Certificate in Quantitative Finance

GLOBAL STANDARD IN FINANCIAL ENGINEERING

3 Differential Equations

3.1 Introduction

2 Types of Differential Equation (D.E)

(i) Ordinary Differential Equation (O.D.E)

Equation involving (ordinary) derivatives

$$x, \; y, \; \frac{dy}{dx}, \; \frac{d^2y}{dx^2}, \; \ldots\ldots.., \; \frac{d^ny}{dx^n} \quad \text{(some fixed } n\text{)}$$

y is some unknown function of x together with its derivatives, i.e.

$$F\left(x,\ y,\ y',\ y'',\ \ldots\ldots,\ y^{(n)}\right) = 0 \tag{1}$$

Note $y^4 \neq y^{(4)}$

Also if $y = y(t)$, where t is time, then we often write

$$\dot{y} = \frac{dy}{dt},\ \ \ddot{y} = \frac{d^2y}{dt^2},\ \ \ldots\ldots,\ \ddddot{y} = \frac{d^4y}{dt^4}$$

(ii) Partial Differential Equation (PDE)

Involve partial derivatives, i.e. unknown function dependent on two or more variables,

e.g.

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial u}{\partial z}-u=0$$

So here we solving for the unknown function $u(x, y, z, t)$.

More complicated to solve - better for modelling real-life situations, e.g. finance, engineering & science.

In quant finance there is no concept of spatial variables, unlike other branches of mathematics.

Order of the highest derivative is the **order of the DE**

An ode is of **degree** r if $\dfrac{d^{\,n}y}{dx^n}$ (where n is the order of the derivative) appears with power r

$\left(r\epsilon\mathbb{Z}^+\right)$ $-$ the definition of n and r is distinct. Assume that any ode has the property that each

$\dfrac{d^\ell y}{dx^\ell}$ appears in the form $\left(\dfrac{d^\ell y}{dx^\ell}\right)^r \longrightarrow \left(\dfrac{d^n y}{dx^n}\right)^r$ order n and degree r.

Examples:

	DE	order	degree
(1)	$y' = 3y$	1	1
(2)	$(y')^3 + 4\sin y = x^3$	1	3
(3)	$\left(y^{(4)}\right)^2 + x^2\left(y^{(2)}\right)^5 + (y')^6 + y = 0$	4	2
(4)	$y'' = \sqrt{y' + y + x}$	2	2
(5)	$y'' + x(y')^3 - xy = 0$	2	1

Note - example (4) can be written as $(y'')^2 = y' + y + x$

We will consider ODE's of degree one, and of the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+....+a_1(x)\frac{dy}{dx}+a_0(x)\,y=g(x)$$

$$\equiv \sum_{i=0}^{n} a_i(x)\,y^{(i)}(x)=g(x) \quad \text{(more pedantic)}$$

Note: $y^{(0)}(x)$ - zeroth derivative, i.e. $y(x)$.

This is a <u>Linear</u> ODE of order n, i.e. $r=1\ \forall$ (for all) terms. Linear also because $a_i(x)$ not a function of $y^{(i)}(x)$ - else equation is Non-linear.

Examples:

	DE	Nature of DE
(1)	$2xy'' + x^2y' - (x+1)\,y = x^2$	Linear
(2)	$yy'' + xy' + y = 2$	$a_2 = y \Rightarrow$ Non-Linear
(3)	$y'' + \sqrt{y'} + y = x^2$	Non-Linear $\because (y')^{\frac{1}{2}}$
(4)	$\dfrac{d^4y}{dx^4} + y^4 = 0$	Non-Linear - y^4

Our aim is to solve our ODE either explicitly or by finding the most general $y\,(x)$ satisfying it or implicitly by finding the function y implicitly in terms of x, via the most general function g s.t $g\,(x, y) = 0$.

Suppose that y is given in terms of x and n arbitrary constants of integration c_1, c_2, ..., c_n.

So $\widetilde{g}\,(x,\ c_1,\ c_2, \ldots\ldots., c_n) = 0$. Differentiating $\widetilde{g}$, n times to get $(n+1)$ equations involving

$$c_1,\ c_2, \ldots\ldots., c_n,\ x,\ y,\ y',\ y'',\ \ldots\ldots,\ y^{(n)}.$$

Eliminating $c_1,\ c_2, \ldots\ldots., c_n$ we get an ODE

$$\widetilde{f}\ \left(x,\ y,\ y',\ y'',\ \ldots\ldots,\ y^{(n)}\right) = 0$$

Examples:

(1) $y = x^3 + ce^{-3x}$ (so 1 constant c)

$\Rightarrow \dfrac{dy}{dx} = 3x^2 - 3ce^{-3x}$, so eliminate c by taking $3y + y' = 3x^3 + 3x^2$, i.e.

$$-3x^2\,(x+1) + 3y + y' = 0$$

(2) $y = c_1 e^{-x} + c_2 e^{2x}$ (2 constant's so differentiate twice)

$y' = -c_1 e^{-x} + 2c_2 e^{2x} \Rightarrow y'' = c_1 e^{-x} + 4c_2 e^{2x}$

Now

$$\left.\begin{array}{ll} y + y' = 3c_2 e^{2x} & \text{(a)} \\ y' + y'' = 6c_2 e^{2x} & \text{(b)} \end{array}\right\}$$

and 2(a)=(b) $\therefore$ 2 $(\, y + y') = y + y'' \rightarrow$

$$y'' - 2y' - y = 0.$$

Conversely it can be shown (under suitable conditions) that the general solution of an n$^{\text{th}}$ order ode will involve n arbitrary constants. If we specify values (i.e. boundary values) of

$$y,\ y',\ \ldots\ldots\ldots\ldots,\ y^{(n)}$$

for values of x, then the constants involved may be determined.

