Interest Rate Modelling - The Mathematics of the Fixed Income World

Basic Finance

A *zero coupon bond* (ZCB) is a contract which pays a known, fixed amount, called the *principal* at some known date in the future called the *maturity*, T. Typically the principal is a normalised quantity of \$1. Usually, but not always, the value of the bond prior to maturity is less than the principal. As we approach T, the value of the bond converges to the principal.

A *coupon bearing bond* is similar to the ZCB, except that in addition to the principal paid at T, it also pays out at various dates prior to the maturity a coupon - which is a known fraction of the principal. These special dates are known as *coupon dates*. As an example consider such a product that pays out a principal value in ten years time and, say, 3% coupons (i.e. \$0.03) every year. Clearly this contract is more valuable than its zero coupon equivalent.

This coupon paying product can be regarded as a collection of ZCBs. That is with a principal equal to the coupon and maturity equal to the coupon date for each coupon date and one with principal equal to the actual principal and maturity equal to the actual maturity.

Yield to Maturity: Consider the price of a ZCB of maturity T and principal \$1 at time $t < T$ to be $Z(t;T)$. The yield to maturity y is defined as

$$Z(t;T) = e^{-y(T-t)} \Longleftrightarrow y = -\frac{1}{T-t}\log Z.$$

It is the constant rate of return which gives the observed bond price (so in a sense like an implied volatility).

For a coupon bearing bond, again with principal \$1and coupons C_i paid on dates t_i we can define the y by

$$V(t;T) = e^{-y(T-t)} + \sum_{\substack{i=1 \\ t_i>t}}^{N} C_i e^{-y(t_i-t)},$$

where N is the total number of coupons paid. We see that the yield is a decreasing function of increasing price and that price is a decreasing function of increasing yield.

The *yield curve*: suppose that today we are observing the prices of ZCBs of varying maturities. So now t is fixed but T varies. So the yield also varies as a function of T,

$$y = y\left(T; t\right),$$

that is it is contingent on today being t.

The yield curve is a plot of the yield against maturity. Typically we plot yield against the time to maturity $(T - t)$ and call the plot the *Term structure of interest rates.*

The *duration* can be thought of in two ways. Here we use parallels with option pricing.

One is to consider this to be the sensitivity of the bond price to yield. In finance terms it is the percentage change in the bond price in response to a one percent change in the yield. Differentiating $V\left(t; T\right)$ above gives

$$\frac{dV}{dy} = -\left(T - t\right) e^{-y(T-t)} - \sum_{\substack{i=1 \\ t_i > t}}^{N} \left(t_i - t\right) C_i e^{-y(t_i - t)}$$

Now consider Taylor's theorem for $V = V(y)$ – a small change in yield δy

$$
\begin{aligned}
V(y+\delta y) &= V(y) + \frac{dV}{dy}\delta y + \tfrac{1}{2}\frac{d^2V}{dy^2}\delta y^2 + \ldots \\
V(y+\delta y) - V(y) &= \frac{dV}{dy}\delta y + O\left(\delta y^2\right)
\end{aligned}
$$

hence to leading order

$$\delta V = \frac{dV}{dy}\delta y$$

so the relative change is

$$\frac{\delta V}{V} = \frac{1}{V}\frac{dV}{dy}\delta y.$$

Of importance is the quantity

$$-\frac{1}{V}\frac{dV}{dy} = \frac{(T-t)\,e^{-y(T-t)} + \sum\limits_{\substack{i=1 \\ t_i>t}}^{N} (t_i - t)\,C_i e^{-y(t_i-t)}}{e^{-y(T-t)} + \sum\limits_{\substack{i=1 \\ t_i>t}}^{N} C_i e^{-y(t_i-t)}}$$

called the duration. The larger the duration, the bigger the response of the bond's relative value to changes in yield.

Interest Rate Modelling

Introduction

When pricing non linear derivatives products a model for the randomness in the underlying is required.

This is as true for fixed-income instruments as it is for equity derivatives.

There are a number of ways to model interest rates:

- Monte Carlo
- Trees
- PDEs

This is an introduction to the PDE framework for interest rate products.

The basis of this section is to derive the bond pricing equation for a one factor model. The underlying state variable is a non-tradeable asset. Hence when pricing derivatives written on such securities a new term called the *market price of risk* appears in the equation. The evolution of this will be discussed in detail.

Unlike for equities, for interest rates there are numerous models. A number of these will be presented. The focus will be on bond prices which can be obtained analytically, i.e. *tractable* and *affine models*.

We start by defining a stochastic differential equations to model the random nature of an interest rate, using one source of randomness. This is the subject of *one-factor interest rate modelling*.

We begin by having one source of randomness, the spot interest rate.

The model will allow the short-term interest rate, the spot rate, to follow a random walk.

This model leads to a parabolic partial differential equation for the prices of bonds and other interest rate derivative products.

Later we will consider modelling the fixed-income world using *two* sources of randomness.

Stochastic interest rates

The 'spot rate' that we will be modelling is a very loosely-defined quantity, meant to represent the yield on a bond of infinitesimal maturity.

In practice one should take this rate to be the yield on a liquid finite-maturity bond, say one of one month.

Bonds with one *day* to expiry do exist but their price is not necessarily a guide to other short-term rates. Since we cannot realistically forecast the future course of an interest rate, it is natural to model it as a random variable.

- We are going to model the behaviour of r, the interest rate received by the shortest possible deposit.

From this we will see the development of a model for all other rates.

The interest rate for the shortest possible deposit is commonly called the **spot interest rate**.

Let us suppose that the interest rate r is governed by a stochastic differential equation of the form

$$dr = u(r,t)\,dt + w(r,t)\,dX. \tag{1}$$

The functional forms of $u(r,t)$ and $w(r,t)$ determine the behaviour of the spot rate r.

For the present we will not specify any particular choices for these functions.

The pricing equation for the general model

When interest rates are stochastic a fixed-income instrument has a price of the form $V(r,t)$. Previously we used $V(S,t)$ to denote an equity derivative price where the underlying stock S was used, to hedge with. Pricing interest rate products presents a different problem, which lies in the hedging. There are now no assets with which to hedge because r is not a traded quantity, which is what makes interest rate derivatives harder.

