Arbitrage free SABR

Patrick S. Hagan, Mathematical Institute,Oxford University

Deep Kumar, AVM, L.P., Boca Raton, FL

Andrew L. Lesniewski, Dept. of Mathematics, Baruch College

Diana E. Woodward, Gorilla Science, London, UK

pathagan1954@yahoo.com

SABR model

• SABR models developed to manage skew/smile risk:

$$d\tilde{F} = \tilde{\alpha} C(\tilde{F}) dW_1,$$
$$d\tilde{\alpha} = \nu \tilde{\alpha} dW_2,$$

with

$$dW_1 dW_2 = \rho dt$$

• Asymptotic analysis yields approximate formulas for the implied normal volatility

– several variations

– of all the $O(\varepsilon^2)$-accurate formulas, our favorite is

$$\sigma_N(K) = \frac{\alpha(f-K)}{\int_K^f \frac{df'}{C(f')}} \cdot \left(\frac{\zeta}{x(\zeta)}\right) \cdot \left\{1 + \left[g\alpha^2 + \tfrac{1}{4}\rho\nu\alpha\frac{C(f)-C(K)}{f-K} + \frac{2-3\nu^2}{24}\right]\tau_{ex} + \cdots\right\}$$

with

$$\zeta = \frac{\nu}{\alpha}\int_K^f \frac{df'}{C(f')}, \qquad x(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2} - \rho + \zeta}{1-\rho}\right)$$

$$g = \log\left(\frac{1}{f-K}\int_K^f \frac{\sqrt{C(f)C(K)}}{C(f')}df'\right) \Big/ \left(\int_K^f \frac{df'}{C(f')}\right)^2$$

– not the simplest, but seems to be the most robust

CEV Backbone

- Most common case is $C(F) = F^{\beta}$

– implied normal vol:

$$\sigma_N(K) = \frac{\alpha(1-\beta)(f-K)}{f^{1-\beta} - K^{1-\beta}} \bullet \left(\frac{\zeta}{x(\zeta)}\right)$$
$$\bullet \left\{1 + \left[g\alpha^2 + \tfrac{1}{4}\rho\nu\alpha\frac{f^{\beta} - K^{\beta}}{f-K} + \frac{2-3\nu^2}{24}\right]\tau_{ex} + \cdots\right\}$$

with

$$\zeta = \frac{\nu}{\alpha}\frac{f^{1-\beta} - K^{1-\beta}}{1-\beta}, \qquad x(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2} - \rho + \zeta}{1-\rho}\right)$$

$$g = \frac{(1-\beta)^2}{(f^{1-\beta} - K^{1-\beta})^2}\log\left((fK)^{\beta/2}\frac{f^{1-\beta} - K^{1-\beta}}{(1-\beta))(f-K)}df'\right)$$

Arbitrage in the SABR model

- Explicit implied vols $\sigma_N(K)$ are usually treated as *exact*

– don't view $\sigma_N(K)$ as an *approximate* solution to the SABR model

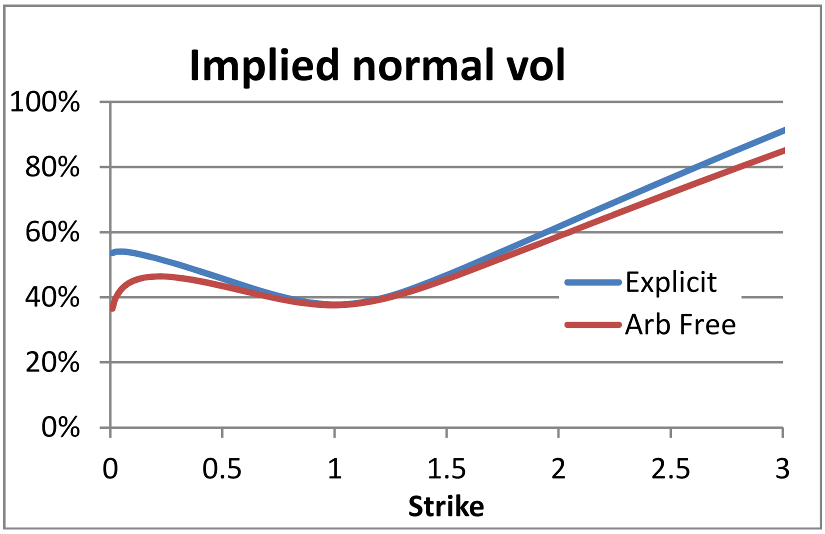
– view $\sigma_N(K)$ as the *exact solution* to some *other model* which is approximated by the SABR model

