Transition Density Function and Partial Differential Equations

In this lecture

Review of Taylor series

Generalised Functions - Dirac delta and heaviside

Invariant Scalings

Transition Density Function - Forward and Backward Kolmogorov Equation

Similarity Reduction Method

Taylor for two Variables

Assuming that a function $f(x,t)$ is differentiable enough, near $x = x_0$, $t = t_0$,

$$
\begin{aligned}
f(x,t) &= \overbrace{f(x_0,t_0)}^{\text{const.}} + \overbrace{(x-x_0)\, f_x(x_0,t_0)}^{\text{linear}} + \\
&\quad (t-t_0)\, f_t(x_0,t_0) \\
&\quad + \frac{1}{2} \underbrace{\left[\begin{array}{c} (x-x_0)^2 f_{xx}(x_0,t_0) \\ +2(x-x_0)(t-t_0)\, f_{xt}(x_0,t_0) \\ +(t-t_0)^2 f_{tt}(x_0,t_0) \end{array} \right]}_{\text{quadratic}} + \;....
\end{aligned}
$$

That is,

$$
\begin{aligned}
f\ (x,t) &= \text{constant} + \ \text{linear} + \text{quadratic} \\
&\quad +....
\end{aligned}
$$

The error in truncating this series after the second order terms tends to zero faster than the included terms. This result is particularly important for Itô's lemma in Stochastic Calculus.

Suppose a function $f = f(x,y)$ and both x, y change by a small amount, so $x \longrightarrow x + \delta x$ and $y \longrightarrow y + \delta y$, then we can examine the change in f using a two dimensional form of Taylor

$$\begin{aligned} f(x+\delta x, y+\delta y) &= f(x,y) + f_x \delta x + f_y \delta y + \\ &\quad \frac{1}{2} f_{xx} \delta x^2 + \frac{1}{2} f_{yy} \delta y^2 + \\ &\quad f_{xy} \delta x \delta y + O\left(\delta x^2, \delta y^2\right). \end{aligned}$$

By taking $f(x,y)$ to the lhs, writing

$$df = f(x+\delta x, y+\delta y) - f(x,y)$$

differential

Total change

and considering only linear terms, i.e.

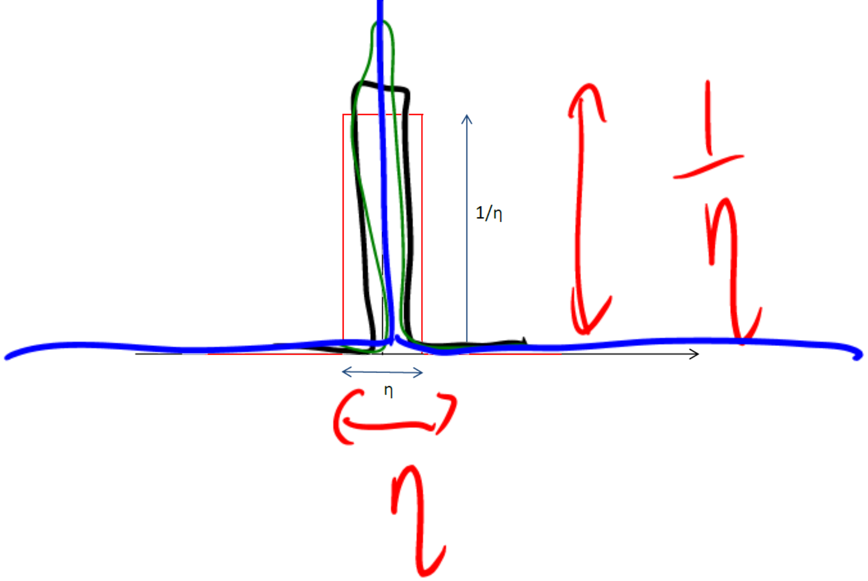
$$\boxed{df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y}$$

we obtain a formula for the *differential* or *total change* in f.

The Dirac delta function

The *delta* function denoted $\delta(x)$; is a very useful 'object' in applied maths and more recently in quant finance. It is the mathematical representation of a point source e.g. force, payment. Although labelled a function, it is more of a distribution or *generalised function*. Consider the following definition for a piecewise function

$$f_\eta(x) = \begin{cases} \frac{1}{\eta}, & x \in \left(-\frac{\eta}{2}, \frac{\eta}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$



Now put the delta function equal to the above for the following limiting value

$$\delta_\eta(x) = \lim_{\eta \longrightarrow 0} f_\eta(x)$$

What is happening here? As η decreases we note the 'hat' narrows whilst becoming taller eventually becoming a spike. Due to the definition, the area under the curve (i.e. rectangle) is fixed at 1, i.e. $\eta \times \frac{1}{\eta}$; which is independent of the value of η. So mathematically we can write in integral terms

$$\begin{aligned}\int_{-\infty}^{\infty} f_\eta(x)\,dx &= \int_{-\infty}^{-\frac{\eta}{2}} f_\eta(x)\,dx + \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} f_\eta(x)\,dx + \int_{\frac{\eta}{2}}^{\infty} f_\eta(x)\,dx \\ &= \eta \times \frac{1}{\eta} = 1 \text{ for all } \eta.\end{aligned}$$

Looking at what happens in the limit $\eta \longrightarrow 0$, the spike like (singular) behaviour at the origin gives the following definition

$$\delta_\eta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

with the property

$$\int_{-\infty}^{\infty} \delta_\eta(x)\, dx = 1.$$

There are many ways to define $\delta(x)$. Consider the Gaussian/Normal distribution with pdf

