Caplet Pricing (with HJM output). Note by Dr Richard Diamond, CQF.

The simple interest rate options are caps and floors. A cap can be treated as a sum of caplets, each **re-setting (expiring) at** T_i and **maturing (paid) at** T_{i+1}. Caplet rate is paid for the time $\tau_i = T_{i+1} - T_i$. The amount of interest accrued is equal to $(T_{i+1} - T_i)L(T_i, T_{i+1}) = \tau_i f_i$.

Interest rate options deliver a cashflow at maturity, ie, interest paid on the notional. The interest payment is unknown until the rate is fixed (re-set). Caplet cashflow is valued as

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_i} \bar{f}(t,\tau)d\tau\right) \times \text{Payoff}_{Cpl}\right]$$

Here, integration over the forward curve (row) at time t gives a discounting factor for the period $[0, T_i]$. Now, for the **forward-starting** caplet, we define the payoff as

$$\text{Payoff}_{Cpl} \quad = \quad \frac{\tau_i}{1+\tau_i f_i} \max\left[f_i - K, 0\right]$$

where $1/(1+\tau_i f_i)$ can be recognised as discount factor for the period $[T_i, T_{i+1}] \equiv \tau_i$.
$f_i \equiv L(T_i, T_{i+1})$ is Forward LIBOR, a simple annualised rate applied to that future period.

Price a caplet simply by

$$\boxed{\text{DF}_{\text{OIS}}(0, T_{i+1}) \times \max\left[L(T_i, T_{i+1}) - K, 0\right] \times \tau}$$

Notice that discounting factor covers both periods, $[0, T_i]$ and $[T_i, T_{i+1}]$. DF can be taken from the OIS spot curve. Proper pricing would require a stochastic DF obtained from the same rolling measure $\mathbb{E}^{\mathbb{Q}}_{i+1}$ as applied to the $f_i = L(T_i, T_{i+1})$. That would require calibrating a separate HJM model for OIS forwards – this is not for implementation.

Once decided on discounting, the question becomes how to obtain that **Forward LIBOR**?

1. FRA formula derived in the previous section gives direct result

$$\begin{aligned} L(T_i, T_{i+1}) \quad &= \quad \boxed{\frac{1}{T_{i+1}-T_i}\left[\exp\left(\int_{T_i}^{T_{i+1}} \bar{f}(T_i,\tau)d\tau\right) - 1\right]} \quad \ddagger \\ &= \quad \frac{1}{T_{i+1}-T_i}\left[\frac{1}{Z(T_i,T_{i+1})} - 1\right] \qquad \text{where} \quad Z(T_i, T_{i+1}) = \frac{Z(0,T_{i+1})}{Z(0,T_i)} \end{aligned}$$

Choice 1. *Yield Curve.xlsm* implements this Forward LIBOR calculation **from the same curve 'today' because no future simulated curves are available.**

Choice 2. Integrate over the simulated curves $\bar{f}(T_i, \tau)$ while decreasing the tenor. Consider cashflows in reverse: the first carries the longest credit risk. For example, $L[0.75, 1]$ is taken from the curve today $T_i = 0$ and reflects the credit risk of 1Y loan; $L[0.5, 0.75]$ from curve $T_i = 0.25$, ..., $L[0, 0.25]$ from $T_i = 0.75$.
The method also gives LIBOR over longer terms, eg, $L[0, 0.75]$ can be obtained by integration over the curve today $T_i = 0$ from 0 to $\tau = 0.75$; $L[0, 0.5]$ from curve $T_i = 0.25$ integrating from 0 to $\tau = 0.5$. This choice combines the logic of taking credit risk as seen today with the idea that over time we work with shorter sections of the curve.

2. Alternatively, we can start with a yield on a future-starting AA-rated bond

$$L(T_i, T_{i+1}) = -\frac{\log Z(T_i, T_{i+1})}{T_{i+1} - T_i}$$

Log discount factors means being linear in rates, so operating on a forward curve

$$L(T_i, T_{i+1}) = \frac{1}{T_{i+1} - T_i} \int_{T_i}^{T_{i+1}} \bar{f}(T_i, \tau) d\tau \quad \Rightarrow \quad \boxed{\frac{1}{n} \sum_{j=1}^{n} \bar{f}_j} \quad \dagger$$

where n is a number of points between tenors $[T_i, T_{i+1}]$ and integration is over the curve(s).

Proper LIBOR is a simple annualised rate, which for short periods can be converted using

$$L' = m \left(e^{\frac{L}{m}} - 1 \right) \quad \dagger$$

where m is compounding frequency per year. *Examples:* 3M LIBOR compounded 4 times a year, 6M LIBOR - 2 times. Calculate † and convert to L'.

Choice 3. The notation $\bar{f}(T_i, \tau)$ implies picking each Forward LIBOR from the future curve as T_i increases. For example, $L(T_i, T_{i+1}) = L(4, 4.25)$ is picked from the curve simulated at time $T_i = 4$ (row 400 with $dt = 0.01$) and the column for tenor $\tau = 4$. Why? Even 3M rate for $[4, 4.25]$ has to carry the credit risk of a longer-term loan.