– For "other model" to be arbitrage free, need:

put-call parity (*automatic from using implied vols*)
option prices must imply positive probability densities:

$$\frac{\partial^2}{\partial K^2} V = \frac{\partial^2}{\partial K^2}\left(\int_K^\infty (F-K)p(\tau_{ex},F)dF\right) = p(\tau_{ex},K) \geq 0 \qquad \text{for all } K$$

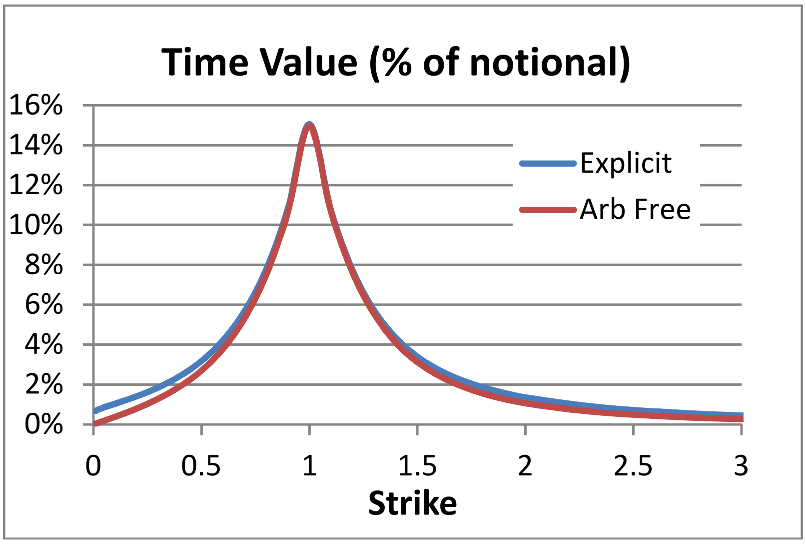
- Can be violated for low strikes, even for nice smiles:



