CQF Module 2 Exercise Solution

Ran Zhao

A. Optimal Portfolio Allocations

1 To solve for the weight in global minimum variance portfolio, we formulate

$$\begin{aligned} \underset{\omega}{\operatorname{argmin}} \quad & \frac{1}{2}\omega'\Sigma\omega \\ \text{subject to} \quad & \omega'\mathbf{1} = 1 \end{aligned}$$

The Lagrangian multiplier of this global minimum variance portfolio is

$$L(\omega, \lambda) = \frac{1}{2}\omega'\Sigma\omega + \lambda(\omega'\mathbf{1} - 1) = 0$$

Set the FOCs to zero yields the optimal solution of the weight:

$$\frac{\partial L}{\partial \omega} = \Sigma\omega + \lambda\mathbf{1} = 0 \tag{1}$$

$$\frac{\partial L}{\partial \lambda} = \omega'\mathbf{1} - 1 = 0 \tag{2}$$

From (5), the optimal weight solution has

$$\omega^* = -\Sigma^{-1}\lambda\mathbf{1} \tag{3}$$

Bring this into (6), we have

$$\omega^{*'}\mathbf{1} = -\lambda\mathbf{1}'\Sigma^{-1}\mathbf{1} = 1 \quad \Rightarrow \quad \lambda^* = -\frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \tag{4}$$

Combine (4) with (3), the analytical solution for optimal allocations ω^* is

$$\omega^* = \frac{\Sigma^{-1}\mathbf{1}'}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}$$

2.a To solve for the minimum variance portfolio under the target return with risk-free asset, we formulate

$$\begin{aligned} \underset{\omega}{\operatorname{argmin}} \quad & \frac{1}{2}\omega'\Sigma\omega \\ \text{subject to} \quad & r + (\mu - r\mathbf{1})\omega' = 0.1 \end{aligned}$$

The Lagrangian multiplier of this global minimum variance portfolio is

$$L(\omega, \lambda) = \frac{1}{2}\omega'\Sigma\omega + \lambda[r + (\mu - r\mathbf{1})\omega' - 0.1] = 0$$

Set the FOCs to zero yields the optimal solution of the weight:

$$\begin{aligned} \frac{\partial L}{\partial \omega} &= \Sigma\omega + \lambda(\mu - r\mathbf{1}) = 0 && (5) \\ \frac{\partial L}{\partial \lambda} &= r + (\mu - r\mathbf{1})\omega' - 0.1 = 0 && (6) \end{aligned}$$

From (1), the optimal weight solution has

$$\omega^* = -\lambda\Sigma^{-1}(\mu - r\mathbf{1}) \tag{7}$$

Bring this into (2), we have

$$(\mu - r\mathbf{1})'\omega^* = -\lambda(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1}) = 0.1 - r \tag{8}$$

which yields

$$\lambda^* = -\frac{0.1 - r}{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})} \tag{9}$$

Combine (7) with (9), the analytical solution for optimal allocations ω^* is

$$\omega^* = \frac{(0.1 - r)\Sigma^{-1}(\mu - r\mathbf{1})'}{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})}$$

2.b First construct the variance-covariance matrix Σ from the correlation matrix. That is,

$$\Sigma = SRS = \begin{pmatrix} 0.0049 & 0.00168 & 0.0063 & 0.00546 \\ 0.00168 & 0.0144 & 0.01512 & 0.01248 \\ 0.0063 & 0.01512 & 0.0324 & 0.04212 \\ 0.00546 & 0.01248 & 0.04212 & 0.0676 \end{pmatrix}$$

Then calculate the optimal weight for the minimum variance portfolio

$$\omega^* = \frac{(0.1 - r)\Sigma^{-1}(\mu - r\mathbf{1})'}{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})} = \begin{pmatrix} 0.3957 & 1.0541 & -0.8268 & 0.7313 \end{pmatrix}'$$

Finally, the standard deviation of the portfolio is

$$\sigma_\Pi = \sqrt{\omega^{*\prime}\Sigma\omega^*} = 0.1321$$

Detailed numerical calculation results could be found in the Appendix Matlab code.

2.c The tangency portfolio is the portfolio that is entirely invested in risky assets and is on the capital market line. The portfolio return μ_T and weight ω_T are shown below (detailed derivation is presented in Page 79-83 in M2S2 slices). Let

$$\left\{\begin{array}{lll} A & = & \mathbf{1}'\Sigma^{-1}\mathbf{1} \\ B & = & \mu'\Sigma^{-1}\mathbf{1} \\ C & = & \mu'\Sigma^{-1}\mu \end{array}\right.$$

and we have

$$\begin{aligned} \mu_t &= \frac{C-Br}{B-Ar} = \frac{\mu'\Sigma^{-1}\mu-\mu'\Sigma^{-1}\mathbf{1}r}{\mu'\Sigma^{-1}\mathbf{1}-\mathbf{1}'\Sigma^{-1}\mathbf{1}r} = 8.17\% \\ \omega_T &= \frac{\Sigma^{-1}(\mu-r\mathbf{1})}{B-Ar} = \frac{\Sigma^{-1}(\mu-r\mathbf{1})}{\mu'\Sigma^{-1}\mathbf{1}-\mathbf{1}'\Sigma^{-1}\mathbf{1}r} = \left(\begin{array}{cccc} 0.2922 & 0.7783 & -0.6105 & 0.5400 \end{array}\right)' \end{aligned}$$

3.a
3.b

B. Value at Risk on FTSE 100

1.a
1.b
1.c
2.a
2.b
2.c

C. Stochastic Calculus

1
2
3
4.a Starting with the lower triangular matrix A, we have

$$\Sigma = AA' = \left(\begin{array}{cc} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{array}\right)\left(\begin{array}{cc} \sigma_1 & \rho\sigma_2 \\ 0 & \sqrt{1-\rho^2}\sigma_2 \end{array}\right) = \left(\begin{array}{cc} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{array}\right)$$

which yields the original covariance matrix.

4.b Given $Y = AX$,

$$\left(\begin{array}{c} Y_1 \\ Y_2 \end{array}\right) = \left(\begin{array}{cc} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{array}\right)\left(\begin{array}{c} X_1 \\ X_2 \end{array}\right) = \left(\begin{array}{c} \sigma_1 X_1 \\ \rho\sigma_2 X_1 + \sqrt{1-\rho^2}\sigma_2 X_2 \end{array}\right)$$

4.c

Matlab code