$$dV(t) = f(.)\,dt + g(.)\,dW(t)$$

Drift

deterministic

like an ODE

scale with dt

Diffusion

stochastic

like a martingale

scales with $\sqrt{dt}$

PDE

Probability

$X(t)$

$\Lambda = \frac{dQ}{dP}$

$X^{Q}(t)$

$Q \sim P$

P

$\$$

Q

$S^*(t)$ is not a martingale in the P-measure

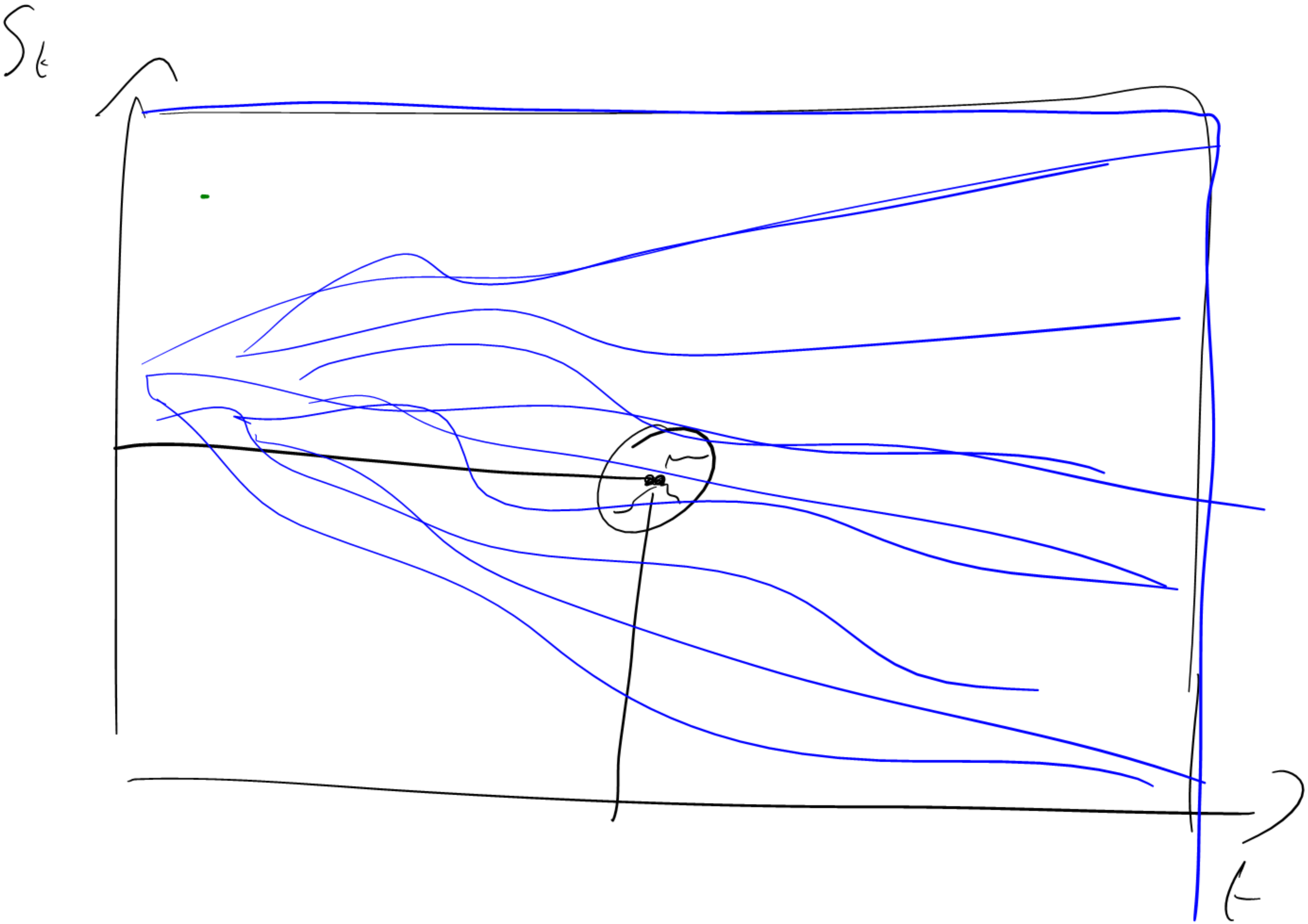
$S^*(t)$ is a martingale in the Q-measure

