CQF Lecture 4.3 Calibration and Data Analysis
Solutions Addendum

Exercises for Calibration and Data Analysis (for the yield curve) lecture:

1. In Exercise 1, the function $A(t;T)$ is given as the result of Chapter 31.2 PWOQF Vol. 2

   The function is used in the bond price solution $Z(r,t;T) = e^{A(t;T)-r(T-t)}$ in the Ho & Lee model, that also gives $\log Z_M(t^*;T) = A(t^*;T) - r^*(T-t^*)$. The fitted function is

$$A^*(t;T) = -\int_t^T \eta^*(s)(T-s)ds + \frac{1}{6}c^2(T-t)^3$$

   where the choice of calibrated parameter

$$\eta^*(t) = c^2(t-t^*) - \frac{\partial^2}{\partial t^2}\log Z_M(t^*;t)$$

   gives

$$A^*(t;T) = -\int_t^T \left(c^2(s-t^*) - \frac{\partial^2}{\partial s^2}\log Z_M(t^*;s)\right)(T-s)ds + \frac{1}{6}c^2(T-t)^3$$

   where

$$\begin{aligned}
\int_t^T \frac{\partial^2}{\partial s^2}\log Z_M(t^*;s)(T-s)ds &= \int_t^T (T-s)\, d\left(\frac{\partial}{\partial s}\log Z_M(t^*;s)\right) \\
&= (T-s)\frac{\partial}{\partial s}\log Z_M(t^*;s)\Big|_{s=t}^{T} + \int_t^T \frac{\partial}{\partial s}\log Z_M(t^*;s)ds \\
&= -(T-t)\frac{\partial}{\partial t}\log Z_M(t^*;t) + \log Z_M(t^*;T) - \log Z_M(t^*;t) \\
&= \log\frac{Z_M(t^*;T)}{Z_M(t^*;t)} - (T-t)\frac{\partial}{\partial t}\log Z_M(t^*;t)
\end{aligned}$$

   and

$$\begin{aligned}
\int_t^T c^2(s-t^*)(T-s)ds + \frac{1}{6}c^2(T-t)^3 &= \frac{1}{6}c^2\, s\left(6Tt^* - 3s(T+t^*) + 2s^2\right)\Big|_{s=t}^{T} + \frac{1}{6}c^2(T-t)^3 \\
&= \frac{1}{6}c^2(T-t)^2(3t^* - 3t + t - T) + \frac{1}{6}c^2(T-t)^3 \\
&= -\frac{1}{2}c^2(t-t^*)(T-t)^2
\end{aligned}$$

The exercises have been edited by CQF Faculty, Richard Diamond, R.Diamond@fitchlearning.com

Combining two results,

$$A^*(t;T) = \log \frac{Z_M(t^*;T)}{Z_M(t^*;t)} - (T-t)\frac{\partial}{\partial t}\log Z_M(t^*;t) - \frac{1}{2}c^2(t-t^*)(T-t)^2$$

An alternative method to derive $A(t^*;T)$ is to solve ODEs obtained by substitution of partial derivatives of $Z(r,t;T)$ into the bond pricing equation. To see examples of that, please use the solutions for Stochastic Interest Rates Modeling Lecture.

2. In Exercise 2, in order to carry the partial differentiation over the integral expression, we use the Leibniz rule

$$\begin{aligned}
\frac{\partial}{\partial T}\left(\int_{t^*}^{T} \eta^*(s)B(s;T)ds\right) &= \int_{t^*}^{T} \eta^*(s)\frac{\partial}{\partial T}B(s;T) + \eta^*(T)B(T;T)\frac{\partial}{\partial T}T - \eta^*(t^*)B(t^*;T)\frac{\partial}{\partial T}t^* \\
&= \text{where } \frac{\partial}{\partial T}T = 1 \text{ and } \frac{\partial}{\partial T}t^* = 0 \\
&= \int_{t^*}^{T} \eta^*(s)\frac{\partial}{\partial T}B(s;T) + \eta^*(T)B(T;T)
\end{aligned}$$

Solution steps for $A(t;T)$ in the Hull & White (extension to Vasicek) are omitted for the purpose of the task being used in exams.