To get round this problem we hedge with IR derivative securities of different maturities - here we cannot do $V(r,t) - \Delta r$.

- We are not modelling a *traded* asset; the traded asset (the bond, say) is a derivative of our independent variable r.

Pricing a fixed-income instrument presents new technical problems, and is in a sense harder than pricing an option since *there is no underlying asset with which to hedge*.

- The only way to construct a hedged portfolio is by hedging one bond with a bond of a different maturity.

We set up a portfolio containing two bonds with different maturities T_1 and T_2.

The bond with maturity T_1 has price $V_1(r,t;T_1)$ and the bond with maturity T_2 has price $V_2(r,t;T_2)$.

We hold one of the former and a number $-\Delta$ of the latter.

We have

$$\Pi = V_1 - \Delta V_2. \tag{2}$$

Again as with equities we exploit correlation (in this theoretical world). One factor modelling implies there is just one source of randomness used for bonds, caps, floors, etc.

It makes no difference which way around V_1 and V_2 are, i.e. we can have $\Pi = V_2 - \Delta V_1$. We will see later why this is unimportant. Then in one time-step dt the change in the portfolio is defined by

$$d\Pi = dV_1 - \Delta dV_2. \tag{3}$$

We hold Δ fixed during the course of the time-step and reset when we come back to hedge again.

The change in this portfolio in a time dt can be obtained using Itô. So we have for $i = 1, 2$

$$dV_i = \frac{\partial V_i}{\partial t}dt + \frac{\partial V_i}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V_i}{\partial r^2}dr^2$$

where $dr^2 = w^2 dt$. So the portfolio can be written as $d\Pi =$

$$\begin{aligned} &\tfrac{\partial V_1}{\partial t}dt + \tfrac{\partial V_1}{\partial r}dr + \tfrac{1}{2}w^2\tfrac{\partial^2 V_1}{\partial r^2}dt - \\ &\Delta\left(\tfrac{\partial V_2}{\partial t}dt + \tfrac{\partial V_2}{\partial r}dr + \tfrac{1}{2}w^2\tfrac{\partial^2 V_2}{\partial r^2}dt\right), \end{aligned} \tag{4}$$

This is not riskless because the dr brings in the random component dX. So if we can eliminate the coefficients of dr then the portfolio is risk-free. That is $\frac{\partial V_1}{\partial r}$ and $-\Delta\frac{\partial V_2}{\partial r}$, so we set

$$\frac{\partial V_1}{\partial r} - \Delta\frac{\partial V_2}{\partial r} = 0,$$

which is rearranged to give

$$\Delta = \frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}\,. \tag{5}$$

This choice of Δ eliminates all the randomness associated with $d\Pi$. So replace Δ with $\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}$ in (4) to give

$$d\Pi = \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}\right)\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right)\right) dt \tag{6}$$

This means that the change in the portfolio is completely riskless, hence it must grow at the risk free rate, i.e. no-arbitrage:

$$\frac{d\Pi}{dt} = r\Pi \longrightarrow d\Pi = r\Pi dt \tag{7}$$

where r is the spot rate at time t. If it was not, then we would have two factors in our model. So $d\Pi = r\Pi dt$ gives

$$d\Pi = r\left(V_1 - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}\right) V_1\right) dt \qquad (8)$$

Hence (6) and (8) are equivalent, so equate them

$$\begin{aligned} & \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}\right)\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right)\right) dt \\ = \;& r\left(V_1 - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}\right) V_1\right) dt. \end{aligned}$$

Now write this equation so that the left hand and ride hand sides are in terms of V_1 and V_2 respectively.

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2}{\frac{\partial V_2}{\partial r}}. \qquad (9)$$

The lhs depends only on V_1, i.e. on T_1. The rhs is dependant upon V_2 only, i.e. on T_2. This is one equation in two unknowns - but more importantly should hold for all

r and t. Hence (9) is a function of (r,t), i.e. a function of 2 common variables to V_1 and V_2. LHS is $f(r,t;T_1)$ and RHS is $g(r,t;T_2)$.

So each side can be a function of r and t but must be independent of T_1 and T_2. Both sides must be equal to a universal constant. *The only way for this to be possible is for both sides to be independent of the maturity date.*

Recall when we solved the simple one dimensional heat equation using the separation of variables method

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}\\ u(X,t) &= X(x)\,T(t)\end{aligned}$$

which simplifies to

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant}$$

i.e. constant indep. of x, t.

So (9) becomes

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2}{\frac{\partial V_2}{\partial r}} = \text{constant}$$

Hedging with V_1 and V_2, or V_3 and V_4 or V_i and V_j etc. makes no difference. The nonlinear operator

$$\frac{\frac{\partial}{\partial t} + \frac{1}{2}w^2\frac{\partial^2}{\partial r^2} - r}{\frac{\partial}{\partial r}}$$

is a universal operator, i.e. some function for all $V_{i.}$ Let's call this $a\left(r,t\right)$ is a universal property of interest rates - but currently does not have a nice finance interpretation.

Both sides of (9) can be functions of the 'variables' in the model, r and t, since these are common to all instruments.

Dropping the subscript from V, we have

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} = a(r,t)$$

for some function $a(r,t)$.

We can write this w.lo.g as

$$a(r,t) = \lambda(r,t)\, w(r,t) - u(r,t)$$

for a given $u(r,t)$ and non-zero $w(r,t)$ this is always possible.

The fixed-income pricing equation or **bond pricing equation** is therefore

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0. \qquad (10)$$

This is another parabolic partial differential equation, very similar to the Black-Scholes equation.

As with equity derivatives, until a final condition is specified the PDE does not know what kind of product is being priced. So we need a time condition corresponding to the payoff at maturity, T.

For example, the final condition for a zero-coupon is

$$V(r, T) = 1.$$

So it is a normalised condition called the redemption value.

The market price of risk?

The term $\lambda(r,t)$ introduced earlier is called the *market price of risk.* This arises because we have to hedge with another bond rather than r itself. This also appears when pricing volatility derivatives. We begin by supposing that we hold an unhedged position in one bond with maturity date T.