RN

Portfolio

$$\frac{dS^*(t)}{S^*(t)} = (\mu - r)\,dt + \sigma\,dX(t)$$

$$dS^*(t) = \ldots\,?\; dX^{Q}(t)$$


S_t
t

Start from

$$dX^{Q}(t) = dX(t) + \theta(t)\,dt$$

$$\Leftrightarrow \quad dX(t) = dX^{Q}(t) - \theta(t)\,dt$$

$X(t) \to BM$ under $\mathbb{P}$

$X^{Q}(t) \to BM$ under Q

Now, under $\mathbb{P}$:

$$dS^*(t) = (\mu - r)\, S^*(t)\,dt + \sigma\, S^*(t)\, dX(t)$$

Using the relation between $dX(t)$ and $dX^{Q}(t)$, we can write the SDE for $S^*(t)$ under Q:

$$dS^*(t) = (\mu - r)\, S^*(t)\,dt + \sigma\, S^*(t)\left[dX^{Q}(t) - \theta(t)\,dt\right]$$

$$dS^*(t) = \underbrace{\left(\mu - r - \theta(t)\,\sigma\right)}_{=0} S^*(t)\,dt + \sigma\, S^*(t)\, dX^{Q}(t)$$

$$\mu - r - \theta(t)\,\sigma = 0 \Leftrightarrow \boxed{\theta = \frac{\mu - r}{\sigma}}$$

$$\theta = \frac{\mu - r}{\sigma} \quad \longrightarrow \quad dS^*(t) = \sigma S^*(t)\, dX^Q(t)$$

$$\Rightarrow \text{martingale !!!}$$

Girsanov applied to $\theta = \frac{\mu - r}{\sigma}$

$$\Lambda = \frac{dQ}{dP} = \exp\left\{ - \frac{\mu - r}{\sigma} X(t) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t \right\}$$

and

$$X^Q(t) = X(t) + \frac{\mu - r}{\sigma} t$$

$$V^*(t) = \frac{V(t)}{B(t)}$$

$$\text{SDE ???} = V(t) \cdot B^{-1}(t)$$

$$dV^*(t) = d\left(V(t) \cdot B^{-1}(t)\right)$$

$$\begin{cases} V(t) = \phi_t^S S(t) + \phi_t^B B(t) \\ dV(t) = \phi_t^S dS_t + \phi_t^B dB_t \end{cases}$$

$$B(t) = e^{rt} \to dB(t) = rB(t)dt$$

$$\to B^{-1}(t) = e^{-rt} \to dB^{-1}(t) = -rB^{-1}dt$$

$$B^{-1}(t) = 1/B(t) \leftarrow$$

$$= dV(t) \cdot B^{-1}(t) + V(t) \cdot dB^{-1}(t)$$

$$= \left[\phi_t^S dS(t) + \phi_t^B dB(t)\right] B^{-1}(t) + \left[\phi_t^S S(t) + \phi_t^B B(t)\right]\left[\underbrace{-rB^{-1}dt}_{dB^{-1}(t)}\right]$$

$$= \phi_t^S \left[\underbrace{B^{-1}(t)dS(t) + S(t) \cdot dB^{-1}(t)}_{d(S(t)B^{-1}(t))} \right] \Rightarrow \phi_t^S dS^*(t)$$

$$+ \cancel{\phi_t^B}\left[\underbrace{r\cancel{B(t)}dt\cancel{B^{-1}}(t)}_{+\cancel{r\,dt}} + \cancel{B(t)} \times (-r\cancel{B^{-1}}(t))dt \right]$$

$$\cancel{-r\,dt}$$

$$dV^*(t) = \phi_t^S \, \cancel{dS^*(t)}$$

$$dV^*(t) = \boxed{\phi_t^S \, \sigma \, S^*(t) \, dX^{\mathbb{Q}}(t)} + 0$$

$\longrightarrow V^*(t)$ is a martingale under $\mathbb{Q}$!!!

