<u>Exercise 1:</u>

The objective of the exercise is to check that the following fact is true:

Fact 1. *If a process $Y(t)$ is a martingale under $\mathbb{Q}$ and $\eta_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$, then the process $M(t) = Y(t)\eta_t$ is a martingale under $\mathbb{P}$.*

We will focus on the case where both $Y(t)$ and $\eta(t)$ are modelled as diffusions processes with respective dynamics

$$dY(t) = f(t, Y(t))dt + g(t, Y(t))dX(t)$$

and

$$\frac{d\eta(t)}{\eta(t)} = -\theta(t)dX(t)$$

where $X(t)$ is a standard Brownian motion under the $\mathbb{P}$ measure.

Questions -

(i). *Knowing that $Y(t)$ is a martingale under $\mathbb{Q}^\theta$, express the drift function $f(\cdot)$ in terms of the diffusion function $g(\cdot)$ and of the process $\theta(t)$.*

(ii). *Apply the Itô product rule to show that $M(t) = Y(t)\eta_t$ is a martingale under $\mathbb{P}$.*

<u>Exercise 2:</u> (Optional)

Derive formula (25) on slide 80

$$C(t) = B(t,U)N\left[d_1(B(t,U),t,T)\right] - KB(t,T)N\left[d_2(B(t,U),t,T)\right] \tag{1}$$

where

$$\begin{aligned} d_1(b,t,T) &= \frac{\ln\left(\frac{b}{K}\right) - \ln B(t,T) + \frac{1}{2}\upsilon_U(t,T)}{\upsilon_U(t,T)} \\ d_2(b,t,T) &= d_1 - \upsilon_U(t,T) \\ \upsilon_U^2(t,T) &= \int_t^T (b(s,U) - b(s,T))^2\, ds \end{aligned}$$

Start from the forward asset pricing formula given in equation (24), on slide 79,

$$C(t) \quad = \quad B(t,T)\mathbf{E}^{\mathbb{P}_T}\left[(F_B(T,T,U)-K)^+|\mathcal{F}_t\right] \tag{2}$$

where the dynamics of the forward price $F_B(t,T,U)$ is given in equations (22) and (23) on slide 78.

Hints:

1. *you could use an approach similar to the derivation of the Black-Scholes formula presented in Section 3.3 of Lecture 3.3 (slides 63-75)*;
2. *Note that the random variable $Y(T) = \int_t^T (b(s,U) - b(s,T))\, dX^T(s)$ is Normally distributed with mean 0 and variance $v_U^2(t,T)$.*

Solutions

1- Exercise 1

(i). By Girsanov, the $\mathbb{Q}$-Brownian motion $X^{\mathbb{Q}}(t)$ is defined as

$$X_t^{\mathbb{Q}} = X_t + \int_0^t \theta(s)ds, \quad t \in [0,T] \tag{3}$$

Hence, the dynamics of $Y(t)$ under the $\mathbb{Q}$-measure is given by

$$\begin{aligned} dY(t) &= f(t,Y(t))dt + g(t,Y(t))dX(t) \\ &= f(t,Y(t))dt + g(t,Y(t))\left(dX^{\mathbb{Q}}(t) - \theta(t)dt\right) \\ &= \left[f(t,Y(t)) - g(t,Y(t))\theta(t)\right]dt + g(t,Y(t))dX^{\mathbb{Q}}(t) \end{aligned}$$