We saw earlier from Itô that in a time step dt this bond changes in value by

$$dV = w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r}\right)dt. \quad (11)$$

Having derived the bond pricing equation (bpe) we know from (10)

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r} - \lambda w\frac{\partial V}{\partial r} - rV = 0$$

which can be arranged to give

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + u \frac{\partial V}{\partial r} = \lambda w \frac{\partial V}{\partial r} + rV. \qquad (12)$$

So substitute (12) in to (11) to give

$$dV = w \frac{\partial V}{\partial r} dX + \left(w\lambda \frac{\partial V}{\partial r} + rV \right) dt,$$

or

$$dV - rV\,dt = w \frac{\partial V}{\partial r} (dX + \lambda dt). \qquad (13)$$

We can interpret (13) financially as the difference between

1. dV - return on an unhedged bond (contains risk) and

2. $rV\,dt$ - risk free return, i.e. the money invested from selling a bond which earns at the risk free rate, i.e. $\frac{dV}{dt} = rV$

which gives $w\frac{\partial V}{\partial r}(dX + \lambda dt)$.

Since the coefficient of $dX \neq 0$, the portfolio is not riskless and the term λdt is the extra return on the portfolio per unit of risk dX. It is the amount by which the market wishes to be compensated, for hedging with a non-tradeable asset.

The sign of λ is generally negative because as r increases the bond price decreases $\Longrightarrow \frac{\partial V}{\partial r} < 0$.

For equities, since S is a tradeable asset

$$\lambda = \frac{\mu - r}{\sigma}$$

More on this later.

We now approach the market price of risk differently, by not assuming the actual form of $a\left(r,t\right)$. Start by writing the BPE as

$$\begin{aligned}\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}-a(r,t)\frac{\partial V}{\partial r}-rV &= 0\\ \frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} &= a(r,t)\frac{\partial V}{\partial r}+rV\end{aligned}$$

add $u\left(r,t\right)\frac{\partial V}{\partial r}$ to both sides

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+u\frac{\partial V}{\partial r}=a(r,t)\frac{\partial V}{\partial r}+rV+u\frac{\partial V}{\partial r} \quad (14)$$

and from earlier the unhedged bond is given by (11). So we substitute (14) into (11), which gives

$$dV=w\frac{\partial V}{\partial r}dX+\left(a(r,t)\frac{\partial V}{\partial r}+rV+u\left(r,t\right)\frac{\partial V}{\partial r}\right)dt.$$

Now do the (unhedged bond) $-$ (risk free return)

$$\begin{aligned}dV-rV\,dt &= w\frac{\partial V}{\partial r}dX+\frac{\partial V}{\partial r}\left(a+u\right)dt\\ &= w\frac{\partial V}{\partial r}\left(dX+\left(\frac{a+u}{w}\right)dt\right)\end{aligned}$$

so put

$$\left(\frac{a+u}{w}\right) = \lambda\,(r,t)\,.$$

Rearranging gives

$$a(r,t) = \lambda\,(r,t)\,w\,(r,t) - u\,(r,t)$$

and justifies our earlier definition of $a(r,t)$.

If we compare the BPE with the BSE

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0.$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

The four terms in the equation represent, in order as written,

- time decay $\frac{\partial V}{\partial t}$,
- diffusion $\frac{1}{2}w^2$,
- drift $(u - \lambda w)$ and
- discounting $-rV$.

The equation is similar to the backward equation for a probability density function except for the final discounting term.

- As such we can interpret the solution of the bond pricing equation as the expected present value of all cashflows.

This idea should be familiar from equity derivatives.

As with equity options, this expectation is not with respect to the *real* random variable, but instead with respect to the *risk-neutral* variable.

- There is this difference because the drift term in the equation is not the drift of the real spot rate u, but the drift of another rate, called the **risk-neutral spot rate**. This rate has a drift of $u - \lambda w$.

Recall that for equities we have a real drift rate denoted by μ. However, we price as if the asset grows with a rate r, the risk-free rate.

For fixed-income products the real growth of the spot interest rate may be $u(r,t)$ but we price as if it were $u(r,t) - \lambda(r,t)w(r,t)$.

The latter is the **risk-adjusted drift rate**.

When pricing interest rate derivatives it is important to model, and price, using the risk-neutral rate.

The risk-neutral spot rate evolves according to

$$dr = (u - \lambda w)dt + w\; dX. \tag{15}$$

We can use Monte Carlo simulations for pricing fixed-income products, but we ensure that we simulate the risk-neutral spot rate process.

The relationship between prices and expectations

The basis of the Monte Carlo method is to price options using expectations. In the equity world we can write

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\text{Payoff}],$$

when interest rates are constant.

We can write

$$V(S,t) = e^{-\int_t^T r(\tau)\,d\tau}\mathbb{E}^{\mathbb{Q}}[\text{Payoff}],$$

when rates are deterministic, $r(t)$,

When we have a fixed-income product and rates are stochastic the present value term must go inside the expectation...

$$V(S,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\tau)\,d\tau}[\text{Payoff}]\right].$$

The value of a derivative is the risk-neutral expectation of the present value of the payoff.

If we wish to price an interest rate derivative then the payoff will be in terms of r, i.e. Payoff(r).

Rule of Thumb:

The new term, market-price-of-risk arises because our modelled underlying variable, r, is not traded.

The general rule is

- If a modelled quantity is traded then the risk-neutral growth rate is r

$$r = \mu - \lambda\sigma$$

- If a modelled quantity is not traded then the risk-neutral growth rate is

$$\text{real growth} - \text{market price of risk} \times \text{volatility}$$

If we start with a model for Geometric Brownian Motion

$$dS = \mu S dt + \sigma S dX$$

then the pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - \lambda\sigma) S \frac{\partial V}{\partial S} - rV = 0 \quad (16)$$

but S is traded which means it satisfies (16) , substituting $V = S$ gives

$$\begin{aligned} (\mu - \lambda\sigma)\, S \cdot 1 - rS &= 0 \\ r &= \mu - \lambda\sigma \end{aligned}$$

which is the risk-neutral growth (called r) $\Longrightarrow \lambda_S = \dfrac{\mu - r}{\sigma}$ which is the Sharpe ratio.

For an interest rate we have

$$\lambda_r = \frac{a + u}{w}$$

Tractable models and solutions of the pricing equation

This forms the basis of obtaining analytical/closed form solutions of the bond pricing equation.

