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A new approximate CVA of interest rate swap in the SABR/LIBOR market model : an asymptotic expansion approach

Masahiro Nisiba[1,*]

Tokyo Institute of Technology

Abstract

The author presents a new approximate pricing formula for the credit valuation adjustment of interest rate swap in the SABR/LIBOR market model using an asymptotic expansion method. He compare it with Monte Carlo simulation. It is shown that the new method makes computations extremely fast.

Keywords: CVA, SABR/LIBOR market model, Asymptotic expansion

1. Introduction

The counterparty risk management is becoming increasingly important after the financial crisis in 2008 and the tightening of the regulations ([3], [12]). Many financial institutions are more interested in the credit valuation adjustment(CVA) of derivatives, but they have problems with the CVA especially in the calculation speed ([12]).

Most institutions cite interest rate products as contributing the most to their overall risk ([12]). In the interest rate derivatives markets, the SABR model is widely used by practitioners in the financial industry. The LIBOR market model has become the market standard, and the SABR model can capture the volatility smile of interest rates ([4], [6]).

In this paper, we consider about the CVA of the interest rate swap. It takes a lot of time to calculate the CVA of the swap by Monte Carlo simulation. Using a new approach, this paper derives an approximate formula for the CVA of the swap. This new formula enable to calculate the CVA much faster than Monte Carlo method. To obtain the formula, we apply an asymptotic expansion method based on infinite dimensional analysis called the Watanabe-Yoshida theory and the Malliavin calculus ([10], [11]). The method was applied to derivative valuation problems ([7], [8], [5]).

The asymptotic expansion requires a huge amount of manipulation of symbolic expressions, so we develop a new library for Maxima in Lisp. With this library, we can obtain high order asymptotic expansions of the solution to SDEs.

In the following sections, after an explanation of the framework of an asymptotic expansion method in Section 2, Section 3 provide a brief description of the new library for maxima. Section 4 applies the method to the forward swap rate. Section 5 gives the new approximate pricing formula for CVA of the interest rate swap and Section 6 provides the accuracy validation by comparing the approximate formula with Monte Carlo simulation. Section 7 concludes.

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Email address: nishiba09@craft.titech.ac.jp (Masahiro Nisiba)

[1]Center for Research in Advanced Financial Technology, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8552 Japan

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2. Asymptotic expansion

We consider $\mathbb{R}^d$-valued diffusion process $X^{(\epsilon)}$ that is the solution to the following stochastic differential equations:

$$\mathrm{d}X^{(\epsilon)} = V_0\left(X^{(\epsilon)}, \epsilon\right) \mathrm{d}t + \epsilon V\left(X^{(\epsilon)}\right) \mathrm{d}W_t \tag{1}$$

$$X_0^{(\epsilon)} = x_0 \tag{2}$$

where x_0 is constant, $W = \left(W^1, \cdots, W^d\right)$ is an d-dimensional standard Brownian process, and $\epsilon \in [0,1]$ is a known parameter. The following theorem is proved in [10].

Theorem 2.1. *Suppose $V_0 : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ are smooth, and these derivatives of any order are bounded. Next, suppose that a function $g : \mathbb{R}^d \to \mathbb{R}$ to be smooth and all derivatives have polynomial growth orders. Then, for $\epsilon \downarrow 0$, $g(X_T^{(\epsilon)})$ has its asymptotic expansion:*

$$g(X_T^{(\epsilon)}) = g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \epsilon^3 g_{3T} + o(\epsilon^3). \tag{3}$$

We explain the calculating method of asymptotic expansion. The coefficients in the expansion, $g_{0T}, g_{1T}, g_{2T}, \cdots$, can be obtained by Taylor's formula and represented by multiple Wiener-Ito integrals. For examples, let $D_t = \frac{\partial X_t^{(\epsilon)}}{\partial \epsilon}\Big|_{\epsilon=0}$, $E_t = \frac{\partial^2 X_t^{(\epsilon)}}{\partial \epsilon^2}\Big|_{\epsilon=0}$, $F_t = \frac{\partial^3 X_t^{(\epsilon)}}{\partial \epsilon^3}\Big|_{\epsilon=0}$, and $g_{0T}, g_{1T}, g_{2T}, g_{3T}$ are represented by

