CQF Value at Risk

Solutions

1. Consider a position of £5 million in a single asset X with daily volatility of 1%. What are the annualised and 10-day standard deviations? Using the Normal factor calculate 99%/10day VaR in money terms.

   Solution:
   In order to annualise volatility we use the additivity of variance,

   $$\sigma_{1Y} = \sqrt{\sigma_{1D}^2 \times 252} = \sigma_{1D}\sqrt{252} = 0.01 \times \sqrt{252} \approx 0.16$$

   Notice that 1% daily volatility equates approximately to 16% volatility per annum.

   In order to calculate Value at Risk we need the value of Factor which corresponds to the $c = 99\%$ confidence. Using tables for the Normal Distribution we identify the factor value that cuts 1% on the left tail as $\Phi(-2.33) = 0.01$.

   $$\text{VaR}_{99\%/10D} = \Phi^{-1}(1-0.99) \times \sigma_{10D} \times \Pi = 2.33 \times 0.01 \times \sqrt{10} \times £5\,\text{million} = £368,405$$

   where Π is portfolio value (we have single asset).

2. Now, consider a portfolio of two assets X and Y, £100,000 investment each. The daily volatilities of both assets are 1% and correlation between their returns is $\rho_{XY} = 0.3$. Calculate 99%/5day Analytical VaR (in money terms) for this portfolio.

   Solution:
   The standard deviation in money terms is $\sigma_X = \sigma_Y = £1000$, which is 1% from £100,000. The variance of the portfolio's daily change is

   $$\begin{aligned} \sigma_\Pi^2 &= \sigma_X^2 + 2\rho_{XY}\,\sigma_X\sigma_Y + \sigma_Y^2 \\ \sigma_\Pi^2 &= 1000^2 + 2 \times 0.3 \times 1000 \times 1000 + 1000^2 = 2.6 \times 10^6 \end{aligned}$$

   which gives the standard deviation for the portfolio (its daily change) $\sigma_\Pi = £1,612.45$.

   Scaling to 5 days and using the factor value for $c = 99\%$ confidence, the result is

   $$\text{VaR}_{99\%/5D} = 2.33 \times 1612.45 \times \sqrt{5} = £8,401.$$

   Question 1 and 2 calculations assume that portfolio value (its cumulative P&L) follows the Normal Distribution. 99% VaR risk measure represents any move beyond 2.33 standard deviations, however we do not know how worse the move (loss) can be.

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3. Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile* to derive analytical expression for VaR calculation.

   Hint: Start with probability argument for the P&L (loss) X exceeding VaR(X) threshold and convert X to a Standard Normal variable ϕ.

   Solution:
   The probability of loss $X < 0$ being worse than VaR < 0 is

   $$\Pr\left(X \le \mathrm{VaR}(X)\right) = 1 - c$$

   Note that if P&L X is a random variable then VaR(X) is also a random variable. In order to use the well-known Normal Distribution functions, we have to work with the Standard Normal variable

   $$\begin{aligned} \Pr(\phi \le \frac{\mathrm{VaR}(X) - \mu}{\sigma}) &= 1 - c \qquad \Longrightarrow \\ \mathrm{VaR}(X) &= \mu + \Phi^{-1}(1-c) \times \sigma \end{aligned}$$

   Inverse CDF for a probability distribution is known as 'percentile function'.

4. Assume 'elliptical markets': asset returns are Normally distributed or close. What percentage of returns are outside two standard deviations from the mean? Consider the left tail.

   Within that tail, what is the mean of standardised returns – that is, what is an average tail loss? Provide analytical solutions for abstract μ, σ using a simplifying assumption of Standard Normal Distribution.

   PDF for Normal Distribution $N(\mu, \sigma^2)$ is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$.

   Solution:
   The percentage of returns outside n standard deviations (in general) on the left tail is the cumulative density function $\Phi(x)$, which is an integral over probability density

   $$\int_{-\infty}^{\mu - n\sigma} f(x)dx = \int_{-\infty}^{\mu - n\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \Phi(\mu - n\sigma)$$

   Note that while $\mathrm{VaR}_{c\%}(X)$ is a percentile and given by ICDF, the percentage 'cut on the tail' $1 - c$ is given by the CDF. Example: for confidence level of 99%, the percentile that 'cuts' 1% of observations is $\Phi^{-1}(1 - 0.99) \approx 2.32635$.