$$\mathbb{E}\left[V^*(T) \mid \mathcal{F}_t \right] = V^*(t)$$

$$\mathbb{E}\left[\underbrace{V^*(T)}_{= \frac{G(S(T))}{B(T)}} \mid \mathcal{F}_t \right] = V_t^*(t) = \frac{\chi(t, S_t)}{B(t)}$$

$$\mathbb{E}^{Q}\left[B_T^{-1}[S_T-K]^{+}\right]$$

$$= \mathbb{E}^{Q}\left[B_T^{-1}\cdot(S_T-K)\,\mathbb{1}\{S_T>K\}\right]$$

$$= \underbrace{\mathbb{E}^{Q}\left[B_T^{-1}\cdot S_T\,\mathbb{1}\{S_T>K\}\right]}_{1} - \underbrace{\mathbb{E}^{Q}\left[B_T^{-1}K\,\mathbb{1}\{S_T>K\}\right]}_{2}$$

$$2 \to \mathbb{E}^{Q}\left[\underbrace{B_T^{-1}\cdot K}_{= e^{-rT}}\cdot\mathbb{1}\{S_T>K\}\right]$$

$$= Ke^{-rT}\underbrace{\mathbb{E}^{Q}\left[\mathbb{1}\{S_T>K\}\right]}_{\int_\Omega \mathbb{1}\{S_T>K\}\,dQ} = \mathbb{P}^{Q}[S_T>K]$$

$$\mathbb{E}^{Q}\left[B_T^{-1} S_T \ \mathbb{1}\{S_T > K\} \right]$$

$$= \mathbb{E}^{Q}\left[S_T^{*} \ \mathbb{1}\{S_T > K\} \right]$$

$$= \mathbb{E}^{Q}\left[S_0 \exp\left\{ -\frac{1}{2}\sigma^2 T + \sigma X_T^{Q} \right\} \mathbb{1}\{S_T > K\} \right]$$

$$= S_0 \mathbb{E}^{Q}\left[\exp\left\{ -\frac{1}{2}\sigma^2 T + \sigma X_T^{Q} \right\} \mathbb{1}\{S_T > K\} \right]$$

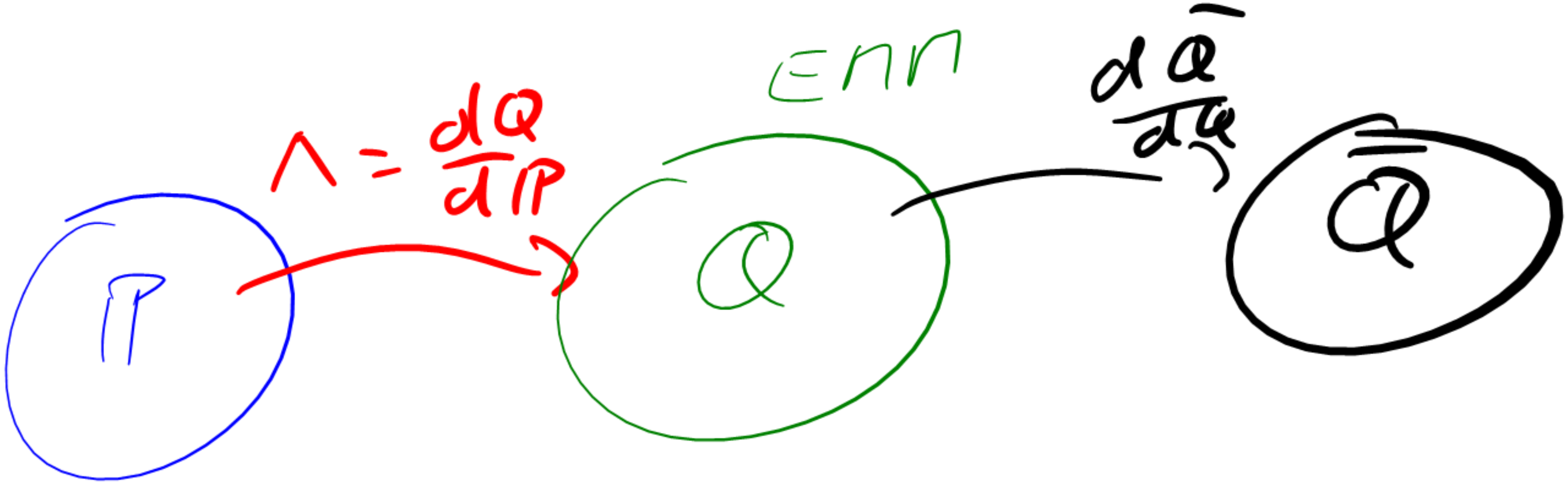
$$\exp\left\{ -\frac{1}{2}\int_0^T \sigma^2 dt + \int_0^T \sigma dX_t^{Q} \right\}$$