We have built up the pricing equation for an arbitrary model.

That is, we have not specified the precise form of the risk-neutral drift, $u - \lambda w$, or the volatility, w.

How can we choose these functions to give us a good model?

There are two ways to proceed:

- Choose a model that matches reality as closely as possible - use numerical schemes for PDEs

- Choose a model which is easy to solve - these are called tractable models

Certain choices for risk-neutral drift and diffusion give analytical solutions of the BPE. Let's examine some, together with the conditions on drift and volatility that lead to tractable models, that is, models for which the solution of the pricing equation for zero-coupon bonds can be found analytically.

Named models

Unlike equities, there are numerous models for interest rates. We have already seen that they possess the property to become negative and a mean reverting behaviour. There are many interest rate models, associated with the names of their inventors. here are just a few

- Vasicek
- Cox, Ingersoll & Ross
- Ho & Lee
- Hull & White

and many more.

Vasicek

The Vasicek model, developed in 1977 (for the risk-neutral spot rate) takes the form

$$dr = (\eta - \gamma r)dt + \beta^{1/2} dX.$$

Thus the pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta\frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r)\frac{\partial V}{\partial r} - rV = 0.$$

And the final condition for a zero-coupon bond is

$$V(r, T) = 1.$$

And look for a solution of the form

$$V(r,t;T) = e^{A(t)-rB(t)}.$$

$$\begin{aligned} A &= A(t;T) \\ B &= B(t;T) \end{aligned}$$

So substitute this expression for V into the BPE. We need three derivative terms

$$\frac{\partial}{\partial t}\left(e^{A(t)-rB(t)}\right) = \left(\dot{A}(t) - r\dot{B}(t)\right)V$$

where $\dot{}$ is a time dependent derivative, i.e. d/dt.

$$\frac{\partial}{\partial r}\left(e^{A(t)-rB(t)}\right) = -B(t)V,$$

and

$$\frac{\partial^2}{\partial r^2}\left(e^{A(t)-rB(t)}\right) = B(t)^2 V.$$

Substituting these expressions into the pricing equation for the Vasicek model and dividing through by V we get

$$\left(\dot{A}(t) - r\dot{B}(t)\right) + \frac{1}{2}\beta B(t)^2 - (\eta - \gamma r)B(t) - r = 0.$$

Grouping the terms, we have an expression that is linear in r:

$$\left(\dot{A}(t) + \frac{1}{2}\beta B(t)^2 - \eta B(t)\right) + r\left(-\dot{B}(t) + \gamma B(t) - 1\right) = 0.$$

Both of the expressions in parentheses must be zero.

We have two first order ordinary differential equations for $A(t)$ and $B(t)$:

$$\begin{aligned} \dot{B}(t) - \gamma B(t) &= -1 \\ \dot{A}(t) + \frac{1}{2}\beta B(t)^2 - \eta B(t) &= 0 \end{aligned}$$

In order for the final condition at $t = T$ to be satisfied we need

$$\begin{aligned} e^{A(T)-rB(T)} &= 1 \Longrightarrow \\ A(T) - rB(T) &= 0 \;\forall r \end{aligned}$$

and so

$$A(T;T) = B(T;T) = 0.$$

So start by solving for $B(t)$. This can be tackled as a linear equation with integrating factor $e^{-\gamma t}$, so multiply through

$$\begin{aligned} e^{-\gamma t}\dot{B}(t) - \gamma e^{-\gamma t}B(t) &= -e^{-\gamma t} \\ \frac{d}{dt}\left(e^{-\gamma t}B(t)\right) &= -e^{-\gamma t} \\ \int_t^T d\left(e^{-\gamma s}B(s)\right) &= -\int_t^T e^{-\gamma s}ds \\ e^{-\gamma s}B(s)\Big|_t^T &= \frac{1}{\gamma}\,e^{-\gamma s}\Big|_t^T \\ e^{-\gamma T}B(T) - e^{-\gamma t}B(t) &= \frac{1}{\gamma}\left(e^{-\gamma T} - e^{-\gamma t}\right) \end{aligned}$$

The solution is

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

To obtain $A(t)$:

$$\begin{aligned}
\frac{dA}{dt} &= -\frac{1}{2}\beta B(t)^2 + \eta B(t) \\
\int_t^T dA\,(s) &= -\frac{1}{2}\frac{\beta}{\gamma^2}\int_t^T (1 - e^{-2\gamma(T-s)} - 2e^{-\gamma(T-s)})ds \\
&\quad +\frac{\eta}{\gamma}\int_t^T (1 - e^{-\gamma(T-s)})ds \\
-A\,(t) &= -\frac{\beta}{2\gamma^2}\left(s - \frac{1}{2\gamma}e^{-2\gamma(T-s)} - \frac{2}{\gamma}e^{-\gamma(T-s)}\right)\Bigg|_t^T \\
&\quad +\frac{\eta}{\gamma}\left(s - \frac{1}{\gamma}e^{-\gamma(T-s)}\right)\Bigg|_t^T
\end{aligned}$$

and

$$A(t;T) = \frac{1}{\gamma^2}(B(t;T) - T + t)(\eta\gamma - \frac{1}{2}\beta) - \frac{\beta B(t;T)^2}{4\gamma}.$$

Cox, Ingersoll & Ross

The CIR model is very similar to Vasicek but has a variable diffusion term to address the Vasicek problem, where rates can become negative. Drift structures for both SDEs are the same

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}\, dX.$$

The spot rate is mean reverting and if $\eta > \alpha/2$ the spot rate stays positive.

