Know Your Weapon

Dr. Espen Gaarder Haug

Market Formula (Bachelier-Thorp)

$$c = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2)$$

$$p = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S/X) + (b + \sigma_{X,T}{}^2/2)T}{\sigma_{X,T}\sqrt{T}}$$

$$d_2 = d_1 - \sigma_{X,T}\sqrt{T}$$

$b = 0$ Bl-76

$b = r$ B-S

$b = \boxed{r - r_f}$ G.-K.83

$b = r - q$ M-73

S = Asset price

X = Strike

T = Years to maturity

r = risk - free - rate

b = cost - of - carry

$\sigma_{X,T}$ = volatility that can be different for each strike and maturity

See Haug 2007 "Derivatives Models on Models" chapter 2

Black-Scholes-Merton

$$c = Se^{(b-r)T} N(d_1) - Xe^{-rT} N(d_2)$$

$$p = Xe^{-rT} N(-d_2) - Se^{(b-r)T} N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S/X) + (b + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S = Asset price

X = Strike

T = Years to maturity

r = risk - free - rate

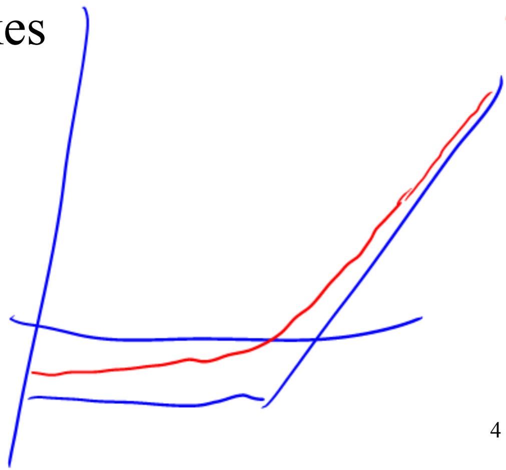
b = cost - of - carry

σ = volatility

Delta Greeks

- Delta
- Delta mirror strikes
- Strike from delta
- Elasticity

$T = \frac{3}{12}$



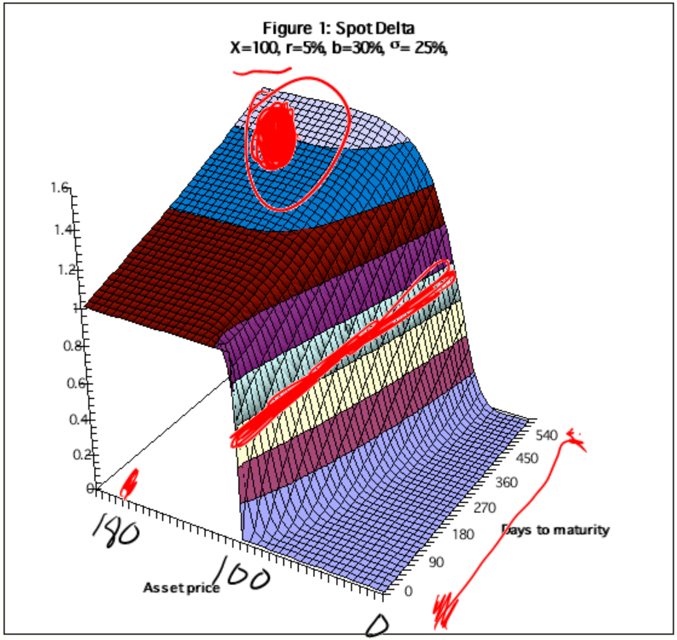
$N(d_1)$

Delta higher than one

$b > r$

$$\Delta_{call} = \frac{\partial c}{\partial S} = e^{(b-r)T} \boxed{N(d_1)}$$

$$\Delta_{put} = \frac{\partial p}{\partial S} = -e^{(b-r)T} N(-d_1)$$



$\Delta_{ATM} = 50\%$

$S=100$

$X_C=110$

Delta Mirror Strikes

$$X_P = \frac{S^2}{X_C} e^{(2b+\sigma^2)T}, \quad \boxed{X_C = \frac{S^2}{X_P} e^{(2b+\sigma^2)T}}$$

Special case delta symmetric straddle (Wystrup(1999)):

$$X_C = X_P = Se^{(b+\sigma^2/2)T}$$

$X_C=F$

Delta symmetric asset: $S = Xe^{(-b-\sigma^2/2)T}$

1W
1M

At this strike the delta is $\Delta_C = \frac{e^{(b-r)T}}{2}, \quad \Delta_P = -\frac{e^{(b-r)T}}{2}$

$$c = \frac{Se^{(b-r)T}}{2} - X^{-rT} N(-\sigma\sqrt{T}), \quad p = X^{-rT} N(\sigma\sqrt{T}) - \frac{Se^{(b-r)T}}{2}$$

1w
1m
3M

Strikes from delta

FX 95-98%

OTC

6M

Wystrup(1999):

