INTEREST RATE MODELING WITH LIBOR MARKET MODEL AND CVA CALCULATION

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by

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ORIGINALITY DECLARATION

I, Ran Zhao, declare that the work in this final report was carried out in accordance with the requirements of Certification in Quantitative Finance and that it has not been submitted for any other academic award or publication. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the final report are those of the author.

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INTEREST RATE MODELING WITH LIBOR MARKET MODEL AND CVA CALCULATION

ABSTRACT

In this paper, I use the swaption volatilities to calibrate the LIBOR market model and generate the interest rate scenarios to price Bermudans swaption. The market risk of vanilla European swaption is analyzed and interest rate related Greeks are calculated using the scenarios. In the second part of the report, I calculate the CVA for a swap portfolio. The expected exposure profile, potential future exposure and other related statistics are also calculated and illustrated.

CHAPTER 1

INTEREST RATE VOLATILITY AND DERIVATIVES

Before introduction of the market models, short rate models are widely used by practitioners for interest rate derivatives pricing. Examples of short-rate models are Vasicek (1977) model, Cox, Ingersoll, and Ross (1985) model and Hull and White (1990) model. These models establish the instantaneous spot interest rate dynamics, using single or multi-dimensional diffusion process(es). However, the interest rate dynamics from short rate models is not compatible with Black's formula for either swap or swaption. In other words, simplified and inexact assumptions are made on interest rate distribution in short rate models, in order to extrapolate the term structure of rates. This knowledge of term structure is vital to interest rate derivatives pricing. The lack of calibration to the whole forward curve is a trade-off with mimicking the Black-Scholes model for stock option in interest rate option, but brings in market inconsistency.

The LIBOR market model (LMM), instead, is based on the discretization of the yield curve into discrete forward rates. And each of these forward rate represents to the market quote of corresponds Forward Rate Agreement (FRA). More importantly, the LIBOR market model prices caps with Black's cap formula (lognormal forward-LIBOR model, LFM) and prices swaption with Black's swaption formula (lognormal forward-swap model, LSM). That is, the interest rate dynamics from the LMM are consistent with caps and swaptions, which are two most standard and basic interest-rate option on the market.

1.1 LMM Framework

1.1.1 Forward Rate

In standard LMM, we assume that the stochastic differential equation of each n spanning forward rates f_i formulates as

$$\frac{df_i}{f_i} = \mu_i(\mathbf{f},t)dt + \sigma_t(t)d\tilde{W}_i \tag{1.1}$$

where $\mathbb{E}[d\tilde{W}_i d\tilde{W}_j] = \rho_{ij}dt$. The lognormal-type model setup ensures positive forward rates. And $i,j = 1,2,\ldots,M$. The derivative of Black's formula for caplets is detailed in Appendix

1.1.2 Numeraire and Measure

Consider the forward (adjusted) probability measure Q^i associated with numeraire $P(\cdot,T_i)$ for maturity T_i, where the price of the bond maturity coincides with the forward rate maturity. With simple compounding, it follows

$$df_iP(t,T_i) = [P(t,T_{i-1}) - P(t,T_{i-1})]/\tau_i$$

Note that $f_iP(t,T_i)$ is a tradable asset's price, where the price divides by the numeraire $P(\cdot,T_i)$ is $f_i(t)$ itself. Therefore, $f_i(t)$ follows a martingale under forward measure. Corresponding driftless dynamics for $f_i(t)$ under this measure with respect to Equation 1.1 is

$$\frac{df_i(t)}{f_i(t)} = \sigma_i(t)dW_i(t)$$

When σ is bounded and using Ito's formula, the unique strong solution of the forward rate dynamic is

$$\log f_i(T) = \log f_i(0) - \int_0^T \frac{\sigma_i(t)^2}{2} dt + \int_0^T \sigma_i(t) dW_i(t)$$

The instantaneous volatility term $\sigma_i(t)$ assumes to be piecewise-constant

$$\sigma_i(t) = \sigma_{i,\beta(t)}(t)$$

where in general $\beta(t) = m$ if $T_{m-2} < t \leq T_{m-1}, m \geq 1$.

