CQF Module 4 Examination Solutions

January 2016 Cohort

dW is the usual increment of a Brownian motion.

1. We know the ZCB price $Z(r,t;T)$ satisfies the BPE

$$\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}+(u-\lambda w)\frac{\partial Z}{\partial r}-rZ=0,\quad Z(r,T;T)=1$$

Expand $Z(r,t;T)$ for small times t to maturity T, i.e. in powers of $(T-t)$

$$Z\sim a(r)+b(r)(T-t)+c(r)(T-t)^2+\ldots\ldots$$

for the unknown coefficients. Firstly, $Z(r,T;T)=1\longrightarrow a(r)=1$

$$\begin{aligned}
Z &\sim 1+b(r)(T-t)+c(r)(T-t)^2+\ldots\ldots\\
\frac{\partial Z}{\partial t} &= -b(r)-2c(r)(T-t)\\
\frac{\partial Z}{\partial r} &= b'(r)(T-t)+c'(r)(T-t)^2;\ \frac{\partial^2 Z}{\partial r^2}=b''(r)(T-t)+c''(r)(T-t)^2
\end{aligned}$$

Substituting these terms into the BPE gives

$$\begin{aligned}
&-b(r)-2c(r)(T-t)+\frac{1}{2}w^2\left(b''(r)(T-t)+c''(r)(T-t)^2\right)+\\
&(u-\lambda w)\left(b'(r)(T-t)+c'(r)(T-t)^2\right)\\
=\ &r\left(1+b(r)(T-t)+c(r)(T-t)^2\right).
\end{aligned}$$

Now compare coefficients of powers of $(T-t)$

$O(1)$:

$$-b(r)=r\longrightarrow b(r)=-r\Longrightarrow b'(r)=-1\Longrightarrow b''(r)=0$$

$O((T-t))$:

$$\begin{aligned}
-2c(r)+(u-\lambda w)(-1) &= -r^2\\
c(r) &= \tfrac{1}{2}\left(r^2-(u-\lambda w)\right)=\tfrac{1}{2}\left(r^2-\eta+\gamma r\right)
\end{aligned}$$

So for $t\longrightarrow T$ we have

$$Z\sim 1-r(T-t)+\tfrac{1}{2}\left(r^2-\eta+\gamma r\right)(T-t)^2+\ldots$$

Now write $\log Z=\log(1+x)$, where $x\equiv -r(T-t)+\frac{1}{2}(r^2-\eta+\gamma r)(T-t)^2$, so

$$\begin{aligned}
\log Z &\sim -r(T-t)+\tfrac{1}{2}\left(r^2-\eta+\gamma r\right)(T-t)^2\\
&\quad -\tfrac{1}{2}\left(-r(T-t)+\tfrac{1}{2}\left(r^2-\eta+\gamma r\right)(T-t)^2\right)^2\\
&\sim -r(T-t)-\tfrac{1}{2}(T-t)^2(\eta-\gamma r)+\ldots.
\end{aligned}$$

The bond price also satisfies $Z=\exp(-r_L(T-t))\longrightarrow r_L=-\frac{1}{T-t}\log Z$, and using the expression for $\log Z$ above

$$\begin{aligned}
r_L &= -\frac{1}{T-t}\left(-r(T-t)-\tfrac{1}{2}(T-t)^2(\eta-\gamma r)\right)\\
&= r+\tfrac{1}{2}(T-t)(\eta-\gamma r)
\end{aligned}$$

So for Vasicek with one month LIBOR, $(T-t)=1/12$, we find

$$r_L\sim r+\frac{1}{2}(\eta-\gamma r)(1/12).$$

2. For the SDE

$$dG = a\left(G,t\right)dt + b\left(G,t\right)dX$$

Itô's Lemma for $V = V\left(G\right)$ gives

$$dV = \left(a\left(G,t\right)\frac{dV}{dG} + \frac{1}{2}b^2\left(G,t\right)\frac{d^2V}{dG^2}\right)dt + b\left(G,t\right)\frac{dV}{dG}dX$$

where

$$\begin{aligned} a &\equiv \theta\left(t\right) + \frac{d\left(\log\sigma\left(t\right)\right)}{dt}\log r \\ b &\equiv \sigma\left(t\right) \end{aligned}$$

Put $V = e^G \longrightarrow \frac{dV}{dG} = e^G = \frac{d^2V}{dG^2}$ where $G = \log r$, hence

$$dr = r\left(\theta\left(t\right) + \frac{d\left(\log\sigma\left(t\right)\right)}{dt}\log r + \frac{1}{2}\sigma^2\left(t\right)\right)dt + \sigma\left(t\right)r\,dX.$$