9M

$$X_C = S\exp[N^{-1}(\Delta_C e^{(r-b)T})\sigma\sqrt{T} + (b+\sigma^2/2)T]$$

$\Delta = 25\%$

12M

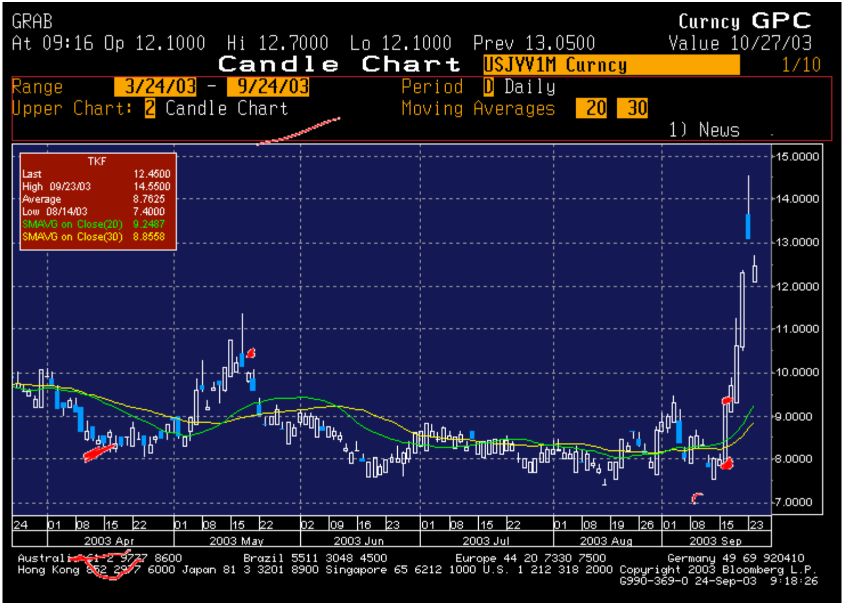
$$X_P = S\exp[N^{-1}(-\Delta_P e^{(r-b)T})\sigma\sqrt{T} + (b+\sigma^2/2)T]$$

Call USD/JPY Put

Robust and accurate approximation of inverse cumulative normal distribution needed, Moro(1995).

8% - 8,5% $T = \frac{3}{12}$

100m USD



GRAB
Curncy GPC
At 09:16 Op 12.1000 Hi 12.7000 Lo 12.1000 Prev 13.0500
Value 10/27/03
Candle Chart
USJYV1M Curncy
1/10
Range 3/24/03 - 9/24/03
Period D Daily
Upper Chart: 2 Candle Chart
Moving Averages 20 30
1) News
TKF
Last 12.4500
High 09/23/03 14.5500
Average 8.7625
Low 08/14/03 7.4000
SMAVG on Close(20) 9.2487
SMAVG on Close(30) 8.8558
15.0000
14.0000
13.0000
12.0000
11.0000
10.0000
9.0000
8.0000
7.0000
2003 Apr
2003 May
2003 Jun
2003 Jul
2003 Aug
2003 Sep
Brazil 5511 3048 4500
Europe 44 20 7330 7500
Germany 49 69 920410
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2003 Bloomberg L.P.
G990-369-0 24-Sep-03 9:18:26

DdeltaDvol $\dfrac{\partial c}{\partial S \partial \sigma} = \dfrac{\partial p}{\partial S \partial \sigma} = \dfrac{-e^{(b-r)T} d_2}{\sigma} n(d_1)$

15%

20%, 25%

Maximal value at

$$S_L = Xe^{-bT-\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

Minimal value at

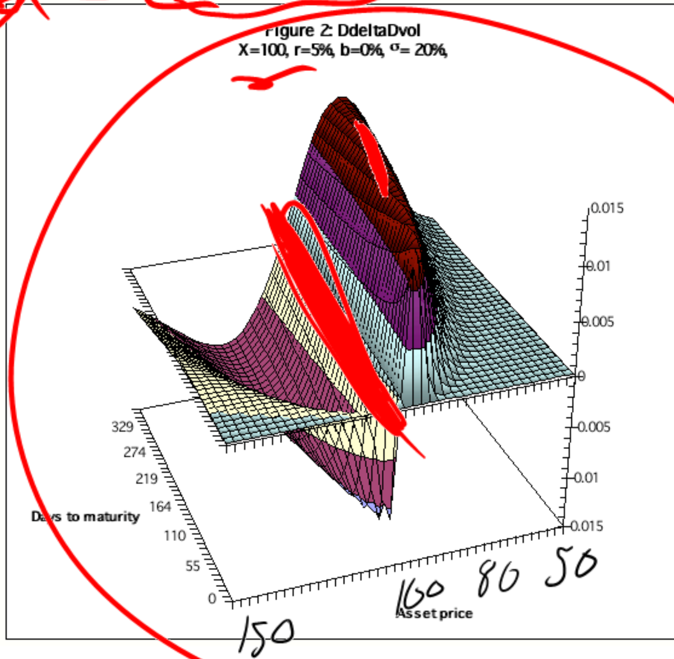
$$S_U = Xe^{-bT+\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

Minimal value at

$$X_L = Se^{bT-\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

Maximal value at

$$X_U = Se^{bT+\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$



Useful Tools

- A library
- Paper and pencil
- Mathematica
- Maple
- Matlab (?)
- Others ?

Implementation:

VBA, VB, C/C++, Java….

you name it

$\Lambda = 5\%$,

$S = C$

Elasticity

$$\Lambda_{call} = \Delta_{call}\frac{S}{call} > 1 \qquad \Lambda_{put} = \Delta_{put}\frac{S}{put} < -1$$

Option volatility: $\sigma_O \approx \sigma\,|\Lambda|$ 20% Compound options

Option Beta, expected return satisfy the CAPM equation (Merton-71):

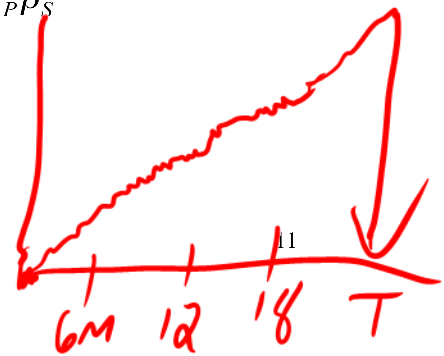
$$E[return] = r + E[r_m - r]\beta_i$$

$$\beta_C = \frac{S}{call}\Delta_C\beta_S = \Lambda_C\beta_S, \quad \beta_P = \frac{S}{put}\Delta_P\beta_S = \Lambda_P\beta_S$$

Option Sharp ratios

$$\frac{\mu_O - r}{\sigma_O} = \frac{\mu_S - r}{\sigma}$$

Smile?