Under this lognormal assumption, it yields that the dynamics of f_k under forward measure Q^i in three cases $i < k, i = k$ and $i > k$ are

$$\begin{aligned}
&i < k, \quad t \leq T_i : df_k(t) = \sigma_k(t) f_k(t) \textstyle\sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t) f_j(t)}{1+\tau_j f_j(t)} dt + \sigma_k(t) f_k(t) dW_k(t) \\
&i = k, \quad t \leq T_{k-1} : df_k(t) = \sigma_k(t) f_k(t) dW_k(t) \\
&i > k, \quad t \leq T_{k-1} : df_k(t) = -\sigma_k(t) f_k(t) \textstyle\sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t) f_j(t)}{1+\tau_j f_j(t)} dt + \sigma_k(t) f_k(t) dW_k(t)
\end{aligned}$$

where $W = W^i$ is a Brownian motion under Q^i.

1.1.3 Risk Neutral Dynamics in LMM

According to Brigo and Mercurio (2006), the risk-neutral dynamics of forward LIBOR rates in the LMM is

$$df_i(t) = \tilde{\mu}_i(t) f_i(t) dt + \sigma_i(t) f_i(t) d\tilde{W}_i(t)$$

where

$$\begin{aligned}
\tilde{\mu}_i(t) &= \sum_{j=\beta(t)}^{i} \frac{\rho_{i,j}\tau_j\sigma_i(t)\sigma_j(t)f_j(t)}{1+\tau_j f_j(t)} + \tilde{\sigma}_i(t)\rho\int_t^{T_{\beta(t)-1}} \tilde{\sigma}_f(t,u)'du \\
&= \sum_{j=\beta(t)}^{i} \frac{\rho_{i,j}\tau_j\sigma_i(t)\sigma_j(t)f_j(t)}{1+\tau_j f_j(t)} + \sum_{j=\beta(t)}^{i} \rho_{i,j}\sigma_i(t)\rho\int_t^{T_{\beta(t)-1}} \tilde{\sigma}_f(t,u)'du
\end{aligned}$$

where $\tilde{\sigma}$ is the horizontal M-vector volatility coefficient for the forward rate $f_i(t)$.

1.2 Calibration of LMM to Caps/Swaptions Prices

Let's assume a unit notional amount, adn the discount payoff at time 0 of a cap with reset date T_α and payment dates $T_{\alpha+1},\ldots,T_\beta$ is given by

$$\sum_{i=\alpha+1}^{\beta} \tau_i D(0,T_i)(f(T_{i-1},T_{i-1},T_i))^+$$

The risk neutral expectation on the cap price described above is

$$\mathbb{E}\left\{\sum_{i=\alpha+1}^{\beta} \tau_i D(0,T_i)(f(T_{i-1},T_{i-1},T_i))^+\right\} = \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)\mathbb{E}^i[(f(T_{i-1},T_{i-1},T_i))^+]$$

Therefore the caplet (the single additive term) is then given by

$$P(0,T_i)\mathbb{E}^i(f_i(T_{i-1})-K)^+$$

According to Proposition 6.4.1 in Brigo and Mercurio (2006), the price of the T_{i-1}-caplet implied by the LMM coincides with that given by the corresponding Black caplet formula

$$
\begin{aligned}
Cpl^{LMM}(0,T_{i-1},T_i,K) &= Cpl^{LMM}(0,T_{i-1},T_i,K,v_i) \\
&= P(0,T_i)\tau_i Bl(K,f_i(0),v_i) \\
Bl(K,f_i(0),v_i) &= \mathbb{E}^i(f_i(T_{i-1})-K)^+ \\
&= f_i(0)\Phi(d_1(K,f_i(0),v_i)) - K\Phi(d_2(K,f_i(0),v_i)) \\
d_{1,2}(K,f,v) &= \frac{\log(f/K) \pm v^2/2}{v}
\end{aligned}
$$

The market quote on the market is typically with first reset date in three months or in six months. An equation is considered between the market price $Cap^{MKT}(0,T_i,K)$ of the cap with $\alpha = 0$ and $\beta = j$ and the sum of the first j caplets prices:

$$Cap^{MKT}(0,T_i,K) = \sum_{i=1}^{j} \tau_i P(0,T_i) Bl(K,f_i(0),\sqrt{T_{i-1}v_{T_j-cap}})$$

where a same average-volatility value v_{T_j-cap} has been put in all caplets up to j. To recover the market cap prices with forward rate dynamics we have

$$\sum_{i=1}^{j} \tau_i P(0,T_i) Bl(K,f_i(0),\sqrt{T_{i-1}v_{T_j-cap}}) = \sum_{i=1}^{j} \tau_i P(0,T_i) Bl(K,f_i(0),\sqrt{T_{i-1}v_{T_{i-1}-caplet}})$$

Recovering the v_{caplet}'s from the market quoted v_{cap}'s using stripping algorithm can be used, which based on the last equality applied to $j = 1,2,3,\ldots$.

