CQF Exercises 4.3 Calibration

1. Substitute the fitted function for $A(t\ ;T)$, using the Ho & Lee model, back into the solution of the bond pricing equation for a zero-coupon bond,

$$Z(r,\ t\ ;T) = \exp\left(A(t\ ;T) - r(T-t)\right).$$

What do you notice?

Solution :
With a Ho & Lee model, the form of the fitted function for $A(t\ ;T)$ *is*

$$A(t\ ;T) = \log\left(\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}\right) - (T-t)\frac{\partial}{\partial t}\log\left(Z_M(t^*;\ t)\right) - \frac{1}{2}c^2(t-t^*)(T-t)^2.$$

Then

$$\begin{aligned} Z(t\ ;T) &= e^{\log\left(\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}\right)-(T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;\ t))-\frac{1}{2}c^2(t-t^*)(T-t)^2-r(T-t)} \\ &= \frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}e^{-(T-t)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;\ t))+\frac{1}{2}c^2(t-t^*)(T-t)+r\right)}. \end{aligned}$$

We note that that when $t = t^*$

$$Z(t^*\ ;T) = \frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}e^{-(T-t^*)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;\ t))+\frac{1}{2}c^2(t^*-t^*)(T-t^*)+r\right)} = Z_M(t^*;\ T).$$

2. Differentiate Equation (2) on page 19 of the lecture notes, twice to solve for the value of $\eta^*(t)$. What is the value of a zero-coupon bond with a fitted Vasicek model for the interest rate?

Solution :
We have

$$\begin{aligned} &-\int_{t^*}^{T}\eta^*(s)\,B(s\ ;T)\ ds + \frac{c^2}{2\gamma^2}\left((T-t^*) + \frac{2}{\gamma}e^{-\gamma(T-t^*)} - \frac{1}{2\gamma}e^{-2\gamma(T-t^*)} - \frac{3}{2\gamma}\right) \\ = &\ \log\left(Z_M(t^*\ ;T)\right) + r^*B(t^*\ ;T). \end{aligned}$$

Differentiating with respect to T,

$$\begin{aligned} &-\int_{t^*}^{T}\eta^*(s)\frac{\partial}{\partial T}B(s\ ;T)\ ds - \eta^*(T)\,B(T\ ;T) + \frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right) \\ = &\ \frac{\partial}{\partial T}\log\left(Z_M(t^*\ ;T)\right) + r^*\frac{\partial}{\partial T}B(t^*\ ;T). \end{aligned}$$

Now

$$B(t\ ;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) \quad \text{so} \quad B(T\ ;T) = 0,$$

and

$$\frac{\partial}{\partial T}B(t\ ;T) = e^{-\gamma(T-t)}.$$

Substituting back into the PDE

$$
\begin{aligned}
& -\int_{t^*}^{T} \eta^*(s)\, e^{-\gamma(T-s)}\ ds + \frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right) \\
= \quad & \frac{\partial}{\partial T}\log\left(Z_M\left(t^*\ ;T\right)\right) + r^* e^{-\gamma(T-t^*)}.
\end{aligned}
$$

Differentiating again with respect to T,

$$
\begin{aligned}
& -\eta^*(T) + \gamma\int_{t^*}^{T} \eta^*(s)\, e^{-\gamma(T-s)}\ ds + \frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)} - 2\gamma e^{-2\gamma(T-t^*)}\right) \\
= \quad & \frac{\partial^2}{\partial T^2}\log\left(Z_M\left(t^*\ ;T\right)\right) - \gamma r^* e^{-\gamma(T-t^*)}.
\end{aligned}
$$

Substituting for the integral from the previous equation, we find

$$
\begin{aligned}
& -\eta^*(T) + \gamma\left(\frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right) - \frac{\partial}{\partial T}\log\left(Z_M\left(t^*\ ;T\right)\right) - r^* e^{-\gamma(T-t^*)}\right) \\
& + \frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)} - 2\gamma e^{-2\gamma(T-t^*)}\right) \\
= \quad & \frac{\partial^2}{\partial T^2}\log\left(Z_M\left(t^*\ ;T\right)\right) - \gamma r^* e^{-\gamma(T-t^*)}.
\end{aligned}
$$

This simplifies to

$$
\eta^*(T) = -\frac{\partial^2}{\partial T^2}\log\left(Z_M\left(t^*\ ;T\right)\right) + \frac{c^2}{2\gamma} - \gamma\frac{\partial}{\partial T}\log\left(Z_M\left(t^*\ ;T\right)\right) - \frac{c^2}{2\gamma}e^{-2\gamma(T-t^*)},
$$

and

$$
\eta^*(t) = -\frac{\partial^2}{\partial t^2}\log\left(Z_M\left(t^*\ ;t\right)\right) - \gamma\frac{\partial}{\partial t}\log\left(Z_M\left(t^*\ ;t\right)\right) + \frac{c^2}{2\gamma}\left(1 - e^{-2\gamma(t-t^*)}\right).
$$

We then have

$$
A(t\ ;T) = -\int_{t}^{T} \eta^*(s)\, B(s\ ;T)\ ds + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right)
$$

and substituting for η^* and integrating, we find

$$
= \log\left(\frac{Z_M\left(t^*;\ T\right)}{Z_M\left(t^*;\ t\right)}\right) - B(t\ ;T)\frac{\partial}{\partial t}\log\left(Z_M\left(t^*;\ t\right)\right) - \frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\right)\left(e^{2\gamma(t-t^*)} - 1\right).
$$

We know the value of a zero-coupon bond is

$$
Z(r,\ t\ ;T) = \exp\left(A(t\ ;T) - rB(t\ ;T)\right),
$$

with $A(t\ ;T)$ given by the above, and

$$
B(t\ ;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right).
$$