A solution $y = y(x)$ of (1) is a function that produces zero upon substitution into the lhs of (1).

Example:

$y'' - 3y' + 2y = 0$ is a 2^{nd} order equation and $y = e^x$ is a solution.

$y = y' = y'' = e^x$ - substituting in equation gives $e^x - 3e^x + 2e^x = 0$. So we can verify that a function is the solution of a DE simply by substitution.

Exercise:

(1) Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$ (c_1, c_2 arbitrary constants) a solution of $y'' + 4y = 0$

(2) Determine whether $y = x^2 - 1$ is a solution of $\left(\frac{dy}{dx}\right)^4 + y^2 = -1$

3.1.1 Initial & Boundary Value Problems

A DE together with conditions, an unknown function $y(x)$ and its derivatives, all given at the same value of independent variable x is called an **Initial Value Problem** (IVP).

e.g. $y'' + 2y' = e^x$; $y(\pi) = 1$, $y'(\pi) = 2$ is an IVP because both conditions are given at the same value $x = \pi$.

A **Boundary Value Problem** (BVP) is a DE together with conditions given at different values of x, i.e. $y'' + 2y' = e^x$; $y(0) = 1$, $y(1) = 1$.

Here conditions are defined at different values $x = 0$ and $x = 1$.

A solution to an IVP or BVP is a function $y(x)$ that both solves the DE and satisfies all given initial or boundary conditions.

Exercise: Determine whether any of the following functions

(a) $y_1 = \sin 2x$ (b) $y_2 = x$ (c) $y_3 = \frac{1}{2}\sin 2x$ is a solution of the IVP

$$y'' + 4y = 0; \quad y(0) = 0, \quad y'(0) = 1$$

3.2 First Order Ordinary Differential Equations

Standard form for a first order DE (in the unknown function $y(x)$) is

$$y' = f(x, y) \tag{2}$$

so given a 1st order ode

$$F\left(x, y, y'\right) = 0$$

can often be rearranged in the form (2), e.g.

$$xy' + 2xy - y = 0 \Rightarrow \quad y' = \frac{y - 2x}{x}$$

3.2.1 One Variable Missing

This is the simplest case

y missing:

$$y' = f(x) \qquad \text{solution is } y = \int f(x)dx$$

x missing:

$$y' = f(y) \qquad \text{solution is } x = \int \frac{1}{f(y)}dy$$

Example:

$$y' = \cos^2 y \ , \quad y = \frac{\pi}{4} \text{ when } x = 2$$

$$\Rightarrow x = \int \frac{1}{\cos^2 y}dy = \int \sec^2 y \ dy \Rightarrow x = \tan y + c \ ,$$

c is a constant of integration.

This is the general solution. To obtain a particular solution use

$$y\,(2) = \frac{\pi}{4} \rightarrow 2 = \tan\frac{\pi}{4} + c \Rightarrow c = 1$$

so rearranging gives

$$y = \arctan\,(x - 1)$$

3.2.2 Variable Separable

$$y' = g(x)\,h(y) \tag{3}$$

So $f(x, y) = g(x)\,h(y)$ where g and h are functions of x only and y only in turn. So

$$\frac{dy}{dx} = g(x)\,h(y) \rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx + c$$

$c -$ arbitrary constant.

Two examples follow on the next page:

$$\frac{dy}{dx} = \frac{x^2+2}{y}$$

$$\int y\ dy = \int \left(x^2+2\right) dx \rightarrow \frac{y^2}{2} = \frac{x^3}{3} + 2x + c$$

$$\frac{dy}{dx} = y \ln x \text{ subject to } y = 1 \text{ at } x = e \ (y(e) = 1)$$

$$\int \frac{dy}{y} = \int \ln x\ dx \quad \text{Recall: } \int \ln x\ dx = x(\ln x - 1)$$

$$\ln y = x(\ln x - 1) + c \rightarrow y = A \exp(x \ln x - x)$$

$A -$ arb. constant

now putting $x = e$, $y = 1$ gives $A = 1$. So solution becomes

$$y = \exp(\ln x^x)\exp(-x) \rightarrow y = \frac{x^x}{e^x} \Rightarrow y = \left(\frac{x}{e}\right)^x$$

3.2.3 Linear Equations

These are equations of the form

$$y' + P(x)\,y = Q(x) \tag{4}$$

which are similar to (3), but the presence of $Q(x)$ renders this no longer separable. We look for a function $R(x)$, called an **Integrating Factor** (I.F) so that

$$R(x)\ y' + R(x)P(x)\,y = \frac{d}{dx}(R(x)y)$$

So upon multiplying the lhs of (4), it becomes a derivative of $R(x)y$, i.e.

$$R\ y' + RPy = Ry' + R'y$$

from (4) .