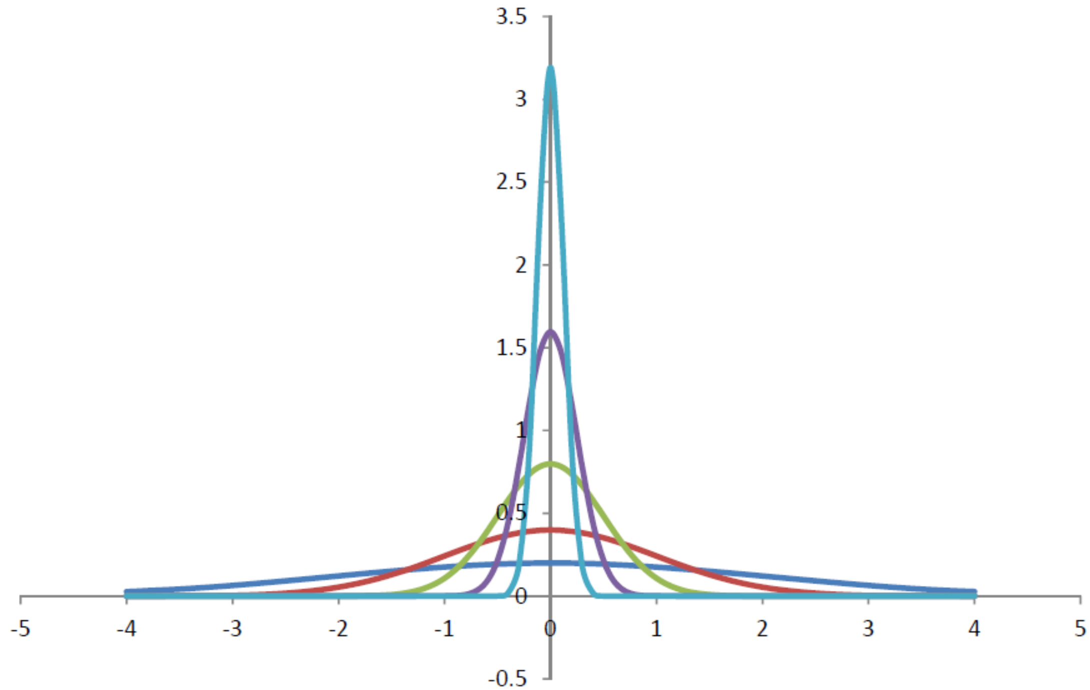
$$G_\eta(x) = \frac{1}{\eta\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\eta^2}\right).$$

The function takes its highest value at $x = 0$; as $|x| \longrightarrow \infty$ there is exponential decay away from the origin. If we stay at the origin, then as η decreases, $G_\eta(x)$ exhibits the earlier spike (as it shoots up to infinity), so

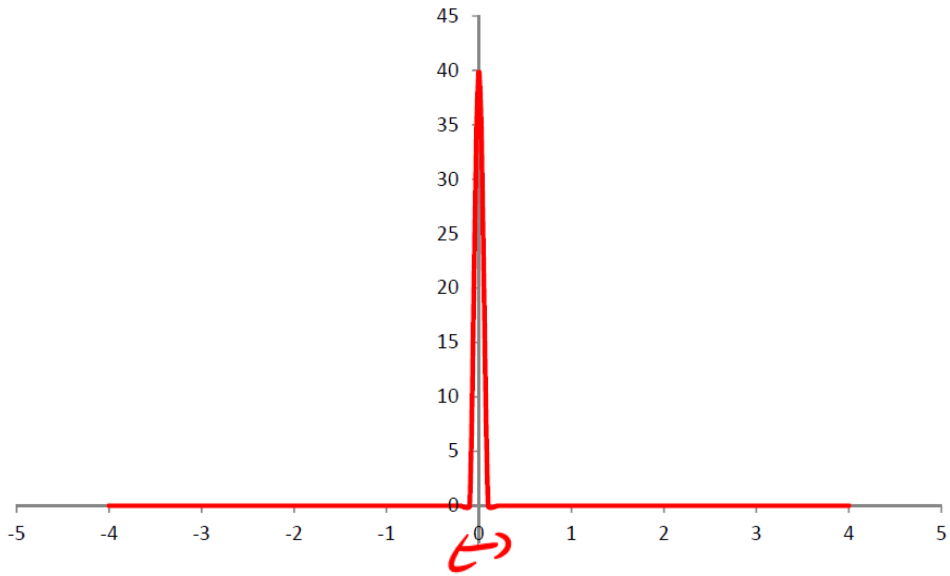
$$\lim_{\eta \longrightarrow 0} G_\eta(x) = \delta(x).$$

The normalising constant $\frac{1}{\eta\sqrt{2\pi}}$ ensures that the area under the curve will always be unity.

The graph below shows $G_\eta(x)$ for values $\eta = 2.0$ (royal blue), 1.0 (red), 0.5 (green), 0.25 (purple), 0.125 (turquoise); the Gaussian curve becomes slimmer and more peaked as η decreases.



$G_\eta(x)$ is plotted for $\eta = 0.01$



Now generalise this definition by centering the function $f_\eta(x)$ at any point x'. So

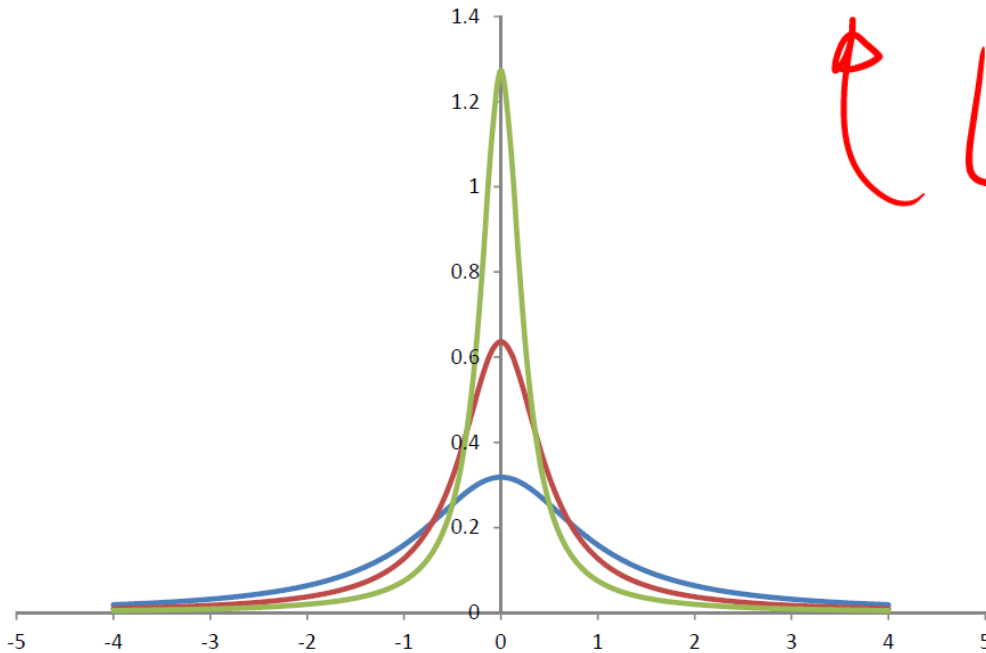
$$\delta\left(x - x'\right) = \lim_{\eta \longrightarrow 0} f_\eta\left(x - x'\right)$$

