial \tau} + \cancel{k\frac{\partial p}{\partial x}} = \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial x^2} + \cancel{\left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial p}{\partial x}}$$

put $k = \mu - \frac{1}{2}\sigma^2$

$$\boxed{\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}}$$

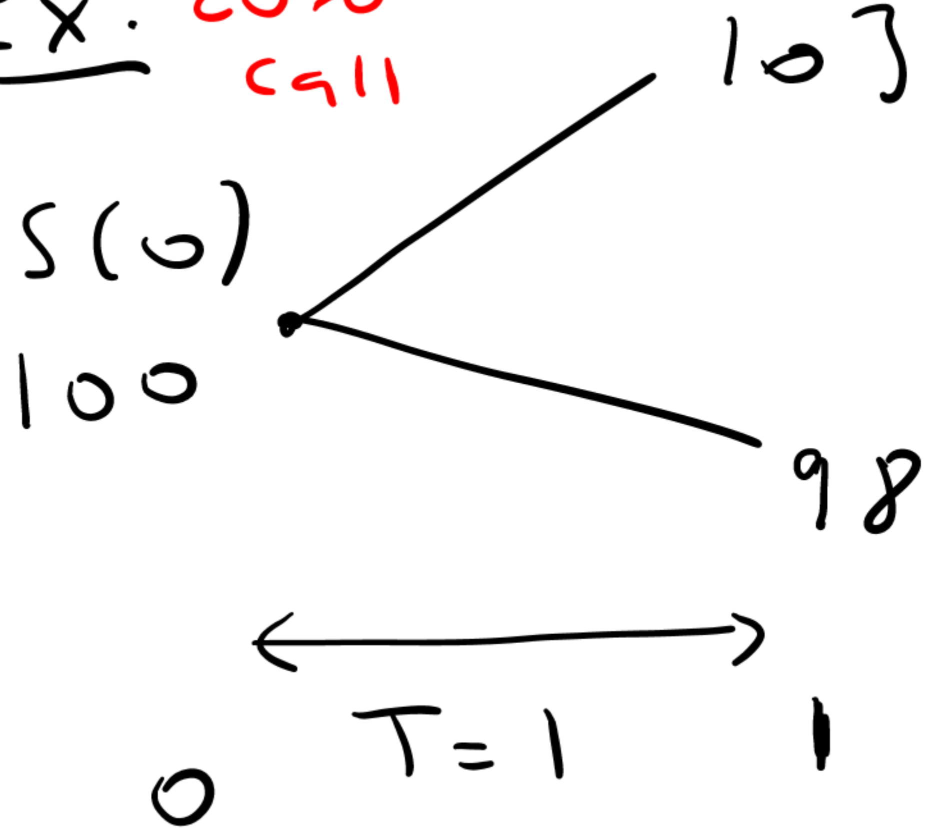
Asset or nothing

Cash or nothing



$$\text{Payoff} = \mathcal{H}(S_T - E) = \begin{cases} 1 & S_T > E \\ 0 & \text{otherwise} \end{cases}$$

