CQF Module Two. Examination Solutions

January 2016 Cohort

Instructions

Please review these Solutions. Answers to all questions are required. Complete mathematical and computational workings must be provided to obtain maximum credit. Mathematical techniques that are appropriate to the given task should have been used.

Solutions to computational tasks are provided in the form of correct numerical answers and charts. No Excel or Matlab/R coding provided with this document, please refer to spreadsheets distributed.

A. Portfolio Allocations [48%]

This computational task is best solved by matrix manipulation on a spreadsheet. Use Excel functions *MMULT(), MINV() and TRANSPOSE()* as necessary. If familiar, use MATLAB, R or Python.

Consider an investment universe composed of the following risky assets:

Asset	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$
A	0.04	0.07
B	0.08	0.12
C	0.12	0.18
D	0.15	0.26

with a correlation structure

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

Solution (all subquestions):

The covariance matrix can be constructed as $\boldsymbol{\Sigma} = diag(\boldsymbol{\sigma})\,\boldsymbol{R}\,diag(\boldsymbol{\sigma})$, where $diag(\boldsymbol{\sigma})$ is a diagonal matrix of standard deviations.

Global Minimum Variance portfolio is obtained by the optimisation s.t. the budget constraint,

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2}\boldsymbol{w'}\,\boldsymbol{\Sigma}\,\boldsymbol{w} \qquad \text{s.t. } \boldsymbol{w'1} = 1$$

To solve the optimization problem we formulate the Lagrange function with one multiplier γ,

$$L(\boldsymbol{w}, \gamma) = \frac{1}{2}\boldsymbol{w'}\,\boldsymbol{\Sigma}\,\boldsymbol{w} + \gamma(1 - \boldsymbol{w'1})$$

$$\begin{aligned}
\frac{\partial L(\boldsymbol{w},\gamma)}{\partial \boldsymbol{w}} &= \boldsymbol{\Sigma}\,\boldsymbol{w} - \gamma\mathbf{1} = 0 \qquad\qquad \text{gives} \quad \boldsymbol{w}^* = \gamma\boldsymbol{\Sigma}^{-1}\mathbf{1} \\
& \quad \text{insert the result for allocations into the next equation} \\
\frac{\partial L(\boldsymbol{w},\gamma)}{\partial \gamma} &= 1 - \boldsymbol{w}'\mathbf{1} = 0 \qquad\qquad \text{gives} \quad \gamma = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \\
& \quad \text{insert the result for the multiplier back into allocations' result!} \\
\boldsymbol{w}^* &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.
\end{aligned}$$

$$\begin{aligned}
1 &= \boldsymbol{w}'\mathbf{1} \qquad\qquad \textbf{Lagrangian multiplier in detail} \\
1 &= \gamma\mathbf{1}'\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1} \\
\gamma &= \frac{1}{\mathbf{1}'\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1}}
\end{aligned}$$

To solve the optimization problem for the targeted return (generalise $0.1 \to m$) we formulate another Lagrange function with single optimisation constraint – because the net of allocations invested (borrowed) in a risk-free asset, there is no separate budget equation.

$$L(\boldsymbol{w},\gamma) = \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \gamma(m - r - (\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{w})$$

First, setting $\frac{\partial L(\boldsymbol{w},\gamma)}{\partial \boldsymbol{w}} = 0$ gives directly $\boldsymbol{w}'\boldsymbol{\Sigma} - \gamma(\boldsymbol{\mu} - r\mathbf{1})' = 0$ and therefore,

$$\boldsymbol{w}^* = \gamma\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$$

Second, to find the multiplier γ analytically, substitute $\boldsymbol{w}^*$ into the derivative $\frac{\partial L(\boldsymbol{w},\gamma)}{\partial \gamma} = 0$,

$$\begin{aligned}
m - r - (\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{w} &= 0 \\
m - r - (\boldsymbol{\mu} - r\mathbf{1})'\,\underbrace{\gamma\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})} &= 0 \qquad (\boldsymbol{w}^* \text{ inserted}) \\
m - r - (\boldsymbol{\mu} - r\mathbf{1})'\gamma\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) &= 0 \\
(\boldsymbol{\mu} - r\mathbf{1})'\gamma\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) &= m - r
\end{aligned}$$

