CQF Module 3: Option Valuation Models

Connecting the Dots

CQF

In this lecture...

We conclude on the first half of the course by bringing together the three valuation models we have seen so far:

- the binomial model;
- the PDE approach;
- the martingale approach.

1. A Review of the Binomial Model

The binomial model is

- neither a great model of asset behaviour,
- ... nor a great numerical method,
- ... but is one of the best ways of understanding how the various asset pricing techniques we have seen so far interact;

1.1. Recall the general 1-period setup:

- stock priced at S_0;
- time period δt;
- stock can only go up to $S_u = uS_0$ with probability p or down to $S_d = dS_0$ with probability $1 - p$. *Formally, we denote by S_T the random variable taking value S_u in an up-move and value S_d in a down-move*;
- condition that $d < 1 < u$;
- discount rate r. Hence discount factor $D = (1 + r\delta t)^{-1}$ and bank account factor $B = \frac{1}{D} = (1 + r\delta t)$;
- Our objective is to find the current value of an option, denoted by V_0.

There are two ways of finding V_0:

- Delta hedging;
- Risk-neutral valuation.

Obviously, both views must yield the same answer, so it is often useful to use one to check the other (especially during exams).

1.2. Delta Hedging

Delta hedging means finding Δ such that the value of a portfolio long a call option and short Δ stocks is independent on the stock value or risk. We can think of delta has the sensitivity of the option to the change in the stock (i.e. same Delta as in Black Scholes).

Since the portfolio is **delta hedged**, the value of the portfolio must be the same in the upper and lower state. Hence,

$$V_u - \Delta S_u = V_d - \Delta S_d \iff \Delta = \frac{V_u - V_d}{(u-d)S_0}$$

With this choice of Δ, the portfolio is deterministic, i.e. **risk-free**. We can use **no-arbitrage valuation** and state that to prevent arbitrage, the return of the portfolio over the time period δt must be the risk-free rate:

$$V_u - \Delta S_u = V_d - \Delta S_d = \frac{1}{D}(V_0 - \Delta S_0)$$

Therefore,

$$\begin{aligned} V_0 &= \frac{V_u - V_d}{u-d} + D\frac{uV_d - dV_u}{u-d} \\ &= \frac{(1-Dd)V_u - (Du-1)V_d}{u-d} \end{aligned} \tag{1}$$

1.3. Risk-Neutral Valuation

The **risk-neutral world** has the following characteristics:

- We don't care about risk and don't expect any extra return for taking unnecessary risk;
- We don't ever need statistics to estimate probabilities of events happening;
- We believe that everything is priced using simple expectations.

The **risk-neutral probabilities** are computed as follows. Since in the risk-neutral world investors are not compensated for the risk they are taking, then

$$S_0 = D\left[p^* S_u + (1-p^*) S_d\right] \iff p^* = \frac{S_0/D - S_d}{S_u - S_d} = \frac{1/D - d}{u - d}$$

Hence,

$$V_0 = D\left[p^* V_u + (1 - p^*) V_d\right]$$

1.4. Tying in Delta Hedging and Risk-Neutral Valuation

Equation (1) can be written more elegantly as

$$V_0 = D\left[p^* V_u + (1 - p^*) V_d\right]$$

where

$$p^* = \frac{\frac{1}{D} - d}{u - d}$$

The parameter p^* can be interpreted as a probability. In fact, the Cox Ross Rubinstein model is built so that the p^* is the probability in use.

2. Martingale Properties of the Binomial Model

In the binomial model, we defined a new parameter, p^* *and identified it as the "risk-neutral probability," while in the martingale method, we performed a change of measure to formally define an equivalent martingale measure* $\mathbb{Q}$.

What is the link here?

2.1. Probability Measures in the Binomial Model

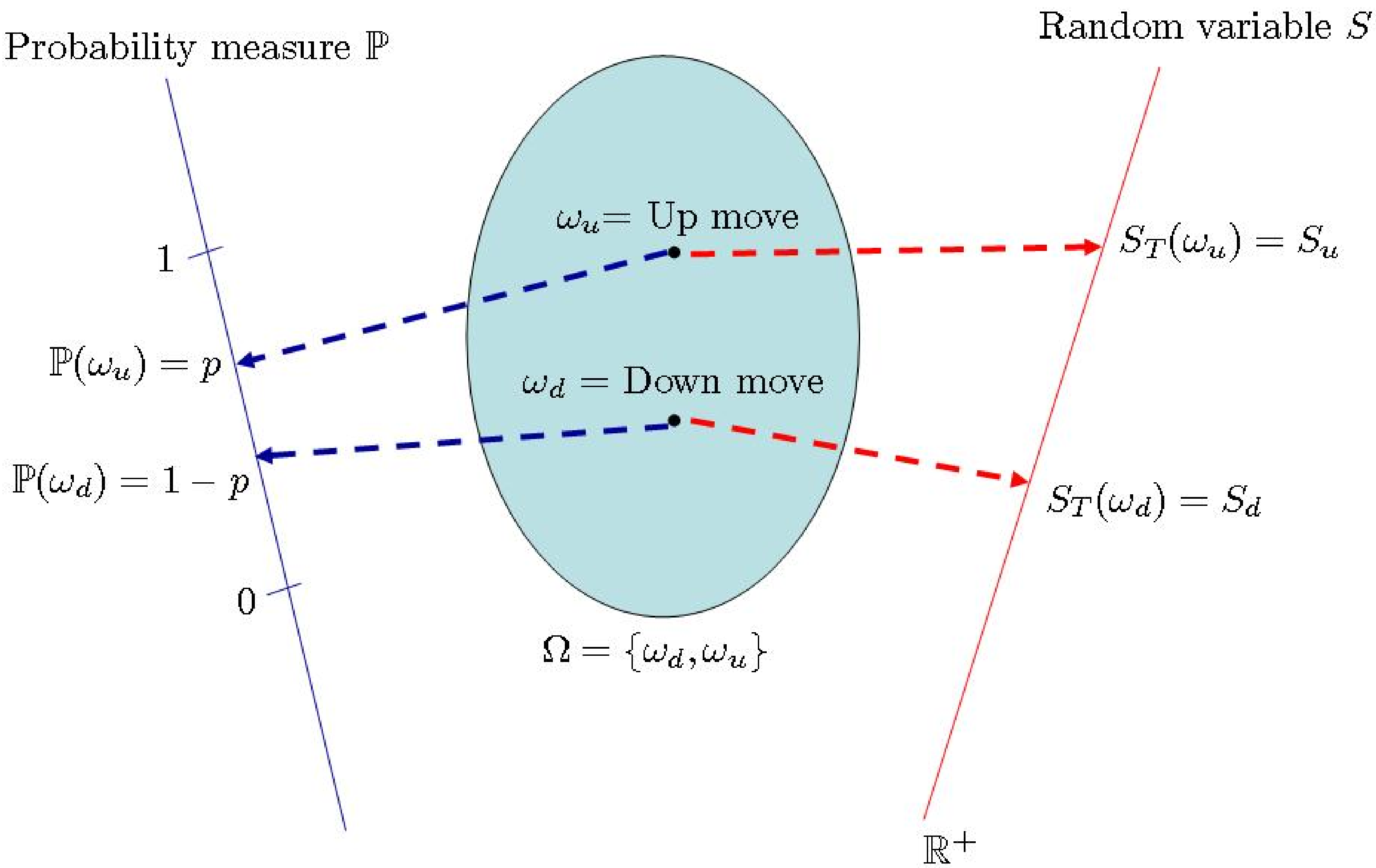
As usual with the binomial model, concepts and their applications are very simple.

Since we just have two possible events[1], namely an up move and a down move, the parameter $p \in (0, 1)$ is enough to uniquely define a probability measure $\mathbb{P}$ as

$$\begin{cases} \mathbb{P}\left[\text{up move}\right] & = & p \\ \mathbb{P}\left[\text{down move}\right] & = & 1 - p \end{cases}$$

[1] Mathematically, $\Omega = \{\text{up move}, \text{down move}\}$

Figure : Probabilistic setting of the binomial model



We have the physical measure $\mathbb{P}$. What about the risk-neutral measure $\mathbb{P}^*$?

Before answering this question, let's try the same approach as in Lecture 3.3, i.e. determine the equivalent martingale measure $\mathbb{Q}$ and price the option in this measure.

2.2. The Equivalent Martingale Measure in the Binomial Model

Recall from Lecture 3.3. that the martingale measure $\mathbb{Q}$ is defined as the equivalent probability measure such that the stock price, when discounted at the risk-free rate, follows a martingale.

Since we are in such a simple setting, we will not need to use any "big" result.

As in the case of the physical measure $\mathbb{P}$, all it takes to define an equivalent measure $\mathbb{Q}$ is a parameter $q \in (0,1)$ such that

$$\begin{cases} \mathbb{Q}\left[\text{up move}\right] & = & q \\ \mathbb{Q}\left[\text{down move}\right] & = & 1-q \end{cases}$$

We can see that since $p \in (0,1)$ and $q \in (0,1)$ the $\mathbb{P}$-measure and $\mathbb{Q}$-measure are indeed equivalent.

