CQF Lecture 3.3 Understanding Volatility

Exercises

1. Explain what actual and implied volatilities are, and what is their relationship? Name three assumptions made in estimation of actual volatility from the market option prices.

2. The market price for a European put with strike 100, expiration one year, interest rate is 5% p.a. is quoted at \$5.57 for stock value at \$100. How do you find its implied volatility?

3. Assume **a time-dependent** volatility function $\sigma(t)$. Consistent with Black-Scholes framework, the implied volatility $\sigma_i(t,T)$ measured at time t of an European option expiring at time T must satisfy

$$\sigma_i(t,T) = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(s)\, ds}$$

   Solve the inverse problem (an integral equation) to show that, at calibration time t^*, the volatility function $\sigma(t)$ must be consistent the implied volatility σ_i as follows:

$$\sigma^2(t) = 2(t-t^*)\,\sigma_i(t^*,t)\,\frac{\partial \sigma_i(t^*,t)}{\partial t} + \sigma_i^2(t^*,t)$$

4. Suppose implied volatilities are observable at $T_i, i = 0,1,2,\cdots,n$, with $T_0 = t^*$ is the date of calibration (fitting). Assuming that the actual volatility function is **piecewise constant**, show that for $T_{i-1} < t < T_i$ the total variance is (this is discretised Q3)

$$\sigma^2(t) = \frac{(T_i - t^*)\,\sigma_i^2(t^*,T_i) - (T_{i-1}-t^*)\,\sigma_i^2\,(t^*,T_{i-1})}{T_i - T_{i-1}}$$

5. Denote the actual volatility by σ_a and implied volatility by σ_i, where subscript 'a' means actual and 'i' means implied. Similarly, Δ_a means Δ is calculated using actual volatility in $N(d_1)$, and Δ_i means Δ is calculated by using implied volatility. Assume that an asset follows the GBM with continuous dividend rate D, and an option written on this asset is denoted by $V(S,t;\sigma)$.

   Within the Black-Scholes framework, what is the Mark-to-Market profit if one hedges the option by using actual volatility to calculate Delta. How is this profit going to be realised (guaranteed or not)?

   What about the Mark-to-Market profit if hedging with the implied volatility?

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