

Libor Market Model: Specification and Calibration

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Abstract

In this paper I implement and calibrate the Lognormal Forward-LIBOR Model (LFM) for the term structure of interest-rates. This model is a subset of the LIBOR Market Model class of stochastic interest-rate models and is characterized by the lognormal distribution of forward LIBOR rates under appropriate numeraires. Specifically, I implemented the LFM under two different instantaneous volatility formulations, Rebonato's 6.21a (2002) and Brigo and Mercurio's Formulation 7 (2006). The implementations are then calibrated to market data for Caps and Swaptions. Finally, the results are analyzed for their accuracy and their correspondence to financial theory and intuition. All implementation and calibration is done in MATLAB.

Keywords: Calibration, Caps, LIBOR Market Model, LFM, Lognormal Forward-LIBOR Model, MATLAB, Stochastic Interest-rate Model, Swaptions.

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Background

Interest-rate models arise for many reasons, including economic forecasting, risk analysis, and trading. As it is commonly believed that interest-rates are at least somewhat (if not entirely) random, stochastic processes provide a natural framework upon which to build these models. The first models were broadly known as Short-Rate Models because they modeled only a single interest, namely the short-rate or the interest-rate applied to smallest available maturity. More intricate formulations of these models are still used today, often with multiple stochastic variables acting as factors to underlie the movement of the short-rate. Examples of these models include the famous Vasicek family of models and Cox, Ingersoll and Ross (CIR) models. A strong limitation of these models is that you must make a series of assumptions regarding the term structure of rates in order to extrapolate a curve across longer maturities. This knowledge of the term structure is extremely important because a vast quantity of derivative products rely on multiple interest-rates at multiple maturities. The desire to integrate more market data into interest-rate models and the pressure to capture the evolution of the entire term structure led to Market Models. Additionally, Market Models have a large degree of practical appeal for trading desks and other practitioners because they allow for rigorous derivation of widely-used price quoting formulas for Caps and Swaptions (which I will discuss below). These pricing formulas are Black's formula for Caps and Black's formula for Swaptions, respectively. The formulas were developed by applying the logic of the Black-Scholes-Merton Option pricing formula to interest-rate markets.¹ Before Market Models, these formulas had no strong underlying theoretical framework. Market Models effectively met the needs of market participants by satisfying their desire for a theoretical justification of their pricing conventions and integrating a wider picture of the interest-rate market into their models.

¹ (Brigo and Mercurio) Pg. 195

While interest-rate models have many uses, the focus of this paper is their relevance to asset pricing. The interest-rate derivative market is the largest security market in the world (by Notional amount). As of 2011, the total, global OTC derivatives market was \$708 trillion in notional (or contract) amount, according the Bank of International Settlements (Figure 1).²

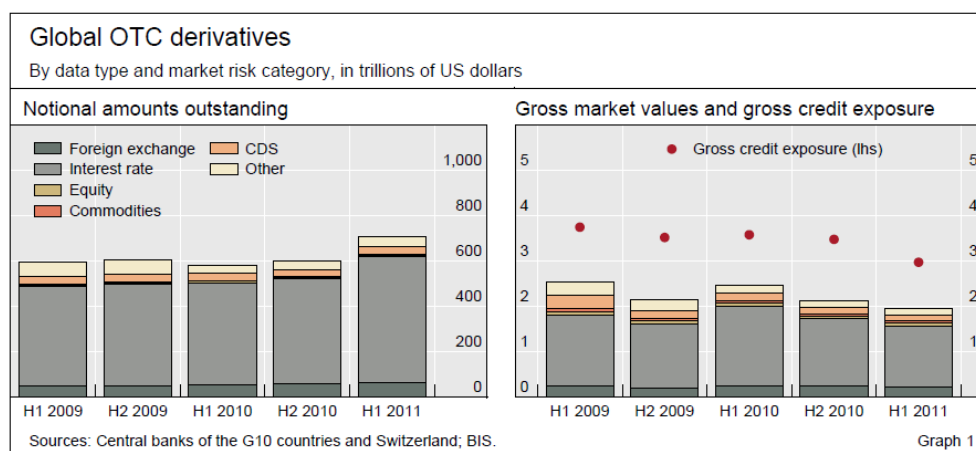


Figure 1: 2011 Bank of International Settlements OTC Derivatives Summary

Of this market, interest-rate derivatives are an overwhelming majority, with a notional amount outstanding of \$554 trillion, or half of one quadrillion dollars (Figure 2).

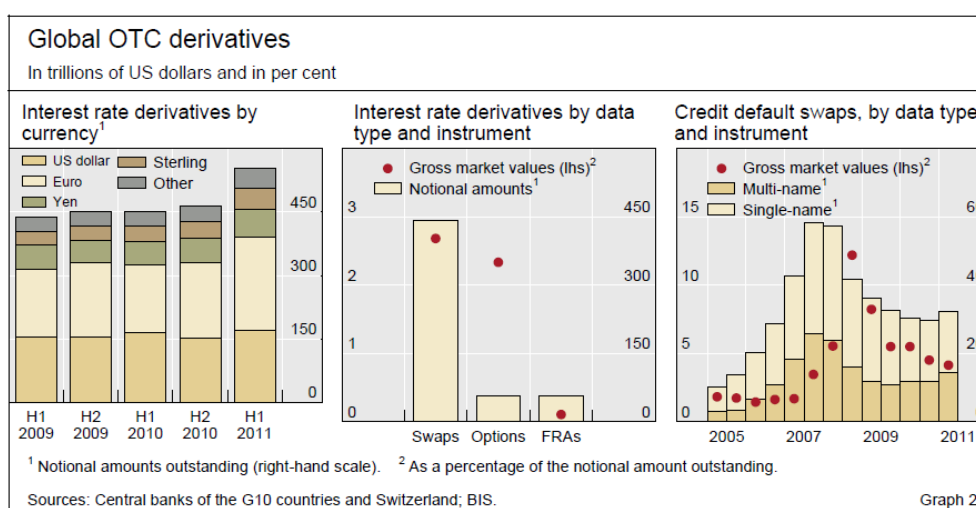


Figure 2: Bank of International Settlements Interest-Rate Derivatives Summary

² (Bank of International Settlements: Monetary and Economic Department) Pg. 1

This paper examines the pricing of two of the most common and liquid interest-rate derivative products, Caps and Swaptions. Both Caps and Swaptions are options on interest-rates. Conceptually, a Cap is an insurance agreement that “caps” the floating interest-rate that the Cap buyer has elsewhere agreed to pay at a series of fixed times in the future. It works by the seller of the Cap agreeing to pay the buyer the difference between the strike rate of the Cap (fixed at its purchase) and the floating-rate, if the floating-rate ever exceeds the strike. Therefore, the buyer is guaranteed to never have to pay more than the strike rate in any given period. For a Cap, all periods are considered independent and exercising (receiving payment from the seller) in one period does not impact the ability to exercise in another. For example, if Company XYZ issues floating-rate bonds which pay LIBOR + 3% coupons quarterly, but they never want to pay more than a total of 5%, they could buy a Cap with a strike rate of 2% from Investment Bank ABC to achieve this protection. If LIBOR moved to 3% after the Cap was purchased, ABC would pay XYZ 1%. Thus the net cash flow for XYZ would be $-(3\% + 3\%) + 1\% = -5\%$. If in the next quarter LIBOR moved back below 2%, ABC would not owe anything that period.