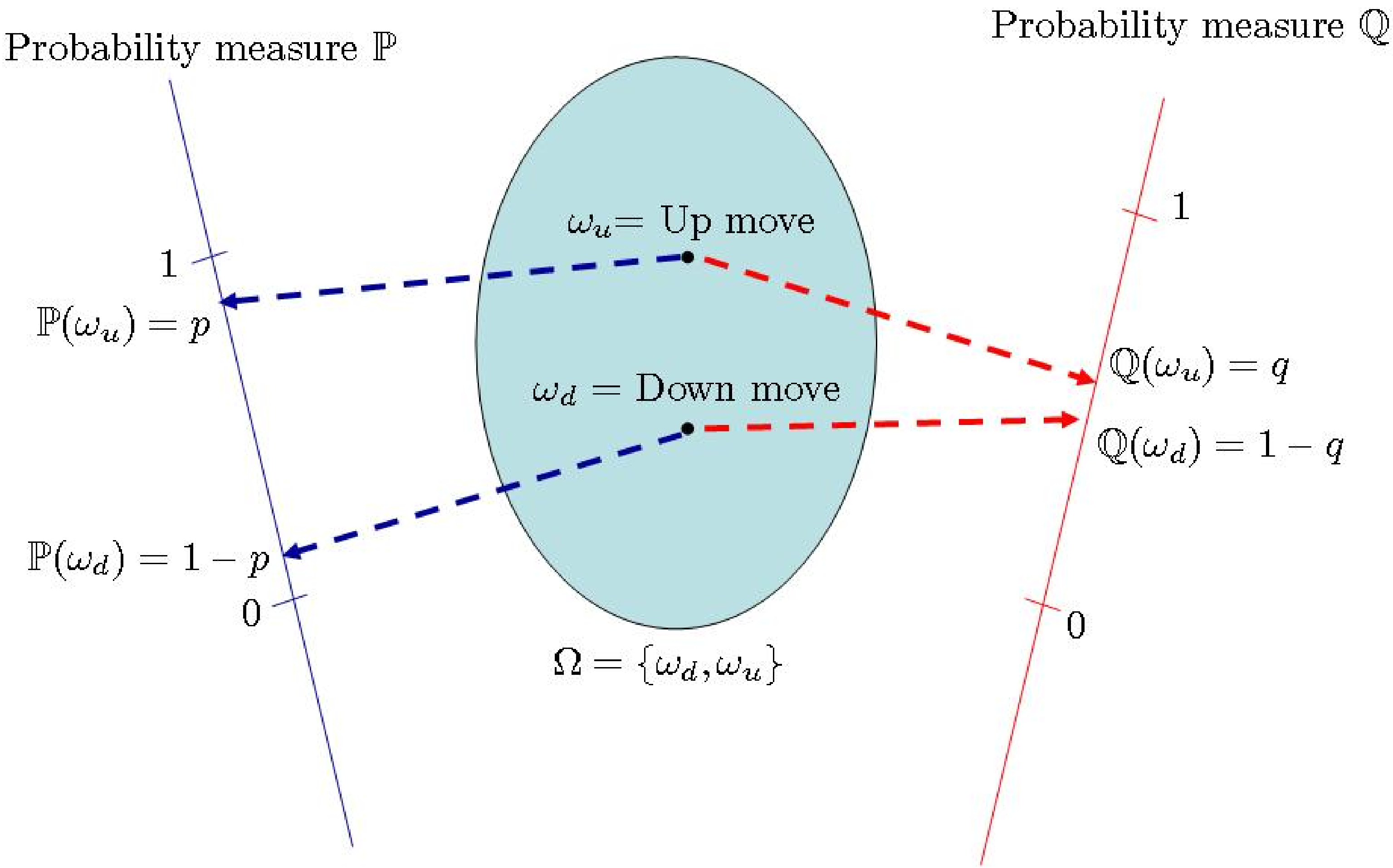


Figure : $\mathbb{P}$ measures and $\mathbb{Q}$ measure in the binomial model

The next step is to find the parameter q such that $\mathbb{Q}$ is the equivalent martingale measure.

Under $\mathbb{Q}$, the discounted stock price is a martingale and hence

$$\begin{aligned} S_0 &= \mathbf{E}^{\mathbb{Q}}\left[DS_T\right] \\ &= D\left(qS_u + (1-q)S_d\right) \qquad (2) \end{aligned}$$

Solving this equation for q, we find that

$$q = \frac{1/D - d}{u - d} = p^* \qquad (3)$$

This derivation leads us to three observations:

1. The equivalent martingale probability q and the risk neutral probability p^* are equal.
2. q is *unique*: there is only one equivalent martingale measure.
3. the risk-neutral valuation formula (2) is exactly the "fundamental" asset pricing formula we derived in Lecture 3.3

$$\begin{aligned} V_0 &= D\left[p^* V_u + (1-p^*) V_d\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[D \max(S_T - E, 0)\right] \end{aligned}$$

Key points of this section...

- There is *one and only one* equivalent martingale measure $\mathbb{Q}$ and it coincides with the "risk-neutral" measure $\mathbb{P}^*$ induced by the parameter p^*.
- The binomial model is equivalent to the martingale approach we developed in Lecture 3.3. The option price can be obtained by evaluating the risk-neutral/equivalent martingale expectation:

$$V_0 = \mathbf{E}^{\mathbb{Q}}\left[D \max\left[S_T - E\right]\right] \tag{4}$$

3. The PDE approach and the Binomial Model

In this section, we will explore the connection between the binomial model and the PDE approach by deducing the Black-Scholes PDE from the Cox-Ross-Rubinstein implementation of the binomial model.

3.1. The Cox-Ross-Rubinstein Implementation

There are different implementations of the binomial model, due to Cox, Ross and Rubinstien (79), Jarrow-Rudd (83) and Leisen-Reiner (96). Each of these implementation distinguishes itself with a specific choice of parameters for u, d and sometimes p.

The implementation we will use in this lecture is the Cox, Ross and Rubinstien (CRR). In the CRR model,

- $u := e^{\sigma\sqrt{\delta t}}$;
- $d := e^{-\sigma\sqrt{\delta t}} = \frac{1}{u}$;
- $D := e^{-r\delta t}$.

where r is the instantaneous risk-free rate and σ is the instantaneous volatility of the stock (log) return.

What is so special about the CRR implementation?

The parameters u, d and D have been chosen so that, in the limit as $\delta t \to 0$, the stock price S_t follows a geometric Brownian motion.

With this choice, our binomial model is consistent with the assumptions made by Black and Scholes.

3.2. From CRR to Black-Scholes PDE

Our starting point is the delta hedging valuation formula (1):

$$V = \frac{V_u - V_d}{u - d} + D\frac{uV_d - dV_u}{u - d}$$

which can be written more conveniently as

$$(1/D)(u - d)V = (1/D)(V_u - V_d) + (uV_d - dV_u) \qquad (5)$$

Next, we perform a Taylor expansion of u, d and $1/D$ up to order δt (anything beyond that is too small) to obtain

$$\begin{aligned} u &= e^{\sigma\sqrt{\delta t}} \approx 1 + \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t \\ d &= e^{-\sigma\sqrt{\delta t}} \approx 1 - \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t \\ 1/D &= e^{r\delta t} \approx 1 + r\delta t \end{aligned}$$

Referring to equation (5) we therefore have

$$
\begin{aligned}
u - d &\approx 2\sigma\sqrt{\delta t} \\
1/D - d &\approx r\delta t + \sigma\sqrt{\delta t} - \frac{1}{2}\sigma^2\delta t \\
u - 1/D &\approx \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t - r\delta t
\end{aligned}
$$

We now perform a Taylor expansion of V_u:

$$V_u \approx V + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\delta S_u + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\delta S_u\right)^2$$

where

$$\begin{aligned} \delta S_u &= S_u - S \\ &= S\left(e^{\sigma\sqrt{\delta t}} - 1\right) \\ &\approx S\left(\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t\right) \end{aligned}$$

and hence

$$\left(\delta S_u\right)^2 \approx S^2\sigma^2\delta t$$

Similarly, for V_d:

$$V_d \approx V + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\delta S_d + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\delta S_d\right)^2$$

where

$$\begin{aligned} \delta S_d &= S_d - S \\ &= S\left(e^{-\sigma\sqrt{\delta t}} - 1\right) \\ &\approx S\left(-\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t\right) \end{aligned}$$

and hence

$$\left(\delta S_d\right)^2 \approx S^2\sigma^2\delta t$$

Substituting all these terms into equation (5), we get

$$
\begin{aligned}
& (2\sigma\sqrt{\delta t})(1 + r\delta t)V \\
\approx \; & \left(V + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S} S \left(\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2 \delta t \right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2 \sigma^2 \delta t \right) \\
& \times (r\delta t + \sigma\sqrt{\delta t} - \frac{1}{2}\sigma^2 \delta t) \\
& + \left(V + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S} S \left(-\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2 \delta t \right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2 \sigma^2 \delta t \right) \\
& \times (\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2 \delta t - r\delta t)
\end{aligned}
$$

Developing (make sure to keep all the terms), rearranging and dividing by $2\sigma\sqrt{\delta t}$, we get

$$+\frac{\partial V}{\partial t}\delta t + rS\frac{\partial V}{\partial S}\delta t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\delta t - rV\delta t \approx 0$$

Dividing by δt and taking the limit as $\delta t \to 0$, we finally obtain the **Black-Scholes PDE**:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Key points of this section...

We have just seen that the Black-Scholes PDE is “contained” in the binomial model’s delta-hedging valuation equation (1).

In fact, we could venture further and assert that **absence of arbitrage**, the necessary condition for delta hedging to be possible, implies the existence of a “Black-Scholes”-type PDE.

In other terms, in a valuation problem, IF you have absence of arbitrage, you can delta-hedge and thus derive a PDE for the dynamics of your derivative.

On a technical note, and as could have been expected, deriving the Black-Scholes PDE from the binomial model made only use of local arguments (Taylor).

4. The binomial model and the Black-Scholes formula

In this section, we will explore the connection between Cox-Ross-Rubinstein implementation of the binomial model and the Black-Scholes formula.