Charm

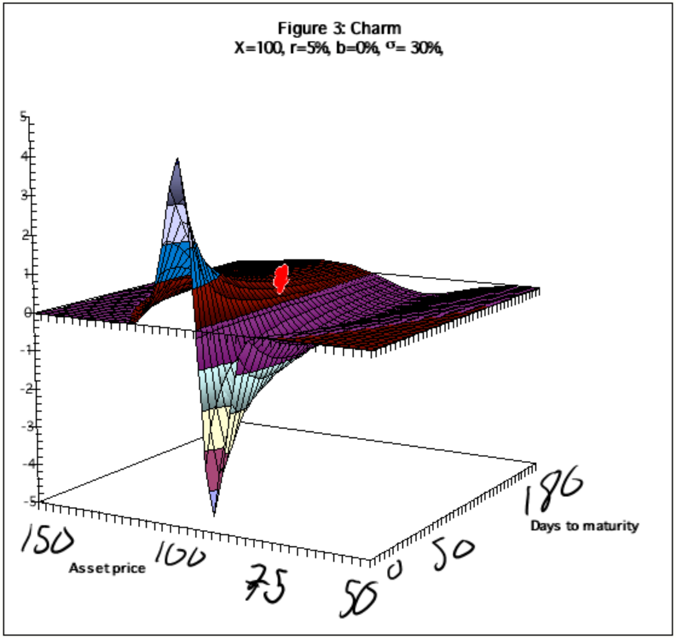
$$\boxed{\frac{\partial \Delta_C}{\partial T}} = -e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_2}{2T}\right) + (b-r)N(d_1)\right]$$

$$\frac{\partial \Delta_P}{\partial T} = -e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_2}{2T}\right) - (b-r)N(-d_1)\right]$$

E 4b

+ $\boxed{\partial w}$

1



Gamma Greeks

- Gamma
- Saddle gamma
- GammaP
- Gamma symmetry
- DGammaDVol
- DGammaDspot
- DGammaDTime

100 ↓ 2 100

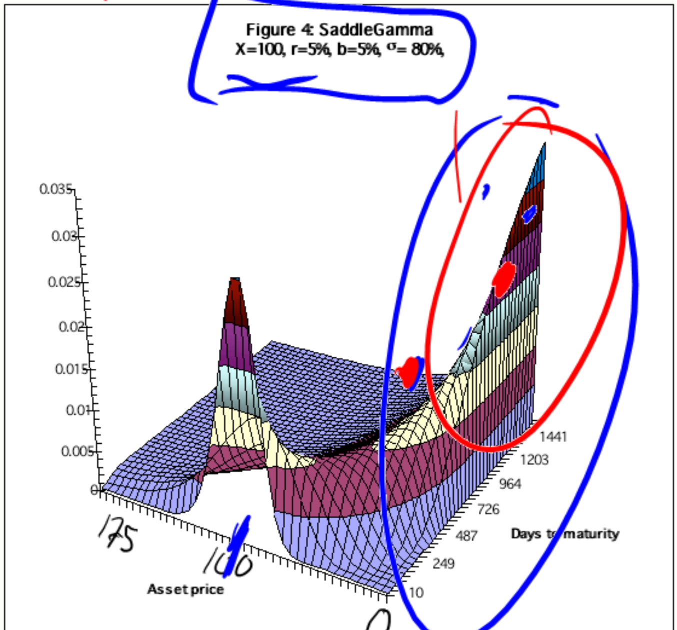
Saddle Gamma

Alexander Adamchuk www.wilmott.com

$$T_{\overline{\Gamma}} = \frac{1}{2(\sigma^2 + b)}$$

$$S_{\overline{\Gamma}} = Xe^{(-b-3\sigma^2/2)T_S}$$

$$\Gamma_S = \frac{e^{(b-r)T}\sqrt{\frac{e}{\pi}}\sqrt{\frac{b}{\sigma^2}+1}}{X}$$