$$\alpha = 35\%,\ \beta = 25\%,\ \rho = -10\%,\ \nu = 100\%$$

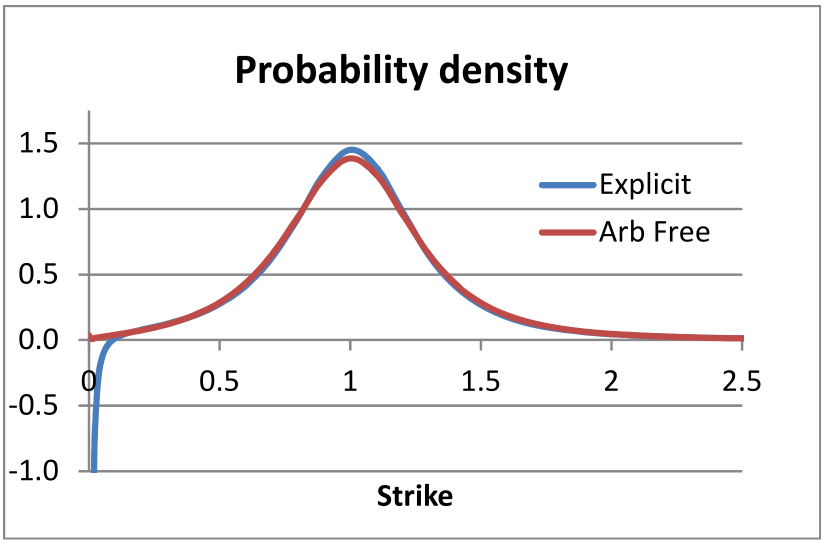
Implied probability density

- Both $\sigma_N(K)$ lead to nearly identical prices



$$\alpha = 35\%,\ \beta = 25\%,\ \rho = -10\%,\ \nu = 100\%$$

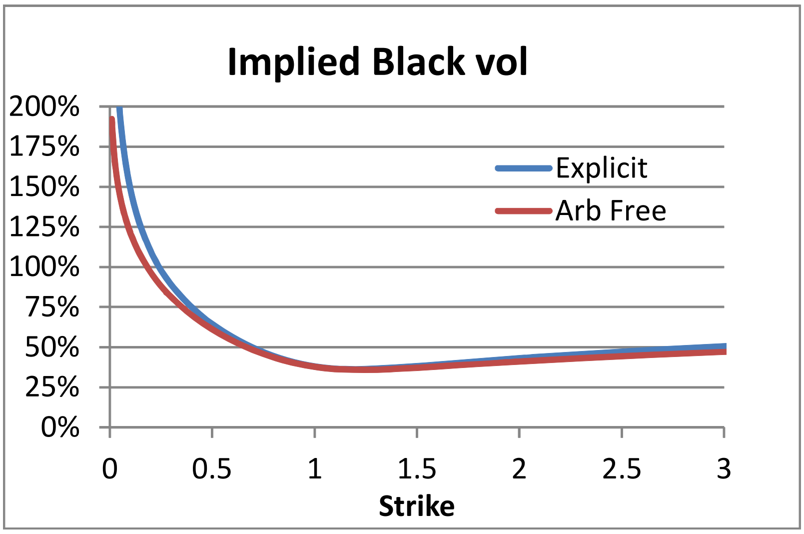
- Yet one leads to negative probability densities, and is not arbitrage free



$$\alpha = 35\%,\ \beta = 25\%,\ \rho = -10\%,\ \nu = 100\%$$

Black vols

- Using log normal vols doesn't help discern which smiles are arbitrage free



$$\alpha = 35\%,\ \beta = 25\%,\ \rho = -10\%,\ \nu = 100\%$$

Arbitrage free approach

- SABR model:

$$d\tilde{F} = \varepsilon\tilde{\alpha}C(\tilde{F})dW_1,$$
$$d\tilde{\alpha} = \varepsilon\nu\tilde{\alpha}dW_2,$$
$$dW_1 dW_2 = \rho dt$$

- Probability density & moments

$$p(T,F,\alpha)dFd\alpha = \mathrm{Prob}\{F < \tilde{F}(T) < F + dF, \alpha < \tilde{\alpha}(T) < \alpha + d\alpha\}$$

$$Q^{(k)}(T,F)dF = \int_0^\infty \alpha^k p(T,F,\alpha)d\alpha$$

- Fökker-Planck equation:

$$p_T = \tfrac{1}{2}\varepsilon^2[\alpha^2 C^2(F)p]_{FF} + \varepsilon^2\rho\nu[\alpha^2 C(F)p]_{F\alpha} + \tfrac{1}{2}\varepsilon^2[\alpha^2 Qp]_{\alpha\alpha}$$

– integrate over all α,

$$\int_0^\infty [\alpha^2 C(F)p]_{F\alpha}d\alpha = [\alpha^2 C(F)p]_F \Big|_0^\infty = 0,$$

$$\int_0^\infty [\alpha^2 p]_{\alpha\alpha}d\alpha = [\alpha^2 p]_\alpha \Big|_0^\infty = 0$$

- Yields conservation law:

$$Q_T^{(0)} = \tfrac{1}{2}\varepsilon^2[C^2(F)Q^{(2)}]_{FF}$$

Effective forward equation

• Conservation law:

$$Q_T^{(0)} = \tfrac{1}{2}\varepsilon^2[C^2(F)Q^{(2)}]_{FF}$$

• Use asymptotic methods to analyze backwards equation for $Q^{(0)}$ and $Q^{(2)}$. Obtain:

$$Q^{(2)} = (\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}Q^{(0)}\{1 + O(\varepsilon^3)\}$$

where

$$z(F) = \int_f^F \frac{df'}{C(f')}, \qquad \Gamma = \frac{C(F) - C(f)}{F - f}$$

• Marginal density

$$Q^{(0)}(T,F)dF = \mathrm{Prob}\{F < \tilde{F}(T) < F + dF\}$$

– satisfies *effective forward equation*

$$Q_T^{(0)} = \tfrac{1}{2}\Big[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\Big]_{FF}$$