This gives $RPy = R'y \Rightarrow R(x)P\left(x\right) = \dfrac{dR}{dx}$, which is a DE for R which is separable, hence

$$\int \frac{dR}{R} = \int Pdx + c \to \ln R = \int Pdx + c$$

So $R(x) = K \exp\left(\int P\ dx\right)$, hence there exists a function $R(x)$ with the required property. Multiply (4) through by $R(x)$

$$\underbrace{R(x)\left[y' + P(x)y\right]}_{=\frac{d}{dx}(R(x)y)} = R(x)Q(x)$$

$$\frac{d}{dx}(Ry) = R(x)Q(x) \to R(x)y = \int R(x)Q(x)dx + B$$

B – arb. constant.

We also know the form of $R(x) \to$

$$yK \exp\left(\int P\ dx\right) = \int K \exp\left(\int P\ dx\right) Q(x)dx + B.$$

Divide through by K to give

$$y\,\mathsf{exp}\left(\int P\ dx\right) = \int \mathsf{exp}\left(\int P\ dx\right) Q(x)dx + \text{ constant.}$$

So we can take $K = 1$ in the expression for $R(x)$.

To solve $y' + P\,(x)\,y = Q\,(x)$ calculate $\boxed{R(x) = \mathsf{exp}\left(\int P\ dx\right)}$, which is the I.F.

Examples:

1. Solve $y' - \frac{1}{x}y = x^2$

In this case c.f (4) gives $P(x) \equiv -\frac{1}{x}$ & $Q(x) \equiv x^2$, therefore

I.F $R(x) = \exp\left(\int -\frac{1}{x}\,dx\right) = \exp\left(-\ln x\right) = \frac{1}{x}$. Multiply DE by $\frac{1}{x} \to$

$$\begin{aligned}\frac{1}{x}\left(y' - \frac{1}{x}y\right) &= x \Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = x \to \int d\left(x^{-1}y\right) \\ &= \int x dx + c\end{aligned}$$

$$\Rightarrow \frac{y}{x} = \frac{x^2}{2} + c \therefore \text{GS is } y = \frac{x^3}{2} + cx$$

2. Obtain the general solution of $(1 + ye^x)\dfrac{dx}{dy} = e^x$

$$\frac{dy}{dx} = (1 + ye^x)\, e^{-x} = e^{-x} + y \Rightarrow$$

$$\frac{dy}{dx} - y = e^{-x}$$

Which is a linear equation, with $P = -1$; $Q = e^{-x}$

$$\text{I.F } R(y) = \exp\left(\int -dx\right) = e^{-x}$$

so multiplying DE by I.F

$$\begin{aligned} e^{-x}\left(y' - y\right) &= e^{-2x} \rightarrow \frac{d}{dx}\left(ye^{-x}\right) = e^{-2x} \Rightarrow \\ \int d\left(ye^{-x}\right) &= \int e^{-2x} dx \end{aligned}$$

$$ye^{-x} = -\frac{1}{2}e^{-2x} + c \therefore y = ce^x - \frac{1}{2}e^{-x} \text{ is the GS.}$$

3.3 Second Order ODE's

Typical second order ODE (degree 1) is

$$y'' = f\left(x, y, y'\right)$$

solution involves two arbitrary constants.

3.3.1 Simplest Cases

A y', y missing, so $\boxed{y'' = f\ (x)}$

Integrate wrt x (twice): $y = \int \left(\int f\ (x)\ dx\right) dx$

Example: $y'' = 4x$

$$\text{GS } y = \int \left(\int 4x \; dx \right) dx = \int \left[2x^2 + C \right] dx = \frac{2x^3}{3} + Cx + D$$

B y missing, so $\boxed{y'' = f \;\; (y', \; x)}$

Put $P = y' \to y'' = \dfrac{dP}{dx} = f \; (P, x)$, i.e. $P' = f \; (P, x)$ - first order ode

Solve once $\to P(x)$

Solve again $\to y(x)$

Example: Solve $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = x^3$

Note: $\boxed{\text{A}}$ is a special case of $\boxed{\text{B}}$

C y' and x missing, so $\boxed{y'' = f\ (y)}$

Put $p = y'$, then

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} \\ &= f\ (y) \end{aligned}$$

So solve 1st order ode

$$p\frac{dp}{dy} = f\,(y)$$

which is separable, so

$$\int p\ dp = \int\ f\ (\ y)\ dy \rightarrow$$

$$\frac{1}{2}p^2 = \int f(y)\,dy + \text{const.}$$

Example: Solve $y^3 y'' = 4$

$\Rightarrow y'' = \dfrac{4}{y^3}$. Put $p = y' \to \dfrac{d^2y}{dx^2} = p\dfrac{dp}{dy} = \dfrac{4}{y^3}$

$\therefore \int p\ dp = \displaystyle\int \frac{4}{y^3}\ dy \Rightarrow p^2 = -\frac{4}{y^2} + D \ \ \therefore\ \ p = \frac{\pm\sqrt{Dy^2 - 4}}{y}$, so from our definition of p,

$$\frac{dy}{dx} = \frac{\pm\sqrt{Dy^2 - 4}}{y} \Rightarrow \int dx = \int \frac{\pm\ y}{\sqrt{Dy^2 - 4}}dy$$