$$\int_{-\infty}^{\infty} \delta\left(x - x'\right) dx = 1.$$

The figure will be as before, except that now centered at x' and not at the origin as before. So we see two definitions of $\delta(x)$. Another is the Cauchy distribution

$$L_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{x^2 + \epsilon^2}$$

Lorentzian



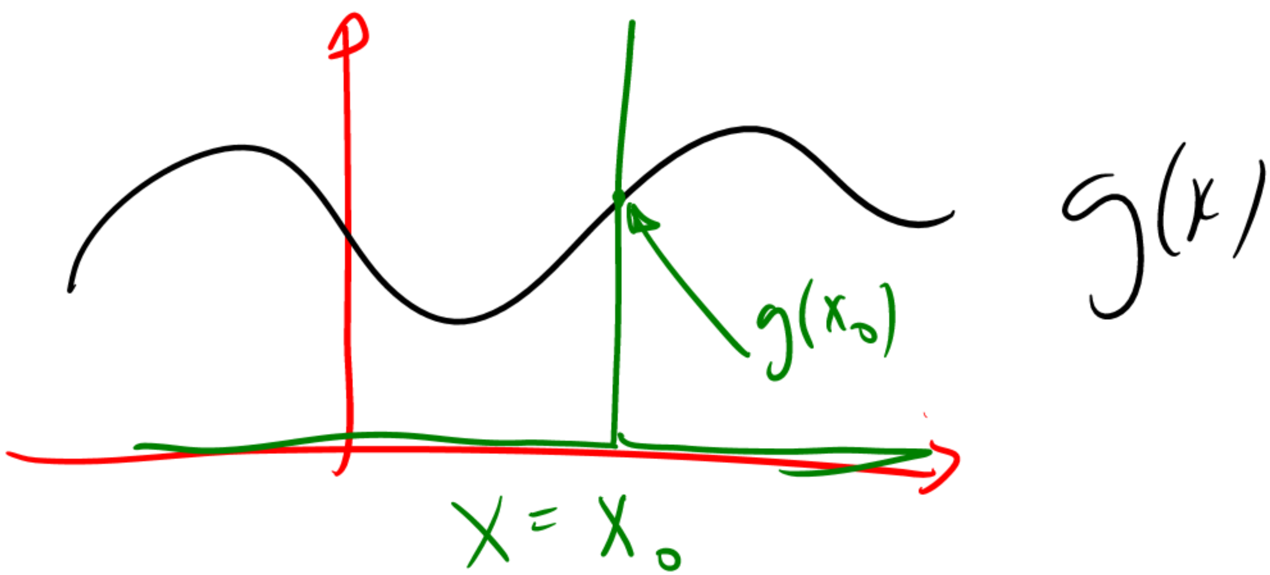
So here

$$\delta(x) = \lim_{\epsilon \longrightarrow 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$$

Now suppose we have a smooth function $g(x)$ and consider the following integral problem

$$\boxed{\int_{-\infty}^{\infty} g(x)\, \delta(x - x_0)\, dx = g(x_0)}$$

This sifting property of the delta function is a very important one.



Heaviside Function / Unit Step function

The **Heaviside function**, denoted by $H()$, is a discontinuous function whose value is zero for negative parameters and one for positive arguments

$$\boxed{\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}}$$

Some definitions have

$$\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$

and

$$\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

It is an example of the general class of step functions.

$x \to \lambda x \quad y \to \lambda y$

Similarity Methods

$f(x,y)$ is **homogeneous of degree** $t \geq 0$ if $f(\lambda x, \lambda y) = \lambda^t f(x,y)$.

1. $f(x,y) = \sqrt{\left(x^2+y^2\right)}$

$$f(\lambda x, \lambda y) = \sqrt{\left[(\lambda x)^2 + (\lambda y)^2\right]} = \lambda\sqrt{\left[\left(x^2+y^2\right)\right]} = \lambda f(x,y)$$

λ^1

2. $g(x,y) = \frac{x+y}{x-y}$ then

$$g(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \lambda^0 \left(\frac{x+y}{x-y}\right) = \lambda^0 g(x,y)$$

3. $h(x,y) = x^2 + y^3$

$$h(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^3 = \lambda^2 x^2 + \lambda^3 y^3 \neq \lambda^t \left(x^2+y^3\right)$$

for any t. So h is not homogeneous.

even $f(-x)=f(x)$

Consider the function

$$F(x,y) = \frac{x^2}{x^2+y^2}$$

If for any $\lambda > 0$ we write

$$x' = \lambda x, \;\; y' = \lambda y$$

$f(-x) = -f(x)$

odd

$\frac{\cancel{\lambda^2} x^2}{\cancel{\lambda^2}(x^2+y^2)}$

then

$$\frac{dy'}{dx'} = \frac{dy}{dx}, \;\; \frac{x^2}{x^2+y^2} = \frac{x'^2}{x'^2+y'^2}.$$

We see that the equation is *invariant* under the change of variables. It also makes sense to look for a solution which is also invariant under the transformation. One choice is to write

$$v = \frac{y}{x} = \frac{y'}{x'}$$

λ^0

$\frac{dy}{dx} = \frac{x^2}{x^2+y^2}$

so write

$$\boxed{y = vx.}$$

Definition The differential equation $\dfrac{dy}{dx} = f(x, y)$ is said to be *homogeneous* when $f(x, y)$ is homogeneous of degree t for some t.

$\lambda^{t} f(x,y)$

Method of Solution

Put $y = vx$ where v is some (as yet) unknown function. Hence we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(vx) = x\frac{dv}{dx} + v\frac{dx}{dx} \\ &= v'x + v\end{aligned}$$

$f(x,y)$

Hence

$$f(x, y) = f(x, vx)$$

$f(x, vx)$

Now f is homogeneous of degree t – so

$$\boxed{f(x, vx) = x^{t} f(1, v)}$$

$x^{t} f(1, v)$

The differential equation now becomes

$$v'x + v = x^t f(1, v)$$

which is not always solvable - the method may not work. But when $t = 0$ (homogeneous of degree zero) then $x^t = 1$. Hence

$$v'x + v = f(1, v)$$

or

$$x\frac{dv}{dx} = f(1, v) - v$$

which is separable, i.e.

$$\int \frac{dv}{f(1, v) - v} = \int \frac{dx}{x} + c$$

and the method is guaranteed to work.