In USD (U.S. Dollar) markets, Caps typically have three month payment periods. In the EUR (Euro) market, payment periods are usually six months. This paper will focus on USD LIBOR markets. Finally, Caps have two additional time parameters beyond payment frequency. These parameters are Expiry and Term. Expiry is most often one period (three months) and is the length of time between the purchase of the Cap and the actual beginning of the Caps’ protection. The Term (or Maturity) is the length of the protection that the Cap offers. For example, a five year Cap purchased today will begin its protection in three months, make its first payment (if necessary) in six months, and make its final payment (again if necessary) five years from today. Each individual period in a Cap is referred to as a Caplet. A final way of thinking about a Cap is as a basket of options on a series of Forward-Rate Agreements (FRAs) with sequential Expiries and Maturities. This last intuition will be important for drawing a parallel, but distinction between Caps and Swaptions.

Swaptions, as their name implies, are options on interest-rate Swaps. Swaps, in turn, are agreements to trade a floating-rate, to be determined in the future, for a fixed-rate determined at contract initiation. This kind of contract allows someone who has a floating-rate obligation to fix their risk today, by converting it to a fixed-rate. A feature of Swaps is that they are worth \$0 at initiation. This is achieved by adjusting the fixed-rate to make the expected payoff of the Swap zero. Swaptions, as options on Swaps, have an inherent premium, because they allow the purchaser to enter into a Swap at a later date, according to their own interests. Similarly, Caps also always have a positive value. The Expiry of a Swaption (assuming European exercise) is the time at which the Swaption will be exercised, therefore entering the purchaser into a Swap, or it expires. European exercise means that the Swaption can only be exercised at Expiry, and not at any other time (before or after). The Tenor of a Swaption is the length of the underlying Swap that would be created if the Swaption is exercised. The Tenor is measured in time after the Expiry date. If, for example, Company FGH purchased a 1x5 (Expiry x Tenor) Swaption today with strike rate 5%, in one year they would have the right to enter into a five year Swap at the 5% fixed-rate set today. In one year, this Swap may be in-, at-, or out-of -the-money, determining whether or not the Swaption will be exercised. If one year later a comparable Swap has a rate of 7%, the Swaption would be exercised because it would have a positive value. On the other hand, if the Swap rate in one year is 4%, the Swaption would not be exercised because a better deal can be found in the market. Like a Cap is basket of options on FRAs, Swaptions can be thought of as a single option on a basket of FRAs.

This last point is the key reason that Swaptions are harder to price than Caps. As Caps are a series of independent options, the correlations of payoffs are not relevant and the value can simply be calculated as the sum of the value of each option. Swaptions are a single option whose value is correlated to all of the relevant forward rates, requiring that their joint movements be considered when pricing. The issues of correlation and pricing will be discussed in more specific terms later.

Model Formulation

The cornerstone of the Lognormal Forward-LIBOR Model is the following Stochastic Differential Equation (SDE) (Equation 1):

$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k(t) .$$

Equation 1: Dynamics of the F_k Forward Rate under T_k forward Measure

This equation shows that a given forward rate, F_k , is a martingale under the measure T_k which is associated with the numeraire of the price of a zero coupon bond maturing at time T_k . It can be shown using Ito's Lemma and Ito's Isometry that this SDE results in lognormally distributed forward rates. This property is extremely useful in that normal distributions are very nice to deal with. Additionally, this distribution function of forward rates is precisely what allows for the recovery of Black's formula for Caplets from the LFM. While this equation is very convenient, it relies on the appropriate choice of measure. Therefore, F_i for $i \neq k$, does not have driftless SDE, as a change of measure maintains the diffusion term, but introduces a new drift. Brigo and Mercurio describe the procedure for modifying the drift of a process under a change of measure in Chapter 2. The process relies on the calculation of the quadratic covariation of the original stochastic process and the process of the quotient of old and new numeraires. The equations below define the full dynamics of all forward-rates F_k in the LFM under a given T_i forward measure (Equation 2).

$$\begin{aligned} i < k, t \leq T_i: \quad dF_k(t) &= \sigma_k(t)F_k(t) \sum_{j=i+1}^k \frac{\rho_{k,j} \tau_j \sigma_j(t)F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t)F_k(t)dZ_k(t) \\ i = k, t \leq T_{k-1}: \quad dF_k(t) &= \sigma_k(t)F_k(t)dZ_k(t) \\ i > k, t \leq T_{k-1}: \quad dF_k(t) &= -\sigma_k(t)F_k(t) \sum_{j=k+1}^i \frac{\rho_{k,j} \tau_j \sigma_j(t)F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t)F_k(t)dZ_k(t) \end{aligned}$$

Equation 2: LFM Forward-Measure Dynamics of Forward-Rates³

³ (Brigo and Mercurio) Pg. 16

In the above equations $\rho_{k,j}$ is the correlation between two forward-rates, $\sigma_j(t)$ is the instantaneous volatility of a given forward-rate, τ_j is time between adjacent forward-rate maturities, Z_k is a Brownian motion under the T_i forward measure.

Now that the forward-rate dynamics have been specified, one can begin pricing payoffs using this model. The first security whose price will be considered is a Cap. Generally, the price of a Cap can be calculated with the following equation (Equation 3).

$$Cap\ Price = \sum_{i=\alpha+1}^{\beta} N\tau_i D(0, T_i) (F(T_{i-1}, T_{i-1}, T_i) - K)^+$$

Equation 3: General Cap Pricing Formula

Here the "+" denotes the $\max(0,*)$ operator, where * is the argument inside parentheses. To explain the other notation, $D(0, T_i)$ is the stochastic discount factor for the time period beginning at time zero (today) and ending at T_i ; tau is the year fraction length of the period between T_{i-1} and T_i ; K is the strike rate of the Cap; α is the number of periods in the expiry of the Cap, minus one (if the Cap has a three month expiry, then α is zero); β is the number of periods between the expiry and the maturity of the Cap (if the Cap has a three month expiry and a two year maturity, then β would be seven); and N is the notional amount of the Cap. The key fact to consider when pricing Caps is the independence of Caplets. This feature allows all Caplets to be modeled separately, and then summed. Modeling Caplets independently allows for a different choice of measure for each Caplet. To elaborate, the price of each Caplet only depends on a single forward-rate. Therefore, for each Caplet, choose the measure T_k that allows the relevant forward-rate F_k to be lognormally distributed. This lognormal distribution allows for a closed-form solution for the price of each Caplet, and thus for the Cap by summing over the Caplets. This closed-form solution for Caplet prices is exactly Black's formula (Equation 4).⁴

$$Caplet\ Price^{LFM}(0, T_{i-1}, T_i, K) = Caplet\ Price^{Black}(0, T_{i-1}, T_i, K)$$

⁴ (Brigo and Mercurio) Pg. 222

$$\begin{aligned}
&= N P(0, T_i) \tau_i BL(K, F_i(0), v_i) \\
BL(K, F_i(0), v_i) &= E^i[(F_i(T_{i-1}) - K)^+] \\
&= F_i(0) \Phi(d_1(K, F_i(0), v_i)) - K \Phi(d_2(K, F_i(0), v_i)) \\
d_1(K, F_i(0), v_i) &= \frac{\ln\left(\frac{F}{K}\right) + \frac{v^2}{2}}{v} \\
d_2(K, F_i(0), v_i) &= \frac{\ln\left(\frac{F}{K}\right) - \frac{v^2}{2}}{v} \\
v_i^2 &= T_{i-1} v_{T_{i-1}-Caplet}^2
\end{aligned}$$

Equation 4: Black's Formula for Caplets

Unlike Caps, Swaptions are very difficult to price under the LFM. As mentioned above, the heart of the issue is the importance of the correlation of forward-rates to pricing of Swaptions. This difficulty manifests itself in the general formula for pricing Swaptions (Equation 5).⁵

$$Swaption\ Price = N D(0, T_\alpha) \left(\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) (F(T_\alpha, T_{i-1}, T_i) - K) \right)^+$$

Equation 5: General Swaption Pricing Formula

The notation in the above formula is the same as in the Cap formula (Equation 3) and the $P(T_\alpha, T_i)$ term refers to the price of a zero coupon bond at time T_α which matures at time T_i . As the max function is outside of the summation, this price cannot be subdivided in the way that Caps were. Therefore, the entire summation must be evaluated together and the correlations of forward-rates in each iteration must be considered.