1.2

200 0-20%

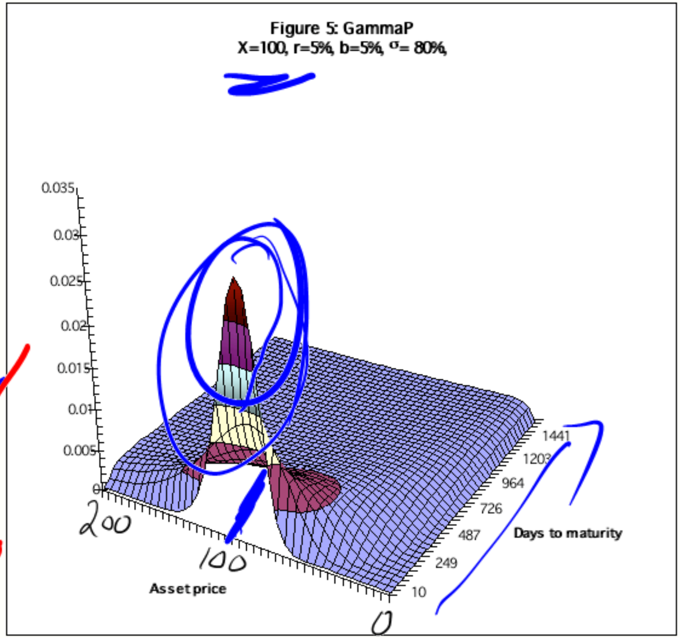
GammaP

$$\Gamma_P = \Gamma \frac{S}{100}$$

Max GammaP at

$$S = Xe^{(-b-\sigma^2/2)T}$$

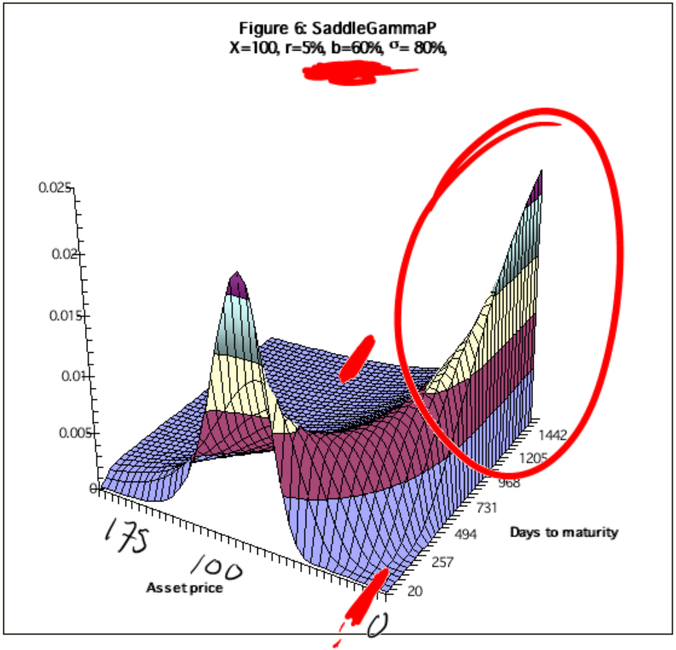
$$X = Se^{(b+\sigma^2/2)T}$$



Saddle GammaP

•Spot gamma

•Forward gamma



Gamma-symmetry

Put-call symmetry Bates(1991) and Carr and Bowie (1994):

$$c(S,X,T,r,b,\sigma)=\frac{X}{Se^{bT}}\,p\!\left(S,\frac{(Se^{bT})^2}{X},T,r,b,\sigma\right)$$

$X_p = \left(\frac{F}{X_c}\right)^2$

Gamma-symmetry

$$\Gamma(S,X,T,r,b,\sigma)=\frac{X}{Se^{bT}}\,\Gamma\!\left(S,\frac{(Se^{bT})^2}{X},T,r,b,\sigma\right)$$

Also gives vega and cost-of-carry symmetry

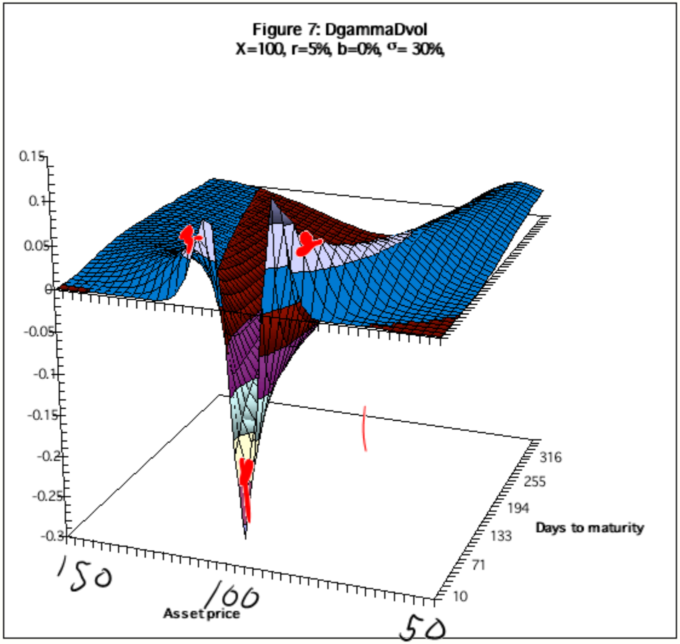
DgammaDvol

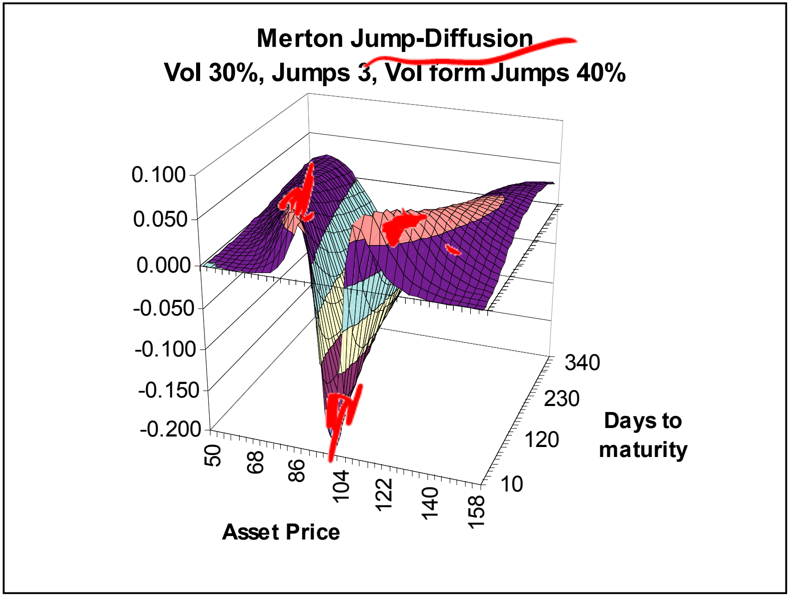
$$\frac{\partial \Gamma}{\partial \sigma} = \Gamma\left(\frac{d_1 d_2 - 1}{\sigma}\right) \qquad \frac{\partial \Gamma_P}{\partial \sigma} = \Gamma_P\left(\frac{d_1 d_2 - 1}{\sigma}\right)$$

