Exercises

CQF

1. Use Itô's formula to determine whether the following are martingales:
   (i) $Y(t) = e^{1/2t} \cos X(t)$;
   (ii) $Y(t) = e^{\alpha t} \sin X(t)$ for some constant α with $0 < \alpha < 1$. Does the answer depend on the value of α?
   (iii) $Y(t) = (X(t) + t) \exp\left\{-\frac{1}{2}t - X(t)\right\}$.

2. **Moments of the Brownian Motion X(t)** - Consider the function $m_n(t)$ defined as

$$m_n(t) = \mathbf{E}[X^n(t)], \qquad n = 1, 2, \ldots \tag{1}$$

where $X(t)$ is a standard Brownian motion.

Applying Itô's formula, show that:

$$m_n(t) = \frac{1}{2}n(n-1)\int_0^t m_{n-2}(s)ds \tag{2}$$

for $n = 2, 3, \ldots$

Deduce from (2) that

$$m_4(t) = 3t^2 \tag{3}$$

compute $m_6(t)$.

3. Let $X_n, n = 1, \ldots$ be i.i.d random variables where $P(X_n = 1) = p$ and $P(X_n = -1) = 1 - p$. You can think of X_n as being the nth coin toss in a sequence. Let $S_n, n = 1, \ldots$ be the associated random walk, defined as

$$S_n = X_1 + X_2 + \ldots + X_n \tag{4}$$

S_n can be viewed as the P&L of the entire coin toss game. We also introduce the filtration $\mathcal{F}_n$ generated by the X_n and such that X_n is $\mathcal{F}_n$-adapted.

Find conditions under which the random walk is (a) a martingale, (b) a submartingale (c) a supermartingale.

4. Let $Y_t = X_t^4$ where X_t is a Brownian motion. Using Itô's lemma, express the SDE for Y_t. Then, deduce the stochastic integral for Y_t over $[0, T]$. Finally, deduce from the stochastic integral an expression for $\mathbf{E}[Y_t]$.

5. **Discrete Time Martingale**: Let $Y_1, \ldots, Y_n$ be a sequence of independent random variables such that $\mathbf{E}[Y_i] = 0$ for $i = 1, \ldots, n$. Let $\mathcal{F}_n$ be the filtration generated by the sequence $Y_1, \ldots, Y_n$. Consider the random variable $S_n = \sum_{i=1}^{n} Y_i$. Prove that S_n is a martingale for all n.

   Reminder - *proving that a process S_n is a martingale involves proving that $\mathbf{E}[|S_n|] < \infty$ and that $\mathbf{E}[S_{n+1}|\mathcal{F}_n] = S_n$*