

3 Differential Equations Problem Sheet

1. For arbitrary constants c_1, c_2, c_3, c_4 find the differential equations satisfied by y when:

a. $y = c_1x + \frac{2}{c_1}$

$$\frac{dy}{dx} = c_1 \therefore y = \frac{dy}{dx}x + \frac{2}{dy/dx} \longrightarrow x(y')^2 - yy' + 2 = 0$$

b. $y = (c_1 + c_2x)e^{-\lambda x}$

$$\begin{aligned} \frac{dy}{dx} &= -\lambda(c_1 + c_2x)e^{-\lambda x} + c_2e^{-\lambda x} = -\lambda ye^{-\lambda x} + c_2e^{-\lambda x} \\ \frac{d^2y}{dx^2} &= -\lambda \frac{dy}{dx} - c_2\lambda e^{-\lambda x} = -\lambda \frac{dy}{dx} - \lambda^2 y - \lambda \frac{dy}{dx} \therefore y'' + 2\lambda y' + \lambda^2 y = 0 \end{aligned}$$

c. $y = c_1 \sin \rho x + c_2 \cos \rho x + c_3 \sinh \rho x + c_4 \cosh \rho x$

$$\begin{aligned} \frac{d^4}{dx^4} \sin \rho x &= \rho^4 \sin \rho x; \quad \frac{d^4}{dx^4} \cos \rho x = \rho^4 \cos \rho x; \\ \frac{d^4}{dx^4} \sinh \rho x &= \rho^4 \sinh \rho x; \quad \frac{d^4}{dx^4} \cosh \rho x = \rho^4 \cosh \rho x \\ \frac{d^4 y}{dx^4} &= \rho^4 y \end{aligned}$$

2. Solve the following differential equations/I.V.P.'s

a. $\left(\frac{dy}{dx}\right)^3 = y^2 \quad y = 1, x = 0$

$$\begin{aligned} \frac{dy}{dx} &= y^{2/3} \longrightarrow dx = y^{-2/3} dy \\ \int_0^x ds &= \int_0^x y^{-2/3} dy \longrightarrow x = 3y^{1/3} \Big|_0^x = 3y^{1/3}(x) - 3y^{1/3}(0) \\ \frac{x}{3} &= y^{1/3} - 1 \therefore y = \left(\frac{x+3}{3}\right)^3 \end{aligned}$$

b. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \quad y = 1, x = 0$

$$\begin{aligned} \int \frac{dy}{1+y^2} &= \int \frac{dx}{1+x^2} \longrightarrow \arctan y = \arctan x + c \text{ and use } \tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \\ y &= \frac{x+C}{1-Cx}; \text{ I.C } y(0) = 1 \implies C = 1 \therefore y = \frac{1+x}{1-x} \end{aligned}$$

c. $\sqrt{1+x^2} \frac{dy}{dx} = xe^{-y} \quad y=0, x=0 \quad \text{Ans: } y = \frac{1}{2} \log(1+x^2)$

$$\int_0^x e^y dy = \int_0^x \frac{s}{\sqrt{1+s^2}} ds$$

The right hand integral is done by substitution $u = 1 + s^2 \longrightarrow du = 2s ds$

$$\begin{aligned} e^y - 1 &= \sqrt{1+s^2} \Big|_0^x = \sqrt{1+x^2} - 1 \\ e^y &= \sqrt{1+x^2} \longrightarrow y = \frac{1}{2} \log(1+x^2) \end{aligned}$$

d. $(1-y)^2 \frac{dy}{dx} + (1+x^2)y = 0 \quad \text{Ans: } x + \frac{x^3}{3} = -\log y + 2y - \frac{1}{2}y^2 + c$

$$\begin{aligned} \int \frac{(1-y)^2}{y} dy &= -\int (1+x^2) dx \\ \int \left(\frac{1}{y} + y - 2\right) dy &= -\int (1+x^2) dx \\ \log y + \frac{y^2}{2} - 2y &= -x - \frac{x^3}{3} + c \end{aligned}$$

which is an implicit solution.

