Fixed Income Products and Analysis

In this lecture. . .

- names and features of the basic and most important fixed-income products
- swaps
- the relationship between swaps and zero-coupon bonds
- an overview of fixed-income modeling

risk

- simple ways to analyze the market value of the instruments: yield, duration and convexity

"bootstrapping"

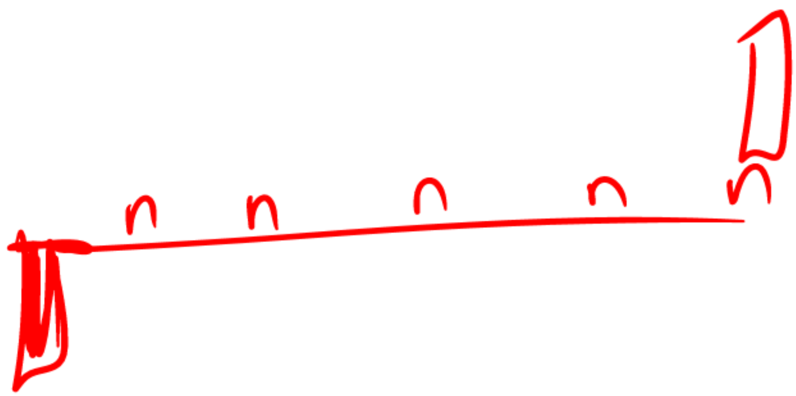
- how to construct yield curves and forward rates

Intro — Black Book: Ch 14, 15.

(3 vol set) → Ch 13, 14 Vol 1

By the end of this lecture you will

- be able to decompose a swap into a portfolio of bonds
- understand the main approaches to interest-rate modeling
- be able to construct the forward curve from simple bonds
- understand the concepts of yield, duration and convexity



Introduction

This lecture is an introduction to some basic instruments and concepts in the world of fixed income, that is, the world of cash-flows that are in the simplest cases independent of any stocks, commodities etc.

We will see the most elementary of fixed-income instruments, the coupon-bearing bond, and how to determine various properties of such bonds to help in their analysis.

Simple fixed-income contracts and features

Principal

The zero-coupon bond

par /face /nominal Price T

- The **zero-coupon bond** is a contract paying a known fixed amount, the **principal**, at some given date in the future, the **maturity** date T.

This promise of future wealth is worth something now: it cannot have zero or negative value.

Furthermore, the amount you pay initially will be smaller than the amount you receive at maturity.

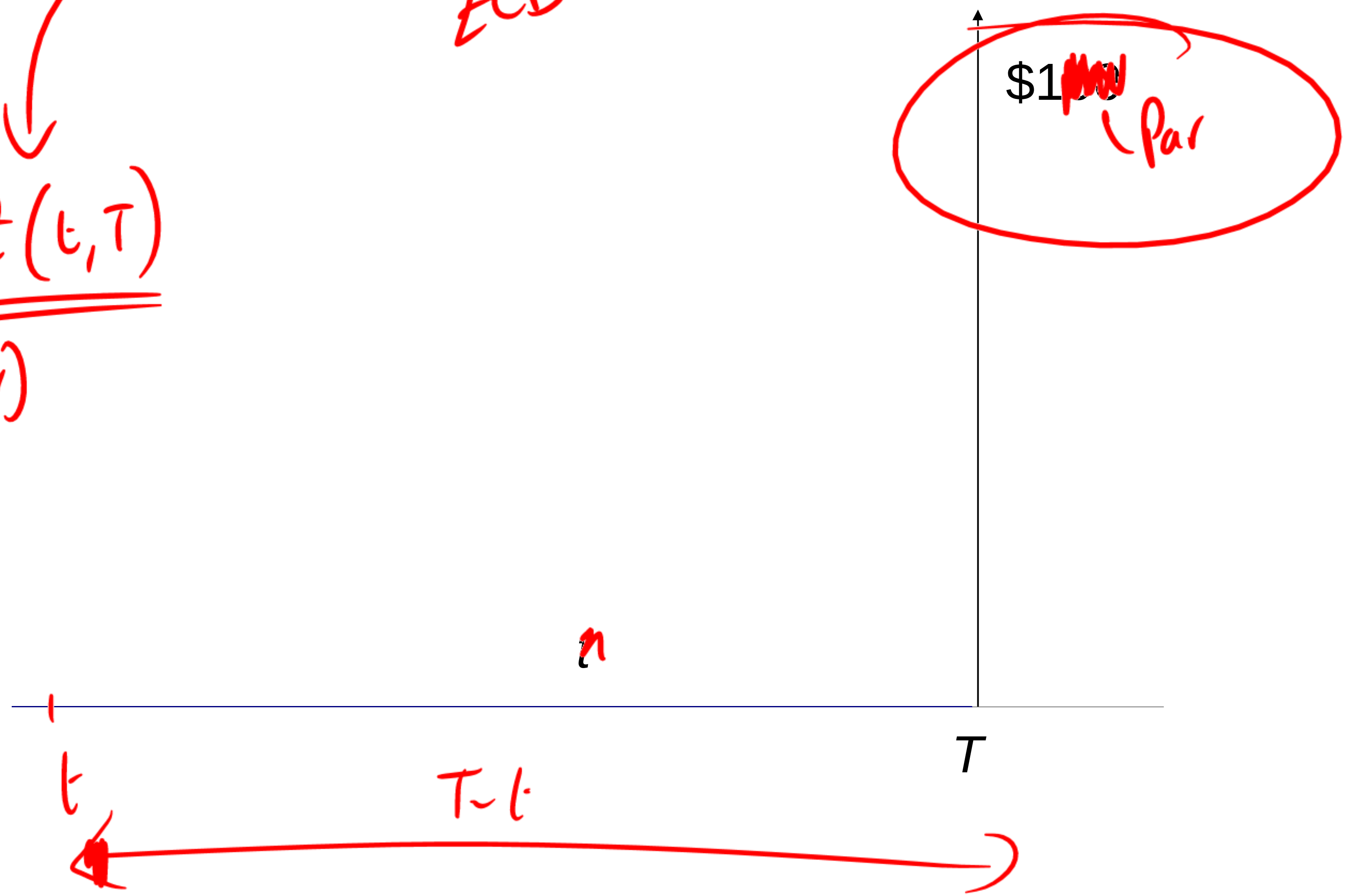
Discount

"Discount Factor"

ZCB

Price $= Z(t,T)$

(of \$1 par)



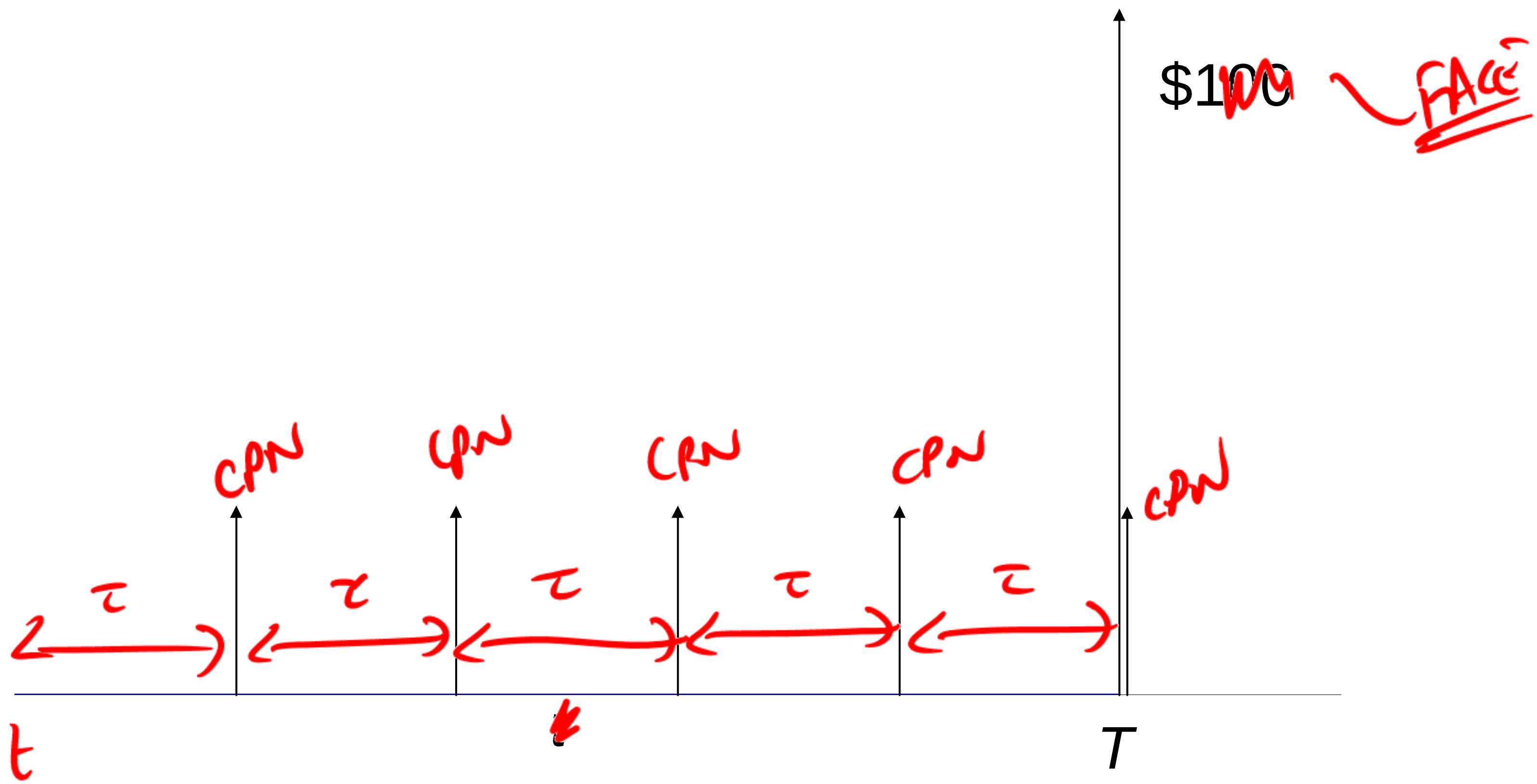
The coupon-bearing bond

- A **coupon-bearing bond** is similar to the above except that as well as paying the principal at maturity, it pays smaller quantities, the coupons, at intervals up to and including the maturity date.

% par

These coupons are usually specified fractions of the principal. For example, the bond pays \$1 in 10 years and 2%, i.e. 2 cents, every six months.

This bond is clearly more valuable than the bond in the previous example because of the coupon payments.



$$\text{Price} = \sum_i Z(t, T_i) \qquad T_i = t + i\tau$$

We can think of the coupon-bearing bond as a portfolio of zero-coupon bearing bonds:

- one zero-coupon bearing bond for each coupon date with a principal being the same as the original bond's coupon, and then a final zero-coupon bond with the same maturity as the original.

A bank account

- Simply an account that accumulates interest compounded at a rate that varies from time to time.

The rate at which interest accumulates is usually a short-term and unpredictable rate.

In the sense that money held in a bank account will grow at an unpredictable rate, such an account is risky when compared with a one-year zero-coupon bond.

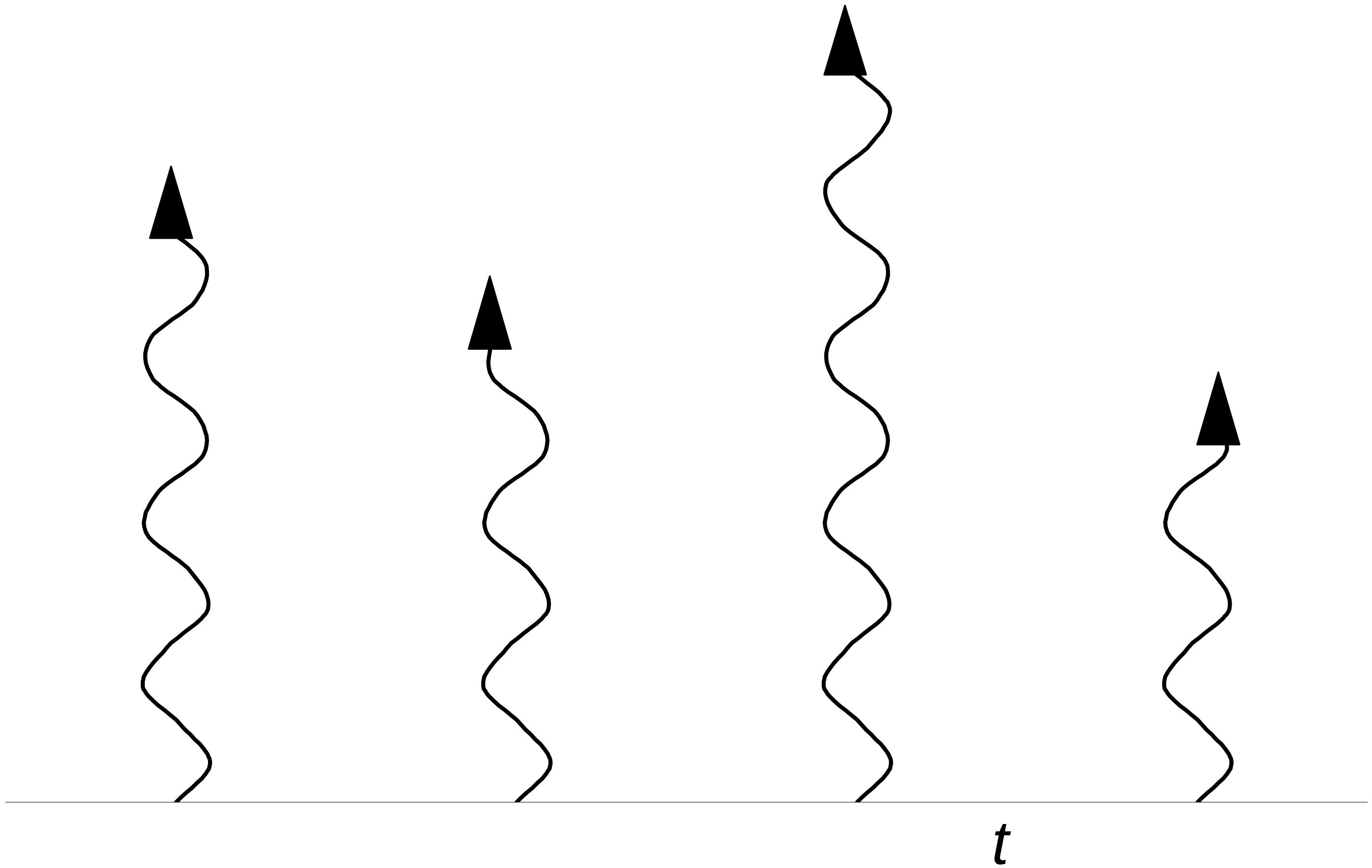
On the other hand, the bank account can be closed at any time but if the bond is sold before maturity there is no guarantee how much it will be worth at the time of the sale.

