CQF Module 1.4 Solutions

Throughout this problem sheet, you may assume that W_t is a Brownian Motion (Weiner Process) and dW_t is its increment. $W_0 = 0$.

1. The change in a share price $S(t)$ satisfies
$$dS = A(S,t)\,dW_t + B(S,t)\,dt,$$
for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following stochastic differential equation
$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dW_t.$$
Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.
A function $g(S)$ will satisfy the shorter SDE
$$dg = \left(B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2}\right)dt + A\frac{dg}{dS}dW.$$
For $g(S)$ to have a zero drift but non-zero diffusion, we require the condition
$$B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2} = 0.$$
i.e.
$$\frac{dg}{dS} + \frac{1}{2}\frac{A^2}{B}\frac{d^2 g}{dS^2} = 0$$
We can find a solution to this problem if $\frac{A^2}{B}$ is independent of time.

2. Show that $F(W_t) = \arcsin(2aW_t + \sin F_0)$ is a solution of the stochastic differential equation
$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dW_t,$$
where F_0 and a is a constant.
$F = \arcsin(2aW(t) + \sin F_0)$ implies $\sin F = 2aW(t) + \sin F_0$ hence
$$\frac{dF}{dW} = \frac{2a}{\sqrt{1-(2aW+\sin F_0)^2}} = 2a\left\{1-(2aW+\sin F_0)^2\right\}^{-1/2}$$
$$\frac{d^2F}{dW^2} = \frac{(2a)^2(2aW(t)+\sin F_0)}{\left\{1-(2aW+\sin F_0)^2\right\}^{3/2}}$$
So Itô gives
$$dF = \frac{2a}{\sqrt{1-(2aW+\sin F_0)^2}}dW + \frac{1}{2}\frac{(2a)^2(2aW(t)+\sin F_0)}{\left\{1-(2aW+\sin F_0)^2\right\}^{3/2}}dt$$
We know $\cos^2 F + \sin^2 F = 1 \Longrightarrow \cos F = \sqrt{1-\sin^2 F} = \sqrt{1-(2aW+\sin F_0)^2}$. Some more trig.
$$\sec F = \frac{1}{\cos F} = \frac{1}{\sqrt{1-(2aW+\sin F_0)^2}}$$
and

$$(\tan F)\left(\sec^2 F\right) = \frac{\sin F}{\cos F}\frac{1}{\cos^2 F} = \frac{\sin F}{\cos^3 F} = \frac{2aW + \sin F_0}{\left\{1 - (2aW + \sin F_0)^2\right\}^{3/2}}$$

which gives

$$dF = 2a^2 (\tan F)\left(\sec^2 F\right) dt + 2a (\sec F)\, dW.$$

3. Show that

$$\int_0^t W_t \left(1 - e^{-W_t^2}\right) dW_t = \overline{F}(W_t) + \int_0^t G(W_\tau)\, d\tau$$

where the functions $\overline{F}$ and G should be determined.
We can do this two ways. Perform Itô's lemma and then integrate or use the stochastic integral version as given in the first question. Comparing

$$\int_0^t W(\tau)\left(1 - e^{-W^2(\tau)}\right) dW(\tau) = \overline{F}(W(t)) + \int_0^t G(W(t))\, d\tau$$

with

$$\int_0^t \frac{\partial F}{\partial W} dW(\tau) = F(W(t), t) - F(W(0), 0) + \int_0^t -\left(\frac{\partial F}{\partial \tau} + \tfrac{1}{2}\frac{\partial^2 F}{\partial W^2}\right) d\tau$$

suggests that

$$\frac{\partial F}{\partial W} = W(\tau)\left(1 - e^{-W^2(\tau)}\right)$$

so integrating over $[0, t]$ gives $\overline{F}(W(t), t)$, which we will do by substitution, i.e. put $u = W^2$ which gives

$$F(W(t), t) - F(W(0), 0) = \tfrac{1}{2}W^2(t) + \tfrac{1}{2}e^{-W^2(t)} - \tfrac{1}{2}.$$

Also knowing $\frac{\partial F}{\partial W}$ allows us to easily obtain $\frac{\partial^2 F}{\partial W^2} = 2W^2(t)\, e^{-W^2(t)} - e^{-W^2(t)} + 1$. Hence

$$G(W(t)) = -\tfrac{1}{2}\frac{\partial^2 F}{\partial W^2} = -\tfrac{1}{2}\left(1 - e^{-W^2(t)}\right) - W^2(t)\, e^{-W^2(t)}$$

and we have shown

$$\int_0^t W(\tau)\left(1 - e^{-W^2(\tau)}\right) dW(\tau) = \overline{F}(W(t)) + \int_0^t G(W(t))\, d\tau$$

where

$$\begin{aligned} \overline{F}(W(t), t) &= \tfrac{1}{2}W^2(t) + \tfrac{1}{2}e^{-W^2(t)} - \tfrac{1}{2} \\ G(W(t)) &= -\tfrac{1}{2}\left(1 - e^{-W^2(t)}\right) - W^2(t)\, e^{-W^2(t)}. \end{aligned}$$

4. Begin by writing a 3D Taylor expansion for $F(t, S_t, v_t)$

$$\begin{aligned} & V(t + dt, S_t + dS, r_t + dr) - V(t, S_t, v_t) \\ = \; & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}dv^2 + \frac{\partial^2 V}{\partial v \partial S}dvdS \end{aligned}$$

Since $dW_i^2 \to dt$ in the mean square limit for $i = 1, 2$, we see that

$$dS_t^2 \to v_t S_t^2 dt,$$

$$dv_t^2 \to \eta^2 v dt,$$

Also, since $dW_1 dW2 = \rho dt$, we see that

$$dS_t dv_t \to \rho\eta v_t S_t dt$$

This gives us a *bivariate* version of Itô's Lemma, the SDE for F is given by

$$\begin{aligned}
dV &= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} - \lambda(v_t - \bar{v})\frac{\partial V}{\partial v_t}\right. \\
&\quad \left. + \frac{1}{2} v_t S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\eta^2 \frac{\partial^2 V}{\partial v_t^2} + \rho\eta v_t S_t \frac{\partial^2 V}{\partial v_t \partial S}\right) dt \\
&\quad + \sqrt{v_t} S_t \frac{\partial V}{\partial S} dW_1 + \eta\sqrt{v_t}\frac{\partial V}{\partial v_t} dW_2
\end{aligned}$$

Integrating over $[0, t]$, we get

$$\begin{aligned}
V(t, S_t, v_t) &= v + \int_0^t \left(\frac{\partial V}{\partial \tau} + \mu S_\tau \frac{\partial V}{\partial S} - \lambda(v_\tau - \bar{v})\frac{\partial V}{\partial v_\tau}\right. \\
&\quad \left. + \frac{1}{2} v_\tau S_\tau^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\eta^2 \frac{\partial^2 V}{\partial v_\tau^2} + \rho\eta v_\tau S_\tau \frac{\partial^2 V}{\partial v_\tau \partial S}\right) d\tau \\
&\quad + \int_0^\tau \sqrt{v_\tau} S_\tau \frac{\partial V}{\partial S} dW_1 + \int_0^\tau \eta\sqrt{v_\tau}\frac{\partial V}{\partial v_\tau} dW_2
\end{aligned}$$