$$g_{0T} = g\left(X_T^{(0)}\right) \tag{4}$$

$$g_{1T} = \sum_{i=1}^{d} \partial_i g\left(X_T^{(0)}\right) D_T^i \tag{5}$$

$$g_{2T} = \frac{1}{2}\sum_{i,j=1}^{d} \partial_i \partial_j g\left(X_T^{(0)}\right) D_T^i D_T^j + \frac{1}{2}\sum_{i=1}^{d} \partial_i g\left(X_T^{(0)}\right) E_T^i \tag{6}$$

$$\begin{aligned} g_{3T} = {} & \frac{1}{6}\sum_{i,j,k=1}^{d} \partial_i \partial_j \partial_k g\left(X_T^{(0)}\right) D_T^i D_T^j D_T^k \\ & + \frac{1}{2}\sum_{i,j=1}^{d} \partial_i \partial_j g\left(X_T^{(0)}\right) E_T^i D_T^j + \frac{1}{6}\sum_{i=1}^{d} \partial_i g\left(X_T^{(0)}\right) F_T^i \end{aligned} \tag{7}$$

where g_{0T} is deterministic, the other terms are stochastic, $D_t^i, E_t^i, F_t^i, (i = 1, \cdots, d)$ denote the i-th elements of D_t, E_t, F_t respectively. D_t, E_t, F_t are represented by

$$D_t = \int_0^t Y_t Y_u^{-1} \left[\partial_\epsilon V_0\left(X_u^{(0)}, 0\right) \mathrm{d}u + V\left(X_u^{(0)}\right) \mathrm{d}W_u\right] \tag{8}$$

$$\begin{aligned} E_t = {} & \int_0^t Y_t Y_u^{-1} \Bigg(\sum_{j,k=1}^{d} \partial_j \partial_k V_0\left(X_u^{(0)}, 0\right) D_u^j D_u^k \,\mathrm{d}u + \partial^2 V_0\left(X_u^{(0)}, 0\right) \mathrm{d}u \\ & + 2\sum_{j=1}^{d} \partial_\epsilon \partial_j V_0\left(X_u^{(0)}, 0\right) D_u^j \,\mathrm{d}u + 2\sum_{j=1} \partial_j V\left(X_u^{(0)}\right) D_u^j \,\mathrm{d}W_u \Bigg) \end{aligned} \tag{9}$$

$$\begin{aligned} F_t = {} & \int_0^t Y_t Y_u^{-1} \Bigg(\sum_{j,k,l=1}^{d} \partial_j \partial_k \partial_l V_0\left(X_u^{(0)}, 0\right) D_u^j D_u^k D_u^l \,\mathrm{d}u + 3\sum_{j,k=1}^{d} \partial_j \partial_k V_0\left(X_u^{(0)}, 0\right) E_u^j D_u^k \,\mathrm{d}u \\ & + 3\sum_{j,k=1}^{d} \partial_j \partial_k \partial_\epsilon V_0\left(X_u^{(0)}, 0\right) D_u^j D_u^k \,\mathrm{d}u + 3\sum_{j=1}^{d} \partial_j \partial_\epsilon V_0\left(X_u^{(0)}, 0\right) E_u^j \,\mathrm{d}u \end{aligned}$$

$$+3\sum_{j=1}^{d}\partial_j\partial_\epsilon^2 V_0\left(X_u^{(0)},0\right)D_u^j\,\mathrm{d}u+\partial_\epsilon^3 V_0\left(X_u^{(0)},0\right)\,\mathrm{d}u$$

$$+3\sum_{j,k=1}\partial_j\partial_k V\left(X_u^{(0)}\right)D_u^j D_u^k\,\mathrm{d}W_u+3\sum_{j=1}^{d}\partial_j V\left(X_u^{(0)}\right)E_u^j\,\mathrm{d}W_u\Bigg) \tag{10}$$

where Y denotes the solution to the differential equation:

$$\mathrm{d}Y_t = \partial V_0\left(X_t^{(0)},0\right)Y_t\,\mathrm{d}t, \tag{11}$$

$$Y_0 = I_d. \tag{12}$$

Here, ∂V_0 denotes the $d\times d$ matrix whose (j,k)-element is $\partial_k V_0^j$, and I_d denotes the $d\times d$ identity matrix. Next, normalize $g\left(X_T^{(\epsilon)}\right)$ to