"Floating rate bonds"

FRN

In its simplest form a **floating interest rate** is the amount that you get on your bank account. This amount varies from time to time, reflecting the state of the economy and in response to pressure from other banks for your business. This uncertainty about the interest rate you receive is compensated by the flexibility of your deposit, it can be withdrawn at any time.

- The most common measure of interest is **London Interbank Offer Rate** or **LIBOR**. LIBOR comes in various maturities, one month, three month, six month etc., and is the rate of interest offered between Eurocurrency banks for fixed-term deposits.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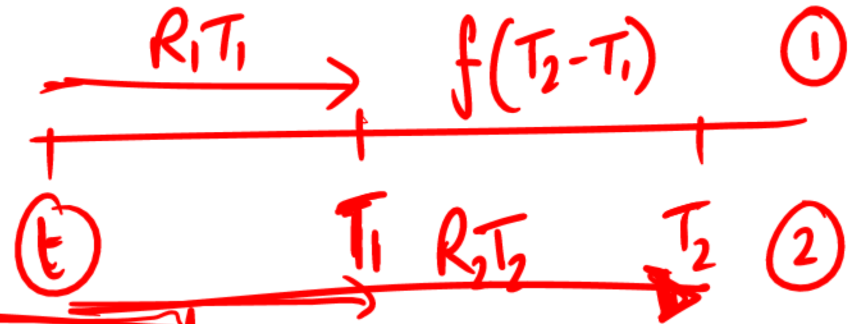


Forward rate agreements

- A **Forward Rate Agreement (FRA)** is an agreement between two parties that a prescribed interest rate will apply to a prescribed principal over some specified period in the future.

The cashflows in this agreement are as follows: party A pays party B the principal at time T_1 and B pays A the principal plus agreed interest at time $T_2 > T_1$.

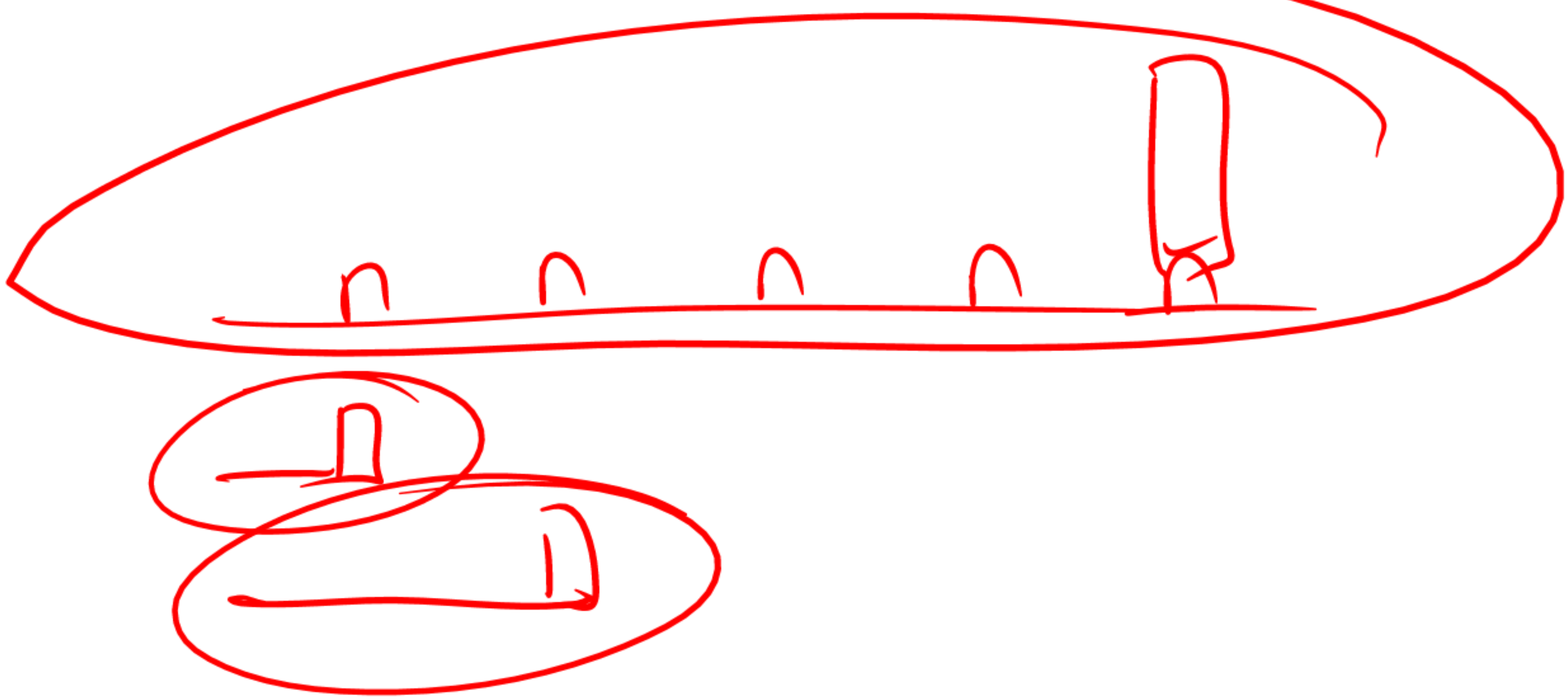
derivative: forward on short term rates

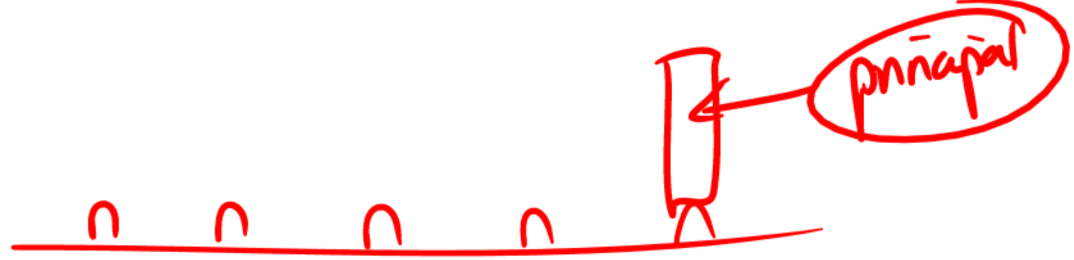
OTC

$$f = \frac{R_2T_2 - R_1T_1}{(T_2 - T_1)}$$

STRIPS

STRIPS stands for ‘Separate Trading of Registered Interest and Principal of Securities’. The coupons and principal of normal bonds are split up, creating artificial zero-coupon bonds of longer maturity than would otherwise be available.





Amortization

In all of the above products we have assumed that the principal remains fixed at its initial level.

- Sometimes this is not the case, the principal can **amortize** or decrease during the life of the contract. The principal is thus paid back gradually and interest is paid on the amount of the principal outstanding.

Such amortization is arranged at the initiation of the contract and may be fixed, so that the rate of decrease of the principal is known beforehand, or can depend on the level of some index, if the index is high the principal amortizes faster for example.

Call provision

Some bonds have a **call provision**. The issuer can call back the bond on certain dates or at certain periods for a prescribed, possibly time-dependent, amount. This lowers the value of the bond.

Callable Bond = option free bond − call.

putable bond = option free bond + put

Swaps

OTC

- A **swap** is an agreement between two parties to exchange, or swap, future cashflows.

Stream of

The size of these cashflows is determined by some formulæ, decided upon at the initiation of the contract. The swaps may be in a single currency or involve the exchange of cashflows in different currencies.

The total notional principal amount is, in US dollars, currently comfortably in *14* figures.

BIS

The vanilla interest rate swap (IRS)

doesn't change hands

In the **interest rate swap** the two parties exchange cashflows that are represented by the interest on a notional principal. Typically,

known at outset 4%

(price!)

- one side agrees to pay the other a fixed interest rate

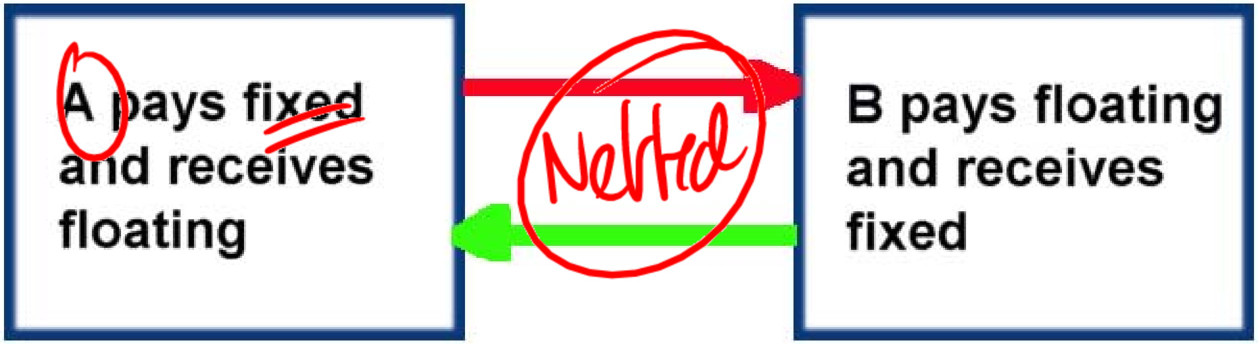
netted

- the cashflow in the opposite direction is a **floating rate**.

One of the commonest floating rates used in a swap agreement is LIBOR, London Interbank Offer Rate.

Fixed cpn

Bond



The parties to a simple interest rate swap.

\$ LIBOR - convention $\times \frac{act}{360}$ to deannualise-

Commonly in a swap, the exchange of the fixed and floating interest payments occur every six months.

In this case the relevant LIBOR rate would be the six-month rate. At the maturity of the contract the principal is *not* exchanged.

Why are swaps so popular?

1. Comparative advantage

2. Hedging

3. Speculation

Swaps: Comparative advantage

Swaps were first created to exploit **comparative advantage**. This is when two companies who want to borrow money are quoted fixed and floating rates such that by exchanging payments between themselves they benefit, at the same time benefitting the intermediary who puts the deal together.

Here's an example.

Two companies A and B want to borrow \$10MM, to be paid back in two years. They each have a choice of a fixed- or floating-rate loan. They are quoted the interest rates for borrowing at fixed and floating rates shown here.

	Fixed	Floating
A	7%	~~six-month LIBOR~~ + 30bps
B	8.2%	six-month LIBOR + 100bps

Note that both must pay a premium over LIBOR to cover risk of default, which is perceived to be greater for company B.

Ideally, company A wants to borrow at floating and B at fixed. This will possibly be because of the cashflows in their business.

If they each borrow directly then they pay the following in total:

$$\text{six-month LIBOR} + 30\text{bps} + 8.2\% = \text{six-month LIBOR} + 8.5\%.$$

A	six-month LIBOR + 30bps (floating)
B	8.2% (fixed)

However, if A borrowed at fixed and B at floating they'd only be paying

$$\text{six-month LIBOR} + 100\text{bps} + 7\% = \text{six-month LIBOR} + 8\%.$$

A	7% (fixed)
B	six-month LIBOR + 100bps (floating)

That's a saving between them of 0.5%.

Let's suppose that A borrows fixed and B floating, even though that's not what they want. Their total interest payments are six-month LIBOR plus 8%.

Now let's see what happens if we throw a swap into the pot.

A is currently paying 7% and B six-month LIBOR plus 1%.

They enter into a swap in which A pays LIBOR to B and B pays 6.95% to A.

They have swapped interest payments.

Looked at from A's perspective they are paying 7% and LIBOR while receiving 6.95%, a net floating payment of LIBOR plus 5bps. Not only is this floating, as A originally wanted, but it is 25bps better than if they had borrowed directly at the floating rate.

There's still another 25bps missing, and, of course, B gets this. B pays LIBOR plus 100bps and also 6.95% to A while receiving LIBOR from A. This nets out at 7.95%, which is fixed, as required, and 25bps less than the original deal.

Swaps: Hedging

Swaps can be used to balance cashflows.

Income might be at a fixed rate with outgoings at a floating rate.