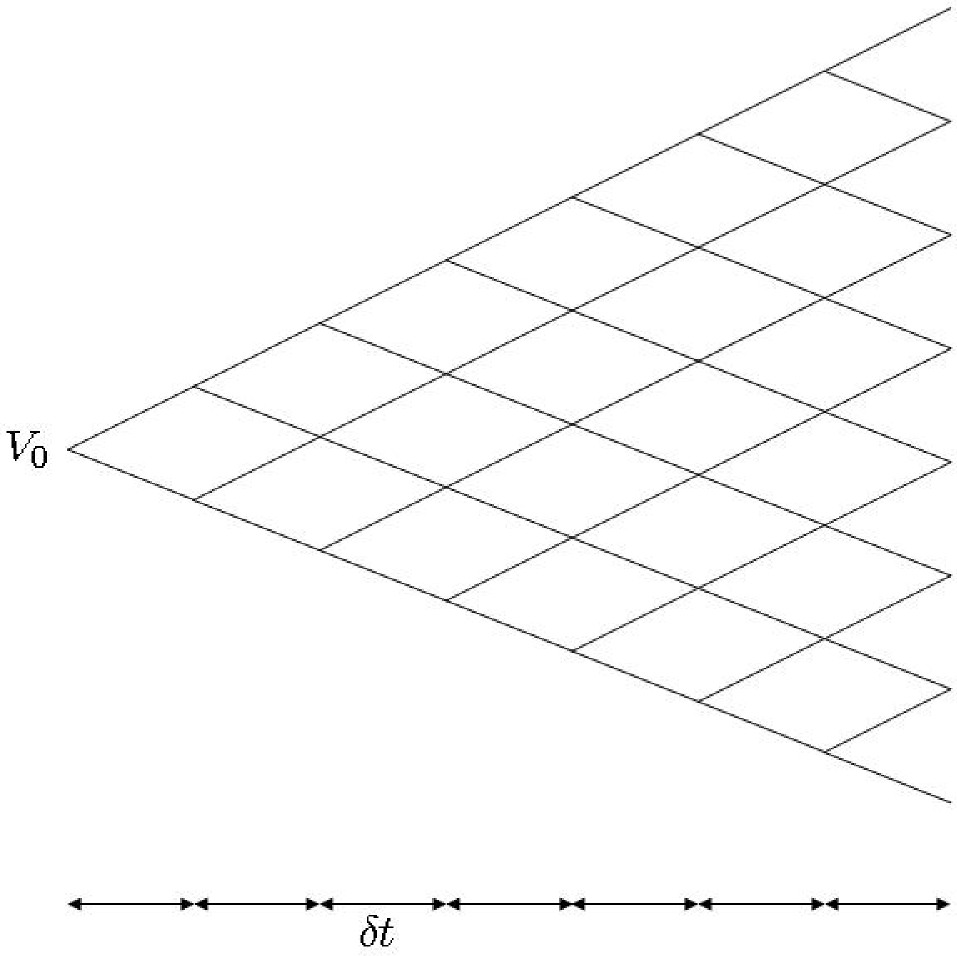
4.1. The N-period binomial model

Let's say that we want to price a European call on a stock using the binomial model. The call has exercise price E and matures at time T. The stock has instantaneous volatility σ and the instantaneous risk-free rate is r.

For pricing purpose, a one-period binomial model is clearly not good enough. We will therefore consider a N-period binomial model by splitting the lifetime T of the option into N steps of "length" δt with

$$\delta t = \frac{T}{N}$$

Figure : N-step binomial tree



At each time step, the stock can go up with probability p and down with probability $1 - p$.

Our model of choice will once again be the CRR, so that the up-move, down-move and discount factor for each time step δt is given by:

- $u := e^{\sigma\sqrt{\delta t}}$;
- $d := e^{-\sigma\sqrt{\delta t}} = \frac{1}{u}$;
- $D := e^{-r\delta t}$.

4.2. Recursive computation of the option price

Since we know the one-step binomial valuation model very well, the natural way of disentangling the N-step pricing problem is to proceed recursively, one step at a time, from expiry backwards in time to time 0.

For each time step, we need to solve the option valuation problem a given number of times. For example, at time step $N - k$, we have $N - k + 1$ option values to compute.

Let's introduce some notation so that we can find our way more easily through the binomial tree.