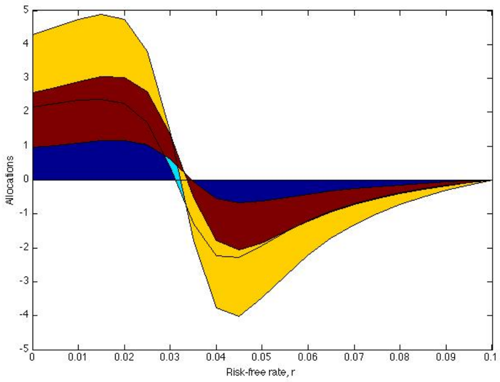
The answer must explain that additional step leading to $\boldsymbol{w}^*$.

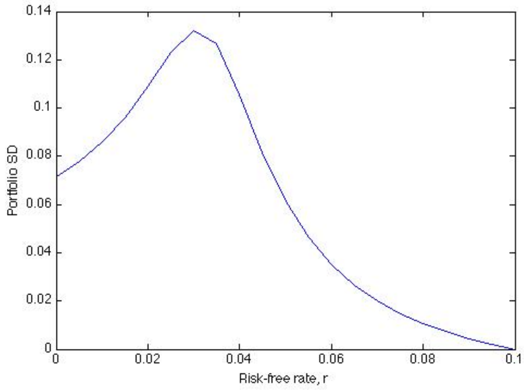
$$\gamma = \frac{m - r}{(\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})} \qquad \text{and so,} \qquad \boldsymbol{w}^* = \frac{(m - r)\;\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{(\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}$$

Computational results for the targeted return $m = 0.1$ and a range of risk-free rates are below.

r	0%	1%	2.5%	**3%**	4%	5%	7.5%
γ	0.0510	0.0818	0.2013	**0.2494**	0.1860	0.0751	0.0088
w_A	216.56%	235.99%	169.09%	**39.57%**	-223.95%	-192.84%	-52.59%
w_B	212.44%	238.79%	210.66%	**105.41%**	-153.48%	-155.75%	-45.71%
w_C	-333.41%	-365.71%	-274.96%	**-82.68%**	323.15%	285.97%	79.04%
w_D	162.34%	181.66%	155.99%	**73.13%**	-124.66%	-122.74%	-35.59%
σ_Π	0.0714	0.0858	0.1229	**0.1321**	0.1056	0.0613	0.0148

The figure on the left shows how allocations into four assets behave as we we vary the interest rate assumption: if the risk-free rate is high enough, the mean-variance optimisation 'suggests' selling assets and investing in money markets. Selling the asset with the largest standard deviation is just as risky as buying it! Getting the carry (fixed interest rate) might not be a sufficient compensation for the risk.





The tangency portfolio is one entirely invested in risky assets, notice the similarity to the Global Minimum Variance. To obtain the solution set up the optimisation to maximise the Sharpe Ratio,

$$\underset{\boldsymbol{w}}{\operatorname{argmax}} \frac{\boldsymbol{w'\mu} - r_f}{\sqrt{\boldsymbol{w'\Sigma w}}} = \frac{\mu_\Pi - r_f}{\sigma_\Pi}$$

$$\text{s.t.} \quad \boldsymbol{w'1} = 1$$

This is solved in the usual way with the Lagrangian method (not exam requirement). The analytical result for $\boldsymbol{w_T}$ in matrix form is below, but the use of ABC formulae from CQF Lecture is acceptable.

$$\boldsymbol{w_T} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})} \quad \Longleftrightarrow \quad \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{B - Ar}$$

where $\mu_T = \boldsymbol{w_T}'\boldsymbol{\mu}$ is portfolio return and $\sigma_T = \sqrt{\boldsymbol{w_T}'\boldsymbol{\sigma}\boldsymbol{w_T}}$ is portfolio risk.