A simple example would be the cashflows associated with a rented out property: Income would be fixed, the rent; Outgoings might vary, the mortgage.

Swaps: Speculation

Because the floating legs of the swap vary with the level of interest rates, swaps can be used to speculate on the future direction of these rates.

The par swap starts life with zero value but gives immediate exposure to interest rates. Once interest rates move the swap will have a non-zero value. This may be positive or negative depending on the direction in which the floating legs move.

The swap can then be closed out resulting in a profit or loss.

Let's see an example of how a swap works.

Example

Suppose that we enter into a five-year swap on 8th July 2008, with semi-annual interest payments.

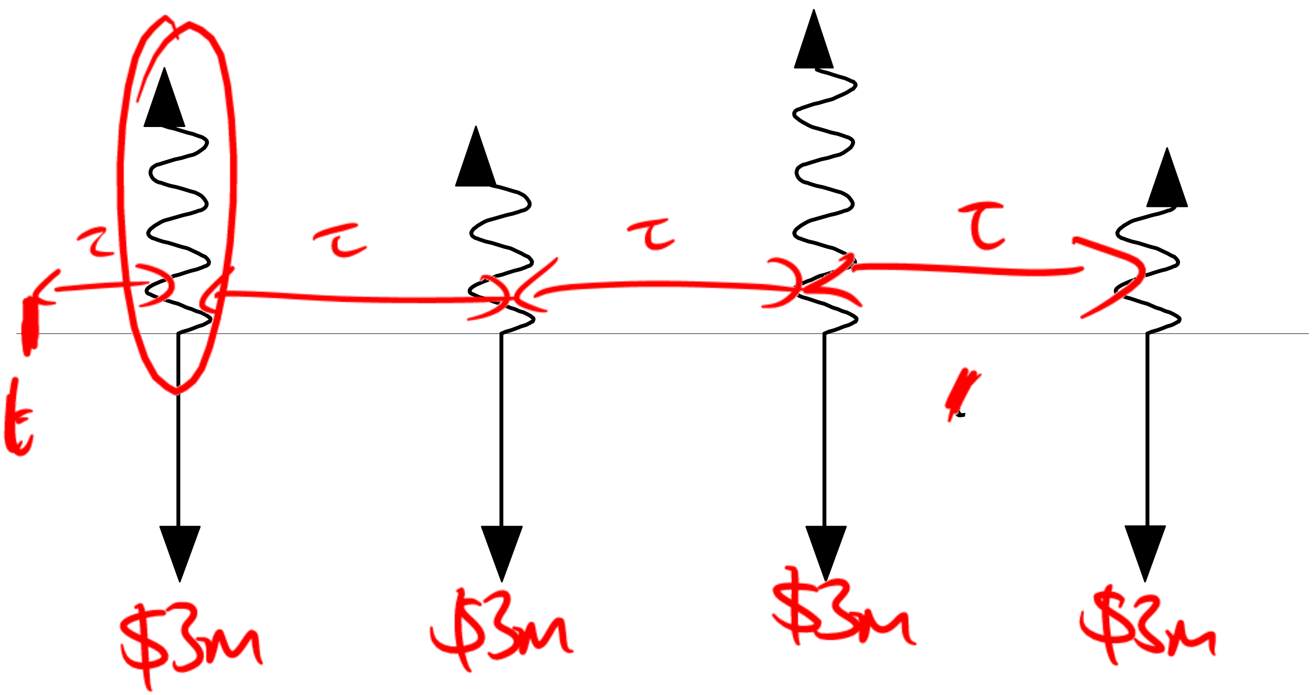
We will pay to the other party a rate of interest fixed at 6% on a notional principal of $100 million, the counterparty will pay us six-month LIBOR.

The cashflows in this contract are shown below.

$\cancel{dr = db + dx}$

The straight lines denote a fixed rate of interest and thus a known amount, the curly lines are floating rate payments.

$\boxed{\tau = 0.5}$



The first exchange of payments is made on 8th January 2009, six months after the deal is signed. How much money changes hands on that first date?

We must pay $0.03 \times \$100{,}000{,}000 = \$3{,}000{,}000$.

The cashflow in the opposite direction will be at six-month LIBOR, *as quoted six months previously* i.e. at the initiation of the contract.

This is a very important point

- The LIBOR rate is set six months before it is paid, so that in the first exchange of payments the floating side is known. This makes the first exchange special.

The second exchange takes place on 8th July 2009. Again we must pay \$3,000,000, but now we receive LIBOR, as quoted on 8th January 2009.

- Every six months there is an exchange of such payments, with the fixed leg always being known and the floating leg being known six months before it is paid.

This continues until the last date, 8th July 2012.

Relationship between swaps and bonds

There are two sides to a swap, the fixed-rate side and the floating-rate side.

- The fixed interest payments, since they are all known in terms of actual dollar amount, can be seen as the sum of zero-coupon bonds.

(eg 6%

If the fixed rate of interest is r_s then the fixed payments add up to

$$r_s \tau \sum_{i=1}^{N} Z(t;T_i),$$

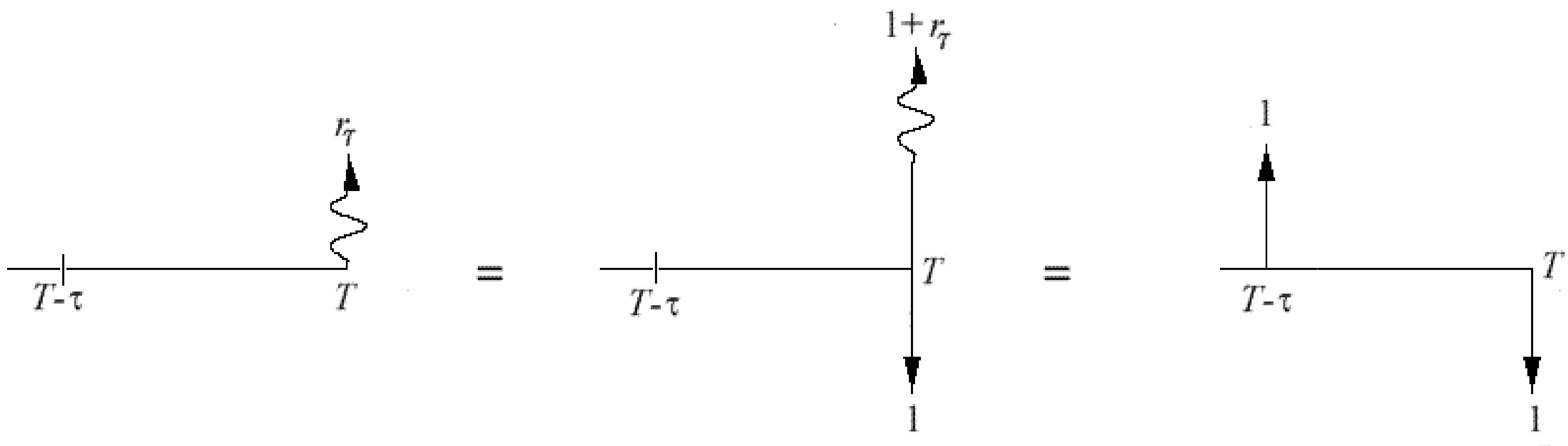
$T_i = t + i\tau$.

0.5,

where τ is the time interval between payments measured in years.

(This assumes a principal of 1.)

To see the simple relationship between the floating leg and zero-coupon bonds we draw some schematic diagrams and compare the cashflows.



At time T_i there is payment of $r_\tau \tau$ of the notional principal, where r_τ is the period τ rate of LIBOR, set at time $T_i - \tau$.

Add and subtract \$1 at time T_i to get the second diagram. The first and the second diagrams obviously have the same present value.

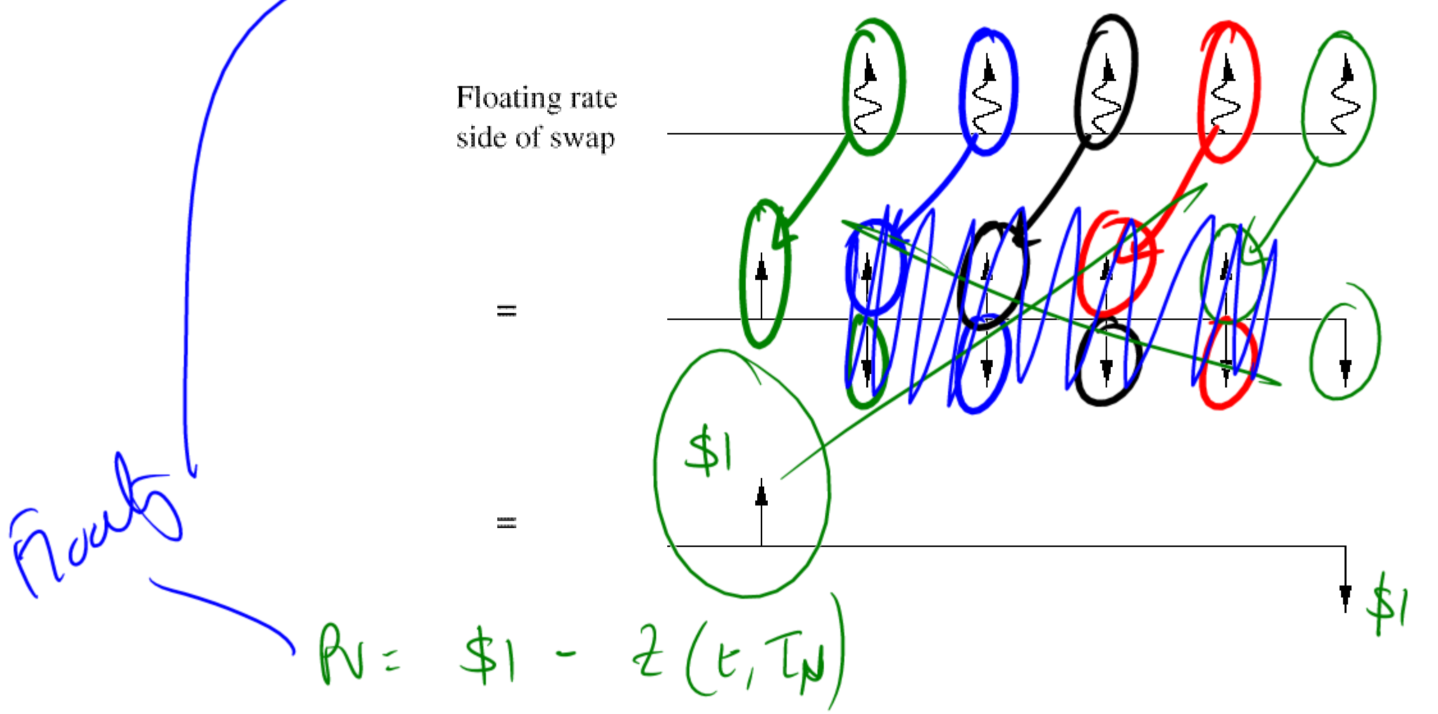
Now recall the precise definition of LIBOR. It is the interest rate paid on a fixed-term deposit. Thus the $\$1 + r_\tau \tau$ at time T_i is the same as \$1 at time $T_i - \tau$. This gives the third diagram.

- It follows that the single floating rate payment is equivalent to two zero-coupon bonds.

A single floating leg of a swap at time T_i is *exactly* equal to a deposit of \$1 at time $T_i - \tau$ and a withdrawal of \$1 at time τ.

Add up all the floating legs, note the cancelation of all cashflows except for the first and last. The floating side has value

$$1 - Z(t; T_N).$$



The floating legs are equivalent to two zero-coupon bonds.

Bring the fixed and floating sides together to find that the value of the swap, to the receiver of the fixed side, is

fixed

$$-1 + Z(t;T_N) + r_s\tau \sum_{i=1}^{N} Z(t;T_i).$$

- This result is *model independent*. This relationship is independent of any mathematical model for bonds or swaps.

don't need

$dr = \underline{\quad\quad} dt + \underline{\quad\quad} dX$

The swap curve

- When the swap is first entered into it is usual for the deal to have no value to either party (i.e. zero value). This is done by a careful choice of the fixed rate of interest r_s.

  Such a swap is called a **par swap**.