Figure : N-step binomial tree

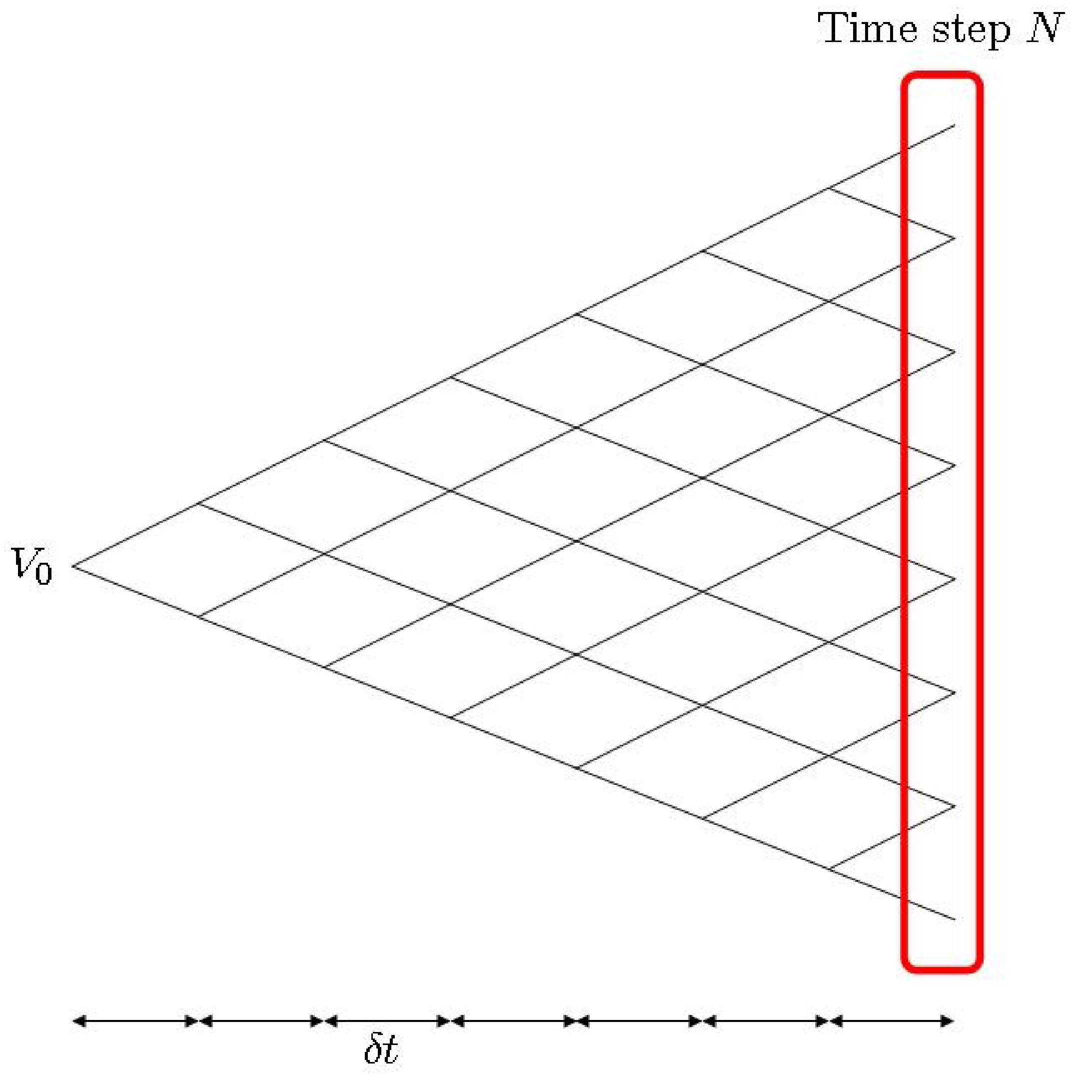


Figure : *N*-step binomial tree

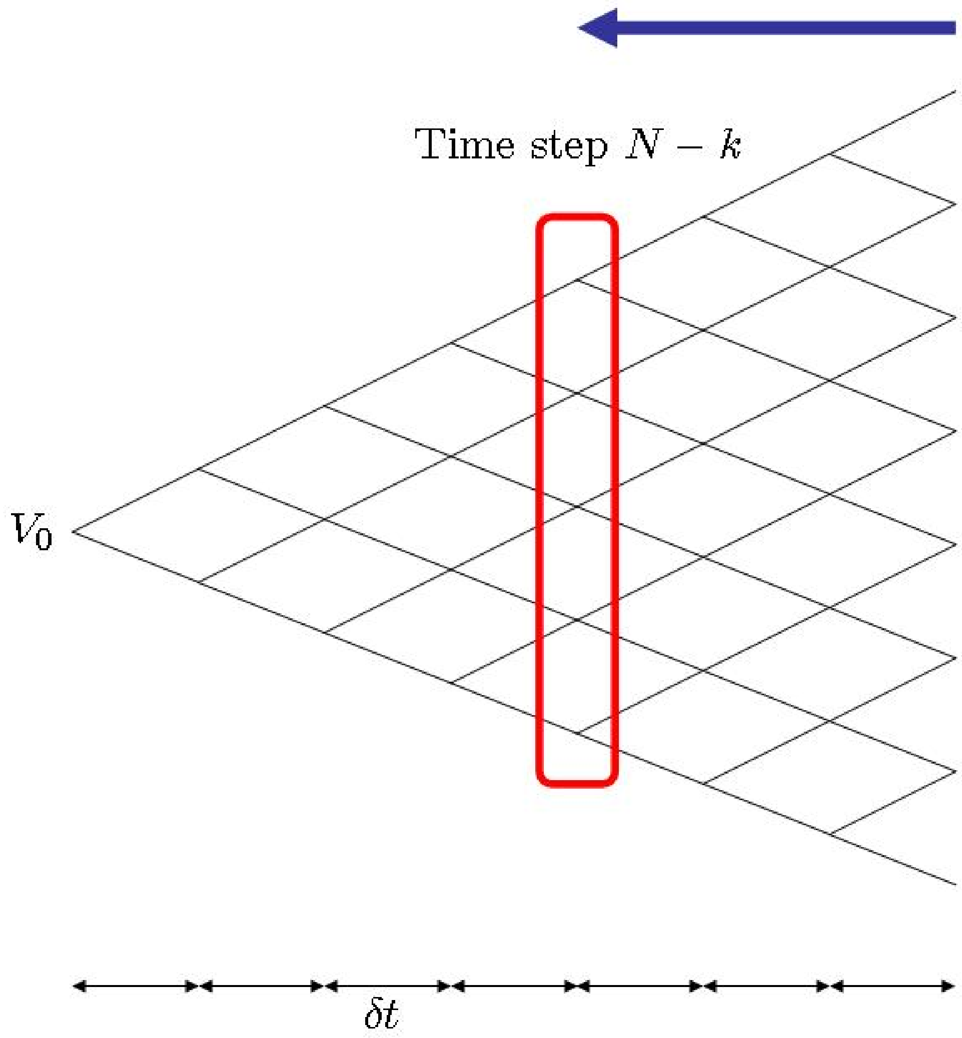


Figure : *N*-step binomial tree

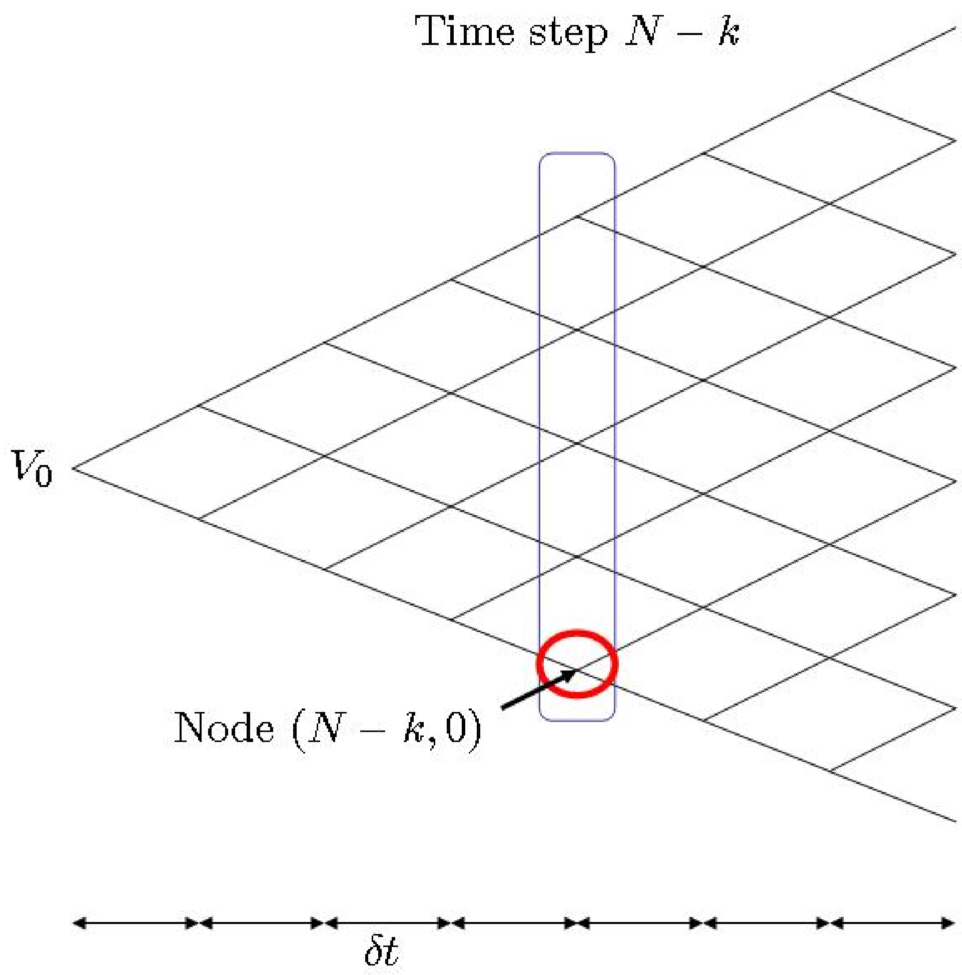


Figure : *N*-step binomial tree

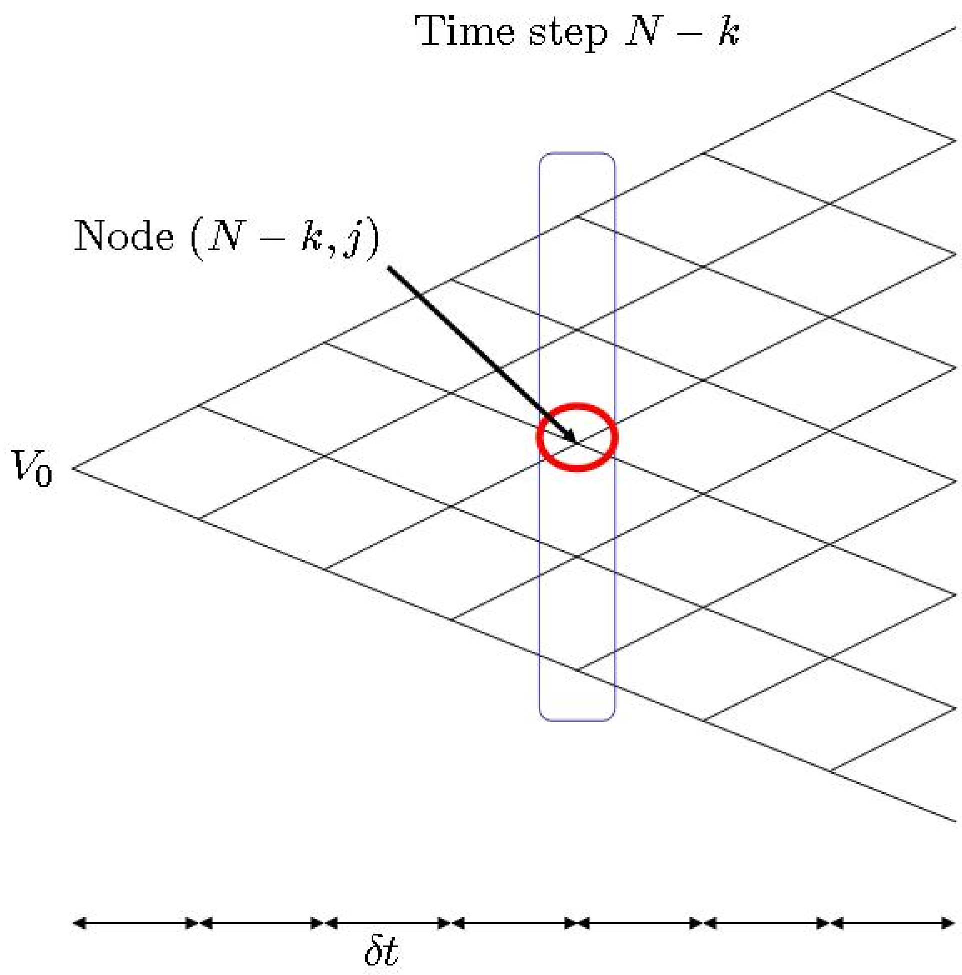
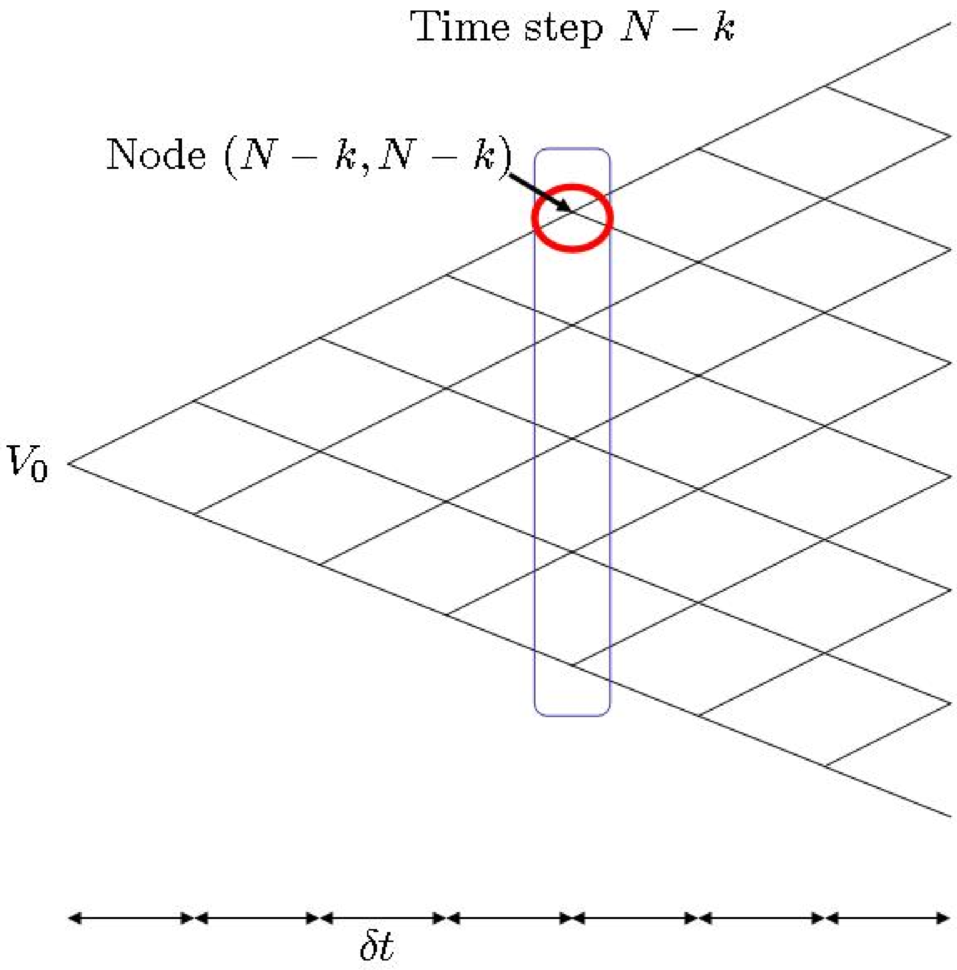


Figure : *N*-step binomial tree



We will use the number of time steps (counted backwards) and the stock price movement as a system of coordinates of the form

(Time Steps, Stock Movement)

to uniquely identify each node in the tree.