In this project, I use the swaption normal volatility to calibration the LMM. The swaption is an option whose under is an interest rate swap (IRS). The discounted payoff of an IRS with a K different from swap rate can be expressed as

$$D(0,T_\alpha)(S_{\alpha,\beta}(T_\alpha - K)\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i))$$

And a swaption is a contract that gives its buyer the right, but not obligation, to enter a future time an interest rate swap. The payer swaption payoff is

$$D(0,T_\alpha)(S_{\alpha,\beta}(T_\alpha - K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i))$$

The coincidence of payer swaption price with Black's formula for swaptions is addressed in Proposition 6.7.1 in Brigo and Mercurio (2006), which states

$$\begin{aligned} PS^{LMM}(0,T_\alpha,[T_\alpha,\ldots,T_\beta],K) &= PS^{Black}(0,T_\alpha,[T_\alpha,\ldots,T_\beta],K) \\ &= C_{\alpha,\beta}(0)Bl(K,S_{\alpha,\beta}(0),v_{\alpha,\beta}(T_\alpha)) \end{aligned}$$

The calibration process of swaptions prices to LMM parameters is to keep adjusting the correlation matrix in the LMM so that the difference between market observed volatility is closest to the model generated volatilities. The Rebonato calibration method is detailed in Rebonato (1999) and Rebonato (2002).

The market date I chose for LMM calibration is 2016-06-30. The U.S. swaption normal volatility date source is Bloomberg. The normal volatilities are from vanilla European swaptions with specific option tenors and swap tenors. Figure 1.1 plots the market observable swaption normal volatilities with respect to various option terms

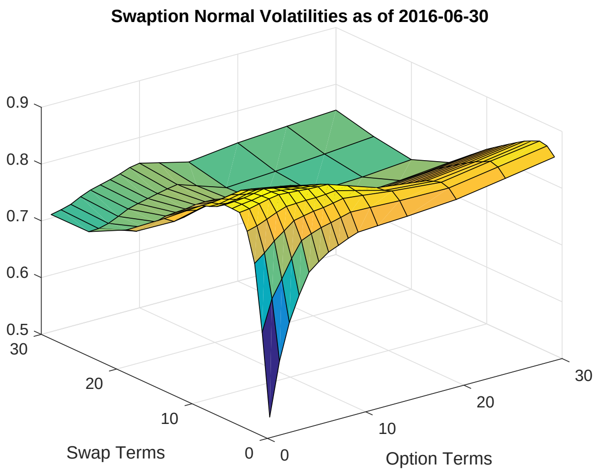

Figure 1.1: The normal volatilities of swaptions as of 2016-06-30. Data source is Bloomberg.

and swap terms. Note that the volatilities of swaption used for calibration are all at-the-money level volatilities.

The U.S. LIBOR OIS curve for the same market date is used to discount the payoff of the swaption. The selected discounting method is dual-curve discounting, where the discount factor comes from the OIS curve. Figure 1.2 plots the zero and forward curves as of market date 2016-06-30.

The LMM calibration has objective function that calculates the difference between LMM generated Black volatilities and the market observed Black volatilities. The market observable Black volatilities is backed out from swaption normal volatilities using the Black model. The LMM projected Black volatilities are calculated from tuned

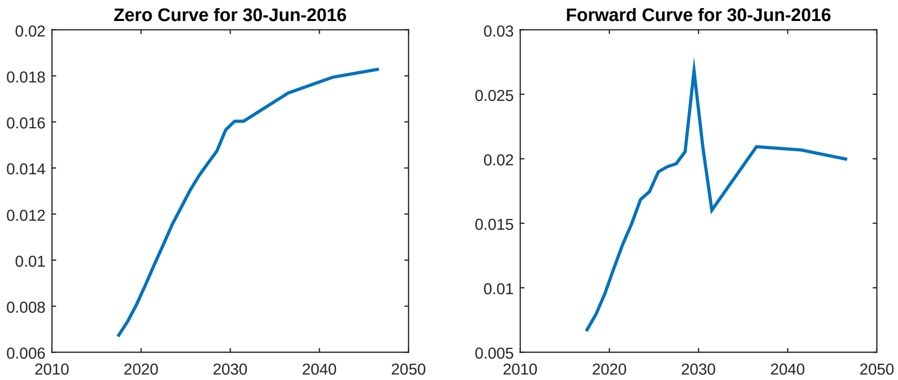

Figure 1.2: The zero curve (left) and forward curves (right) as of 2016-06-30.

