dX is the usual increment of Brownian motion

1. The bond pricing equation, derived in class is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\omega^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda\omega)\frac{\partial V}{\partial r} - rV = 0.$$

   A bond has payoff at maturity $t = T$ of one unit, i.e.

$$V(r, T) = 1$$

   Solve the above equation for $V(r, T)$ given that ω is constant and $(u - \lambda\omega) = 1$.

   [Hint: we know the solution has the form $V(r, t) = \exp(A(t) - rB(t))$.]

Solution:

Now $V = Z = e^{A-rB}$. We know $Z(r, T\,; T) = 1 \Rightarrow$

$$\exp(A(T\,;T) - rB(T\,;T)) = 1.$$

This will only happen when $A(T\,;T) - rB(T\,;T) = 0 \Rightarrow A(T\,;T) = B(T\,;T) = 0$

$$Z_t = \left(\dot{A} - r\dot{B}\right) Z \qquad Z_r = -BZ, \qquad Z_{rr} = B^2 Z \qquad \text{where } \cdot \equiv \frac{d}{dt}$$

Substituting in the BPE gives

$$\begin{aligned} &\Rightarrow \quad \dot{A} - r\dot{B} + \frac{1}{2}\omega^2 B^2 - B - r = 0 \\ &\Rightarrow \quad \left(\dot{A} + \frac{1}{2}\omega^2 B^2 - B\right) - r\left(\dot{B} + 1\right) = 0 \end{aligned}$$

Now have two equations,

$$\begin{aligned} \dot{B} + 1 &= 0 \\ \dot{A} + \frac{1}{2}\omega^2 B^2 - B &= 0 \end{aligned}$$

$$\begin{aligned} \frac{dB}{dt} &= -1 \rightarrow \int_t^T dB = -\int_t^T d\tau \\ &\rightarrow \underbrace{B(T\,;T)}_{=0} - B(t\,;T) = -(T - t) \\ &\therefore \quad B(t\,;T) = (T - t) \end{aligned}$$

Now the second equation becomes

$$\begin{aligned}
\dot{A} &= -\frac{1}{2}\omega^2 B^2 + B \\
\frac{dA}{dt} &= -\frac{1}{2}\omega^2 (T-t)^2 + (T-t) \rightarrow \\
\int_t^T dA &= -\frac{1}{2}\omega^2 \int_t^T (T-\tau)^2 \, d\tau + \int_t^T (T-\tau) \, d\tau \\
\underbrace{A(T\,;T)}_{=0} - A(t\,;T) &= -\frac{1}{2}\omega^2 \int_t^T (T-\tau)^2 \, d\tau + \int_t^T (T-\tau) \, d\tau
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & A = \frac{\omega^2}{2} \int_t^T (T-\tau)^2 \, d\tau - \int_t^T (T-\tau) \, d\tau \\
= \quad & \frac{\omega^2}{6} (T-t)^3 - \frac{1}{2} (T-t)^2
\end{aligned}$$

2. The interest rate r is assumed to be satisfied by a SDE $dr = dX$. By hedging with a bond of different maturity derive the bond pricing equation. Consider a one-factor risk-neutral world in which the spot rate, r, evolves according to the SDE

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} - a(r,t) \frac{\partial V}{\partial r} - rV = 0,$$

where $a(r,t)$ is an arbitrary function. Assuming a is a function of t only and a bond has payoff at maturity $t = T$ of one unit, i.e.

$$V(r, T; T) = 1$$

find a solution of the form

$$V(r,t) = \exp\left(A(t) + rB(t)\right)$$

where $A(t)$ can be written as

$$A(t) = -\int_t^T \left[a(s)(s-T) + \beta (s-T)^2\right] ds$$

and determine the constant β.

Solution:

one-factor risk neutral world spot rate: $dr = dX$
Construct a hedged portfolio:

$$\Pi = V_1 - \Delta V_2 \qquad \text{where } \ Z = \text{zero-coupon bond price}$$

Change in portfolio $d\Pi$

$$d\ \Pi = dV_1 - \Delta dV_2$$

with Itô's Lemma:

$$\begin{aligned} \Rightarrow \quad d\Pi = & \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\frac{\partial^2 V_1}{\partial r^2} \right) dt + \frac{\partial V_1}{\partial r} dX \\ -\Delta & \left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\frac{\partial^2 V_2}{\partial r^2} \right) dt - \Delta \frac{\partial V_2}{\partial r} dX \end{aligned}$$

Choose Δ in a way that the risk in $d\Pi$ vanishes $\Rightarrow \Delta = \frac{\partial V_1 / \partial r}{\partial V_2 / \partial r}$ and hence $d\Pi = r\Pi dt$ for no arbitrage. This gives

$$\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\frac{\partial^2 V_1}{\partial r^2} - rV_1 \right) \Big/ \frac{\partial V_1}{\partial r} = \left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\frac{\partial^2 V_2}{\partial r^2} - rV_2 \right) \Big/ \frac{\partial V_2}{\partial r}$$

