$$\underline{W \equiv X}$$

1. $F = F(W)$

$$dF = \underbrace{\frac{dF}{dW}\, dW}_{\text{diffusion}} + \underbrace{\frac{1}{2}\frac{d^2F}{dW^2}\, dt}_{\text{drift}}$$

2. $F = F(t, W)$

$$dF = \underbrace{\left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}\right)}_{\text{drift}} dt + \overbrace{\frac{\partial F}{\partial W}}^{\text{diffusion}} dW$$

$$d(\log S) = \left[\mu \cancel{S}\left(\frac{1}{\cancel{S}}\right) + \frac{1}{2}\sigma^2 \cancel{S^2}\left(-\frac{1}{\cancel{S^2}}\right)\right] dt$$

$$+ \; \sigma \cancel{S}\left(\frac{1}{\cancel{S}}\right) dX$$

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \, dX$$

Integrate over t and T

$$\int_t^T d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right)\int_t^T d\tau + \sigma \int_t^T dX$$

$$\int dx$$

$$\log \frac{S(T)}{S(t)} = \left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left[X(T) - X(t)\right]$$

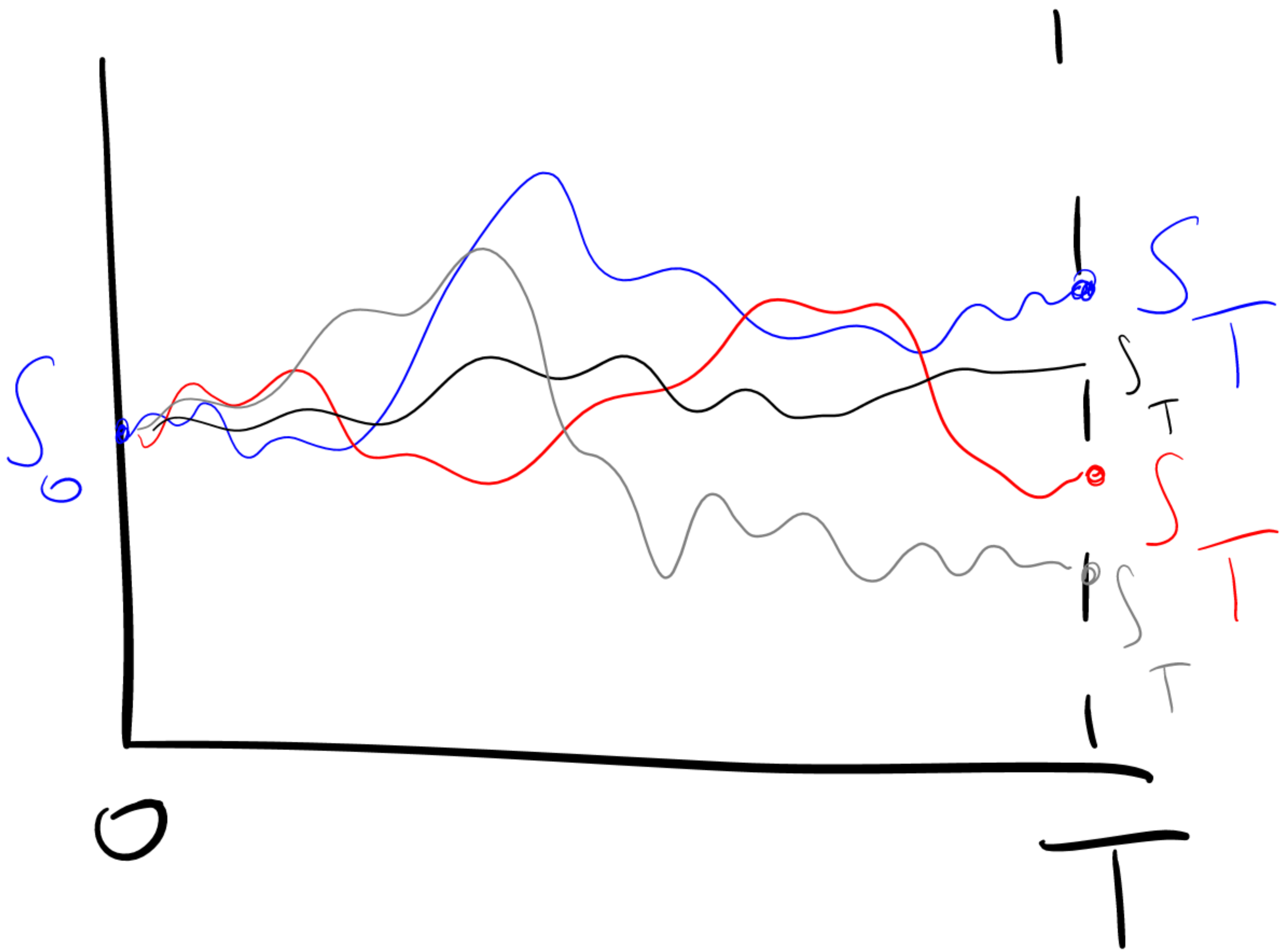
$$S(T) = S(t)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(X(T) - X(t)\right)\right]$$