Hence, $(N-k, j)$ refers to the node

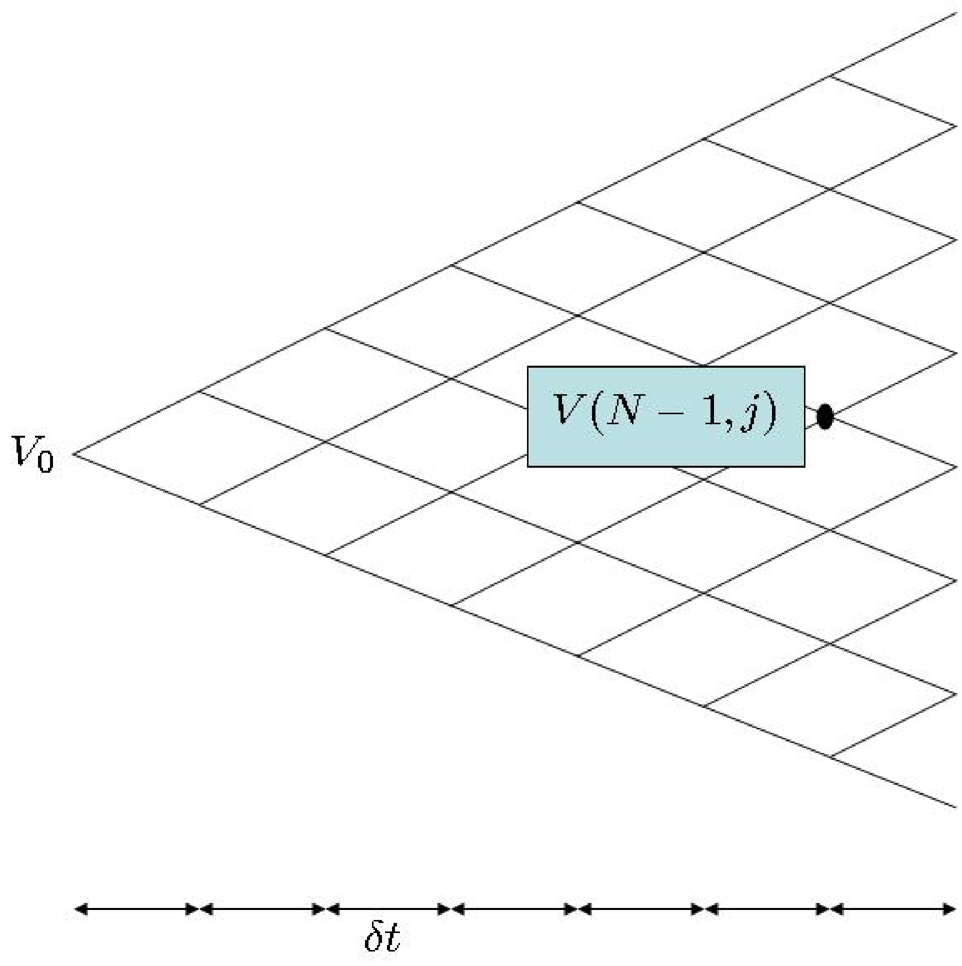
- k steps back in time from expiry;
- j asset steps up from the lowest stock price for this time step;

4.2.1 Pricing the Option at Step $N-1$

Let's pick a decision node $(N-1, j)$ with $0 \leq j \leq N-1$ to illustrate.

At time step $N-1$, we just have one hedging decision to make before expiry (corresponding to time N).

Figure : Pricing the Option at Step $N-1$



Considering our binomial option pricing formula (2), and adapting it to our N-period setting we get the value of the option 1 step from expiry.

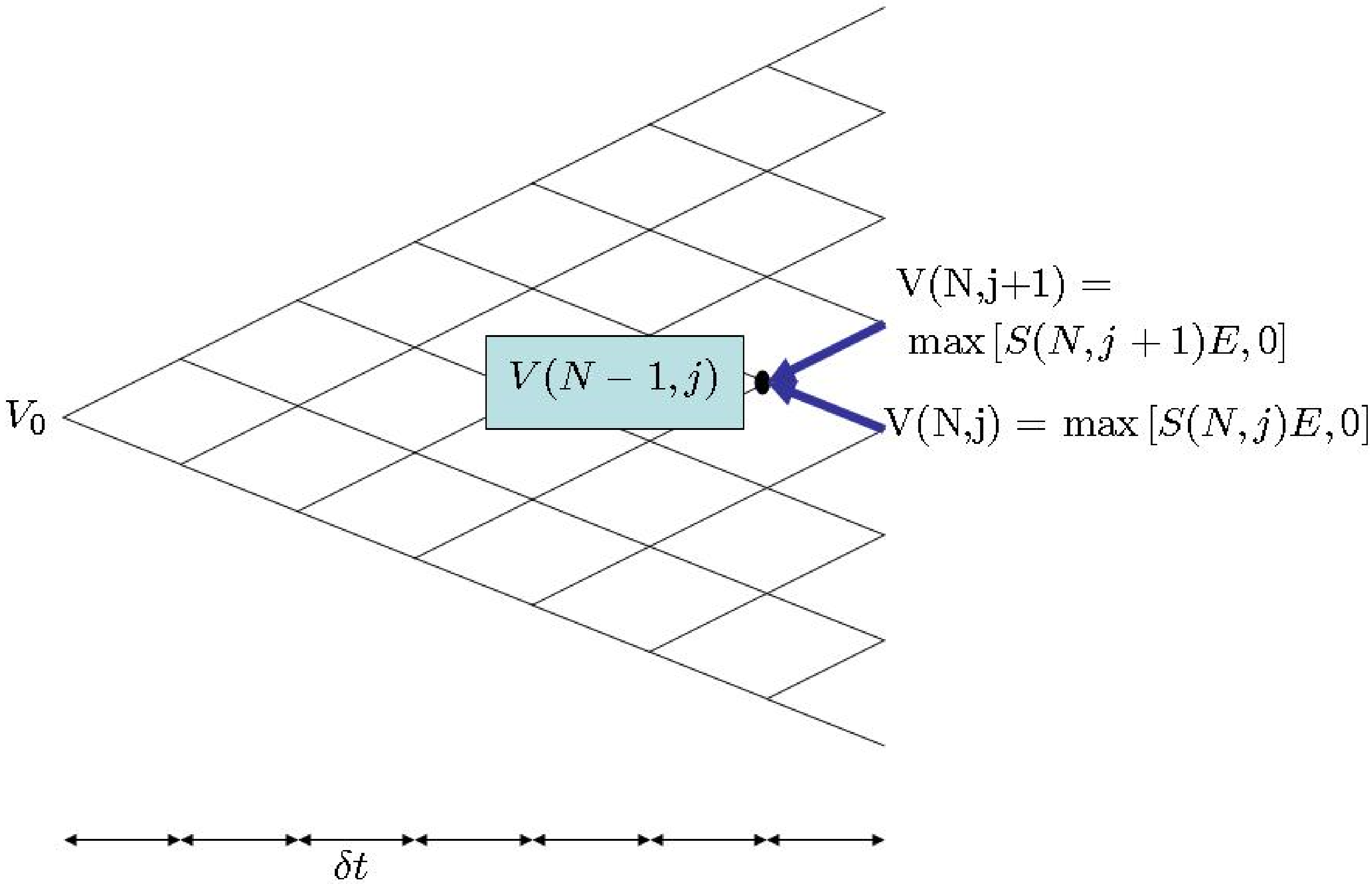
$$V(N-1,j) = D\left[p^* V(N,j+1) + (1-p^*)V(N,j)\right]$$

where

$$p^* = \frac{e^{r\delta t} - d}{u-d}$$

Hence, to compute $V(N-1,j)$, we need to know the value of $V(N,j+1)$ and $V(N,j)$

Figure : Pricing the Option at Step $N-1$

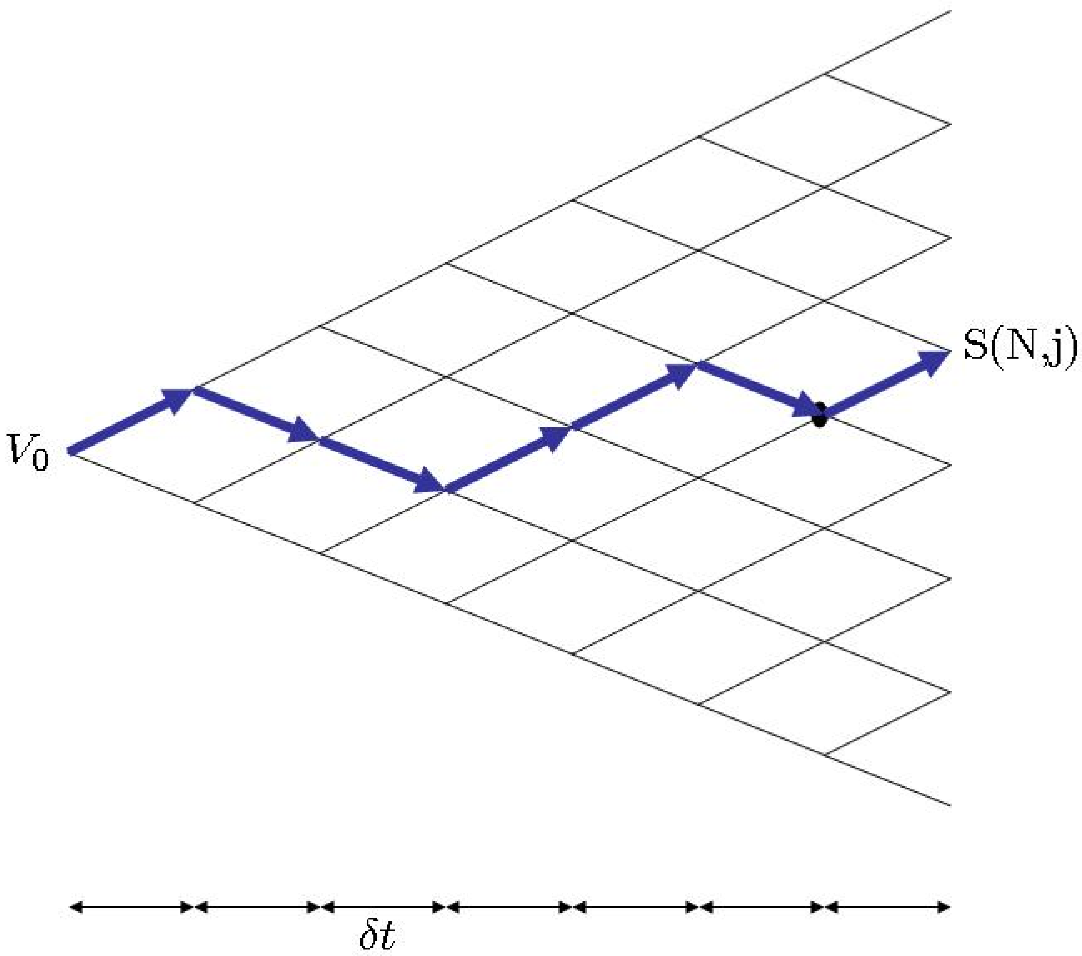


This is rather simple since time step N represents the terminal time T, and hence $V(N, j+1)$ and $V(N, j)$ are respectively the option payoffs:

$$\begin{aligned} V(N, j+1) &= \max\left[S(N, j+1) - E, 0\right] \\ V(N, j) &= \max\left[S(N, j) - E, 0\right] \end{aligned}$$

The next question is: how do we compute $S(N, j)$?

