CQF Exercises The Black Scholes Model

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$\begin{aligned} d_1 &= \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(S/E) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\ N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-\phi^{\,2}/2)d\phi; \quad N'(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2) \end{aligned}$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$

the expiry date, $r \geq 0$ the interest rate, D is the dividend yield and σ is the volatility of S.

1. The Black–Scholes formula for a European call option $C(S,t)$ is given by

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

By differentiating with respect to S and σ show that the delta and vega are given by

$$\Delta = \exp(-D(T-t))N(d_1), \quad \text{and} \quad \upsilon = \sqrt{\frac{T-t}{2\pi}}S\exp(-D(T-t))\exp(-{d_1}^2/2).$$

You may find the following relationship useful:

$$Se^{(-D(T-t))}\exp\left(-\frac{{d_1}^2}{2}\right) = Ee^{(-r(T-t))}\exp\left(-\frac{{d_2}^2}{2}\right)$$

2. The Black–Scholes formula for a European call option $C(S,t)$ is

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2)$$

From this expression, find the Black–Scholes value of the call option in the following limits:

(a) (time tends to expiry) $t \to T^{\,-}$, $\sigma > 0$ (*this depends on* S/E);

(b) (volatility tends to zero) $\sigma \to 0^+$, $t < T$; (*this depends on* $S\exp(-D(T-t))/E\exp(-r(T-t))$)

(c) (volatility tends to infinity) $\sigma \to \infty$, $t < T$;

3. Consider an option which pays a continuous cash-flow to the holder at a rate proportional to the square of the underlying asset's price, so that during a time interval dt the holder receives $S^2 dt$. Suppose that at expiry the value of the option is

$$V(S,T) = S^2.$$

The underlying evolution follows geometric Brownian motion

$$dS = \mu S dt + \sigma S dX.$$

Derive the Black-Scholes partial differential equation for this "power" option and show that it is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -S^2.$$

By assuming a solution of the form

$$V(S,t) = \phi(t) S^2$$

show that

$$\phi(t) = \frac{1}{\sigma^2 + r}\left(\left(\sigma^2 + r + 1\right) e^{\left(\sigma^2 + r\right)(T-t)} - 1\right).$$