$$S_T = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma X_T}$$

$$\phi\sqrt{T}$$

$$dS = \mu\, dt + \sigma\, dX$$

$F(\omega_t)$



$$\int \frac{dS}{S} = \mu \int dt + \sigma \int dW$$

$$\log S = \mu t + \sigma dW$$

$$\int d(\quad) \qquad\qquad \int dW$$

$$du = -\gamma u\, dt + \sigma\, dX$$

$$du + \gamma u\, dt = \sigma\, dX$$

I.F is $e^{\gamma t}$

$$\underbrace{e^{\gamma t}\left[du + \gamma u\, dt\right]}_{d(e^{\gamma t} u)} = \sigma e^{\gamma t}\, dX$$

$$dG = A\,dt + B\,dW$$

$$V(G)$$

$$dV = \left(A \frac{dV}{dG} + \frac{1}{2} B^2 \frac{d^2V}{dG^2} \right) dt + B \frac{dV}{dG}\, dW$$

$$y' + Py = Q$$

$$I.F \qquad R(x) = e^{\int P \, dx}$$

$$\cancel{\int d}\left(R(x) y \right) = \int R Q$$

$$d\left(u e^{rt}\right) = \sigma e^{rt}\, dX$$

$$\int_0^t d\left(u e^{rs}\right) = \sigma \int_0^t e^{rs}\, dX_s$$

$$u_t e^{rt} - u_0 = \sigma \int$$

$$e^{rt} u_t = u_0 + \sigma \int$$

$$U_t = U_0 e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)}\, dX_s$$

$$\int_0^t \frac{\partial F}{\partial X}\, dX = F(X_t, t) - F(X_0, 0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} \right) ds$$

④ Itô for $V(S,t)$

where S evolves according to G.B.M.

$$S \to S + dS$$
$$t \to t + dt$$

2D TSE $V(S+dS,\, t+dt) =$

$$V(t,S) + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 \quad \frac{\partial V}{\partial t} dt$$

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dX$$

General Itô $dG = A\,dt + B\,dX$

$V = V(t, G)$

$$dV = \left(\frac{\partial V}{\partial t} + A \frac{\partial V}{\partial G} + \frac{1}{2} B^2 \frac{\partial^2 V}{\partial G^2} \right) dt + B \frac{\partial V}{\partial G} \, dX$$

$$dy = A\,dt + B\,dX$$

$$\mathbb{E}[dy] = A\,dt \quad \because \mathbb{E}[dX] = 0$$

$$\mathbb{V}[dy] = \mathbb{V}[B\,dX]$$

$$= B^2\,\mathbb{V}[dX]$$

$$= B^2\,dt$$

$$\frac{1}{2}\frac{d^2V}{ds^2} \quad ds^2 \qquad \frac{1}{2}\frac{d^2F}{dw^2}$$

$$dX$$

$$\phi \sim N(0,1)$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} \, ds$$