The value of a zero-coupon bond is again of the form

$$V = e^{A(t) - rB(t)},$$

for different (and more complicated) functions A and B. The BPE becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\alpha r\frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r)\frac{\partial V}{\partial r} - rV = 0.$$

giving two ODEs

$$\begin{aligned} \dot{B}(t) &= \frac{1}{2}\alpha B(t)^2 + \gamma B - 1 \\ \dot{A}(t) &= \eta B(t) \end{aligned}$$

Although the first ODE is a Riccati type equation,

$$\frac{dy}{dx} = a\,(x)\,y^2 + b\,(x)\,y - c\,(x)$$

we don't have a particular solution from which to obtain a more general one. The following integrals should be used as a hint (although the working is a little messy):

$$\begin{aligned} \int \frac{dx}{x^2 - c^2} &= \frac{1}{2c}\ln\left(\frac{c-x}{c+x}\right) \\ \int \frac{dx}{b + ce^{ax}} &= \frac{1}{ab}\ln\left(\frac{e^{ax}}{b + ce^{ax}}\right) \end{aligned}$$

Ho & Lee

Ho & Lee introduced some time dependence into the coefficient of their random walk. This allows the yield curve to be fitted.

$$dr = \eta(t)dt + \beta^{1/2}dX.$$

The value of zero-coupon bonds is again given by

$$V = e^{A(t) - rB(t)}$$

The BPE becomes

$$\begin{aligned} \dot{B}(t) &= -1 \\ \dot{A}(t) &= -\frac{1}{2}\beta B^2 + \eta(t)B(t) \end{aligned}$$

which we integrate over t and T

$$\begin{aligned} \int_t^T dB &= -\int_t^T ds \longrightarrow \\ \underbrace{B(T)}_{=0} - B(t) &= -(T - t) \end{aligned}$$

so

$$B(t) = (T - t)$$

and for $A(t)$

$$\begin{aligned}
\int_t^T dA(s) &= -\frac{1}{2}\beta \int_t^T B^2 ds + \int_t^T \eta(s)B(s)ds \\
& \quad A(T) - A(t) \\
&= -\frac{1}{2}\beta \int_t^T (T-s)^2 ds + \int_t^T \eta(s)(T-s)\,ds \\
-A(t) &= -\frac{1}{6}\beta (T-s)^3 + \int_t^T \eta(s)(T-s)\,ds
\end{aligned}$$

$$A(t) = -\int_t^T \eta(s)(T-s)ds + \frac{1}{6}\beta(T-t)^3.$$

This model was the first 'no-arbitrage model' of the term structure of interest rates.

By this is meant that the careful choice of the function $\eta(t)$ will result in theoretical zero-coupon bonds prices, output by the model, which are the same as market prices.

- This technique is also called **yield curve fitting**.

This careful choice is

$$\eta(t) = -\frac{\partial^2}{\partial t^2} \log Z_M(t^*; t) + \beta(t - t^*)$$

where today is time $t = t^*$.

In this $Z_M(t^*; T)$ is the market price today of zero-coupon bonds with maturity T.

Hull & White

This process of generalisation (i.e. time dependency) continues with the Hull & White. They extended both the Vasicek and the CIR models to incorporate time-dependent parameters:

$$1) \quad dr = (\eta(t) - \gamma(t)r)dt + \beta(t)^{1/2}dX$$

$$2) \quad dr = (\eta(t) - \gamma(t)r)dt + \sqrt{\alpha(t)r}\, dX$$

This time dependence again allows the yield curve (and even a volatility structure) to be fitted.

In general consider we can write:

$$dr = (\eta(t) - \gamma(t)r)dt + \beta(t)r^{\nu}dX$$

We then have the following models

$\nu = 1$: Black Derman & Toy

$\nu = \frac{1}{2}$: CIR

$\nu = 0$: Extended Vasicek (contains Ho and Lee as a special case)

Black, Derman & Toy

This model is popular because it is relatively easy to calibrate and, with two functions to be fitted - this allows the initial yield curve volatilities to also be calibrated. The risk-neutral random walk is

$$d\,(\log r) = \left(\theta\,(t) - \frac{d\,(\log \sigma\,(t))}{dt}\log r\right)dt + \sigma\,(t)\,dX$$

This BDT model is not in the affine class.

A more general model

We have seen *Affine Models*, i.e. ones which give ZCB prices of the form:

$$V\left(r,t;T\right)=\mathsf{exp}\left(A\left(t;T\right)-rB\left(t;T\right)\right)$$

What are the conditions of the drift and diffusion for an analytical solution of the above form to exist?

The most general form of risk-adjusted random walk for which this is possible has structure given by

$$u(r,t)-\lambda(r,t)w(r,t)=\eta(t)-\gamma(t)r,$$

$$w(r,t)=\sqrt{\alpha(t)r+\beta(t)}.$$

So this choice of u and w ensures a "nice" form for the price.

Before we derive these conditions let us consider some important properties:

- Interest rates should remain positive

- Large rates eventually decrease: mean reversion

- The model should be relatively easy to calibrate and use.

Substitute $V(r, t; T)$ into the BPE to give

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}w^2B^2 - (u - \lambda w)B - r = 0.$$

Some of these terms are functions of t and T (i.e. A and B) and others are functions of r and t (i.e. u and w).

Since we require a rule for r we differentiate with respect to r once:

$$-\frac{\partial B}{\partial t}+\frac{1}{2}B^2\frac{\partial}{\partial r}(w^2)-B\frac{\partial}{\partial r}(u-\lambda w)-1=0.$$

Differentiate again with respect to r and divide through by B:

$$\frac{1}{2}B\frac{\partial^2}{\partial r^2}(w^2)-\frac{\partial^2}{\partial r^2}(u-\lambda w)=0.$$

In this, only B can depend on the bond maturity T, therefore we must have each term equal to zero:

$$\begin{aligned}\frac{\partial^2}{\partial r^2}(w^2) &= 0\\ -\frac{\partial^2}{\partial r^2}(u-\lambda w) &= 0.\end{aligned}$$

$$\begin{array}{cc}\int\frac{\partial^2}{\partial r^2}(w^2)dr=0 & -\int\frac{\partial^2}{\partial r^2}(u-\lambda w)dr=0\\ \frac{\partial}{\partial r}(w^2)=\alpha(t) & \frac{\partial}{\partial r}(u-\lambda w)=-\gamma(t)\\ \int\frac{\partial}{\partial r}(w^2)=\alpha(t)\int dr & \int\frac{\partial}{\partial r}(u-\lambda w)=-\gamma(t)\int dr\\ w^2=\alpha(t)r+\beta(t) & (u-\lambda w)=-\gamma(t)r+\eta(t)\end{array}$$

Therefore $u - \lambda w$ and w^2 must be linear in r.