⁵ (Brigo and Mercurio) Pg. 19

There is no exact, analytical solution for Swaption prices under the LFM. There are two options for pricing Swaptions in the LFM model, simulation and approximation. Simulation can be initially attractive, but the process can be slow and influenced by Monte Carlo error. Especially in a trading environment, speed is highly valued, even at the expense of small variation from true value. A true value late, is worth infinitely less than a close value when needed. Approximation thus becomes the typically preferred method for pricing Swaptions. Riccardo Rebonato, a patriarch of fixed-income derivative mathematics and quantitative finance, developed a high-quality, closed-form, and theoretically appealing approximation for pricing Swaptions in the LFM. This is a good time to mention a market quoting convention that will be important throughout the rest of this paper. When one refers to the “price” of a Cap or Swaption, what is actually quoted is the Black implied volatility. This implied volatility is determined by inverting Black’s formula for Caps and Swaptions, respectively, with volatility as the unknown and the price a trader is willing to buy or sell a security as an input, to infer a volatility. This is the market convention for a variety of reasons, most significantly the ubiquity of Black’s formulas. Rebonato’s approximation for the Swaption price in the LFM actually approximates the Black Swaption volatility, not the dollar price of the Swaption. This volatility can then be compared to market quotes. Rebonato’s Formula, as the approximation is known, is as follows (Equation 6).⁶

$$(v_{\alpha,\beta}^{LFM})^2 = \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt$$

$$w_i(t) = \frac{\tau_i FP(t, T_\alpha, T_i)}{\sum_{k=\alpha+1}^{\beta} \tau_k FP(t, T_\alpha, T_k)}$$

$$FP(t, T, S) = \frac{P(t, S)}{P(t, T)}$$

Equation 6: Rebonato’s Formula

⁶ (Brigo and Mercurio) Pg. 283

In this formula, $S_{\alpha,\beta}(0)$ is the swap rate for a swap beginning at period α and maturing at period β . At this point, the central aspect of the LFM formulation that remains is the specification of the instantaneous volatility function, σ , and the instantaneous correlation matrix, ρ . The main instantaneous volatility used in this paper is Brigo and Mercurio's Formulation 7, after Rebonato's time-homogeneous formulation (Equation 7).⁷⁸

$$\sigma_i(t) = \Phi_i \psi(T_{i-1} - t; a, b, c, d) := \Phi_i ([a(T_{i-1} - t) + d] e^{-b(T_{i-1} - t)} + c)$$

Equation 7: Formulation 7 Instantaneous Volatility

Above, Φ_i , are time-invariant constants and a, b, c, and d are fit parameters. For instantaneous correlation, another time-homogeneous specification is used. This form was also formulated by Rebonato (Equation 8).⁹

$$\rho_{i,j} = e^{-\beta|T_i - T_j|}$$

Equation 8: Instantaneous Correlation Matrix

Here β is a parameter which can be manipulated for model calibration. After the formulation is finished, the next step is calibration of the model to market quotes.

Calibration

The first step for calibrating a model to market data is deciding what fits to prioritize. This decision impacts the formulation of the model, the data needed for calibration, and the steps taken to optimize the solution. An additional tradeoff when fitting parametric models is between the optimality of fit to the data and the economic and theoretical intuition behind the derived values. To illustrate this point, the calibration procedure may find that a β of 50 produces the best fit to market data, given other parameters. This value is immediately suspect as it would imply that forward-rates as little as one period

⁷ (Brigo and Mercurio) Pg. 212

⁸ (Rebonato) Pg. 167

⁹ (Rebonato) Pg. 177

apart in maturity have essentially zero correlation, which makes little economic sense and fails to satisfy the market-based intuition of many practitioners.

Rebonato describes two possible goals for calibration under the LFM.¹⁰ The first goal is to capture the current Caplet market and calibrate the model to evolve with a similar term structure into the future. The second goal would be to again fit Caplet quotes, but calibrate the model's parameters to capture the Swaption market as well, at the cost of desirable evolution over time. Both of these forms of calibration are considered in this paper. Before the details of calibration are discussed, another dimension of market quoting must be considered.

This market convention is to quote a single, constant, and annualized implied volatility for a Cap, v_{T_j-Cap} , even though each underlying Caplet has its own, distinct implied volatility. This chart from

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90) Actions		91) Products		93) Data & Settings		94) Help		Swap Manager	
3) Main	4) Curves	5) Cashflow	7) Details	10) Resets	11) Risk	13) Scenario	17) Matrix		
21) Cashflow Table		22) Cashflow Graph							
Cap	Cpty	CAP CNTRPARTY	CCP	OTC	Ticker / CAP	Series	Deal ID	20) Properties	
Cashflow	Cap				Currency	USD		99) Export to Excel	
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07/30/12	11/01/12	92	10,000,000.00	0.90266	73.81	0.98977	0.66	4,738.25	0.996291 2218
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01/30/13	05/01/13	89	10,000,000.00	0.90266	73.71	1.53728	0.87	17,732.15	0.989494 1552
04/29/13	08/01/13	92	10,000,000.00	0.90266	70.28	0.41692	0.23	721.41	0.988441 0.00
07/30/13	11/01/13	92	10,000,000.00	0.90266	70.28	0.16741	0.05	41.59	0.988018 0.00
10/30/13	02/03/14	94	10,000,000.00	0.90266	70.28	-0.08433	0.00	0.00	0.988236 0.00
01/30/14	05/01/14	87	10,000,000.00	0.90266	70.24	-0.26852	0.00	0.00	0.989023 0.00
04/29/14	08/01/14	92	10,000,000.00	0.90266	68.82	0.81933	0.64	7,180.70	0.986956 0.00
07/30/14	11/03/14	94	10,000,000.00	0.90266	68.82	0.88040	0.68	8,883.59	0.984693 0.00
10/30/14	02/02/15	91	10,000,000.00	0.90266	68.82	0.94101	0.71	10,127.01	0.982357 952.
01/29/15	05/01/15	88	10,000,000.00	0.90266	68.76	1.01134	0.73	11,462.44	0.979965 2603
04/29/15	08/03/15	94	10,000,000.00	0.90266	65.35	1.34075	0.79	18,932.85	0.976546 1117
07/30/15	11/02/15	91	10,000,000.00	0.90266	65.35	1.44239	0.81	20,875.39	0.973000 1327
10/29/15	02/01/16	91	10,000,000.00	0.90266	65.35	1.54081	0.81	23,376.69	0.969226 1563
01/28/16	05/03/16	92	10,000,000.00	0.90266	65.32	1.63810	0.82	26,153.61	0.965186 1814

Zoom - + 100%

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Figure 3: Bloomberg SWPM Showing a Four Year Cap Broken Down Into Caplets

¹⁰ (Rebonato) Pg. 226

Bloomberg (Figure 3) illustrates the cash flow and implied volatility structure of a Cap broken down into Caplets.

The convention is convenient because it allows for a single number to convey the price of an entire Cap, with maturity T_j (Equation 9). The problem with this method is an apparent contradiction for the implied volatility of Caplets. Specifically, a five year Cap and a three year Cap (given the same strike) would have identical Caplets for the first three years, but they would have different implied volatilities. However, this apparent contradiction is not actually a problem. When a trader constructs a quote, she solves a non-linear equation for the constant implied volatility v_{T_j-Cap} that, when assigned to all Caplets, prices the Cap the same as the sum of the Caplets with individual volatilities $v_{T_{i-1}-Caplet}$ (Equation 10).¹¹

$$Cap\ Price^{MKT}(0, T_j, K) = \sum_{i=1}^j N P(0, T_i) \tau_i BL \left(K, F_i(0), \sqrt{T_{i-1}} v_{T_j-Cap} \right)$$

Equation 9: Cap Price for a Given Market Quoted Implied Volatility

$$\begin{aligned} \sum_{i=1}^j N P(0, T_i) \tau_i BL \left(K, F_i(0), \sqrt{T_{i-1}} v_{T_j-Cap} \right) \\ = \sum_{i=1}^j N P(0, T_i) \tau_i BL \left(K, F_i(0), \sqrt{T_{i-1}} v_{T_{i-1}-Caplet} \right) \end{aligned}$$

Equation 10: Equation for Inferring Cap Implied Volatility from Caplet Implied Volatilities

The volatilities in Equation 10 are scaled by $\sqrt{T_{i-1}}$ because their squares are annualized for quoting and the original value must be recovered for pricing.