Integrate rhs by substitution (i.e. $u = Dy^2 - 4$) to give

$$x = \frac{\pm\sqrt{Dy^2 - 4}}{D} + E \rightarrow \left[D\left(x - E\right)^2\right] = Dy^2 - 4$$

$$\therefore \text{ GS is } Dy^2 - D^2\left(x - E\right)^2 = 4$$

D x missing: $\quad y'' = f\ \left(y',\ y\right)$

Put $P = y'$, so $\dfrac{d^2y}{dx^2} = P\dfrac{dP}{dy} = f\ \left(P, y\right)$ - 1$^{\text{st}}$ order ODE

3.3.2 Linear ODE's of Order at least 2

General n$^{\text{th}}$ order linear ode is of form:

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \ldots\ldots + a_1(x)\,y' + a_0(x)\,y = g(x)$$

Use symbolic notation:

$$D \equiv \frac{d}{dx}\;;\;\; D^r \equiv \frac{d^{\,r}}{dx^{\,r}} \quad \text{so } D^r y \equiv \frac{d^{\,r} y}{dx^{\,r}}$$

$\therefore\;\; a_r D^r \equiv a_r(x)\dfrac{d^{\,r}}{dx^{\,r}}$ so

$$a_r D^r y = a_r(x)\frac{d^{\,r} y}{dx^{\,r}}$$

Now introduce

$$L = a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots\ldots\ldots\ldots + a_1 D + a_0$$

so we can write a linear ode in the form

$$L\;y = g$$

$L-$ <u>Linear Differential Operator of order n</u> and its definition will be used throughout.

If $g(x) = 0\ \forall x$, then $L\ y = 0$ is said to be **HOMOGENEOUS**.

$L\ y = 0$ is said to be the homogeneous part of $L\ y = g$.

L is a linear operator because as is trivially verified:

(1) $L(y_1 + y_2) = L\ (y_1) + L(y_2)$

(2) $L(cy) = cL(y) \qquad c \in \mathbb{R}$

GS of $Ly = g$ is given by

$$\boxed{y = y_c + y_p}$$

where y_c- Complimentary Function & y_p- Particular Integral (or Particular Solution)

$$\left.\begin{array}{l} y_c \text{ is solution of } \; Ly = 0 \\ y_p \text{ is solution of } \; Ly = g \end{array}\right\} \therefore \text{ GS } y = y_c + y_p$$

Look at homogeneous case $Ly = 0$. Put Ⓢ $=$ all solutions of $Ly = 0$. Then Ⓢ forms a vector space of dimension n. Functions $y_1(x)$, ..., $y_n(x)$ are LINEARLY DEPENDENT if $\exists\ \lambda_1$, ..., $\lambda_n \in \mathbb{R}$ (not all zero) s.t

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) \; + \ldots\ldots\ldots\ldots + \; \lambda_n y_n(x) = 0$$

Otherwise y_i's ($i = 1$, ..., n) are said to be LINEARLY INDEPENDENT (Lin. Indep.) $\Rightarrow$ whenever

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) \; + \ldots\ldots\ldots\ldots + \; \lambda_n y_n(x) = 0 \;\; \forall x$$

then $\lambda_1 = \lambda_2 =$ .... $= \lambda_n = 0$.

FACT:

(1) $L-$ n^{th} order linear operator, then $\exists$ n Lin. Indep. solutions y_1, ..., y_n of $Ly = 0$ s.t GS of $Ly = 0$ is given by

$$y = \lambda_1 y_1 + \lambda_2 y_2 + \dots\dots\dots + \lambda_n y_n \qquad \underset{1 \le i \le n}{\lambda_i \in \mathbb{R}} .$$

(2) Any n Lin. Indep. solutions of $Ly = 0$ have this property.

To solve $Ly = 0$ we need only find by "hook or by crook" n Lin. Indep. solutions.

3.3.3 Linear ODE's with Constant Coefficients

Consider Homogeneous case: $Ly = 0$.

All basic features appear for the case $n = 2$, so we analyse this.

$$L\ y = a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad a,\ b,\ c \in \mathbb{R}$$

Try a solution of the form $\qquad y = \exp(\lambda x)$

$$L\left(e^{\lambda x}\right) = \left(aD^2 + bD + c\right)e^{\lambda x}$$

hence $a\lambda^2 + b\lambda + c = 0$ and so λ is a root of the quadratic equation

$a\lambda^2 + b\lambda + c = 0 \qquad$ **AUXILLIARY EQUATION (A.E)**

There are three cases to consider:

(1) $b^2 - 4ac > 0$

So $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, so GS is

$$\boxed{y = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x)}$$

c_1, c_2 – arb. const.

(2) $b^2 - 4ac = 0$

So $\lambda = \lambda_1 = \lambda_2 = -\dfrac{b}{2a}$

Clearly $e^{\lambda x}$ is a solution of $L\ y = 0$ - but theory tells us there exist two solutions for a 2nd order ode. So now try $y = x\,\mathsf{exp}\,(\lambda x)$

$$\begin{aligned} L\left(xe^{\lambda x}\right) &= \left(aD^2+bD+c\right)\left(xe^{\lambda x}\right) \\ &= \underbrace{\left(a\lambda^2+b\lambda+c\right)}_{=0}\left(xe^{\lambda x}\right)+\underbrace{(2a\lambda+b)}_{=0}\left(e^{\lambda x}\right) \\ &= 0 \end{aligned}$$