The equations for A and B are

$$\frac{\partial A}{\partial t} = \eta(t)B - \frac{1}{2}\beta(t)B^2$$

and

$$\frac{\partial B}{\partial t} = \frac{1}{2}\alpha(t)B^2 + \gamma(t)B - 1.$$

In order to satisfy the final data that $V(r, T; T) = 1$ we must have

$$A(T; T) = 0 \quad \text{and} \quad B(T; T) = 0.$$

Popular one-factor spot-rate models

The real spot rate r satisfies the stochastic differential equation

$$dr = u(r,t)dt + w(r,t)dX.$$

Model	$u(r,t) - \lambda(r,t)w(r,t)$	$w(r,t)$
Vasicek	$a - br$	c
CIR	$a - br$	$cr^{1/2}$
Ho & Lee	$a(t)$	c
Hull & White I	$a(t) - b(t)r$	$c(t)$
Hull & White II	$a(t) - b(t)r$	$c(t)r^{1/2}$
General affine	$a(t) - b(t)r$	$(c(t)r - d(t))^{1/2}$

Here $\lambda(r,t)$ denotes the market price of risk. The function $u - \lambda w$ is the risk-adjusted drift.

For all of these models the zero-coupon bond value is of the form $Z(r,t;T) = e^{A(t,T)-rB(t,T)}$.

The time-dependent coefficients in all of these models allow for the fitting of the yield curve and other interest-rate instruments.

Multi-factor interest rate modelling

The simple one-factor stochastic spot interest rate models cannot hope to capture the rich yield-curve structure found in practice: from a given spot rate at a given time they will predict the whole yield curve.

Generally speaking, the one source of randomness, the spot rate, may be good at modelling the overall level of the yield curve but it will not necessarily model shifts in the yield curve that are substantially different at different maturities.

For some instruments this may not be important. For example, for instruments that depend on the *level* of the yield curve it may be sufficient to have one source of randomness, i.e. one factor.

More sophisticated products depend on the difference between yields of different maturities and for these products it is important to model the tilting of the yield curve.

One way to do this is to invoke a second factor, a second source of randomness.

Theoretical framework

We now use two state equations to model simple interest rate. One captures the evolution of the spot interest rate r, and another independent variable l. We use

$$dr = u\,dt + w\,dX_1$$

and

$$dl = p\,dt + q\,dX_2.$$

Simple instruments will then have prices which are functions of r, l and t, $V(r, l, t)$.

All of u, w, p and q are allowed to be functions of r, l and t. The correlation coefficient ρ between dX_1 and dX_2 may also depend on r, l and t.

Possible choices for l :

- Another interest rate

- long rate

- volatility of spot rate

- stochastic MPOR

Since we have two sources of randomness now, in pricing one bond we must hedge with *two* others to eliminate the risk:

$$\Pi = V(r,l,t;T) - \Delta_1 V_1(r,l,t;T_1) - \Delta_2 V_2(r,l,t;T_2).$$

As before the distinguishing features of V_1 and V_2 are the maturities T_1 and T_2, in turn.

The change in the value of this portfolio is given by

$$d\Pi = dV(r,l,t;T) - \Delta_1 dV(r,l,t;T_1) - \Delta_2 dV(r,l,t;T_2).$$

We know from an extension of Itô that

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl + \\ &\quad \frac{1}{2}\frac{\partial^2 V}{\partial r^2}dr^2 + \frac{1}{2}\frac{\partial^2 V}{\partial l^2}dl^2 + \frac{\partial^2 V}{\partial r \partial l}drdl \end{aligned}$$

Now

$$\begin{aligned} dr^2 &= w^2 dt \\ dl^2 &= q^2 dt \\ drdl &= ? \end{aligned}$$

For $drdl$ we use the correlation

$$\begin{aligned} \mathbb{E}\left[dX_1 dX_2\right] &= \mathbb{E}\left[\phi_1\sqrt{dt}\phi_2\sqrt{dt}\right] = dt\mathbb{E}\left[\phi_1\phi_2\right] \\ &= \rho dt \end{aligned}$$

So

$$drdl = w\rho q dt$$

Substituting all into dV and rearranging gives

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2} + \rho w q \frac{\partial^2 V}{\partial r \partial l}\right)dt + \\ &\quad \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl. \end{aligned}$$

We will define the coefficient of dt as

$$\mathcal{L}(V) = \frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2} + \rho w q\frac{\partial^2 V}{\partial r \partial l}$$

where

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + \frac{1}{2}w^2\frac{\partial^2}{\partial r^2} + \frac{1}{2}q^2\frac{\partial^2}{\partial l^2} + \rho w q\frac{\partial^2}{\partial r \partial l}$$

is the Black-Scholes operator, therefore

$$dV = \mathcal{L}(V) + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl$$

Similarly for V_1 and V_2 we have

$$dV_1 = \mathcal{L}(V_1) + \frac{\partial V_1}{\partial r}dr + \frac{\partial V_1}{\partial l}dl$$

$$dV_2 = \mathcal{L}(V_2) + \frac{\partial V_2}{\partial r}dr + \frac{\partial V_2}{\partial l}dl$$

Hence

$$\begin{aligned} d\Pi &= \mathcal{L}(V)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl \\ &\quad -\Delta_1\left(\mathcal{L}(V_1)dt + \frac{\partial V_1}{\partial r}dr + \frac{\partial V_1}{\partial l}dl\right) \\ &\quad -\Delta_2\left(\mathcal{L}(V_2)dt + \frac{\partial V_2}{\partial r}dr + \frac{\partial V_2}{\partial l}dl\right) \end{aligned}$$

and grouping similar terms together gives

$$d\Pi = \left(\mathcal{L}(V) - \Delta_1\mathcal{L}(V_1) - \Delta_2\mathcal{L}(V_2)\right)dt$$

$$+\left(\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} - \Delta_2 \frac{\partial V_2}{\partial r}\right) dr + \left(\frac{\partial V}{\partial l} - \Delta_1 \frac{\partial V_1}{\partial l} - \Delta_2 \frac{\partial V_2}{\partial l}\right) dl. \tag{17}$$