LMM parameters and Black's formula for swaption pricing. A set of market consistent parameters, especially the correlation matrix from the LMM, is optimized using least square technique. A least square type of error is minimized between the market volatility surface and LMM projected surface. The calibration process implementation is detailed in the code submitted among with this paper.

The option and swap tenors used for calibration are 1-Yr, 3-Yr, 5-Yr. 7-Yr, 10-Yr, 20-Yr and 30-Yr, in total 49 data points from the volatility surface. The parametric correlation function takes the form $\rho_{i,j} = \exp(-\beta(t)|i-j|)$, and a piecewise constant instantaneous volatility assumption is selected. One useful approximation (Rebonato 2002) follows as below, which computes the Black volatility for a European swaption, given an LMM with a set of volatility functions and a correlation matrix.

$$(v_{\alpha_\beta}^{LMM})^2 = \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)f_i(0)f_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt$$

where

$$w_i(t) = \frac{\tau_i P(t,T_i)}{\sum_{k=\alpha+1}^{\beta} \tau_k P(t,t_k)}$$

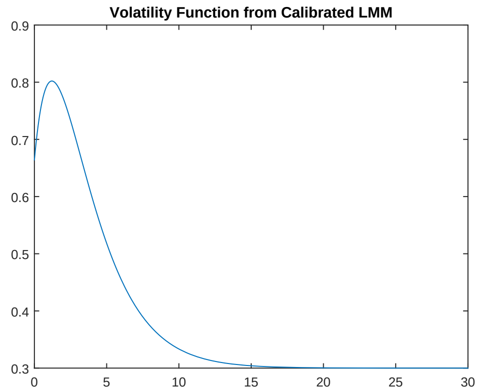

Figure 1.3: The term structure of volatility from calibrated LMM.

The calibration is achieved via optimization Rebonato method. Figure 1.3 plots the term structure of the volatility from the calibrated LMM parameters.

Another benefit of calibrating market consistent model parameters is to populate market consistent interest rate scenarios for exotic option pricing. In this project, I use the LMM generated scenarios to price vanilla swaption (as discussed in 1.3) and Bermudans swaption. For a Bermudans swaption with maturity date in 5 years and a strike of 0.045, the averaged price is $3.250, over 500 scenarios. Figure 1.4 plots the simulated zero curves and forward curves in one scenario (out of 1000) from the calibrated LMM.

1.3 Price Sensitivity Analysis

The pricing vanilla European swaption is done in the calibration process, where the Black volatility of a swaption is calculated (and compares to the market quote). A

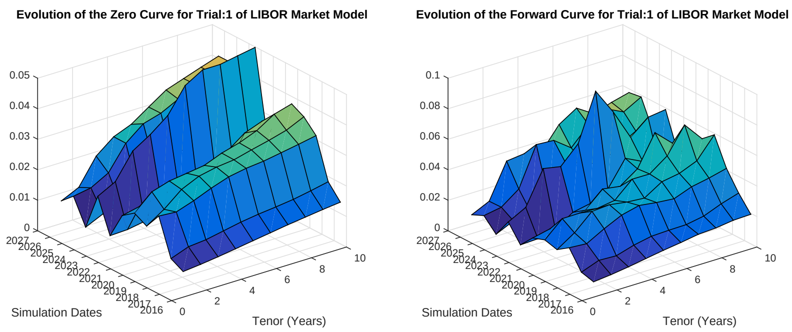

Figure 1.4: Simulated zero curves and forward curves in one scenario from calibrated LMM.

swaption price is then calculated using either Black formula for swaption or a Monte Carlo simulation using the scenarios generated from LMM. The Black formula pricing method uses the Black volatility of the swaption, which is approximated by Rebonato method. Just as an example, I price the 5-Yr-into-5-Yr (5-Yr option tenor and 5-Yr swap tenor) vanilla payer swaption with strike price 2.5%. The fair value from the LMM is \$ 1.8236, using Monte Carlo simulation with 200 scenarios.

