CQF Lecture 5.6 CDO and Copula Models

Solutions

1. A **synthetic CDO**, based on balance sheet information, is structured as follows:

Assets:	125 single-name CDS
Principal:	0.8 million per name (EAD_i)
Maturity:	5 years
CDS spread:	200 bps
Payments:	Act/360 quarterly in arrears

Tranche	Attachment point	Expected Loss	Fair Spread	Rating
Senior	7%-10%	0.002%	L+45	AAA
Class A	5%-7%	0.1%	L+70	AA-
Class B	2%-5%	2.3%	L+20	BBB-
Class C	0%-2%	26.27%	Excess spread	NR

Table 1: CDO Capital Structure

(a) Holders of which trache are long correlation and why? Which tranche is the most sensitive to changes in default correlation?

(b) What about exposure of mezzanine noteholders to default correlation?

(c) Assuming 0% recovery, how many defaults should occur before Senior tranche experiences a capital loss? If we assume 40% recovery how much more protection does that afford to Senior noteholder?

(d) A downgrade is triggered when the entire Equity tranche is lost. Assuming 0% recovery, how many defaults should occur before the implied ratings are downgraded.

Solution:

(a) **Equity Tranche investors are long correlation** thus, *equity note value increases with correlation rising.* In fact, higher correlation implies higher probabilities of both outcomes: 'less defaults' and 'more defaults'. Equity Tranche investors are sensitive to even one default so they desire higher probability of less defaults.
Senior Tranche is sensitive to large *changes in correlation.* This tranche suffers loss in extreme scenarios associated with very high correlation ≈ 1 and multiple defaults.

(b) Mezzanine notes are the least sensitive to change in correlation, so investors in these notes should be less concerned with correlation parameter when valuing the notes.

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(c) Senior note has 7% subordination, therefore it is affected once the portfolio suffers 9 defaults or more – the number of defaults has to be an a whole integer above $0.07 * 125 = 8.75$ without recovery.

Assuming 40% recovery $LGD = 1 - RR = 0.6$, the loss per name is $0.6 * 0.8 = 0.48$ million. 7% subordination from 100 million portfolio notional gives the proportion of $\frac{\sum EAD}{LGD \times EAD_i} = 7,000,000/480,000 = 14.58333$ or 15 defaults. Recovery assumption affords additional protection for six **further** defaults $(15 - 9 = 6)$.

(d) We assume an implied rating downgrade after the first tranche is wiped out. That will be after 2% of the portfolio has suffered default, or 2.5 defaults, assuming no recovery value. In practice, downgrade will be triggered by 2 defaults.

2. Consider a random variable X that provides information about default time, conditional on intensity parameter θ. X follows the exponential distribution with *cdf*:

$$\Pr(X \leq x|\theta) \equiv F(x|\theta) = 1 - e^{-\theta x}$$

Assuming that intensity follows Gamma distribution, i.e., $\theta \sim \Gamma(\alpha, \beta)$ with *pdf*:

$$g(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

Use the result below to show that the unconditional **marginal cdf** of X follows Pareto distribution–that is,

$$\Pr(X \leq x) \equiv F(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}$$

Hint: In order to recover a *cdf* for the unconditional distribution, we need to integrate over the conditional distribution as follows:

$$F(x) = \int_0^\infty F(x|\theta)\, g(\theta)\, d\theta.$$

Solution:

$$\begin{aligned}
F(x) &= \int_0^\infty F(x|\theta)\, g(\theta)\, d\theta \\
&= \int_0^\infty (1 - e^{\theta x}) \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta \\
&= 1 - \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-(x+\beta)\theta} d\theta \qquad \text{(integration over } \Gamma \text{ } pdf \text{ gave 1)} \\
&= 1 - \left(\frac{\beta}{\beta + x}\right)^\alpha \int_0^\infty \frac{(\beta + x)^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-(\beta+x)\theta} d\theta \qquad \text{(notice change of variable to } \beta + x\text{)} \\
&= 1 - \left(\frac{\beta + x}{\beta}\right)^{-\alpha} \qquad \text{(have integrated over } \Gamma \text{ } pdf \text{ again)} \\
&= 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}.
\end{aligned}$$

This is a *cdf* for Pareto distribution with intensity $\frac{\alpha}{x + \beta}$.