To eliminate risk, we want the coefficients of dr and dl to be zero (as they contain the uncertainty). That is

$$d\Pi = (\mathcal{L}(V) - \Delta_1 \mathcal{L}(V_1) - \Delta_2 \mathcal{L}(V_2))\, dt$$

which means no arbitrage, hence the change in the portfolio occurs at the risk-free rate

$$\begin{aligned} d\Pi &= (\mathcal{L}(V) - \Delta_1 \mathcal{L}(V_1) - \Delta_2 \mathcal{L}(V_2))\, dt \\ &= r\Pi dt \\ &= r\underbrace{(V - \Delta_1 V_1 - \Delta_2 V_2)}_{=\Pi} dt. \end{aligned}$$

Rearranging

$$(\mathcal{L}(V) - \Delta_1 \mathcal{L}(V_1) - \Delta_2 \mathcal{L}(V_2)) = r\,(V - \Delta_1 V_1 - \Delta_2 V_2)$$

gives

$$(\mathcal{L}(V) - rV) - \Delta_1 (\mathcal{L}(V_1) - rV_1) - \Delta_2 (\mathcal{L}(V_2) - rV_2) = 0$$

Now choose Δ_1 and Δ_2 to make the coefficients of dr and dl in (17) equal to zero. The corresponding portfolio is risk free and should earn the risk-free rate of interest, r.

We thus have the three equations

$$\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} - \Delta_2 \frac{\partial V_2}{\partial r} = 0,$$

$$\frac{\partial V}{\partial l} - \Delta_1 \frac{\partial V_1}{\partial l} - \Delta_2 \frac{\partial V_2}{\partial l} = 0$$

and

$$\mathcal{L}'(V) - \Delta_1 \mathcal{L}'(V_1) - \Delta_2 \mathcal{L}'(V_2) = 0$$

where

$$\mathcal{L}'(V) = \mathcal{L}(V) - rV.$$

N.B The primes are not derivatives.

So we have 3 equations in 2 unknowns.

N equations in $(N+1)$ unknowns is called an over-prescribed system. The linear system can be expressed as a matrix problem

$$\underbrace{\begin{pmatrix} \mathcal{L}'(V) & \mathcal{L}'(V_1) & \mathcal{L}'(V_2) \\ \partial V/\partial r & \partial V_1/\partial r & \partial V_2/\partial r \\ \partial V/\partial l & \partial V_1/\partial l & \partial V_2/\partial l \end{pmatrix}}_{=M} \begin{pmatrix} 1 \\ -\boldsymbol{\Delta}_1 \\ -\boldsymbol{\Delta}_2 \end{pmatrix} = \underline{\mathbf{0}}$$

and for the system to be consistent (i.e. solution exists) we require

$$\det M = 0$$

If $\det M = 0$ then the three rows are linearly dependent so we can write the first row as a linear combination of the other two, i.e.

$$\mathcal{L}'(V) = \alpha_1 \frac{\partial V}{\partial r} + \alpha_2 \frac{\partial V}{\partial l}$$

for each $V,\ V_1,\ V_2$.

$\alpha_1,\ \alpha_2$ are arbitrary functions of r, l, t but not T, T_1, T_2.

Put

$$\begin{aligned} \alpha_1 &= \lambda_r w - u \\ \alpha_2 &= \lambda_l q - p \end{aligned}$$

We can therefore write

$$\mathcal{L}'(V) = (\lambda_r w - u)\frac{\partial V}{\partial r} + (\lambda_l q - p)\frac{\partial V}{\partial l}$$

where the two functions $\lambda_r(r, l, t)$ and $\lambda_l(r, l, t)$ are the market prices of risk for r and l respectively, and are again independent of the maturity of any bond.

In full, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w q\frac{\partial^2 V}{\partial r \partial l} + \frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2} +$$

$$(u - \lambda_r w)\frac{\partial V}{\partial r} + (p - \lambda_l q)\frac{\partial V}{\partial l} - rV = 0. \qquad (18)$$

The model for interest rate derivatives is defined by the choices of w, q, ρ, and the risk-adjusted drift rates $u - \lambda_r w$ and $p - \lambda_l q$.

This is yet another parabolic partial differential equation, a multi-factor model

General affine model

If r and l satisfy the following:

- the risk-adjusted drifts of both r and l are linear in r and l (but can have an arbitrary time dependence)

- the random terms for both r and l are both square roots of functions linear in r and l (but can have an arbitrary time dependence)

- the stochastic processes for r and l are uncorrelated

then the two-factor pricing equation (18) has a solution for a zero-coupon bond of the form

$$V\left(r,l,t;T\right)=e^{A(t)-B(t)r-C(t)l}.$$

Substituting this into the 2 factor BPE gives a set of ordinary differential equations for A, B and C

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \left(\dot{A}(t) - \dot{B}(t)r - \dot{C}(t)l\right) V \\
\frac{\partial V}{\partial r} &= -BV \longrightarrow \frac{\partial^2 V}{\partial r^2} = B^2 V \\
\frac{\partial V}{\partial l} &= -CV \longrightarrow \frac{\partial^2 V}{\partial l^2} = C^2 V \\
\frac{\partial^2 V}{\partial r \partial l} &= BCV = \frac{\partial^2 V}{\partial l \partial r}
\end{aligned}
$$

$$
\begin{aligned}
&\dot{A}(t) - \dot{B}(t)r - \dot{C}(t)l + \frac{1}{2}w^2 B^2 + \rho wqBC \\
&+\frac{1}{2}q^2 C^2 - B(u - \lambda_r w) - C(p - \lambda_l q) - r \\
=\ & 0
\end{aligned}
$$

$$
\begin{aligned}
&\left(-\dot{B}(t) - 1\right) r - \left(-\dot{C}(t)\right) l \\
&+\dot{A}(t) + \frac{1}{2}w^2 B^2 + \rho wqBC + \frac{1}{2}q^2 C^2 \\
&-B(u - \lambda_r w) - C(p - \lambda_l q) \\
=\ & 0
\end{aligned}
$$

Calibration

Introduction

The common approach used is to choose a favourite model and pick parameters so that the price of ZCBs are correct. The theoretical price can be wrong so we go back and change the parameters to obtain the correct value from market traded instruments. The methodology is called *CALIBRATION* or *FITTING.* This is what banks do!

