CQF Binomial Model

Essentials of Delta Hedging, Risk Neutrality and No Arbitrage

Solutions

1. A share price is currently \$92, at the end of a three-month period it will be either \$86 or \$98. Present a single-step binomial tree and use it to calculate the value of a European call option with strike \$90. The applicable risk-free rate is 2%.

   Additional questions: Is the call option in-the money or out-of-the money? Find the price of the matching European put option by using Put-Call Parity.

   Solution:

   The binomial tree for the share price is

$$
\begin{array}{ccc}
 & & 98 \\
 & 92 & \\
 & & 86
\end{array}
$$

   The binomial tree for the option payoff is

$$
\begin{array}{ccc}
 & & 8, \quad \max(98-90,0) \\
\text{V} & & \\
 & & 0, \quad \max(86-90,0)
\end{array}
$$

   The binomial tree for the delta hedged portfolio uses $V - \Delta S$

$$
\begin{array}{ccc}
 & & 8-\Delta 98 \\
V-\Delta 92 & & \\
 & & -\Delta 86
\end{array}
$$

   At each level of the share price (= each timestep), we can find the hedging ratio *Delta* by choosing $8 - \Delta 98 = -\Delta 86$, which gives $\Delta = 2/3$.

   - Given the share price is $S_t = \$92$ and strike $K = \$90$, this call option is in-the money (ITM) $S_t > K$. For an at-the money (ATM) call $S_t = K$, $\Delta = 0.5$.
   - $-\Delta 86$ is a quantity to reduce our previously bought hedge portfolio (shares) but we do not enter the short position in the stock. For an out-of-the money option we hold some minimal amount of shares to protect against a sudden move (jump) upwards.

The exercises have been edited by CQF Faculty, Richard Diamond, rdiamond@fitchlearning.com

It remains for us to calculate the option price as discounted payoff ($r = 0.02$, $T = 1/4$ of the year)

$$\begin{aligned} V - \Delta 92 &= e^{-rT}(8 - \Delta 98) \\ &\text{or} \\ V - \Delta 92 &= e^{-rT}(-\Delta 86) \\ V &= \frac{2}{3}\left(92 - 86\, e^{-0.02\times 0.25}\right) \\ V &= \$4.3. \end{aligned}$$

While not formally derived until we see the Black-Scholes, Put-Call Parity allows to calculate the price

$$C - P = S - Ke^{-rT}$$

$$\begin{aligned} 4.3 - P &= 92 - 90\, e^{-0.02\times 0.25} \\ P &= 4.3 + 90\, e^{-0.02\times 0.25} - 92 \\ P &= \$1.85 \quad \text{out-of-the money put option.} \end{aligned}$$

and find the relationships between the Greeks, particularly Delta for call and put,

$$\begin{aligned} C(S,t) - P(S,t) &= S - Ke^{-rT} \\ &\qquad \text{differentiate } wrt\ S \\ \frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} &= 1 \qquad \Rightarrow \qquad \Delta_C = 1 + \Delta_P \end{aligned}$$

It follows that for an at-the money (ATM) put option $\Delta_P = 0.5 - 1 = -0.5$, and the put option Delta is always negative.

2. A European put option is being sold for \$4 with one month expiry and the strike \$100, while the share price is \$95. The risk-free interest rate is 3% per annum.

   If you detect an arbitrage opportunity which strategy can you devise to utilise it. What is the minimum profit? Consider put option as the future sale of the asset.

   Solution:

   By purchasing a put option, we sell asset at time T for the value today of Ke^{-rT} We know that the lower boundary for European put option is

$$\begin{aligned} Ke^{-rT} - S &= P - C \\ 100 \times e^{-0.03/12} - 95 &> 0 \\ \$4.75 & \end{aligned}$$

   So even if the call's price is zero, the minimum put price is \$4.75 (lower boundary).

We implement arbitrage by **buying this put option and hedging it by holding the positive position in shares** (buy shares) at full $\Delta_P = -1$ (or 100%. The hedge portfolio is

$$\begin{aligned} P - \Delta 95 &= \Pi r dt \\ 4 + 95 &= \Pi r dt \quad \Rightarrow \\ \Pi &= 99\, e^{0.03/12} \quad \text{continuously compounded.} \end{aligned}$$

We have to finance these two purchases (1 put and 1 share) by borrowing the funds \$99 at the risk-free rate.