Thus

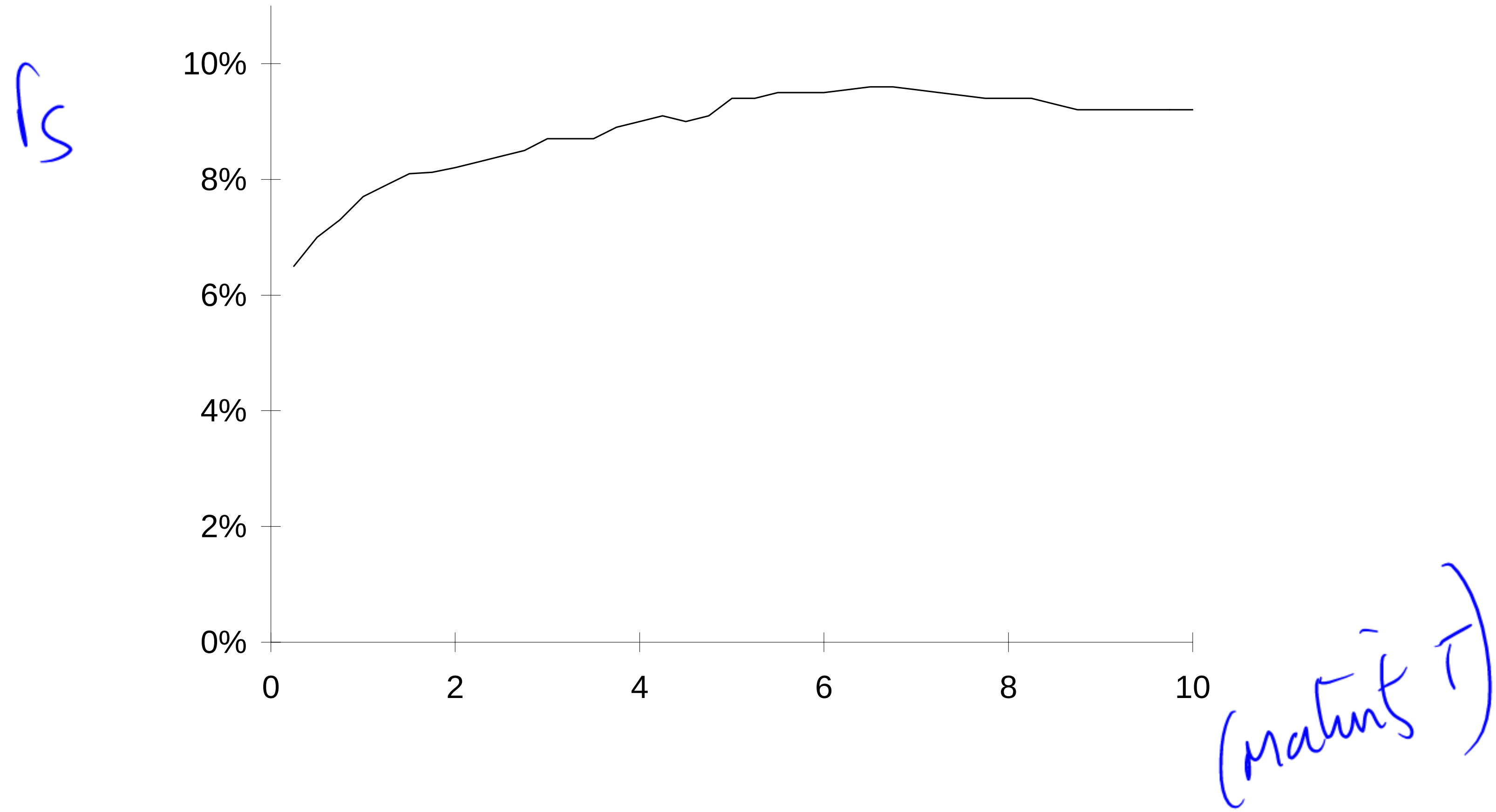
$$r_s = \frac{1 - Z(t; T_N)}{\tau \sum_{i=1}^{N} Z(t; T_i)}. \tag{1}$$

This is the quoted swap rate.

In other words, the present value, let us say, of the fixed side and the floating side both have the same value, netting out to zero.

r_S

The rates of interest in the fixed leg of a swap are quoted at various maturities. These rates make up the **swap curve**.



Swaps are now so liquid and exist for an enormous range of maturities that their prices determine the yield curve and not *vice versa*.

In practice one is given $r_s(T_i)$ for many maturities T_i and one uses (1) to calculate the prices of zero-coupon bonds and thus the yield curve.

Other features of swaps contracts

The above is a description of the vanilla interest rate swap. There any many features that can be added to the contract that make it more complicated, and most importantly, model dependent.

Callable and puttable swaps

A **callable** or **puttable swap** allows one side or the other to close out the swap at some time before its natural maturity. If you are receiving fixed and the floating rate rises more than you had expected you would want to close the position.

Mathematically we are in the early exercise world of American-style options.

$dr!$

Extendible swaps

The holder of an **extendible swap** can extend the maturity of a vanilla swap at the original swap rate.

Index amortizing rate swaps

The principal in the vanilla swap is constant. In some swaps the principal declines with time according to a prescribed schedule. The index amortizing rate swap is more complicated still with the amortization depending on the level of some index, say LIBOR, at the time of the exchange of payments.

Currency swaps

A **currency swap** is an exchange of interest payments in one currency for payments in another currency. The interest rates can both be fixed, both floating or one of each. As well as the exchange of interest payments there is also an exchange of the principals (in two different currencies) at the beginning of the contract and at the end.

The key point about swaps is that they can be priced in terms of bonds (and vice versa).

Most financial instruments require a model for interest rates, however.

dr

And if the instrument has some convexity in rates, then we need a model for the randomness in rates.

There now follows an overview of interest-rate modeling...

The different approaches to interest-rate modeling

1. Determinististic — Tonight. $r(t)$ not random "bootstrapping"
2. Black '76
3. Stochastic spot rate — Vasicek — $dr = \_\_\,dt + \_\_\,dX$ — mean reverting!
4. Multi factor — $dr = \_\_\,dt + \_\_\,dX_1 + \_\_\,dX_2$
5. Heath, Jarrow & Morton (HJM)
6. LIBOR Market Model (LMM)

Deterministic rates

The history of interest-rate modeling begins with deterministic rates, and the ideas of yield to maturity, duration etc. We will see this later in this lecture.

Briefly, one assumes that there is a quantity called the **spot interest rate**, this being the interest paid over a very short period of time. This quantity could be constant in the simplest case (and we often assume this when pricing equity options), or a time-dependent function

- $r(t)$

How do we know what is the function $r(t)$?

This is the subject of **bootstrapping**.

The assumption of determinism is not at all satisfactory for pricing derivatives.

Black ’76

In 1976 Fischer Black introduced the idea of treating bonds as underlying assets so as to use the Black–Scholes equity option formulas for fixed-income instruments.

- All you need to know is the volatility of the ‘underlying’ bond.

This is not entirely satisfactory since there can be contradictions in this approach. On one hand bond prices are random, yet on the other hand interest rates used for discounting from expiration to the present are assumed to be deterministic.

However, the main advantage of this approach is that for simple fixed-income contracts there are simple formulas for their prices.

For more complicated, less liquid instruments an internally consistent stochastic rates approach was needed.

Stochastic spot rate

The first step on the stochastic interest rate path used the very short-term interest rate, the spot rate, as the random factor driving the entire yield curve.

The mathematics of these spot-rate models is identical to that for equity models, and the fixed-income derivatives satisfied similar equations as equity derivatives.

- Diffusion equations governed the prices of derivatives, and derivatives prices could be interpreted as the risk-neutral expected value of the present value of all cashflows as well.

And so the solution methods of finite-difference methods for solving partial differential equations, trees and Monte Carlo simulation carried over.

Models of this type are **Vasicek**, **Hull & White**.

The advantages of these models are

1. they are internally consistent

2. they are easy to solve numerically by many different methods

But there are several aspects to the downside:

1. the spot rate does not exist, it has to be approximated in some way

2. with only one source of randomness the yield curve is very constrained in how it can evolve, essentially parallel shifts

3. the yield curve that is output by the model will not match the market yield curve. To some extent the market thinks of each maturity as being semi independent from the others, so a model should match all maturities otherwise there will be arbitrage opportunities

Multi factor

Models were then designed to get around the second and third of these problems.

- A second random factor was introduced, sometimes representing the long-term interest rate (**Brennan & Schwartz**), and sometimes the volatility of the spot rate (**Fong & Vasicek**). This allowed for a richer structure for yield curves.

- And an arbitrary time-dependent parameter (or sometimes two or three such) was allowed in place of what had hitherto been constant(s). The time dependence allowed for the yield curve (and other desired quantities) to be instantaneously matched. Thus was born the idea of calibration, the **Ho & Lee** model.

Heath, Jarrow & Morton

The business of calibration in such models was rarely straightforward.

The next step in the development of models was by Heath, Jarrow & Morton (HJM) who modeled the evolution of the *entire* yield curve directly so that calibration simply became a matter of specifying an initial curve.

The model is easy to implement via simulation.

Because of the non-Markov nature of the general HJM model it is not possible to solve these via finite-difference solution of partial differential equations, the governing partial differential equation would generally be in an infinite number of variables, representing the infinite memory of the general HJM model.

Since the model is usually solved by simulation it is straightforward having any number of random factors and so a very, very rich structure for the behaviour of the yield curve.

The only downside with this model, as far as implementation is concerned, is that it assumes a continuous distribution of maturities and the existence of a spot rate.

LIBOR Market Model

The LIBOR Market Model (LMM) as proposed by Miltersen, Sandmann, Sondermann, Brace, Gatarek, Musiela and Jamshidian in various combinations and at various times, models *traded* forward rates of different maturities as correlated random walks.

- The key advantage over HJM is that only prices which exist in the market are modelled, the LIBOR rates.

Each traded forward rate is represented by a stochastic differential equation model with a drift rate and a volatility, as well as a correlation with each of the other forward rate models.

For the purposes of pricing derivatives we work as usual in a risk-neutral world. In this world the drifts cannot be specified independently of the volatilities and correlations.

Again, the LMM is solved by simulation, with the yield curve 'today' being the initial data. Calibration to the yield curve is therefore automatic.

The LMM can also be made to be consistent with the standard approach for pricing caps, floors and swaptions using Black 1976. Thus calibration to volatility- and correlation-dependent liquid instruments can also be achieved.

Whether the LMM is a good model in terms of scientific accuracy is another matter, but for its ease of use and calibration and its relationship with standard models make it very appealing to practitioners.

And now... deterministic interest rate modeling...

(The other, more interesting, models will be seen in other lectures.)

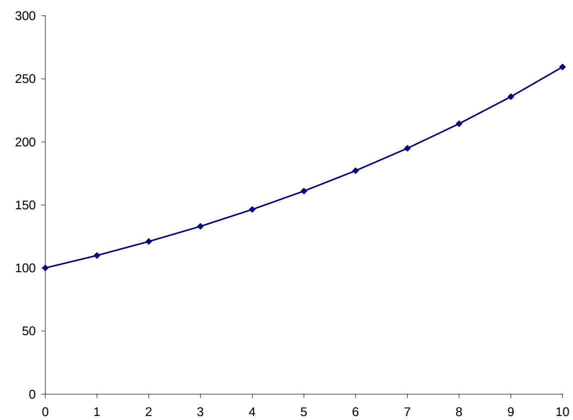
Discretely and continuously compounded interest

Put \$100 in the bank. Suppose that the one-year interest rate is 10%.

After one year you have \$100 $\times(1+0.1) = \$110$.

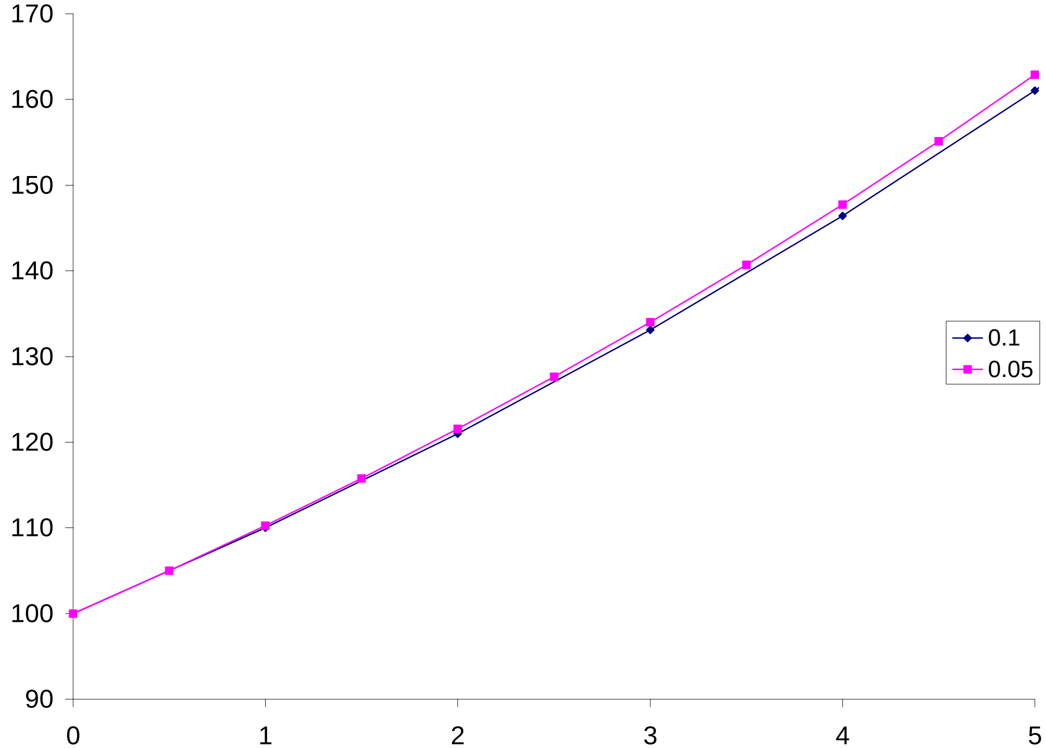
After two years, \$100 $\times(1+0.1)\times(1+0.1) = \$121$.

After ten years, \$100 $\times(1+0.1)^{10} = \$259.4$.



Would you rather have 10% paid once a year or 5% twice a year?

In the latter case after one year you have \$100 $\times 1.05^2 =$ \$110.25.



