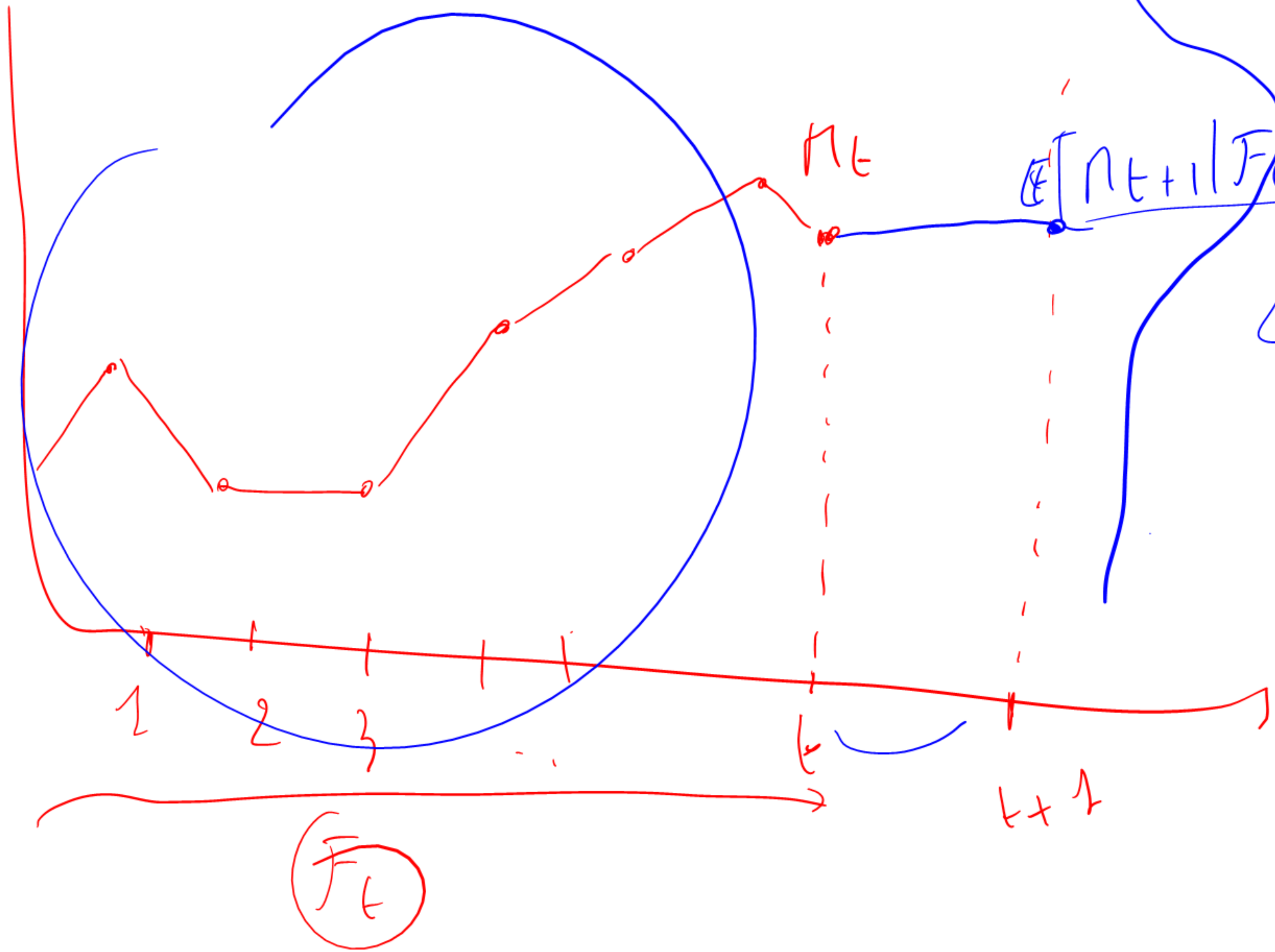
M_t

$\mathcal{F}_t \rightarrow$ filtration

$\mathcal{F}_t$ - adapted process.