One month later, at the time of expiration the stock can be above or below strike,

$$\text{P} \quad \begin{matrix} \text{Payoff} = 0 \\ \\ \text{Payoff} = K - S_T \end{matrix}$$

The total P&L is obtained from the option payoff and the hedge portfolio cash value – we hold the stock so its cash value is positive S_T,

$$\Pi \quad \begin{matrix} 0 + S_T - 99\, e^{0.03/12} \\ \\ K - S_T + S_T - 99\, e^{0.03/12} \end{matrix}$$

- if $S_T > K$ the put payoff is zero but the hedge portfolio value is <u>at least</u> $S_T - 99.25 = 100 - 99.25 = 0.75$.
- if $S_T < K$ then the profit is $100 - 99\, e^{0.03/12} = 0.75$.

The total P&L is positive in both cases, with the minimum profit of \$0.75.

The put option with strike \$100 while share price is \$95 is in-the money (ITM). On top, it must have time value (called intrinsic value) because the share price can move further down – the less time to expiry the less magnitude of move is expected by $\sigma\sqrt{\delta t}$.

Even though we have not seen the Black-Scholes equation and its greeks, we can say that Theta greek quantifies time value, each day the option value loses $\Theta = \partial V / \partial t$.

3. A share price is currently \$15. At the end of three months, it will be either \$13 or \$17. Use a single-step binomial tree to value a European (call) option with payoff $max(S^2 - 159, 0)$. Assume zero rates.

Solution:

The binomial tree for the share price is

$$\begin{array}{ccc} & & 17 \\ 15 & & \\ & & 13 \end{array}$$

The option payoff depends on the squared price S^2 but calculated as is usual for a call

$$\begin{array}{ccc} & & 130, \quad \max(17^2 - 159, 0) \\ \text{V} & & \\ & & 10, \quad \max(13^2 - 159, 0) \end{array}$$

The binomial tree for the delta hedged portfolio $V - \Delta S$ is also as usual, given that we know numerical values of the payoff for the down and up moves $\{10, 130\}$.

$$\begin{array}{ccc} & & 130 - \Delta 17 \\ V - \Delta 15 & & \\ & & 10 - \Delta 13 \end{array}$$

To satisfy the risk-neutrality, we choose Δ such that $130 - \Delta 17 = 10 - \Delta 13$, so $\Delta = 30$.

With the interest rate at zero (no discounting) $V - \Delta 15 = 10 - \Delta 13$, so $V = 70$.

4. A share price is currently \$75, at the end of a three-month period it will be either \$59 or \$92. The risk-free interest rate is 4%. What are the risk-neutral probabilities that the share price rises or falls?

   Use the risk-neutral probabilities to value European call option with the strike \$85. What about a put option with the same strike?

   Solution:

   To calculate the risk-neutral probability we write down the mathematical expected value with p as the probability stock of going up and $1 - p$ as the probability of going down. Then, re-arrange and use $r = 0.04, T = 1/4$ (three months).

$$\begin{aligned} 92p + 59(1-p) &= 75e^{rT} \\ &= \\ p = \frac{75\,e^{0.01} - 59}{33} &= 0.5077 \end{aligned}$$

The binomial tree for the share price is

$$\begin{array}{ccc} & & 92 \\ 75 & & \\ & & 59 \end{array}$$

The binomial tree for the option payoff is

$$\begin{array}{ll} & V^u = 7, \quad \max(92 - 85, 0) \\ \text{V} & \\ & V^d = 0, \quad \max(59 - 85, 0) \end{array}$$

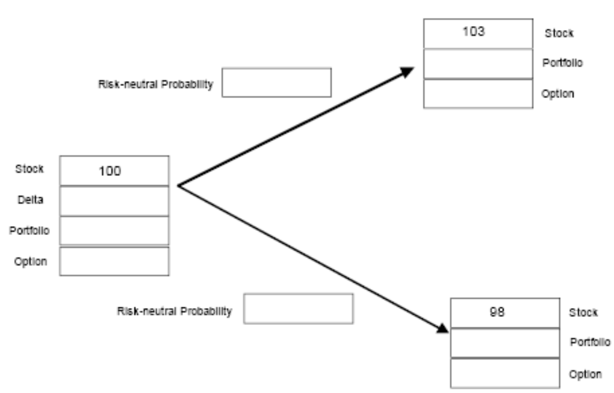
$$\begin{aligned} V &= e^{-rT}\left(pV^u + (1-p)V^d\right) \\ V &= e^{-0.01} \times 0.5077 \times 7 \approx \$3.52 \\ &\ \text{out-of-the money call option.} \end{aligned}$$