$$G^{(\epsilon)}=\frac{g\left(X_T^{(\epsilon)}\right)-g_{0T}}{\epsilon} \tag{13}$$

for $\epsilon\in(0,1]$. Moreover, let

$$a_t=a_t^{(0)}={}^t\!\left(\partial g\left(X_T^{(0)}\right)\right)\left[Y_T Y_t^{-1}V\left(X_t^{(0)}\right)\right] \tag{14}$$

where ${}^tA$ denotes the inverse matrix of A. We make the following assumption:
(Assumption 1)

$$\Sigma_T=\int_0^T a_t{}^t a_t\,\mathrm{d}t>0. \tag{15}$$

Note that g_{1T} follows a normal distribution with variance Σ_T and hence Assumption 1 means that the distribution of g_{1T} does not degenerate. Next, let $\psi_{G^{(\epsilon)}}(\xi)$ be a characteristic function of $G^{(\epsilon)}$. Then, $\psi_{G^{(\epsilon)}}(\xi)$ is expanded around $\epsilon=0$ as follows:

$$\begin{aligned}
\psi_{G^{(\epsilon)}}(\xi) &= \mathbb{E}\left[\exp\left(\sqrt{-1}\xi G^{(\epsilon)}\right)\right]\\
&= \mathbb{E}\left[\exp\left(\sqrt{-1}\xi g_{1T}\right)\right]+\epsilon\left(\sqrt{-1}\xi\right)\mathbb{E}\left[\exp\left(\sqrt{-1}\xi g_{1T}\right)g_{2T}\right]\\
&\quad+\epsilon^2\left(\sqrt{-1}\xi\right)\mathbb{E}\left[\exp\left(\sqrt{-1}\xi g_{1T}\right)g_{3T}\right]+\frac{\epsilon^2}{2}\left(\sqrt{-1}\xi\right)^2\mathbb{E}\left[\exp\left(\sqrt{-1}\xi g_{1T}\right)g_{2T}^2\right]+o(\epsilon^2)\\
&= \exp\left(\frac{\left(\sqrt{-1}\xi\right)^2\Sigma_T}{2}\right)+\epsilon\left(\sqrt{-1}\xi\right)\mathbb{E}\left[\exp\left(\sqrt{-1}\xi g_{1T}\right)\mathbb{E}\left[g_{2T}|g_{1T}\right]\right]\\
&\quad+\epsilon^2\left(\sqrt{-1}\xi\right)\mathbb{E}\left[\exp\left(\sqrt{-1}\xi g_{1T}\right)\mathbb{E}\left[g_{3T}|g_{1T}\right]\right]\\
&\quad+\frac{\epsilon^2}{2}\left(\sqrt{-1}\xi\right)^2\mathbb{E}\left[\exp\left(\sqrt{-1}\xi g_{1T}\right)\mathbb{E}\left[g_{2T}^2|g_{1T}\right]\right]+o(\epsilon^2).
\end{aligned} \tag{16}$$

Then, we obtain the below proposition.

Lemma 2.1. *We assume (Assumption 1). Then, approximate probability density function of $G^{(\epsilon)}$ is represented as follows:*

$$\begin{aligned}
f_{G^{(\epsilon)}} &= n[x;0,\Sigma_T]+\epsilon\left[-\frac{\partial}{\partial x}\left\{h_2(x)n[x;0,\Sigma_T]\right\}\right]\\
&\quad+\epsilon^2\left[-\frac{\partial}{\partial x}\left\{h_3(x)n[x;0,\Sigma_T]\right\}\right]+\frac{1}{2}\epsilon^2\left[\frac{\partial^2}{\partial x^2}\left\{h_{22}(x)n[x;0,\Sigma_T]\right\}\right]+o(\epsilon^2).
\end{aligned} \tag{17}$$

where $G^{(\epsilon)}$ is defined by the equation (13), $h_2(x)=\mathbb{E}\left[g_{2T}|g_{1T}=x\right]$, $h_{22}(x)=\mathbb{E}\left[g_{2T}^2|g_{1T}=x\right]$, $h_3(x)=\mathbb{E}\left[g_{3T}|g_{1T}=x\right]$, and $n[x;0,\Sigma_T]$ denotes the pdf of the normal distribution with average 0 and variance Σ_T.