Computational results are offered in the table below. Tangency allocations $\boldsymbol{w_T}$ omitted here but from your workings it must be possible to see them as more sensible than in the presence of the risk-free asset. When global minimum variance portfolio return is below the risk-free rate $\mu_G < r$, the SR is negative.

r	0%	1%	2.5%	**3%**	4%	5%	7.5%
μ_T	0.0388	0.0410	0.0538	**0.0817**	0.0065	0.0230	0.0294
σ_T	0.0277	0.0295	0.0471	**0.0976**	0.0590	0.0331	0.0270
SR (CML slope)	1.40	1.05	0.61	**0.53**	-0.57	-0.82	-1.69
VaR_{NF}	-2.56%	-2.77%	-5.58%	**-14.53%**	-13.09%	-5.39%	-3.35%
VaR_{tF}	-2.93%	-3.16%	-6.20%	**-15.80%**	-13.86%	-5.82%	-3.70%

$$\text{SR} = \frac{\mu_T - r}{\sigma_T} \qquad \Rightarrow \qquad \mu_T = r + \text{SR} \times \sigma_T \quad \text{is the equation for CML}$$

The slope of Capital Market Line is equal to the Sharpe Ratio of the tangency portfolio. Such portfolio has two properties: **a.** SR is at the maximum and **b.** marginal contributions to risk from each individual asset are equal. Capital Market Line (CML) is **the** Efficient Frontier for the optimisation problem with the risk-free asset – the other assets get shorted and proceeds invested in the risk-free asset.

Portfolio value Π can make a move of Factor $\times\, \sigma$ downwards and result in the loss above the VaR (as a percentage, on negative scale) given above.

1. the Normal distribution gives Factor $= -\mathbf{2.32635}$.

2. the Student's t distribution with $\nu = 30$ gives Factor $= -\mathbf{2.45726}$.

B. Value at Risk on FTSE 100 [20%]

$\boxed{\textbf{Solution:}}$ Below is a comparison of two VaR measures calculated using sample standard deviation (blue) and GARCH-filtered standard deviation (red) according to VaR $= \mu + \sigma \times$ Factor. **While not required for the exam**, VaR based on GARCH is brought as more robust approach that translates to lesser variation in capital required.

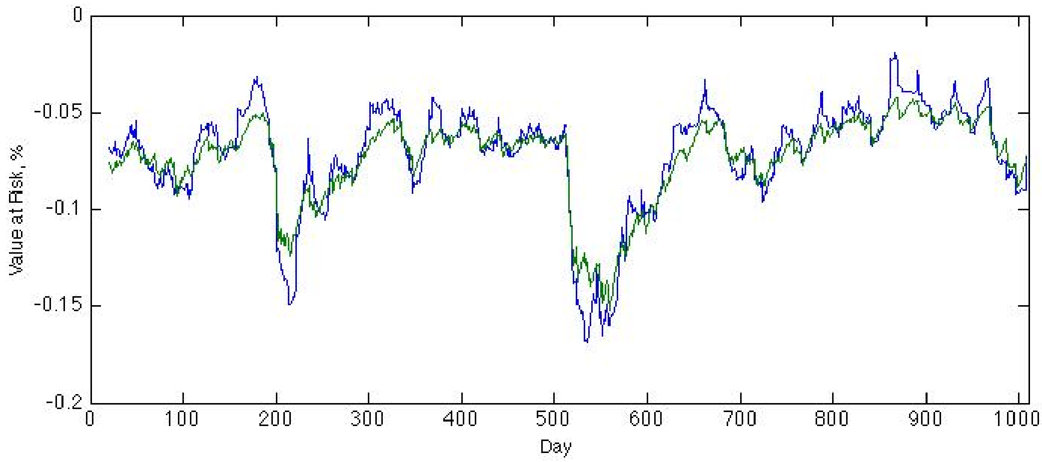


Figure 1: VaR time series plot, FTSE100 index return

If the actual losses exceed the VaR number too often (here in percentage terms) the calculation methodology is rejected. How many is too often? 99% confidence level means that a well-calibrated VaR model would produce exactly 1% breaches, no more and no less. Risk managers will have to develop a more nuanced calculation that one based on the common models, such as GARCH(1,1).