For the put option, we can use risk-neutral probabilities since we have them

$$\begin{aligned} V &= e^{-rT}\left(pV^u + (1-p)V^d\right) \\ V &= e^{-0.01}\left(0 + (1 - 0.5077) \times 26\right) \approx \$12.67 \\ &\ \text{deep in-the money put option.} \end{aligned}$$

5. A share price is currently \$100 and can either fall to \$98 or rise to \$103. Calculate the hedge portfolio value, option value and inferred risk-neutral probabilities for an at-the-money call option. Fill in the blanks on the diagram.

   Assume the risk-free rate at zero. If the risk-free rate is not zero but gives the discounting factor of $e^{-rT} = 0.99$ how does that affect the results?



Solution:

The option payoff is on the left. The matching delta hedged portfolio delivers cash $V - \Delta S$ (have to buy shares to hedge a sold call)

$$\begin{array}{lll} & 3, \quad \max(103-100,0) & \\ \text{V} & & \\ & 0, \quad \max(98-100,0) & \end{array} \qquad\qquad \begin{array}{ll} & 3-\Delta 103 \\ V-100 & \\ & -\Delta 98 \end{array}$$

Note that our first hedge was at $\Delta = 1$ or 100% (we bought one share at \$100), while Delta tells us the hedge for the next step.

$$\begin{aligned} 3 - \Delta 103 = -\Delta 98 & \qquad \text{gives } \Delta = 0.6 \\ V - 100 = -\Delta 98 & \qquad \text{gives } V = \$1.2 \\ \Pi_{u,d} = 3 - \Delta 103 = -\Delta 98 \quad = & \qquad \text{gives } \Pi_{u,d} = -58.8 \end{aligned}$$

$$p103 + (1-p)98 = 100 \qquad \text{gives } p = 0.4, \quad (1-p) = 0.6.$$

e.g., we only have to have 0.6 of a share or 60 shares if trading in the multiple of $\times 100$.

With the discounting factor DF=0.99,

$$\begin{aligned} 3 - \Delta 103 = -\Delta 98 & \qquad \text{gives } \Delta = 0.6 \\ V - 100 = 0.99(-\Delta 98) & \qquad \text{gives } V = \$1.788 \\ \Pi = V - \Delta 100 \quad = & \qquad \text{gives } \Pi = -58.212 \end{aligned}$$

$$p103 + (1-p)98 = 100/0.99 \qquad \text{gives } p = 0.602, \quad (1-p) = 0.398.$$

6. Consider a binomial model that approximates the GBM SDE $dS = S\mu dt + S\sigma dX$. It assumes that the asset with initial value S can either rise to uS with probability p (here $u > 1$) or fall to vS with probability $(1-p)$ (here $0 < v < 1$). The move occur over the small timestep δt. The tree is subject to the condition $uv = 1$.

   Multiple parametrisations are possible for u and v – you have seen an example of $u, v = 1 \pm \sigma\sqrt{\delta t}$ in the lecture.[1] Now, consider the following implicit parametrisation:

$$pu + (1-p)v \quad = \quad e^{\mu\,\delta t}$$

$$pu^2 + (1-p)v^2 \quad = \quad e^{(2\mu+\sigma^2)\,\delta t}$$

[1]Parametrisation term refers to how we choose to represent model quantities using auxiliary parameters. For the binomial model, the sensible quantities are move up and move down which we parametrise using either volatility or risk-neutral probability.

