CQF Module 1.3 Solutions

Throughout this problem sheet, you may assume that W_t is a Brownian Motion (Weiner Process) and dW_t is its increment; and $W_0 = 0$.

1. Use Itô's lemma to obtain a SDE for each of the following functions

   (a) $y(W_t) = \exp(W_t)$

$$\begin{aligned} dy &= \exp(W_t)\,dX + \exp(W_t)\,\frac{1}{2}dt \text{ or} \\ \frac{df}{y} &= \frac{1}{2}dt + dX \end{aligned}$$

   (b) $g(W_t) = \ln W_t$

$$dg = -\frac{1}{2W_t^2}dt + \frac{1}{W_t}dX$$

   (c) $h(W_t) = \sin W_t + \cos W_t$

$$dh = (\cos W_t - \sin W_t)\,dW_t - \frac{1}{2}(\sin W_t + \cos W_t)\,dt$$

   (d) $f(W_t) = a^{W_t}$, where the constant $a > 1$

$$\begin{aligned} f(W_t) &= a^{W_t} \Rightarrow \ln f = W_t \ln a \Rightarrow \frac{1}{f}f'(W_t) = \ln a \Rightarrow f'(W_t) = (\ln a)\, f \\ \text{therefore } f'(W_t) &= (\ln a)\, a^{\,W_t} \text{ and hence } f''(W_t) = (\ln a)^2\, a^{\,W_t} \\ df &= (\ln a)\, a^{\,W_t}\, dX + \frac{1}{2}(\ln a)^2\, a^{\,W_t} dt \\ \text{or } \frac{df}{f} &= \frac{1}{2}(\ln a)^2\, dt + (\ln a)\; dW_t \end{aligned}$$

   (e) $f(W_t) = (W_t)^n$

$$df = nW_t^{n-1}dX + \frac{1}{2}n(n-1)W_t^{n-2}dt$$

2. Using the formula below for stochastic integrals, for a function $F(W_t, t)$,

$$\int_0^t \frac{\partial F}{\partial W_t} dW_t = F(W_t, t) - F(W_0, 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W_\tau^2} \right) d\tau$$

show that we can write

a. $\int_0^t W_\tau^3 \, dW_\tau = \frac{1}{4} W^4(t) - \frac{3}{2} \int_0^t W_\tau^2 d\tau$; here we have ordinary derivatives and no $\frac{\partial F}{\partial t}$

$$\frac{dF}{dW} = W^3(t) \longrightarrow F(W(t)) = \frac{1}{4} W^4(t) \longrightarrow \frac{d^2 F}{dX^2} = 3W^2(t)$$

b. $\int_0^t \tau dW_\tau = tW_t - \int_0^t W_\tau d\tau$

$$\frac{\partial F}{\partial W} = t \longrightarrow F(W(t), t) = tW(t) \Rightarrow \frac{\partial^2 F}{\partial W^2} = 0 \text{ and } \frac{\partial F}{\partial t} = W(t)$$

substituting all of these terms in to the formula

$$\begin{aligned} \int_0^t \tau dW(\tau) &= tW(t) - 0 - \int_0^t \left(W(\tau) + \frac{1}{2}.0 \right) d\tau \\ &= tW(t) - \int_0^t W(\tau) \, d\tau \end{aligned}$$

c. $\int_0^t (W_\tau + \tau) \, dW_\tau = \frac{1}{2} W_t^2 + tW_t - \int_0^t \left(W_\tau + \frac{1}{2} \right) d\tau$

$$\frac{\partial F}{\partial W} = W(t) + t \longrightarrow F(W(t)) = \frac{1}{2} W^2(t) + tW(t) \longrightarrow \frac{\partial F}{\partial t} = W(t)$$

and $\frac{\partial^2 F}{\partial W^2} = 1$, therefore

$$\int_0^t (W(\tau) + \tau) \, dW(\tau) = \frac{1}{2} W^2(t) + tW(t) - \int_0^t \left(W(\tau) + \frac{1}{2} \right) d\tau$$