$\mathbb{E}[M_{t+1} | \mathcal{F}_t] = M_t$

$\mathbb{E}[M_{t+1} | \mathcal{F}_t]$



$Y(t) = X^2(t)$

$y = f(x) = x^2$

$\frac{df}{dx} = 2x \qquad \frac{d^2 f}{dx^2} = 2$

Apply Itô to the function f and the BM $X(t)$

$$dY(t) = \frac{df}{dx}\Big|_{X(t)} dX(t) + \frac{1}{2}\frac{d^2 f}{dx^2}\Big|_{X(t)} \cdot \underbrace{dt}_{dX^2 \to dt}$$

$$= 2X(t)\,dX(t) + \frac{1}{2} \times 2\,dt$$

$$\underbrace{\int_0^T dY(t)}_{Y(t) - Y(0)} = \int_0^T dt + 2\int_0^T X(t)\,dX(t)$$

Itô integral

$$Y(t) - Y(0) = T + \boxed{2\int_0^T X(t)\,dX(t)}$$

$y(T) = x^2(T) \qquad y(0) = x^2(0) = 0$

$$x^2(T) = T + 2\int_0^T x(t)\,dx(t)$$

$$\mathbb{E}[x^2(T)] = T$$

Take the expectation

$$\mathbb{E}[x^2(T)] = T + 2\underbrace{\mathbb{E}\left[\int_0^T x(t)\,dx(t)\right]}_{=0}$$

$= ?$

$$\mathbb{E}\left[h(X)\right] = \int_{-\infty}^{+\infty} h(x)\, \boxed{p(x)\,dx}$$

R.V.