• Reduction accurate through $O(\varepsilon^2)$; same as original SABR analysis

• *No corresponsing 1-d local volatility model*

Boundary conditions

- Numerically solve the effective forward equation:

$$Q_T^{(0)} = \tfrac{1}{2}\Big[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\Big]_{FF}$$

over the domain $0 < F < F_{\max}$.

– initial condition:

$$Q^{(0)}(0,F) = \delta(F-f) \qquad \text{at } T = 0.$$

- *Absorbing* boundary conditions are *required* for $\tilde{F}(T)$ to be a Martingale:

$$Q^{(0)} = 0 \qquad \text{at } F = 0$$
$$Q^{(0)} = 0 \qquad \text{at } F = F_{\max}$$

- Conservation requires:

$$Q(T,F) = \begin{cases} Q^L(T)\delta(F) & \text{at } F = 0 \\ Q^{(0)}(T,F) & \text{for } 0 < F < F_{\max} \\ Q^R(T)\delta(F-F_{\max}) & \text{at } F = F_{\max} \end{cases}$$

with

$$\frac{dQ^L}{dT} = \tfrac{1}{2}\Big[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\Big]_F\Big|^{F=0^+}$$

$$\frac{dQ^R}{dT} = -\tfrac{1}{2}\Big[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\Big]_F\Big|^{F=F_{\max}^-}$$

Option prices

- Numerically solve the PDE

$$Q_T^{(0)} = \tfrac{1}{2}\left[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\right]_{FF},$$

over $0 < F < F_{\max}$, with

$$Q^{(0)} = 0 \quad \text{at } F = 0, \qquad Q^{(0)} = 0 \quad \text{at } F = F_{\max}$$

and

$$Q^{(0)}(0,F) = \delta(F - f) \qquad \text{at } T = 0$$

– δ-functions at $F = 0$ and $F = F_{\max}$:

$$\frac{dQ^L}{dT} = \tfrac{1}{2}\left[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\right]_F\Big|^{F=0^+}$$

$$\frac{dQ^R}{dT} = -\tfrac{1}{2}\left[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\right]_F\Big|^{F=F_{\max}^-}$$

- Option prices:

$$V_{call}(\tau_{ex},K) = \int_K^{F_{\max}}(F-K)Q^{(0)}(\tau_{ex},F)dF + (F_{\max}-K)Q^R(\tau_{ex})$$

$$V_{put}(\tau_{ex},K) = \int_0^K (K-F)Q^{(0)}(\tau_{ex},F)dF + KQ^L(\tau_{ex})$$

– reduced problem has 1 space dimension

numerical solution is essentially instantaneous!

– solving the PDE for $0 < T < \tau_{ex}$ yields option prices for all strikes K at τ_{ex}

Numerical method

$$Q_T^{(0)} = \tfrac{1}{2}\Big[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\Big]_{FF}$$

$$Q^{(0)} = 0 \quad \text{at } F = 0, \qquad Q^{(0)} = 0 \quad \text{at } F = F_{\max}$$

$$Q^{(0)}(0,F) = \delta(F - f) \qquad \text{at } T = 0$$

$$\frac{dQ^L}{dT} = \tfrac{1}{2}\Big[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\Big]_F\Big|^{F=0^+}$$

$$\frac{dQ^R}{dT} = -\tfrac{1}{2}\Big[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2)e^{\varepsilon^2\rho\nu\alpha\Gamma T}C^2(F)Q^{(0)}\Big]_F\Big|^{F=F_{\max}^-}$$