B.2 Backtesting must be an objective part of risk model validation. Here, VaR breaches are clustered and **not** independent of the level and in time. The use of GARCH-filtered variance improves the model marginally. To make % of breaches closer to 1% requires improved forecasting of the short-term volatility.

N_{obs}	VaR_{SD}	VaR_{GARCH}	**Expected**
978	2.76%	2.25%	1%

Table 1: % of breaches when the loss (10D realised return) was worse than VaR

Methodology Notes:

- We use the mean of the entire dataset (daily average return) as a predictor of the daily move. Then, we scale the quantity to relate to moves over 10 days as $\mu_{1D} \times 10$.

  Empirically, the daily average return on a market index is a very small quantity, eg, $\mu = 0.000386$ for our dataset. It makes a small impact on squared differences $(u_t - \mu)^2$ and therefore could be dropped from calculation for both, standard deviation and GARCH.

  Variance is an average of squared returns (residuals) $\sigma^2 \approx \sum u_t^2/(n-1)$.

- We use 99%/10day VaR as a predictive, forward-looking measure.

- There are $N = 978$ comparisons between VaR_{SD} and VaR_{GARCH} because we need the first 20 days to calculate $\text{VaR}_{SD,t}$ and then have to 'wait' for the realised return $ln(S_{t+10}/S_t)$ to be available.

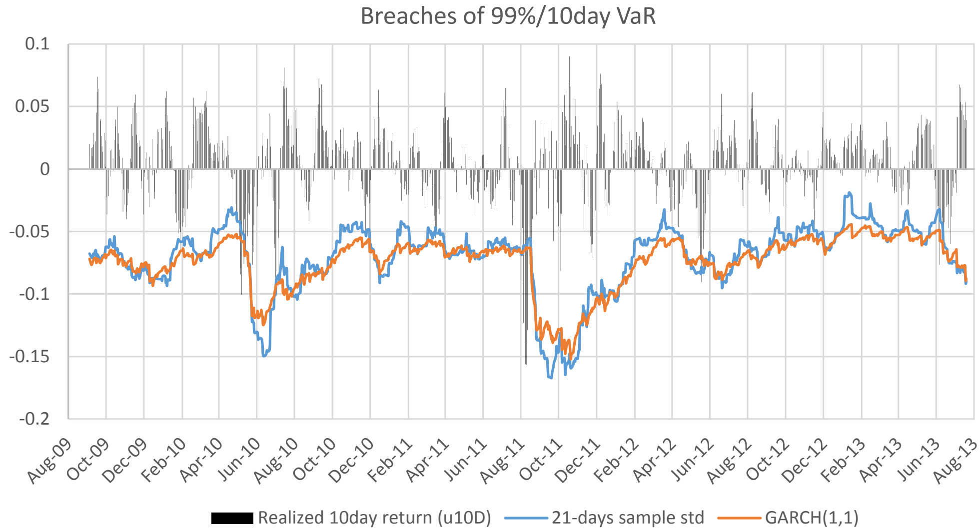


Figure 2: Chart credit to Lu Zhao, CQF Delegate (Hong Kong)

- Rounding matters. Using a rounded value of the Standard Normal 1% percentile (Factor), such as 2.32635 vs. *NORMSINV(0.01)* leads to a slightly different percentage of breaches.

- To evaluate the independence of occurrences we can use the simple tools: check for clustering, conditional probability, and lag plot. (There is no single common test *iid*-ness of such data as VaR breaches or regression residuals.)

- To calculate the conditional probability of a breach in VaR, given that a breach was observed for the previous period we count cases of two consecutive breaches. A column with $1, 1, 1$ gives two.