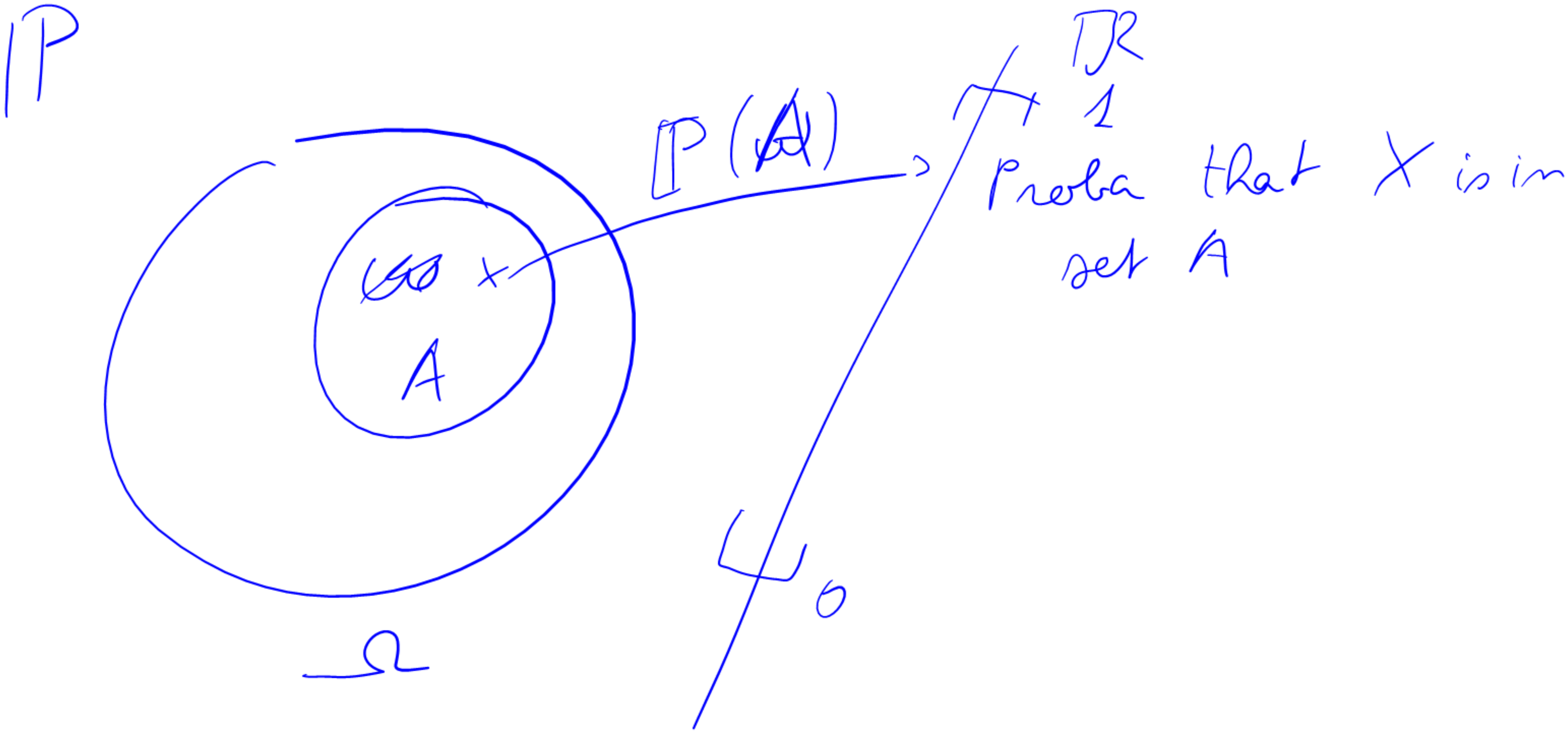
PDF of X

Cumulative D. F (CDF)

$$P[X \le x] \rightarrow \boxed{p(x) = \frac{dP}{dx}}$$

$$p(x)\,dx = dP$$

$$\mathbb{E}\left[h(X)\right] = \int_{-\infty}^{+\infty} h(x)\, \underbrace{dP}_{\text{differential of the CDF}}$$



ℙ
ℙ(A)
ℝ
1
Proba that X is in set A
A
0
Ω

Before

$$\mathbb{E}[h(x)] = \int_{-\infty}^{+\infty} h(x)\, p(x)\, dx$$

$$= \boxed{\int_{-\infty}^{+\infty} h(x)\, dP(x)}$$

Now (since 1933!)

$$\mathbb{E}[h(x)] = \int_{\Omega} h(X(\omega))\, d\mathbb{P}(\omega)$$

events

$$\mathbb{E}\left[\int_0^T f(t,X_t)\,dt\right]$$

$$= \int_\Omega \int_0^T f(t,X_t)\,dt\,d\mathbb{P}$$

Fubini

$$= \int_0^T \boxed{\int_\Omega f(t,X_t)\,d\mathbb{P}}\,dt$$

$$= \int_0^T \underbrace{\mathbb{E}\left[f(t,X_t)\right]}\,dt$$

$$\circledast \quad \mathbb{E}\left[\int_0^T f(t)\,dX_t \,\middle|\, \mathcal{F}_s\right] = \int_0^s f(t)\,dX_t$$