Figure : A possible path leading to $S(N, j)$



Note that $S(N, N)$ corresponds to N up-moves and zero down-moves.

Conversely, $S(N, 0)$ corresponds to zero up-moves and N down-moves.

Thus, $S(N, j)$ corresponds

- j up-moves;
- $N - j$ down-moves.

which implies that

$$S(N, j) = S_0 u^j d^{N-j}$$

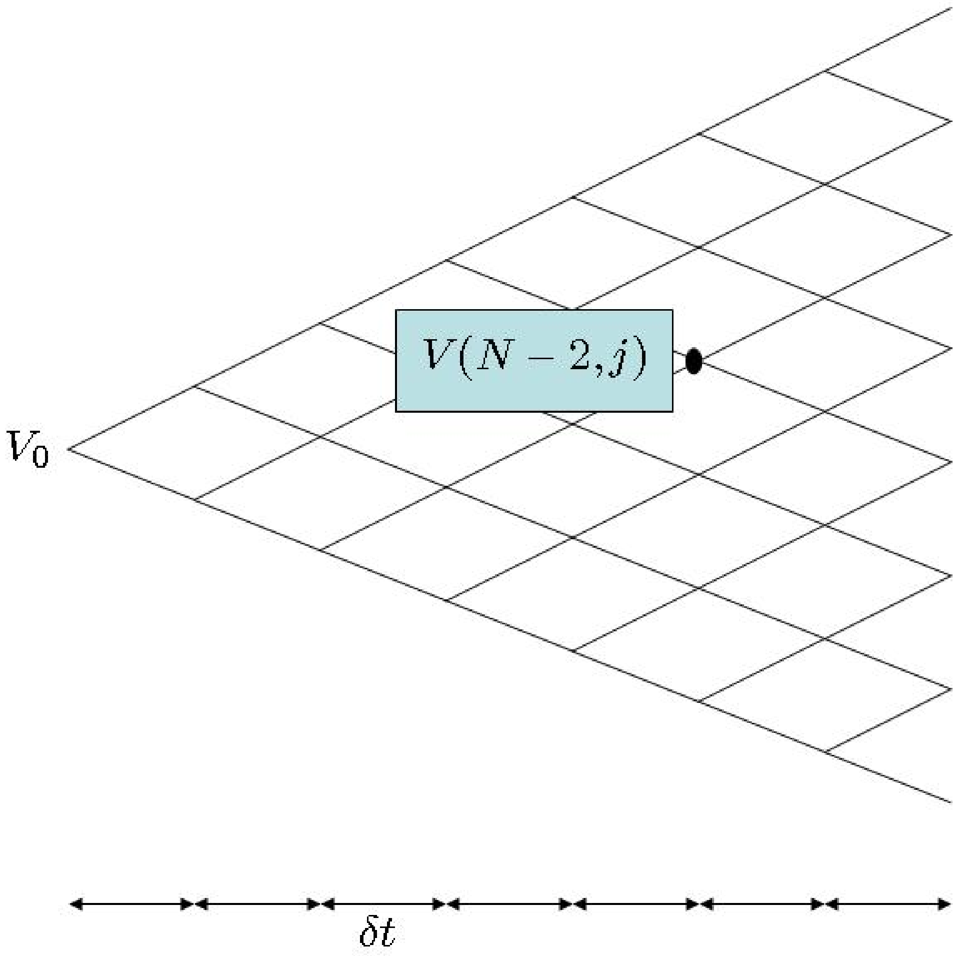
To conclude for this step, the option value $V(N-1, j)$ for $0 \leq j \leq N$ can be computed as:

$$\begin{aligned} V(N-1, j) &= D\left[p^* \max\left[S(N, j+1) - E, 0\right]\right. \\ &\quad \left. + (1-p^*) \max\left[S(N, j) - E, 0\right]\right] \end{aligned}$$

4.2.2 Pricing the Option at Step $N-2$

Let's pick a decision node $(N-2, j)$ with $0 \leq j \leq N$ to illustrate.

Figure : Pricing the Option at Step $N-2$

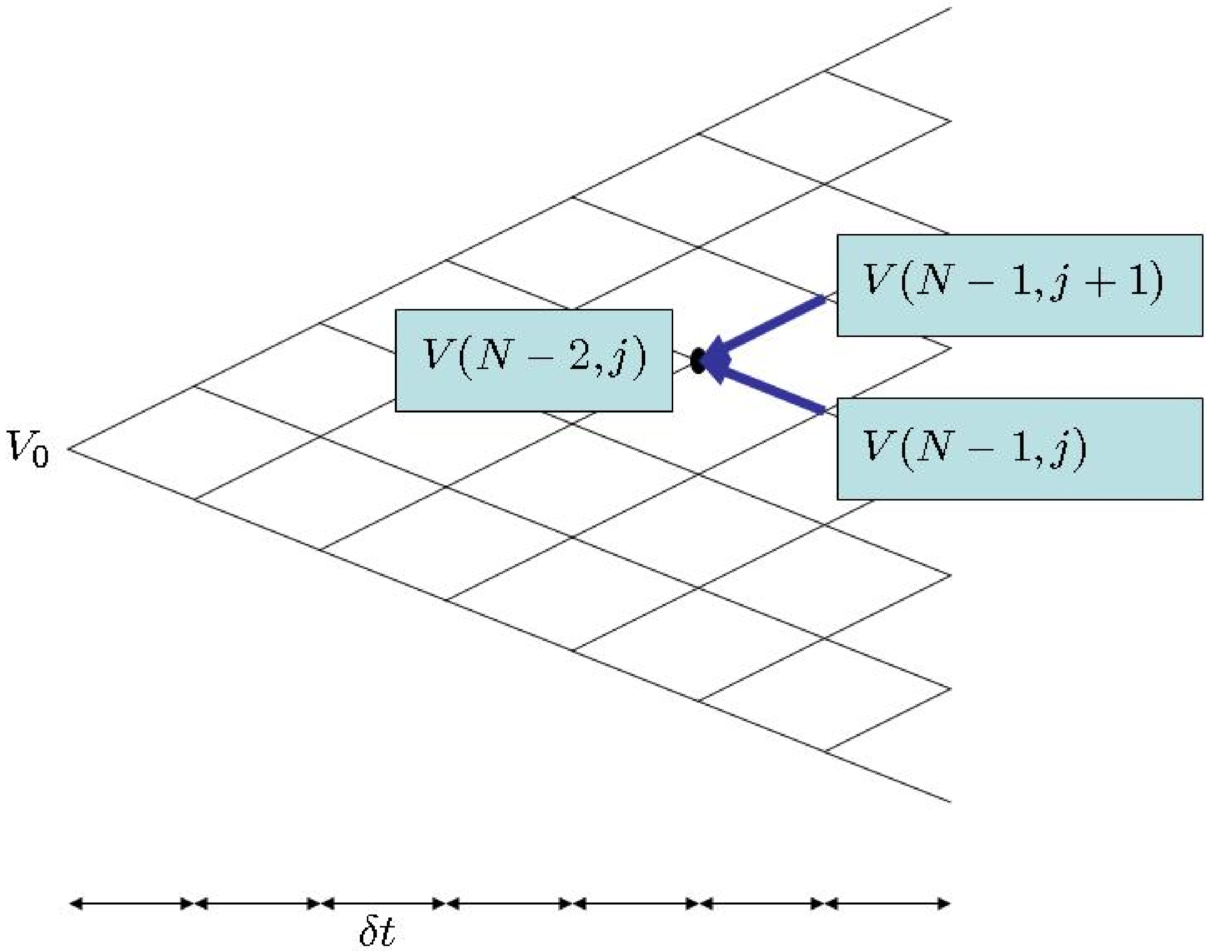


Again, we use our binomial option pricing formula (2). Adapting it to our N-period setting, we get

$$V(N-2,j) = D\left[p^* V(N-1,j+1) + (1-p^*)V(N-1,j)\right]$$

Hence, to compute $V(N-2,j)$, we need to know the value of $V(N-1,j+1)$ and $V(N-1,j)$...

Figure : Pricing the Option at Step $N-2$



... and we have just computed these two values in the previous step. Developing our expression for the option value:

$$\begin{aligned}
V(N-2,j) &= D\left[p^* V(N-1,j+1) + (1-p^*)V(N-1,j)\right] \\
&= D\left[p^* \left(D\left[p^* \max\left[S(N,j+2)-E,0\right]\right.\right.\right. \\
&\quad + \left.\left.(1-p^*)\max\left[S(N,j+1)-E,0\right]\right]\right) \\
&\quad +(1-p^*)\left(D\left[p^* \max\left[S(N,j+1)-E,0\right]\right.\right. \\
&\quad \left.\left.\left.+(1-p^*)\max\left[S(N,j)-E,0\right]\right]\right)\right] \\
&= D^2\left((p^*)^2 \max\left[S(N,j+2)-E,0\right]\right. \\
&\quad +2p^*(1-p^*)\max\left[S(N,j+1)-E,0\right] \\
&\quad \left.+(1-p^*)^2\max\left[S(N,j)-E,0\right]\right) \qquad (6)
\end{aligned}$$

As far as intuition goes, this development hints at a binomial expansion of the form:

$$(x+y)^n = x^n + nx^ny + \ldots = \sum_{i=0}^{n} C_n^i x^i y^{(n-i)}$$

where N is an integer and

$$C_n^i := (^n_i) := \frac{n!}{i!(n-i)!}$$

is the number of combinations (i.e. the number of ways one can pick i objects from a bag of n objects when the order in which the objects are picked does not matter).