VaR_{SD}	17/27	63%
VaR_{GARCH}	16/22	72.7%

Table 2: VaR breaches: conditional probability (on past breach $t-1$)

- Observations from an *iid* dataset would remain within a circle on the Lag Plot of returns:

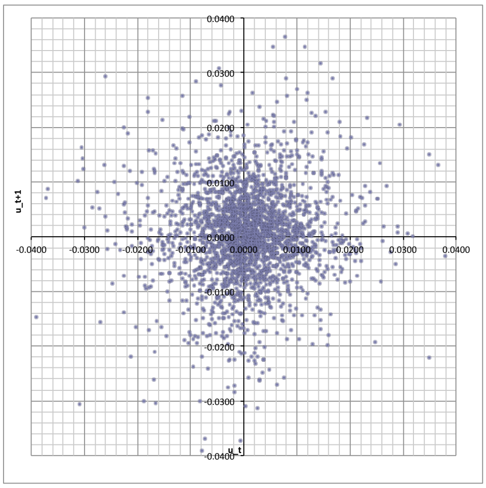


Figure 3: Examining location-dispersion ellipsoid (u_t plotted vs. u_{t-1}) gives a robust check for *iid*-ness.

Arbitrary sampling is a plague of every VaR and volatility forecasting model including GARCH: until the recent observations causing high volatility drop of the sample there is nothing to do.

C. Stochastic Calculus [32%, 8% each]

1. Consider a basket of N assets each following the Geometric Brownian Motion SDE

$$dS_i = S_i\mu_i dt + S_i\sigma_i dX_i \qquad \text{for } 1 \le i \le N$$

The price changes are correlated as measured by the linear correlation coefficients ρ_{ij}.
Invoke the multi-dimensional Itô Lemma to write down the SDE for $F(S_1, S_2, \dots, S_N)$ *in the most compact form possible* but with distinct drift and diffusion terms. Apply $dX_i dX_j \to \rho_{ij}dt$.

$\boxed{\textbf{Solution:}}$ With changes in the underlying asset price(s) S_i, the value of derivative F is re-priced as follows (change for $t + dt$ omitted for now):

$$\begin{aligned} F(S_1 + dS_1, S_2 + dS_2, \dots, S_N + dS_N) &= F(S_1, S_2, \dots, S_N) + \\ &= +\sum_{i=1}^{N}\frac{\partial F}{\partial S_i}dS_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2 F}{\partial S_i \partial S_j}(dS_i dS_j) + \cdots \end{aligned}$$

For $(dS_i dS_j)$ or dS_i^2 for $j = i$, terms smaller than $O(dt^2)$ dropped

$$\begin{aligned} dF &= \sum_{i=1}^{N}\frac{\partial F}{\partial S_i}dS_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2 F}{\partial S_i \partial S_j}S_i S_j \sigma_i \sigma_j dX_i dX_j \\ & \quad \text{Using } dX_i dX_j \to \rho_{ij}dt \qquad \rho_{ii} = 1 \text{ terms remain inside summation} \\ dF &= \sum_{i=1}^{N}\frac{\partial F}{\partial S_i}\underbrace{dS_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2 F}{\partial S_i \partial S_j}S_i S_j \rho_{ij}\sigma_i \sigma_j dt \end{aligned}$$

To obtain the distinct drift and diffusion, we have to insert GBM SDE $dS_i = \underbrace{S_i\mu_i dt + S_i\sigma_i dX_i}$,
When evaluating $(dS_i dS_j)$ or dS_i^2 for $j = i$, we simplify by dropping terms $O(dt^2)$

$$\begin{aligned} dF &= \sum_{i=1}^{N}\frac{\partial F}{\partial S_i}(S_i\mu_i dt + S_i\sigma_i dX_i) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2 F}{\partial S_i \partial S_j}S_i S_j \rho_{ij}\sigma_i \sigma_j dt \\ &= \left(\sum_{i=1}^{N}\mu_i S_i\frac{\partial F}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}\sigma_i\sigma_j S_i S_j\frac{\partial^2 F}{\partial S_i \partial S_j}\right)dt + \sum_{i=1}^{N}\sigma_i S_i\frac{\partial F}{\partial S_i}dX_i. \end{aligned}$$

Insert a derivative *wrt* time to account for change over $t + dt$,

$$dF = \left(\frac{\partial F}{\partial t} + \sum_{i=1}^{N}\mu_i S_i\frac{\partial F}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}\sigma_i\sigma_j S_i S_j\frac{\partial^2 F}{\partial S_i \partial S_j}\right)dt + \sum_{i=1}^{N}\sigma_i S_i\frac{\partial F}{\partial S_i}dX_i.$$

2. Construct an SDE for the process $Y(t) = e^{\sigma X(t) - \frac{1}{2}\sigma^2 t}$ and show it is, in fact, an Exponential Martingale of the form $dY(t) = Z(t)\,g(t)\,dX(t)$. Identify the terms $g(t)$ and $Z(t)$.