Continuously paid interest

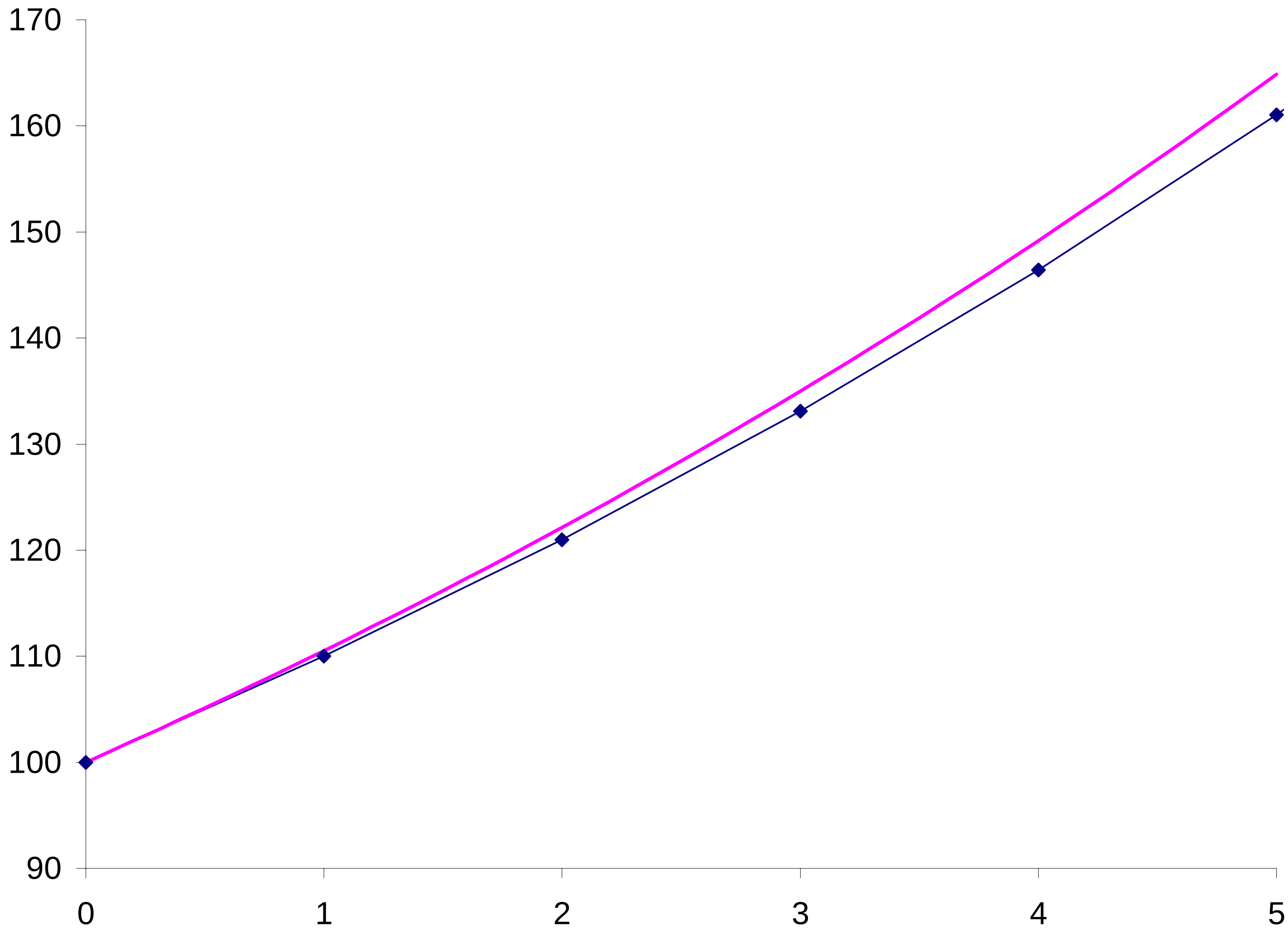
We've seen what happens when interest is paid once or twice a year. But what happens if interest is paid every week, day, hour, millisecond? Is there a 'continuous-time limit'?

Interest rate r paid m times a year for t years, multiply initial value by

$$\left(1+\frac{r}{m}\right)^{mt}.$$

As $m \to \infty$

$$\left(1+\frac{r}{m}\right)^{mt} = e^{mt \log\left(1+\frac{r}{m}\right)} \approx e^{mt\frac{r}{m}} = e^{rt}.$$



170
160
150
140
130
120
110
100
90
0
1
2
3
4
5

This expression also follows from the cash-in-the-bank or money market account equation

$$dM = rM dt.$$

This is the convention used in the options world.

Note on Conventions

In fixed income it is more common to talk about discretely paid interest.

In the world of options it is more common to use continuously paid interest.

It is possible to translate from one convention to another...

$$r_{\text{Continuous}} = m \ \log\left(1 + \frac{r_{\text{Discrete}}}{m}\right).$$

What are we trying to find?

Given an interest rate you can calculate the present value of a cashflow.

- In practice we are more likely to be given the value of a bond and asked to back out an interest rate.

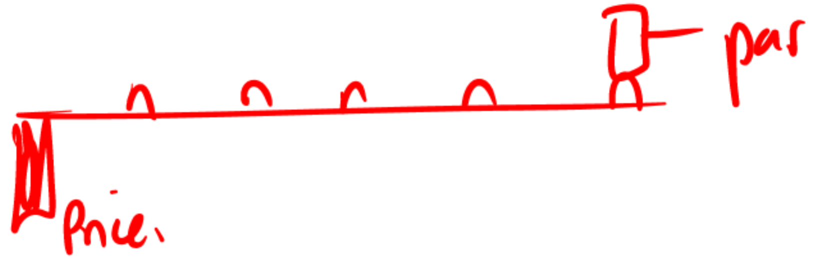
This is straightforward if there is just a single cashflow, but more complicated if there is more than one.

Measures of yield

rehin

There is such a variety of fixed-income products, with different coupon structures, amortization, fixed and/or floating rates, that it is necessary to be able to consistently compare different products.

One way to do this is through measures of how much each contract earns, there are several measures of this all coming under the name **yield**.



Current yield

The simplest measurement of how much a contract earns is the **current yield**. This measure is defined by

Income only

- $$\text{current yield} = \frac{\text{annual \$ coupon income}}{\text{bond price}}.$$

For example, consider a 10-year bond that pays 2 cents every six months and $1 at maturity. This bond has a total income per annum of 4 cents. Suppose that the quoted market price of this bond is 88 cents.

The current yield is simply

$$\frac{0.04}{0.88} = 4.5\%.$$

The current yield does not take into account the principal repayment at the maturity of the bond.

The next measure of interest is far more important and meaningful.

Yield to maturity (YTM) or internal rate of return (IRR)

Suppose that we have a zero-coupon bond maturing at time T when it pays one dollar.

- At time t it has a value $Z(t;T)$.

Compare the following two investments:

1. Buy one zero-coupon bond that matures at time T. So you spend $Z(t;T)$ now to get \$1 at T.

2. Put an amount (cash) of $Z(t;T)$ in a bank account earning a fixed, continously compounded, interest rate of y. So you spend $Z(t;T)$ now to get $Z(t;T)e^{y(T-t)}$ at T.

Question: What y makes the two amounts at time T the same?

$$\boxed{Z(t;T)e^{y(T-t)} = 1.}$$

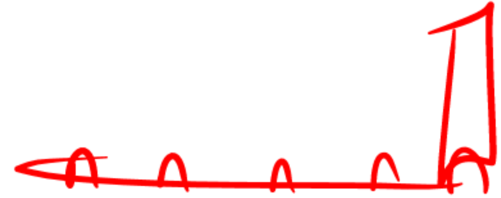
$$Z(t;T) = e^{-y(T-t)}.$$

It follows that

$$y = -\frac{\log Z}{T-t}.$$

This is the simple relationship between the value of a bond with a single cashflow and a continuously compounded rate of interest.

Let us generalize this.



Suppose that we have a coupon-bearing bond.

- Discount all coupons and the principal to the present by using some interest rate y. *And we use the same rate of interest for present valuing each cashflow.*

If P is the principal, N the number of coupons, C_i the coupon paid on date t_i then the present value of the bond, at time t, is

$$V = Pe^{-y(T-t)} + \sum_{i=1}^{N} C_i e^{-y(t_i-t)}. \tag{2}$$

Bond price.

Principal.

$t_i = t + \tau_i$

If the bond is a traded security then we know the price at which the bond can be bought.

If this is the case then we can calculate the **yield to maturity** or **internal rate of return** as the value y that we must put into Equation (2) to make V equal to the traded price of the bond.

This calculation must be performed by some trial and error/iterative procedure.

Example:

A five-year bond with principal of \$1 has twice-yearly payments of 3 cents.

It has a market value of 96 cents.

What is its yield to maturity? (y)

We ask

- ‘What is the rate of return we must use to give these cash flows a total present value of 96 cents?’

This value is the yield to maturity.

We must solve

$$0.96 = 1 \times e^{-5y} + \sum_{i=1}^{10} 0.03 \times e^{-0.5\, i\, y}$$

for y.

Time	Coupon	Principal repayment	PV (discounting at 6.8406%)
0			0
0.5	.03		0.0290
1.0	.03		0.0280
1.5	.03		0.0270
2.0	.03		0.0262
2.5	.03		0.0253
3.0	.03		0.0244
3.5	.03		0.0236
4.0	.03		0.0228
4.5	.03		0.0220
5.0	.03	1.00	0.7316
		Total	0.9600

An example of a coupon-bearing bond.

In the fourth column in this table is the present value (PV) of each of the cashflows using a rate of 6.8406%: since the sum of these present values is 96 cents the YTM or IRR is 6.8406%.

This yield to maturity is a valid measure of the return on a bond if we intend to hold it to maturity.

	A	B	C	D	E	F	G	H	I	J	K
1					Date	Coupon	Principal	PVs	Time	Time^2	
2					0				wtd	wtd	
3		YTM	4.95%		0.5	2%		0.0195	0.0098	0.0049	
4		Mkt price	0.921		1	2%		0.0190	0.0190	0.0190	
5		Th. Price	0.921		1.5	2%		0.0186	0.0279	0.0418	
6		Error	1.4E-08		2	2%		0.0181	0.0362	0.0725	
7		Duration	8.2544		2.5	2%		0.0177	0.0442	0.1104	
8		Convexity	76.8728		3	2%		0.0172	0.0517	0.1551	
9					3.5	2%		0.0168	0.0589	0.2060	
10					4	2%		0.0164	0.0656	0.2625	
11					4.5	2%		0.0160	0.0720	0.3241	
12					5	2%		0.0156	0.0781	0.3903	
13					5.5	2%		0.0152	0.0838	0.4607	
14					6	2%		0.0149	0.0892	0.5349	
15					6.5	2%		0.0145	0.0942	0.6124	
16					7	2%		0.0141	0.0990	0.6929	
17					7.5	2%		0.0138	0.1035	0.7760	
18					8	2%		0.0135	0.1077	0.8613	
19					8.5	2%		0.0131	0.1116	0.9485	
20					9	2%		0.0128	0.1153	1.0374	
21					9.5	2%		0.0125	0.1187	1.1276	
22					10	2%	1	0.6216	6.2161	62.1614	
23											
24											
25											
26											
27											
28											
29											
30											
31											
32											
33											
34											
35											
36											
37											

= SUM(H3:H22)

= C4-C5

= SUM(I3:I22)/C5

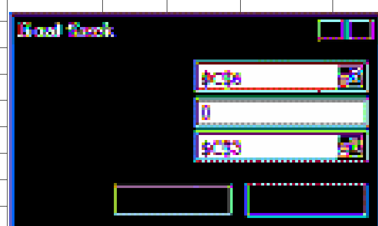
= SUM(J3:J22)/C5

= F20*EXP(-E20*C3)

= E20*H20

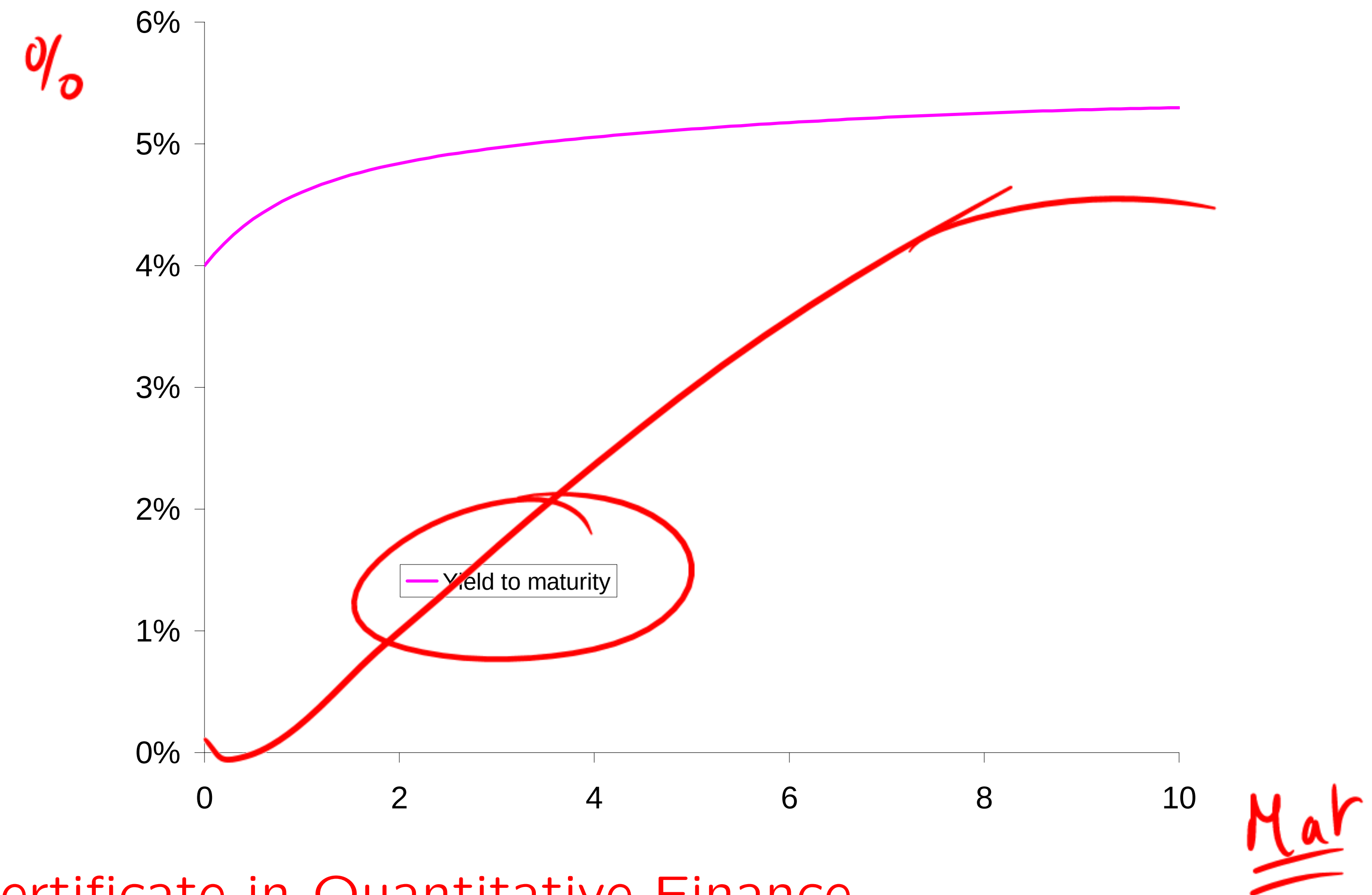
= (G22+F22)*EXP(-E22*C3)

= E20*I20



Every bond has an associated yield to maturity.

