CQF Module 2 Exercise Solution

Ran Zhao

A. Optimal Portfolio Allocations

1 To solve for the weight in global minimum variance portfolio, we formulate

$$\begin{aligned} \underset{\omega}{\operatorname{argmin}} \quad & \frac{1}{2}\omega'\Sigma\omega \\ \text{subject to} \quad & \omega'\mathbf{1} = 1 \end{aligned}$$

The Lagrangian multiplier of this global minimum variance portfolio is

$$L(\omega, \lambda) = \frac{1}{2}\omega'\Sigma\omega + \lambda(\omega'\mathbf{1} - 1) = 0$$

Set the FOCs to zero yields the optimal solution of the weight:

$$\frac{\partial L}{\partial \omega} = \Sigma\omega + \lambda\mathbf{1} = 0 \tag{1}$$

$$\frac{\partial L}{\partial \lambda} = \omega'\mathbf{1} - 1 = 0 \tag{2}$$

From (5), the optimal weight solution has

$$\omega^* = -\Sigma^{-1}\lambda\mathbf{1} \tag{3}$$

Bring this into (6), we have

$${\omega^*}'\mathbf{1} = -\lambda\mathbf{1}'\Sigma^{-1}\mathbf{1} = 1 \quad \Rightarrow \quad \lambda^* = -\frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \tag{4}$$

Combine (4) with (3), the analytical solution for optimal allocations ω^* is

$$\omega^* = \frac{\Sigma^{-1}\mathbf{1}'}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}$$

2.a To solve for the minimum variance portfolio under the target return with risk-free asset, we formulate

$$\begin{aligned} \underset{\omega}{\operatorname{argmin}} \quad & \frac{1}{2}\omega'\Sigma\omega \\ \text{subject to} \quad & r + (\mu - r\mathbf{1})\omega' = 0.1 \end{aligned}$$

The Lagrangian multiplier of this global minimum variance portfolio is

$$L(\omega, \lambda) = \frac{1}{2}\omega'\Sigma\omega + \lambda[r + (\mu - r\mathbf{1})\omega' - 0.1] = 0$$

Set the FOCs to zero yields the optimal solution of the weight:

$$\begin{aligned} \frac{\partial L}{\partial \omega} &= \Sigma\omega + \lambda(\mu - r\mathbf{1}) = 0 & (5) \\ \frac{\partial L}{\partial \lambda} &= r + (\mu - r\mathbf{1})\omega' - 0.1 = 0 & (6) \end{aligned}$$

From (1), the optimal weight solution has

$$\omega^* = -\lambda\Sigma^{-1}(\mu - r\mathbf{1}) \tag{7}$$

Bring this into (2), we have

$$(\mu - r\mathbf{1})'\omega^* = -\lambda(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1}) = 0.1 - r \tag{8}$$

which yields

$$\lambda^* = -\frac{0.1 - r}{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})} \tag{9}$$

Combine (7) with (9), the analytical solution for optimal allocations ω^* is

$$\omega^* = \frac{(0.1 - r)\Sigma^{-1}(\mu - r\mathbf{1})'}{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})}$$

2.b First construct the variance-covariance matrix Σ from the correlation matrix. That is,

$$\Sigma = SRS = \begin{pmatrix} 0.0049 & 0.00168 & 0.0063 & 0.00546 \\ 0.00168 & 0.0144 & 0.01512 & 0.01248 \\ 0.0063 & 0.01512 & 0.0324 & 0.04212 \\ 0.00546 & 0.01248 & 0.04212 & 0.0676 \end{pmatrix}$$

Then calculate the optimal weight for the minimum variance portfolio

$$\omega^* = \frac{(0.1 - r)\Sigma^{-1}(\mu - r\mathbf{1})'}{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})} = \begin{pmatrix} 0.3957 & 1.0541 & -0.8268 & 0.7313 \end{pmatrix}'$$

Finally, the standard deviation of the portfolio is

$$\sigma_\Pi = \sqrt{\omega^{*\prime}\Sigma\omega^*} = 0.1321$$

Detailed numerical calculation results could be found in the Appendix Matlab code.
3.a
3.b

B. Value at Risk on FTSE 100

1

2.a

2.b

2.c

2.d

3.a

3.b

4

5.a

5.b

5.c

C. Stochastic Calculus

1

2

3

4.a Starting with the lower triangular matrix A, we have

$$\Sigma = AA' = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

which yields the original covariance matrix.

4.b Given $Y = AX$,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 X_1 \\ \rho\sigma_2 X_1 + \sqrt{1-\rho^2}\sigma_2 X_2 \end{pmatrix}$$

4.c

Matlab code