CQF Module 1.2 Exercises

1. Find the general solution of the differential equation

$$x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}.$$

2. By solving the initial value problem

$$\frac{dy}{dx} - 2xy = 2, \; y(0) = 1$$

show that the solution can be written as

$$y(x) = e^{x^2}\left(1 + 2\int_0^x e^{-t^2}dt\right).$$

3. The integral on the right hand side of the last solution cannot be simplified any further if we wish this to remain as a closed form solution. Note the following very important non-elementary integrals, namely the *error function* and *complimentary error function* in turn,

$$\boxed{\operatorname{erf}(x) = \tfrac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds}$$

$$\boxed{\operatorname{erf}c(x) = \tfrac{2}{\sqrt{\pi}}\int_x^\infty e^{-s^2}ds}$$

Write the solution of the last problem in terms of $\operatorname{erf}(x)$. Verify that

$$\operatorname{erf}(x) + \operatorname{erf}c(x) = 1.$$

4. Using a binomial (2 step symmetric) random walk where the probability of an up move or down move is $\frac{1}{2}$, derive both the forward and backward Kolmogorov equations in turn, given by

$$\begin{aligned} \frac{\partial p}{\partial t'} &= c^2\frac{\partial^2 p}{\partial y'^2} \\ \frac{\partial p}{\partial t} + c^2\frac{\partial^2 p}{\partial y^2} &= 0 \end{aligned}$$

for the transition density function $p(y,t\,;y',t')$. The states (y,t) are past /current while (y',t') refer to future ones.
By simple substitution show that

$$\frac{1}{2c\sqrt{\pi(t'-t)}}\exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right)$$

<u>satisfies</u> the backward Kolmogorov equation.