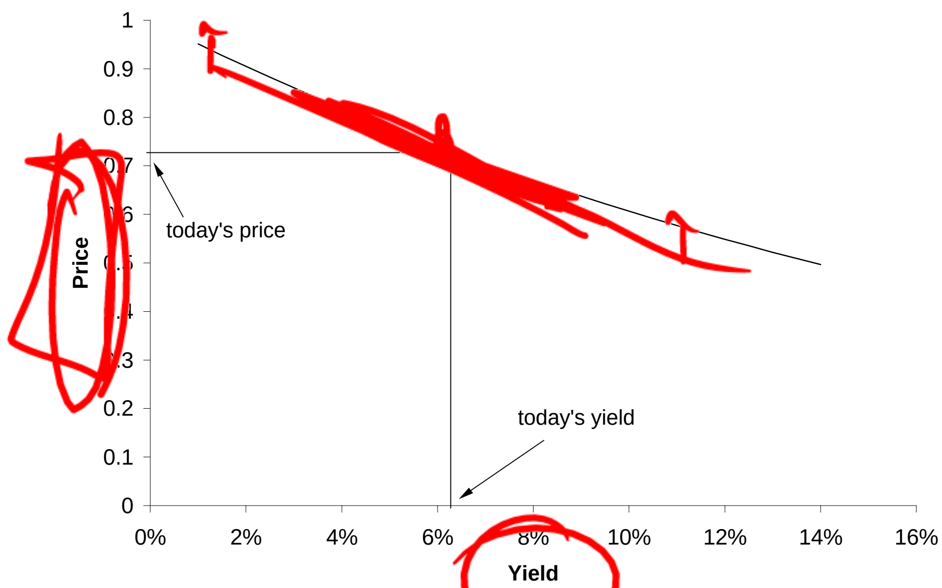
We can therefore plot yield versus maturity, the so called **yield curve**.



Price/yield relationship

~~nsk~~

From Equation (2) we can easily see that the relationship between the price of a bond and its yield is of the form shown below (assuming that all cash flows are positive).



Δy

Since we are often interested in the sensitivity of instruments to the movement of certain underlying factors it is natural to ask how does the price of a bond vary with the yield, or vice versa.

To a first approximation this variation can be quantified by a measure called the **duration**.

Duration

units of $\frac{dV}{dy} = \frac{\$}{1/\text{time}}$

From Equation (2) we find that

$\$\,\text{time}$

$$\frac{dV}{dy} = -(T-t)Pe^{-y(T-t)} - \sum_{i=1}^{N} C_i(t_i - t)e^{-y(t_i-t)}.$$

This is the slope of the Price/Yield curve.

The quantity

units = time

- $$-\frac{1}{V}\frac{dV}{dy}$$

is called the **Macaulay duration**.

In the expression for the duration the time of each coupon payment is weighted by its present value.

The higher the value of the present value of the coupon the more it contributes to the duration.

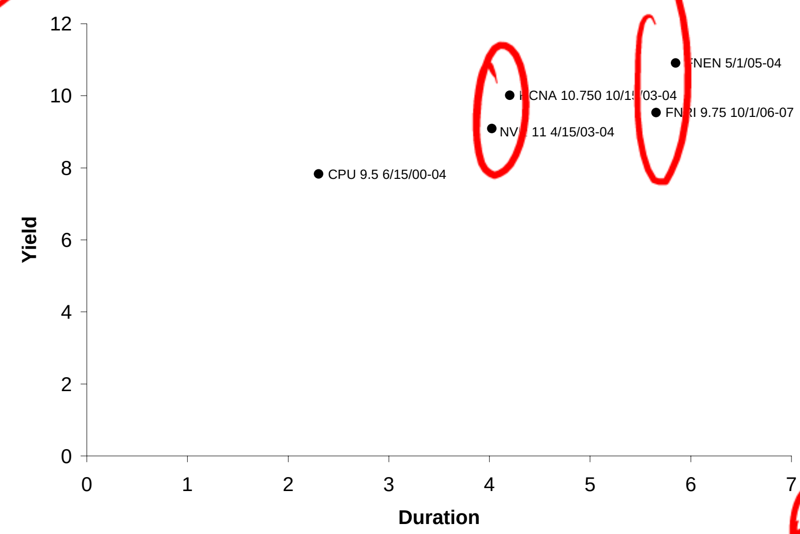
Also, since y is measured in units of inverse time, the units of the duration are time.

- The duration is a measure of the **average life of the bond**.

One of the most common uses of the duration is in plots of yield *versus* duration for a variety of instruments.

- We can group together instruments with the same or similar durations and make comparisons between their yields.

Retn



Risk

- Two bonds having the same duration but with one bond having a higher yield might be suggestive of value for money in the higher-yielding bond, or of credit risk issues.

However, such indicators of relative value must be used with care.

It is possible for two bonds to have vastly different cashflow profiles yet have the same duration; one may have a maturity of 30 years but an average life and hence a duration of seven years, whereas another may be a seven-year zero-coupon bond.

Clearly, the former has twenty-three years' more risk than the latter.

Taylor series

For small movements in the yield, the duration gives a good measure of the change in value with a change in the yield.

For larger movements we need to look at higher order terms in the Taylor series expansion of $V(y)$.

Convexity

The Taylor series expansion of V gives

$$dV = \frac{dV}{dy}\delta y + \frac{1}{2}\frac{d^2V}{dy^2}(\delta y)^2 + \cdots,$$

where δy is a change in yield.

For very small movements in the yield, the change in the price of a bond can be measured by the duration.

For larger movements we must take account of the curvature in the Price/Yield relationship.

The **dollar convexity** is defined as

$$\frac{d^2V}{dy^2} = (T-t)^2 P e^{-y(T-t)} + \sum_{i=1}^{N} C_i (t_i - t)^2 e^{-y(t_i - t)}.$$

and the **convexity** is

avg time²

- $\frac{1}{V}\frac{d^2V}{dy^2}.$ = "C"

A problem. . .

Every bond has its own yield.

Is it possible to construct an interest rate model that is simultaneously consistent with *all* bonds?

Yes.

- The simplest way to do this is to introduce one time-dependent interest rate, representing the interest received over an infinitesimal period of time.

Time-dependent interest rate

Let's look at bond pricing when we have an interest rate that is a **known function of time**. The interest rate we consider will be what is called the **short-term interest rate** or **spot interest rate** $r(t)$.

- This means that the rate $r(t)$ is to apply at time t: interest is compounded at this rate at each moment in time.

We begin with a zero-coupon bond example.

The bond price is a function of time: $Z = Z(t; T)$, with maturity T being a parameter.

Because we receive \$1 at time $t = T$ we know that $Z(T; T) = 1$.

Let's derive an equation for the value of the bond at a time before maturity, $t < T$.

The change in the value of the bond in a time-step dt (from t to $t + dt$) is

$$dZ.$$

Arbitrage considerations—there is no risk or randomness in this model—again lead us to equate this with the return from a bank deposit receiving interest at a rate $r(t)$:

$$dZ = r(t)\ Z\ dt.$$

The solution of this equation is

- $$Z(t;T) = e^{-\int_t^T r(\tau)d\tau}.$$

So, given an $r(t)$ we can find prices of zero-coupon bonds.

Unfortunately, there is no one to tell us what $r(t)$ is!

Instead we are usually given the market prices of zero-coupon bonds. The question still remains, can we find *one* function $r(t)$ that is consistent with all bond prices? . . .

Forward rates and bootstrapping

Let us suppose that we are in a perfect world in which we have a continuous distribution of zero-coupon bonds with all maturities T. Call the prices of these at time t, $Z(t;T)$.

The **forward rate** is the curve of a time-dependent spot interest rate that is consistent with the market price of instruments.

If this rate is $r(t)$ at time t then it satisfies

$$Z(t;T) = e^{-\int_t^T r(\tau)d\tau}.$$

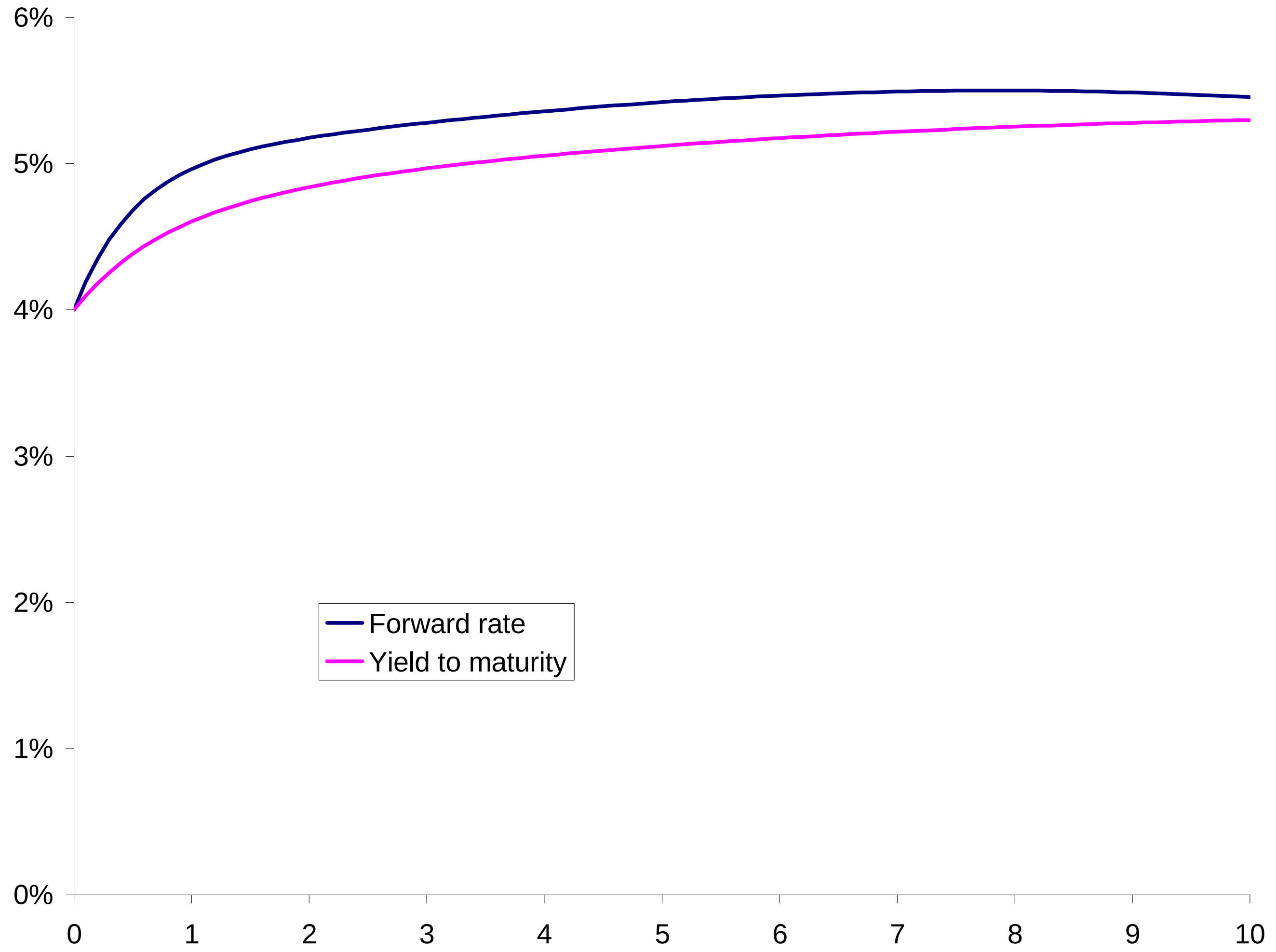
On rearranging and differentiating with respect to T this gives

$$r(T) = -\frac{\partial}{\partial T}(\log Z(t;T)).$$

Of course, this should really be quoted as a function of calendar time t, so we just change variables and write

$$r(t) = -\frac{\partial}{\partial T}(\log Z(t;T))|_{T=t}$$

- Forward rates are interest rates that are assumed to apply over given periods *in the future* for *all* instruments. This contrasts with yields which are assumed to apply up to maturity, with a different yield for each bond.


