CQF Module 4 Exercise Solution

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1 The zero coupon bonds satisfy

$$\frac{\partial Z}{\partial t} + \frac{1}{2}\omega(r,t)^2\frac{\partial^2 Z}{\partial r^2} + [u(r,t) - \lambda(r,t)\omega(r,t)]\frac{\partial Z}{\partial r} - rZ = 0 \tag{1}$$

As time being close to the maturity ($T - t \to 0$), expand the zero coupon bond with the form

$$Z \sim 1 + a(r)(T-t) + b(r)(T-t)^2 + \dots \tag{2}$$

substitute Z in Equation 2 into Equation 1, and equating powers of $(T - t)$ yields

$$Z(r,t;T) \sim 1 - r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda\omega) + \dots \qquad \text{as } t \to T$$

So the shape of the yield curve near the short end becomes

$$-\frac{\ln Z}{T-t} \sim r + \frac{1}{2}(u - \lambda\omega)(T-t) + \dots \qquad \text{as } t \to T$$

In Vasicek model, the risk-neutral spot rate take the form

$$dr = (\eta - \gamma r)dt + \sqrt{\beta}dX$$

and we have

$$u - \lambda\omega = \eta - \gamma r$$

Therefore, for the Vasicek model with one month Libor, we find

$$r_L \sim r + \frac{1}{2}(\eta - \gamma r)\frac{1}{12}$$

Finally, a floorlet cashflow has approximate value

$$\max(r_f - r_L, 0) \sim \max\left(r_f - r - \frac{1}{24}(\eta - \gamma r), 0\right)$$

as claimed.

2 The Black-Derman & Toy short-rate model is

$$d(\log r) = \left(\theta(t) + \frac{d(\log\sigma(t))}{dt}\log r\right)dt + \sigma(t)dW \tag{3}$$

Let $f(x) = \exp x$ and $X = \log r$. Apply Itô's Lemma on $f(X)$ we obtain

$$d(f(X)) = \frac{\partial f}{\partial X} dX + \frac{1}{2}\frac{\partial f^2}{\partial^2 X} dX^2$$

where $\partial f / \partial X = \exp(X) = r$ and $\partial f^2 / \partial^2 X = \exp(X) = r$. Given dX from Equation 3, we then have

$$\begin{aligned} d(f(X)) &= dr = \frac{\partial f}{\partial X} dX + \frac{1}{2}\frac{\partial f^2}{\partial^2 X} dX^2 \\ &= r(t)\left(\theta(t) + \frac{d(\log \sigma(t))}{dt}\log r\right)dt + \sigma(t)r(t)dW + \frac{1}{2}\sigma^2(t)dt \\ &= r(t)\left(\theta(t) + \frac{d(\log \sigma(t))}{dt}\log r + \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)r(t)dW \end{aligned}$$

Using the notation of

$$dr = A(r,t)dt + B(r,t)dW$$

yields

$$\begin{cases} A(r,t) = r(t)\left(\theta(t) + \frac{d(\log \sigma(t))}{dt}\log r + \frac{1}{2}\sigma^2(t)\right) \\ B(r,t) = \sigma(t)r(t) \end{cases}$$

The terminal condition, $Z(r, T; T) = 1$, must hold for all r, and this implies that $A(T,T) = B(T,T) = 0$.

3 With model specification

$$dr = [\eta(t) - \gamma r]dt + c dW$$

where $\eta(t)$ is an arbitrary function of time t and γ and c are constants. The partial differential equation for the zero coupon bond price becomes

$$\frac{\partial Z}{\partial t} + [\eta(t) - \gamma r]\frac{\partial Z}{\partial r} + \frac{1}{2}c^2\frac{\partial^2 Z}{\partial r^2} = rZ \tag{4}$$

Initially guess and subsequently verify that the solution has the form

$$Z(r,t;T) = \exp(A(t;T) - rB(t;T))$$

for some nonrandom functions $A(t;T)$ and $B(t;T)$ to be determined. Furthermore,

$$\begin{aligned} \frac{\partial Z}{\partial t} &= \left[\frac{\partial A(t,T)}{\partial t} - r\frac{\partial B(t,T)}{\partial t}\right] Z(r,t;T) \\ \frac{\partial Z}{\partial r} &= -B(t,T)Z(r,t;T) \\ \frac{\partial^2 Z}{\partial r^2} &= B(t,T)^2 Z(r,t;T) \end{aligned}$$

