Intensity Based (Reduced Form) Models

Dr. SiYi Zhou

CQF, FitchLearning

2016

The main topics of this lecture are the following:

- Modeling default by using Poisson Process with constant intensity.
- Derivation of pricing PDE for risky bonds with stochastic interest rate and/or stochastic intensity (hazard rate).
- Risky bond pricing with different recovery assumptions.
- Fundamental pricing formula for general contingent claims subject to default risk.
- Theory of affine intensity models.
- Example two factor Vasicek intensity model.

By the end of this lecture you will be able to

- Understand mathematics of intensity and intensity based model.
- Understand the pros and cons of intensity models relative to structural model.
- Derive risky bond pricing equations according to assumptions made on interest rate, intensity, recovery and hedging strategy.
- Calibrate default probability on bond prices.
- Solve affine intensity models analytically.

What is intensity based (reduced form) model

- In previous lecture we have studied structural approach to the modeling default risk. In this lecture we will introduce a different approach. These models are named "reduced-form" or "intensity based" models, in which default is treated as an unpredictable event governed by an exogenous intensity (hazard-rate) process and a jump process.
- Intensity governs the likelihood of default which is assumed to be exogenous. In addition, the mathematics of intensity and short interest rate are so similar which conveniently allows us to recycle existing term structure models. The intensity based models, therefore, are convenient and are one of the most popular credit risk models.

Poisson Process: a simplest model for default risk

Before embark on pricing any credit risky instruments using this type of model, we need to understand the basic concept of intensity first and see how it can be used to model default. Like many other discrete and countable events, such as the number of buses will arrive in the next 10 min, a default event can be modeled as the first jump in a Poisson process.

Definition of Poisson Process

A Poisson Process with intensity $\lambda > 0$ is a stochastic process

$$N_t : t \geq 0$$

taking values in $S = \{0, 1, 2, \cdots\}$ such that

1. $N_0 = 0$
2. if $s < t$, then $N_s \leq N_t$
3. if $0 < s < t$, then the increment $N_t - N_s$ is independent of what happened during $[0, s]$

Definition of Poisson Process: continue

4. let $h \to 0^+$

$$Pr(N_{t+h} = n + m | N_t = n) = \begin{cases} \lambda h + o(h), & m = 1 \\ o(h), & m > 1 \\ 1 - \lambda h + o(h), & m = 0 \end{cases}$$

In words, it means within an infinitesimal time interval h, maximum only one event can happen.

Distribution function of N_t

N_t follows Poisson distribution with parameter λt,

$$Pr(N_t = i) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

where

$$i = 0, 1, 2, \cdots$$

N_t is the number of events occurred by time t. It is also called "counting process". We can use Poisson process as a starting point to model default event.

Arrival and Inter-arrival Times

Denote T_n the arrival time of the nth event, i.e.,

$$T_n = \inf\{t : N_t = n\}, \quad T_0 = 0,$$

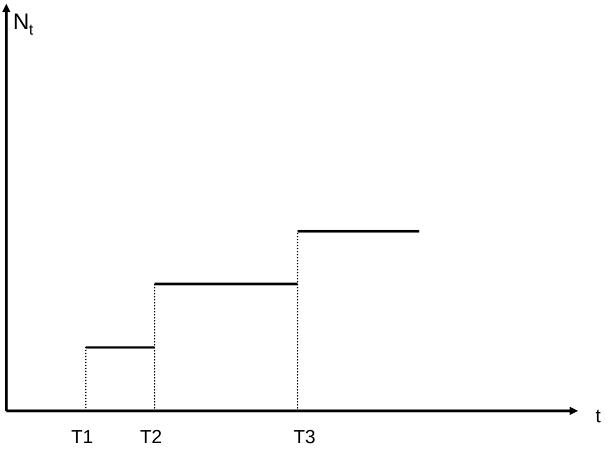
and τ_n is the inter arrival time, which is given by

$$\tau_n = T_n - T_{n-1}.$$

So T_n can be written as

$$T_n = \sum_{i=1}^{n} \tau_i$$

Sample Path of Poisson Process



Distribution of τ

Define τ the arrival time of the first event in the counting process N_t, i.e.,

$$\tau = \inf\{t : N_t > 0\}.$$

One can think of τ as the default time of a company, the question is what the distribution of τ should be.