$$\sum_{1}^{N} \mathrm{RAND}() \qquad R \equiv \mathrm{RAND}()$$

$$\mathbb{E}\left[\sum_{1}^{N} R\right] = \sum_{1}^{N} \mathbb{E}(R) = \frac{N}{2}$$

$$\sum_{1}^{N} \mathrm{RAND}() - \frac{N}{2}$$

$$\mathbb{V}\left[\sum_{1}^{N} R - \frac{N}{2}\right] = \sum_{1}^{N} \mathbb{V}(R) = \frac{N}{12}$$

$$\mathbb{V}\left[\alpha\left(\sum_{1}^{N} R - \frac{N}{2}\right)\right] = 1$$

$$\alpha^2 \underbrace{\mathbb{V}\left[\quad\right]}_{N/12} = 1$$

$$\alpha = \sqrt{\frac{12}{N}}$$

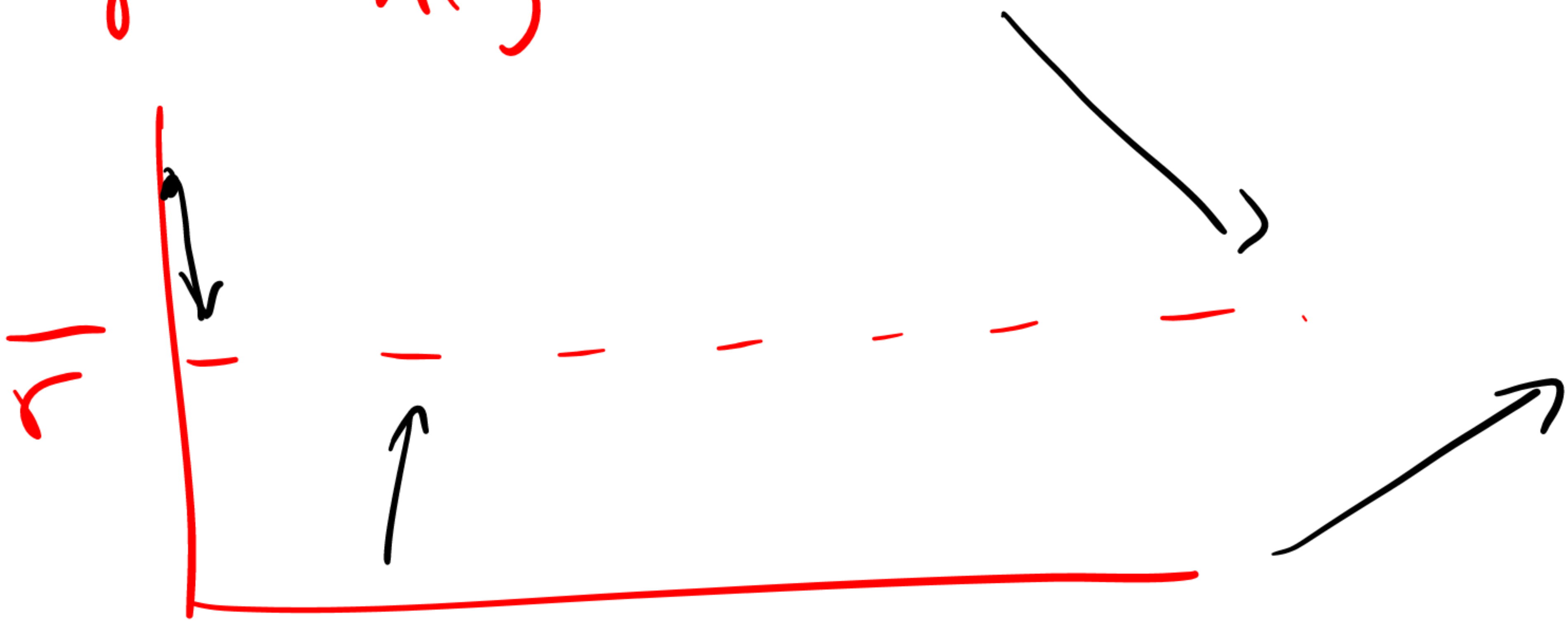
$$\boxed{\sqrt{\frac{12}{N}}\left[\sum_{1}^{N} \mathrm{RAND}(\,) - \frac{N}{2}\right]}$$