¹¹ (Brigo and Mercurio) Pg. 225-226

Therefore, before a model can be fit to market quote data, that data must be modified to recover the Caplet implied volatilities necessary for calibration for the Cap volatilities outlined above. This process involves a bootstrapping procedure across increasing Cap maturities and Cap strikes.

Another difficulty that presents itself in market data is the phenomenon known as skew, or volatility smile. Skew is the presence of a non-flat implied volatility across strike rates, namely that as strike rates move further into or out-of-the-money the implied volatility increases. On an intuitive level, this structure points to the fat-tailed underlying distribution of rate movements. In essence, a normal distribution applied to larger moves requires a higher standard deviation, according to the market. Thus, a normal distribution that uses the standard deviation of small price movements underprices the likelihood of large jumps in interest-rates (there is a rich literature of stochastic volatility [notably SABR] models and jump-diffusion models that capture this smile in a more systematic fashion). While this paper does not use these more complex volatility schemes, the calibration procedure incorporates the volatility smile present in market quotes.

The bootstrapping of Caplet implied volatilities from market quotes requires two steps. The first step is to bootstrap all maturities for each quoted strike rate, and then inter/extrapolating a smile. The second step is the adjustment to at-the-money (ATM) quotes. These quotes are treated separately because they have non-standard strike rates that change at each maturity, thus additional knowledge of the volatility smile is needed to be able to successfully bootstrap ATM Caplets. The reason for taking the time to use ATM Caps is that they tend to be the most quoted and most liquid Caps, therefore they have the freshest and most accurate data for the market.

The market, when accessed through a Bloomberg Terminal (the standard market data service) gives the following information for Caps at a given point in time (Figure 4).

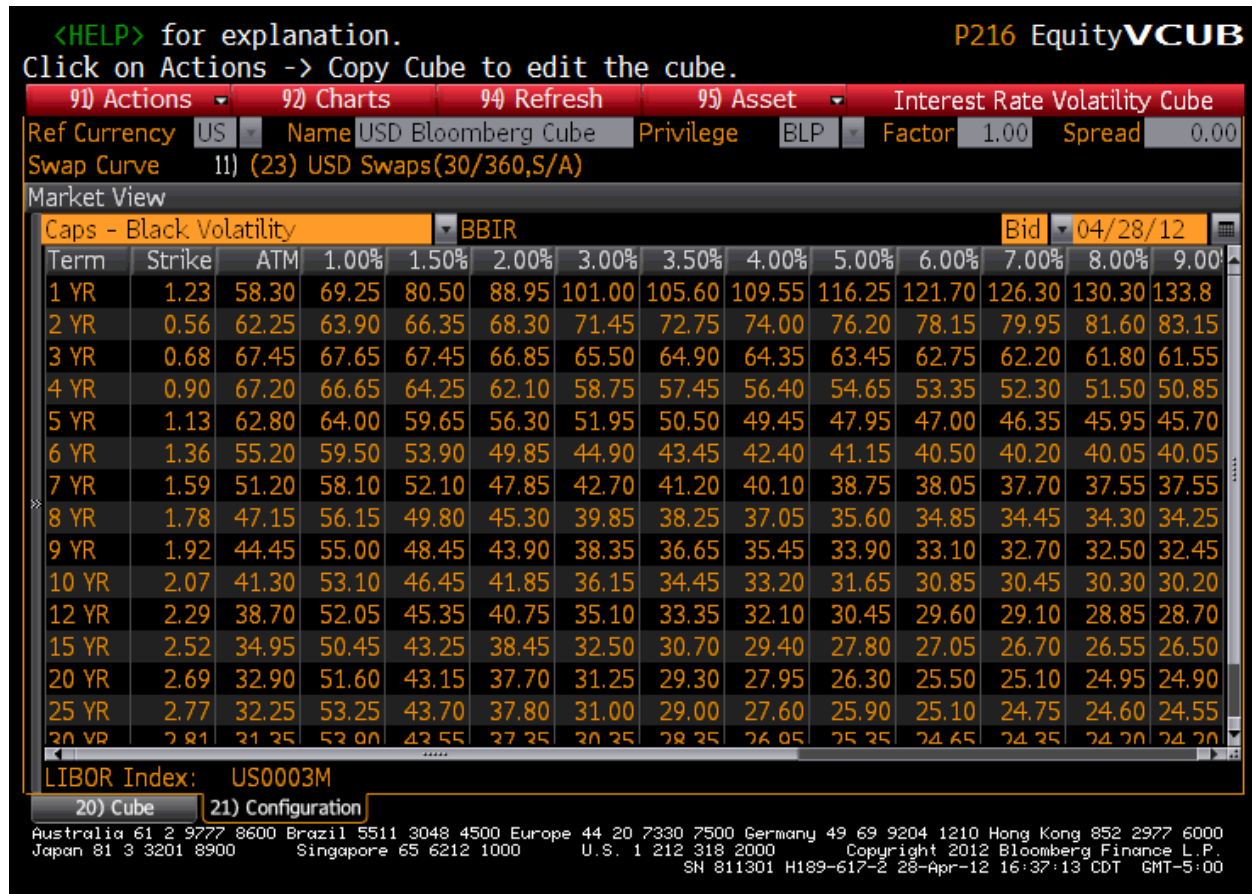


Figure 4: Bloomberg VCUB Cap Market

The final obstacle to Caplet bootstrapping is an assumption about the term structure of implied volatility of Caplets. As the Cap quote density is typically in one year (or greater) intervals, there is not sufficient information to uniquely determine quarterly Caplet implied volatilities from market information alone. A common, and typically robust, solution is to assume that volatility is piece-wise constant in-between quoted Cap maturities.¹² This is the assumption which will be used in this paper.

To implement the first part of bootstrapping stated above, the following procedure is used¹³:

1. Iterate over all available strike rates K.

¹² (Levin) Pg. 8

¹³ (Levin) Pg. 8

2. For each K, bootstrap Caplet implied volatilities, starting with the first Cap maturity, using the assumption of piece-wise constant intervals and Equation 11 below.
3. After a Caplet volatility is found for all Caplet maturities at all quoted strikes, interpolate the volatility smile across strikes, this time iterating over all Caplet maturities.
4. Now a mesh of Caplet implied volatilities, consistent with market quotes, has been built for all strikes and maturities.

$$\begin{aligned}
 & Cap\ Price^{MKT}(0, T_{j+1}, K) - Cap\ Price^{MKT}(0, T_j, K) \\
 &= \sum_{T_i = T_j \text{ for each } \tau_i}^{T_{j+1} - \tau_j} Caplet\ Price(0, T_i, T_{i+1}, K)
 \end{aligned}$$

Equation 11: Equation for Bootstrapping Caplet Implied Volatilities

Equation 11 utilizes the additivity of Caplets to equate the difference between the prices of two adjacent Caps to the sum of the intermediate Caplets.