We can calculate the conditional expectations in the equation (17) by applying Ito's lemma and the following proposition which in [9].

Proposition 2.1. *Let $J_n(f_n)$ denote the n-times iterated Ito integral of $f_n \in L^2(\mathbb{T}^n)$:*

$$J_n(f_n) = \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, t_2, \cdots, t_n)\, \mathrm{d}W_{t_n} \cdots \mathrm{d}W_{t_2}\, \mathrm{d}W_{t_1} \tag{18}$$

for $n \in Z^+$, $J_0(f_0) = f_0$ where f_0 is constant.
Then, its expectation conditional on $J_1(q) = x$ is given by

$$\begin{aligned} &\mathbb{E}\left[J_n(f_n) | J_1(q) = x\right] \\ = &\left(\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, t_2, \cdots, t_n) q(t_1) q(t_2) \cdots q(t_n)\, \mathrm{d}t_n \cdots \mathrm{d}t_2\, \mathrm{d}t_1\right) \frac{H_n\left(x; \|q\|^2_{L^2(\mathbb{T})}\right)}{\left(\|q\|^2_{L^2(\mathbb{T})}\right)^n} \end{aligned} \tag{19}$$

where $T > 0$, $\mathbb{T} = [0, T]$, $i \in \{1, 2, \cdots, n\}$, $t_i \in \mathbb{T}$ and $H_n(x; \Sigma)$ is the Hermite polynomial of degree n, that is to say

$$H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/(2\Sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/(2\Sigma)}. \tag{20}$$

3. Asymptotic expansion library for Maxima

We develop a library for the asymptotic expansion of the solution to SDEs in Maxima using lisp, the programming language. So, we explain about Maxima, a computer algebra system, and the library in this section.

3.1. Maxima

Maxima is a system for the manipulation of symbolic and numerical expressions, including differentiation and integration, etc ([13]).

3.2. Asymptotic expansion library

To obtain the symbolic expressions of the asymptotic expansion, we extend Maxima to manipulate Ito's integral, Ito's formula, Fubini's theorem, the formula to calculate conditional expectations(Proposition 2.1), formulas to solve SDEs, and the inversion formula of characteristic functions, etc.

We define these theorems and formulas in the library as follows.

Definition 3.1. *(Ito's formula in the library)*

$$\begin{aligned} \int_0^T f_1(t)\, \mathrm{d}W1(t) \int_0^T f_2(t)\, \mathrm{d}W_2(t) \quad = \quad & \int_0^T f_1(t) \int_0^t f_2(s)\, \mathrm{d}W_2(s)\, \mathrm{d}W_1(t) \\ & + \int_0^T f_2(t) \int_0^t f_1(s)\, \mathrm{d}W_1(s)\, \mathrm{d}W_2(t) \\ & + \int_0^T f_1(t) f_2(t)\, \mathrm{d}W_1(t)\, \mathrm{d}W_2(t) \end{aligned}$$

where

$$\mathrm{d}W_1(t)\, \mathrm{d}W_2(t) = \begin{cases} 0 & W_1 \neq W_2 \\ \mathrm{d}t & W_1 = W_2. \end{cases}$$

Definition 3.2. *(Fubini's theorem in the library)*

$$\int_0^T f_1(t) \int_0^t f_2(s)\, \mathrm{d}W(s)\, \mathrm{d}t \quad = \quad \int_0^T f_1(t)\, \mathrm{d}t \int_0^T f_2(t)\, \mathrm{d}W(t) - \int_0^T f_2(t) \int_0^t f_1(s)\, \mathrm{d}s\, \mathrm{d}W(t).$$

Definition 3.3. *(The formula to calculate conditional expectations in the library)*

