CQF Module 1.3 Exercises

Throughout this problem sheet, you may assume that W_t is a Brownian Motion (Weiner Process) and dW_t is its increment; and $W_0 = 0$.

1. Use Itô's lemma to obtain a SDE for each of the following functions

   (a) $y(W_t) = \exp(W_t)$

   (b) $g(W_t) = \ln W_t$

   (c) $h(W_t) = \sin W_t + \cos W_t$

   (d) $f(W_t) = a^{W_t}$, where the constant $a > 1$

   (e) $f(W_t) = (W_t)^n$

2. Using the formula below for stochastic integrals, for a function $F(W_t, t)$,

$$\int_0^t \frac{\partial F}{\partial W_t} dW_t = F(W_t, t) - F(W_0, 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W_\tau^2} \right) d\tau$$

   show that we can write

   a. $\displaystyle\int_0^t W_t^3 \, dW_\tau = \tfrac{1}{4} W^4(t) - \tfrac{3}{2} \int_0^t W_\tau^2 d\tau$

   b. $\displaystyle\int_0^t \tau dW_\tau = tW_t - \int_0^t W_\tau d\tau$

   c. $\displaystyle\int_0^t (W_\tau + \tau) \, dW_\tau = \tfrac{1}{2} W_t^2 + tW_t - \int_0^t \left(W_\tau + \tfrac{1}{2} \right) d\tau$