The second part of bootstrapping, using ATM data, uses these steps¹⁴:

1. Iterate over ATM Cap maturities.
2. For each maturity, find the term structure in the mesh of Caplet volatilities previously derived that corresponds to the strike rate of the particular ATM Cap.
3. Use this synthetic term structure to price a Cap with a maturity one quoting interval less than that of the ATM Cap being bootstrapped.
4. Finally, use Equation 11 to calculate an implied Caplet volatility, again piece-wise constant, which correctly prices the difference between the ATM quoted Cap and the synthetic Cap priced in step 3.

¹⁴ (Levin) Pg. 9

The results of part one are displayed below in Figure 5.

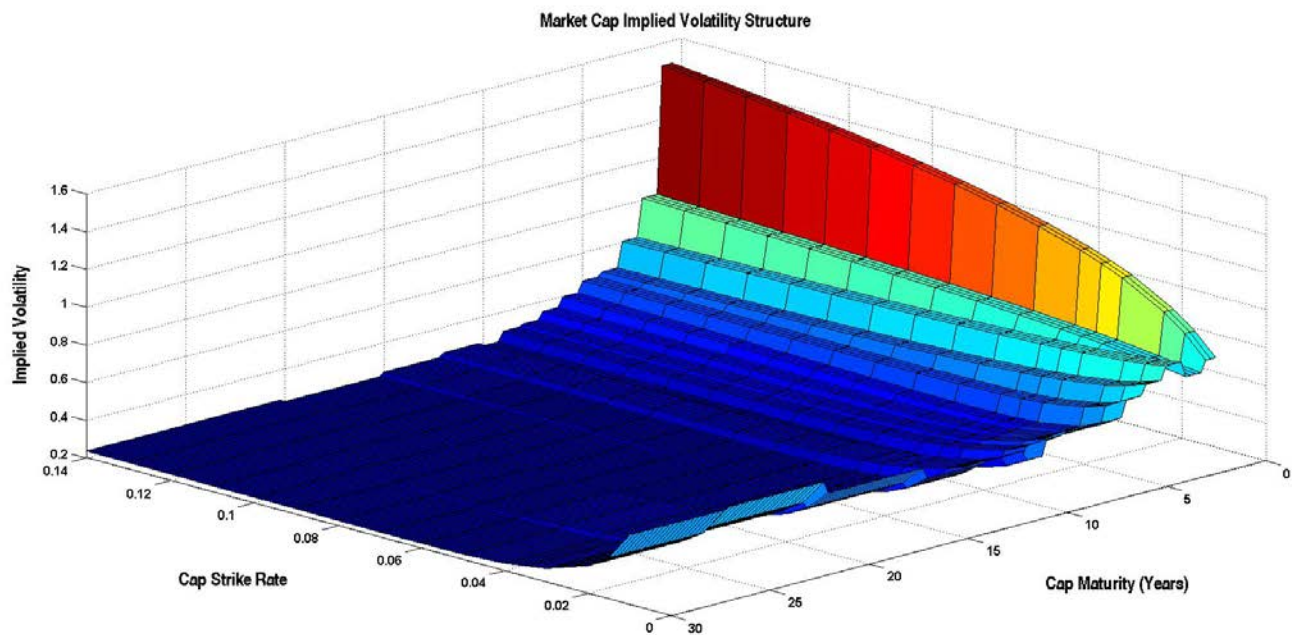


Figure 5: Market Cap Implied Volatility Surface

The volatility skew is the curvature in the strike dimension of the graph.

Now that the necessary market data has been extracted from quotes, the model can now be calibrated. The procedure general for calibration is the following: using the model parameters as input variables, minimize the objective function equal to the sum of squared errors between model and market prices. Under this problem specification, a variety of optimization schemes can be used to determine the model parameters. In this project, MATLAB's `fmincon` constrained optimization with an active-set algorithm was used.

Under the specific instantaneous volatility and correlation specifications made in this paper thus far, the following model parameters were used in the optimization: a , b , c , d , and β . Additionally, the primary focus of the project was the second of the two calibration goals stated at the beginning of this section. Therefore, the optimization was done to minimize error between model and market Swaption prices. A convenient feature of the instantaneous volatility Formulation 7 is its incorporation of the

time-invariant constants Φ_i . These constants allow the model to exactly fit the initial term structure of Caps by construction, regardless of the choice of model parameters. The tradeoff, as already highlighted, is that the evolution of the term structure over time tends to be less desirable.

A second formulation of the LFM was also implemented, using the first calibration goal to provide a comparison with the results of the first formulation. This formulation adds a time-variant term (Equation 12).¹⁵

$$\epsilon(t) = \left[\sum_{i=1}^3 \epsilon_i \sin\left(\frac{t\pi i}{Mat} + \epsilon_{i+1}\right) \right] e^{-\epsilon_7 t}$$

Equation 12: Time-Variant Instantaneous Volatility Component

In this equation, Mat , is the maturity in years of the longer tenor Cap.

Then the instantaneous volatility has the form of Equation 13.

$$\begin{aligned} \sigma_i(t) &= \Phi_i \psi(T_{i-1} - t; a, b, c, d) \epsilon(t) : \\ &= \Phi_i ([a(T_{i-1} - t) + d] e^{-b(T_{i-1} - t)} + c) \left[\sum_{i=1}^3 \epsilon_i \sin\left(\frac{t\pi i}{Mat} + \epsilon_{i+1}\right) \right] e^{-\epsilon_7 t} \end{aligned}$$

Equation 13: Rebonato's 6.21a Instantaneous Volatility Formulation

This formulation adds five additional, time-varying parameters ($\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_7$) for optimization of the model. The choice of sinusoidal functions for the time-variant term was motivated by Fourier analysis. Additionally, the number of frequencies is limited to balance having additional fit flexibility and having a too loosely specified model which will incorporate too much market noise.¹⁶ This formulation is then optimized with respect to Cap prices alone, without considering Swaptions. A close reader would question the purpose of this optimization, as the model is already guaranteed to exactly fit current market data. The motivation is to optimize the evolution of the term structure. Under the previous

¹⁵ (Rebonato) Pg. 233

¹⁶ (Rebonato) Pg. 169

formulation, the Φ_i constants are made whatever is necessary to fit Cap prices after the model is fit to Swaptions. As these Φ_i largely determine the model's transition over time, the resulting term structure can be highly erratic. In the present formulation however, the majority of the calibration "work" is done by the time-homogenous and time-variant portions of the instantaneous volatility formula. Therefore the Φ_i 's that arise from this formulation tend to be close to one, with much smaller variation. These values result in a much more behaved evolution.¹⁷ The second formulation utilized the same instantaneous correlation formula, but fixed β at 0.1 (see the Results for a justification of this choice).

The following section will give the calibration results of these two model formulations.

Results

For the code used to arrive at the following results, please consult the MATLAB Code Appendix.

As predicted, both formulations were able to precisely capture the current market Cap price, as measured by Black implied volatility. Figure 6 shows the market and model Cap implied volatilities for the Formulation 7. Figure 8 shows the same results for Rebonato's 6.21a formulation.