With this in mind, and noting that $C_m^0 = C_m^m = 1$, we could express formula (6) as

$$\begin{aligned} & V(N-2, j) \\ = \;& D^2 \left(\sum_{i=0}^{2} C_2^i (p^*)^i (1-p^*)^{2-i} \max\left[S(N, j+i) - E, 0\right] \right) \end{aligned} \quad (7)$$

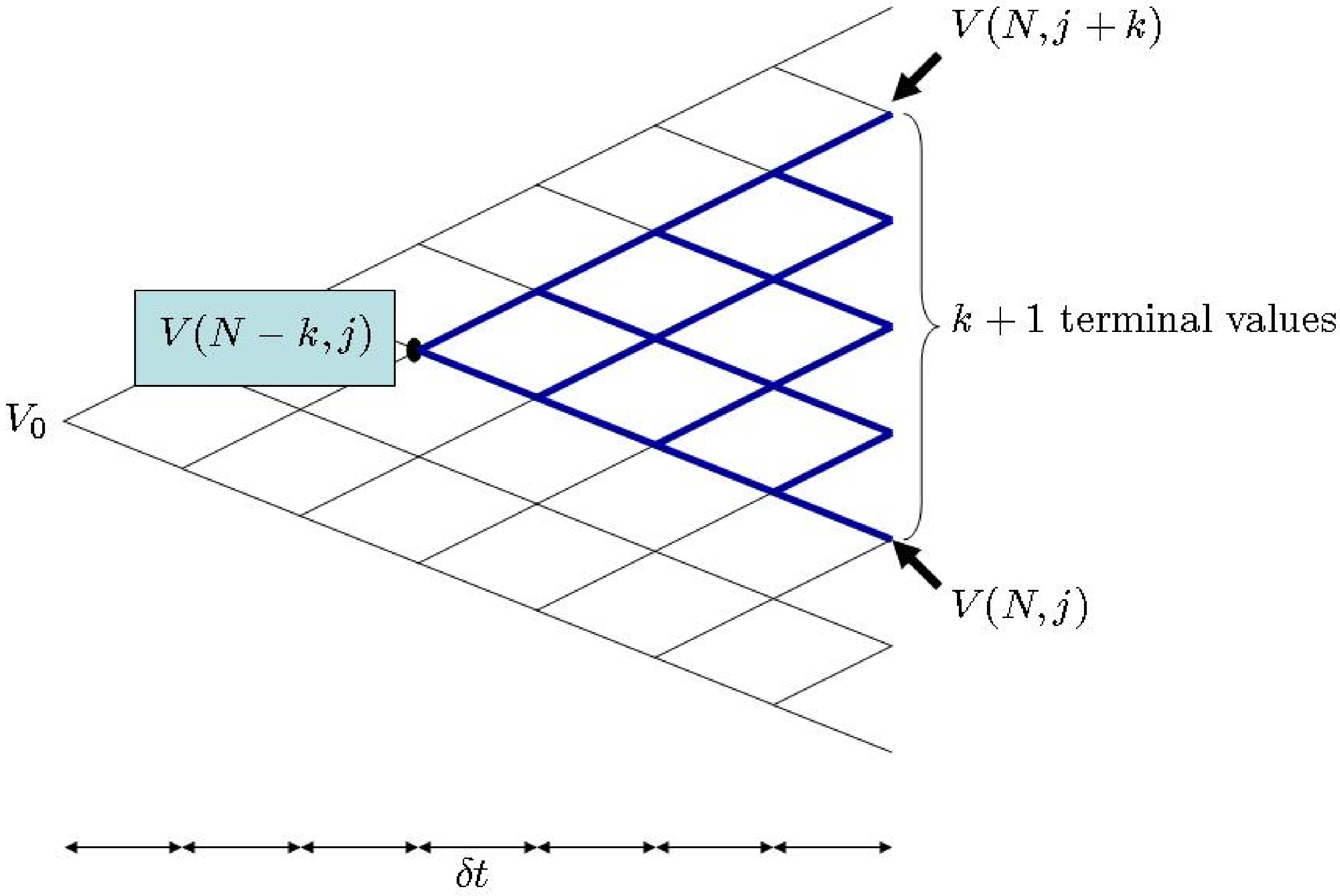
4.2.4 Step $N-k$

If our intuition is correct, then the option value $V(N-k, j)$ at the time step $N-k$ and asset step j with $0 \leq j \leq N$ can be expressed as:

$$
\begin{aligned}
& V(N-k, j) \\
= \quad & D^k \left(\sum_{i=0}^{k} C_k^i (p^*)^i (1-p^*)^{k-i} \max\left[S(N, j+i) - E, 0\right] \right) \qquad (8)
\end{aligned}
$$

Let's verify that this statement is true.

Figure : Intuition for the option value at time step $N - k$



Using the one-step formula for the binomial model (2), we know that the option value $V(N-k,j)$ at the time step $N-k$ and asset step j with $0 \leq j \leq N$ is equal to:

$$\begin{aligned} V(N-k,j) &= D\left(p^* V(N-(k-1),j+1)\right. \\ &\quad \left. +(1-p^*)V(N-(k-1),j)\right) \end{aligned} \tag{9}$$

If our intuition is correct, then the same type of formula as (8) applies to the option values at nodes $(N-(k-1), j+1)$ and $(N-(k-1), j)$. Hence, we conjecture that

$$\begin{aligned} & V(N-(k-1), j+1) \\ = \quad & D^{k-1}\left(\sum_{i=0}^{k-1} C_{k-1}^{i}(p^{*})^{i}(1-p^{*})^{k-1-i} \max\left[S(N, j+1+i)-E, 0\right]\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} & V(N-(k-1), j) \\ = \quad & D^{k-1}\left(\sum_{i=0}^{k-1} C_{k-1}^{i}(p^{*})^{i}(1-p^{*})^{k-1-i} \max\left[S(N, j+i)-E, 0\right]\right) \end{aligned} \tag{11}$$

Substituting (10) and (11) into (9), we get

$$
\begin{aligned}
& V(N-k,j) \\
= \; & D\left[p^* D^{k-1}\left(\sum_{i=0}^{k-1} C_{k-1}^{i}(p^*)^{i}(1-p^*)^{k-1-i}\max\right.\right. \\
& \times[S(N,j+1+i)-E,0]) \\
& +(1-p^*)D^{k-1} \\
& \left.\left.\times\left(\sum_{i=0}^{k-1} C_{k-1}^{i}(p^*)^{i}(1-p^*)^{k-1-i}\max\left[S(N,j+i)-E,0\right]\right)\right]\right.
\end{aligned}
$$

This can be rewritten as

$$
\begin{aligned}
& V(N-k, j) \\
= \quad & D^{k}\left(\sum_{i=0}^{k-1} C_{k-1}^{i}\left(p^{*}\right)^{i+1}\left(1-p^{*}\right)^{k-1-i} \max \left[S(N, j+1+i)-E, 0\right]\right. \\
& \left.+\sum_{i=0}^{k-1} C_{k-1}^{i}\left(p^{*}\right)^{i}\left(1-p^{*}\right)^{k-i} \max \left[S(N, j+i)-E, 0\right]\right) \qquad (12)
\end{aligned}
$$

The first term on the right-hand side is a term in

$$\max\left[S(N, j + 1 + i) - E, 0\right]$$

while the second term on the right-hand side is a term in

$$\max\left[S(N, j + i) - E, 0\right]$$

To be able to add these two terms together, we will need to change the indexing of the first term from

$$\sum_{i=0}^{k-1} C_{k-1}^{i} (p^*)^{i+1} (1-p^*)^{k-1-i} \max\left[S(N, j+1+i) - E, 0\right]$$

to

$$\sum_{i=1}^{k} C_{k-1}^{i-1} (p^*)^{i} (1-p^*)^{k-i} \max\left[S(N, j+i) - E, 0\right]$$

by setting "$i+1 \to i$".

With the new indexing, equation (12) can be written as

$$\begin{aligned} & V(N-k,j) \\ = \quad & D^k \left(\sum_{i=1}^{k} C_{k-1}^{i-1} (p^*)^i (1-p^*)^{k-i} \max\left[S(N, j+i) - E, 0\right] \right. \\ & \left. + \sum_{i=0}^{k-1} C_{k-1}^{i} (p^*)^i (1-p^*)^{k-i} \max\left[S(N, j+i) - E, 0\right] \right) \end{aligned}$$

Developing, we get

$$
\begin{aligned}
& V(N-k, j) \\
= \quad & D^k \Big((p^*)^k \max\left[S(N, j+k) - E, 0\right] \\
& + \sum_{i=1}^{k-1} C_{k-1}^{i-1} (p^*)^i (1-p^*)^{k-i} \max\left[S(N, j+i) - E, 0\right] \\
& + \sum_{i=1}^{k-1} C_{k-1}^{i} (p^*)^i (1-p^*)^{k-i} \max\left[S(N, j+i) - E, 0\right] \\
& + (1-p^*)^k \max\left[S(N, j) - E, 0\right] \Big)
\end{aligned}
$$

Recalling that

- $C_m^{j-1} + C_m^j = C_{m+1}^j$;
- $C_m^0 = C_m^m = 1$

we can now add term by term to obtain:

$$\begin{aligned} & V(N-k,j) \\ = \quad & D^k \left(\sum_{i=0}^{k} C_k^i (p^*)^i (1-p^*)^{k-i} \max\left[S(N,j+i)-E,0\right] \right) \end{aligned}$$

which confirms our intuition.