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, t_2, \cdots, t_n)\, \mathrm{d}W_{t_n} \cdots \mathrm{d}W_{t_2}\, \mathrm{d}W_{t_1} \middle| \int_0^T q(t)\, \mathrm{d}W_t = x\right] \\
= &\left(\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, t_2, \cdots, t_n) q(t_1) q(t_2) \cdots q(t_n)\, \mathrm{d}t_n \cdots \mathrm{d}t_2\, \mathrm{d}t_1\right) \frac{H_n\left(x; \|q\|^2_{L^2(\mathbb{T})}\right)}{\left(\|q\|^2_{L^2(\mathbb{T})}\right)^n}
\end{aligned}
$$

where $\mathbb{T} = [0, T](T > 0)$*,* $t_i \in \mathbb{T}(i = 1, 2, \cdots, n)$ *and* $H_n(x; \Sigma)$ *is defined as*

$$H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/(2\Sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/(2\Sigma)}.$$

Definition 3.4. *(The formulas to solve SDEs in the library)*
We consider following SDE:

$$
\begin{aligned}
\mathrm{d}S_t &= f(t, S_t)\, \mathrm{d}t + g(t, S_t)\, \mathrm{d}W, \\
S_0 &= s.
\end{aligned}
$$

Formula 1.
If $g(t, x) = 0$ $(\forall t \geq 0, \forall x \in \mathbb{R})$ *then,* S_t *is the solution to the following ODE:*

$$
\begin{aligned}
\mathrm{d}S_t &= f(t, S_t)\, \mathrm{d}t, \\
S_0 &= s.
\end{aligned}
$$

Formula 2.
If $f(t, x) = a(t)\, x$ *and* $g(t, x) = b(t)\, x$ $(\forall t \geq 0, \forall x \in \mathbb{R})$ *then,*

$$S_t = s \exp\left(\int_0^t \left(a(s) - \frac{1}{2} b(s)^2\right) \mathrm{d}t + \int_0^t b(s)\, \mathrm{d}W\right).$$

Formula 3.
If $f(t, x) = a(t)\, x$ *and* $g(t, x) = b(t)$ $(\forall t \geq 0, \forall x \in \mathbb{R})$ *then,*

$$S_t = s + \exp\left(\int_0^t a(s)\, \mathrm{d}s\right) \int_0^t \exp\left(-\int_0^s a(u)\, \mathrm{d}u\right) b(s)\, \mathrm{d}s$$

4. Asymptotic expansion of forward swap rate

According to [6], forward swap rate in SABR/LMM market model is represented by the solution to the following stochastic differential equation :

$$
\begin{aligned}
\mathrm{d}S_t &= (S_t)^B \sigma_t\, \mathrm{d}W_t^1 && (21) \\
\mathrm{d}\sigma_t &= v_1 \sigma_t\, \mathrm{d}W_t^1 + v_2 \sigma_t\, \mathrm{d}W_t^2 && (22)
\end{aligned}
$$

where $W = \left(W^1, W^2\right)$ is 2-dimensional standard Wiener process, and $B \in [0, 1]$. Let the correlation between forward swap rate and its volatility be $\rho \in [-1, 1]$, then $v_1 = v\rho$, $v_2 = v\sqrt{1 - \rho^2}$ where $v \geq 0$.
Now, we calculate the asymptotic expansion of the solution to above SDE. (21), (22) are represented as follows:

$$
\begin{aligned}
S^{(\epsilon)}(T) &= S(0) + \epsilon \int_0^T \left(S^{(\epsilon)}(t)\right)^B \sigma^{(\epsilon)}(t)\, \mathrm{d}W_t^1 && (23) \\
\sigma^{(\epsilon)}(T) &= \sigma(0) + \epsilon \int_0^T v_1 \sigma^{(\epsilon)}(t)\, \mathrm{d}W_t^1 + \epsilon \int_0^T v_2 \sigma^{(\epsilon)}(t)\, \mathrm{d}W_t^2 && (24)
\end{aligned}
$$

Then, for $\epsilon \downarrow 0$, $S^{(\epsilon)}(T)$ has its asymptotic expansion:

$$\epsilon^n S^{(\epsilon)}(T) = \sum_{n=0}^{m} \epsilon^n S_n(T) + o(\epsilon^m) \tag{25}$$