¹⁷ (Rebonato) Pg. 234

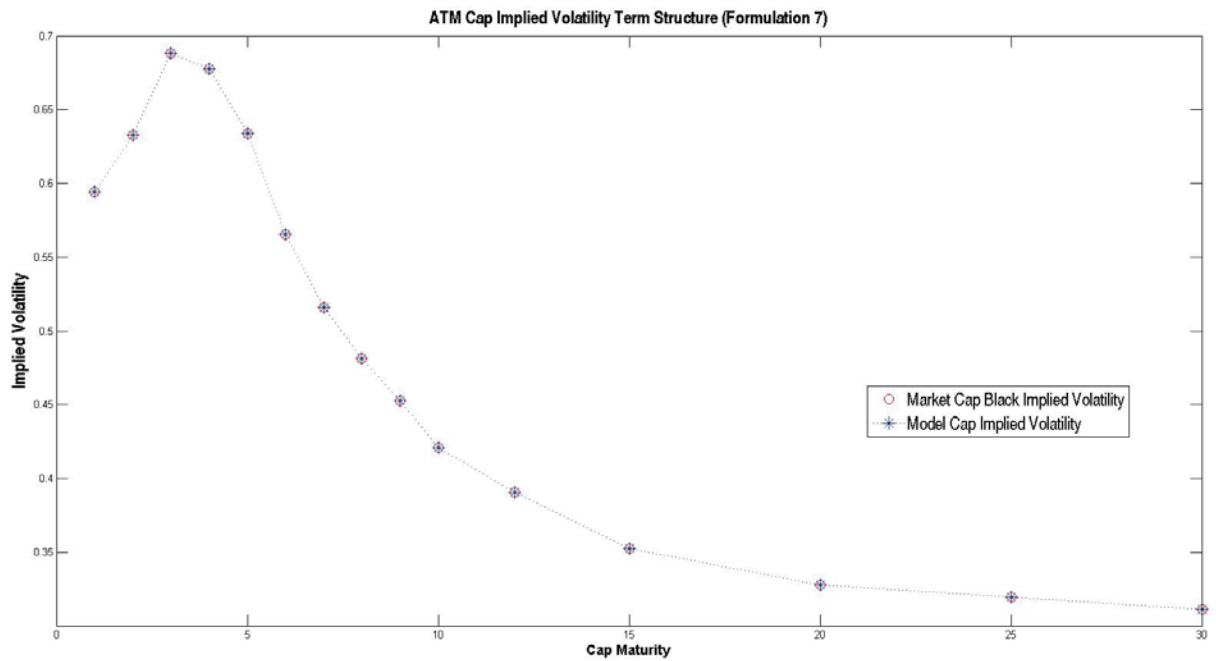


Figure 6: Formulation 7 Cap Implied Volatility Term Structure

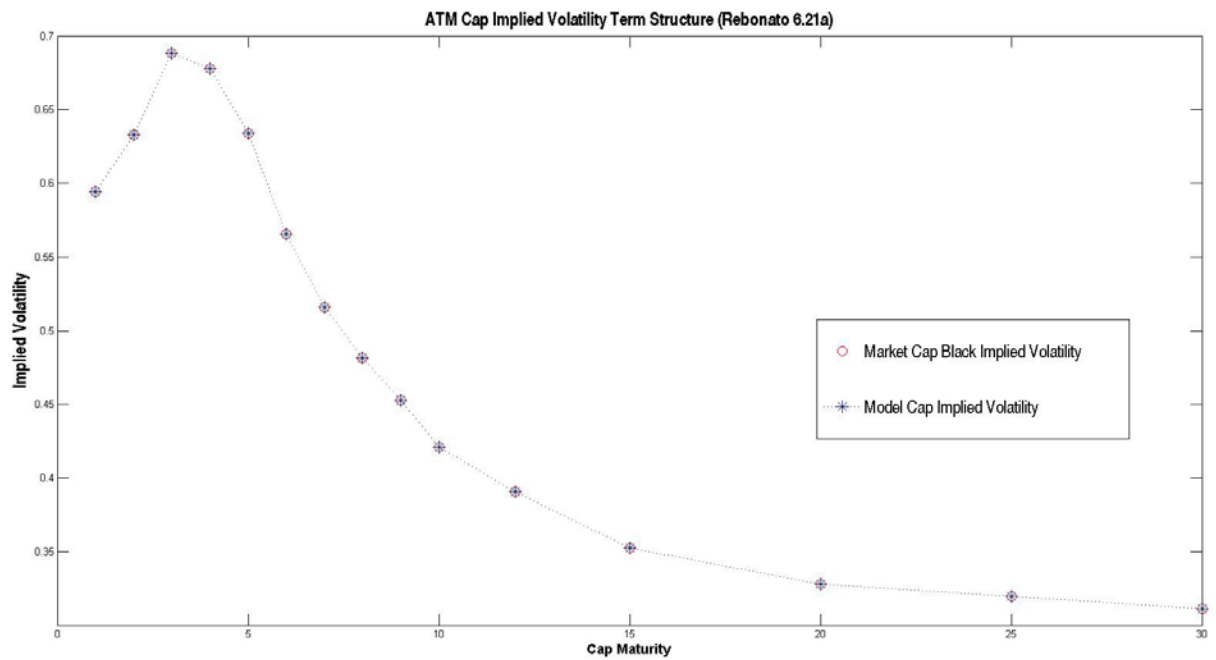


Figure 8: Rebonato's 6.21a Cap Implied Volatility Term Structure

The optimal model parameters for each formulation are summarized in Table 1.

Parameter	Formulation 7	Rebonato 6.21a
a	12.2690	-20
b	1.7798	6.3973
c	0.8290	1.3830
d	7.7659	0.6914
β	0.108	0.1 (Set)
ϵ_1	-	-0.3534
ϵ_2	-	2.1037
ϵ_3	-	1.4645
ϵ_4	-	3.8375
ϵ_7	-	0.1068

Table 1: Summary of Optimal Model Parameters

The above optimal values were found using the following constraints, shown in Table 2.¹⁸

Formulation 7	Rebonato 6.21a
$d + c > 0$	$d + c > 0$
$c, d > 0$	$c, d > 0$
$-20 < a, b, c, d < 20$	$-20 < a, b, c, d < 20$
$0.00005 < \beta < 0.108$	-
-	$-\infty < \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 < \infty$
-	$0.00000000001 < \epsilon_7 < \infty$

Table 2: Model Optimization Constraints

¹⁸ (Rebonato) Pg. 168 for the first two of these constraints.

The first two constraints come from Rebonato as recommended bounds for maintaining the shape of the instantaneous volatility function. The others are made to keep the optimization algorithm in the general region of desirable solutions. The objective functions are both non-linear, fairly “ugly” functions to optimize, therefore it is necessary to provide some guidance to the algorithm. The limits of β are implicitly recommended by Rebonato who states that the correlation of forward-rates should be kept reasonably high to maintain resemblance to the financial intuition underlying the model.¹⁹ As a general rule, β has historically been set ≈ 0.1 , thus I chose that value as the constant under my second formulation.²⁰ The resulting correlation structures are very similar for both formulations, therefore (Figure 9), the surface for Rebonato’s 6.21a, is shown to illustrate both.