This gives a 2nd solution $\therefore$ GS is $y = c_1\,\mathsf{exp}\,(\lambda x) + c_2 x\,\mathsf{exp}\,(\lambda x)$, hence

$$\boxed{y = (c_1 + c_2 x)\,\mathsf{exp}\,(\lambda x)}$$

(3) $b^2 - 4ac < 0$

So $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ - Complex conjugate pair $\lambda = p \pm iq$ where

$$p = -\frac{b}{2a}, \qquad q = \frac{1}{2a}\sqrt{\left|b^2-4ac\right|} \quad (\neq 0)$$

Hence

$$\begin{aligned} y &= c_1 \exp(p+iq)x + c_2 \exp(p-iq)x \\ &= c_1 e^{px} e^{iq} + c_2 e^{px} e^{-iq} = e^{px}\left(c_1 e^{iqx} + c_2 e^{-iqx}\right) \end{aligned}$$

Eulers identity gives $\exp(\pm i\theta) = \cos\theta \pm i \sin\theta$

Simplifying (using Euler) then gives the GS

$$y(x) = e^{px}(A\cos qx + B\sin qx)$$

Examples:

(1) $y'' - 3y' - 4y = 0$

Put $y = e^{\lambda x}$ to obtain A.E

A.E: $\lambda^2 - 3\lambda - 4 = 0 \rightarrow (\lambda - 4)(\lambda + 1) = 0 \qquad \Rightarrow \lambda = 4 \ \& \ -1$ - 2 distinct $\mathbb{R}$ roots

$$\text{GS} \quad y(x) = Ae^{4x} + Be^{-x}$$

(2) $y'' - 8y' + 16y = 0$

A.E $\quad \lambda^2 - 8\lambda + 16 = 0 \to (\lambda - 4)^2 = 0 \Rightarrow \lambda = 4\ ,\ 4$ (2 fold root)

'go up one', i.e. instead of $y = e^{\lambda x}$, take $y = xe^{\lambda x}$

$$\text{GS} \quad y\,(x) = (C + Dx)\,e^{4x}$$

(3) $y'' - 3y' + 4y = 0$

A.E: $\lambda^2 - 3\lambda + 4 = 0 \to \lambda = \dfrac{3 \pm \sqrt{9 - 16}}{2} = \dfrac{3 \pm i\sqrt{7}}{2} \equiv p \pm iq$

$\left(p = \dfrac{3}{2},\ \ q = \dfrac{\sqrt{7}}{2}\right)$

$$y = e^{\frac{3}{2}x}\left(a\cos\frac{\sqrt{7}}{2}x + b\sin\frac{\sqrt{7}}{2}x\right)$$

3.4 General n^{th} Order Equation

Consider

$$Ly = a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

then

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + ... + a_1 D + a_0$$

so $Ly = 0$ and the A.E becomes

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + ... + a_1 \lambda + a_0 = 0$$

Case 1 (Basic)

n distinct roots λ_1, ..., λ_n then $e^{\lambda_1 x}$, $e^{\lambda_2 x}$, ..., $e^{\lambda_n x}$ are n Lin. Indep. solutions giving a GS

$$y = \beta_1 e^{\lambda_1 x} + \beta_2 e^{\lambda_2 x} + \ldots\ldots + \beta_n e^{\lambda_n x}$$

β_i- arb.

Case 2

If λ is a real $r-$ fold root of the A.E then $e^{\lambda x}$, $xe^{\lambda x}$, $x^2 e^{\lambda x}$,.........., $x^{r-1}e^{\lambda x}$ are r Lin. Indep. solutions of $Ly = 0$, i.e.

$$y = e^{\lambda x}\left(\alpha_1 + \alpha_2 x + \alpha_3 x^2 \ldots\ldots + \alpha_r\, x^{r-1}\right)$$

α_i- arb.

Case 3

If $\lambda = p + iq$ is a r - fold root of the A.E then so is $p - iq$

$$\left.\begin{array}{c} e^{px}\cos qx,\ \ xe^{px}\cos qx,\ \ldots\ldots\ldots\ldots, x^{r-1}e^{px}\cos qx \\ e^{px}\sin qx,\ xe^{px}\sin qx,\ \ldots\ldots\ldots\ldots, x^{r-1}e^{px}\sin qx \end{array}\right\}$$

$\longrightarrow$ $2r$ Lin. Indep. solutions of $L\ y = 0$

$$\begin{aligned} \text{GS } y &= e^{px}\left(c_1 + c_2x + c_3x^2 + \ldots\ldots\ldots\ldots\right)\cos qx + \\ & \quad e^{px}\left(C_1 + C_2x + C_3x^2 + \ldots\ldots\ldots\ldots\right)\sin qx \end{aligned}$$

Examples: Find the GS of each ODE

(1) $y^{(4)} - 5y'' + 6y = 0$

A.E: $\lambda^4 - 5\lambda^2 + 6 = 0 \rightarrow \left(\lambda^2 - 2\right)\left(\lambda^2 - 3\right) = 0$

So $\lambda = \pm\sqrt{2}$, $\lambda = \pm\sqrt{3}$ - four distinct roots

$\therefore$ GS $y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x} + Ce^{\sqrt{3}x} + De^{-\sqrt{3}x}$ (Case 1)

(2) $\dfrac{d^{\,6}y}{dx^6} - 5\dfrac{d^{\,4}y}{dx^4} = 0$

A.E: $\lambda^6 - 5\lambda^4 = 0$ roots: 0, 0, 0, 0, $\pm\sqrt{5}$

GS $y = Ae^{\sqrt{5}x} + Be^{-\sqrt{5}x} + \left(C + Dx + Ex^2 + Fx^3\right)$ ($\because \exp(0) = 1$)

(3) $\dfrac{d^{\,4}y}{dx^4} + 2\dfrac{d^{\,2}y}{dx^2} + y = 0$

A.E: $\lambda^4 + 2\lambda^2 + 1 = \left(\lambda^2 + 1\right)^2 = 0$ $\quad\lambda = \pm i$ is a 2 fold root.