$$\frac{\sum \text{RAND()} - N \times \frac{1}{2}}{\sqrt{N} \times \sqrt{\frac{1}{12}}}$$

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N} Y_i - N\mu}{\sigma \sqrt{N}} \sim N(0,1)$$

$$dr = -\gamma(r - \bar{r})\,dt + \sigma\sqrt{dt}$$

γ high $\qquad\qquad$ γ low



$$dr = -\gamma(r - \bar{r})\,dt + \sigma\,dY$$

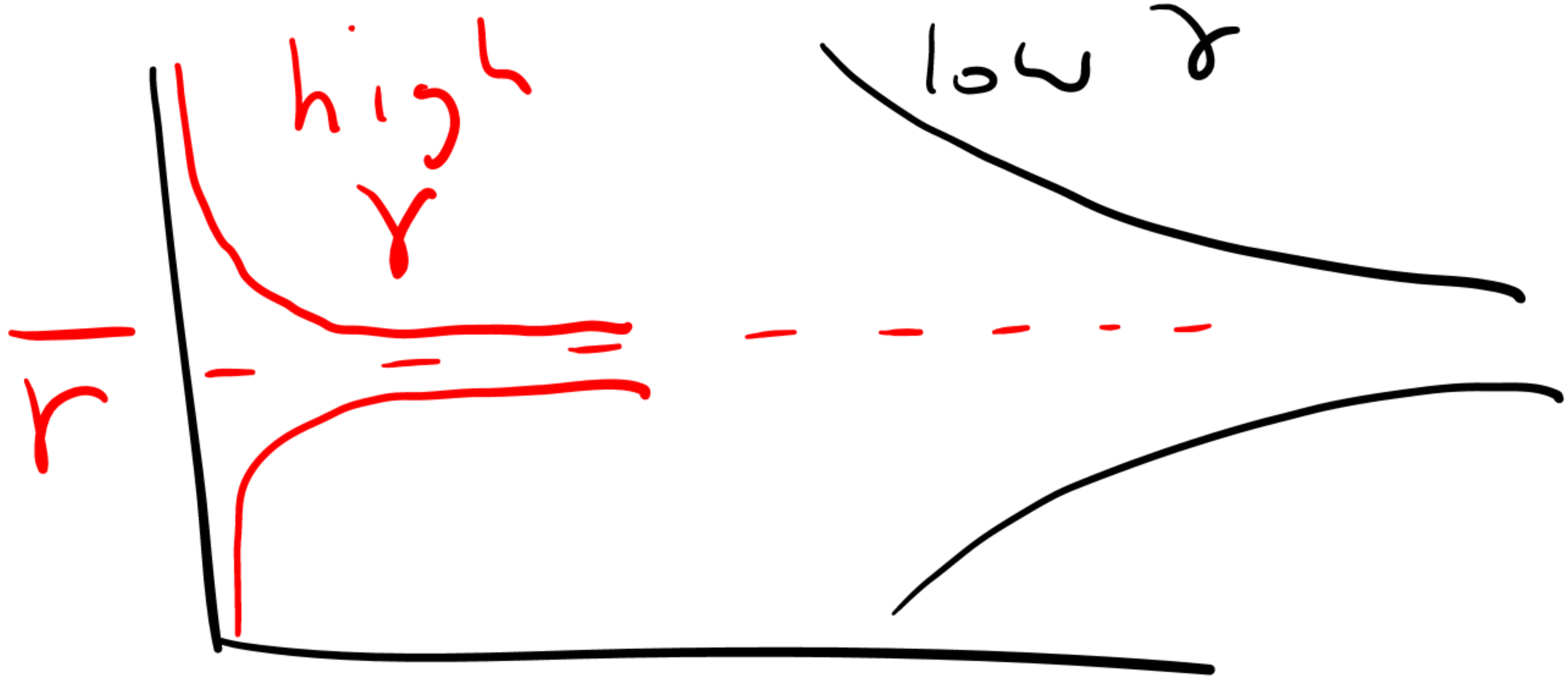
Put $\sigma = 0$

$$dr = -\gamma(r - \bar{r})\,dt$$

$$\int \frac{dr}{r - \bar{r}} = -\gamma \int dt$$

$$\log(r - \bar{r}) = -\gamma t + C$$

$$r = \bar{r} + Ae^{-\gamma t}$$



$$\phi_1 = \varepsilon_1$$

$$\phi_2 = \alpha \varepsilon_1 + \beta \varepsilon_2$$

$$\mathbb{E}[\phi_1 \phi_2] = \rho$$

$$\mathbb{E}[\varepsilon_1 (\alpha \varepsilon_1 + \beta \varepsilon_2)] = \rho$$

$$\alpha \underbrace{\mathbb{E}[\varepsilon_1^2]}_{=1} + \beta \underbrace{\mathbb{E}[\varepsilon_1 \varepsilon_2]}_{=0} = \rho$$

$$\alpha = \rho$$

$$\mathbb{E}[\phi_2^2] = 1$$

$$\mathbb{E}[(\alpha\varepsilon_1 + \beta\varepsilon_2)^2] = 1$$

$$\alpha^2 \underbrace{\mathbb{E}[\varepsilon_1^2]}_{=1} + 2\alpha\beta \underbrace{\mathbb{E}[\varepsilon_1\varepsilon_2]}_{=0} + \beta^2 \underbrace{\mathbb{E}[\varepsilon_2^2]}_{=1} = 1$$

$$\rho^2 + \beta^2 = 1$$

$$\beta = \sqrt{1-\rho^2}$$

$$\phi_1 = \varepsilon_1$$

$$\phi_2 = \rho\varepsilon_1 + \sqrt{1-\rho^2}\,\varepsilon_2$$