6%
5%
4%
3%
2%
1%
0%
0
1
2
3
4
5
6
7
8
9
10
Forward rate
Yield to maturity

What should happen to this forward rate curve when we calculate it in one week's time?

What actually happens?

(Hmmm. . . we will see a 'stochastic' model soon.)

Joining the dots

In the less-than-perfect real world we must do with only a discrete set of data points.

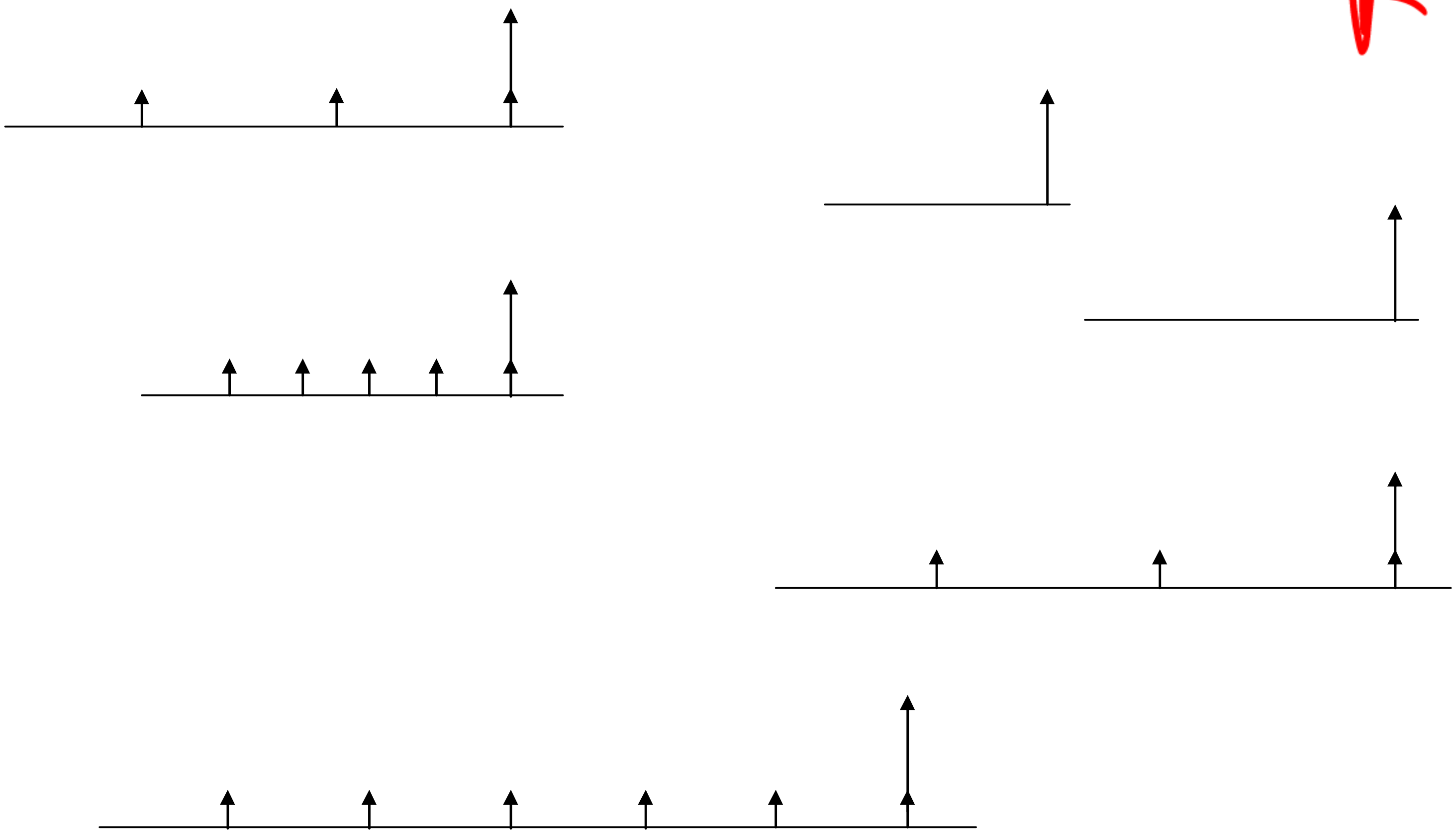
We continue to assume that we have zero-coupon bonds but now we will only have a discrete set of them.

We can still find an implied forward rate curve as follows.

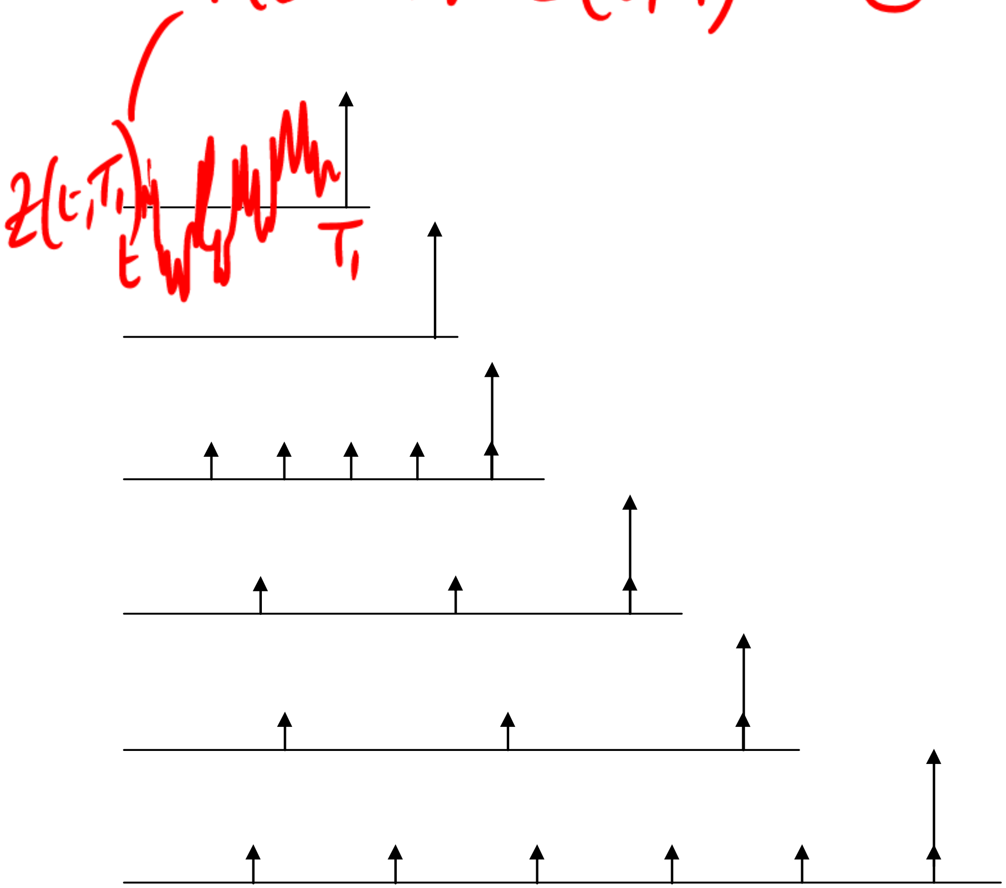
- Rank the bonds according to maturity, with the shortest maturity first. The market prices of the bonds will be denoted by Z_i^M where i is the position of the bond in the ranking.

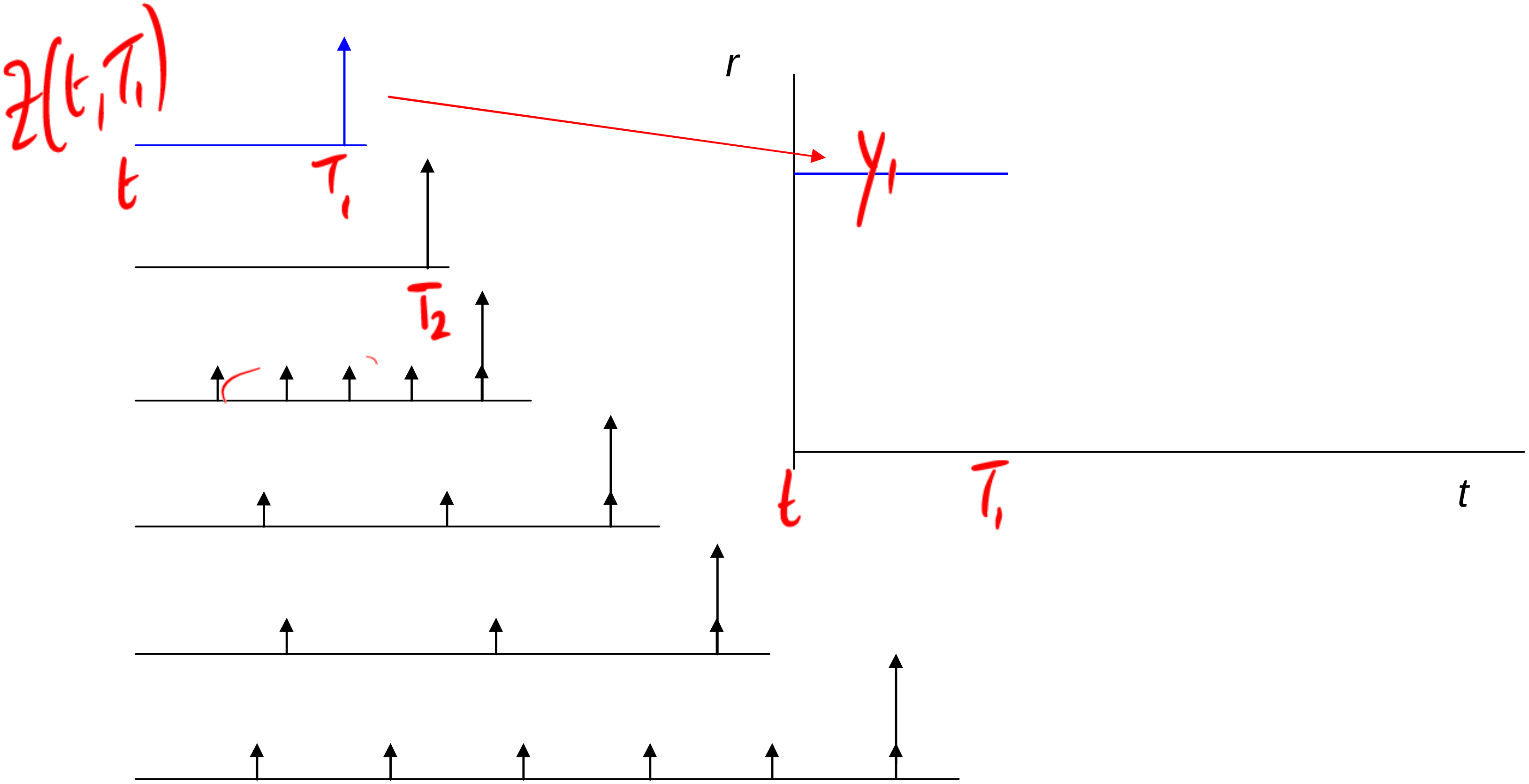
First, the idea . . .

