

1 Calculus Problem Sheet

1. Consider two functions $f(x) = 9x + 2$ and $g(x) = \frac{x}{9} - \frac{2}{9}$. Show that they are functions of one another.
2. Obtain the inverse of the function $f(x) = x^{1/3} + 2$.
3. Calculate the following limits:

$$\begin{array}{llll} \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} & \lim_{x \rightarrow 1} \frac{x^2 - x}{2x^2 + 5x - 7} & \lim_{x \rightarrow -25} \frac{\sqrt{x} + 5}{x - 25} & \lim_{h \rightarrow 0} \frac{(x + h)^3 - x^3}{h} \\ \lim_{h \rightarrow -2} \frac{h^3 + 8}{h + 2} & \lim_{t \rightarrow 1} \frac{(1/t) - 1}{t - 1} & \lim_{x \rightarrow \sqrt{2}} (x^2 + 3)(x - 4) & \end{array}$$

4. Using the definition of the derivative, show that for

$$\begin{array}{ll} y &= 2x + 1, \quad y' = 2 \\ f(x) &= \frac{1}{x - 2}, \quad f'(x) = -\frac{1}{(x - 2)^2} \\ g(x) &= |x - 5|, \quad \text{no derivative exists at } x = 5 \end{array}$$

5. Differentiate the following functions y , to obtain $\frac{dy}{dx}$ where :

$$\begin{array}{llll} y = (x^2 - 4x + 2)^5 & y = \frac{1}{(4x^2 + 6x - 7)^3} & y^4 + 3y - 4x^3 = 5x + 1 & y = \ln \sqrt[3]{(2x + 5)^2} \\ y = \cos(4 - 3x) & y = x^2 \exp(x) & y = \frac{3x^2 - x + 2}{4x^2 + 5} & \end{array}$$

6. Calculate the following

$$\begin{array}{lll} \int \sqrt{x} (x^2 - 4x + 2) dx & \int_4^1 (3\sqrt{x} + 1)(\sqrt{x} - 2) dx & \int_{-1}^{-2} \frac{2s - 7}{s^3} ds \\ \int_3^2 \frac{x^2 - 1}{x - 1} dx & \int_{-1}^5 |2x - 3| dx & \int \frac{5x - 12}{x(x - 4)} dx \end{array}$$

7. By using suitable substitutions (change of variable), evaluate the following

$$\begin{array}{lll} \int (3 - x^4)^3 x^3 dx & \int \frac{x^2 + x}{(4 - 3x^2 - 2x^3)^4} dx & \int \frac{(\sqrt{u} + 3)^4}{\sqrt{u}} du \\ \int \left(1 + \frac{1}{u}\right)^{-3} \left(\frac{1}{u^2}\right) du & \int x \exp(x^2) dx & \int \sin x \exp(\cos x) dx \end{array}$$

8. If $f(x, y) = (x - y) \sin(3x + 2y)$, determine f_x , f_y , f_{xx} , f_{yy} , f_{xy} , f_{yx} .
Now evaluate these expressions at $(0, \pi/3)$.

9. Show that $z = \ln((x - a)^2 + (y - b)^2)$ satisfies

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

except at (a, b) .

10. Obtain Taylor series expansions for the following functions about the given point x_0 . If no point is given, then expand about the point 0 (in which case you can use standard Taylor series expansions)

$$\begin{array}{lll} f(x) = x^2 \sin x & f(x) = \cos x; \ x_0 = \pi/3 & f(x) = \exp x; \ x_0 = -3 \\ f(x) = \frac{1}{1 - 4x} & f(x) = \frac{3}{2x + 5} & f(x) = \frac{x^2 + 1}{x - 1} \end{array}$$

11. If $U(x, y, z) = 2x^2 - yz + xz^2$, where $x = 2 \sin t$, $y = t^2 - t + 1$, $z = 3 \exp(-t)$,

find $\frac{dU}{dt}$ at $t = 0$.

12. Given $w = f(x, y)$; $x = r \cos \theta$, $y = r \sin \theta$; show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$