Example of Case (3)

$$y = A\cos x + Bx\cos x + C\sin x + Dx\sin x$$

3.5 Non-Homogeneous Case - Method of Undetermined Coefficients

$$\text{GS} \quad y = \text{C.F} + \text{P.I}$$

C.F comes from the roots of the A.E

There are three methods for finding P.I

(a) "Guesswork" - which we are interested in

(b) Annihilator

(c) D-operator Method

(a) Guesswork Method

If the rhs of the ode $g(x)$ is of a certain type, we can guess the form of P.I. We then try it out and determine the numerical coefficients.

The method will work when $g(x)$ has the following forms

i. Polynomial in x $\quad g(x) = p_0 + p_1 x + p_2 x^2 + \ldots\ldots\ldots + p_m x^m$.

ii. An exponential $\quad g(x) = Ce^{kx}$ $\quad$ (Provided k is <u>not</u> a root of A.E).

iii. Trigonometric terms, $g(x)$ has the form $\sin ax$, $\cos ax$ $\quad$ (Provided ia is <u>not</u> a root of A.E).

iv. $g(x)$ is a combination of i. , ii. , iii. provided $g(x)$ does not contain part of the C.F (in which case use other methods).

$\boxed{\textbf{Examples:}}$

(1) $y'' + 3y' + 2y = 3e^{5x}$

The homogeneous part is the same as in (1), so $y_C = Ae^{-x} + Be^{-2x}$. For the non-homog. part we note that $g(x)$ has the form e^{kx}, so try $y_p = Ce^{5x}$, and $k = 5$ is not a solution of the A.E.

Substituting y_p into the DE gives

$$C\left(5^2 + 15 + 2\right)e^{5x} = 3e^{5x} \ \rightarrow C = \frac{1}{14}$$

$$\therefore y = Ae^{-x} + Be^{-2x} + \frac{1}{14}e^{5x}$$

(2) $y'' + 3y' + 2y = x^2$

GS $y =$ C.F + P.I $= y_c + y_p$

C.F: A.E gives

$$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -1,\ -2 \therefore y_c = ae^{-x} + be^{-2x}$$

P.I Now $g(x) = x^2$,

so try $y_p = p_0 + p_1 x + p_2 x^2 \quad \rightarrow y_p' = p_1 + 2p_2 x \quad \rightarrow y_p'' = 2p_2$

Now substitute these in to the DE, ie

$2p_2 + 3(p_1 + 2p_2 x) + 2\left(p_0 + p_1 x + p_2 x^2\right) = x^2$ and equate coefficients of x^n

$O\left(x^2\right)$: $\quad 2p_2 = 1 \Rightarrow p_2 = \frac{1}{2}$

$$O\left(x\right): \qquad 6p_2 + 2p_1 = 0 \Rightarrow p_1 = -\tfrac{3}{2}$$

$$O\left(x^0\right): \qquad 2p_2 + 3p_1 + 2p_0 = 0 \Rightarrow p_0 = \tfrac{7}{4}$$

$$\therefore \text{GS} \quad y = ae^{-x} + be^{-2x} + \frac{7}{4} - \frac{3}{2}x + \frac{1}{2}x^2$$

(3) $y'' - 5y' - 6y = \cos 3x$

A.E: $\lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = -1,\ 6 \Rightarrow y_c = \alpha e^{-x} + \beta e^{6x}$

Guided by the rhs, i.e. $g(x)$ is a trigonometric term, we can try $y_p = A\cos 3x + B\sin 3x$,and calculate the coefficients A and B.

How about a more sublime approach? Put $y_p = \mathrm{Re}\, K e^{i3x}$ for the unknown coefficient K.

$\rightarrow\ y_p' = 3\,\mathrm{Re}\, iKe^{i3x} \rightarrow\ {y_p}'' = -9\,\mathrm{Re}\, Ke^{i3x}$ and substitute into the DE, dropping Re

$$\begin{aligned} (-9 - 15i - 6)\, Ke^{i3x} &= e^{i3x} \\ -15\,(1+i)\, K &= 1 \\ -15K &= \frac{1}{1+i} \longrightarrow K = \frac{1}{2}(1-i) \end{aligned}$$

Hence $K = -\frac{1}{30}(1 - i)$ to give

$$\begin{aligned} y_p &= -\frac{1}{30}\,\mathrm{Re}\,(1-i)(\cos 3x + i\sin 3x) \\ &= -\frac{1}{30}(\cos 3x + i\sin 3x - i\cos 3x + \sin 3x) \end{aligned}$$

so general solution becomes

$$y = \alpha e^{-x} + \beta e^{6x} - \frac{1}{30}(\cos 3x + \sin 3x)$$

3.5.1 Failure Case

Consider the DE $y'' - 5y' + 6y = e^{2x}$, which has a CF given by $y(x) = \alpha e^{2x} + \beta e^{3x}$. To find a PI, if we try $y_p = Ae^{2x}$, we have upon substitution

$$Ae^{2x}[4 - 10 + 6] = e^{2x}$$

so when k $(= 2)$ is also a solution of the C.F , then the trial solution $y_p = Ae^{kx}$ fails, so we must seek the existence of an alternative solution.