$$S_0 \, \mathbb{E}^{Q}\Big[\underbrace{\exp\{\sigma X_T^{Q} - \tfrac{1}{2}\sigma^2 T\}}_{\Lambda = \frac{d\bar{Q}}{dQ}} \mathbb{1}\{S_T > K\} \Big]$$

$$= S_0 \int_\Omega \frac{d\bar{Q}}{\cancel{dQ}} \, \mathbb{1}\{S_T > K\} \, \cancel{dQ}$$

$$= S_0 \int_\Omega \mathbb{1}\{S_T > K\} \, d\bar{Q}$$

$$= S_0 \, \underbrace{\mathbb{E}^{\bar{Q}}\big[\mathbb{1}\{S_T > K\} \big]}_{S_0 \, P^{\bar{Q}}[S_T > K)}$$


$\mathbb{P}$
$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$
Emm
$\mathbb{Q}$
$\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}}$
$\bar{\mathbb{Q}}$

$\mathbb{P}\left[S_T > K\right]$ equation for S_T under $\mathbb{P}$

$= \mathbb{P}\left[S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma X_T\right\} > K\right\}$

$= \mathbb{P}\left[\ln\left(\frac{S_0}{K}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T > -\sigma X_T\right]$

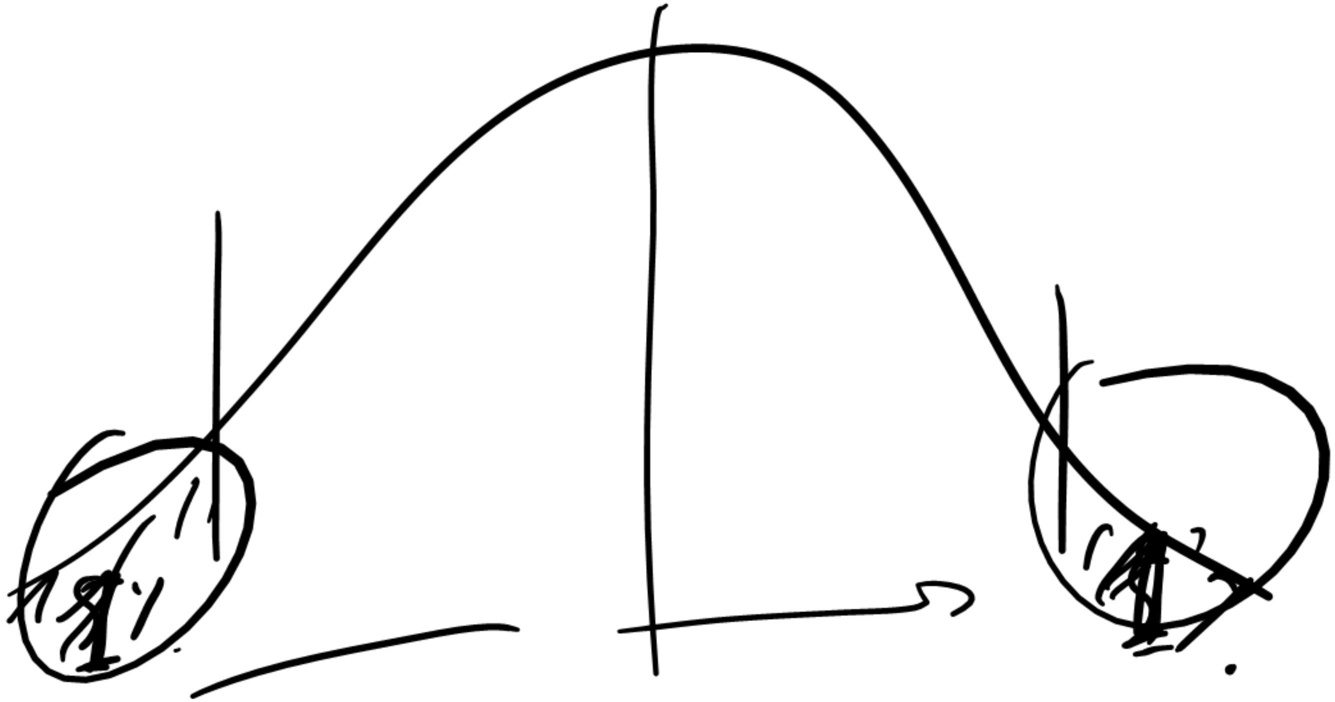