- Use moment preserving Crank-Nicholson scheme

– guarantees probability is conserved exactly, and that $\tilde{F}(T)$ is exactly a Martingale:

$$Q^L(T) + \int_0^{F_{\max}} Q^{(0)}(T,F)dF + Q^R(T) = 1,$$

$$\int_0^{\infty} FQ^{(0)}(T,F)dF + F_{\max}Q^R(T) = f.$$

– Maximum principle guarantees that

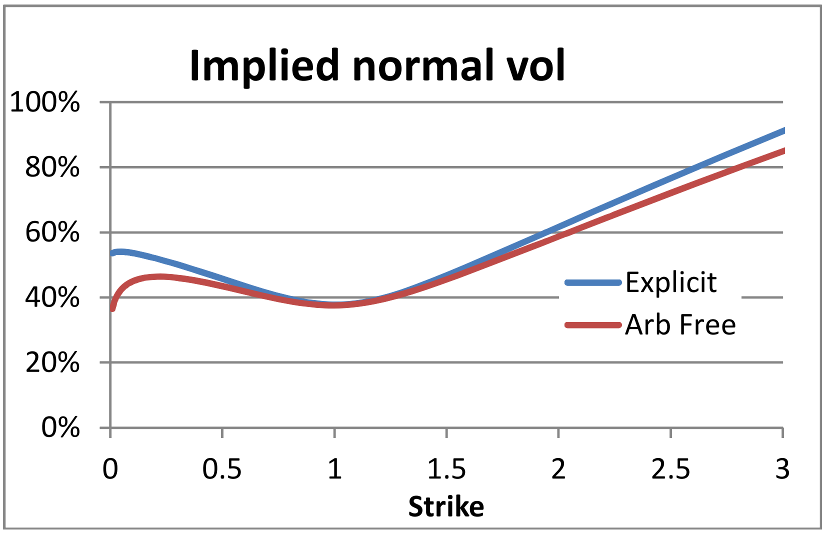
$$Q^{(0)}(T,F) \ge 0 \qquad \text{for all } 0 < F < F_{\max},\ \text{all } T > 0,$$

$$Q^L(T) \ge 0, \qquad Q^L(T) \ge 0 \qquad \text{for all } T > 0$$

– Numerical solution is an exactly arbitrage free model!

Boundary layer

- Arbitrage free approach yields nearly the same values as the explicit SABR formulas $\sigma_N(K)$, except for low strikes and forwards



$$\alpha = 35\%, \ \beta = 25\%, \ \rho = -10\%, \ \nu = 100\%$$

- Effective forward equation:

$$Q_T^{(0)} = \tfrac{1}{2}\left[(\alpha^2 + 2\varepsilon\rho\nu\alpha z + \varepsilon^2\nu^2 z^2) e^{\varepsilon^2 \rho\nu\alpha\Gamma T} C^2(F) Q^{(0)} \right]_{FF}$$

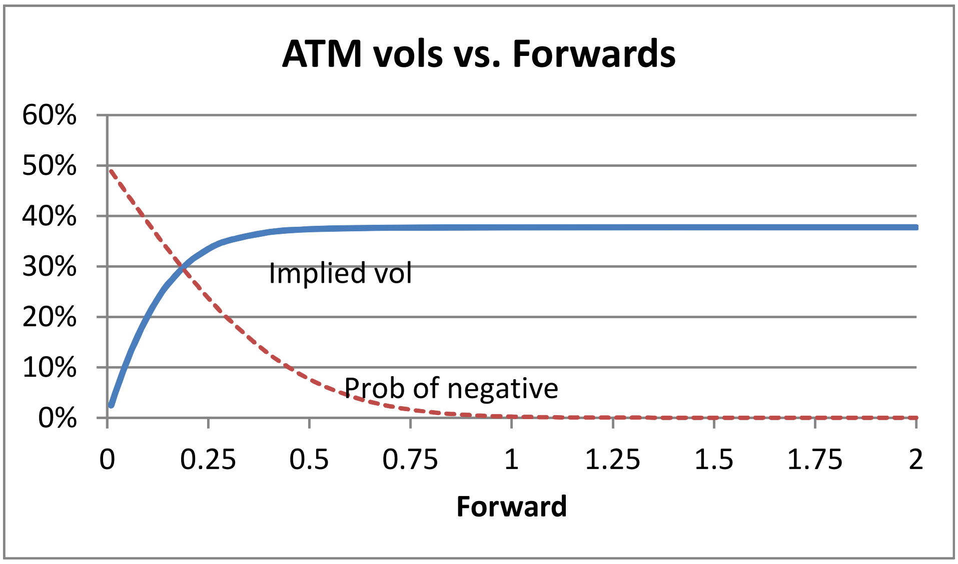
– using asymptotic methods to solve the effective forward equation leads to the *same explicit formulas* for $\sigma_N(K)$ as in the original analysis, unless the forward or strike is near zero