For $Y(t)$ to be a $\mathbb{Q}$-martingale, we need it to be driftless, which implies that

$$f(t,Y(t)) - g(t,Y(t))\theta(t) = 0$$

Therefore, we can express the drift function $f(t,Y(t))$ in terms of the diffusion function $g(\cdot)$ and of the process $\theta(t)$ as

$$f(t,Y(t)) = g(t,Y(t))\theta(t)$$

(ii). We apply the Itô product rule to derive the dynamics of $M(t)$ under $\mathbb{P}$:

$$\begin{aligned} dM(t) &= d\left(Y_t\eta_t\right) \\ &= dY_t \cdot \eta_t + Y_t \cdot d\eta_t - \theta(t)\eta(t)g(t,Y(t))dt \\ &= \left(f(t,Y(t))dt + g(t,Y(t))dX(t)\right)\eta(t) \\ &\quad -\theta(t)\eta(t)Y(t)dX(t) - \theta(t)\eta(t)g(t,Y(t))dt \\ &= \left(f(t,Y(t))\eta(t) - \theta(t)\eta(t)g(t,Y(t))\right)dt \\ &\quad +\eta(t)\left[g(t,Y(t)) - \theta(t)Y(t)\right]dX(t) \end{aligned}$$

Now, $Y(t)$ is $\mathbb{Q}$-martingale, which implies that $f(t,Y(t)) = g(t,Y(t))\theta(t)$. Substituting in the previous equation, we find that

$$dM(t) = \left(g(t,Y(t)) - \theta(t)Y(t)\right)\eta(t)dX(t)$$

The dynamics of $M(t)$ is driftless. Therefore $M(t)$ is a martingale under $\mathbb{P}$.

2- Derivation of Formula (24) on Slide 79

We start from the forward asset pricing formula

$$C(t) \quad = \quad B(t,T)\mathbf{E}^{\mathbb{P}_T}\left[(F_B(T,T,U)-K)^+|\mathcal{F}_t\right] \tag{4}$$

which we rewrite in the (now) usual way as

$$\begin{aligned} C(t) \quad = \quad & B(t,T)\Big(\mathbf{E}^{\mathbb{P}_T}\left[F_B(T,T,U)\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] \\ & -K\mathbf{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right]\Big) \end{aligned} \tag{5}$$

Step 1: Evaluating the Second Expectation: $\mathbf{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right]$

$$\begin{aligned} & \mathbf{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] \\ = \quad & \mathbb{P}_T\left[F_B(T,T,U)\geq K|\mathcal{F}_t\right] \\ = \quad & \mathbb{P}_T\left[F_B(t,T,U)\exp\left\{\int_t^T (b(s,U)-b(s,T))\,dX^T(s)-\frac{1}{2}\upsilon_U^2(t,T)\right\}\geq K\right] \\ = \quad & \mathbb{P}_T\left[\int_t^T (b(s,U)-b(s,T))\,dX^T(s)\geq \ln\frac{K}{F_B(t,T,U)}+\frac{1}{2}\upsilon_U^2(t,T)\right] \end{aligned}$$

Note that the random variable $Y(T)=\int_t^T (b(s,U)-b(s,T))\,dX^T(s)$ is Normally distributed with mean 0 and variance $\upsilon_U^2(t,T)$.

We can then define a standard Normal random variable $Z\sim N(0,1)$ as $Z=\frac{Y}{\upsilon_U(t,T)}$ and express the expectation as

$$\begin{aligned} & \mathbf{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] \\ = \quad & \mathbb{P}_T\left[Z\geq \frac{\ln\frac{K}{F_B(t,T,U)}+\frac{1}{2}\upsilon_U^2(t,T)}{\upsilon_U(t,T)}\right] \end{aligned}$$

By symmetry of the Normal distribution, we conclude that:

$$\begin{aligned} & \mathbf{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] \\ = \quad & \mathbb{P}_T\left[Z\leq \frac{\ln\frac{F_B(t,T,U)}{K}-\frac{1}{2}\upsilon_U^2(t,T)}{\upsilon_U(t,T)}\right] \\ = \quad & N\left[d_2(B(t,U),t,T)\right] \end{aligned}$$

where

$$d_2(b,t,T) \quad = \quad \frac{\ln\left(\frac{b}{K}\right) - \ln B(t,T) - \frac{1}{2}\upsilon_U(t,T)}{\upsilon_U(t,T)}$$