People like calibration because we choose a well known model and ensure that theory and market prices match up, then use it for the pricing of other instruments. So in other words firstly do a fit so that the theoretical model gives the correct ZCB price and then use it for pricing other complex interest rate products.

In simple terms take a popular model and ensure the parameter which was/were previously constant is/are now time dependent - thus introducing more flexibility. So every time you see a constant, make it a function of time. This gives an extra degree of freedom - so introduce time dependency in the model itself.

Easiest to do with the Ho and Lee model. Historically this was the first finance model specifically developed for fitting. This class of model is called *No Arbitrage* as opposed to equilibrium models.

The beauty is its simple form as a SDE.

So in using any model we have to decide how to choose the parameters.

Should the parameters be chosen to match

- the market yield curve? Or

- historical interest rate data?

The former is **calibration** to a snapshot of the market at one instant in time.

The latter is fitting to time series data.

Because of this need to correctly price liquid instruments, the idea of **yield curve fitting** or **calibration** has become popular.

When stochastic models are used in practice they are almost always fitted.

To match a theoretical yield curve to a market yield curve requires a model with enough degrees of freedom. (You are matching a curve, i.e. an 'infinite' number of points, so you need infinite degrees of freedom!)

This is done by making one or more 'parameters' time dependent.

- This functional dependence on time is then carefully chosen to make an output of the model, the price of zero-coupon bonds, exactly match the market prices for these instruments.

Ho & Lee

The Ho & Lee spot interest rate model is the *simplest that can be used to fit the yield curve*.

We saw earlier that the process for the *risk-neutral* spot rate evolves according to

$$dr = \eta(t)dt + c dX.$$

This is an affine model with solution of the BPE of the form

$$Z(r, t; T) = e^{A(t;T) - r(T-t)},$$

where we calculated

$$\begin{aligned} B(t;T) &= (T-t) \\ A(t;T) &= -\int_t^T \eta(s)(T-s)ds + \frac{1}{6}c^2(T-t)^3. \end{aligned}$$

(Note that the variables are r and t, but we are also explicitly referring to the parameter T, the bond maturity.)

The normal way of progressing is that knowing $\eta(t)$ gives $A(t;T)$, hence we can determine $Z(r,t;T)$.

We are not going to do this.

Working forwards: If we know $\eta(t)$ then the above gives us the theoretical value of zero-coupon bonds of all maturities. That is start with model ($\eta(t)$) and find answer (Z).

An inverse problem: But what if we know Z from the market, but don't know the unobservable η? Turn this relationship around and ask the question

- 'What functional form must we choose for $\eta(t)$ to make the theoretical value of the discount rates for all maturities equal to the market values?'

That is calibration.

Suppose we want to calibrate our model today, time t^*. Today's spot interest rate is r^* (which tomorrow will be different) and the discount factors *in the market* are $Z_M(t^*; T)$. Suppose today is t^* =Wednesday 20 May 2008.

Call the special, calibrated, choice for η, $\eta^*(t)$.

We look at the prices of ZCBs on our screens on date t^*. Each ZCB has a certain maturity T.

Today we can say what the price is of a one year bond $(T = 1)$, or 10 year bond $(T = 10)$, etc. These are all prices on 20 May 2008.

To match the market and theoretical bond prices, we must solve

$$Z_M(t^*;T) = e^{A(t^*;T) - r^*(T-t^*)}. \tag{1}$$

Taking logarithms of this

$$\log(Z_M(t^*;T)) = A - rB$$

We know the forms of A and B

$$\log Z_M = \frac{1}{6}c^2(T-t^*)^3 - \int_{t^*}^{T} \eta^*(s)(T-s)ds - r^*(T-t^*)$$

rearranging gives

$$\begin{aligned} & \int_{t^*}^{T} \eta^*(s)(T-s)ds \\ = \; & -\log(Z_M(t^*;T)) - r^*(T-t^*) + \frac{1}{6}c^2(T-t^*)^3. \end{aligned}$$

We know everything on the right-hand side. So this is an **integral equation** for $\eta^*(t)$.

It is called an integral equation because the unknown term (which we are solving for) η^* is under the integral sign.

Fortunately it is a simple equation and can be solved by a technique called *differentiation under the integral sign.* The method is known as *Leibniz Rule.*

The shortened version of this is

$$\frac{\partial}{\partial x}\int_a^x F(y,x)dy = F(x,x) + \int_a^x \frac{\partial F(y,x)}{\partial x}dy$$

Note that we differentiate w.r.t. the upper limit.

Observe what happens if we differentiate the integral term with respect to T.

$$\left.\begin{array}{c} x \equiv T \\ y \equiv s \end{array}\right\} \quad \begin{array}{rcl} F(y,x) & \equiv & (T-s)\,\eta^*(s); \\ \dfrac{\partial F}{\partial x} & \equiv & \dfrac{\partial F}{\partial T} = \eta^*(s) \\ F(x,x) & \equiv & (T-T)\,\eta^*(T) = 0 \end{array}$$

First differentiate the l.h.s once with respect to T

$$\frac{d}{dT}\int_{t^*}^{T}\eta^*(s)(T-s)ds = \int_{t^*}^{T}\eta^*(s)ds + 0$$

Differentiate again

$$\begin{aligned}\frac{d^2}{dT^2}\int_{t^*}^{T}\eta^*(s)(T-s)ds &= \frac{d}{dT}\int_{t^*}^{T}\eta^*(s)ds\\ &= \eta^*(T).\end{aligned}$$

So, differentiating the rhs twice with respect to T we get

$$-\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;T)) + c^2(T-t^*).$$

The parameters are not functions of the derivative maturity. Interest rates can depend on time, but cannot have $\eta(T)$ in the SDE for the spot rate. That is we can't have a short term interest rate depend on the maturity of a bond. Hence the solution is

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)) + c^2(t-t^*).$$

With this choice for the time-dependent parameter $\eta(t)$ the theoretical and actual market prices of zero-coupon bonds are the same.

Notes:

- Now that we know $\eta(t)$ we can price other fixed income instruments.
- We say that our prices are **consistent with the yield curve**.
- The same idea can be applied to other spot interest rate models.
- This is an inverse problem, and will typically be sensitive to input date (the Z).