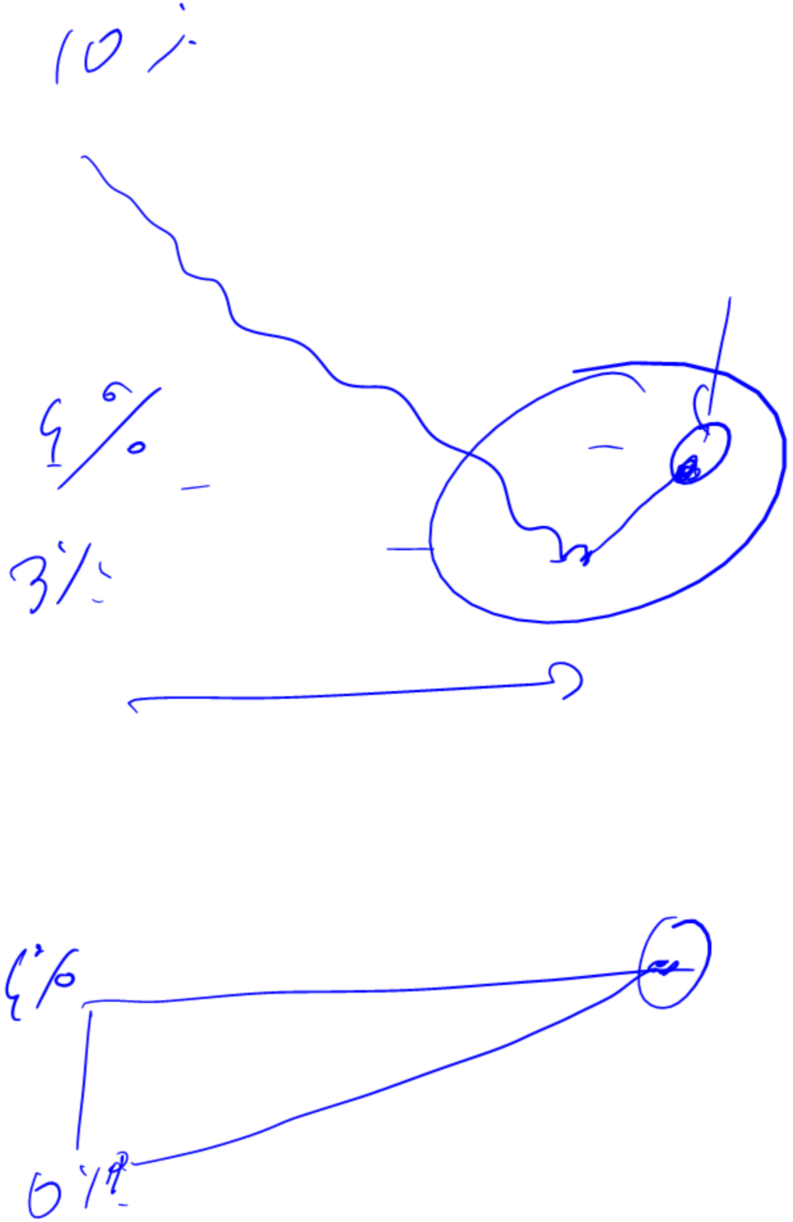
$$Y(u) = \int_0^u f(t)\,dX_t)$$

$$\mathbb{E}\left[\,Y(T) \,\middle|\, \mathcal{F}_s\right] = Y(s)$$

Martingale condition!

set $s = 0$

$$\underbrace{\mathbb{E}\left[\int_0^T f(t)\,dX(t) \,\middle|\, \cancel{\mathcal{F}_0}\right]}_{\mathbb{E}\left[\int_0^T f(t)\,dX(t)\right]} = \underbrace{\int_0^0 f(t)\,dX(t)}_{=0}$$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0%

$$dY(t) = f(t)\,dt + g(t)\,dX(t) \qquad Y(0) = Y_0$$

can be random

$$\boxed{\int_s^t \cancel{d}Y(u)} = \int_s^t f(u)\,du + \int_s^t g(u)\,dX(u)$$

$$Y(t) - Y(s)$$

$$\boxed{Y(t) = Y(s) + \int_s^t f(u)\,du + \int_s^t g(u)\,dX(u)}$$

To check whether $Y(t)$ is a martingale, I need to check if

$$E\left[Y(t) \mid F_s\right] = Y(s)$$

$$\mathbb{E}[Y(t) \mid \mathcal{F}_s] = \mathbb{E}\left[Y(s) + \int_s^t f(u)\,du + \int_s^t g(u)\,dX_u \,\Big|\, \mathcal{F}_s\right]$$

1 2 3

$$= \mathbb{E}[Y(s) \mid \mathcal{F}_s] + \mathbb{E}\left[\int_s^t f(u)\,du \,\Big|\, \mathcal{F}_s\right] + \mathbb{E}\left[\int_s^t g(u)\,dX(u) \,\Big|\, \mathcal{F}_s\right]$$

