CQF Module 4 Exercise Solution

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2 The Black-Derman & Toy short-rate model is

$$d(\log r) = \left(\theta(t) + \frac{d(\log\sigma(t))}{dt}\log r\right)dt + \sigma(t)dW \tag{1}$$

Let $f(x) = \exp x$ and $X = \log r$. Apply Itô's Lemma on $f(X)$ we obtain

$$d(f(X)) = \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial f^2}{\partial^2 X}dX^2$$

where $\partial f/\partial X = \exp(X) = r$ and $\partial f^2/\partial^2 X = \exp(X) = r$.
Given dX from Equation 1, we then have

$$\begin{aligned} d(f(X)) &= dr = \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial f^2}{\partial^2 X}dX^2 \\ &= r(t)\left(\theta(t) + \frac{d(\log\sigma(t))}{dt}\log r\right)dt + \sigma(t)r(t)dW + \frac{1}{2}\sigma^2(t)dt \\ &= r(t)\left(\theta(t) + \frac{d(\log\sigma(t))}{dt}\log r + \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)r(t)dW \end{aligned}$$

Using the notation of

$$dr = A(r,t)dt + B(r,t)dW$$

yields

$$\begin{cases} A(r,t) = r(t)\left(\theta(t) + \frac{d(\log\sigma(t))}{dt}\log r + \frac{1}{2}\sigma^2(t)\right) \\ B(r,t) = \sigma(t)r(t) \end{cases}$$