- Explicit formulas for $\sigma_N(K)$ do not hold in a boundary layer around zero

– boundary layer occurs where a significant fraction of the paths get absorbed at 0 before expiry

Boundary layer effects

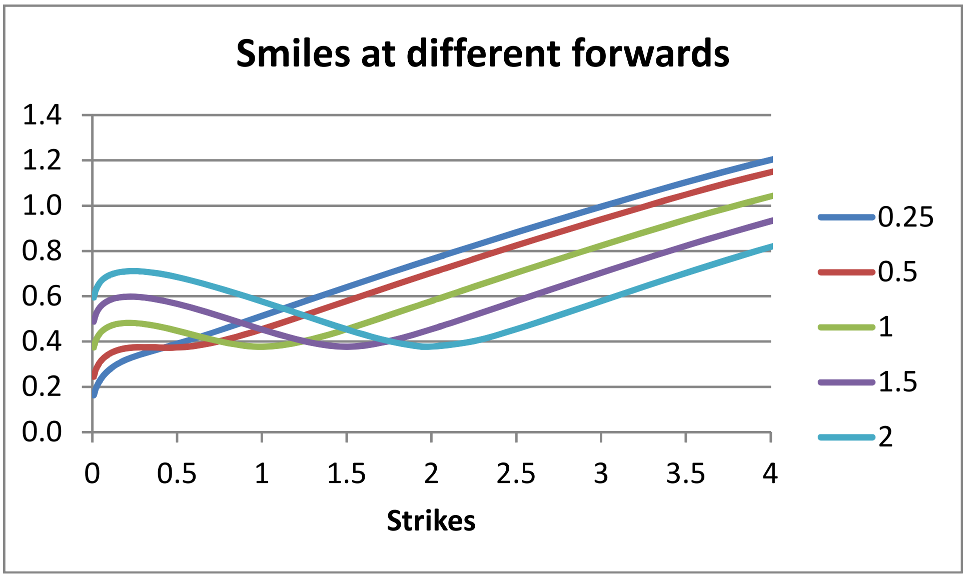
• At the money vols decrease linearly for small rates



$$\alpha = 35\%,\ \beta = 0\%,\ \rho = 0\%,\ \nu = 100\%$$

• Knee is often incorrectly ascribed to market switching from normal to log normal behavior in ultra-low rate environments

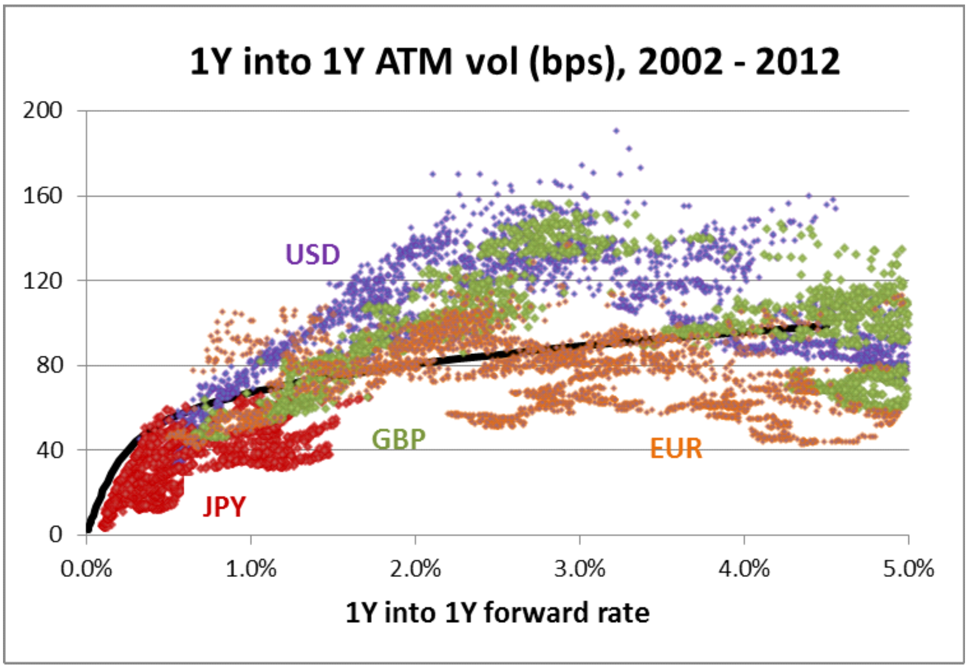
– this leads to mispricing *high* strike options in low rate environments



