CQF Binomial Model

Essential of Delta Hedging, Risk Neutrality and No Arbitrage

Exercises

1. A share price is currently \$92, at the end of a three-month period it will be either \$86 or \$98. Present a single-step binomial tree and use it to calculate the value of a European call option with strike \$90. The applicable risk-free rate is 2%.

   Additional questions: Is the call option in-the money or out-of-the money? Find the price of the matching European put option by using Put-Call Parity.

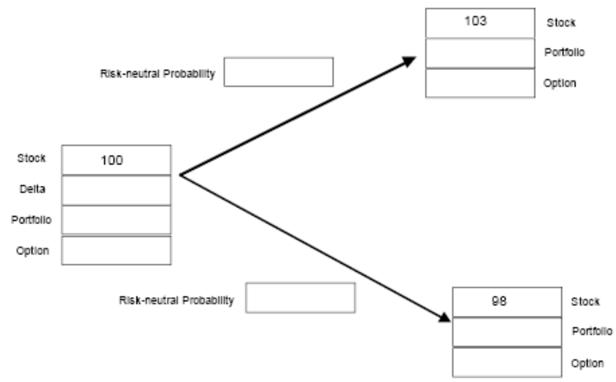
2. A European put option is being sold for \$4 with one month expiry and the strike \$100, while the share price is \$95. The risk-free interest rate is 3% per annum.
   If you detect an arbitrage opportunity which strategy can you devise to utilise it. What is the minimum profit? Consider put option as the future sale of the asset.

3. A share price is currently \$15. At the end of three months, it will be either \$13 or \$17. Use a single-step binomial tree to value a European (call) option with payoff $max(S^2 - 159, 0)$. Assume zero rates.

4. A share price is currently \$75, at the end of a three-month period it will be either \$59 or \$92. The risk-free interest rate is 4%. What are the risk-neutral probabilities that the share price rises or falls?
   Use the risk-neutral probabilities to value European call option with the strike \$85. What about a put option with the same strike?

5. A share price is currently \$100 and can either fall to \$98 or rise to \$103. Calculate the hedge portfolio value, option value and inferred risk-neutral probabilities for an at-the-money call option. Fill in the blanks on the diagram.
   Assume the risk-free rate at zero. If the risk-free rate is not zero but gives the discounting factor of $e^{-rT} = 0.99$ how does that affect the results?

The exercises have been edited by CQF Faculty, Richard Diamond, rdiamond@fitchlearning.com



6. Consider a binomial model that approximates the GBM SDE $dS = S\mu dt + S\sigma dX$. It assumes that the asset with initial value S can either rise to uS with probability p (here $u > 1$) or fall to vS with probability $(1-p)$ (here $0 < v < 1$). The move occur over the small timestep δt. The tree is subject to the condition $uv = 1$.

   Multiple parametrisations are possible for u and v – you have seen an example of $u, v = 1 \pm \sigma\sqrt{\delta t}$ in the lecture.[1] Now, consider the following implicit parametrisation:

$$
\begin{aligned}
pu + (1-p)v &= e^{\mu\,\delta t} \\
pu^2 + (1-p)v^2 &= e^{(2\mu+\sigma^2)\,\delta t}
\end{aligned}
$$

   Show that this parametrisation implies the following relationships:

$$
\begin{aligned}
u + v &= e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t} \\
u &= \frac{1}{2}\left(e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t}\right) + \frac{1}{2}\sqrt{\left(e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t}\right)^2 - 4}
\end{aligned}
$$

7. The same as the previous exercise but instead of now $uv = 1$, use the risk-neutral probability $p = \frac{1}{2}$. The condition $uv = 1$ means an expectation that the tree returns where it started.

[1] Parametrisation term refers to how we choose to represent model quantities using auxiliary parameters. For the binomial model, the sensible quantities are move up and move down which we parametrise using either volatility or risk-neutral probability.

8. **Computational task** on option pricing by binomial tree. Implement the binomial method (up and down moves parametrised as in the lecture) with the following parameters: stock $S = 100$, interest rate $r = 0.05$ (continuously compounded) for a call option with strike $K = 100$, and expiry $T = 1$. The implementation should allow multi-step binomial trees.

   (a) For the constant number of time steps, NTS=4, calculate the value of the option for a range of volatilities and plot the result.

   (b) Then, fix volatility at $\sigma = 0.2$ and plot the value of the option as the number of time steps of the tree increases $NTS = 1, 2, \ldots, 50$. You will need a different tree for each NTS.

   Note: Preferred solution method is with a function written in VBA but Excel spreadsheets with binomial trees will be accepted given that correct plots are produced. For simplicity, use compounded rate as a proxy to discrete.

   VBA code by Paul Wilmott to price a European option using the binomial model can be found for Chapter 15 Binomial Model, *Paul Wilmott on Quantitative Finance* textbook (hardcover).