$$dS_i = \mu_i S_i \, dt + \sigma_i S_i \, dX_i$$

$$dX_1 \, dX_2 = \rho \, dt$$

$i = 1, 2$

$$V = V(t, S_1, S_2)$$

$t \longrightarrow t + dt$

$S_1 \longrightarrow S_1 + dS_1$

$S_2 \longrightarrow S_2 + dS_2$

2D TSE: $V(t+dt, S_1+dS_1, S_2+dS_2) =$

$$V(S_1, S_2, t) + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2$$

$$+ \frac{1}{2} \frac{\partial^2 V}{\partial S_1^2} dS_1^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_2^2} dS_2^2 + \frac{\partial^2 V}{\partial S_1 \partial S_2} dS_1 dS_2$$

$$dV = \left(\frac{\partial V}{\partial t} + \mu_1 S_1 \frac{\partial V}{\partial S_1} + \mu_2 S_2 \frac{\partial V}{\partial S_2} + \right.$$

$$\frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} +$$

$$\left. \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \right) dt +$$

$$\sigma_1 S_1 \frac{\partial V}{\partial S_1} dX_1 + \sigma_2 S_2 \frac{\partial V}{\partial S_2} dX_2$$

$$p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \, p(x) \, dx = \int_0^1 x \, dx$$

$$= \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}$$

$$\mathbb{V}[X] = \underbrace{\int_0^1 x^2 \, dx}_{1/3} - \left(\frac{1}{2}\right)^2$$

$$1/3 - \frac{1}{4} = \frac{1}{12}$$

$$r_{i+1} = r_i + (\eta - \gamma r_i)\,\delta t + \sqrt{\beta}\,\phi\,\sqrt{\delta t}$$