$Ly = y'' + ay' + b = \alpha e^{kx}$ - trial function is normally $y_p = Ce^{kx}$.

If k is a root of the A.E then $L\left(Ce^{kx}\right) = 0$ so this substitution does not work. In this case, we try $y_p = Cxe^{kx}$ - so 'go one up'.

This works provided k is not a repeated root of the A.E, if so try $y_p = Cx^2e^{kx}$, and so forth ....

3.6 Linear ODE's with Variable Coefficients - Euler Equation

In the previous sections we have looked at various second order DE's with constant coefficients. We now introduce a 2^{nd} order equation in which the coefficients are variable in x. An equation of the form

$$L\ y = ax^2\frac{d^2y}{dx^2} + \beta x\frac{dy}{dx} + cy = g(x)$$

is called a Cauchy-Euler equation. Note the relationship between the coefficient and corresponding derivative term, ie $a_n(x) = ax^n$ and $\dfrac{d^{\,n}y}{dx^n}$, i.e. both power and order of derivative are n.

The equation is still linear. To solve the homogeneous part, we look for a solution of the form

$$y = x^{\lambda}$$

So $y' = \lambda x^{\lambda - 1} \rightarrow y'' = \lambda (\lambda - 1) x^{\lambda - 2}$, which upon substitution yields the quadratic, A.E.

$$a\lambda^2 + b\lambda + c = 0$$

[where $b = (\beta - a)$] which can be solved in the usual way - there are 3 cases to consider, depending upon the nature of $b^2 - 4ac$.

$\boxed{\text{Case 1:}}$ $b^2 - 4ac > 0 \to \lambda_1,\ \lambda_2 \in \mathbb{R}$ - 2 real distinct roots

$$\text{GS } y = Ax^{\lambda_1} + Bx^{\lambda_2}$$

$\boxed{\text{Case 2:}}$ $b^2 - 4ac = 0 \to \lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$ - 1 real (double fold) root

$$\text{GS } y = x^{\lambda}\left(A + B\ln x\right)$$

$\boxed{\text{Case 3:}}$ $b^2 - 4ac < 0 \to \lambda = \alpha \pm i\beta \in \mathbb{C}$ - pair of complex conjugate roots

$$\text{GS } y = x^{\alpha}\left(A\cos\left(\beta\ln x\right) + B\sin\left(\beta\ln x\right)\right)$$

$\boxed{\text{Example 1}}$ Solve $\qquad x^2y'' - 2xy' - 4y = 0$

Put $y = x^\lambda \Rightarrow y' = \lambda x^{\lambda-1} \Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2}$ and substitute in DE to obtain (upon simplification) the A.E. $\quad \lambda^2 - 3\lambda - 4 = 0 \rightarrow (\lambda-4)(\lambda+1) = 0$

$\Rightarrow \lambda = 4$ & -1 : 2 distinct $\mathbb{R}$ roots. So GS is

$$y(x) = Ax^4 + Bx^{-1}$$

$\boxed{\text{Example 2}}$ Solve $\qquad x^2y'' - 7xy' + 16y = 0$

So assume $y = x^\lambda$

A.E $\qquad \lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda = 4\ ,\ 4$ (2 fold root)

'go up one', i.e. instead of $y = x^\lambda$, take $y = x^\lambda \ln x$ to give

$$y(x) = x^4(A + B\ln x)$$

$\boxed{\text{Example 3}}$ Solve $\qquad x^2y'' - 3xy' + 13y = 0$

Assume existence of solution of the form $y = x^\lambda$

A.E becomes $\qquad \lambda^2 - 4\lambda + 13 = 0 \rightarrow \lambda = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm 6i}{2}$

$\lambda_1 = 2 + 3i, \;\; \lambda_2 = 2 - 3i \;\; \equiv \alpha \pm i\beta \qquad (\alpha = 2, \;\; \beta = 3)$

$$y = x^2 \left(A \cos\left(3 \ln x\right) + B \sin\left(3 \ln x\right)\right)$$

3.6.1 Reduction to constant coefficient

The Euler equation considered above can be reduced to the constant coefficient problem discussed earlier by use of a suitable transform. To illustrate this simple technique we use a specific example.

Solve $\quad x^2y'' - xy' + y = \ln x$

Use the substitution $x = e^t$ i.e. $t = \ln x$. We now rewrite the the equation in terms of the variable t, so require new expressions for the derivatives (chain rule):

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\begin{aligned}\frac{d^{\,2}y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dx}\frac{dy}{dt} - \frac{1}{x^2}\frac{dy}{dt} \\ &= \frac{1}{x}\frac{dt}{dx}\frac{d}{dt}\frac{dy}{dt} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\end{aligned}$$

$\therefore$ the Euler equation becomes

$$x^2\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right) - x\left(\frac{1}{x}\frac{dy}{dt}\right) + y = t \qquad \longrightarrow$$

$$y''(t) - 2y'(t) + y = t$$

The solution of the homogeneous part , ie C.F. is $y_c = e^t(A + Bt)$.