1 $\quad \mathbb{E}[Y(s) \mid \mathcal{F}_s] = Y(s)$

3 $\quad \mathbb{E}\left[\int_s^t g(u)\,dX_u \,\Big|\, \mathcal{F}_s\right] \overset{\text{Because mart}}{=} \int_s^s g(u)\,dX(u) = 0$

$$\boxed{\mathbb{E}[Y(t) \mid F_0] = Y(0)} + \underbrace{\mathbb{E}\left[\int_0^t f(u)\,du \mid F_0\right]}_{\equiv 0}$$

if $Y(t)$ is a martingale then

$$\Rightarrow \boxed{f(u) = 0 \quad \text{for all } u}$$

$Y(t)$ solves the SDE

$$\boxed{dY(t) = g(t)\,dX(t)} \qquad Y(0) = Y_0$$

$$dY(t) = f(t)\,dt + g(t)\,dX(t)$$

drift

- deterministic
- scales with dt

diffusion

- random
- $\sqrt{dt}$

(1) $Y(t) = X(t) + 4t$

$\to dY(t) = dX(t) + 4\,dt$

Drift

$Y(t)$ is NOT a martingale

(2) $Y(t) = X^2(t) + k$

$dY(t) = dX^2(t) + 0$

$dY(t) = dt + 2X(t)\,dX(t)$

Drift

$Y(t)$ is NOT a martingale.

(3). $Y(t) = \underbrace{t^2 X(t)}_{Z(t)} - 2\int_0^t s\,X(s)\,ds$ Y is a martingale

Define $Z(t) = t^2 X(t)$

$$Z = f(s,x) = s^2 x$$

$$\frac{\partial f}{\partial s} = 2sx$$

$$\frac{\partial f}{\partial x} = s^2$$

$$\frac{\partial^2 f}{\partial x^2} = 0$$

$$dZ(t) = \frac{\partial f}{\partial s}\Big|_{(t,X_t)} dt + \frac{\partial f}{\partial x}\Big|_{(t,X_t)} dX(t) + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{(t,X_t)} dt}^{\,=0}$$

$$dZ(t) = 2t\,X(t)\,dt + t^2\,dX(t)$$

$$Z(t) = \underbrace{Z(0)}_{0} + 2\int_0^t s\,X(s)\,ds + \int_0^t s^2\,dX(s)$$

$Z(t) = e^{Y(t)}$ $\qquad z = \exp(y)$

Apply Itô to the function $f = \exp(y)$ and the process $Y(t)$

$$dZ(t) = \frac{df}{dy}\Big|_{Y(t)} dY(t) + \frac{1}{2}\frac{d^2 f}{dy^2}\Big|_{Y(t)} \cdot \underbrace{dY^2(t)}_{= g^2(t)\,dt}$$

$$= e^{Y(t)}\left(f(t)dt + g(t)\,dX(t)\right) + \frac{1}{2} e^{Y(t)} g^2(t)\,dt$$

first recall $e^{Y(t)} = Z(t)$, then

$$dZ(t) = Z(t)\left[\left(f(t) + \frac{1}{2}g^2(t)\right)dt + g(t)\,dX(t)\right]$$

If we want $Z(t)$ to be a martingale, ~~we need~~

$$f(t) + \frac{1}{2}g^2(t) = 0 \iff \boxed{f(t) = -\frac{1}{2}g^2(t)}$$

$\mathbb{P}$ $\xrightarrow{\text{Girsanov}}$ $\mathbb{Q}$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \Lambda$$

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \Lambda^{-1}$$

$

$X(t) \longrightarrow X^{\mathbb{Q}}(t)$

Girsanov → continuous process

Radon-Nikodym

$\mathbb{Q} \ll \mathbb{P}$

$$\underbrace{\mathbb{Q}(A)}_{\text{Proba of A with Q}} = \int_A \Lambda \, d\mathbb{P} = \int_A \frac{d\mathbb{Q}}{\cancel{d\mathbb{P}}} \cancel{d\mathbb{P}} = \int_A d\mathbb{Q} = \mathbb{Q}(A)$$

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

$$Y(t) = f(t, S_1(t), S_2(t))$$

$$Y(t) = f(S_1, S_2) = S_1(t)\, S_2(t)$$

<u>Itô Product Rule</u>

$$f(x, y) = xy$$