e. $x \frac{dy}{dx} + 3y = 8x^5 \quad \text{Ans: } y = x^5 + \frac{c}{x^3}$

$$\frac{dy}{dx} + \frac{3}{x}y = 8x^4$$

linear equation with IF: $e^3 \int 1/x dx = x^3$

$$\begin{aligned} x^3 \frac{dy}{dx} + 3x^2 y &= 8x^7 \\ \frac{d}{dx} (yx^3) &= 8x^7 \longrightarrow \int d(yx^3) = 8 \int x^7 dx \\ yx^3 &= x^8 + c \longrightarrow y = x^5 + c/x^3. \end{aligned}$$

f. $\frac{dy}{dx} - 2y \tan x = x^2 \sec^2 x$ when $x=0$ and $y=0$
So comparing with standard form we have $P = -2 \tan x$, so

$$\text{I.F } R(x) = e^{-2 \int \tan x dx} = e^{-2 \ln \sec x} = e^{\ln(\sec x)^{-2}} = (\sec x)^{-2}.$$

Note: apart from the few basic integrals, you need not worry about remembering others - always consult a list of integrals in a book. So the differential equation is multiplied by the I.F

$$\begin{aligned} (\sec x)^{-2} (y' - 2y \tan x) &= x^2 \sec^2 x (\sec x)^{-2} \\ y (\sec x)^{-2} &= \int x^2 dx \longrightarrow y = \frac{x^3}{3} \sec^2 x + c \end{aligned}$$

the initial condition gives $c = 0$, so the particular solution becomes $y = \frac{x^3}{3} \sec^2 x$

1. **g.** $\sin x \frac{dy}{dx} + 2y \cos x = \cos x$

$$\frac{dy}{dx} + 2y \cot x = \cot x$$

which is a linear equation with IF: $e^{\int 2 \cot x} = e^{2 \log \sin x} = \sin^2 x$

$$\begin{aligned} \sin^2 x \frac{dy}{dx} + 2(\sin x \cos x) y &= \sin x \cos x \\ \frac{d}{dx} (y \sin^2 x) &= \sin x \cos x \\ \int d(y \sin^2 x) &= \int \sin x \cos x dx \end{aligned}$$

The right hand integral is solved by writing $I = \int \sin x \cos x dx$ and solving by parts to give $I = \frac{1}{2} \sin^2 x$

$$\begin{aligned} y \sin^2 x &= \frac{1}{2} \sin^2 x + c \\ y &= \frac{1}{2} + c \csc^2 x \end{aligned}$$

h. $(x+1)y' - 2y = 3(x+1)^3$ Ans: $y = (3x+c)(x+1)^2$
start by putting in standard form, divide through by $(x+1)$ to express as a linear equation

$$y' - \frac{2}{(x+1)}y = 3(x+1)^2$$

so $P(x) = -\frac{2}{(x+1)}$, hence I.F

$$R(x) = \exp\left(-\int \frac{2}{(x+1)} dx\right) = \exp\left(\ln(x+1)^{-2}\right) = \frac{1}{(x+1)^2}$$

multiply DE through by $R(x)$

$$y' \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}y = 3$$

$$\begin{aligned} \frac{d}{dx} \left(y (x+1)^{-2} \right) &= 3 \\ y (x+1)^{-2} &= 3 \int dx + c \\ &= 3x + c \\ y &= (3x + c)(x+1)^2 \end{aligned}$$

3. Solve the 2nd order equations

a. $\frac{d^2y}{dx^2} = 2y^3 + 8y$ where $y = 2$, $y' = -8$ when $x = \frac{\pi}{4}$
 Put $p = y' \longrightarrow p' = y''$

$$y'' = \frac{dp}{dx} = \frac{dy}{dx} \frac{dp}{dy} = p \frac{dp}{dy}$$

$$p \frac{dp}{dy} = 2y^3 + 8y \text{ which is variable separable}$$

$$\frac{1}{2}p^2 = \frac{1}{2} \left(\frac{dy}{dx} \right)^2 = \frac{y^4}{2} + 4y^2 + c$$