4.2.5 Option Price at time 0

We can get the pricing formula for time 0 by setting $k = N$ and $j = 0$ into (13):

$$\begin{aligned} & V(0,0) \\ = \; & D^N \left(\sum_{i=0}^{N} C_N^i (p^*)^i (1-p^*)^{N-i} \max\left[S(N,i) - E, 0\right] \right) \end{aligned}$$

4.3. From binomial model to Black-Scholes formula

Looking at formula (13) closely, we notice that the terms

$$C_N^i (p^*)^i (1 - p^*)^{N-i}$$

are probabilities under the Binomial distribution with N trials and probability of success p^*, $\mathscr{B}(N, p^*)$. Specifically, if $X \sim \mathscr{B}(N, p^*)$, then

$$P[X = i] = C_N^i (p^*)^i (1 - p^*)^{N-i}$$

Since the expectation of a function $f(x)$ with respect to a *discrete* random variable X is defined as

$$\mathbf{E}[f(X)] = \sum_{i=1}^{N} p_i f(X_i)$$

then formula (13) can be viewed as the following expectation under a Binomial distribution:

$$V(0,0) \quad = \quad D^N \mathbf{E}^{\mathscr{B}(N,p^*)} \left[\max \left[S_T - E, 0 \right] \right] \qquad (13)$$

Equation (13) is simply an application of the "fundamental" asset pricing formula we derived in Lecture 3.3 and which state that

$$\text{Value of a Derivative} \quad = \quad \mathbf{E}^{\substack{\text{equivalent} \\ \text{martingale} \\ \text{measure}}} \left[\text{PV of Future Cash Flows} \right]$$

Since $S(N,i) = S_0 u^i d^{N-i}$, we can go even further by writing formula (13) as:

$$
\begin{aligned}
& V(0,0) \\
= \quad & S_0 D^N \sum_{i=0}^{N} C_N^i (p^*)^i (1-p^*)^{N-i} u^i d^{N-i} \mathbf{1}_{S(N,i)>E} \\
& -ED^N \sum_{i=0}^{N} C_N^i (p^*)^i (1-p^*)^{N-i} \mathbf{1}_{S(N,i)>E}
\end{aligned}
$$

where $\mathbf{1}_{y>a}$ is the indicator function returning 1 if $y > a$ and 0 otherwise.

Regrouping,

$$
\begin{aligned}
& V(0,0) \\
= \quad & S_0 \sum_{i=0}^{N} C_N^i (Dp^* u)^i (D\left[1-p^*\right] d)^{N-i} \mathbf{1}_{S(N,i)>E} \\
& -ED^N \sum_{i=0}^{N} C_N^i (p^*)^i (1-p^*)^{N-i} \mathbf{1}_{S(N,i)>E}
\end{aligned}
$$

Define

$$\bar{p} := Dup^* = Du\frac{\frac{1}{D} - d}{u - d}$$

and note that

$$\begin{aligned} Dd(1 - p^*) &= Dd\frac{u - \frac{1}{D}}{u - d} \\ &= 1 - \bar{p} \end{aligned}$$

then,

$$
\begin{aligned}
& V(0,0) \\
= \quad & S_0 \sum_{i=0}^{N} C_N^i (\bar{p})^i (1-\bar{p})^{N-i} \mathbf{1}_{S(N,i)>E} \\
& -ED^N \sum_{i=0}^{N} C_N^i (p^*)^i (1-p^*)^{N-i} \mathbf{1}_{S(N,i)>E}
\end{aligned}
$$

and the parameter $\bar{p}$ uniquely defines a new measure $\bar{\mathbb{P}}$ (and a new Binomial distribution $\mathscr{B}(N, \bar{p})$).

The previous equation can then be written as

$$\begin{aligned} V(0,0) &= S_0 \mathbf{E}^{\mathscr{B}(N,\bar{p})}\left[\mathbf{1}_{S(N,i)>E}\right] - ED^N \mathbf{E}^{\mathscr{B}(N,p^*)}\left[\mathbf{1}_{S(N,i)>E}\right] \\ &= S_0 \bar{\mathbb{P}}\left[S_T > E\right] - ED^N \mathbb{P}^*\left[S_T > E\right] \end{aligned} \tag{14}$$

since $\mathbf{E}^{\mathbb{P}}\left[\mathbf{1}_{X \in A}\right] = \mathbb{P}\left[X \in A\right]$.

We are now very close to the Black-Scholes formula...

It turns out that it can be shown that as $N \to \infty$ (i.e. $\delta t \to 0$), the formula (14) converges to the Black-Scholes formula:

$$V(0,0) \quad = \quad S_0 N(d_1) - E e^{-rT} N(d_2)$$

The intuition for this convergence is that by the Central Limit Theorem, the Binomial distribution converges to the Normal distribution as $N \to \infty$. Most of the remaining work consists in verifying that we effectively get d_1 and d_2 out of $\bar{p}$ and p^*.

Key points of this section...

We have just seen that the binomial model's risk-neutral valuation equation (2) tends to the Black-Scholes formula in the limit as the number of time steps tends to infinity.

While deriving and interpreting the risk-neutral valuation equation proved relatively easy, obtaining the Black-Scholes formula as a result proved more difficult.

The reason is that the derivation of the risk-neutral valuation equation is relatively model-independent (subject to probabilistic assumptions), while the Black-Scholes formula is very much model-dependent. Adapting a model-independent result to a specific model often require a significant amount of work because you need to impose all of the model assumptions and structure onto the result.

5. The **BIG** idea...

So far, we have seen that the binomial model connects two seemingly different aspects of derivatives valuation:

- the binomial model and the martingale approach coincide regardless of the specific implementation we choose for the binomial model.
- when we adopt the CRR implementation, the binomial model's *no-arbitrage pricing formula* (1) converges to the Black-Scholes *PDE*;
- when we adopt the CRR implementation, the binomial model's *risk-neutral pricing formula* (2) converges to the Black-Scholes *formula*;

So, what is the **BIG** idea here?

The BIG Idea...

... is that in the Black-Scholes setting, the no-arbitrage approach and the martingale measure approach are strictly equivalent!

In other words, if one of the two methods works on a specific valuation problem then the other method works as well. It is then up-to-you to choose the appropriate method and/or to combine both methods in order to get the most out of the valuation problem.

5.1. No Arbitrage implies that we can find a Martingale Measure

At the begining of the lecture (slide 14), we noted that the no-arbitrage valuation equation (1) could be rewritten as the risk-neutral valuation equation (2) by defining a probability p^*

This probability p^* uniquely defines the risk-neutral measure $\mathbb{P}^*$ which coincide with the martingale measure $\mathbb{Q}$.

Hence, **no arbitrage leads to the existence of a unique equivalent martingale measure**.

5.2. Finding a Martingale Measure implies that there is no arbitrage

For an equivalent martingale measure to exist, we need $0 < p^* < 1$. Indeed, if $p^* = 0$, then $\mathbb{P}^*[S_T = uS] = 0$, but $\mathbb{P}[S_T = uS] > 0$. Hence $\mathbb{P}$ and $\mathbb{P}^*$ are not equivalent. A similar conclusion arises if $p^* = 1$.

To generalize a little, let's just imagine that no equivalent martingale measure exists because $p^* \geq 1$. This implies that

$$p^* = \frac{\frac{1}{D} - d}{u - d} \geq 1 \Leftrightarrow u \leq \frac{1}{D}$$

Hence $d < u \leq \frac{1}{D}$.

Since D is risk-free, an arbitrage opportunity may exist: shortselling the stock and investing in the bank account guarantees a minimum return equal to $\frac{1}{D} - u \geq 0$. And even if academic arbitrage does not exist, then the "probabilistic" arbitrage does exists: why invest in a risky asset that can at best give you a risk-free return when you could get the risk-free return for sure[2]!

[2] remember that p^* and p are not the same thing

Similarly, let's imagine that no equivalent martingale measure exists because $p^* \leq 0$. This implies that

$$p^* = \frac{\frac{1}{D} - d}{u - d} \leq 0 \Leftrightarrow \frac{1}{D} \leq d$$

Hence $\frac{1}{D} \leq d < u$.

An arbitrage opportunity may also exist: buying the stock and financing the purchase by borrowing from the bank account guarantees a minimum return equal to $d - \frac{1}{D} \geq 0$ with no risk of loss.

Now, what happens if $0 < p^* < 1$?

This implies that

$$0 < \frac{\frac{1}{D} - d}{u - d} < 1 \Leftrightarrow d < \frac{1}{D} < u$$

and no arbitrage opportunity exists.

Hence, **the existence of a unique equivalent martingale measure leads to the absence of arbitrage opportunities**.

6. Complete and Incomplete Markets

The BIG idea that the No Arbitrage and Equivalent Martingale Measure approaches are equivalent works well enough in the (continuous) Black-Scholes setting and the binomial model because they are examples of **complete markets**.

6.1. Complete Markets

Broadly speaking, a complete market is a market in which we have enough tradeable instruments to perfectly hedge all of the risk(s) of the derivative we are intending to price.

For example, in the Black-Scholes setting, we are trying to replicate a European option using the underlying stock and the risk-free rate.

The option is exposed to *one* source of risk, the risk of the underlying share modelled as the Brownian motion $X(t)$. Luckily, we can trade and hedge this risk by trading the stock.

Complete markets are so special because in a complete market the derivative is a *redundant security* in the sense that

1. it does not add any new source of risk to the market;
2. it could be fully and efficiently replicated by trading the other securities available on this market.

This is exactly the case in the Black-Scholes world: assuming no trading costs or other market friction, the European option is redundant...

... and this is precisely why No-Arbitrage pricing works so flawlessly.

6.2. Incomplete Markets

Incomplete markets are markets in which you cannot find enough tradeable securities to hedge the risk(s) of the derivative you are trying to price.

The simplest example of an incomplete market is the bond market. One can view bond as a derivative on some (stochastic) interest rates. Since interest rates are not directly tradeable, the only way we have to hedge/replicate a bond B_1 is by trading a related bond B_2 exposed to the same interest rate risk.

The problem here is since we do not know how to price a bond to start with, using bond B_2 as a hedging instrument for bond B_1 is far from solving all our problems, and we will have to make some assumptions if we want to solve the pricing problem.

More on this in **Module 4**!

In this lecture, we have seen...

- the binomial model and the martingale approach coincide regardless of the specific implementation we choose for the binomial model.
- when we adopt the CRR implementation, the binomial model's *risk-neutral pricing formula* (2) converges to the Black-Scholes *formula*;
- when we adopt the CRR implementation, the binomial model's *no-arbitrage pricing formula* (1) converges to the Black-Scholes *PDE*;
- In complete markets, the no-arbitrage approach and the martingale measure approach are strictly equivalent.