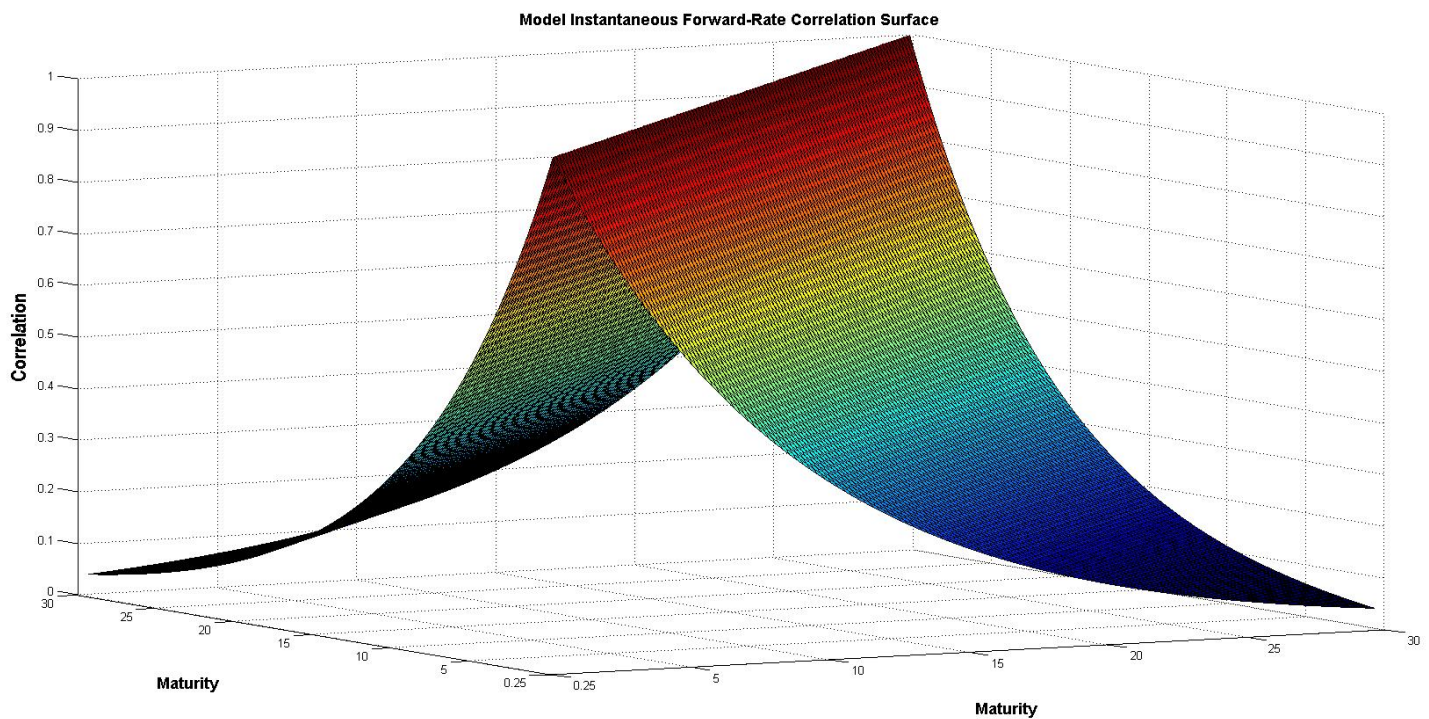


Figure 9: Model Instantaneous Forward-Rate Correlation Surface (Rebonato’s 6.21a Formulation)

¹⁹ (Rebonato) Pg. 222

²⁰ (Rebonato) Pg. 243

For each formulation, the calibrated evolution of Caplet implied volatilities, as shown by their Φ_i 's is depicted in (Figure 10) for Formulation 7 and 6.21a, respectively.

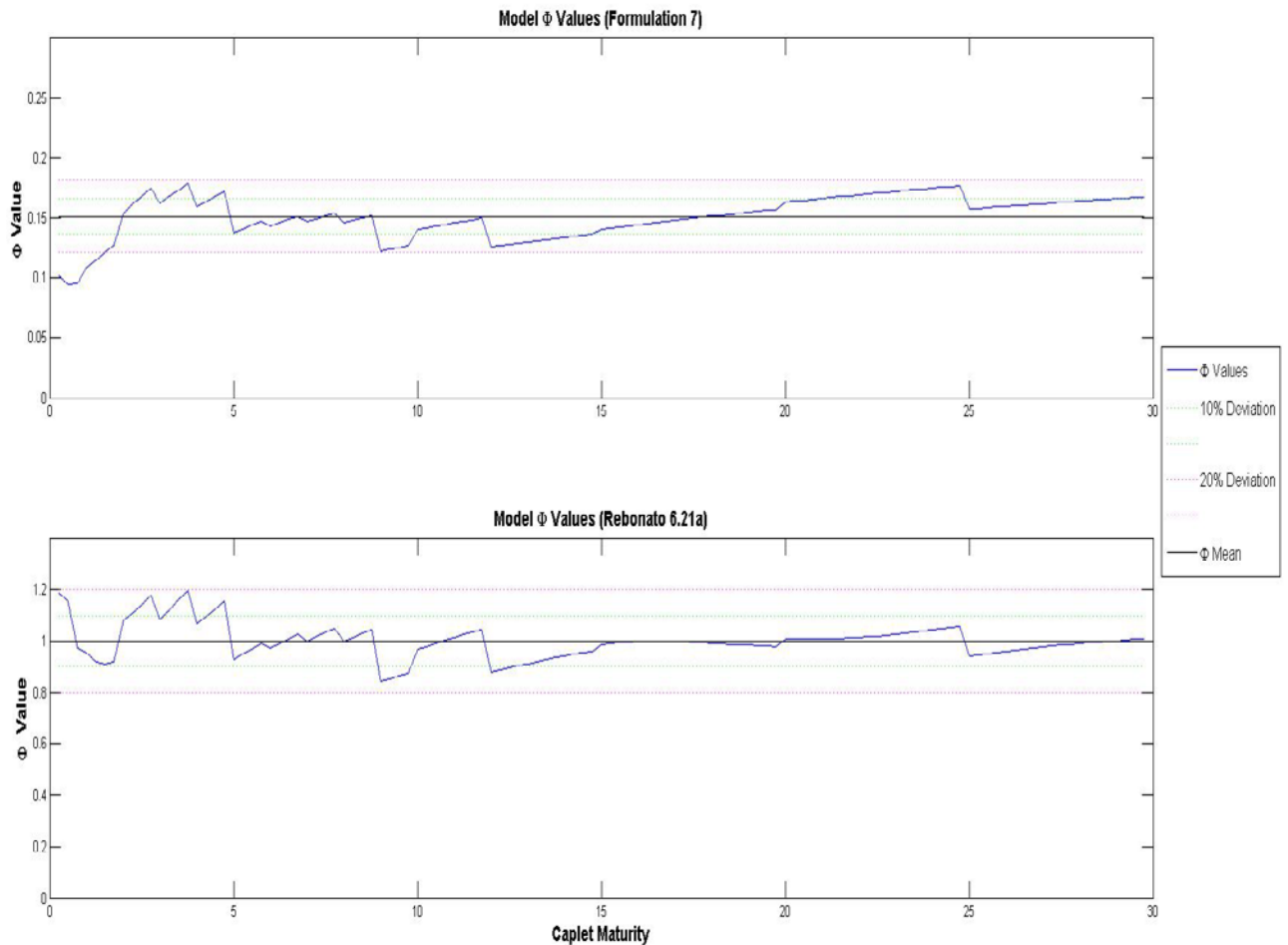


Figure 10: Model Φ Values for Formulation 7 (Top) and Rebonato 6.21a (Bottom)

As expected, the Φ_i values for formula 6.21a are both much closer to one (mean = 1.0002) and exhibit less percentage variation (with all values within 20% of the mean). Formulation 7 resulted in Φ_i 's within 40% of the mean (0.1513).

The Formulation 7 of the model was also fit to Swaption data. For this calibration, I used the three month expiry edge (across all quoted maturities) of the Swaption volatility surface. Figure 11 shows the model and market implied volatility curve, along with a chart of the error.