The coefficients, $S^{(n)}(T)(n = 0, 1, 2, 3, \ldots)$, are represented as follows:

$$\begin{aligned}
S_0(T) &= S(0) && (26)\\
S_1(T) &= S(0)^B \sigma(0) \int_0^T \mathrm{d}W^1(t) && (27)\\
S_2(T) &= \left(B\sigma(0)^2 S(0)^{2B-1} + \sigma(0)v_1 S(0)^B\right) \int_0^T \int_0^t \mathrm{d}W^1(s)\,\mathrm{d}W^1(t)\\
&\quad + \sigma(0)v_2 S(0)^B \int_0^T \int_0^t \mathrm{d}W^2(s)\,\mathrm{d}W^1(t) && (28)\\
S_3(T) &= \left(B^2\sigma(0)^3 S(0)^{3B-2} + B\sigma(0)^2 v_1 S(0)^{2B-1} + \sigma(0)v_1^2 S(0)^B\right) \int_0^T \int_0^t \int_0^s \mathrm{d}W^1(u)\,\mathrm{d}W^1(s)\,\mathrm{d}W^1(t)\\
&\quad + \left(\frac{1}{2}B^2\sigma(0)^3 S(0)^{3B-2} - \frac{1}{2}B\sigma(0)^3 S(0)^{3B-2} + B\sigma(0)^2 v_1 S(0)^{2B-1}\right) \int_0^T \left(\int_0^t \mathrm{d}W^1(s)\right)^2 \mathrm{d}W^1(t)\\
&\quad + \left(B\sigma(0)^2 v_2 S(0)^{2B-1} + \sigma(0)v_1 v_2 S(0)^B\right) \int_0^T \int_0^t \int_0^s \mathrm{d}W^2(u)\,\mathrm{d}W^1(s)\,\mathrm{d}W^1(t)\\
&\quad + B\sigma(0)^2 v_2 S(0)^{2B-1} \int_0^T \int_0^t \mathrm{d}W^1(s) \int_0^t \mathrm{d}W^2(s)\,\mathrm{d}W^1(t)\\
&\quad + \sigma(0)v_2^2 S(0)^B \int_0^T \int_0^t \int_0^s \mathrm{d}W^2(u)\,\mathrm{d}W^2(s)\,\mathrm{d}W^1(t)\\
&\quad + \sigma(0)v_1 v_2 S(0)^B \int_0^T \int_0^t \int_0^s \mathrm{d}W^1(u)\,\mathrm{d}W^2(s)\,\mathrm{d}W^1(t) && (29)
\end{aligned}$$

The coefficients of higher order are represented in a same way.

By lemma 2.1, the approximate pdf of $G^{(\epsilon)}_{SABR} = \frac{S^{(\epsilon)}(T) - S^{(0)}(T)}{\epsilon}$ is represented as follows:

$$\begin{aligned}
f_{G^{(\epsilon)}_{SABR}}(x) &= n[x; 0, \Sigma_T] + \epsilon\left[-\frac{\partial}{\partial x}\{\mathbb{E}[S_2(T)|S_1(T) = x]\, n[x; 0, \Sigma_T]\}\right]\\
&\quad + \epsilon^2\left[-\frac{\partial}{\partial x}\{\mathbb{E}[S_3(T)|S_1(T) = x]\, n[x; 0, \Sigma_T]\}\right]\\
&\quad + \frac{1}{2}\epsilon^2\left[\frac{\partial^2}{\partial x^2}\left\{\mathbb{E}\left[(S_2(T))^2\, |S_1(T) = x\right] n[x; 0, \Sigma_T]\right\}\right] + o(\epsilon^2) && (30)
\end{aligned}$$

where $\Sigma_T = \mathbb{E}\left[(S_1(T))^2\right]$ and $n[x; 0, \Sigma_T]$ denotes the pdf of the normal distribution with average 0 and variance Σ_T.

5. CVA of interest rate swap

Consider a payer swap such that a fixed interest rate K is paid and a floating interest rate $F(T_i; T_i, T_{i+1})$, $i = a, \ldots, b-1$, is received at the consecutive dates $T_{a+1}, \ldots, T_b$.

5.1. Basic setup

This subsection defines basic concepts such as tenor structures, discount bond price, the money market account, forward Libor rates, the spot measure, the forward measure and the forward swap measure. First a tenor structure is given by a finite set date:

$$0 = T_0 < T_1 < \cdots < T_N$$

where T_N is a pre-specified date and $\delta = T_j - T_{j-1}$ for $j = 1, 2, \ldots, N$. $P(t, T_j)$ denotes the price of the the discount bound with maturity T_j at time t, where $P(T_j, T_j) = 1$ and $P(t, T_j) = 0$ for $t \in (T_j, T_N]$.