Show that this parametrisation implies the following relationships:

$$\begin{aligned} u+v &= e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t} \\ u &= \frac{1}{2}\left(e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t}\right) + \frac{1}{2}\sqrt{\left(e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t}\right)^2 - 4} \end{aligned}$$

$\boxed{\textbf{Solution:}}$

Let's pre-multiply the first relationship by $(u+v)$ – these intuitive steps can be unclear but this is the quantity we would like to find, so

$$\begin{aligned} (u+v)e^{\mu\,\delta t} &= (u+v)(pu+(1-p)v) = pu^2 + uv - \cancel{puv} + \cancel{pvu} + v^2 - pv^2 \\ &= \underbrace{pu^2 + (1-p)v^2}_{} + uv \\ & \text{we know what the underlined term equals to, and } uv = 1 \\ &= e^{(2\mu+\sigma^2)\,\delta t} + 1 \quad \Rightarrow \\ (u+v) &= e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t} \; \dagger \end{aligned}$$

With the right first step, this was almost easy.

Think of u and v as the roots of a quadratic equation.

$$ax^2 + bx + c = 0 \tag{1}$$

$$(x-u)(x-v) = 0 \tag{2}$$

The first equation gives

$$\begin{aligned} u+v = -\frac{b}{a} &\qquad = \quad \dagger \\ uv = \frac{c}{a} &\qquad = \quad 1 \end{aligned}$$

The second equation unfolds

$$\begin{aligned} & x^2 - (u+v)x + uv = 0 \text{ where} \\ x_{1,2} &= \frac{(u+v) \pm \sqrt{(u+v)^2 - 4uv}}{2} \end{aligned}$$

We had the condition $u > 1$, and so

$$u = \frac{1}{2}\left(e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t}\right) + \frac{1}{2}\sqrt{\left(e^{-\mu\,\delta t} + e^{(\mu+\sigma^2)\,\delta t}\right)^2 - 4}.$$

7. The same as the previous exercise but instead of now $uv = 1$, use the risk-neutral probability $p = \frac{1}{2}$. The condition $uv = 1$ means an expectation that the tree returns where it started.

Solution:

Our system of equations becomes

$$\begin{aligned} p &= \frac{1}{2} \\ pu + (1-p)v &= e^{\mu\,\delta t} \\ pu^2 + (1-p)v^2 &= e^{(2\mu+\sigma^2)\,\delta t} \end{aligned}$$

Substitution for constant p gives

$$\begin{aligned} u + v &= 2e^{\mu\,\delta t} \\ u^2 + v^2 &= 2e^{(2\mu+\sigma^2)\,\delta t} \end{aligned}$$

If we would like to find $u > 1$ then we substitute for $v^2 = \left(2e^{\mu\,\delta t} - u\right)^2$

$$u^2 - 2ue^{\mu\,\delta t} + e^{2\mu\,\delta t}\left(2 - e^{\sigma^2\,\delta t}\right) = 0$$

solving this quadratic equation *wrt* $u > 1$ (exercise algebra skills) gives a viable answer

$$u = e^{\mu\,\delta t}\left(1 + \sqrt{e^{\sigma^2\,\delta t} - 1}\right).$$

8. **Computational task** on option pricing by binomial tree. Implement the binomial method with the following parameters: stock $S = 100$, interest rate $r = 0.05$ (continuously compounded) for a call option with strike $K = 100$, and expiry $T = 1$. The implementation should allow multi-step binomial trees.

Note: When parametrising up and down moves, include the second moment $\sigma^2\delta t$ and create a recombinable tree by keeping the relationship $uv = 1$.

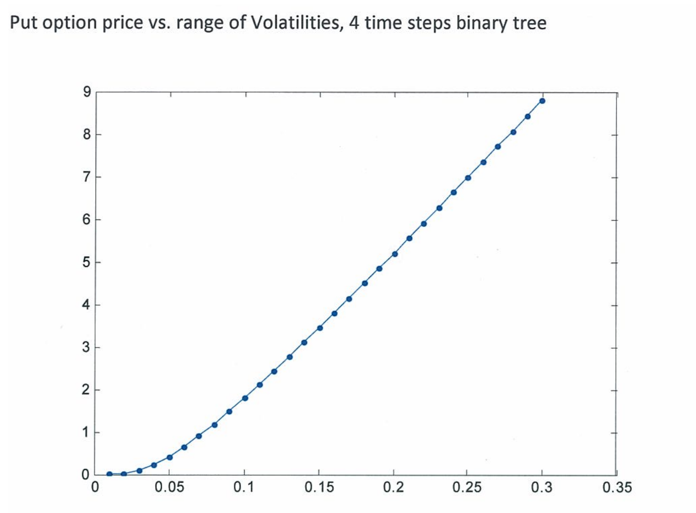
(a) For the constant number of time steps, NTS=4, calculate the value of the option for a range of volatilities and plot the result.

