CQF Module 2 Examination

June 2013 Cohort

Instructions

All questions must be attempted. Complete workings must be provided to obtain maximum credit. Each plot must have a brief explanation.

Books and lecture notes may be referred to. Help from others is not permitted. If you are unclear about any question or what is required, contact the tutor only on Richard.Diamond@fitchlearning.com.

A. Strategies and Pricing [36%]

1. A European put option is traded at \$4 with the underlying price at \$95. Time to expiry is one month and the strike is \$100. The risk-free interest rate is 3% per annum. Devise a trading strategy to explore the arbitrage opportunity. What is the minimum profit of the strategy?

2. To hedge a Binary Call option B, a trader creates a call spread by short-selling a vanilla call option with the lower strike (below Binary's) $\lambda_1 < 0$ and buying a call option with the higher strike $\lambda_2 > 0$.

$$B + \lambda_1 C_1 + \lambda_2 C_2 = 0$$

   (a) Construct a payoff diagram for a call spread that offsets a Binary Call with the strike \$100. **Note:** use $\lambda_1 = -1$ and $\lambda_2 = 1$ to see the shape of the payoff.

   (b) For the option prices listed in the table below, find symmetric values $|\lambda_1| = |\lambda_2|$ that offset a payoff of exactly one Binary Call quoted in the market at $B = \$0.5$.

Option	Strike	Maturity	Price
C_1	\$90	1	\$14.81
C_2	\$110	1	\$4.94

   Construct a payoff diagram for a basket of all three options using the updated values of λ_1, λ_2.

3. Implement the multi-step Binomial Method to price a European put with the following parameters: strike $K = 100$ and maturity $T = 1$. Asset price level $S_0 = 100$ and interest rate $r = 0.05$.

   (a) For the constant number of time steps in the tree NTS=4, calculate the value of the option for a range of volatilities and plot the result.

   (b) Then, fix volatility at $\sigma = 0.2$ and plot the value of the option as a function of the number of time steps in the tree, NTS=$1, 2, \ldots, 50$. You will need a different tree for each NTS value.

Note: This is a computational task. Preferred solution method is a function written in VBA but Excel spreadsheets with binomial trees will be accepted given the plots are correct. For simplicity, use compound rate as a substitute to simple (discrete) rate.

B. Portfolio Optimisation and Risk [44%]

Note: Part of this task is computational. It is best to present matrix manipulations on a spreadsheet. You can use the necessary Excel functions such as *MMULT(), MINV() and TRANSPOSE()*. Alternatively, you can use MATLAB or R if familiar with those environments.

Consider an investment universe composed of the following assets:

Asset	μ	σ
A	0.04	0.07
B	0.08	0.12
C	0.12	0.18
D	0.15	0.26

with a correlation structure

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

Denote the column vector of asset weights by $\boldsymbol{w}$, the column vector of asset returns by $\boldsymbol{\mu}$, and the covariance matrix by $\boldsymbol{\Sigma}$.

1. Compute the covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{SRS}$, where $\boldsymbol{S}$ is a diagonal matrix of standard deviations.

2. Consider the following optimization task

$$\min_{\mathbf{w}} \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\Sigma} \boldsymbol{w}$$

   subject to constraints

$$\begin{aligned} \boldsymbol{w}^{\boldsymbol{T}} \mathbf{1} &= 1 \\ \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\mu} &= 0.1 \end{aligned}$$

   (a) Explain in plain English the purpose of the optimization.

   (b) Solve this optimization task **analytically** – provide workings for the Lagrangian method.

   (c) Compute the standard deviation σ_Π of this optimal portfolio allocation $\mathbf{w}^*$.

   (d) Simulate the Efficient Frontier for this investment universe and identify this portfolio on it . The number of points to simulate the Frontier is your choice.

3. Use the optimal allocations $\boldsymbol{w}^*$ to estimate the Analytical VaR for the portfolio with $c = 99\%$ confidence

$$\mathrm{VaR}(X) = \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\mu} + \sqrt{\boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\Sigma} \boldsymbol{w}} \times \text{Factor}$$

   where for the Factor is drawn from

   (a) the Normal distribution Factor $= \Phi^{-1}(1 - c)$.

   (b) the Student's t distribution Factor $= T_\nu^{-1}(1 - c)$ with $\nu = 30$.

   Student's t distribution allows more realistic modelling of 'fatter tails' of returns distribution. The distribution changes its shape depending on the degrees of freedom parameter ν. You can use Excel function *TINV()* but note that it provides an answer for the two-tail Factor.

4. Calculate contributions of individual assets to Value at Risk, estimated with Student's t Factor, using $\boldsymbol{w^T}\text{VaR}(X)$. Construct and bar chart.

5. Download the spreadsheet built during the Value at Risk lecture. For the column with the Estimated P&L result, find the numerical answer to the Value at Risk by the following method:

   (a) Sort the column in an ascending order. Create a parallel column with an index number from 1 to the N, i.e., $i = 1, 2, ..., 1009$

   (b) $\frac{i}{N}$ represents the cumulative probability of each P&L figure $\Pr(X \le \text{Sorted P\&L})$. Use this property to find the first percentile (i.e., 99% VaR figure).

   (c) Check your result using Excel's *PERCENTILE()* function.

C. Useful Processes and Martingales [20%]

A diffusion process $Y(t)$ is a martingale if its SDE does not have a drift term. The SDE can be constructed by evaluating partial derivatives of a function $F(t, X)$ and substituting into

$$dY(t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right) dt + \frac{\partial F}{\partial X} dX(t)$$

Assume that all $Y(t)$ satisfy the integrability condition.

1. (a) Construct an SDE for the process $Y(t) = e^{\sigma X(t) - \frac{1}{2}\sigma^2 t}$ and (b) show that the process is, in fact, an Exponential Martingale.
   Note: if the generic Exponential Martingale follows $dZ(t) = Z(t)g(t)dX(t)$ as defined in Martingales Fundamentals lecture, to show (b) identify the $g(t)$ and $Z(t)$ for this expression $Y(t)$.

2. $Y(t) = wX_1(t) + \sqrt{1 - w^2}X_2(t)$ is an expression to create a correlated BM from two uncorrelated BMs, where w is factor loading that links to correlation. Assume that $X_1(t)$ and $X_2(t)$ are independent Brownian Motions <u>with drifts</u>, such that

$$\begin{aligned} dX_1(t) &= \mu_1 dt + \sigma_1 dW_1(t) \\ dX_2(t) &= \mu_2 dt + \sigma_2 dW_2(t) \end{aligned}$$

   Construct an SDE for $Y(t)$ and answer the following questions:

   (a) Is the process $Y(t)$ a martingale?

   (b) How does answer to (a) depends on the value of constant w? Consider the drift term.

   (c) Will $Y(t)$ keep the properties of the Brownian Motion? Consider the increment $Y(t) - Y(s)$.

3. A Poisson random variable $N(t) \sim Po(\lambda)$ creates a jump process with the fixed increment Δ. The probability of the jump between times s and t is $\Pr(\Delta) \approx \lambda(t - s)$.

   Now, consider a compensated Poisson process $M(t) = N(t) - \lambda t$ where $\mathbb{E}[N(t)] = \lambda t$. Use this property in order to provide <u>a discrete-time proof that $M(t)$ is a martingale</u>.
   Note: to explain both conditions,

$$\begin{aligned} \mathbb{E}[|M(t)|] &< \infty \qquad \text{(the process is bounded)} \\ \mathbb{E}[M(t)|\mathcal{F}_s] &= M(s) \qquad \forall t > s \end{aligned}$$