Example

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$f(x,y)$

First we check:

$$\frac{\lambda y - \lambda x}{\lambda y + \lambda x} = \lambda^0 \left(\frac{y-x}{y+x} \right)$$

$$\boxed{v = \frac{y}{x}}$$

which is homogeneous of degree zero. So put $y = vx$

$$v'x + v = f(x, yx) = \frac{vx - x}{vx + x} = \frac{v-1}{v+1} = f(1, v)$$

therefore

$$\begin{aligned} v'x &= \frac{v-1}{v+1} - v \\ &= \frac{-\left(1+v^2\right)}{v+1} \end{aligned}$$

$v = y/x$

and the D.E is now separable

$$
\begin{aligned}
\int \frac{v+1}{v^2+1} dv &= -\int \frac{1}{x} dx \\
\int \frac{v}{v^2+1} dv + \int \frac{1}{v^2+1} dv &= -\int \frac{1}{x} dx \\
\frac{1}{2} \ln\left(1+v^2\right) + \arctan v &= -\ln x + c \\
\frac{1}{2} \ln x^2 \left(1+v^2\right) + \arctan v &= c
\end{aligned}
$$

Now we turn to the original problem, so put $v = \dfrac{y}{x}$

$$
\frac{1}{2} \ln x^2 \left(1 + \frac{y^2}{x^2}\right) + \arctan\left(\frac{y}{x}\right) = c
$$

which simplifies to

$$
\frac{1}{2} \ln\left(x^2 + y^2\right) + \arctan\left(\frac{y}{x}\right) = c.
$$

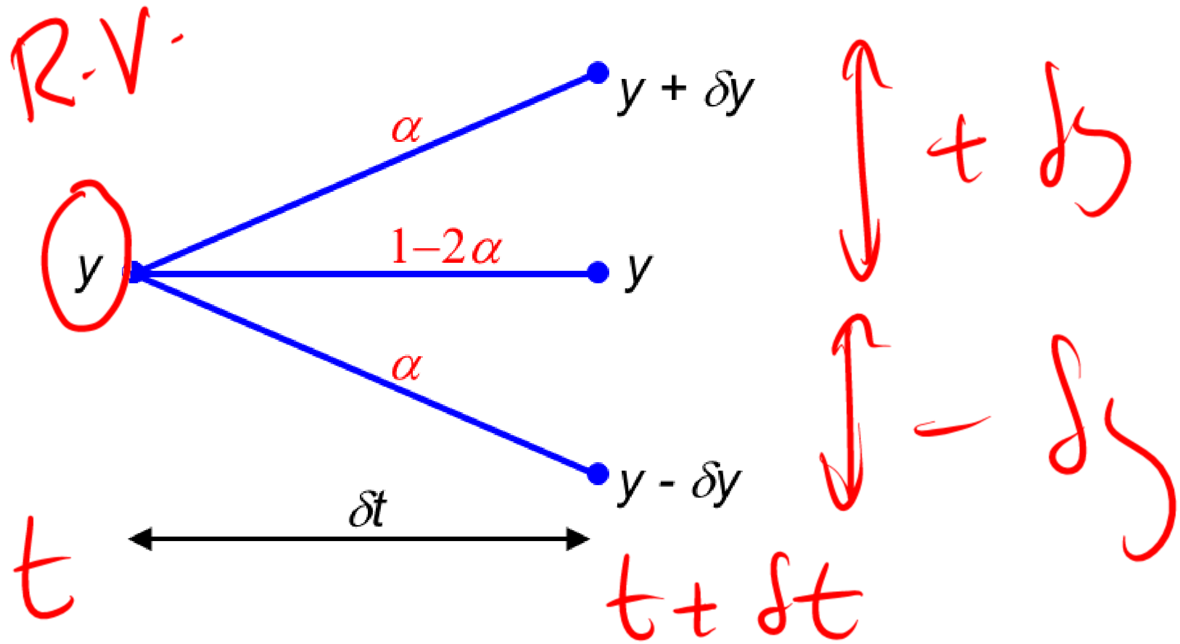
Probability Distributions $\frac{dS}{S} = \mu dt + \sigma dW$

At the heart of modern finance theory lies the uncertain movement of financial quantities. For modelling purposes we are concerned with the evolution of random events through time.

A *diffusion process* is one that is continuous in space, while a *random walk* is a process that is discrete. The random path followed by the process is called a realization. Hence when referring to the path traced out by a financial variable will be termed as an asset price realization.

The mathematics can be achieved by the concept of a transition density function and is the connection between probability theory and differential equations.

Trinomial Random Walk



A trinomial random walk models the dynamics of a random variable, with value y at time t. α is a probability. δy is the size of the move in y.

$S(t)$ S_t

The Transition Probability Density Function

The transition pdf is denoted by

$$p\left(y, t; y', t'\right)$$

dX_t

We can gain information such as the centre of the distribution, where the random variable might be in the long run, etc. by studying its probabilistic properties. So the density of particles diffusing from (y, t) to (y', t') .

Think of (y, t) as current (or backward) variables and (y', t') as future ones.

The more basic assistance it gives is with

$$\mathbb{P}\left(a < y' < b \text{ at } t' \middle|\ y \text{ at } t\right) = \int_a^b p\left(y, t; y', t'\right) dy'$$

i.e. the probability that the random variable y' lies in the interval a and b,at a future time t', given it started out at time t with value y.

$p(y,t;y',t')$ satisfies two equations:

Forward equation involving derivatives with respect to the future state (y',t') . Here (y,t) is a starting point and is 'fixed'.

Backward equation involving derivatives with respect to the current state (y,t) . Here (y',t') is a future point and is 'fixed'. The backward equation tells us the probability that we were at (y,t) given that we are now at (y',t') , which is fixed.

The mathematics: Start out at a point (y,t) . We want to answer the question, what is the probability density function of the position y' of the diffusion at a later time t'?

This is known as the **transition density function** written $p(y,t;y',t')$ and represents the density of particles diffusing from (y,t) to (y',t') . How can we find p?