A diffusion process $Y(t)$ is a martingale if its SDE has no drift term. The SDE can be constructed by evaluating partial derivatives of a function $F(X(t), t) = Y(t)$ and constructing as follows:

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)dt + \frac{\partial F}{\partial X}dX(t).$$

$\boxed{\textbf{Solution:}}$ Seeing $Y(t)$ as a function $F(X(t),t)$ of two variables, time t and stochastic $X(t)$,

$$\begin{aligned}
\frac{\partial F}{\partial t} &= -\frac{1}{2}\sigma^2 e^{\sigma X(t)-\frac{1}{2}\sigma^2 t} \\
\frac{\partial F}{\partial X} &= \sigma e^{\sigma X(t)-\frac{1}{2}\sigma^2 t} \\
\frac{\partial^2 F}{\partial X^2} &= \sigma^2 e^{\sigma X(t)-\frac{1}{2}\sigma^2 t}
\end{aligned}$$

By Itô, the dynamics of the process $Y(t) = F(X(t),t)$ is given by

$$\begin{aligned}
dY(t) &= \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right) dt + \frac{\partial F}{\partial X} dX(t) \\
&= \underbrace{\left(-\frac{1}{2}\sigma^2 e^{\sigma X(t)-\frac{1}{2}\sigma^2 t} + \frac{1}{2}\sigma^2 e^{\sigma X(t)-\frac{1}{2}\sigma^2 t}\right)} dt + \sigma e^{\sigma X(t)-\frac{1}{2}\sigma^2 t} dX(t) \\
&= \underline{e^{\sigma X(t)-\frac{1}{2}\sigma^2 t}}\, \sigma\, dX(t) \equiv \underline{Z(t)}\, g(t)\, dX(t).
\end{aligned}$$

We eliminated the drift, therefore, $Y(t)$ **is a martingale**. In terms of the Exponential Martingale formalisation, $Z(t)$ is the same as $Y(t)$ and $g(t) = \sigma$. Soluable SDEs are scarce. This process $Y(t)$ is called Doleans Exponential of the Brownian Motion.

3. What about the process $Y(t) = \sqrt{t}X(t) - \int_0^t \frac{X(s)}{2\sqrt{s}} ds$, is it a martingale? This <u>must not</u> be solved by differentiation. Instead, think which function to apply the Itô Lemma to (from $Y(t)$ expression), and show that Y(t) is equal to an Itô Integral. **Solution:** the integral version of Itô Lemma is

$$F(t) - F(0) = \int_0^t \frac{dF}{dX} dX(s) + \frac{1}{2}\int_0^t \frac{d^2F}{dX^2} ds$$

$\boxed{\textbf{Solution:}}$ The answer path that avoids the pitfalls of brute force differentiation over $Y(t)$ is at first, to invoke the Itô Lemma in order to construct an SDE for $F(X(t),t) = \sqrt{t}X(t)$,

$$\frac{\partial F}{\partial t} = \frac{X(t)}{2\sqrt{t}}, \qquad \frac{\partial F}{\partial X} = \sqrt{t}, \qquad \frac{\partial^2 F}{\partial X^2} = 0.$$

$$dF = \left(\frac{X(t)}{2\sqrt{t}} + \frac{1}{2}\times 0\right) dt + \sqrt{t}dX(t) = \frac{X(t)}{2\sqrt{t}}dt + \sqrt{t}dX(t)$$