Step 2: Evaluating the First Expectation: $\mathbf{E}^{\mathbb{P}_T}\left[F_B(T,T,U)\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right]$

$$\begin{aligned}
& \mathbf{E}^{\mathbb{P}_T}\left[F_B(T,T,U)\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] \\
= \;& F_B(t,T,U)\mathbf{E}^{\mathbb{P}_T}\left[\exp\left\{\int_t^T (b(s,U)-b(s,T))\,dX^T(s) - \frac{1}{2}\upsilon_U^2(t,T)\right\}\mathbf{1}_{\{F_B(T,T,U)\geq K\}}\right]
\end{aligned}$$

The process

$$\Lambda(t) \quad = \quad \exp\left\{\int_t^T (b(s,U)-b(s,T))\,dX^T(s) - \frac{1}{2}\upsilon_U^2(t,T)\right\}$$

is an exponential martingale which we can use to define a new measure $\mathbb{Z}$ via the Radon-Nikod'ym derivative:

$$\frac{d\mathbb{Z}}{d\mathbb{P}_T} \quad = \quad \Lambda(t) \tag{6}$$

Under the $\mathbb{Z}$-measure,

$$X^{\mathbb{Z}}(t) = X^T(t) - \int_t^T (b(s,U)-b(s,T))\,ds$$

is a standard Brownian motion and

$$F_B(T,T,U) = F_B(t,T,U)\exp\left\{\int_t^T (b(s,U)-b(s,T))\,dX^T(s) + \frac{1}{2}\upsilon_U^2(t,T)\right\}$$

Therefore,

$$\begin{aligned}
& \mathbf{E}^{\mathbb{P}_T}\left[F_B(T,T,U)\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] \\
= \;& F_B(t,T,U)\mathbf{E}^{\mathbb{Z}}\left[\mathbf{1}_{\{F_B(T,T,U)\geq K\}}\right] \\
= \;& F_B(t,T,U)\mathbb{Z}\left[F_B(T,T,U)\geq K\right] \\
= \;& F_B(t,T,U)\mathbb{Z}\left[\int_t^T (b(s,U)-b(s,T))\,dX^{\mathbb{Z}}(s) \geq \ln\frac{K}{F_B(t,T,U)} - \frac{1}{2}\upsilon_U^2(t,T)\right]
\end{aligned}$$

After a few additional manipulations similar to what was done in ***Step 1***, we obtain:

$$
\begin{aligned}
& \mathbf{E}^{\mathbb{P}_T}\left[F_B(T,T,U)\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] \\
= \;& F_B(t,T,U)\mathbb{Z}\left[Z \leq \frac{\ln\frac{F_B(t,T,U)}{K} + \frac{1}{2}\upsilon_U^2(t,T)}{\upsilon_U(t,T)}\right] \\
= \;& F_B(t,T,U)N\left[d_1(B(t,U),t,T)\right]
\end{aligned}
$$

where

$$
d_1(b,t,T) \quad = \quad \frac{\ln\left(\frac{b}{K}\right) - \ln B(t,T) + \frac{1}{2}\upsilon_U(t,T)}{\upsilon_U(t,T)}
$$

Step 3: Concluding

Putting it all together,

$$
\begin{aligned}
C(t) &= B(t,T)\left(\mathbf{E}^{\mathbb{P}_T}\left[F_B(T,T,U)\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right] - K\mathbf{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{F_B(T,T,U)\geq K\}}|\mathcal{F}_t\right]\right) \\
&= B(t,T)F_B(t,T,U)N\left[d_1(B(t,U),t,T)\right] - KN\left[d_2(B(t,U),t,T)\right] \\
&= B(t,U)N\left[d_1(B(t,U),t,T)\right] - KN\left[d_2(B(t,U),t,T)\right]
\end{aligned}
$$

where in the last line, we have used the fact that $F_B(t,T,U) = \frac{B(t,U)}{B(t,T)}$ (see equation (19) on slide 74).