Forward Equation

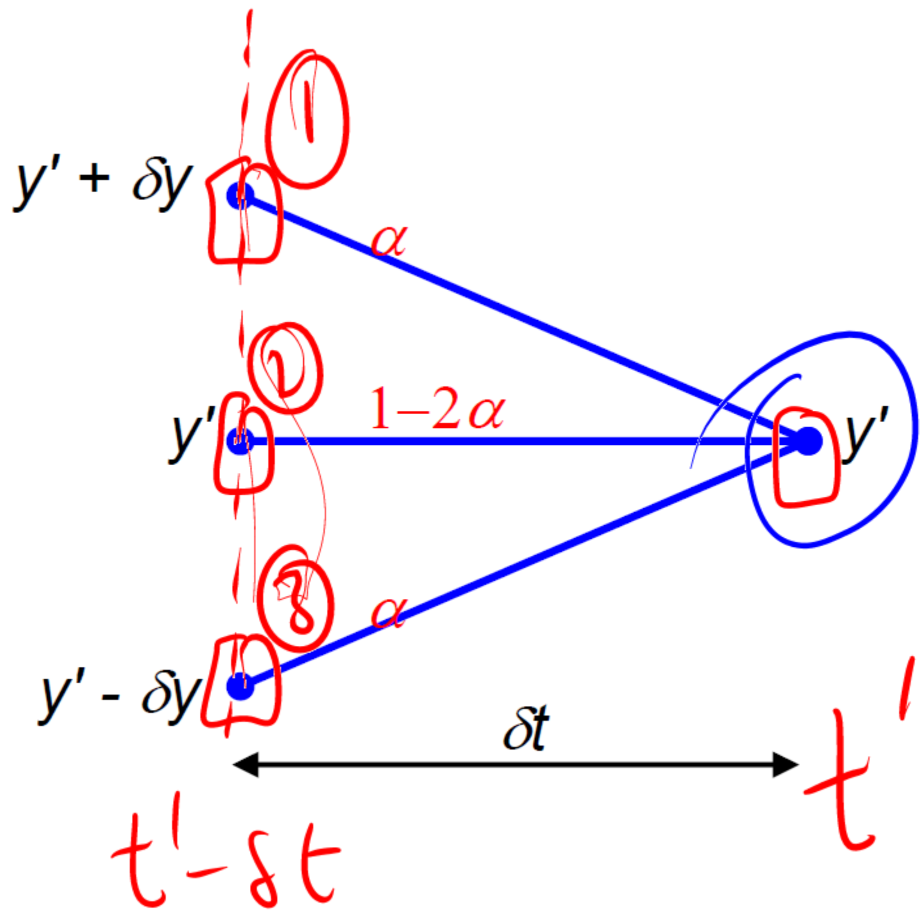
Starting with a trinomial random walk which is discrete we can obtain a continuous time process to obtain a partial differential equation for the transition probability density function (i.e. a time dependent PDF).

So the random variable can either rise or fall with equal probability $\alpha < \frac{1}{2}$ and remain at the same location with probability $1 - 2\alpha$.

Suppose we are at (y', t') , how did we get there?

(y, t)

(y,t)



At the previous step time step we must have been at one of $(y'+\delta y, t'-\delta t)$ or $(y'-\delta y, t'-\delta t)$ or $(y', t'-\delta t)$.

So

$$p\left(y,t,y',t'\right) = \underbrace{\alpha p\left(y,t,y'+\delta y,t'-\delta t\right)}_{1} + \underbrace{(1-2\alpha)\, p\left(y,t,y',t'-\delta t\right)}_{2} + \underbrace{\alpha p\left(y,t,y'-\delta y,t'-\delta t\right)}_{3}$$

Taylor series expansion gives (omit the dependence on (y,t) in your working as they will not change)

$$
\begin{aligned}
p\left(y'+\delta y, t'-\delta t\right) &= p\left(y',t'\right) - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \ldots \qquad (1)\\
p\left(y', t'-\delta t\right) &= p\left(y',t'\right) - \frac{\partial p}{\partial t'}\delta t + \ldots \qquad (2)\\
p\left(y'-\delta y, t'-\delta t\right) &= p\left(y',t'\right) - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \ldots \qquad (3)
\end{aligned}
$$

Substituting into the above

$$
\begin{aligned}
p\left(y',t'\right) &= \alpha\left(p\left(y',t'\right) - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right) + \\
&\quad (1-2\alpha)\left(p\left(y',t'\right) - \frac{\partial p}{\partial t'}\delta t +\right) \\
&\quad +\alpha\left(p\left(y',t'\right) - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right)
\end{aligned}
$$

3×10^{-4}

$$0 = -\frac{\partial p}{\partial t'}\delta t + \alpha \frac{\partial^2 p}{\partial y'^2}\delta y^2$$

c^2

$$\frac{\partial p}{\partial t'} = \alpha \frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y'^2}$$

O – order of magnitude

Now take limits. This only makes sense if $\alpha \frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2 \sim O(\delta t)$ and letting $\delta y,\ \delta t \longrightarrow 0$ gives the equation

$$\boxed{\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2},}$$

where $c^2 = \alpha \frac{\delta y^2}{\delta t}$. This is called the **forward Kolmogorov equation.** Also called Fokker Planck equation.

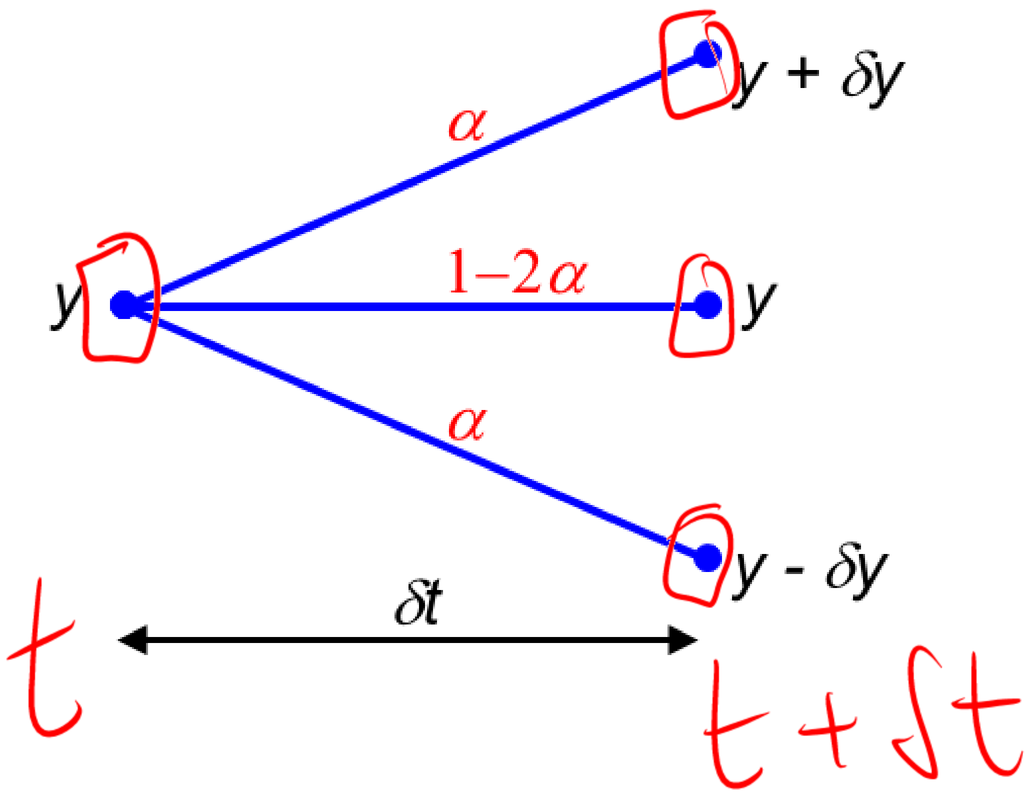
It shows how the probability density of future states evolves, starting from (y, t).

