# CQF Module 4 Examination

## January 2016 Cohort

**Instructions**

All questions must be attempted. Books and lecture notes may be referred to. Spreadsheets and VBA (or other recognised programing language) may be used.
**Any queries should be e-mailed to riaz.ahmad@fitchlearning.com for questions 1-4 (only) and richard.diamond@fitchlearning.com for question 5**

$dW$ is the usual increment of a Brownian motion.

1. We wish to find the approximate value of a cashflow for a floorlet on the one month LIBOR, when using the Vasicek model. Show that this is given by

$$\max\left(r_f - r - \tfrac{1}{24}\left(\eta - \gamma r\right), 0\right),$$

where $r_f$ is the floor rate and $r$ the spot rate. **[10 Marks] You must start by considering the yield curve power series expression given in the calibration and data analysis lecture. Full working should be given for the series expansion**.

2. Consider the Black-Derman & Toy (BDT) short-rate model given by

$$d\left(\log r\right) = \left(\theta\left(t\right) + \frac{d\left(\log\sigma\left(t\right)\right)}{dt}\log r\right)dt + \sigma\left(t\right)dW.$$

Using Itô, write down the BDT model as

$$dr = A\left(r,t\right)dt + B\left(r,t\right)dW.$$

**[5 Marks]**

3. Consider the spot rate $r$, which evolves according to the SDE

$$dr = u\left(r,t\right)dt + w\left(r,t\right)dW.$$

The extended Hull and White model has drift and diffusion

$$u\left(r,t\right) = \eta\left(t\right) - \gamma r, \qquad w\left(r,t\right) = c,$$

in turn, where $\eta\left(t\right)$ is an arbitrary function of time $t$ and $\gamma$ and $c$ are constants. Deduce that the value of a zero coupon bond, $Z\left(r,t\;;T\right)$ which has

$$Z\left(r,T;T\right) = 1$$

in the extended Hull and White model is given by

$$Z\left(r,t\;;T\right) = \exp\left(A\left(t;T\right) - rB\left(t;T\right)\right),$$

where

$$\begin{aligned} B\left(t\;;T\right) &= \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) \\ A\left(t;T\right) &= -\int_t^T \eta\left(\tau\right)B\left(\tau;T\right)\,d\tau + \frac{c^2}{2\gamma^2}\left(\left(T-t\right) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right). \end{aligned}$$

**[8 Marks] Note: You are required to solve the Bond Pricing Equation for this model.**

4. Consider the process given by

$$dU_t = -\gamma U_t dt + \sigma dW_t,\ U_0 = u,$$

where $\gamma$, $\sigma$ are constants. Solve this equation for $U_t$ and hence obtain the expectation $\mathbb{E}[U_t]$ and variance $\mathbb{V}[U_t]$. **[12 Marks]**

5. HJM model evolves the whole forward curve. To obtain an expectation of LIBOR rate in the future, Forward LIBOR $L(t; T_i, T_{i+1})$, select the rate from corresponding tenor column $\tau = T_{i+1} - T_i$ of the HJM output, from the correct simulated time (future curves are in rows). Convert to the simple annualised rate using $L = m\left(e^{f/m} - 1\right)$ where $m$ is compounding frequency per year.

   Forward LIBOR re-sets (expires) at $T_i$ and matures (paid on some cashflow) at $T_{i+1}$. This future payment has to be discounted.

   Use the *HJM Model - MC.xlsm* spreadsheet from the HJM Model Lecture in order to price **a caplet option** written on 6M LIBOR starting six months from today:

   $$\mathrm{DF}_{\mathrm{OIS}}(0,1) \times \max\left(L(0; 0.5, 1)\ -\ K, 0\right) \times \tau \times N$$

   with the following parameters $K = 3.5\%$, $N = 100,000$, $\tau$ as follows from the task, and $\mathrm{DF}_{\mathrm{OIS}}(0,1) = 0.996$, a discount factor taken from a curve built from traded OIS swaps. **[35 Marks]**

   To satisfy the risk-neutral expectation conduct Monte-Carlo with and without the use of antithetic variance reduction technique.

   - Two convergence diagrams must be provided together with a brief error analysis.
   - Forward LIBOR from the given forward curve calculation examples are in *Yield Curve v2.xlsm*
   - An annotated caplet payoff calculation is given on Caplet tab in *HJM Model - MC.xlsm*