3. If $Z(r,t) = \exp(A(t\ ;T) - rB(t\ ;T))$ then $Z_r = -BV \qquad Z_{rr} = B^2 Z$
Solution with $Z(r,T\ ;\ T) = 1$ implies $B(T;T) = A(T;T) = 0$
$Z_t = (A'(t) - rB'(t))V$ so subst. these in to the PDE above

$$\begin{aligned} A'(t\ ;T) - rB'(t\ ;T) + \tfrac{1}{2}c^2B^2 - (\eta(t) - \gamma r)B - r &= 0 \quad \forall r \\ \left(A' + \tfrac{1}{2}c^2B^2 - \eta(t)B\right) - r(B' - \gamma B + 1) &= 0 \end{aligned}$$

$$\Rightarrow A' = -\tfrac{1}{2}c^2B^2 + \eta(t)B, \qquad\qquad B' = \gamma B - 1$$

From $Z(r,t\ ;T) = \exp(A(t) - rB(t))$ we note that as $r \to \infty$, $Z \to 0$. Solving for $B(t;T)$

$$\frac{dB}{dt} = \gamma B - 1$$

You can solve this by the variable sep. method: $\dfrac{dB}{\gamma B - 1} = dt$. Now recall:

$$\int \frac{dx}{ax+1} = \frac{1}{a}\ln|ax+1| + K$$

therefore we have

$$\begin{aligned} \int_t^T \frac{dB}{\gamma B - 1} &= \int_t^T d\tau = \frac{1}{\gamma}\ln|\gamma B(\tau\ ;T) - 1|_t^T = (T-t) \\ \ln\left|\frac{\gamma B(T\ ;T) - 1}{\gamma B(t\ ;T) - 1}\right| &= \gamma(T-t) \end{aligned}$$

we know $B(T\ ;T) = 0$ so

$$\begin{aligned} \ln\left|\frac{\gamma B(t\ ;T) - 1}{-1}\right|^{-1} &= \gamma(T-t) = -\ln|1 - \gamma B(t\ ;T)| = \gamma(T-t) \\ 1 - \gamma B(t\ ;T) &= \exp[-\gamma(T-t)] = \gamma B(t\ ;T) = 1 - \exp[-\gamma(T-t)] \\ B(t\ ;T) &= \frac{1}{\gamma}(1 - \exp[-\gamma(T-t)]) \end{aligned}$$

Then

$$A(t\ ;T) = \tfrac{1}{2}c^2\int_t^T B^2(\tau\ ;T)\,d\tau - \int_t^T B(\tau\ ;T)\,\eta(\tau)\,d\tau$$

and

$$\begin{aligned} \int_t^T B^2(\tau;T)\,d\tau &= \frac{1}{\gamma^2}\int_t^T \left(1 - 2e^{-\gamma(T-\tau)} + e^{-2\gamma(T-\tau)}\right)d\tau \\ &= \frac{1}{\gamma^2}\left((T-t) - \frac{2}{\gamma}e^{-\gamma(T-\tau)}\Big|_t^T + \frac{1}{2\gamma}e^{-2\gamma(T-\tau)}\Big|_t^T\right) \\ &= \frac{1}{\gamma^2}\left((T-t) - \frac{2}{\gamma} + \frac{2}{\gamma}e^{-\gamma(T-t)} + \frac{1}{2\gamma} - \frac{1}{2\gamma}e^{-2\gamma(T-t)}\right) \\ &= \frac{1}{\gamma^2}\left((T-t) - \frac{3}{2\gamma} + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)}\right) \end{aligned}$$

Hence

$$A(t\ ;T) = -\int_t^T B(\tau;T)\,\eta(\tau)\,d\tau + \frac{c^2}{2\gamma^2}\left((T-t) - \frac{3}{2\gamma} + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)}\right)$$

4. This is an Ornstein-Uhlenbeck process and an analytic solution for this equation exists. The SDE can be written as $dU_t + \gamma U_t dt = \sigma dX_t$.
Multiply both sides by an integrating factor $e^{\gamma t}$

$$\begin{aligned} e^{\gamma t}\left(dU_t + \gamma U_t\right)dt &= \sigma e^{\gamma t} dX_t \\ d\left(e^{\gamma t} U_t\right) &= \sigma e^{\gamma t} dX_t \end{aligned}$$

Integrating over $[0, t]$ gives

$$U_t = ue^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dX_s$$

where $U(0) = u = U_0$. Now taking expectations

$$\begin{aligned} \mathbb{E}[U_t] &= \mathbb{E}\left[ue^{-\gamma t}\right] + \sigma \int_0^t e^{\gamma(s-t)} \mathbb{E}[dX_s] \\ &= ue^{-\gamma t}. \end{aligned}$$