Backward Equation

The **backward equation** tells us the probability that we are at (y', t') given that we were at an initial state (y, t) .

(y', t')

(y', t') are now fixed and (y, t) are variables.



So the probability of being at (y', t') given we are at y at t in the past is linked to the probabilities of being at $(y + \delta y, t + \delta t, y', t')$, $(y, t + \delta t, y', t')$ and $(y - \delta y, t + \delta t, y', t')$.

The backward equation is particularly important in the context of finance, but also a source of much confusion. Illustrate with the 'real life' example that Wilmott uses. Wilmott uses a *Trinomial* Random Walk.

So a concrete example!

- At 7pm you are at the office - this is the point (y, t)

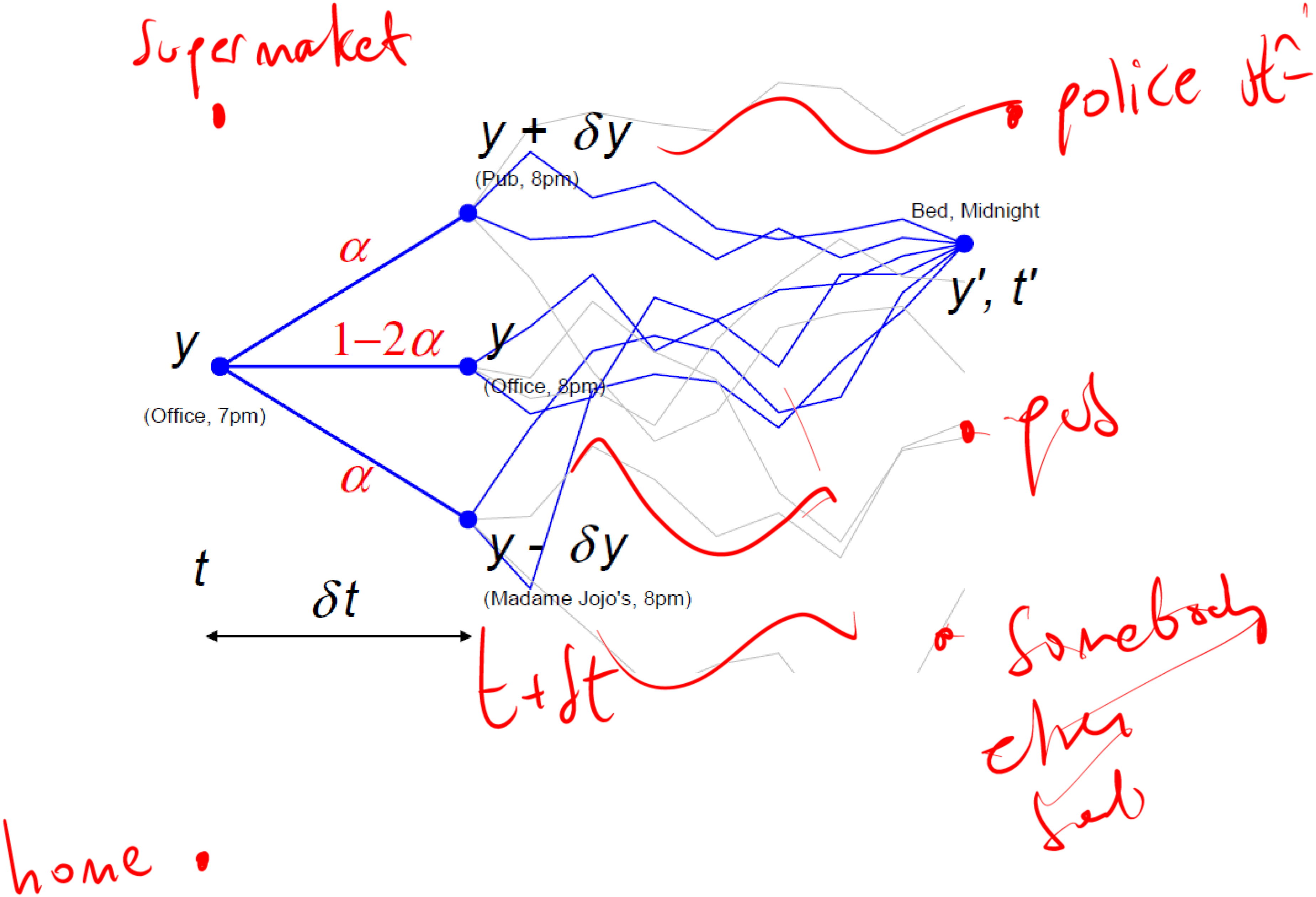
- At 8pm you will be at one of three places:

  § The Pub - the point $(y + \delta y, t + \delta t)$;

  § Still at the office - the point $(y, t + \delta t)$;

  § Madame Jojo's - the point $(y - \delta y, t + \delta t)$

We are interested in the probability of being tucked up in bed at midnight (y', t') , given that we were at the office at 7pm.



Supermarket
police st
$y + \delta y$
(Pub, 8pm)
Bed, Midnight
α
y', t'
y
$1-2\alpha$
y
(Office, 8pm)
(Office, 7pm)
pub
α
$y - \delta y$
t
δt
(Madame Jojo's, 8pm)
$t+\delta t$
somebody else's bed
home

Remember $p(y, t, y', t')$ represents the probability of being at the future point (y', t'), i.e. bed at midnight, given that you started at (y, t), the office at 7pm.