   At times, $\Phi(\mu - n\sigma) = 1 - c$, is substituted in integration limits using any of the following:

   $$\mu - n\sigma = \Phi^{-1}(1 - c) = \mathrm{VaR}_c$$

The mean of the values that fall within that tail (ie, cut by the percentile threshold) is

$$
\begin{aligned}
\frac{\int_{-\infty}^{\mu-n\sigma} x\, f(x)dx}{\int_{-\infty}^{\mu-n\sigma} f(x)dx} &= \frac{1}{\Phi(\mu-n\sigma)}\int_{-\infty}^{\mu-n\sigma}\frac{x}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \quad \dagger \\
& \quad \text{to simplify assume Standard Normal, } \Phi(\mu-n\sigma)=\Phi(-n) \\
&= \frac{1}{\Phi(-n)}\int_{-\infty}^{-n}\frac{x}{\sqrt{2\pi}}\,e^{-x^2/2}dx \quad \text{ready for calculation using } n=2\text{, see below.} \\
& \quad \text{To derive a general solution lets } \textbf{swap variables} \text{ in original problem } \dagger \\
& \quad \mu-n\sigma = x \quad \Leftrightarrow \quad -n=\frac{x-\mu}{\sigma}=z \quad \Leftrightarrow \quad -dn=\frac{dx}{\sigma}=dz \\
&= \frac{1}{\Phi(x)}\int_{-\infty}^{x}\frac{\mu+z\sigma}{\sqrt{2\pi}}\,e^{-z^2/2}\,dz \qquad \Leftrightarrow \qquad \frac{1}{\Phi(x)}\int_{-\infty}^{x}\frac{\mu-n\sigma}{\sqrt{2\pi}}\,e^{-n^2/2}\,(-dn) \\
&= \frac{1}{\Phi(x)}\left(\mu\,\Phi(x)+\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{x} ze^{-z^2/2}\,dz\right) \\
&= \mu+\frac{\sigma}{\Phi(x)}\,\frac{-1}{\sqrt{2\pi}}e^{-x^2/2} \\
& \quad \text{which relates to} \qquad \mathrm{ES}_c=\mu+\sigma\frac{\phi(\mathrm{VaR}_c)}{1-c} \qquad \text{cited in textbooks,} \\
& \quad \text{where standartised } \mathrm{VaR}_c=\Phi^{-1}(1-c)=x \quad \text{is in line with the above.}
\end{aligned}
$$

For the Standard Normal Distribution $N(\mu=0,\sigma^2=1)$ with density $\phi(z)$, we find that the percentage of returns outside two standard deviations on the left tail is

$$\int_{-\infty}^{-2}\phi(z)=\int_{-\infty}^{-2}\frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}\,dz=\Phi(-2)=0.02275$$

The mean of the values falling in this tail is

$$\frac{1}{\Phi(-2)}\int_{-\infty}^{-2}\frac{z}{\sqrt{2\pi}}\,e^{-z^2/2}\,dz=-\frac{1}{\Phi(-2)\sqrt{2\pi}}e^{-2}\approx -2.37$$

$$\text{using} \qquad \int ze^{-z^2/2}=-e^{-z^2/2}$$

The -2.37 figure is the mean of *standartised returns* and therefore, itself is a Standard Normal variable or Z-score. It also has a definition of Expected Shortfall (ES), *an average tail loss* given that the loss is *below* the VaR threshold. 'Given' reveals conditionality and ES is mathematically known as Conditional Value at Risk.

5. Recall the example of three bonds A,B and C from the Market Risk Measurement (Value at Risk) lecture: each bond has a face value of £1,000 payable at maturity and the independent probability of default 0.5%, when the loss is the face value in full.

   (a) Calculate the Expected Shortfall (within the 1% tail) of bonds A, B and C.

   (b) Calculate the Expected Shortfall (within the 1% tail) of a portfolio equally invested in bonds A, B and C.

   (c) Compare results from (a) and (b) to conclude whether ES is *sub-additive*.

   Solution:

   (a) Within the 1% tail, there is a 0.5%/1% likelihood of losing £1,000, so the ES of each bond is £500 calculated as expected value $0.5/1 \times$ £1,000. The same principle at work here as $\frac{\Pr}{1-c}$ textbook formula for ES. We have two separate events 'default' and 'being on the tail' which we condition on one another.

   (b) If we are already in a tail event situation (conditional loss) there is no likelihood of losing nothing (no defaults); 0.9925%/1% likelihood of one default, 0.0074625%/1% likelihood of two defaults and finally, 0.0000375%/1% likelihood of three defaults. Calculating expected value (probability-weighted averaging)

$$0.9925 \times 1000 + 0.0074625 \times 2000 + 0.0000375 \times 3000 = \pounds 1007.54.$$

   (c) The result in (b) is considerably less than 3 × ES of each bond = £1,500, therefore in this case ES is sub-additive. This holds for most cases, except a few that are only of academic interest.