To calculate the variance we have $\mathbb{V}[U_t] = \mathbb{E}[U_t^2] - \mathbb{E}^2[U_t]$

$$\begin{aligned} &= \mathbb{E}\left[\left(ue^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dW_s\right)^2\right] - u^2 e^{-2\gamma t} \\ &= \mathbb{E}\left[u^2 e^{-2\gamma t}\right] + \sigma^2 \mathbb{E}\left[\left(\int_0^t e^{\gamma(s-t)} dW_s\right)^2\right] + 2\sigma u e^{-\gamma t} \mathbb{E}\underbrace{\left[\int_0^t e^{\gamma(s-t)} dW_s\right]}_{\text{Itô integral}} - u^2 e^{-2\gamma t} \\ &= \sigma^2 \mathbb{E}\left[\left(\int_0^t e^{\gamma(s-t)} dW_s\right)^2\right] \end{aligned}$$

Now use Itô's Isometry

$$\mathbb{E}\left[\left(\int_0^t Y_s dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t Y_s^2 ds\right],$$

So

$$\begin{aligned} \mathbb{V}[U_t] &= \sigma^2 \mathbb{E}\left[\int_0^t e^{2\gamma(s-t)} ds\right] \\ &= \frac{\sigma^2}{2\gamma} e^{2\gamma(s-t)}\Big|_0^t = \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right) \end{aligned}$$

5.

$$\begin{aligned}
\frac{dZ}{Z} &= r(t)dt \\
\int_t^T d(\log Z(s;T)) &= \int_t^T r(s)ds \\
-\log Z(t;T) &= \int_t^T r(s)ds \qquad \text{because } \log Z(T;T) = \log(1) = 0 \\
Z(t;T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right]
\end{aligned}$$

This is consistent with Feynman-Kac for a bond as a derivative *wrt* the stochastic variable $r(t)$ with boundary condition (payoff) $Z(r,T;T)$ and pricing equation (since we are not given any coefficients of $dr(t)$)

$$\frac{\partial Z}{\partial t}(r,t;T) - r(t)Z(r,t;T) = 0$$

$$\begin{aligned}
Z(r,t;T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\, Z(r,T;T)|\mathcal{F}_t\right] \qquad Z(r,T;T) = 1 \quad \forall\, r_T \\
&= \qquad \text{discounting with stochastic } r(t) \text{ remains inside the expectation} \\
Z(r,t;T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right]
\end{aligned}$$

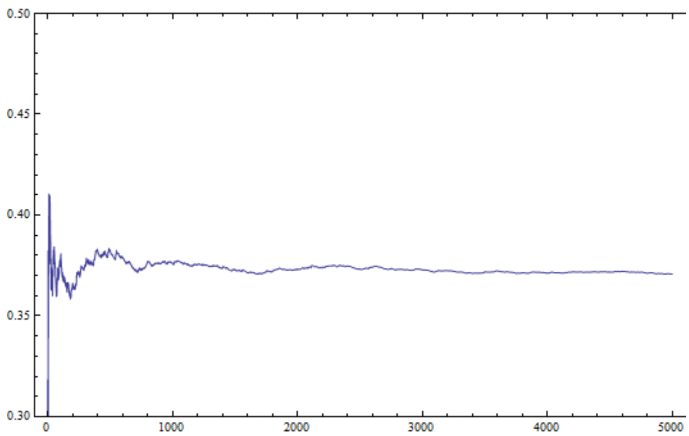
In order to satisfy the expectation, the pricing is carried out by Monte-Carlo. Within discrete implementation of HJM framework, use **the first column** of the simulated rates matrix – the shortest maturity instantaneous forward rate – as a proxy to short rate $r(t) = f(t,t)$:

$$\begin{aligned}
Z(t;T) &= \mathbb{E}\left[e^{-\int_t^T r(s)ds}\right] \\
&= \exp\left(-\sum r_t\, \Delta t\right)
\end{aligned}$$

The answer to the computational side of the question is a convergence diagram like one below. The longer maturity T selected, the smaller ZCB price will be compared to 1.

Excel Notes: Use $SUM()$ on the first column to include rows up to the timestep that correspond to maturity, e.g., $\delta t = 0.01$ so for $T = 0.5$ there are 50 rows to add up.

Press *F9* to update the random numbers. Record the history of bond prices and plot the running average for $N = 1, 2, 3 \ldots 1000$ simulations – this average should converge to a stable value with minimal variance. Writing a macro that saves the history of simulations would be useful.


0.50
0.45
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