$$y = 2, \quad y' = -8 \implies c = 8$$

$$\frac{dy}{dx} = \sqrt{y^4 + 8y^2 + 16} = \sqrt{(y^2 + 4)^2} = -(y^2 + 4)$$

we have taken the negative sign to satisfy the IC $y'(2) = -8$

$$\begin{aligned} \int dx &= - \int \frac{dy}{(y^2+4)} \\ x &= -\frac{1}{2} \arctan(y/2) + d \end{aligned}$$

using the IC $y(\frac{\pi}{4}) = 2$ gives $d = 3\pi/8$, so the PS becomes

$$y = 2 \tan \left(\frac{3\pi}{4} - 2x \right)$$

b. $\frac{d^2y}{dx^2} + 2x \left(\frac{dy}{dx} \right)^2 = 0$ where $y = 0$, $y' = 1$ when $x = 0$.

$$p = y'; \quad p' = y''$$

the ODE becomes $\frac{dp}{dx} = -2xp^2 \longrightarrow \int p^{-2} dp = -2 \int x dx$

$$\frac{1}{p} = x^2 + c : y' = 1, \quad x = 0 \implies c = 1$$

$$\frac{1}{dy/dx} = x^2 + 1 \longrightarrow \frac{dy}{dx} = \frac{1}{x^2 + 1} \longrightarrow \int dy = \int \frac{dx}{x^2 + 1}$$

$$y = \arctan x + d : y(0) = 0 \implies d = 0$$

therefore the PS is $y = \arctan x$

4. For each of the following constant coefficient differential equations,

$$y'' + by' + cy = g(x)$$

find the complimentary function and state which function you would use to try and find a Particular Solution by the method of undetermined coefficients.

- a. $b = 3, c = 2, g(x) = e^{5x}$ Ans: C.F: $y = Ae^{-2x} + Be^{-x}$ PS
 $y = Ce^{5x}$.
- b. $b = 1, c = -6, g(x) = 2e^{2x} + \sin 3x$ Ans: C.F: $y = Ae^{-3x} + Be^{2x}$ PS: $y_1 = Cxe^{2x}$, because -2 is a root of the A.E. $y_2 = (D \sin 3x + E \cos 3x)$.
- c. $b = 7, c = 0, g(x) = 4x^2 + x + 2$ Ans: C.F: $y = A + Be^{-7x}$ PS
 $y = (p_2x^2 + p_1x + p_0)x$ because 0 is a root of the A.E.
- d. $b = 1, c = 1, g(x) = 2e^{-x}$ Ans: C.F: $y = e^{-x/2} \left(A \sin \frac{\sqrt{3}}{2}x + B \cos \frac{\sqrt{3}}{2}x \right)$ PS
 $y = Ce^{-x}$.
- e. $b = 4, c = 4, g(x) = 3e^{-2x} + 2e^{3x} + \sin x$ Ans: C.F: $y = e^{-2x}(A + Bx)$ PS $y_1 = Cx^2e^{-2x}$ because -2 is a two fold root of the A.E, $y_2 = De^{3x}, y_3 = (E \sin x + F \cos x)$.

5. By converting the Euler equation

$$x^2 y''(x) - 2xy'(x) + 2y(x) = 4x^3$$

to a constant coefficient problem show that the solution is given by

$$y(x) = Ax + Bx^2 + 2x^3.$$

The change of variable is $t = \log x$, with the derivatives represented as

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x} \frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \frac{1}{x^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt} \right). \end{aligned}$$

The ODE becomes

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{3t}$$

$$\text{A.E } \lambda^2 - 3\lambda + 2 = 0 \longrightarrow y_c = Ae^t + Be^{2t}.$$

For the PI look for a solution of the form $y_p = Ce^{3t}$: substitute in ODE

$$(9C - 9C + 2C)e^{3t} = 4e^{3t} \implies C = 2$$

General Solution $y(t) = Ae^t + Be^{2t} + 2e^{3t} \longrightarrow$

$$y(x) = Ae^x + Be^{2x} + 2x^3$$