3. Exercise 3 is intended as Excel work using one's own data. Follow steps to estimate β as done in the Calibration lecture. The exercise has no ready-made solution.

4. In Chapter 31 Yield Curve Fitting of PWOQF Volume 2 on page 528 (page 377 of the paperback single-volume PWIQF), the bond pricing equation (BPE) was typeset as follows:

$$\begin{gathered}
-a - 2b(T-t) - 3c(T-t)^2 + \frac{1}{2}\left(w^2 - 2(T-t)w\frac{\partial w}{\partial t}\right)\left((T-t)\frac{\partial^2 a}{\partial r^2} + (T-t)^2\frac{\partial^2 b}{\partial r^2}\right) \\
+\left((u-\lambda w) - (T-t)\frac{\partial(u-\lambda w)}{\partial t}\right)(T-t)\left(\frac{\partial a}{\partial r} + (T-t)^2\frac{\partial b}{\partial r}\right) \\
-r\left(1 + a(T-t) + c(T-t)^2\right) + \ldots = 0.
\end{gathered}$$

The equation was obtained using the following expression for bond price:

$$Z(r,t;T) \approx 1 + a(r)(T-t) + b(r)(T-t)^2 + c(r)(T-t)^3 + \ldots$$

Analyse the structure of the BPE expression and identify errors. How many times Taylor series expansion was used? **Hint:** There are two main errors.

$\boxed{\textbf{Solution:}}$

Bond pricing equation for the general model is given as Equation (30.4) on p.512, with the substitution of $V \equiv Z$

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0$$

(a) The diffusion w^2 and risk-neutral drift $(u - \lambda w)$ are expanded at r and T using Taylor series, given that $w(r,t)$, $u(r,t)$ and $\lambda(r,t)$ are all functions of (r,t). These two expansions are easy to recognise in the BPE expression.

(b) The partial derivatives of bond value Z *w.r.t.* t and r can be derived from the third Taylor series expansion for $Z(r,t;T)$ as follows:

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= -a - 2b(T-t) - 3c(T-t)^2 \\
\frac{\partial Z}{\partial r} &= (T-t)\frac{\partial a}{\partial r} + (T-t)^2\frac{\partial b}{\partial r} \\
\frac{\partial^2 Z}{\partial r^2} &= (T-t)\frac{\partial^2 a}{\partial r^2} + (T-t)^2\frac{\partial^2 b}{\partial r^2}
\end{aligned}$$

(c) An easy to spot error is in substitution for $-rZ$ which should read $-r\left(1 + a(T-t) + \mathbf{b}(T-t)^2\right)$.

After substituting the expansions for w^2 and $(u - \lambda w)$ and partial derivatives of bond value Z (marked with underbraces) into the general bond pricing equation, we obtain the correct expression with **a bracket fixed** for the substitution of $\frac{\partial Z}{\partial r}$, in line two.

$$\begin{gathered}
\underbrace{-a - 2b(T-t) - 3c(T-t)^2} + \frac{1}{2}\left(w^2 - 2(T-t)w\frac{\partial w}{\partial t}\right)\underbrace{\left((T-t)\frac{\partial^2 a}{\partial r^2} + (T-t)^2\frac{\partial^2 b}{\partial r^2}\right)} \\
+ \left((u - \lambda w) - (T-t)\frac{\partial(u-\lambda w)}{\partial t}\right)\underbrace{\left((T-t)\frac{\partial a}{\partial r} + (T-t)^2\frac{\partial b}{\partial r}\right)} \\
-r\underbrace{\left(1 + a(T-t) + b(T-t)^2\right)} + \ldots = 0
\end{gathered}$$

Compare with the BPE Equation (30.4)

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0.$$

Note that because of reliance on Taylor series expansion, a solution for the expression is only valid for short times to expiry–that is, for small $T - t$. The benefit of the solution is that it allows to relate the risk-neutral short term rate to the slope and curvature of the yield curve. See PWOQF Volume 2, page 529 for details.