- Since Poisson Process has independent increment, the associated inter arrival times are also independent, i.e., $\forall i \neq j$, τ_i and τ_j are independent.
- Further more, every inter arrival time has an exponential distribution with intensity λ. To summarize τ_i are i.i.d. $\exp(\lambda)$ distribution.
- To show that we now need to formally introduce the definition of intensity.

Definition of Intensity

Denote $F(\cdot)$ the Cumulative Distribution Function (CDF) of τ, and the survival function

$$S(\tau) = 1 - F(\tau).$$

According to the definition of Poisson process point (4) when $h \to 0^+$

$$Pr(t < \tau \leq t + h | \tau > t) = \lambda h + o(h).$$

So

$$\lambda = \lim_{h \to 0^+} \frac{Pr(t < \tau \leq t + h | \tau > t)}{h}$$

Finding distribution of τ

$$
\begin{aligned}
\lambda &= \lim_{h \to 0^+} \frac{Pr(t < \tau \leq t+h)}{hPr(\tau > t)} \\
&= \lim_{h \to 0^+} \frac{S(t) - S(t+h)}{hS(t)} \\
&= -\frac{d \log S(t)}{dt}
\end{aligned}
$$

Solve the above ODE for $S(t)$ with boundary condition $S(0) = 1$, we get

$$S(t) = e^{-\lambda t},$$

which implies

$$S(t) \sim \exp(\lambda).$$

Distribution of τ_i

So far we know distribution for the 1st inter arrival time, what about the rest of them? By the third property of Poisson process we know

$$S_2(t) = Pr(\tau_2 > t|\tau_1 = s) = Pr(\tau_2 > t).$$

Then follow the same argument for τ_1 we have

$$\tau_2 \sim \exp(\lambda),$$

and so on for all τ_i.
As a result, simulating Poisson process is equivalent to simulating i.i.d. exponential random variables. We will see how to do this in subsequent M5 lecture.

Inhomogenous Poisson process

What if the intensity isn't constant but deterministic? Then the counting process is called inhomogenous Poisson process. The analysis are almost the same as before. The the survival function in this case becomes

$$S(t) = \exp\left(-\int_0^t \lambda_s \, ds\right).$$

Cox Process: Basic Concepts

Later on we will derive pricing PDEs in which the intensity is not deterministic but a stochastic process. **Cox Process** is an inhomogenous Poisson Process with a stochastic intensity. In more rigorous mathematical language we assume

- The stochastic intensity process λ_t which drives the probability of default essentially is adapted to a filtration $\mathcal{G}_t$. This filtration can also contain other market variable such as interest rate, FX and so on.
- The Cox jump process N_t is adapted to a filtration $\mathcal{H}_t$.
- So the full filtration is obtained as $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$.

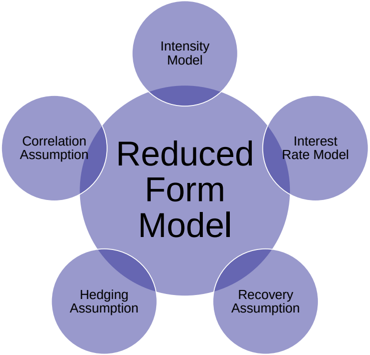
The process N_t is called a Cox Process with stochastic intensity h_t, however, conditional on the background filtration $\mathcal{G}_t$, N_t becomes an inhomogenous Poisson Process with deterministic intensity.

Cox Process: Survival Probability

The pure probabilistic approach will not be pursued much further in this lecture. The general idea is to use conditional expectation, i.e., conditional on the filtration to which the path of the intensity is adapted, then the survival function is known. The survival probability in this case is equal to

$$S(t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s\, ds\right)\right].$$

Input for Reduced form model



Plan for the risky bond pricing equation

In the next a few slides, we will derive several different BPEs for a risky bond based on different assumptions made on hazard rate, hedging and recovery rate.

In the order of complexity these are

1. Constant hazard rate and zero recovery without the hedging of default risk.
2. Stochastic hazard rate and zero recovery with the hedging of default risk.
3. Stochastic hazard rate and positive recovery with hedging of default risk.

We will always assume stochastic interest rate and hedge interest rate risk.