Positive outside interval

$$S_L = Xe^{-bT-\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

$$S_U = Xe^{-bT+\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$



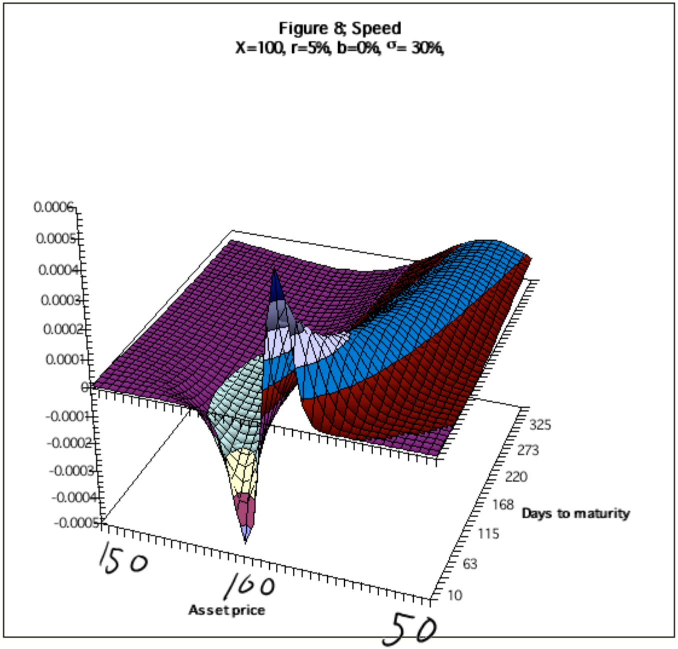


Call	
Asset price (S)	80.00
Strike price (X)	100.00
Time to maturity (T)	0.25
Risk-free rate (r)	5.00%
Volatility (σ)	30.00%
Jumps per year (λ)	3.00
Percent of total volatility (γ)	40.00%
Value	0.5255

Speed (DgammaDspot)

$$\frac{\partial^3 c}{\partial S^3} = -\frac{\Gamma\left(1+\frac{d_1}{\sigma\sqrt{T}}\right)}{S}$$

$$SpeedP = -\Gamma\frac{d_1}{S}$$

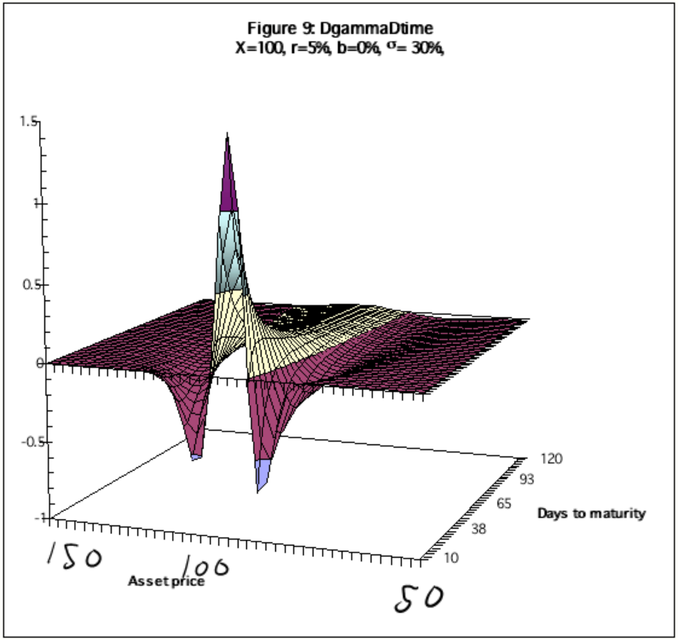


Speed is used by Fouque, Papanicolaou, and Sircar (2000) as part of stochastic vol model

DgammaDtime

$$\frac{\partial \Gamma}{\partial T} = \Gamma\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1 - d_1 d_2}{2T}\right)$$

$$\frac{\partial \Gamma_P}{\partial T} = \Gamma_P\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1 - d_1 d_2}{2T}\right)$$



Numerical Greeks

- More robust (?)
- Model independent
- Faster to implement (?)

ΔS = 0.1

S = 100 + ΔS

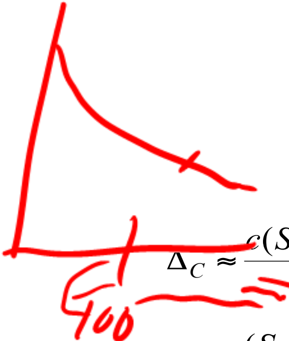
S = 100 − ΔS

Two-sided finite difference

$$\Delta_C \approx \frac{c(S+\Delta S, X, T, r, b, \sigma) - c(S-\Delta S, X, T, r, b, \sigma)}{2\Delta S}$$

Backward derivative

$$\Theta \approx \frac{c(S, X, T, r, b, \sigma) - c(S, X, T-\Delta T, r, b, \sigma)}{\Delta T}$$



Numerical Greeks

$$\Delta_C \approx \frac{c(S+\Delta S, X, T, r, b, \sigma_1) - c(S-\Delta S, X, T, r, b, \sigma_2)}{2\Delta S}$$

$$\Theta \approx \frac{c(S, X, T, r, b, \sigma_1) - c(S, X, T-\Delta T, r, b, \sigma_2)}{\Delta T}$$

Gamma and other second derivatives, central finite difference

$$\Gamma \approx \frac{c(S+\Delta S, \ldots) - 2c(S, \ldots) + c(S-\Delta S, \ldots)}{\Delta S^2}$$

Speed and other third order derivatives, central finite difference

$$Speed \approx \frac{1}{\Delta S^3}[c(S+2\Delta S, \ldots) - 3c(S+\Delta S, \ldots) + 3c(S, \ldots) - c(S-\Delta S, \ldots)]$$

Know Your Weapon
Part 2

Numerical Greeks

What about mixed derivatives? For example DdeltaDvol and Charm

$$DdeltaDvol \approx \frac{1}{4\Delta S\Delta\sigma}[c(S+\Delta S,...,\sigma+\Delta\sigma)-c(S+\Delta S,...,\sigma-\Delta\sigma)$$
$$-c(S-\Delta S,...,\sigma+\Delta\sigma)+c(S-\Delta S,...,\sigma-\Delta\sigma)]$$