$\sim \sqrt{T}$

$= \mathbb{P}\left[\frac{\ln\left(S_0/K\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} > \xi\right)$

d_0

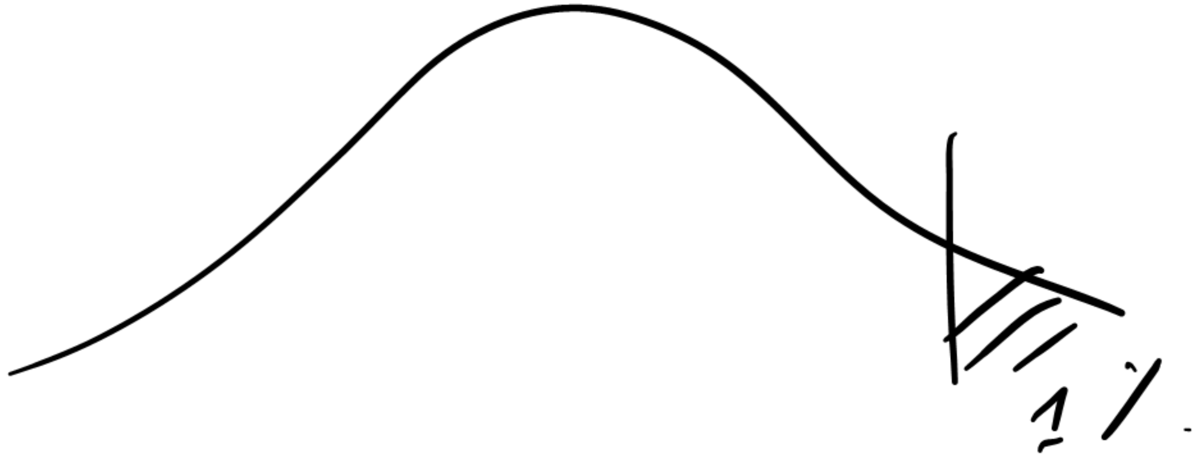
$= N(d_0)$

$$\mathbb{1}_{\{x \in A\}} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$



2 tail @ 10%
2

10%
2



$$Ke^{-rT}\,\underbrace{P^{Q}\left[S_T > K\right]}$$

$$= Ke^{-rT}\,P^{Q}\left[\ln\left(S_0\, e^{\left\{\sigma X_T^{Q} + \left(r-\frac{1}{2}\sigma^2\right)T\right\}}\right) > \ln(K)\right]$$

$$= Ke^{-rT}\,P^{Q}\left[\ln\left(\frac{S_0}{K}\right) + \left(r-\frac{1}{2}\sigma^2\right)T > -X_t^{Q}\right]$$

BM

$\sim N(0, T)$

$\xi\sqrt{T}$ where

$\xi \sim N(0,1)$

$$= Ke^{-rT}\,\underbrace{P^{Q}\left[\frac{\ln\left(\frac{S_0}{K}\right) + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} > \xi\right]}_{N\left(\underbrace{\frac{\ln\frac{S_0}{K} + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}_{=d_2}\right)}$$