(b) Then, fix volatility at $\sigma = 0.2$ and plot the value of the option as the number of time steps of the tree increases $NTS = 1, 2, \ldots, 50$. You will need a different tree for each NTS.

Solution: VBA or any other code for this task simply implements the binomial formulae to move asset up and down. The advantage of computational implementation (vs. manual tree-drawing) is that a tree can have any number of steps which increases precision. One can also set a sufficiently large number of timesteps (NTS) and vary the volatility.

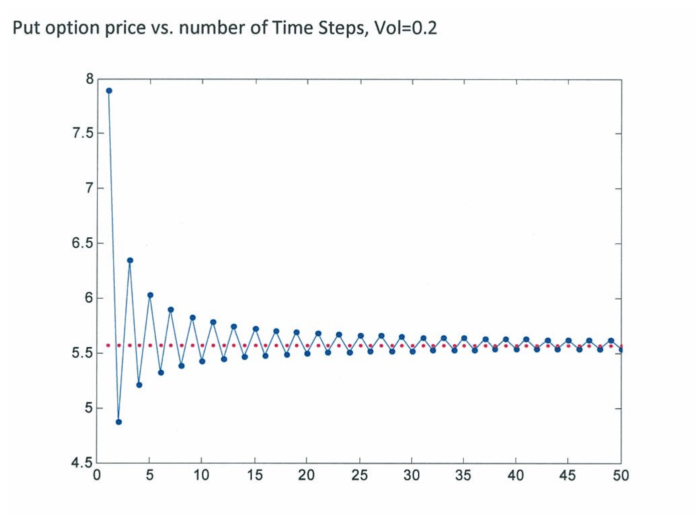
(a) With increasing volatility, option price increases. The relationship is near flat for small volatility $\sigma < 0.05$ and can be linearly approximated when $\sigma > 0.2$. Such a spline is linked by a curved corner, convexity of which depends on moneyness.

Moneyness of an option is equal to $\log(S/K)$ and is a quantity favoured by the practitioners because it is positive for in-the-money options $S > K$ and negative otherwise.



Note: pricing a call option will produce the same relation.

(b) Increasing the amount of time steps will lead to decaying sine-wave pattern that converges around the theoretical (Black-Scholes) price for the option.



VBA code to price a European option using the binomial model. From: Chapter 15 Binomial Model, *Paul Wilmott on Quantitative Finance* textbook.

The code sets up the binomial tree to approximate the GBM as derived in Exercise 6. The condition $uv = 1$ allows for the fast calculation $v = 1/u$ and creates a recombinable tree. Please see the appendix to Chapter 15 on page 293 for the formulae.

```
Function Price(Asset As Double, Volatility As Double, IntRate As Double, _
                       Strike As Double, Expiry As Double, NoSteps As Integer)
ReDim s(0 To NoSteps)
ReDim V(0 To NoSteps)
timestep = Expiry / NoSteps
DiscountFactor = Exp(-IntRate * timestep)
temp1 = Exp((IntRate + Volatility * Volatility) * timestep)
temp2 = 0.5 * (DiscountFactor + temp1)
u = temp2 + Sqr(temp2 * temp2 - 1)
d = 1 / u
p = (Exp(IntRate * timestep) - d) / (u - d)

s(0) = Asset
For n = 1 To NoSteps
    For j = n To 1 Step -1
        s(j) = u * s(j - 1)
    Next j
        s(0) = d * s(0)
Next n

For j = 0 To NoSteps
    V(j) = Payoff(s(j), Strike)
Next j

For n = NoSteps To 1 Step -1
    For j = 0 To NoSteps - 1
        V(j) = (p * V(j + 1) + (1 - p) * V(j)) _
            * DiscountFactor
    Next j
Next n
Price = V(0)
End Function

Function Payoff(s, e)
If s < e Then Payoff = e - s
End Function
```