The market risks of this vanilla swaption are mainly *R*ho and *C*onvexity. *R*ho represents the first-order market exposure of swaption with respect to the interest rate moves. *C*onvexity, instead, represents the second-order market exposure of swaption with respect to the interest rate moves. The Greeks *R*ho and *C*onvexity can be calculated from bumping the forward rate curves parallel, re-calibration on the LMM parameters, and repricing the swaption. The *R*ho calculation method is

$$Rho = \frac{\text{Price when IR curve up } \Delta \text{ bps} - \text{Price when IR curve down } \Delta \text{ bps}}{2\Delta}$$

where the unit the calculation is per \$ price change given 1 basis point interest rate input move. Similarly, the *R*ho calculation method is

$$Convexity = \frac{\text{Price IR up } \Delta \text{ bps} - 2 \times \text{Price no bump} - \text{Price IR down } \Delta \text{ bps}}{\Delta^2}$$

The *R*ho of the swaption based on 25 basis points interest rate shocks (up and down) is 0.0331. The *C*onvexity of the swaption based on 25 basis points interest rate shocks (up and down) is 0.0006516. To see clearer the price sensitivity of swaption to different sizes of market interest rate shock, the *R*ho profile is calculated and shown in Table 1.3.

Interest Rate Parallel Shock (bps)	-50	-25	-10	0	10	25	50	100	300
Swaption Price	1.1925	1.1993	1.4819	1.8236	1.7226	2.8551	3.2337	4.4265	10.2215

Figure 1.5 plots the price sensitivity of swaption with respect to the interest rate curve (OIS curve) bumpings. The bumpy behavior at +10 bps shock may come from the calibration noise.

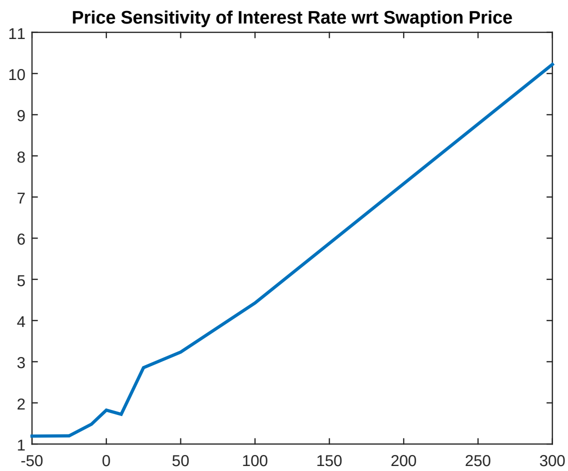

Figure 1.5: The swaption prices with respect to the OIS curve bumpings.

CHAPTER 2

CVA CALCULATION FOR AN INTEREST RATE SWAP

The credit value adjustment (CVA) measures the counterparty credit risk (CCR) in aims of regulation and accounting and pricing purposes. The needs for CVA lie on

- Existence of variation on couterparty's credit rating;
- The variation causes uncertainty to future expected value of the instrument's accounting CVA;
- The uncertainty causes potential MtM losses.

In this project, I calculate the CVA of a swap portfolio of an imaginary investment bank and particularly the CVA of a swap given in context.

The formula of CVA is

$$CVA(t,T) = LGD \int_t^T EE(u) dPD_C(c)$$

where $LGD = (1 - RR)$ is the loss given default, EE is the discounted expected exposure, and PD is the default probability.

2.1 Swap Portfolio

Consider a trading book of an investment book that contains 31 swaps with 6 counterparties. The swap book specifications are shown in Table 2.1. The swap position with our particular interest is the last one, with 1 million dollar notional value and 5-Yr maturity. The unique couterparty of this swap is 6, different with all other swaps.