Model Assumptions

- Suppose a corporate's default follows a homogenous Poisson process with intensity p, and a ZCB with maturity T is issued by this company, the value of the bond is denoted by $V(t, r; p)$.
- Suppose $Z(t, r)$ is the value of a riskless ZCB with exactly the same maturity where the short interest rate dynamics follows a diffusion process

$$dr = u(r, t)\, dt + w(r, t) dX.$$

- For simplicity we will assume that there is no correlation between the diffusive change in the short interest rate and the Poisson process.

Hedging interest rate risk

Now following our convention when pricing fixed income product, we construct a ‘hedged’ portfolio:

$$\Pi = V(r, t; p) - \Delta Z(r, t).$$

Note here only the interest rate risk is hedged.

Case A: No default occurs

There is a probability of $(1 - p\,dt)$ that the bond does not default. Then the change in the value of the portfolio during an infinitesimal time step dt is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right)dt + \frac{\partial V}{\partial r}dr$$
$$-\Delta\left(\left(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}dr\right).$$

- Choose Δ to eliminate the risky dr term.

Case B: Default occurs

On the other hand, if the bond defaults, with a probability of $p\,dt$, then the change in the value of the portfolio is

$$d\Pi = -V + O(dt^{1/2}).$$

This is due to the default loss of the risky bond, the second term represents the changes in the riskless bond.

Bond pricing PDE

Taking expectation and using the bond-pricing equation of the riskless bond, we find that the value of the risky bond satisfies

$$\frac{\partial V}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - (r + p)V = 0.$$

Feynman Kac:1st call

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s+p)\,ds} | \mathcal{F}_t\right)$$

Very similar analysis can be carried out with deterministic hazard rate, the result will be almost identical apart from changing p to p_t, so that

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s+p_s)\,ds} | \mathcal{F}_t\right)$$

Yield Spread

The yield to maturity on this bond is now given by

$$y = -\frac{\log(Z(t,T)S_t(T))}{T-t} = y_f + \frac{1}{T-t}\int_t^T p_s ds,$$

where y_f is the yield to maturity of a risk free bond with the same maturity as the risky bond.
Thus the effect of the risk of default on the yield is to add a spread on riskless yield. In this simple model, the spread will be the average of the hazard rate from t to T.

Forward rate spread

If one calculates the forward rate implied by the risky bond

$$-\frac{\partial}{\partial T}\log(V(t, T)) = f(t, T) + p_T.$$

The spread is simply the hazard rate.

Implied default probability:no recovery

Given term structure of risk free bond and risky bond, one can extract implied default probability by using

$$S_t(T) = \frac{V(t,T)}{Z(t,T)} = \exp(-(T-t)(y-y_f)).$$

One can also calculate implied hazard rate by using forward spread.

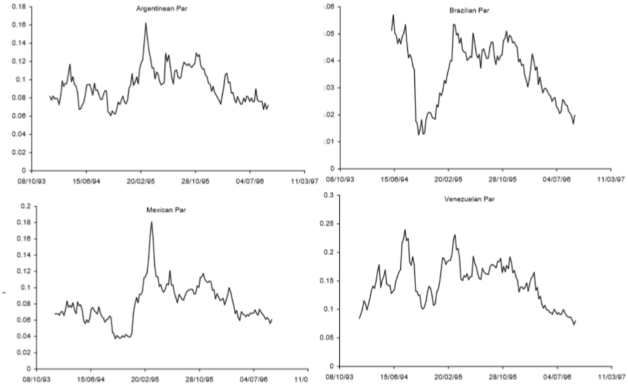
PD calibration on bond price: example

Year	Riskfree zero rate	Risky bond zero rate	Cummulative PD	Marginal PD
1	5%	5.25%	0.2497%	0.2497%
2	5%	5.50%	0.9950%	0.7453%
3	5%	5.70%	2.0781%	1.0831%
4	5%	5.85%	3.3428%	1.2647%
5	5%	5.95%	4.6390%	1.2961%