Ex: Euro call

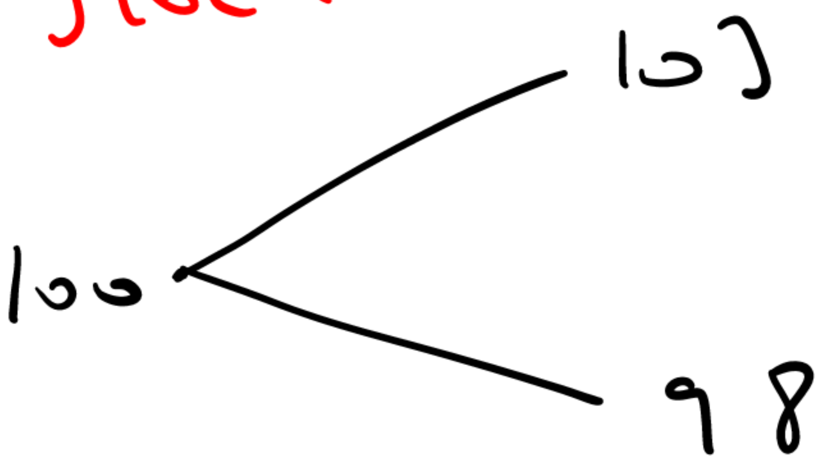


$r = 0$

$E = 100$

ω	$S(0)$	$S(1)$
ω_1	100	103
ω_2	100	98

Stock



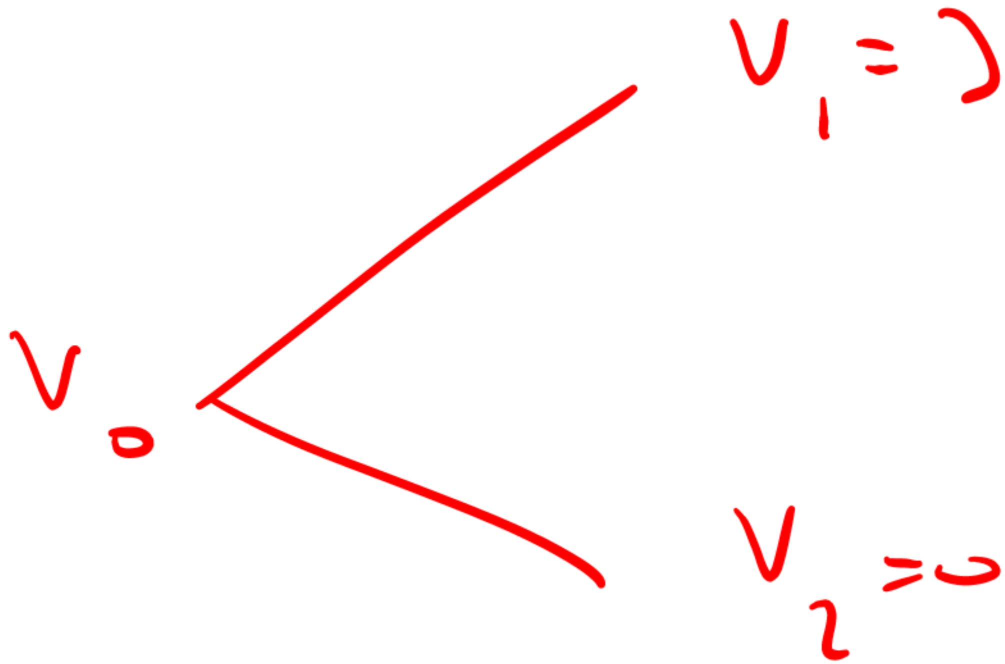
$S(1) > 100$

Payoff = 3

$98 < 100$

Payoff = 0

option



ϕ stocks

ψ bonds

$$\left.\begin{aligned} 103\phi + \psi &= 3 \\ 98\phi + \psi &= 0 \end{aligned}\right\} \quad \begin{aligned} \phi &= \frac{3}{5} \\ \psi &= -\frac{294}{5} \end{aligned}$$

$$V_0 = S(0) \times \phi + \psi \times 1 = \boxed{1.2}$$

$$q(up) = \frac{e^{rt} S - S_d}{S_u - S_d} \qquad q(down) = \frac{S_u - e^{rt} S}{S_u - S_d}$$

$r = 0$

ω_1	100	103
ω_2	100	98

$$\omega_1 = \frac{100 - 98}{103 - 98} = \frac{2}{5}$$

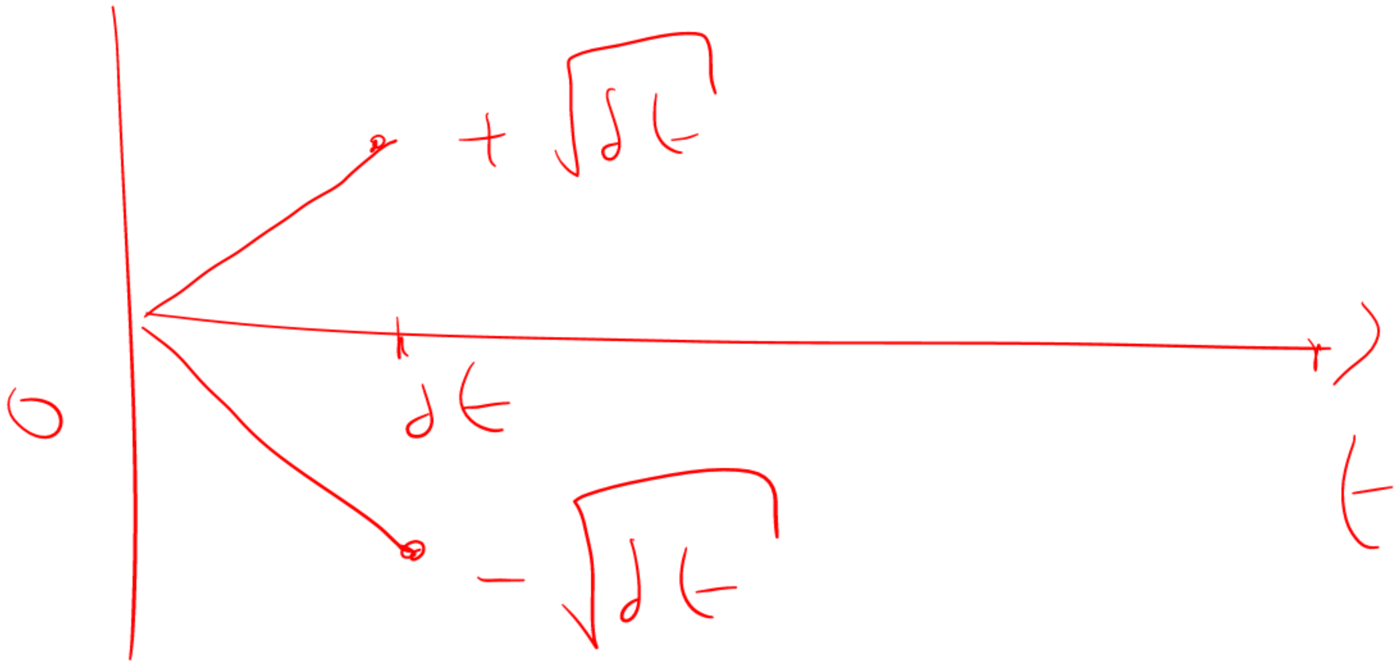
$$\omega_2 = \frac{103 - 100}{103 - 98} = \frac{3}{5}$$

$$\left(\frac{2}{5}, \frac{3}{5}\right) \qquad \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} X\right]$$

$$r=0, \quad t=0, \quad T=1$$

$$\mathbb{E}^{\mathbb{Q}}[X] = \sum_{\omega} q(\omega) X(\omega)$$

$$= \frac{2}{5}\times 3 + \frac{3}{5}\times 0 = \boxed{1.2}$$



$+\sqrt{dt}$
0
dt
t
$-\sqrt{dt}$