Vega “Greeks”

- Vega
- Vega maximum
- VegaP
- Vega symmetry
- Vega Leverage
- DVegaDvol
- DVegaDtime

$X = F$

$c = p \approx F \cdot 0.4 \cdot \sigma\sqrt{T} e^{-rT}$

Vega $\quad \dfrac{\partial c}{\partial \sigma} = Se^{(b-r)T} n(d_1)\sqrt{T}$

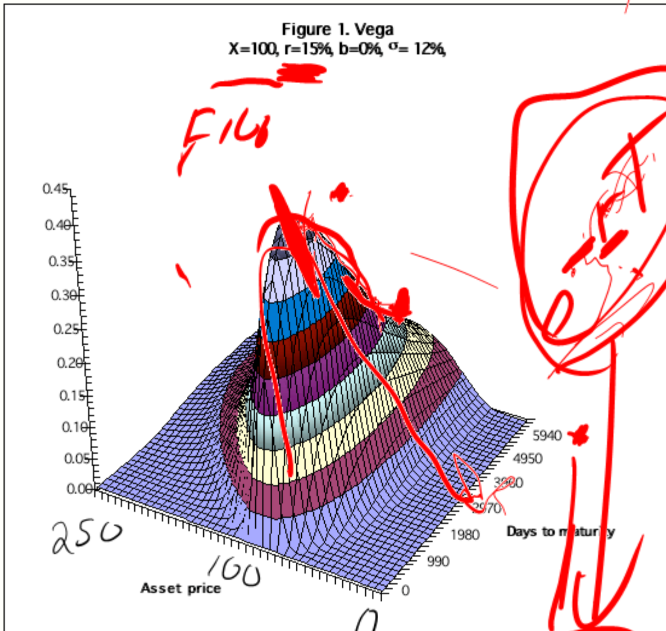


Figure 1. Vega
X=100, r=15%, b=0%, σ= 12%

Vega local max

$$S = Xe^{(-b+\sigma^2/2)T}$$

$$X = Se^{(b+\sigma^2/2)T}$$

Global maximum

$$T_{\overline{V}} = \frac{1}{2r}$$

$$S_{\overline{V}} = Xe^{(-b+\sigma^2/2)T_{\overline{V}}}$$

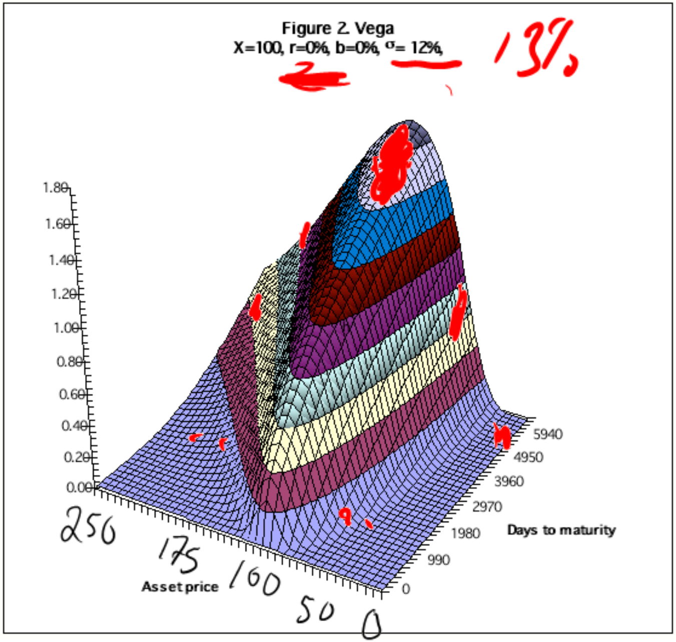
$$= Xe^{\frac{-b+\sigma^2/2}{2r}}$$

$$Vega(S_{\overline{V}}, T_{\overline{V}}) = \frac{X}{2\sqrt{re\pi}}$$

$S\sqrt{T}$ $\quad \sigma\sqrt{T}$ $\quad \boxed{\sigma\sqrt{T}}$

Why the Vega top?

Discounting at some point will dominate over volatility effect (Vega).



Vega-symmetry

Put-call symmetry Bates(1991) and Carr and Bowie (1994):

$$c(S,X,T,r,b,\sigma) = \frac{X}{Se^{bT}}\, p\left(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma\right)$$

Vega-symmetry

$$Vega(S,X,T,r,b,\sigma) = \frac{X}{Se^{bT}}\, Vega\left(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma\right)$$

Also gives gamma and cost-of-carry symmetry

Vega-gamma relationship

Taleb(1997):

$$Vega = \Gamma \sigma S^2 T$$

Vega from delta

$$Vega = Se^{(b-r)T}\sqrt{T}n[N^{-1}(e^{(r-b)T}\,|\,\Delta\,|)]$$

Gamma from delta

$$\Gamma = \frac{e^{(b-r)T}n[N^{-1}(e^{(r-b)T}\,|\,\Delta\,|)]}{S\sigma\sqrt{T}}$$

VegaP

Vega gives dollar change in option value for one percent point change in implied volatility. VegaP gives dollar change in option value for percentage move in volatility.

$$VegaP = \frac{\sigma}{10} S e^{(b-r)T} n(d_1)\sqrt{T}$$

VegaP makes much more sense when comparing sensitivity to changes in Implied volatility.

Long D st AT OTM Call

If you want to speculate on an increase in implied volatility what type of options offers the most bang for the bucks?