To obtain the probabilities given in the lecture one has to solve a bootstrapping task numerically. The boostrapping approach might produce several solutions that satisfy conditions, such as $0.9925 + 0.0074625 + 0.0000375 = 1$. We know that

$$\begin{aligned} 1 - \text{PrSurv} = 1 - 3 \times 0.995 &\approx 0.0149 \\ 0.9925 \times 0.015 + 0.0074625 \times 0.0075 + 0.0000375 \times 0.00001 &\approx 0.0149 \end{aligned}$$

where we multiplied a conditional $\Pr(k \text{ defaults}|k \geq 1)$ by its respective marginal likelihood.[1] The bootstrapping task is set as follows:

$$\begin{aligned} \Pr(\text{one default}|) \times 0.015 + \Pr(\text{two defaults}|) \times 0.0075 + \Pr(\text{three defaults}|) \times 0.00001 &\approx 0.0149 \\ \Pr(\text{one default}|) + \Pr(\text{two defaults}|) + \Pr(\text{three defaults}|) &= 1 \end{aligned}$$

Inside the boostrapping task, there is a combinatorial logic of how several defaults can occur. Imagine *a decision tree*: certain outcomes mean the same event, e.g., one default can occur on either A, B or C, two defaults mean AB, AC, or BC. These combinations are already reflected in the marginal likelihoods $0.015, 0.0075, 0.00001$.

[1] Calculation is on lecture slides. Take into account that results are rounded up, e.g., 0.015 is approximate.

6. VaR calculation for a portfolio of derivatives often done as a breakdown of the P&L into contributions from Delta, Gamma and Vega greeks. The contributions sum up linearly (algebraically) across greeks for each option. The total VaR is also calculated as a simple sum across positions.

   However, it might be necessary to base VaR calculation on the asset price and take into account cross-asset movement (correlation). Consider a formula for Analytical VaR

$$\text{Factor} \times \sqrt{\delta t} \sqrt{\sum_{j=1}^{N} \sum_{i=1}^{N} \rho_{ij}\, \sigma_i \sigma_j\, \Delta_i S_i\, \Delta_j S_j}$$

   where Delta approximates the change in value over δt.

   What are the key assumptions of this calculation?

   $\boxed{\textbf{Solution:}}$

   (a) We assume the asset follows the log-random walk defined by $dS = \mu S dt + \sigma S dX$ SDE where $dX \to \phi\sqrt{\delta t}$ requires random Normal variable generation. If we study sample asset returns data, we are likely to see that an empirical returns distribution has fatter tails and a higher peak when compared to the Normal distribution.

   (b) All relevant volatilities and correlations are known and robust. Robustness means the low variance of a statistical estimate, ie, if you throw out/add in some observations, the change in robust estimate would be insignificant. Empirical studies find strong heteroskedasticity of volatility (it goes up and down wildly over time), while correlation changes in sine-wave pattern between upper and lower bounds. These stylised facts are true for samples of various length, eg, 30D, 60D, 6M, etc.

   (c) This analytical calculation is *linear*. It assumes that the *sensitivity* of options portfolio to the change in the underlying is approximated linearly, in this case by delta of the option.

7. What are the two main numerical methods used for the Empirical VaR estimation? What are their drawbacks?

Solution:

- **Monte Carlo** method requires generation of Normally distributed random numbers and relies on their low latency (evenness).
- **Bootstrapping** (or Historic Simulation) method uses actual asset price movements taken from historical data (eg, the last two years). More precisely, bootstrapping means sampling from *standardised historic residuals*

$$Z_t^* = \frac{u_{t,Hist}}{\sqrt{\sigma^2_{t,GARCH}}}$$

where $u_{t,Hist}$ is econometric notation for return and standard deviation $\sigma_{t,GARCH}$ is smoothed by the application of a GARCH filter.

The criticisms of the Monte-Carlo method often return to the assumption of a log-normal random walk. To introduce correlation, a solution from multi-factor PDE or factorisation (eg, by Cholesky decomposition on covariance matrix) would be required. Coupled together with the need to run tens to hundreds thousands of simulations (to secure a reliable projection) it means that the MC method can be computationally slow.

The main criticism of the Bootstrapping method is that it too requires a large amount of data. Historical data include values that correspond to very different economic conditions. Often, precisely the data required for adequate estimation suffers from 'structural breaks', such as missing prices/assets not being traded.