Looking at the earlier figure, we can only get to bed at midnight via either

- the pub

- the office

- Madame Jojo's

at 8pm.

What happens after 8pm doesn't matter - we don't care, you may not even remember! We are only concerned with being in bed at midnight.

The earlier figure shows many different paths, only the ones ending up in 'our' bed are of interest to us.

In words: The probability of going from the office at 7pm to bed at midnight is

- the probability of going to the pub from the office and then to bed at midnight plus

- the probability of staying in the office and then going to bed at midnight plus

- the probability of going to Madame Jojo's from the office and then to bed at midnight

$$
\begin{aligned}
p\left(y,t;y',t'\right) \;&=\; \alpha p\left(y+\delta y, t+\delta t; y', t'\right) + (1-2\alpha)\, p\left(y, t+\delta t; y', t'\right) + \\
&\quad\; \alpha p\left(y-\delta y, t+\delta t; y', t'\right).
\end{aligned}
$$

Now since (y', t') do not change, drop these for the time being and use a TSE on the right hand side

$$
\begin{aligned}
&p\,(y,t) = \\
&\qquad \alpha\left(p\,(y,t) + \frac{\partial p}{\partial t}\delta t + \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + \ldots\right) + \\
&\qquad (1-2\alpha)\left(p\,(y,t) + \frac{\partial p}{\partial t}\delta t + \ldots\right) \\
&\qquad \alpha\left(p\,(y,t) + \frac{\partial p}{\partial t}\delta t - \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + \ldots\right)
\end{aligned}
$$

$\delta y^2 \sim \delta t$

which simplifies to

$$
0 = \frac{\partial p}{\partial t} + \tfrac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y^2}.
$$

Putting $\frac{\delta y^2}{\delta t} = O(1)$ and taking limit gives the **backward equation**

$$-\frac{\partial p}{\partial t} = \tfrac{1}{2}c^2\frac{\partial^2 p}{\partial y^2}.$$

or commonly written as

$$\frac{\partial p}{\partial t} + \frac{1}{2}c^2\frac{\partial^2 p}{\partial y^2} = 0.$$

The above can be expressed mathematically as

$$
\begin{aligned}
p\left(y,t;y',t'\right) \;&=\; \alpha p\left(y+\delta y,t+\delta t;y',t'\right)+(1-2\alpha)\,p\left(y,t+\delta t;y',t'\right)+\\
&\quad\;\alpha p\left(y-\delta y,t+\delta t;y',t'\right).
\end{aligned}
$$

Performing a Taylor expansion gives

$$0=\delta t\frac{\partial p}{\partial t}+\alpha\delta y^2\frac{\partial^2 p}{\partial y^2}+\ldots$$

which becomes

$$0=\frac{\partial p}{\partial t}+\alpha\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y^2}+\ldots$$

and letting $\alpha\frac{\delta y^2}{\delta t}=c^2$ where c is non-zero and finite as $\delta t,\delta y\longrightarrow 0$, we have

$$\frac{\partial p}{\partial t}+c^2\frac{\partial^2 p}{\partial y^2}=0$$

Diffusion Equation

y, t

The model equation is

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2} \qquad (\ast)$$

for the unknown function $p = p\,(y', t')$. The idea is to obtain a solution in terms of Gaussian curves. Let's drop the primed notation.

We assume a solution of the following form exists:

$$p\,(y, t) = t^{\alpha} f\left(\frac{y}{t^{\beta}}\right)$$

new var.

where α, β are constants to be determined. So put

$$\boxed{\xi = \frac{y}{t^{\beta}}}$$

$$p = t^{\alpha} f(\xi)$$

which allows us to obtain the following derivatives

$$\boxed{\frac{\partial \xi}{\partial y} = \frac{1}{t^{\beta}}; \quad \frac{\partial \xi}{\partial t} = -\beta y t^{-\beta-1}}$$

Chain rule
product rule

we can now say

$$p(y,t) = t^{\alpha} f(\xi)$$

therefore

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial \xi}\frac{\partial \xi}{\partial y} = t^{\alpha} f'(\xi) \cdot \frac{1}{t^{\beta}} = t^{\alpha-\beta} f'(\xi)$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial y}\left(t^{\alpha-\beta} f'(\xi)\right) \\
&= \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}\left(t^{\alpha-\beta} f'(\xi)\right) \\
&= t^{\alpha-\beta}\frac{1}{t^{\beta}}\frac{\partial}{\partial \xi} f'(\xi) = t^{\alpha-2\beta} f''(\xi)
\end{aligned}$$

A

$$\frac{\partial p}{\partial t} = t^{\alpha}\frac{\partial}{\partial t} f(\xi) + \alpha t^{\alpha-1} f(\xi)$$

we can use the chain rule to write

$$\frac{\partial}{\partial t} f(\xi) = \frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} = -\beta y t^{-\beta-1} f'(\xi)$$

so we have

$$\frac{\partial p}{\partial t} = \alpha t^{\alpha-1} f(\xi) - \beta y t^{\alpha-\beta-1} f'(\xi)$$

B

and then substituting these expressions in to the pde gives

$$\alpha t^{\alpha-1} f(\xi) - \beta y t^{\alpha-\beta-1} f'(\xi) = c^2 t^{\alpha-2\beta} f''.$$

subst A, B into *

We know from ξ that

$$y = t^{\beta}\xi$$

$\xi = \frac{y}{t^{\beta}}$

hence the equation above becomes

$$\alpha t^{\alpha-1} f(\xi) - \beta \xi t^{\alpha-1} f'(\xi) = c^2 t^{\alpha-2\beta} f''.$$

For the similarity solution to exist we require the equation to be independent of t, i.e. $\alpha - 1 = \alpha - 2\beta \implies \beta = 1/2$, therefore

$$\boxed{\alpha f - \frac{1}{2}\xi f' = c^2 f''}$$

thus we have so far

$$p = t^{\alpha} f\left(\frac{y}{\sqrt{t}}\right)$$

which gives us a whole family of solutions dependent upon the choice of α.