LHS depends on T_1, and RHS on T_2, so both sides must equal a function that is independent of T, call this $a(r,t)$. Dropping subscripts

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} - a(r,t)\frac{\partial V}{\partial r} - rV = 0$$

If $a = a(t)$ and $V(r,t;T) = 1$, then substitution of $V(r,t) = e^{A(t)+rB(t)}$ in BPE gives:

$$\dot{A} + \frac{1}{2}B^2 - aB - r + r\dot{B} = 0$$

which is true for all r. Gives two equations $\dot{B} = 1$; $\dot{A} + \frac{1}{2}B^2 - aB = 0$

$$\dot{B} = 1 \Longrightarrow B = t - T$$

$$\begin{aligned} \dot{A} &= a(t)(t-T) - \frac{1}{2}(t-T)^2 \Longrightarrow \\ \int_t^T dA &= \int_t^T \left(a(s)(s-T)\,ds - \frac{1}{2}(s-T)^2 \right) ds \\ A(t) &= -\left(\int_t^T a(s)(s-T)\,ds - \frac{1}{6}\left.(s-T)^3\right|_t^T \right) \\ A(t) &= -\int_t^T [a(s)(s-T)]\,ds - \frac{(t-T)^3}{6} \end{aligned}$$

hence $\beta = -1/2$.

3. What final condition (payoff) should be applied to the bond pricing equation for a swap, cap, floor, zero-coupon bond and a bond option?

Solution:
Final condition for a swap:

$$V(r,T) = (r - r_s)\, P,$$

where r_s is the fixed rate and P is the principal.
Final condition for a cap:

$$V(r,T) = \max(r - r_c, 0)\, P,$$

where r_c is the cap rate and P is the principal.
Final condition for a floor:

$$V(r,T) = \max(r_f - r, 0)\, P,$$

where r_f is the floor rate and P is the principal.
Final condition for a zero-coupon bond:

$$V(r,T) = P,$$

where P is the principal.
Final condition for a coupon bond:

$$V(r,T) = (1 + c)\, P,$$

where c is the (discrete) coupon rate and P is the principal.
Final condition for a bond option:

$$V(r,T) = \max(Z(r,T) - E, 0)\,,$$

where E is the exercise price and $Z(r,t)$ is the value of the underlying bond at time t .

4. Consider the bond pricing equation

$$\frac{\partial B}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 B}{\partial r^2} + (u - \lambda w) \frac{\partial B}{\partial r} - rB = 0,$$

where $dr = (u - \lambda w)\, dt + wdX$ is the risk-neutral spot rate. Suppose this risk-neutral model is defined by

$$dr = ar^2 dt + br^{3/2} dX.$$

Suppose we wish to use this to price a new type if interest rate derivative called a "perpetual bond" whose value is where a and b are constants. Show that the value of a zero coupon bond can be written in the form

$$\max(r - E, 0)$$

and which can be exercised at any time , where $E > 0$ is the exercise price. Show that this price is given by

$$B = \frac{E}{\lambda_1 - 1}$$

where

$$\lambda_1 = \frac{-\left(a - b^2/2\right) + \sqrt{\left(a - b^2/2\right)^2 + 2b^2}}{b^2}.$$

Solution
The BPE for this risk-adjusted spot rate is

$$\frac{\partial B}{\partial t} + \frac{1}{2} b^2 r^3 \frac{\partial^2 B}{\partial r^2} + ar^2 \frac{\partial B}{\partial r} - rB = 0$$

so perpetual here means independent of time t, and we have an Euler equation

$$\frac{1}{2} b^2 r^3 \frac{d^2 B}{dr^2} + ar^2 \frac{dB}{dr} - rB = 0$$

with solution of form

$$B(r) = r^\lambda$$

so

$$\frac{1}{2} b^2 \lambda(\lambda - 1) + a\lambda - 1 = 0 \longrightarrow \frac{1}{2} b^2 \lambda^2 + \left(a - \frac{1}{2} b^2\right) \lambda - 1 = 0$$

and

$$\lambda_{1,2} = \frac{-\left(a - b^2/2\right) \pm \sqrt{\left(a - b^2/2\right)^2 + 2b^2}}{b^2}.$$

so general solution is

$$B(r) = \alpha r^{\lambda_1} + \beta r^{\lambda_2}$$

$\lambda_1 > 0 > \lambda_2$, and if $r = 0$ then $B = 0$ so we must have $\beta = 0$. Hence exercise the option when $r = r^* > E$, then

$$B(r) = (r^* - E) \left(\frac{r}{r^*}\right)^{\lambda_1}$$

where $\lambda_1 = \dfrac{-\left(a - b^2/2\right) \pm \sqrt{\left(a - b^2/2\right)^2 + 2b^2}}{b^2}$. Choose r^* to maximise $B(r)$, B is max at r^* for fixed r :

