Asian Option Pricing using Monte Carlo Simulation

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1 Stock Price Simulation

1.1 Milstein Scheme Simulation

The underlying stock price follows the Geometric Brownian Motion (GBM), whose dynamic is

$$dS_t = r_t S_t dt + \sigma_t dW_t \tag{1}$$

where r_t is the short rate at time t, and σ_t is the implied volatility at time t. W_t is the Wiener process.

The Forward Euler-Maruyama methods for the GBM gives

$$S_{t+\Delta t} - S_t = r_t S_t \Delta t + \sigma_t \phi \sqrt{\Delta t}$$

where Δt refers to as the time step in discrete time. And ϕ is a standard normal random number. That is, $\phi \sim N(0, 1)$.

The Milstein method corrects the Forward Euler-Maruyama with a term on error level of $O(\delta t)$. Given a stochastic process Y_t with

$$dY_t = A(Y_t, t)dt + B(Y_t, t)dW_t$$

the discretization using Milstein scheme is

$$Y_{t+\Delta t} - Y_t = A\Delta t + B\phi\sqrt{\Delta t} + \frac{1}{2} B \frac{\partial B}{\partial Y_t}(\phi^2 - 1)\Delta t$$

where $\frac{1}{2}(\phi^2 - 1)\Delta t$ is the Milstein correction term. For GBM, the Milstein scheme yields

$$S_{t+\Delta t} - S_t = r_t S_t \Delta t + \sigma_t S_t \phi \sqrt{\Delta t} + \frac{1}{2} S_t \sigma^2 (\phi^2 - 1)\Delta t$$

1.2 Antithetic Variance Reduction

The antithetic variable technique attempts to reduce the variance by introducing negatively correlated random numbers between pair of observations. While simulating the GBM, one set of standard normal random numbers is generated, labeled as $\phi^n \sim N(0, 1)$. Then $-\phi^n \sim N(0, 1)$ also has a standard normal distribution.

The pairs $\{(\phi^n, -\phi^n)\}$ are distributed more desirable than $2n$ independent samples, since the samples with antithetic variable have mean 0 and negative correlation.

The whole simulation procedure is shown in 1.

Algorithm 1 Stock Price Generation

procedure S_T PROJECTION$(S_0, E, T-t, \sigma, r)$
    $S_1(0,:) = S_0$, $S_2(0,:) = S_0$
    $N = (T-t)/\Delta t$ ▷ Define number of time steps
    for $i = 1$ to N **do**
        for $\omega = 1$ to M **do** ▷ Define number of scenarios
            Generate ϕ
            $S_1(i,\omega) = S_1(i-1,\omega) * (1 + r * \Delta t + \sigma * \phi * \sqrt{\Delta t} + \frac{1}{2} * \sigma^2 * (\phi^2 - 1) * \Delta t)$
            $S_2(i,\omega) = S_2(i-1,\omega) * (1 + r * \Delta t + \sigma * (-\phi) * \sqrt{\Delta t} + \frac{1}{2} * \sigma^2 * (\phi^2 - 1) * \Delta t)$
    return $\{S_1, S_2\}$

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