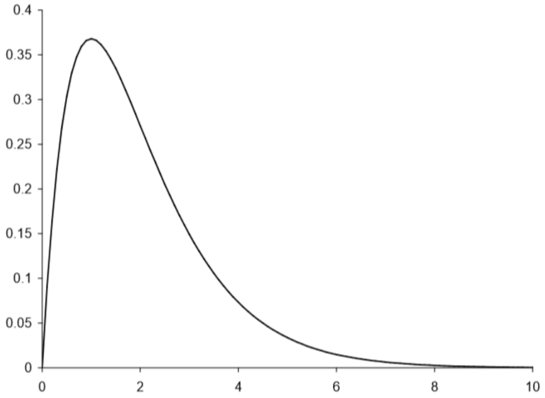
To find the risky bond value:

1. Find the risk-free yield for the maturity of each cashflow in the risky bond;
2. Add a constant spread, p, to each of these yields;
3. Use this new yield to calculate the present value of each cashflow;
4. Sum all the present values.

Implied default rates for 4 Latin American Brady bonds



Implied time dependent hazard rate



Stochastic risk of default

Now consider a model in which the default intensity is stochastic per se:

$$dp = \gamma(r, p, t)dt + \delta(r, p, t)dX_1,$$

with the interest rate still given by

$$dr = u(r, t)dt + w(r, t)dX_2,$$

where

$$dX_1\, dX_2 = \rho\, dt.$$

Hedging default intensity

- In the previous model we used riskless bonds to hedge the random movements in the spot interest rate.
- Can we introduce another risky bond or bonds into the portfolio to help with the hedging of the default risk?
- To do this we must assume that default in one bond automatically triggers default in the other.

Hedged portfolio

To value our risky zero-coupon bond we construct a portfolio with one of the risky bond, with value $V(r, p, t)$, and delta hedged by shorting Δ unit of a riskless bond, with value $Z(r, t)$, and Δ_1 shorting another risky bond issued by the same company with different maturity, with value $V_1(r, p, t)$:

$$\Pi = V(r, p, t) - \Delta Z(r, t) - \Delta_1 V_1(r, p, t).$$

Case A: No default

Suppose that the bond does not default, the change in the value of the portfolio during an infinitesimal time step is

$$d\Pi = dV - \Delta\, dZ - \Delta_1 dV_1.$$

By using Itô's lemma, above can be written as

$$\begin{aligned} d\Pi &= \left(\mathcal{L}'(V) - \Delta\mathcal{L}(Z) - \Delta_1\mathcal{L}'(V_1)\right) dt \\ &+ \left(\frac{\partial V}{\partial r} - \Delta\frac{\partial Z}{\partial r} - \Delta_1\frac{\partial V_1}{\partial r}\right) dr \\ &+ \left(\frac{\partial V}{\partial p} - \Delta_1\frac{\partial V_1}{\partial p}\right) dp \end{aligned}$$

No default continue

where

$$\mathcal{L}'(V) = \frac{\partial V}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \rho w \delta \frac{\partial^2 V}{\partial r \partial p} + \tfrac{1}{2} \delta^2 \frac{\partial^2 V}{\partial p^2}$$

$$\mathcal{L}(Z) = \frac{\partial Z}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2}$$

and ρ is the correlation between dX_1 and dX_2.

No default continue

Choose Δ and Δ_1 to eliminate the risky terms.

$$\Delta_1 = \frac{\partial V}{\partial p} \Big/ \frac{\partial V_1}{\partial p}$$

and

$$\Delta = \frac{\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r}}{\frac{\partial Z}{\partial r}}$$

Case B: Default

If the bond defaults then the change in the value of the portfolio is

$$d\Pi = -V + \Delta_1 V_1 + O(dt^{1/2}).$$

Taking expectations and using the bond-pricing equation for the riskless bond, we find that the value of the risky bond satisfies

$$\frac{\partial V}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \rho w \delta \frac{\partial^2 V}{\partial r \partial p} + \tfrac{1}{2} \delta^2 \frac{\partial^2 V}{\partial p^2} +$$
$$(u - \lambda w)\frac{\partial V}{\partial r} + (\gamma - \lambda' \delta)\frac{\partial V}{\partial p} - (r + p)V = 0.$$

Feynman Kac:2nd call

Similar to the interest rate risk, λ' is called the market price of default intensity risk. So the fundamental pricing formula for the risky bond under risk neutral measure is

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s+p_s)ds}|\mathcal{F}_t\right).$$

Positive recovery

In default there is usually *some* payment, not all of the money is lost. In the table are shown the mean and standard deviations for recovery according to the seniority of the debt. This emphasizes the fact that the rate of recovery is itself very uncertain.