CounterpartyID	NettingID	Principal	Maturity	LegType	LegRateReceiving	LegRatePaying	LatestFloatingRate	Period
5	5	813450	13-Dec-17	1	0.036134726	10	0.03462651	1
5		441321	26-Oct-17	0	87	0.039251637	0.033598155	1
1		629468	4-Sep-20	1	0.038682219	0	0.035674961	1
5		774308	2-Mar-22	0	70	0.046303151	0.035364042	1
4		918177	4-Feb-23	1	0.047524758	74	0.034985981	1
1	1	969469	10-Apr-18	0	78	0.040000628	0.03497566	1
2	2	660412	27-Nov-20	0	8	0.0395239	0.034441285	1
3		353968	23-Apr-20	1	0.041441865	36	0.03571308	1
5	5	361971	26-Jul-17	1	0.036346283	23	0.034906748	1
5		443131	8-Jul-19	0	72	0.042262413	0.034499829	1
1		880538	20-Jun-18	1	0.038769776	39	0.03594597	1
5		440712	4-Apr-22	1	0.047340175	82	0.03484266	1
5	5	860714	13-May-19	0	16	0.038394576	0.03458768	1
3	3	432644	30-Aug-20	0	24	0.040345451	0.034794372	1
5		946948	28-Jun-18	0	13	0.03672928	0.034406452	1
1		512488	7-Feb-21	0	12	0.040163882	0.03421591	1
3		397446	27-Jan-19	1	0.042759613	78	0.0367682	1
5	5	438313	1-Jun-21	1	0.044307315	52	0.036158848	1
4	4	712034	17-Aug-21	1	0.043550797	49	0.035215275	1
5	5	604967	24-Dec-21	1	0.041084165	13	0.034120017	1
4		513745	13-Mar-20	1	0.043913829	77	0.034394172	1
1	1	873121	31-Dec-17	0	56	0.038559439	0.034876426	1
5	5	688948	13-Nov-18	1	0.039015461	32	0.035553991	1
5		662293	21-Dec-22	1	0.044799911	46	0.034067597	1
4		937895	30-May-18	0	36	0.037257157	0.033369093	1
4		464379	13-Jun-22	1	0.041390133	7	0.033985632	1
4	4	817900	21-Sep-20	1	0.040423141	22	0.035233458	1
2		815297	21-Jun-23	1	0.04294236	11	0.035273948	1
4	4	535334	18-Dec-17	1	0.036839018	17	0.035316176	1
1		675866	24-Feb-20	1	0.039916611	22	0.034908801	1
6	6	1.00E+06	30-Jun-21	1	0.036134726	10	0.03462651	1

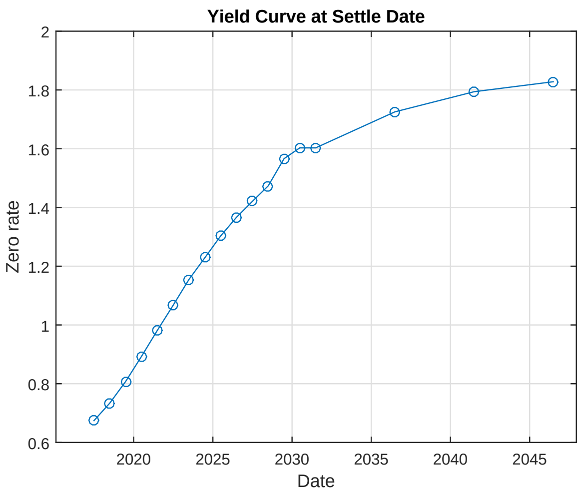

Figure 2.1: The zero rate as of 2016-06-30 from Bloomberg.

The credit spread dynamics over the next 5 years are shown in Table 2.1. The unit of these credit spreads is per basis point. For the 5-Yr swap of our interest, I assume a flat credit spread with 200 basis points over the risk-free rate.

Date	cp1	cp2	cp3	cp4	cp5	cp6
6/30/2017	140	85	115	170	140	200
6/30/2018	185	120	150	205	175	200
6/30/2019	215	170	195	245	210	200
6/30/2020	275	215	240	285	265	200
6/30/2021	340	255	290	320	310	200

2.2 Market Environment

I chose the market date 2016-06-30 and download the swap rates from Bloomberg. The zero rates from the market is shown in Figure 2.1.

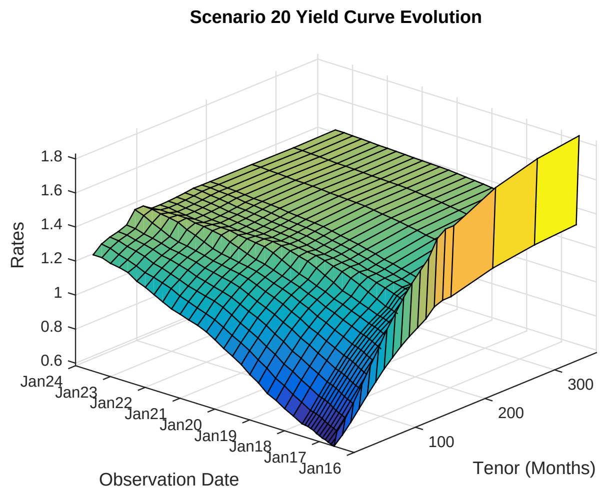

Figure 2.2: Yield curve evolution at scenario 20.

