# CQF Module 4 Examination

## Instructions

All questions must be attempted. Books and lecture notes may be referred to. Spreadsheets and VBA may be used. Help from other people is not permitted.

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$dX$ is the usual increment of a Brownian motion.

1. In this problem use continuous compounding.

   (a) A zero-coupon bond has a principal of 100 and matures in 3 years. The market price of the bond is 82. Calculate the *yield to maturity*, *duration* and *convexity* of the bond.

   (b) A coupon bond pays out 3% each year on a principal of 100. The bond matures in 5 years and is currently priced at 90. Find the yield to maturity, duration and convexity of the bond.

2. Consider the Black-Derman & Toy (BDT) short-rate model given by

$$d(\log r) = \left(\theta(t) + \frac{d(\log \sigma(t))}{dt}\log r\right)dt + \sigma(t)\,dX.$$

   Using Itô, write down the BDT model as

$$dr = A(r,t)\,dt + B(r,t)\,dX.$$

3. Consider the spot rate $r$, which evolves according to the SDE

$$dr = u(r,t)\,dt + w(r,t)\,dX.$$

   The extended Hull and White model has drift and diffusion

$$u(r,t) = \eta(t) - \gamma r, \qquad w(r,t) = c,$$

   in turn, where $\eta(t)$ is an arbitrary function of time $t$ and $\gamma$ and $c$ are constants. Deduce that the value of a zero coupon bond, $Z(r,t\,;T)$ which has

$$Z(r,T;T) = 1$$

   in the extended Hull and White model is given by

$$Z(r,t\,;T) = \exp\left(A(t;T) - rB(t;T)\right),$$

   where

$$B(t\,;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right)$$

   and

$$A(t;T) = -\int_t^T \eta(\tau)\,B(\tau;T)\;d\tau + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right).$$

   **Note: You are required to solve the Bond Pricing Equation for this model.**

4. Consider the process given by

$$dU_t = -\gamma U_t dt + \sigma dX_t, \; U_0 = u,$$

where $\gamma$, $\sigma$ are constants. Solve this equation for $U_t$ and hence write down the expectation $\mathbb{E}[U_t]$ and variance $\mathbb{V}[U_t]$.

5. Consider the process $dZ = r(t) Z dt$, where $r(t)$ is stochastic and $Z = Z(r, t\,; T)$ is a zero coupon bond. Provide a bond pricing partial differential equation and invoke Feynman-Kac to show the solution requires the risk-neutral measure $\mathbb{Q}$:

$$Z(r, t\,; T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\, ds\right)\right]$$

Use the **HJM** model - *MC.xlsm* spreadsheet from the HJM lecture and the fact that $r(t) = f(t, t)$ in order to price a zero coupon bond with $t = 0$ and maturity of your choice from the range $T = 0.5, ..., 2$ years; using more than 1000 simulations. **Note: this is a computational task and to obtain maximum credit a convergence diagram must be included.**

## Mark Scheme

1. 10
2. 5
3. 10
4. 10
5. 15