Class	**Mean (%)**	**Std Dev. (%)**
Senior secured	53.80	26.86
Senior unsecured	51.13	25.45
Senior subordinated	38.52	23.81
Subordinated	32.74	20.18
Junior subordinated	17.09	10.90

There is also a statistical relationship between rate of recovery and default rates. (Years with low default rates have higher recovery when there is default.)

Recovery of market value

Suppose that on default we know that we will loss l percent of pre-default value. This will change the partial differential equation .
On default we have

$$d\Pi = -lV + l\Delta_1 V_1 + O(dt^{1/2});$$

we suffer a loss from the first bond but gain l from the second bond. The pricing equation becomes

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w\delta\frac{\partial^2 V}{\partial r\partial p} + \tfrac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}$$

$$+(u-\lambda w)\frac{\partial V}{\partial r} + (\gamma - \lambda'\delta)\frac{\partial V}{\partial p} - (r+lp)V = 0.$$

Recovery of treasury

- Although the assumption of recovery on market value is convenient for the purpose of mathematical modeling and makes economic sense since it measures the loss in value associated with default, it is impossible to give an immediate expression for implied default probability.
- Recovery on treasury assumes that, if a corporate bond defaults, its value will be replaced by a treasury bond with the same maturity. Under this assumption and with the independence of interest rate and hazard rate, the bond price will be

$$V(0,t) = (1 - F(t))\, Z(0,t) + F(t)\theta Z(0,t).$$

- Thus implied default probability can be easily extracted from the above relationship.

Calibration with recovery rate

Year	Riskfree zero rate	Risky bond zero rate	Cummulative PD	Marginal PD
1	5%	5.25%	0.4161%	0.4161%
2	5%	5.50%	1.6584%	1.2422%
3	5%	5.70%	3.4635%	1.8051%
4	5%	5.85%	5.5714%	2.1079%
5	5%	5.95%	7.7316%	2.1602%
	Recovery	40%		

Feynman Kac:3rd call

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s + lp_s)ds} | \mathcal{F}_t\right).$$

It can be rewritten as

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T R_s ds} | \mathcal{F}_t\right).$$

Where

$$R_t = r_t + l\,p_t,$$

is called the risk adjusted discount rate.

Introduction

- In M4, we have seem affine short rate models lead to explicit solutions for bond prices, e.g. Vasicek, CIR and Ho& Lee etc.
- Similarly, affine intensity based models can lead to analytical solutions for risk bond prices.

Pricing for general contingent claim

For a general contingent claim $g(X_T)$ at time T, where

$$X_t = (x_{1t}, x_{2t}, \cdots, x_{nt})$$

is a vector of state variable, The fundamental pricing formula is

$$V(t, T) = \mathbb{E}\left(e^{-\int_t^T R(X_s)ds} g(X_T) | \mathcal{F}_t\right).$$

Solution of general affine model

If the model is affine, i.e.,

- X_t: affine process
- $R(X_s)$: affine in X_t
- $g(X_s)$: affine is X_t

then the general solution is

General Affine Model Solution

$$V(t, T) = e^{\alpha(t,T)+\beta(t,T)X_t}.$$

Affine conditions

In previous section while we deriving risky bond pricing equation, we have the stochastic interest rate

$$dr = u(r,t)dt + w(r,t)dX_1,$$

and the stochastic intensity

$$dp = \gamma(p,t)dt + \delta(p,t)dX_2.$$

To be affine intensity model

- we must choose the functions $u - \lambda w$, w, $\gamma - \lambda' \delta$, δ and ρ carefully.
- We must choose $u - \lambda w$ and w^2 to be linear in state variables, same for $\gamma - \lambda' \delta$ and δ^2.
- The form of the correlation coefficient is assumed to be constant, i.e., $dX_1 dX_2 = \rho\, dt$.

Solution for risky ZCB

Suppose recovery rate is a constant θ, and $s_t = (1-\theta)p_t$.
With appropriate choices of the functions in the two stochastic differential equations we find that the solution with final condition $V(r, s, T) = 1$ is

$$V = \exp\{A(t, T) - B(t, T)r - C(t, T)s\}$$

where A, B and C satisfy non-linear first-order ordinary differential equations.