The forward LIBOR rate at time t ($\leq T_j$) with term $[T_{j-1}, T_j]$ is defined as

$$F(t; T_{j-1}, T_J) = \frac{1}{\delta}\left(\frac{P(t; T_{j-1})}{P(t; T_j)} - 1\right), \tag{31}$$

for any $j = 1, 2, \ldots, N$.

The money market account's price $B(t)$ is defined as

$$B(t) = \frac{P(t; T_{\gamma(t)})}{\Pi_{j=1}^{\gamma(t)} P(T_{j-1}; T_j)} \tag{32}$$

where $T_{\gamma(t)}$ denotes the first tenor after time t.

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ denote a complete probability space satisfying the usual conditions where $\mathbb{Q}$ is the spot measure which is the risk neutral measure with numeraire $B(t)$. Then, let $\mathbb{Q}^T$ and $\mathbb{Q}^{(a,b)}$ be the T-forward measure and the forward swap measure which are equivalent martingale measures with numeraire $P(t; T)$ and $N(t) = \sum_{i=a+1}^{b} P(t, T_i)$ respectively.

5.2. Interest rate swaption

Let $S^{(a,b)}(t)$ denote the forward rate with term $[T_a, T_b]$ at time t ($t \in [0, T_a]$), and it is represented as follows:

$$S^{(a,b)}(t) = \frac{P(t, T_a) - P(t, T_b)}{\delta \sum_{i=a+1}^{b} P(t, T_i)}. \tag{33}$$

The price of a payer swaption with nominal 1, strike K, maturity t and underlying tenor $T_a, T_{a+1} \ldots T_b$ (T_a is the first reset date and T_b the maturity of the underlying swap) is represented as follows:

$$\begin{aligned}
\pi(0; t, T_a, T_b) &= \delta \sum_{i=a+1}^{b} P(0, T_i) \mathbb{E}^{Q^{(a,b)}} \left[(S(t) - K)^+ \right] \\
&= \delta \sum_{i=a+1}^{b} P(0, T_i) \int_{\frac{K - S^{(0)}(t)}{\epsilon}}^{\infty} \left(\epsilon x - \left(K - S^{(0)}(t)\right)\right) f_{G^{(\epsilon)}_{SABR}}(x)\, \mathrm{d}x + o(\epsilon^2)
\end{aligned} \tag{34}$$

where $\mathbb{E}^{Q^{(a,b)}}$ is expectation under $Q^{(a,b)}$.

5.3. CVA

Definition 5.1. *The CVA of a derivative which value at time t is $V(t)$ is defined as follows:*

$$CVA := (1 - R)\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_b} B(t)^{-1} \max(V(t), 0)\ \mathrm{d}DP(t)\right] \tag{35}$$

where R ($R \in [0, 1]$) and $DP(t)$ denote the recovery rate and default probability respectively.

5.4. CVA of interest rate swap

We define the default probability $DP(t)$ as

$$DP(t) = 1 - \exp\left(-\int_0^t \lambda\, \mathrm{d}t\right) = 1 - \exp(-\lambda t). \tag{36}$$

Let $V(t)$ denote the price of the interest rate swap at time t. According to [1], its CVA is represented as follows:

$$\begin{aligned} CVA &= (1-R)\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_b} B(t)^{-1}\max\left(V(t),0\right)\,\mathrm{d}DP(t)\right] \\ &\sim (1-R)\int_0^{T_b}\pi(0;t,T_{\gamma(t)},T_b)\lambda e^{-\lambda t}\,\mathrm{d}t \\ &= (1-R)\left\{\int_0^{T_a}\pi(0;t,T_a,T_b)\lambda e^{-\lambda t}\,\mathrm{d}t + \sum_{i=a+1}^{b-1}\int_{T_{i-1}}^{T_i}\pi(0;t,T_i,T_b)\lambda e^{-\lambda t}\,\mathrm{d}t\right\} \end{aligned} \tag{37}$$

where R ($R \in [0,1]$) denotes the recovery rate. Hence, we assumed the correlation between the interest rate swap and default is 0.

