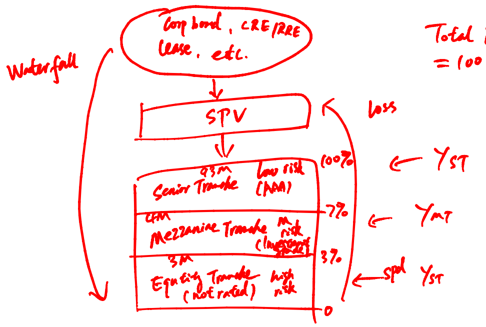
Total portfolio = 100 m



	Apt	Dpt	Size	Spd	Cash flow ①	Cash flow ②
ET	0	3%	3m	Y_{ET}	$3m \cdot Y_{ET}$	$2m \cdot Y_{ET}$
MT	3%	7%	4m	Y_{MT}	$4m\ Y_{MT}$	$4m\ Y_{MT}$
ST	7%	100%	93m	Y_{ST}	$93m\ Y_{ST}$	$93m\ Y_{ST}$

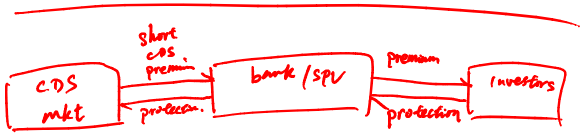
	← ③
ET	0
MT	$3m\ Y_{MT}$
ST	$93m\ Y_{ST}$

① PL = 0%

② PL = 1%

③ PL = 4%

synthetic CDO



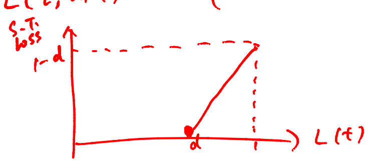
Tranche Payoff f^n

$$L(t; d, u) = \max(\underline{\min(L(t), u) - d}, 0)$$

$$// \\ L(t)$$

S.T. $u = 1$

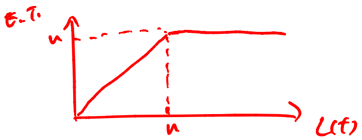
$$L(t; d, 1) = \max(L(t) - d, 0)$$



Call $\max(S_T - K, 0)$

E.T. $\quad d=0$

$$L(t;0,u) = \max\left[\min(L(t),u)-d,\ 0\right]$$
$$= \min\left\{L(t),\ u\right]$$



M.T. $\quad$ ① $\quad L(t) < d < u$

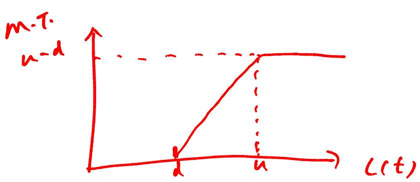
$$L(t;d,u) = \max(\min(L,u)-d,0)$$
$$= \max(L-d,\ 0)$$
$$= 0$$

② $\quad d < L(t) < u$

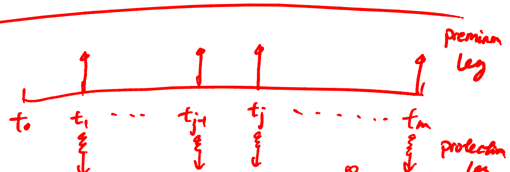
$$L(t;d,u) = \max(\min(L,u)-d,0)$$
$$= \max(L-d,\ 0)$$
$$= L-d$$

③ $\quad L(t) > u > d$

$$L(t;d,u) = \max(\min(L,u)-d,0)$$
$$= \max(u-d,\ 0)$$
$$= u-d.$$



CDO pricing



At inception $\quad \overset{\mathbb{Q}}{E}\left\{\text{P.V. premium leg}\right] = \overset{\mathbb{Q}}{E}\left\{\text{P.V. protection leg}\right]$

① premium leg

def : S spd

δ temporal freq.

$$S \cdot \delta \cdot \sum_{j=1}^{m} (u-d) - L(t_j ; d, u) \, Z(t_0, t_j)$$

② protection leg.

$$\sum_{j=1}^{m} \left\{ L(t_j ; d, u) - L(t_{j-1} ; d, u) \right] Z(t_0, t_j)$$

$$S = f\left(L(t_j ; d, u) \quad ; u, d, z \right)$$

$$E\left\{ L(t_j ; d, u) \right]$$

$$L(t_j , d, u) = \max\left(\min(L(t), u) - d, 0 \right)$$

Loss Dist^n of portfolio

$L(t)$

$$L(t) \sim f(z_1, z_2, \cdots z_n, \ EAD_i, LGD_i, etc)$$

target

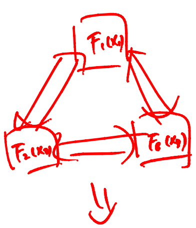
$\longrightarrow$ Dis^n: $f(z_1, z_2, \cdots z_n)$

$$F(x_1, \cdots x_n)$$

$$= C(F_1(x_1), \cdots, F_n(x_n))$$

$$u_i = F_i(x_i)$$

$$= \boxed{C(u_1, \cdots, u_n)}$$



$F(x_1, x_2, x_3)$

$Y = F(x)$ any r.v. X

$Y \sim$ uniform$(0, 1)$ hold all dis[?]