We know that p represents a pdf, hence

for all time $$\boxed{\int_{\mathbb{R}} p\,(y,t)\,dy = 1} = \int_{\mathbb{R}} t^{\alpha} f\left(\frac{y}{\sqrt{t}}\right) dy$$

change of variables $u = y/\sqrt{t} \longrightarrow du = dy/\sqrt{t}$ so the integral becomes

$$\boxed{t^{\alpha+1/2} \boxed{\int_{-\infty}^{\infty} f\,(u)\,du = 1}}$$

which we need to normalize independent of time t. This is only possible if $\alpha = -1/2$.

$$\frac{d}{d\xi}(\xi f)$$

So the D.E becomes

$$-\frac{1}{2}\left(f+\xi f'\right)=c^2 f''.$$

We have an exact derivative on the lhs, i.e. $\dfrac{d}{d\xi}(\xi f)=f+\xi f'$, hence

$$-\frac{1}{2}\frac{d}{d\xi}(\xi f)=c^2 f''$$

and we can integrate once to get

$$-\frac{1}{2}(\xi f)=c^2 f'+K.$$

$\xi\to\infty$

$f(\xi)\to 0$

$f'(\xi)\to 0$

We set $K=0$ in order to get the correct solution, i.e.

$$-\frac{1}{2}(\xi f)=c^2 f'$$

which can be solved as a simple first order variable separable equation:

$$f(\xi)=A\exp\left(-\frac{1}{4c^2}\xi^2\right)$$

$y/4tc^2$ $\quad = -\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}$

A is a normalizing constant, so write

$$\boxed{A\int_{\mathbb{R}} \exp\left(-\frac{1}{4c^2}\xi^2\right) d\xi = 1.}$$

$-\frac{1}{2}\frac{(y-0)^2}{2tc^2}$

Now substitute $x = \xi/2c$, so $2c\,dx = d\xi$

$$2cA\underbrace{\int_{\mathbb{R}} \exp\left(-x^2\right) dx}_{=\sqrt{\pi}} = 1,$$

which gives $A = 1/2c\sqrt{\pi}$. Returning to

$A = \frac{1}{2c\sqrt{\pi}}$

$$p\left(y,t\right) = t^{-1/2} f\left(\xi\right)$$

becomes

$\xi = y/\sqrt{t}$

$$\boxed{p\left(y',t'\right) = \frac{1}{2c\sqrt{\pi t'}} \exp\left(-\frac{y'^2}{4t'c^2}\right).}$$

This is a pdf for a variable y that is normally distributed with mean zero and standard deviation $c\sqrt{2t}$, which we ascertained by the following comparison:

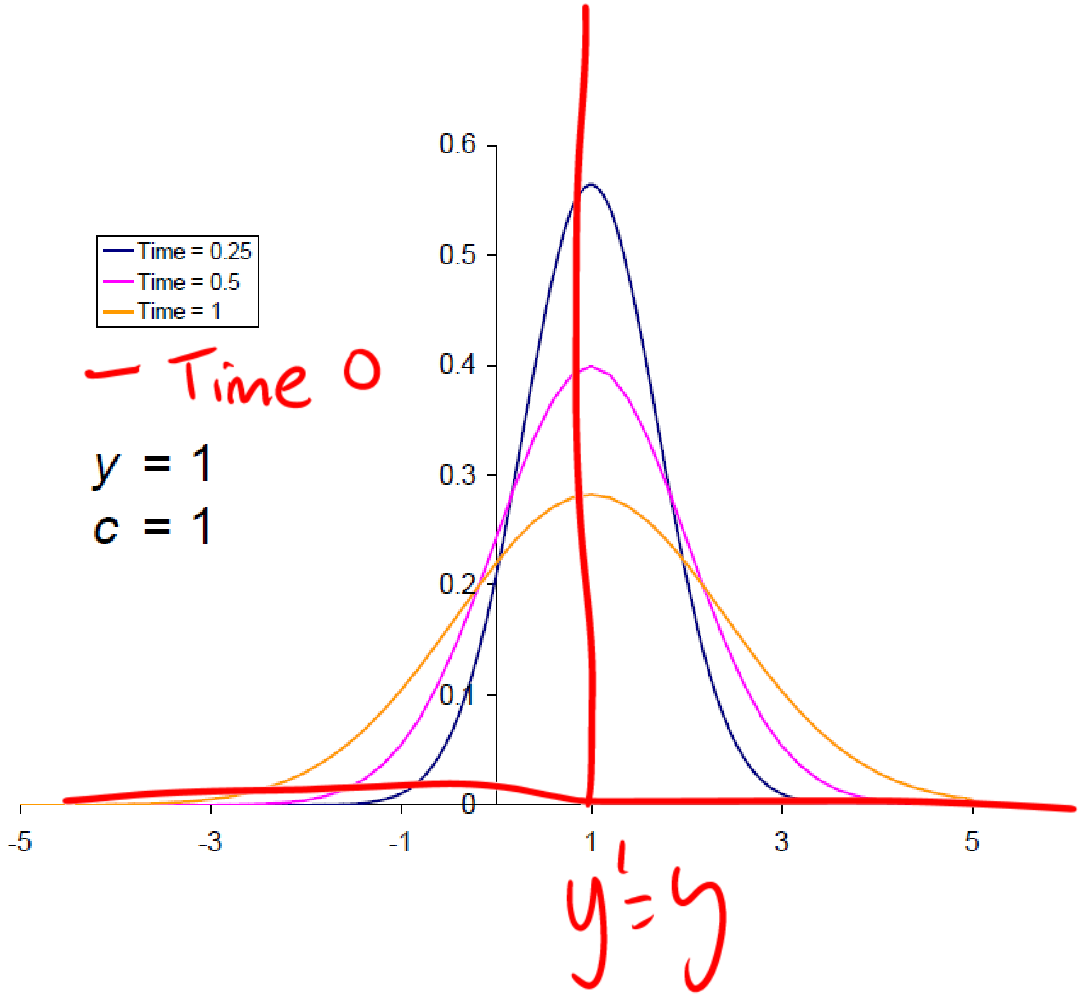
$$-\frac{1}{2}\frac{{y'}^2}{2t'c^2} : -\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}$$

i.e. $\mu \equiv 0$ and $\sigma^2 \equiv 2t'c^2$.

If the random variable y' has value y at time t then we can generalize to

$$p\left(y, t; y', t'\right) = \frac{1}{2c\sqrt{\pi\,(t'-t)}} \exp\left(-\frac{\left(y'-y\right)^2}{4c^2\,(t'-t)}\right)$$

(y, t)



At $t' = t$ this is now a Dirac delta function $\delta\left(y' - y\right)$. This particle is known to start from (y, t) and diffuses out to (y', t') with mean y and standard deviation $c\sqrt{2\left(t' - t\right)}$.