Calibration

- In M4 lectures, we have seen when there is a time dependent parameter in the dynamics of the spot interest rate model, then this model have the freedom to fit the entire term structure of the initial yield curve.
- Similarly, if there is time dependence in the model of the intensity, and the model is sufficiently tractable, then we can also fit risky bond term structure.

Standard two-factor Vasicek model

Suppose there are two state variables X and Y whose dynamics can be written as

$$\begin{aligned} dX &= (a_1 - b_{11}X - b_{12}Y)dt + \sigma_1 dW_1 \\ dY &= (a_2 - b_{21}X - b_{22}Y)dt + \sigma_2 dW_2 \end{aligned}$$

where

$$dW_1 dW_2 = \rho dt$$

The risk adjusted discount rate is

$$R = g_0 + g_1 X + g_2 Y.$$

Modified Canonical form

To simplify the parameterizations(standard form doesn't usually have unique solution) we can work with canonical form

$$\begin{aligned} dX &= -a\,Xdt + \sigma dW_1 \\ dY &= -b\,Ydt + \eta dW_2 \end{aligned}$$

and

$$R(t) = \phi(t) + X(t) + Y(t).$$

Note in order to calibrate on the risky bond yield we employ a time dependent parameter $\phi(t)$ into the risk adjusted rate. Here we correspond the state variable X to the short rate r and Y to the spread $s = (1-\theta)p$.

Two-factor Vasicek risky BPE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \tfrac{1}{2}\eta^2 \frac{\partial^2 V}{\partial y^2} + \sigma\eta\rho \frac{\partial^2 V}{\partial x \partial y} - ax\frac{\partial V}{\partial x} - by\frac{\partial V}{\partial y} - RV = 0$$

Risky bond solution

Plug the general affine solution into the BPE, come up with 3 ODEs w.r.t A, B and C respectively, solve them one by one to obtain the price of risky bond.

$$
\begin{aligned}
V(t,T) \quad = \quad & \exp\left\{-\int_t^T \phi(s)ds - \frac{1-e^{-a(T-t)}}{a}X(t) - \right. \\
& \left. \frac{1-e^{-b(T-t)}}{b}Y(t) + \tfrac{1}{2}M(t,T)\right\}
\end{aligned}
$$

Risky bond solution: continue

where

$$
\begin{aligned}
M(t,T) &= \frac{\sigma^2}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}\right] \\
&+ \frac{\eta^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right] \\
&+ 2\rho\frac{\sigma\eta}{ab}\left[T-t-\frac{1-e^{-a(T-t)}}{a}-\frac{1-e^{-b(T-t)}}{b}\right. \\
&- \left.\frac{e^{-(a+b)(T-t)-1}}{a+b}\right]
\end{aligned}
$$

Compare to structural approach

1. Structural models assume that the modeler has the same information set as the firms manager-complete knowledge of all the firm's assets and liabilities. In contrast, reduced form models assume that the modeler has the same information set as the market-incomplete knowledge of the firms condition.
2. Structural models use a firm's asset and debt values to determine the time of default, thus defaults are endogenously generated within the model. Hence default is locally predictable within the model. In contrast, the time of default in intensity models is determined by the first jump of an jump process whose hazard rate is given by exogenous stochastic process. Default can not be predicted in this case.

Pros and Cons

1. In contract to Structural approach, reduced form models do not need to model firm's asset and liabilities. Given its analytical tractability inherited from similar term structure models, one if its most appealing feature is that it is market consistent and arbitrage-free through the ability of calibration on live market data. Hence it is the top modelling choice for the valuation and market risk analysis for most traded assets and portfolio.
2. Although the model is very popular, it excludes several plausible situations. For example, the fact that the counting process is not adopted to market information, the default event has no influence on all market variable including the default intensity itself. One of the consequences is the model is inadequate to capture default correlation.

Please take away the following important ideas

- Poisson process assumes constant intensity and Cox Process assumes stochastic intensity.
- Risky bond is discounted by risk-adjusted interest rate.
- Stochastic default intensity increases dimension of bond pricing partial differential equation .
- Risky bond pricing partial differential equation are consistent with fundamental pricing formula through Feynman Kac.
- Reduced-form models are tractable if they satisfy affine structure.