$$\alpha = 35\%,\ \beta = 0\%,\ \rho = 0\%,\ \nu = 100\%$$

Historical market data

- Arbitrage free SABR closely matches market data

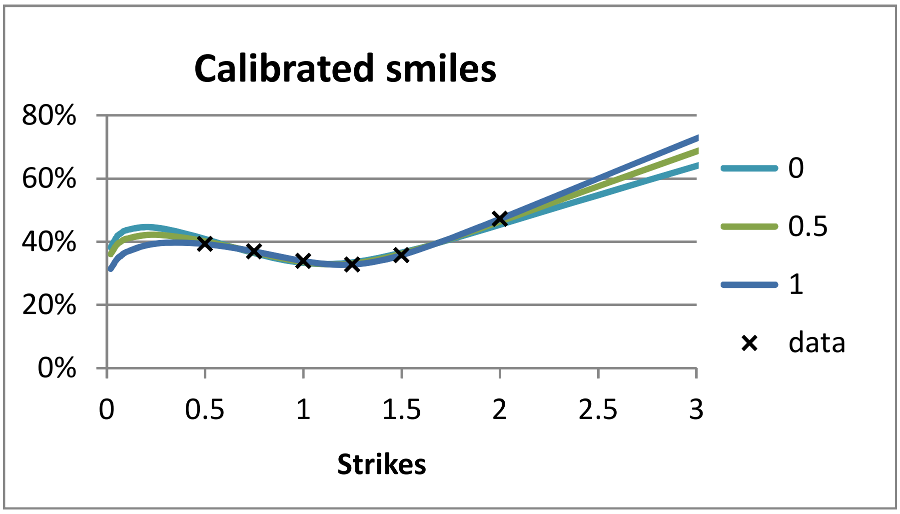


Historic swaption vols for 2002 through 2012

Calibrating the SABR model

- α controls the at-the-money vol, ν controls the smile, but both ρ and β control the skew

- SABR model calibrated to same market data with β chosen to be 0, $\frac{1}{2}$, and 1



SABR model calibrated with β of 0, $\frac{1}{2}$, and 1.

– calibrated parameters:

α	31.8%	32.9%	35.1%
β	0	0.5	1
ρ	-18.3%	-45.5%	-64.4%
ν	0.777	0.867	0.985

– although tails are somewhat different, all three sets of parameters fit the actual market data well within market noise