KNOWN PRICES

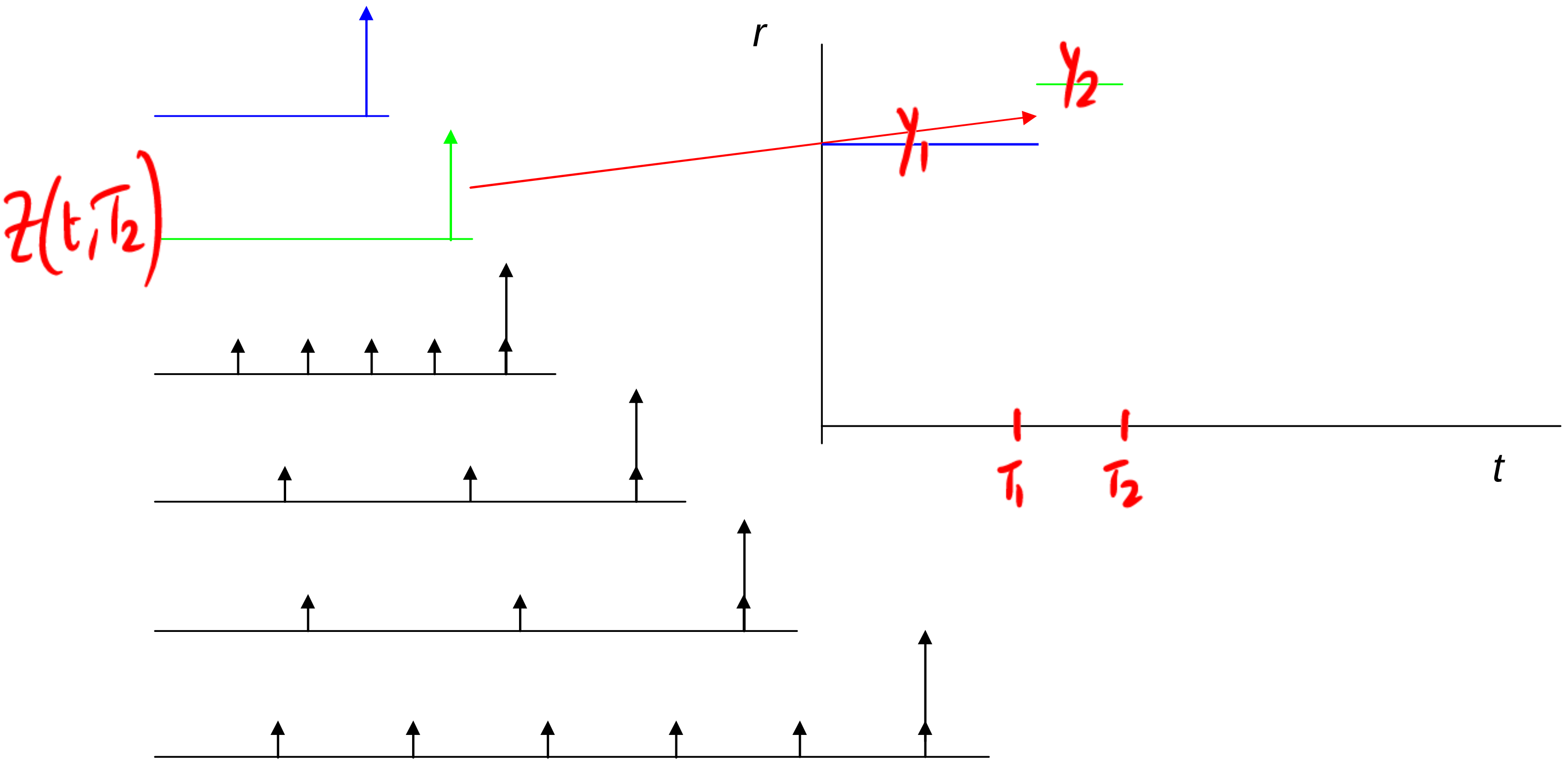


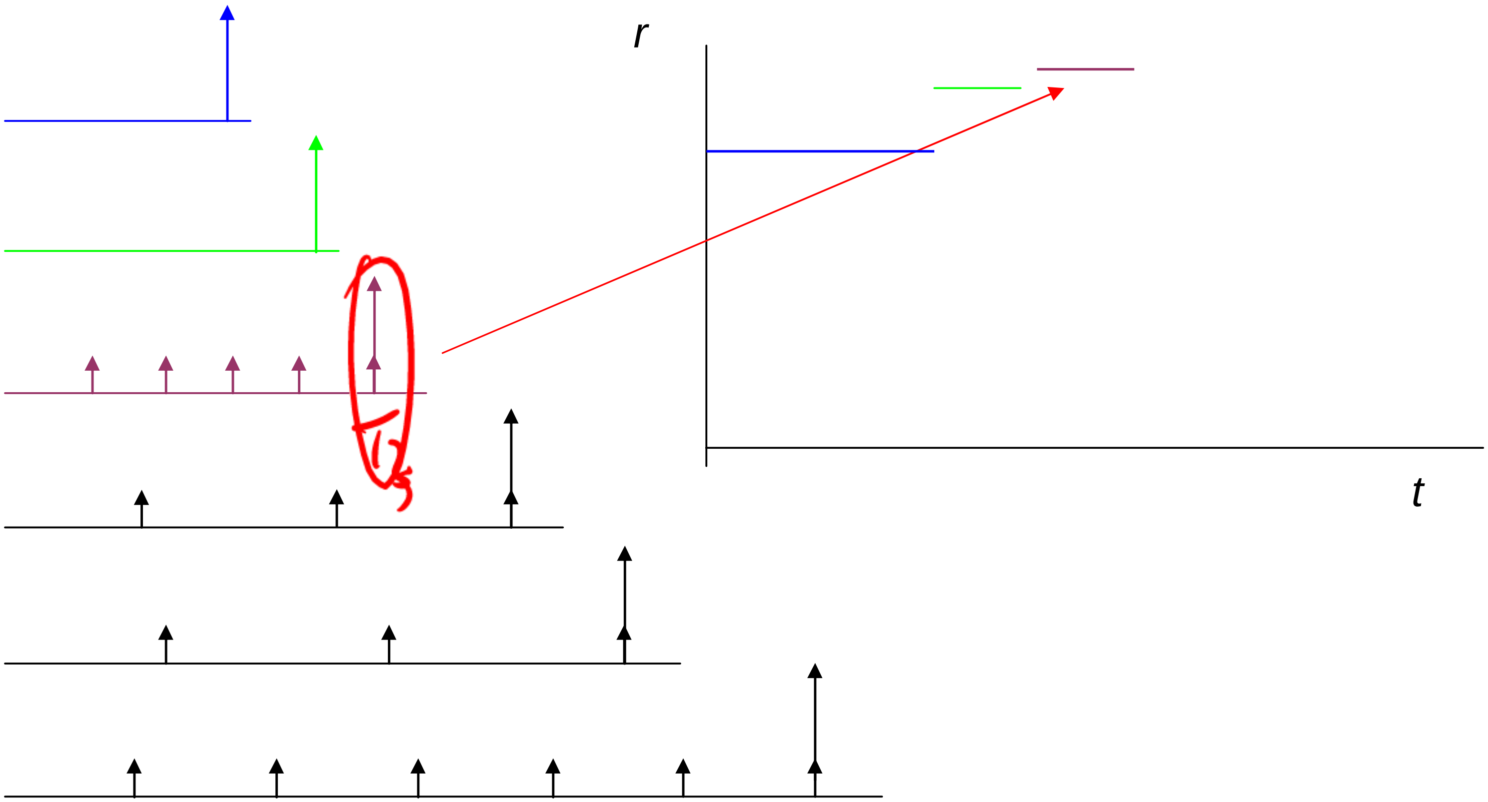
$r(t)$ s.t.: $Z(t,T_1) = e^{-\int_t^{T_1} r(\tau)\, d\tau}$

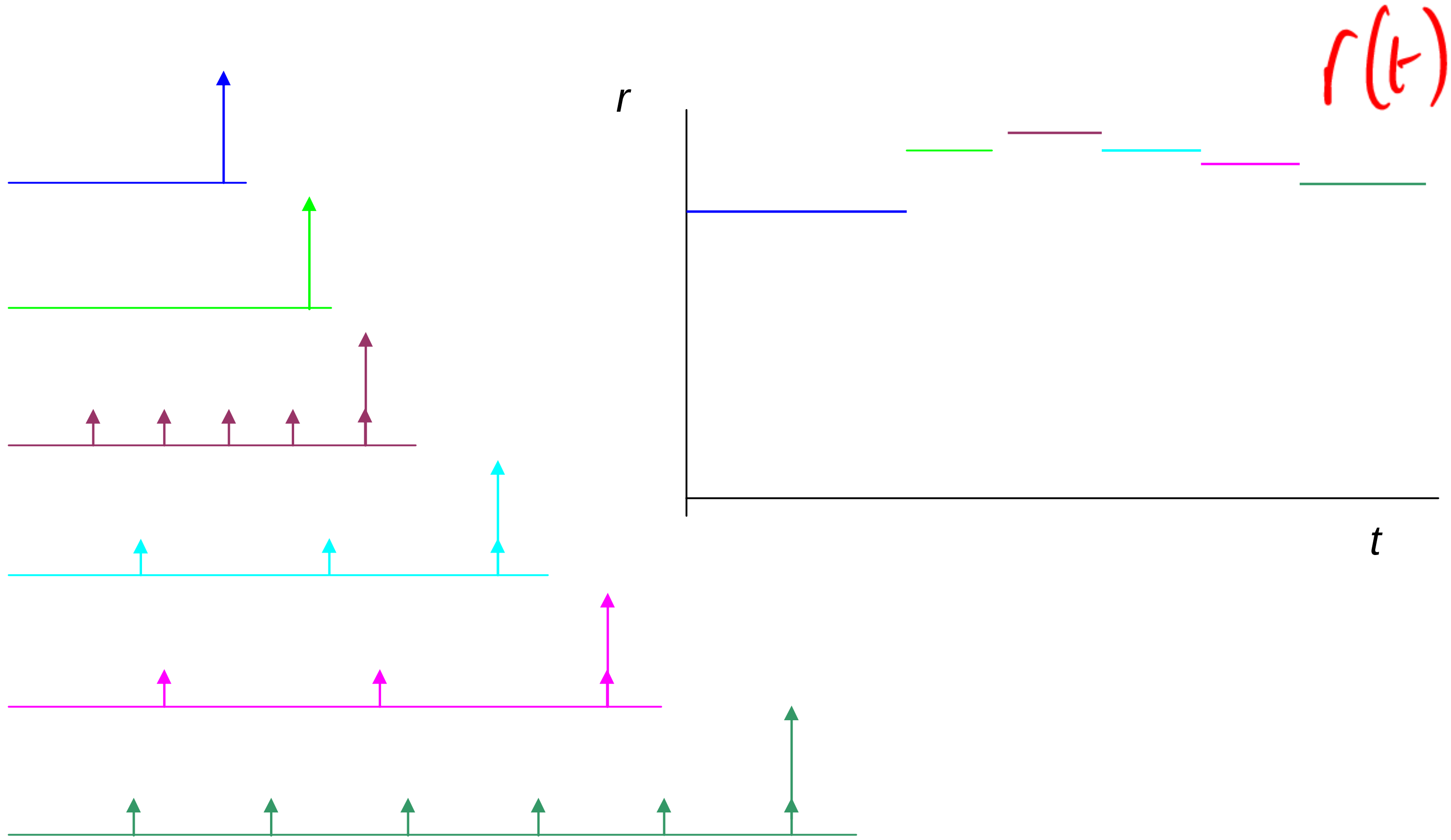




$Z(t, T_1)$
t
T_1
T_2
r
y_1
t
T_1
t


r
t
$Z(t;T_2)$
y_1
y_2
T_1
T_2


r
t


r
t
r(t)

Then the maths . . .

What interest rate is implied by the market price of the **first bond**? The answer is y_1, the solution of

$$Z_1^M = e^{-y_1(T_1-t)},$$

i.e.

$$y_1 = -\frac{\log(Z_1^M)}{T_1 - t}.$$

This rate will be the rate that we use for discounting between the present and the maturity date T_1 of the first bond.

- It will be applied to *all* instruments whenever we discount over this period.

Now move on to the **second bond**, having maturity date T_2.

We know the rate to apply between now and time T_1, but at what interest rate must we discount between dates T_1 and T_2 to match the theoretical and market prices of the second bond?

The answer is y_2 which solves the equation

$$Z_2^M = e^{-y_1(T_1-t)} e^{-y_2(T_2-T_1)},$$

t T_1 T_2

i.e.

$$y_2 = -\frac{\log\left(Z_2^M / Z_1^M\right)}{T_2 - T_1}.$$

Z_1

- By this method of **bootstrapping** we build up the forward curve. Note how the forward rates are applied between two dates, for which period we have assumed they are constant.

	A	B	C	D	E
1	**Time to**	**Market**	**Yield to**	**Forward**	
2	**maturity**	**price z-c b**	**maturity**	**rate**	
3	0.25	0.9809	7.71%	7.71%	
4	0.5	0.9612	7.91%	8.12%	
5	1	0.9194	8.40%	8.89%	
6	2	0.8436	8.50%	8.60%	
7	3	0.7772	8.40%	8.20%	
8	5	0.644	8.80%	9.40%	
9	7	0.5288	9.10%	9.85%	
10	10	0.3985	9.20%	9.43%	
11					
12					
13					
14					
15					
16					

= -LN(B10)/A10

= (C10*A10-C9*A9)/(A10-A9)

Interpolation

We have explicitly assumed in the previous section that the forward rates are piecewise constant, jumping from one value to the next across the maturity of each bond. Other methods of **interpolation** are also possible.

For example, the forward rate curve could be made continuous, with piecewise constant gradient. Some people like to use cubic splines.

Whatever interpolation method you use you would expect the resulting curve to have certain nice properties (such as being non negative, perhaps with continuity and smoothness).

Finding the forward curve with these properties amounts to deciding on a way of interpolating 'between the points,' the 'points' meaning the constraints on the integrals of the r function.

There have been many proposed interpolation techniques such as. . .

- linear in discount factors
- linear in spot rates
- linear in the logarithm of rates
- piecewise linear continuous forwards
- cubic splines
- Bessel cubic spline
- Monotone preserving cubic spline
- quartic splines

and others.

It should be emphasized that there is no ‘correct’ way to join the dots.

If you want to know what rate to apply to a two-and-a-half-year cashflow and the nearest bonds are at two and three years then you will have to make some assumptions; there is no ‘correct’ value.

Finally, the method should result in a forward rate function that is not too sensitive to the input data, the bond prices and swap rates, it must be fast to compute and must not be too local in the sense that if one input is changed it should only impact on the function nearby.

- Because of the relative liquidity of the instruments it is common to use deposit rates in the very short term, bonds and FRAs for the medium term and swaps for longer end of the forward curve.

Summary

Please take away the following important ideas

- A vanilla swap can be decomposed exactly into a portfolio of bonds

- The main ideas behind interest-rate modeling

- Yield, duration and convexity and important measures of interest rate and sensitivities

- The forward curve can be constructed from simple bonds and swaps