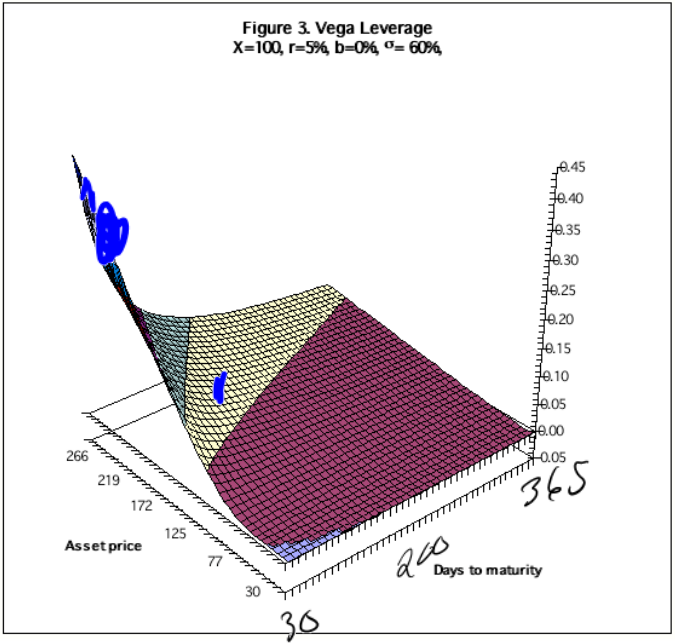
$\sigma = 20\% \nearrow 30\%$ ✓

Vega leverage

20%

Percent change in option value for percent point change in implied volatility.

$$Vega\frac{\sigma}{call}, \quad Vega\frac{\sigma}{put}$$



DvegaDvol Vomma/Volga $$\frac{\partial^2 c}{\partial \sigma^2} = Vega\left(\frac{d_1 d_2}{\sigma}\right)$$

Positive outside

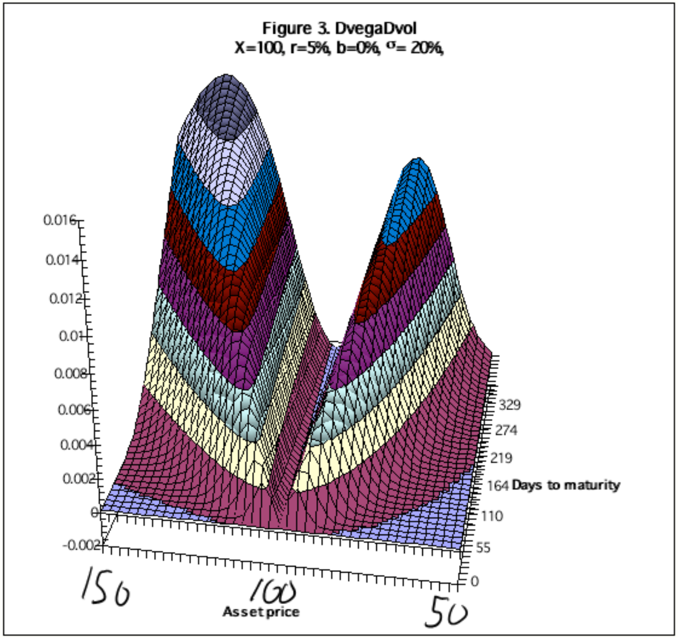
$$S_L = Xe^{(-b-\sigma^2/2)T}$$

$$S_U = Xe^{(-b+\sigma^2/2)T}$$

Positive outside

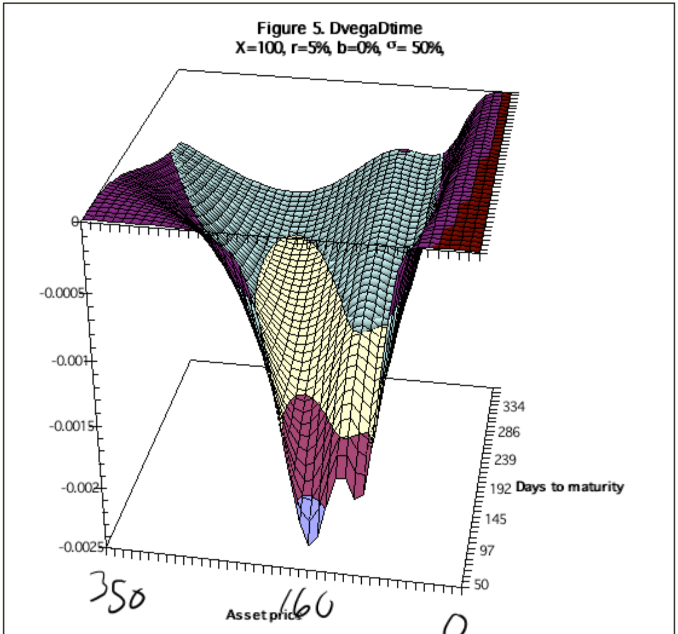
$$X_L = Se^{(b-\sigma^2/2)T}$$

$$S_U = Se^{(b+\sigma^2/2)T}$$



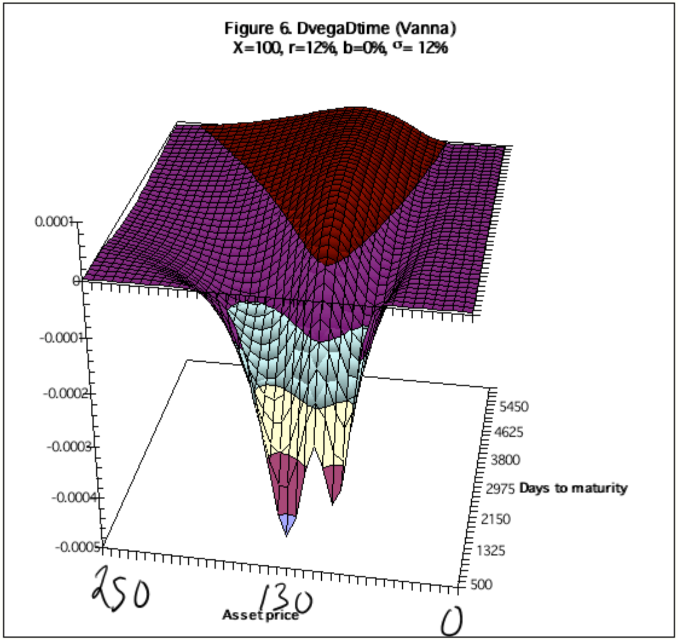
DvegaDtime

$$\frac{\partial^2 c}{\partial\sigma\partial T} = Vega\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} - \frac{1 + d_1 d_2}{2T}\right)$$