To populate market consistent interest rate scenarios, a Hull-White one factor model is fitted with $a = 0.2$ and $\sigma = 0.015$. Therefore, the model specification is

$$dr = [\theta(t) - ar]dt + \sigma dW(t)$$

where

$$\theta(t) = f_t(0,t) + af_t(0,t) + \frac{\sigma^2}{a}(1 - e^{-2at})$$

For a randomly selected scenario, the yield curve evolution is shown in Figure 2.2.

2.3 CVA Calculation

Using the Hull-White model projected scenarios to price the swap, the mark-to-market portfolio value on the 5-Yr swap can be calculated. Note that for easier comparison and illustration, I make the notional of the swap to 1,000,000. Figure 2.3.

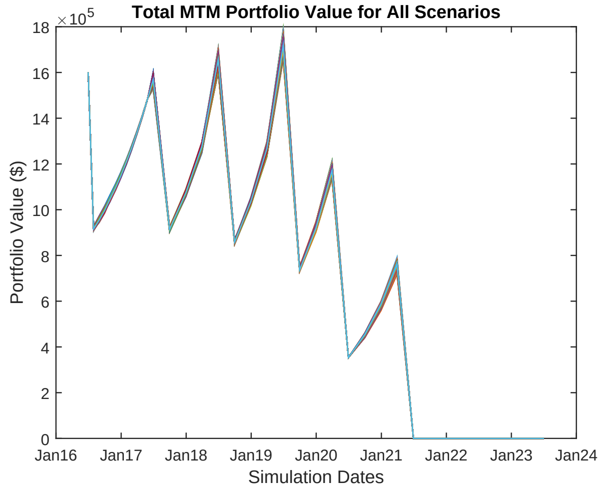

Figure 2.3: Mark-to-market value of the swap over projection periods.

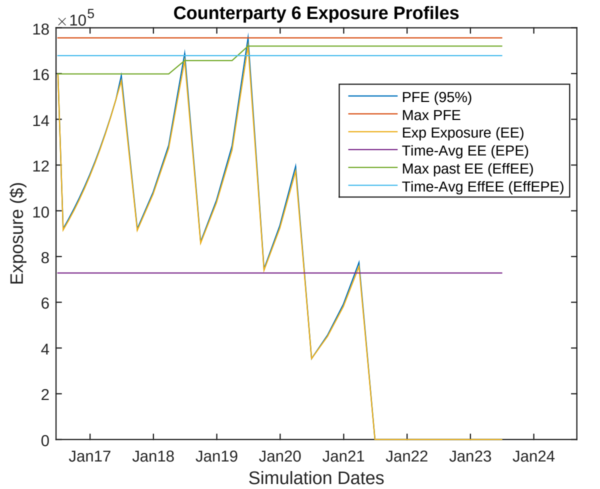

Figure 2.4: Mark-to-market value of the swap over projection periods.

The exposure profile of the swap is plotted in Figure 2.4. The exposure over time, location of maximum exposure and potential future exposure at 95% are calculated and shown in the graph.

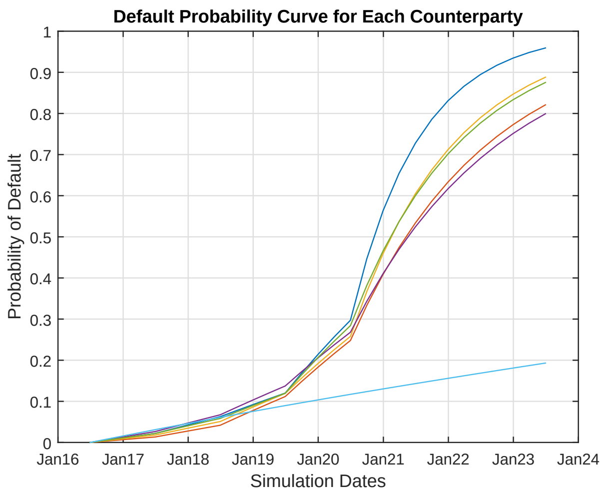

Figure 2.5: Probability probabilities bootstrapped from the credit spreads.