The particular integral (P.I.) is obtained by using $y_p = p_0 + p_1 t$ to give $y_p = 2 + t$

The GS of this equation becomes

$$y(t) = e^{t}(A + Bt) + 2 + t$$

which is a function of t . The original problem was $y = y(x)$, so we use our transformation $t = \ln x$ to get the GS

$$y = x(A + B\ln x) + 2 + \ln x.$$

3.7 Partial Differential Equations

The formation (and solution) of PDE's forms the basis of a large number of mathematical models used to study physical situations arising in science, engineering and medicine.

More recently their use has extended to the modelling of problems in finance and economics.

We now look at the second type of DE, i.e. PDE's. These have partial derivatives instead of ordinary derivatives.

One of the underlying equations in finance, the Black-Scholes equation for the price of an option $V(S,t)$ is an example of a linear PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = 0$$

providing σ, D, r are not functions of V or any of its derivatives.

If we let $u = u(x, y)$, then the general form of a linear 2nd order PDE is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where the coefficients A, ..., G are functions of x & y.

When

$$G(x, y) = \begin{cases} 0 & \quad\quad (1) \text{ is homogeneous} \\ \text{non-zero} & \quad\quad (1) \text{ is non-homogeneous} \end{cases}$$

hyperbolic $\quad B^2 - 4AC > 0$

parabolic $\quad B^2 - 4AC = 0$

elliptic $\quad B^2 - 4AC < 0$

In the context of mathematical finance we are only interested in the 2nd type, i.e. parabolic.

There are several methods for obtaining solutions of PDE's.

We look at a simple (but useful) technique:

3.7.1 Method of Separation of Variables

Without loss of generality, we solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{*}$$

for the unknown function $u(x,t)$.

In this method we assume existence of a solution which is a product of a function of x (only) and a function of y (only). So the form is

$$u(x,t) = X(x)\,T(t).$$

We substitute this in (*), so

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}(XT) = XT' \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(XT)\right) = \frac{\partial}{\partial x}\left(X'T\right) = X''T
\end{aligned}$$

Therefore (*) becomes

$$X\ T\ ' = c^2 X\ '' \ T$$

dividing through by $c^2 X\ T$ gives

$$\frac{T'}{c^2 T} = \frac{X''}{X}.$$

The RHS is independent of t and LHS is independent of x.

So each equation must be a constant. The convention is to write this constant as λ^2 or $-\lambda^2$.

There are possible cases:

Case 1: $\lambda^2 > 0$

$$\frac{T\ '}{c^2 T} = \frac{X\ ''}{X} = \lambda^2 \text{ leading to } \left.\begin{array}{c} T\ ' - \lambda^2 c^2 T = 0 \\ X\ '' - \lambda^2 X = 0 \end{array}\right\}$$

which have solutions, in turn

$$\left.\begin{array}{r} T\left(t\right)=k\exp\left(c^2\lambda^2 t\right) \\ X\left(x\right)=A\cosh\left(\lambda x\right)+B\sinh\left(\lambda x\right) \end{array}\right\}$$

So solution is

$$u\left(x,t\right)=X\ T=k\exp\left(c^2\lambda^2 t\right)\left\{A\cosh\left(\lambda x\right)+B\sinh\left(\lambda x\right)\right\}$$

Therefore $\qquad u=\exp\left(c^2\lambda^2 t\right)\left\{\alpha\cosh\left(\lambda x\right)+\beta\sinh\left(\lambda x\right)\right\}$

$(\alpha = Ak; \quad \beta = Bk)$

Case 2: $-\lambda^2 < 0$

$$\frac{T\,'}{c^2 T} = \frac{X\,''}{X} = -\lambda^2 \;\text{ which gives }\; \left.\begin{array}{c} T\,' + \lambda^2 c^2 T = 0 \\ X\,'' + \lambda^2 X = 0 \end{array}\right\}$$

resulting in the solutions

$$\left.\begin{array}{c} T = \overline{k}\,\exp\left(-c^2\lambda^2 t\right) \\ X = \overline{A}\cos(\lambda x) + \overline{B}\sin(\lambda x) \end{array}\right\}$$

respectively.

Hence

$$u(x,t) = \exp\left(-c^2\lambda^2 t\right)\{\gamma\cos(\lambda x) + \delta\sin(\lambda x)\}$$

where $\left(\gamma = \overline{k}\overline{A};\quad \delta = \overline{k}\overline{B}\right)$.

<u>Case 3</u>: $\lambda^2 = 0$

$$\left.\begin{array}{c} T\,' = 0 \\ X\,'' = 0 \end{array}\right\} \longrightarrow \left.\begin{array}{c} T\,(t) = \widetilde{A} \\ X = \widetilde{B}x + \widetilde{C} \end{array}\right\}$$

which gives the simple solution

$$u\,(x, y) = \widehat{A}x + \widehat{C}$$

where $\left(\widehat{A} = \widetilde{A}\widetilde{B}; \quad \widehat{C} = \widetilde{B}\widetilde{C}\right)$.