DvegaDtime

$$\frac{\partial^2 c}{\partial \sigma \partial T} = Vega\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} - \frac{1 + d_1 d_2}{2T} \right)$$



Theta

$$\Theta_C = -\frac{\partial c}{\partial T} = -\frac{Se^{(b-r)T} n(d_1)}{2\sqrt{T}} - (b-r)Se^{(b-r)T} N(d_1) - rXe^{-rT} N(d_2)$$

$$\Theta_C = -\frac{\partial c}{\partial T} = -\frac{Se^{(b-r)T} n(d_1)\sigma}{2\sqrt{T}} + (b-r)Se^{(b-r)T} N(-d_1) + rXe^{-rT} N(-d_2)$$

Drift-less theta

$$\theta_C = \theta_P = -\frac{Sn(d_1)}{2\sqrt{T}}$$

Theta symmetry

$$\theta(S, X, T, 0, 0, \sigma) = \frac{X}{S}\theta(S, \frac{S^2}{X}, T, 0, 0, \sigma)$$

Bleed-offset volatility

$$\frac{\Theta}{Vega}$$

500%

Rho

0%

$$\rho_C = \frac{\partial c}{\partial r} = TXe^{-rT}N(d_2), \quad \rho_P = \frac{\partial p}{\partial r} = -TXe^{-rT}N(-d_2)$$

14%

In case of options on futures (b=0)

15%

-0.5%

$$\rho_C = \frac{\partial c}{\partial r} = -Tc, \quad \rho_P = \frac{\partial p}{\partial r} = -Tp$$

25%

60%

15%

12%

20%

Probability “Greeks”

32%

36%

38%

Risk neutral probability of ending up in-the-money

$$\zeta_C = \boxed{N(d_2)} > 0, \quad \zeta_P = N(-d_2) > 0$$

$N(d_1)$

Strike-delta

$$\frac{\partial c}{\partial X} = -e^{-rT} N(d_2), \quad \frac{\partial p}{\partial X} = e^{-rT} N(-d_2)$$

$d_2 = d_1 - \sigma\sqrt{T}$

Probability mirror strikes

$$X_P = \frac{S^2}{X_C} e^{(2b-\sigma^2)T}, \quad X_C = \frac{S^2}{X_P} e^{(2b-\sigma^2)T}$$

Probability neutral straddle

$$X_C = X_P = Se^{(b-\sigma^2/2)T}$$

Probability "Greeks"

Strikes from probability

$$X_C = S\exp[-N^{-1}(p_i)\sigma\sqrt{T} + (b-\sigma^2/2)T]$$

$$X_P = S\exp[N^{-1}(-p_i)\sigma\sqrt{T} + (b-\sigma^2/2)T]$$

Risk neutral probability density

$$RND = \boxed{\frac{\partial^2 c}{\partial X^2}} = \frac{\partial^2 p}{\partial X^2} = \frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}}$$

Probability neutral straddle

$$X_C = X_P = Se^{(b-\sigma^2/2)T}$$

Risk neutral probability density

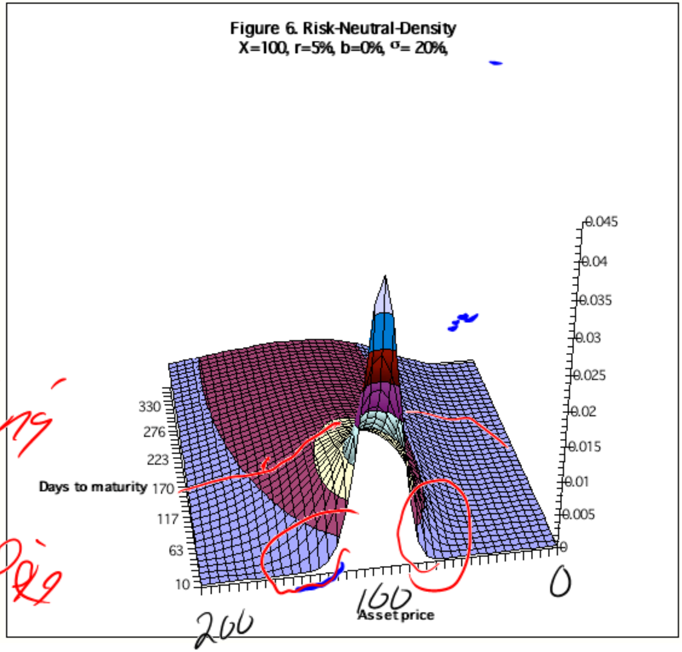
$$RND = \frac{\partial^2 c}{\partial X^2} = \frac{\partial^2 p}{\partial X^2} = \frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}}$$

Breeden and Litzenberger (1978)

1994 Rub

94 Der-Kani

$\sigma\sqrt{T}$ Br Dupire



Probability "Greeks"

Risk neutral probability of ever being in-the-money

$$p_C = (X/S)^{\mu+\lambda} N(-z) + (X/S)^{\mu-\lambda} N(-z + 2\lambda\sigma\sqrt{T})$$

$$p_p = (X/S)^{\mu+\lambda} N(z) + (X/S)^{\mu-\lambda} N(z - 2\lambda\sigma\sqrt{T})$$

where

$$z = \frac{\ln(X/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \quad \mu = \frac{b - \sigma^2/2}{\sigma^2}, \quad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}$$

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