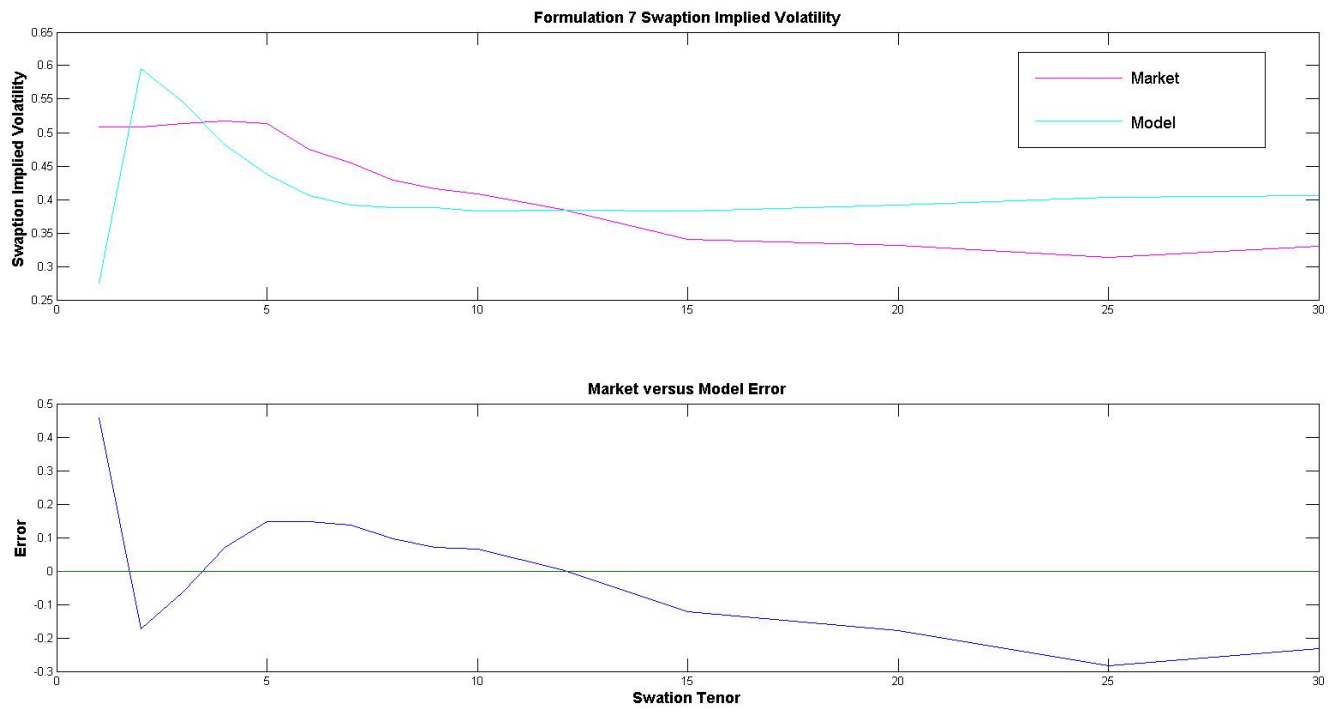


Figure 11: Formulation 7 Model Swaption Implied Volatility Term Structure

Finally, by relaxing the constraint on β to an upper limit of 20, better Swaption calibration was achieved.

This result is summarized in Table 3 and Figure 12 below.

Parameter	Formulation 7
a	20.000
b	0
c	0.0002
d	8.3989
β	19.999

Table 3: Formulation 7 Relaxed Optimization Parameters

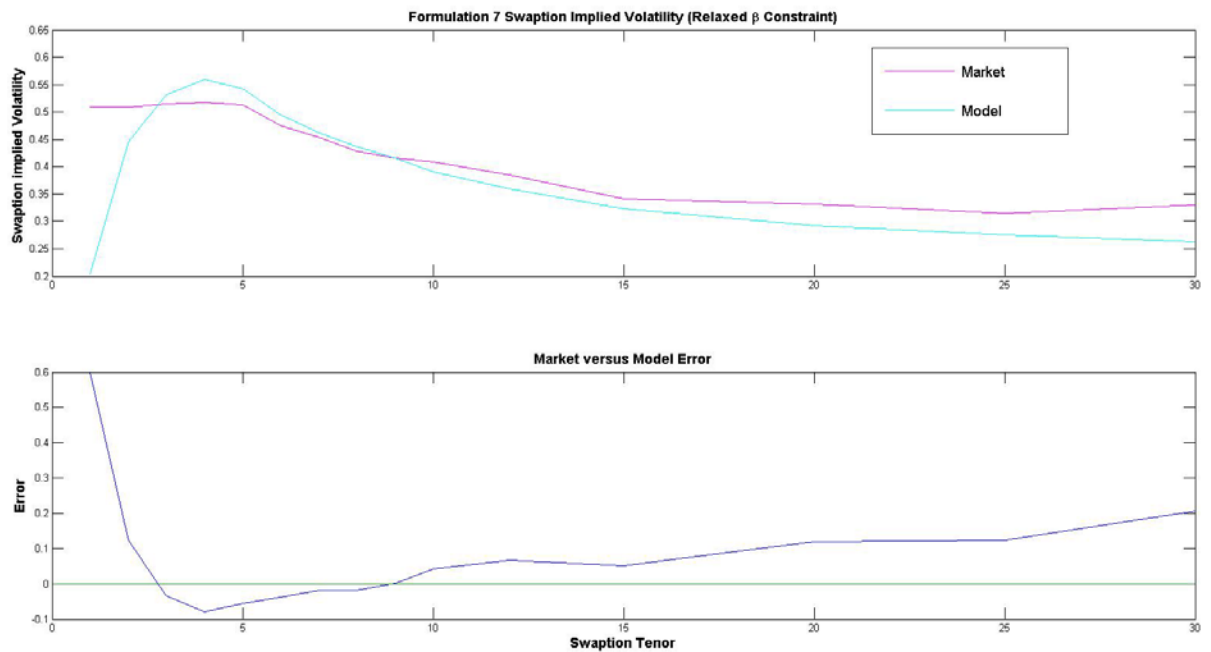


Figure 12: Formulation 7 Model Swaption Implied Volatility Term Structure (Relaxed β Constraint)

Analysis

While the process of model calibration is very much a science, consisting of a formulaic process, it is equally an art. In practice, models are not typically used to price products as liquid and relatively simple as Swaptions and Caps. Rather, models aid in the analysis of the risk and valuation of exotic derivatives and other one-off contracts for which market data is not available. Therefore, the decision of how to calibrate a model, and what it should be calibrated to, is largely dependent on what the end purpose of the model will be. For example, if the exotic derivative more closely resembles a Cap than a Swaption, it would be more important for the model to fit the Cap market than the Swaption price data. Other aspects of the art of calibration include the inconsistent quality of market data. For at-the-money Cap and Swaptions, and those in popular maturities, price data is usually high quality because they tend to be quoted often and by many traders. Therefore, judging which market points to consider is another important step for refining model fit. The paper does not endeavor to incorporate these market nuances, but they are worth noting for completeness.

Another aspect of optimizing calibration is the difficulty of finding globally optimal parameters in an acceptable value range. This paper used the `fmincon` MATLAB optimization framework with an active-set algorithm. This method was recommended by a practitioner as typically producing the best results. Something that deserves further exploration if this model were to be implemented for trading purposes is the use of other optimization algorithms that may produce better results for these non-linear, not “nice” objective functions. These optimizations also tend to be sensitive to the choice of seed values. This project’s optimization used potential optimal parameter values listed by Brigo and Mercurio as starting points for `fmincon`.²¹

The Lognormal Forward-LIBOR Model, and the LIBOR Market Model generally, appears promising based on the results of this implementation. The unique combination of theoretical rigor and correspondence to market practices makes the LFM an attractive framework for pricing interest-rate derivative products. The model performed very well with respect to Caps and satisfactory results were achieved for Swaptions. When the model was calibrated strictly to Caps, the resulting Φ_i values being close to one indicated that the model was capturing the term structure of Cap implied volatilities well and required minimal scaling. The calibration process is much more complex for Swaption than it is for Caps, and is expected to be less accurate. The Rebonato approximation discussed above was used to calculate Swaption implied volatilities that were compared with the market. This approximation has been proven to be fairly accurate, but it still presents a source of error. Additionally, with only five parameters, exact calibration to the Swaption market is impossible. That said, the model showed promising results which indicated that it could be refined using the considerations described above to be viable for pricing. The relaxation of the β constraint beyond economically sensible bounds illustrates the tension between the science and art of calibration; it is less theoretically sound, yet produces a better fit to the data. Additionally, model extensions, namely stochastic volatility, and the addition of

²¹ (Brigo and Mercurio) Pg. 320

new market developments, such as OIS (Overnight Indexed Swap) discounting and CVA (Credit Value Adjustment), show important steps for improving the fit of the LFM to market data going forward.

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