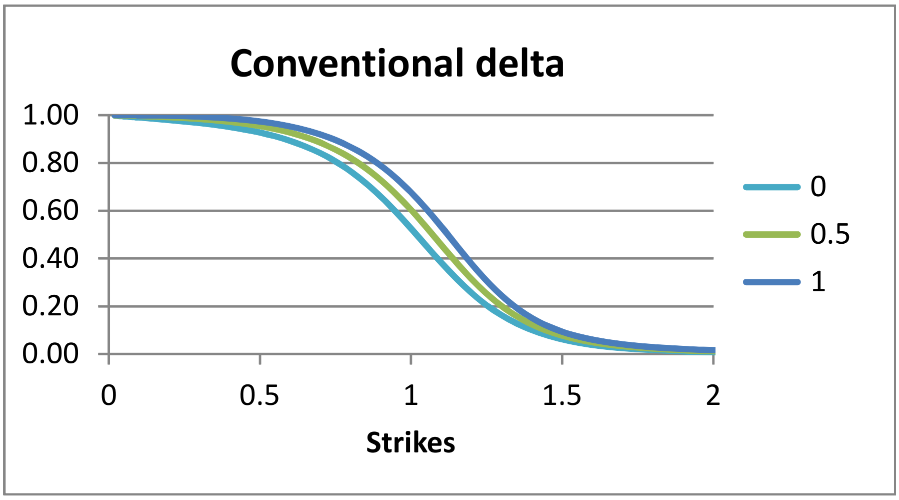
– ρ can largely compensate for β

Conventional hedging

• Conventional delta, $\partial V/\partial F$, based on the scenario

$$\tilde{F} \to \tilde{F} + \Delta F, \qquad \tilde{\alpha} \to \tilde{\alpha}$$

• conventional delta for the same three sets of SABR parameters



$\partial V/\partial F$ against the strike K for β of 0, $\frac{1}{2}$, and 1.

• Even though all three sets of parameters closely fit the market smile, they lead to different conventional hedges, even near the money

– choosing the incorrect beta can lead to good fits of the smile, but relatively poor delta hedges

Alternative delta hedges

$$d\tilde{F} = \tilde{\alpha} C(\tilde{F}) dW_1,$$
$$d\tilde{\alpha} = v\tilde{\alpha} dW_2,$$
$$dW_1 dW_2 = \rho dt.$$

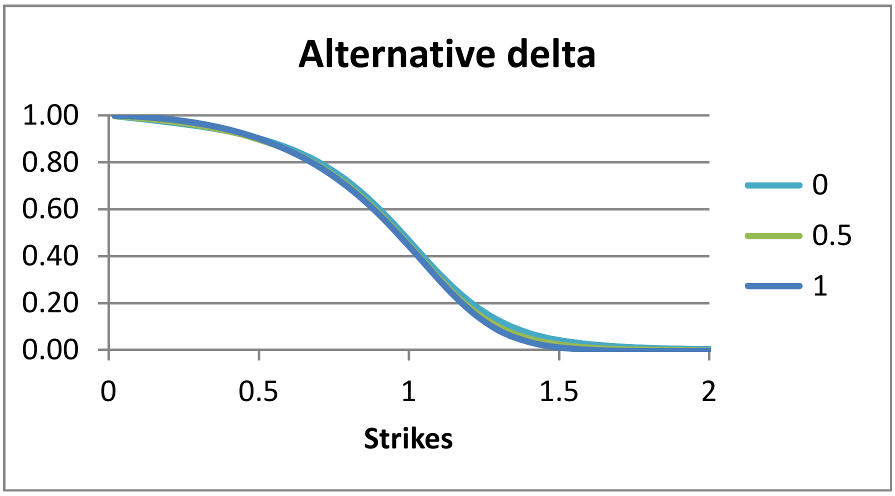
- When $\tilde{F}$ changes, $\tilde{\alpha}$ should also change, at least on average

$$d\tilde{\alpha} = v\tilde{\alpha}\left\{\rho dW_1 + \sqrt{1-\rho^2}\, dW_\perp\right\} = \left\{\rho v \frac{d\tilde{F}}{C(\tilde{F})} + \sqrt{1-\rho^2}\, v\tilde{\alpha} dW_\perp\right\}$$

- Alternative delta based on scenario:

$$\tilde{F} \to \tilde{F} + \Delta F, \quad \tilde{\alpha} \to \tilde{\alpha} + \rho v \frac{\Delta F}{C(\tilde{F})}$$

– alternative delta for the same three sets of parameters



Alternative delta, $\partial V/\partial f + [\rho v/C(f)]\partial V/\partial \alpha$

- Alternative delta is nearly independent of β. It depends mainly on the actual market skew/smile, and not on how the smile is parameterized

– alternative deltas are believed to provide much better hedges