CQF Module 1.2 Exercises

1. Find the general solution of the differential equation

$$x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}.$$

$$\begin{aligned} y' &= \frac{y}{x} + \frac{1}{x}\sqrt{(x^2+y^2)} = \frac{ty}{tx} + \frac{1}{tx}\sqrt{(t^2x^2+t^2y^2)} \\ &= t^0\left[\frac{y}{x} + \frac{1}{x}\sqrt{(x^2+y^2)}\right] \end{aligned}$$

$\therefore$ homogeneous of degree 0. Put

$$y = vx$$

$\Rightarrow$

$$y' = v + v'x$$

Now

$$\begin{aligned} y' &= \frac{y}{x} + \frac{1}{x}\sqrt{(x^2+y^2)} \\ v + v'x &= \frac{vx}{x} + \frac{1}{x}\sqrt{(x^2+x^2v^2)} \\ v + v'x &= v + \frac{x}{x}\sqrt{(1+v^2)} \\ v + v'x &= v + \sqrt{(1+v^2)} \text{ now separable} \\ v'x &= \sqrt{(1+v^2)} \\ x\frac{dv}{dx} &= \sqrt{(1+v^2)} \\ \int \frac{dv}{\sqrt{(1+v^2)}} &= \int \frac{dx}{x} \end{aligned}$$

Now

$$\int \frac{du}{\sqrt{(u^2 \pm a^2)}} = \ln\left|u + \sqrt{(u^2 \pm a^2)}\right|$$

$\therefore$

$$\int \frac{dv}{\sqrt{(1+v^2)}} = \ln\left|v + \sqrt{(v^2+1)}\right| + c, \quad c \text{ is an arbitrary constant}$$

$\Rightarrow$

$$\begin{aligned} \ln\left|v + \sqrt{(v^2+1)}\right| &= \ln x + c \\ \ln\left|v + \left(v^2+1\right)^{\frac{1}{2}}\right| - \ln x &= c \\ \ln\left|\frac{1}{x}[v + \left(v^2+1\right)^{\frac{1}{2}}]\right| &= c \\ \frac{1}{x}\left[v + \sqrt{(v^2+1)}\right] &= e^c \end{aligned}$$

But $e^c = k$, so

$$\frac{1}{x}\left[v + \sqrt{(v^2+1)}\right] = k$$

Put

$$v = \frac{y}{x}$$

$$\begin{aligned}
\frac{1}{x}\left[\frac{y}{x} + \sqrt{\left(\frac{y^2}{x^2} + 1\right)}\right] &= k \\
\frac{1}{x}\left[\frac{y}{x} + \frac{1}{x}\sqrt{(y^2 + x^2)}\right] &= k \\
\frac{1}{x^2}\left[y + \sqrt{(y^2 + x^2)}\right] &= k
\end{aligned}$$

which is the general solution to the problem. k is a constant.

2. By solving the initial value problem

$$\frac{dy}{dx} - 2xy = 2, \; y(0) = 1$$

show that the solution can be written as

$$y(x) = e^{x^2}\left(1 + 2\int_0^x e^{-t^2} dt\right).$$

This is clearly a linear equation. The integrating factor is $R(x) = \exp(-x^2)$ which multiplying through gives

$$\begin{aligned}
e^{-x^2}\left(\frac{dy}{dx} - 2xy\right) &= 2e^{-x^2} \\
\frac{d}{dx}\left(e^{-x^2} y\right) &= 2e^{-x^2} \\
\int_0^x d\left(e^{-t^2} y\right) &= 2\int_0^x e^{-t^2} dt
\end{aligned}$$

Concentrate on the lhs and noting the IC $y(0) = 1$

$$e^{-t^2} y\Big|_0^x = e^{-x^2} y(x) - y(0) = e^{-x^2} y(x) - 1$$

hence

$$\begin{aligned}
e^{-x^2} y(x) - 1 &= 2\int_0^x e^{-t^2} dt \\
y(x) &= e^{x^2}\left(1 + 2\int_0^x e^{-t^2} dt\right)
\end{aligned}$$

3. The integral on the right hand side of the last solution cannot be simplified any further if we wish this to remain as a closed form solution. Note the following very important non-elementary integrals, namely the *error function* and *complimentary error function* in turn,

$$\boxed{\operatorname{erf}(x) = \tfrac{2}{\sqrt{\pi}}\int_0^x e^{-s^2} ds}$$

$$\boxed{\operatorname{erf} c(x) = \tfrac{2}{\sqrt{\pi}}\int_x^\infty e^{-s^2} ds}$$