$C(u_1, \cdots, u_n)$

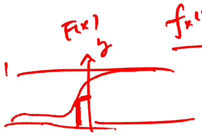
$= C(F_1(x_1), \cdots, F_n(x_n))$

why $Y = F(x)$ is uniform

① $Y \in [0, 1]$

$Y = F(x) = Pr(X \le x) \in [0, 1]$

② $f_Y(y) = 1$



$f_X(x)\,dx = f_Y(y)\,dy$

$f_Y(y) = f_X(x) \frac{dx}{dy}$

$= f_X(x) \,/\, \frac{dy}{dx} = f_X(x) \cdot \frac{1}{f_X(x)} = 1$

$y = 2x$

$Pr(x \le X \le x + dx)$

$= Pr(y \le Y \le y + dy)$

sklar Theorem Compa $\Rightarrow F(x_1 \cdots x_n)$

$C(u_1, \cdots u_n) = Pr\{U_1 \le u_1, \cdots U_n \le u_n\}$

$u_i = F_i(x_i)$

$= Pr\{F_1(X_1) \le u_1, \cdots F_n(X_n) \le u_n\}$

$= Pr(X_1 \le F_1^{-1}(u_1), \cdots, X_n \le F_n^{-1}(u_n))$

$= Pr(X_1 \le x_1, \cdots, X_n \le x_n)$

$= F_n(x_1, \cdots, x_n)$

Copula Density f^n

$f(x_1, \cdots, x_n) = \frac{\partial F_n(x_1, \cdots, x_n)}{\partial x_1 \cdots \partial x_n}$

$= \frac{\partial C(F_1(x_1), \cdots F_n(x_n))}{\partial x_1 \cdots \partial x_n}$

$= \frac{\partial C(u_1, \cdots u_n)}{\partial u_1 \cdots \partial u_n} \prod_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$ $\qquad u_i = F_i(x_i)$

$= c^d(u_1, \cdots, u_n) \prod_{i=1}^{n} f_i(x_i)$

$c^d(u_1, \cdots, u_n) = \frac{f(x_1, \cdots x_n)}{\prod_{i=1}^{n} f_i(x_i)}$

Bi-Variate Gaussian Copula

$$C(u_1, u_2) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho)$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right\} dx\, dy$$

A $\quad p_A = 1\%$ $\quad \Rightarrow Pr(\text{A\&B default}; \rho^{c})$

B $\quad p_B = 5\%$

$u_1 = 1\% \quad \Phi^{-1}(u_1) = -2.3$

$u_2 = 5\% \quad \Phi^{-1}(u_2) = -1.96$

Simulate default times using Gaussian Copula

$$C(u_1, \cdots u_n) = \Phi_n(\Phi^{-1}(u_1) \cdots \Phi^{-1}(u_n))$$

$u_i = F_i(\tau_i)$ — marginal dis^{n} for τ_i

$$x_i = \Phi^{-1}(u_i) = \Phi^{-1}(F_i(\tau_i))$$

$$C(u_1, \cdots, u_n) = \Phi_n(x_1, \cdots x_n; \rho)$$

① generate x_i's

② $\quad u_i = \Phi(x_i)$

③ $\quad \tau_i = F_i^{-1}(u_i)$

$\Rightarrow \tau_i = F_i^{-1}(\Phi(x_i))$

Example: Simulate Bi-Variate Gaussian Copula

$$\tau_i \sim \exp(\lambda_i)$$

$$u_i = F_i(t) = 1 - e^{-\lambda_i t}$$

$$\rightarrow \quad t = -\frac{\log(1-u_i)}{\lambda}$$

$$\begin{cases} x_1 = z_1 \\ x_2 = \rho z_1 + \sqrt{1-\rho^2}\, z_2 \end{cases}$$

ρ: Correlation

z_1 & $z_2 \sim$ iid $N(0,1)$

$$A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

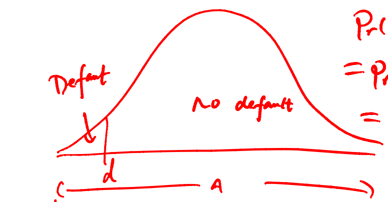
$$\Sigma_\rho = AA^T$$

t- Copula

$$t = \frac{z}{\sqrt{\frac{\chi_v}{v}}}$$

$$C(u_1 \cdots u_n) = t_v(t_v^{-1}(u_1), \dots, t_v^{-1}(u_n))$$

χ_v - chi-square dis d.f. v.



Pr(default) $= \Pr(A < d) = \Phi(d)$

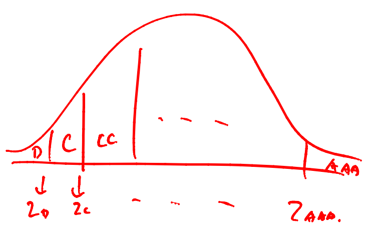
$= PD.$

$d = \Phi^{-1}(PD).$

Asset Value $(A) \sim N(0,1)$

credit quality

high A $\rightarrow$ high credit quality.



Copula

$$C(u_1, u_2) = \Phi_2(\Phi^{-1}(F_1(z_1)), \Phi^{-1}(F_2(z_2)); \rho)$$
$$= \Phi_2(\Phi^{-1}(PD_1), \Phi^{-1}(PD_2); \rho)$$
$$= \Phi_2(d_1, d_2, \rho)$$

factor model

$$\Pr(A_1 < d_1, A_2 < d_2, \rho_A')$$
$$= \Phi_2(d_1, d_2, \rho_A')$$

if $\rho_A' = \rho$ $\Rightarrow$ Copula $\equiv$ factor model.

$r = 0.$

$$\text{premium} = E\{\text{Tloss}\}$$