We already see similarities between dF and $Y(t)$ but the important distinction is the $dX(t)$ term (diffusion), which is a stochastic increment. Move from dF to $F(t) - F(0)$ by the integral Itô Lemma

$$\begin{aligned}
F(t) - F(0) &= \int_0^t \frac{X(s)}{2\sqrt{s}} ds + \int_0^t \sqrt{s} dX(s) \\
&= \text{remember} \qquad F = \sqrt{t}X(t) \\
\sqrt{t}X(t) - \sqrt{t}X(0) &= \int_0^t \frac{X(s)}{2\sqrt{s}} ds + \int_0^t \sqrt{s} dX(s)
\end{aligned}$$

By BM's property $X(0) = 0$ and re-arranging, we spot the expression that matches $Y(t)$

$$\begin{aligned}
\underbrace{\sqrt{t}X(t) - \int_0^t \frac{X(s)}{2\sqrt{s}} ds} &= \int_0^t \sqrt{s} dX(s) \\
Y(t) &= \int_0^t \sqrt{s} dX(s)
\end{aligned}$$

The process $Y(t)$ is an Itô integral over the increment of BM $dX(s)$ and therefore, is a martingale.

4. Covariance matrix can be decomposed as $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A'}$ (Cholesky decomposition). The result, lower triangular matrix $\boldsymbol{A}$, is used for imposing correlation on a vector of random Normal variables $\boldsymbol{X}$.

$$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

- For the two-variate case, show analytically what $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A'}$ is equal to.
- Write down the results for $Y_1(t)$ and $Y_2(t)$ which are correlated via $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$.
- Briefly discuss, does $Y_2(t)$ keep the properties of the Brownian Motion if $X_1(t), X_2(t)$ are random Normal? **Note:** consider distribution/variance of the increment of $Y_2(t)$.

Solution:

- The answer reconstructs the covariance matrix as $\boldsymbol{A}\boldsymbol{A'} = \boldsymbol{\Sigma}$

$$\begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \times \begin{pmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

where

$$\mathrm{Cov}[Y_1, Y_2] \equiv \mathrm{Cov}[dY_1, dY_2] = \rho\sigma_1\sigma_2$$

Each Brownian Motion process is regarded as independent, and so two Brownian Motions are called orthogonal to each other, the notation $dX, dX_\perp$ is used. Very important to note that when operating with the Brownian Motion/Wiener Process the references refer to the Random Variable $X(t)$ but imply the distribution for its increment dX. Hence $Y \to dY$ above.

- Matrix multiplication (linear algebra) $\boldsymbol{A}\boldsymbol{X}$ gives

$$\begin{aligned} Y_1 &= \sigma_1 X_1 \\ Y_2 &= \rho\,\sigma_2 X_1 + \sqrt{1-\rho^2}\,\sigma_2 X_2 \end{aligned}$$

- As a linear combination of two Brownian Motions, $Y_2(t)$ **keeps the properties of BM**:
  - simplify $\sigma_1 = \sigma_2 = 1$, the increment of such standardised $Y_2(t) - Y_2(s), \forall s < t$ follows the distribution $\sim N(0, \tau)$, which is the sum

    $$N(0, \tau\rho^2) \qquad \text{and} \qquad N(0, \tau(1-\rho^2))$$

    it is also said that the Normal distribution is closed under sum; and
  - in further detail, consider the variance of a random variable algebra ($\mathbb{V}\mathrm{ar}$ operator)

    $$\begin{aligned} \mathbb{V}\mathrm{ar}[Y_2(t)] &= \mathbb{V}\mathrm{ar}\left[\rho\, X_1(t) + \sqrt{1-\rho^2}\, X_2(t)\right] \\ &= \rho^2\, \mathbb{V}\mathrm{ar}[X_1(t)] + (1-\rho^2)\, \mathbb{V}\mathrm{ar}[X_2(t)] \\ &= \rho^2\tau + (1-\rho^2)\tau = \tau \quad \text{variance scales with time; Standard BM has } \sigma^2 = 1 \end{aligned}$$

    It follows that the increment $Y_2(t) - Y_2(s)$ will be distributed as $\sim N(0, t-s) \equiv N(0, \tau)$.