6. Numerical results

This section provides the accuracy validation of the approximate CVA of the swap in section 5. Benchmark values are computed by Monte Carlo simulations. Let ϵ be 1, $B = 0.5$, $\delta = 0.5$, $a = 10$, $b = 20$, $K = (3.5\%, 4\%, 4.5\%)$ and $R = 0$. Other parameters used in the test are given in Table 1. The parameters are calibrated from market date in Table 2 using the method of [6].

Table 1: Parameters

first reset date	$S(0)$	$\sigma(0)$	ν	ρ
5	0.0477	0.012	0.158	0.000145
5.5	0.0481	0.011	0.149	0.000409
6	0.0486	0.011	0.142	0.000741
6.5	0.049	0.011	0.135	0.000472
7	0.0495	0.011	0.13	0.000113
7.5	0.0498	0.011	0.125	-0.000121
8	0.0502	0.01	0.12	-0.000472
8.5	0.0504	0.01	0.116	-0.000181
9	0.0509	0.01	0.112	0.000401
9.5	0.0509	0.01	0.109	0.000401

In Monte Carlo simulations for benchmark values, we use Euler-Maruyama scheme as a discretization scheme with 10,000 time steps and generate 5,000,000 paths in each simulation.

Table 2: Euro cap prices (in basis points) on 18 November 2008

Y(year)-K(%)	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
2.0	25.0	11.0	5.0	2.5	1.5	1.0	0.5	0.0	0.0
3.0	77.0	40.5	21.5	12.0	7.0	4.0	2.5	1.5	1.5
4.0	148.5	86.0	48.5	27.0	16.0	10.0	6.5	4.5	4.0
5.0	230.5	140.5	82.0	47.5	28.5	17.5	11.5	8.0	7.5
6.0	325.5	206.0	125.5	74.5	45.5	29.0	19.0	13.5	12.5
7.0	431.5	283.5	178.0	109.0	68.0	44.5	29.5	21.0	20.5
8.0	545.5	368.5	238.0	149.0	95.0	62.5	42.5	30.0	29.0
9.0	664.0	459.0	304.5	196.5	127.0	85.0	58.5	42.0	40.0
10.0	786.0	554.5	376.5	248.5	164.0	111.0	77.0	56.0	53.0

Table 3: Numerical result

fixed rate	MC	A.E.(2nd)	A.E.(3rd)	A.E.(4th)	A.E.(5th)
3.5%	0.018513735	0.018500255	0.0185033665	0.0185030400	0.0185036872
4.0%	0.01160205	0.011557854	0.0115733700	0.0115722193	0.0115735701
4.5%	0.005348109	0.005317969	0.0053641099	0.0053632918	0.0053640888

Table 4: Difference between MC and Asymptotic expansion

fixed rate	A.E.(2nd)-MC	A.E.(3rd)-MC	A.E.(4th)-MC	A.E.(5th)-MC
3.5%	-1.34797E-05	-1.03684E-05	-1.06949E-05	-1.00477E-05
4.0%	-4.41953E-05	-2.86798E-05	-2.98305E-05	-2.84797E-05
4.5%	-3.01402E-05	1.60005E-05	1.51823E-05	1.59793E-05

Table 5: #Partition, #Sample, and CPU time required for 3 digits accuracy in the case of $K = 3.5\%$

Method	#Partition	#Sample	CPU time(sec)
E-M + MC	2000	2×10^6	196.808669
AE 2nd	-	-	0.000221
AE 3rd	-	-	0.000417
AE 4th	-	-	0.000587
AE 5th	-	-	0.001039

The results (Table 3, Table 4) show the approximation error of the asymptotic expansions are relatively small to nominal value 1. Moreover, the computational time of the new formula is much faster than that of the Monte Carlo simulation (Table 5).

7. Conclusion

We have presented a new approximate formula for the CVA of the interest rate swap in the SABR/LIBOR market model using an asymptotic expansion method. Numerical results show that our approximate formula is reasonably-accurate compared with Monte Carlo method. And new formula can be calculated much faster then Monte Carlo method.

Our future research topics are as follows: First, we apply this approach to more complicated default probability models. Second, we apply the asymptotic expansion to the forward swap rate without freezing techniques. Third, we derive the formulas for the other derivatives such as an American option.

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