$$\frac{\partial B}{\partial r^*} = \left(\frac{r}{r^*}\right)^{\lambda_1} - \lambda_1 \frac{(r^* - E)}{r^*} \left(\frac{r}{r^*}\right)^{\lambda_1} = \left(\frac{r}{r^*}\right)^{\lambda_1} \left(1 - \lambda_1 \frac{(r^* - E)}{r^*}\right) = 0$$

when

$$r^* = \frac{E\lambda_1}{\lambda_1 - 1}$$

then

$$\left.\frac{\partial B}{\partial r}\right|_{r=r^*} = \lambda_1 \frac{(r^* - E)}{r^*} = 1$$

and

$$B = \frac{E}{\lambda_1 - 1}.$$

5. Consider the Vasicek model for the spot rate r with mean rate $\overline{r}$ and reversion rate γ Suppose $\gamma = 0.1$, $\overline{r} = 0.1$. and standard deviation $\sigma = 20\%$. Price a Zero Coupon Bond that matures in year 10, if the spot rate is 10%. (Very much a spreadsheet based problem).

Solution:
Vasicek model is given by:

$$dr = \gamma\left(\overline{r} - r\right) dt + \sigma dX$$

Zero coupon bond ZCB is given by $Z(r, t; T) = \exp(A - rB)$
where

$$B(t;T) = \frac{1}{\gamma}\left(1 - \exp\left(-\gamma\left(T - t\right)\right)\right)$$

$$A(t;T) = \frac{1}{\gamma^2}\left(B - (T - t)\right)\left(\overline{r}\gamma^2 - \frac{1}{2}\sigma^2\right) - \frac{\sigma^2 B^2}{4\gamma}$$

(given in Wilmott). Remember $\overline{r} = \eta/\gamma$.

We are given the reversion rate $\gamma = 0.1$, mean rate $\overline{r} = 0.1$ and standard deviation (or diffusion) $\sigma = 0.2$. Maturity $T = 10$. These quantities substituted in give $B = 6.321$ and $A = 2.994$.

Therefore the bond price is now given as a function of r

$$Z = 19.97 \exp\left(-6.321r\right).$$

The spot rate r is 10%, therefore we have a bond price $Z = 10.61$.

6. In class we derived a two factor interest rate model with the BPE given by

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+\rho wq\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2}+(u-\lambda_r w)\frac{\partial V}{\partial r}+(p-\lambda_l q)\frac{\partial V}{\partial l}-rV=0.$$

where the two state variables evolve according to

$$\begin{aligned} dr &= udt+wdX_1 \\ dl &= pdt+qdX_2. \end{aligned}$$

Given that $u-\lambda_r w=0=p-\lambda_l q$ and $w=q=\sqrt{a+br+cl}$, where $a,\ b$ and c are constants, derive a set of equations and boundary conditions for $A,\ B$ and C such that a bond V is of the form

$$V=\exp\left(A(t)+rB(t)+lC(t)\right)$$

is a solution of the BPE with redemption value

$$V(r,l,T;T)=1.$$

You are not required to solve these equations.

Solution: The information given reduces the BPE to

$$\frac{\partial V}{\partial t}+\frac{1}{2}(a+br+cl)\frac{\partial^2 V}{\partial r^2}+\rho(a+br+cl)\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}(a+br+cl)\frac{\partial^2 V}{\partial l^2}=rV.$$

We are given $V=\exp\left(A(t)+rB(t)+lC(t)\right)\longrightarrow$

$$\begin{aligned} \dot{V} &= \left(\dot{A}(t)+r\dot{B}(t)+l\dot{C}(t)\right)V \\ V_r &= BV\longrightarrow V_{rr}=B^2V \\ V_l &= CV\longrightarrow V_{ll}=C^2V \\ V_{rl} &= BCV \end{aligned}$$

and substitute in BPE to give

$$\left(\dot{A}(t)+r\dot{B}(t)+l\dot{C}(t)\right)+\frac{1}{2}B^2(a+br+cl)+\rho BC(a+br+cl)+\frac{1}{2}C^2(a+br+cl)=r$$

and now equation coefficients of $O(1)\,,\ O(r)\,,\ O(l)$ to give in turn the following ODE's

$$\begin{aligned} \dot{A}(t)+\frac{1}{2}B^2a+\rho BCa+\frac{1}{2}aC^2 &= 0 \\ \dot{B}(t)+\frac{1}{2}B^2b+\rho BCb+\frac{1}{2}bC^2 &= 1 \\ \dot{C}(t)+\frac{1}{2}B^2c+\rho BCc+\frac{1}{2}cC^2 &= 0 \end{aligned}$$

which are solved together with the final condition

$$A(T)=B(T)=C(T)=0.$$