The probability of default for each counterparty is bootstrapped from the credit spreads, and its dynamic over time is plotted in Figure 2.5.

Finally, the CVA is calculated by a finite sum over the valuation dates.

$$CVA = (1 - RR) \sum_{i=2}^{n} EE(T_i)[PD(T_i) - PD(T_{i-1})]$$

The CVA of the swap with 1,000,000 notional and credit spread assumption is 29054.76. The CVA for each counterparty is shown in Figure 2.6.

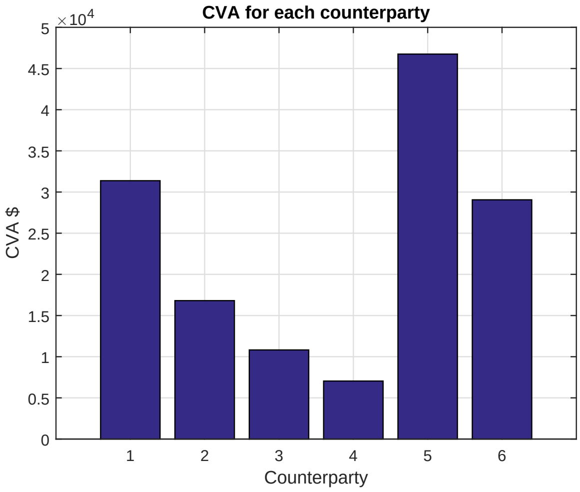

Figure 2.6: The CVA calculation for each counterparty in the swap book.

APPENDICES

APPENDIX A

BLACK'S FORMULA FOR CAPLETS

Denote F_2 as martingale under Q^2. Then

$$df(t;T_1,T_2) = \sigma_2(t)f(t;T_1,T_2)dW(t)$$

where σ_2 is the corresponding instantaneous volatility, and W is one Brownian motion under measure Q^2. The caplet price term $\mathbb{E}^{Q^2}[(f_2(T_1)-X)^+]$ can be computed using Ito's formula

$$\begin{aligned}
d\log(f_2(t)) &= \frac{\partial \log(f_2)}{\partial f_2}df_2 + \frac{1}{2}\frac{\partial^2 \log(f_2)}{\partial f_2^2}df_2df_2 \\
&= \frac{1}{f_2}df_2 - \frac{1}{2f_2^2}df_2df_2 \\
&= \sigma_2(t)dW(t) - \frac{1}{2}\sigma_2^2(t)dt
\end{aligned}$$

Integrate both sides and we yields

$$f_2(T) = f_2(0)\exp\left(\int_0^T \sigma_2(t)dW(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt\right)$$

To calculate the distribution of the Gaussian random variable in the exponent, we have the expectation

$$\mathbb{E}\left[\int_0^T \sigma_2(t)dW(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt\right] = -\frac{1}{2}\int_0^T \sigma_2^2(t)dt$$

and the variance

$$\begin{aligned}\mathbb{V}\left[\int_0^T \sigma_2(t)dW(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt\right] &= \mathbb{V}\left[\int_0^T \sigma_2(t)dW(t)\right] \\ &= \mathbb{E}\left[\left(\int_0^T \sigma_2(t)dW(t)\right)^2\right] - 0^2 \\ &= \int_0^T \sigma_2(t)dt\end{aligned}$$

Recall that

$$f_2(T) = f_2(0)\exp(I(T)) = f_2(0)e^{m+VN(0,1)}$$

Use the classic Black-Scholes framework,

$$\begin{aligned}\mathbb{E}^{Q^2}[(f_2(T_1) - X)^+] &= \mathbb{E}^{Q^2}[(f_2(0)e^{m+VN(0,1)} - X)^+] \\ &= \int_{-\infty}^{+\infty} (f_2(0)e^{m+VN(0,1)} - X)^+ p_{N(0,1)}(y)dy \\ &= \dots \\ &= f_2(0)\Phi(d_1) - X\Phi(d_2) \\ d_{1,2} &= \frac{\log\frac{f_2(0)}{X} \pm \frac{1}{2}\int_0^{T_1}\sigma_2^2(t)dt}{\sqrt{\int_0^{T_1}\sigma_2^2(t)dt}}\end{aligned}$$

When including the inital discount factor $P(0,T_2)$ and the year fraction τ this is the classic Black's formula for the $T_2 - T_1$ caplet. $\square$

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