Substitution into the partial differential equation 4 gives

$$\left[\left(-\frac{\partial B(t,T)}{\partial t}+\gamma B(t,T)-1\right)r\ +\ \frac{\partial A(t,T)}{\partial t}-\eta(t)B(t,T)+\frac{1}{2}c^2B(t,T)^2\right]Z(r,t;T)=0 \tag{5}$$

This equation must hold for all r, therefore the term that multiplies r in this equation must be zero. Otherwise, changing the value of r would change the value of the left-hand side of equation 5, and hence it could not always be equal to zero. This gives us an ordinary differential equation in t as

$$\frac{\partial B(t,T)}{\partial t}=\gamma B(t,T)-1 \tag{6}$$

Setting the term 6 to zero in equation 5, we have

$$\frac{\partial A(t,T)}{\partial t}=\eta(t)B(t,T)-\frac{1}{2}c^2B(t,T)^2 \tag{7}$$

From the equation 6, it is not hard to solve

$$\begin{aligned}
\frac{\partial B(t,T)}{\partial t} &= \gamma B(t,T)-1\\
\frac{1}{\gamma}\frac{d(\gamma B(t,T)-1)}{\gamma B(t,T)-1} &= d\tau\\
\frac{1}{\gamma}\int_t^T\frac{d(\gamma B(t,T)-1)}{\gamma B(t,T)-1} &= \int_t^T d\tau\\
\frac{1}{\gamma}\log(\gamma B(t,T)-1) &= -(T-t)\\
B(t,T) &= \frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)
\end{aligned}$$

Substitution the solution of $B(t,T)$ into 7 yields

$$\begin{aligned}
A(t,T) &= -\left[\int_t^T\eta(\tau)B(\tau,T)-\frac{1}{2}c^2B(\tau,T)^2\right]d\tau\\
&= -\int_t^T\eta(\tau)B(\tau,T)+\frac{1}{2}c^2\int_t^T\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)^2d\tau\\
&= \int_t^T\eta(\tau)B(\tau,T)+\frac{c^2}{2\gamma^2}\left((T-t)+\frac{2}{\gamma}e^{-\gamma(T-t)}-\frac{1}{2\gamma}e^{-\gamma(T-t)}-\frac{3}{2\gamma}\right)
\end{aligned}$$

And we solved $A(t,T)$ and $B(t,T)$ as claimed.

4 Given the process

$$dU_t=-\gamma U_t dt+\sigma dW_t$$

we have

$$
\begin{aligned}
d(e^{\gamma}U_t) &= \gamma e^{\gamma t}U_t dt + e^{\gamma t}dU_t \\
&= \gamma e^{\gamma t}U_t dt - \gamma e^{\gamma t}U_t dt + \sigma e^{\gamma t}dW_t \\
&= \sigma e^{\gamma t}dW_t \\
e^{\gamma t}U_t &= U_0 + \sigma\int_0^t e^{\gamma s}dW_s \\
&= u + \sigma\int_0^t e^{\gamma s}dW_s \\
U_t &= ue^{-\gamma t} + \sigma\int_0^t e^{-\gamma(t-s)}dW_s
\end{aligned}
$$

Then the moments of U_t should be

$$
\begin{aligned}
\mathbb{E}[U_t] &= \mathbb{E}[ue^{-\gamma t}] + \mathbb{E}[\sigma\int_0^t e^{-\gamma(t-s)}dW_s] \\
&= ue^{-\gamma t} \\
\mathbb{V}[U_t] &= \mathbb{E}\left[\sigma^2\left(\int_0^t e^{-\gamma(t-s)}dW_s\right)^2\right] \\
&= \sigma^2\int_0^t e^{-2\gamma(t-s)}ds \\
&= \frac{\sigma^2}{2\gamma}(1-e^{-2\gamma t})
\end{aligned}
$$

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