Using the forward curve simulated for $t = 4$ is awkward. The notation $\bar{f}(t = T_i, \tau)$ is best suited to **the constant maturity case: when we are not exposed to the long-term credit risk but only to 3M or 6M LIBOR paid over and over**. For example, for an IRS (swap) we can use the same column of HJM output to pick Forward LIBOR in $t = 0.5$ (row 50), in $t = 1$ (row 100) and so on.

Market risk factors that affect caplet price are the inputs into pricing formula. The terminology comes from the formal risk management, which regards any input as 'a risk factor'.

1. The standard risk factors are strike, maturity and 'bucket risks', ie, bumping rates at particular tenors and re-pricing. In a sense, HJM does that by introducing shocks $\sqrt{\lambda}\phi\delta t$.

2. Semi-annual vs. quarterly expiry. BOE forward curve is already smoothed for 6M increments – re-interpolation not advisable.

3. Choice of discounting factor: from the model vs. OIS vs. simulated OIS forwards to match the forward risk measure.

Caplet-floorlet parity is given by

$$\mathbf{Cpl}(0; T_i, T_{i+1}, K) - \mathbf{Flr}(0; T_i, T_{i+1}, K) = Z(0, T_{i+1}) \times (L(T_i, T_{i+1}) - K) \times \tau$$

where $t = 0$ means that we are pricing from today's curve but the caplets (floorlets) are forward-starting at time T_i. Discount factor $Z(0, T_{i+1})$ can be substituted with one taken from OIS spot curve $\mathrm{DF}_{OIS}(0, T_{i+1})$ (or simulated OIS forwards to match the expectation).

Black Formula

The formula converts the discounted cashflow of caplet into implied volatility quotation and applies. Market volatility for 3M caplets $\mathbf{Cpl}^{LMM}$ requires bootstrapping from traded caps of longer maturity. As usual, $t = 0$ means that we are pricing from today's curve but the caplets (floorlets) are forward starting at time T_i.

$$\begin{aligned}\mathbf{Cpl}^{LMM}(0;T_i,T_{i+1},K) &= \mathbf{Cpl}^{Bl}(0;T_i,T_{i+1},K,\zeta_i)\\ &= Z(0,T_i)\times Bl(K,L(T_i,T_{i+1}),\zeta_i)\end{aligned}$$

- Complete notation $L(t;T_i,T_{i+1})$ is equivalent to $L_i(t)$ and $F_i(0)$ means Forward LIBOR fixed from today's curve $t = 0$. $F(T_i)$ notation also used, referring to the same quantity.
- Discount factor $Z(0,T_i)$ is separated from $Z(T_i,T_{i+1})$ (which is technically under the risk-neutral expectation).

$$\begin{aligned}Bl(K,L(T_i,T_{i+1}),\zeta_i) &= \left[L(T_i,T_{i+1})\,\Phi(d_1) - K\Phi(d_2)\right]\frac{\tau}{1+\tau L(T_i,T_{i+1})}\\ &= \left[f_i\,\Phi(d_1) - K\Phi(d_2)\right]\frac{\tau}{1+\tau f_i}\end{aligned}$$

$$\text{where}\quad d_{1,2} = \frac{\ln\frac{f_i}{K} \pm \frac{\zeta^2}{2}}{\zeta}$$

in these formulae $\zeta^2 = \zeta_i^2 T_i$ where ζ_i is an implied volatility of the caplet that expires at T_i and pays off at T_{i+1}. Once the rate is fixed at T_i, there is no more volatility!

The implied variance of forward rate f_i (also referred to as FRA rate) is an integrated instantaneous variance $\zeta_i^2 = \dfrac{1}{T_i}\displaystyle\int_0^{T_i} \sigma^2(s)ds$.

LMM SDE

Discretised LMM SDE to implement looks confusing. **Notation t_{j-k-1} just means that we refer to the previous time step $k-1$.**

We can express the SDE in simpler terms by changing $f(t_{k+1}) \to f(t+1) \to f(t+\delta t)$ and $f(t_k) \to f(t) \to f$. There is duplicate index i in $\sigma_i(t_{i-k-1})$, while $i-k$ relates to how we bootstrapped the instantaneous volatility of f_i from caplets, given that after time t_i the forward rate is fixed and there is no volatility. Lets write $\sigma_i(t_{i-k-1}) \to \sigma_{i,t-1} \to \sigma_i$ (can also say, we take sigmas 'from the previous row'). LMM SDE must look simpler,

$$f_i(t+\delta t) = f_i \exp\left[\left(\sigma_i \sum_{j=0}^{i} \frac{\tau_j f_j\,\sigma_j \rho_{ij}}{1+\tau_j f_j} - \frac{1}{2}\sigma_i^2\right)\delta t\right] + \sigma_i \phi_i \sqrt{\delta t} \tag{1}$$

Note that this is a single-factor SDE. With the discrete market model dt can be large.