Write the solution of the last problem in terms of $\text{erf}\,(x)$. Verify that

$$\text{erf}\,(x) + \text{erf}\,c\,(x) = 1.$$

The solution to the IVP can now be written

$$y\,(x) = e^{x^2}\left(1 + \sqrt{\pi}\,\text{erf}\,(x)\right).$$

$$\begin{aligned}
\text{erf}\,(x) + \text{erf}\,c\,(x) &= \frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-s^2}ds + \int_x^\infty e^{-s^2}ds\right) \\
&= \frac{2}{\sqrt{\pi}}\left(\int_0^\infty e^{-s^2}ds\right) \\
&= \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1.
\end{aligned}$$

4. Using a binomial (2 step symmetric) random walk where the probability of an up move or down move is $\frac{1}{2}$, derive both the forward and backward Kolmogorov equations in turn, given by

$$\begin{aligned}
\frac{\partial p}{\partial t'} &= c^2\frac{\partial^2 p}{\partial y'^2} \\
\frac{\partial p}{\partial t} + c^2\frac{\partial^2 p}{\partial y^2} &= 0
\end{aligned}$$

for the transition density function $p(y,t\ ;y',t')$. The states (y,t) are past /current while (y',t') refer to future ones.
By simple substitution show that

$$\frac{1}{2c\sqrt{\pi(t'-t)}}\exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right)$$

<u>satisfies</u> the backward Kolmogorov equation.
Staring with the forward equation we ask the usual question - suppose we are at (y',t') in the future (starting at (y,t)), how did we get there? At the previous time step we must have been at one of $(y'+\delta y, t'-\delta t)$ or $(y'-\delta y, t'-\delta t)$.
So

$$p\,(y',t') = \tfrac{1}{2}p\,(y'+\delta y, t'-\delta t) + \tfrac{1}{2}p\,(y'-\delta y, t'-\delta t)$$

Taylor series expansion gives

$$\begin{aligned}
p\,(y'+\delta y, t'-\delta t) &= p\,(y',t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + ... \\
p\,(y'-\delta y, t'-\delta t) &= p\,(y',t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + ...
\end{aligned}$$

Substituting into the above

$$\begin{aligned}
p\,(y',t') &= \tfrac{1}{2}\left(p\,(y',t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right) \\
&\quad + \tfrac{1}{2}\left(p\,(y',t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right)
\end{aligned}$$

$$0 = -\frac{\partial p}{\partial t'}\delta t + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2$$

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y'^2}$$

Now take limits. This only makes sense if $\frac{1}{2}\frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2 \sim O(\delta t)$ and letting $\delta y,\ \delta t \longrightarrow 0$ gives the equation

$$\frac{\partial p}{\partial t'} = c^2\frac{\partial^2 p}{\partial y'^2}$$

The **backward equation** tells us the probability that we are at (y', t') in the future given that we were at (y', t') earlier. So (y', t') are now fixed and (y, t) are variables. So the probability of being at (y', t') given we are at y at t is linked to the probabilities of being at $(y + \delta y, t + \delta t)$ and $(y - \delta y, t + \delta t)$.

$$p(y, t; y', t') = \tfrac{1}{2}p(y + \delta y, t + \delta t; y', t') + \tfrac{1}{2}p(y - \delta y, t + \delta t; y', t')$$

Since (y', t') do not change, drop these for the time being and use a TSE on the right hand side $p(y, t) =$

$$\tfrac{1}{2}\left(p(y, t) + \frac{\partial p}{\partial t}\delta t + \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + \ldots\right) +$$
$$\tfrac{1}{2}\left(p(y, t) + \frac{\partial p}{\partial t}\delta t - \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + \ldots\right)$$

which simplifies to

$$0 = \frac{\partial p}{\partial t} + \tfrac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y^2}.$$

Putting $\frac{1}{2}\frac{\delta y^2}{\delta t} = O(1)$ and taking limit gives the **backward equation**

$$-\frac{\partial p}{\partial t} = c^2\frac{\partial^2 p}{\partial y^2}.$$

or commonly written as $\dfrac{\partial p}{\partial t} + c^2\dfrac{\partial^2 p}{\partial y^2} = 0$

To verify that

$$p = \frac{1}{2c\sqrt{\pi(t'-t)}}\exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right) \tag{1}$$

is a solution of this equation, obtain the terms $\frac{\partial p}{\partial t}$ and $\frac{\partial^2 p}{\partial y^2}$ from (1); y' and t' are treated like constants. Substitute the derivatives in to the backward equation.