Exercise 1:

The objective of the exercise is to check that the following fact is true:

Fact 1. *If a process $Y(t)$ is a martingale under $\mathbb{Q}$ and $\eta_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$, then the process $M(t) = Y(t)\eta_t$ is a martingale under $\mathbb{P}$.*

We will focus on the case where both $Y(t)$ and $\eta(t)$ are modelled as diffusions processes with respective dynamics

$$dY(t) = f(t, Y(t))dt + g(t, Y(t))dX(t)$$

and

$$\frac{d\eta(t)}{\eta(t)} = -\theta(t)dX(t)$$

where $X(t)$ is a standard Brownian motion under the $\mathbb{P}$ measure.

Questions -

(i). Knowing that $Y(t)$ is a martingale under $\mathbb{Q}^{\theta}$, express the drift function $f(\cdot)$ in terms of the diffusion function $g(\cdot)$ and of the process $\theta(t)$.

(ii). Apply the Itô product rule to show that $M(t) = Y(t)\eta_t$ is a martingale under $\mathbb{P}$.

Exercise 2: (Optional)

Derive formula (25) on slide 80

$$C(t) = B(t,U)N\left[d_1(B(t,U),t,T)\right] - KB(t,T)N\left[d_2(B(t,U),t,T)\right] \tag{1}$$

where

$$\begin{aligned} d_1(b,t,T) &= \frac{\ln\left(\frac{b}{K}\right) - \ln B(t,T) + \frac{1}{2}\upsilon_U(t,T)}{\upsilon_U(t,T)} \\ d_2(b,t,T) &= d_1 - \upsilon_U(t,T) \\ \upsilon_U^2(t,T) &= \int_t^T (b(s,U) - b(s,T))^2\, ds \end{aligned}$$

Start from the forward asset pricing formula given in equation (24), on slide 79,

$$C(t) \quad = \quad B(t,T)\mathbf{E}^{\mathbb{P}_T}\left[(F_B(T,T,U)-K)^{+}|\mathcal{F}_t\right] \tag{2}$$

where the dynamics of the forward price $F_B(t,T,U)$ is given in equations (22) and (23) on slide 78.

Hints:

1. you could use an approach similar to the derivation of the Black-Scholes formula presented in Section 3.3 of Lecture 3.3 (slides 63-75);

2. Note that the random variable $Y(T) = \int_t^T (b(s,U) - b(s,T))\, dX^T(s)$ is Normally distributed with mean 0 and variance $v_U^2(t,T)$.