3. Consider two identically distributed variables X_1 and X_2. When conditioned on θ, same as in the previous exercise, they are independent. Their unconditional joint *cdf* is

$$\begin{aligned} F(x_1, x_2) &\equiv \Pr(X_1 \le x_1, X_2 \le x_2) \\ &= 1 - \Pr(X_1 > x_1) - \Pr(X_2 > x_2) + \Pr(X_1 > x_1, X_2 > x_2) \end{aligned}$$

In a practical context, X_1 and X_2 represent default times τ_1 and τ_2 respectively, so that $F(t_1, t_2) = \Pr(\tau_1 \le t_1, \tau_2 \le t_2)$ but let's continue working in 'random variable X' notation.

(a) Express the **joint *cdf*** $F(x_1, x_2)$ as a function of the isolated marginal *cdf*s $F(x_1)$ and $F(x_2)$ (also called 'marginals').

Hint: We can spot $F(x) = 1 - \Pr(X > x)$ but the unconditional term is unknown: $\Pr(X_1 > x_1, X_2 > x_2)$? We need to calculate it by integration over the product of $\Pr(X_1 > x_1 \,|\theta) \Pr(X_2 > x_2 \,|\theta)\, g(\theta)$, treating *conditional* X_1 and X_2 as independent.

(b) By substituting uniform variables u_1, u_2 instead of marginals $F(x_1)$ and $F(x_2)$ show that the associated **copula function** is

$$\begin{aligned} C(u_1, u_2) &\equiv \Pr(U_1 \le u_1, U_2 \le u_2) \\ &= u_1 + u_2 - 1 + \left((1-u_1)^{-\frac{1}{\alpha}} + (1-u_2)^{-\frac{1}{\alpha}} - 1\right)^{-\alpha} \end{aligned}$$

Solution:

(a) First, we need define the probabilities for the inverted inequality sign

$$\begin{aligned} \Pr(X \le x|\theta) &= 1 - e^{-\theta x} \Longrightarrow \\ \Pr(X > x\,|\theta) &= e^{-\theta x} \end{aligned}$$

Then, the unknown unconditional term is found by integration (see previous exercise)

$$\begin{aligned} \Pr(X_1 > x_1, X_2 > x_2) &= \int_0^\infty \Pr(X_1 > x_1, X_2 > x_2|\theta) g(\theta) d\theta \\ &= \int_0^\infty \Pr(X_1 > x_1|\theta) \Pr(X_2 > x_2|\theta) g(\theta) d\theta \\ &= \int_0^\infty e^{-\theta x_1} e^{-\theta x_2} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta \\ &= \left(\frac{\beta}{\beta + x_1 + x_2}\right)^\alpha \\ &= \left(1 + \frac{x_1}{\beta} + 1 + \frac{x_2}{\beta} - 1\right)^{-\alpha} \end{aligned}$$

In this function, we need to spot the isolated marginals $F(x)$

$$\left(1 + \frac{x}{\beta}\right)^{-\alpha} = 1 - F(x) \quad \Longrightarrow \quad \left(1 + \frac{x}{\beta}\right) = \underbrace{(1 - F(x))^{-\frac{1}{\alpha}}}$$

and therefore,

$$\Pr(X_1 > x_1, X_2 > x_2) = \left(\underbrace{(1 - F(x_1))^{-\frac{1}{\alpha}}} + \underbrace{(1 - F(x_2))^{-\frac{1}{\alpha}}} - 1\right)^{-\alpha}.$$

We expressed $\Pr(X_1 > x_1, X_2 > x_2)$ in terms of the marginals $F(x_1)$ and $F(x_2)$ that were, otherwise, hidden. Now, we can we can express the entire joint *cdf* in terms of the marginals:

$$\begin{aligned} F(x_1, x_2) &= 1 - \Pr(X_1 > x_1) - \Pr(X_2 > x_2) + \Pr(X_1 > x_1, X_2 > x_2) \\ &= 1 - (1 - F(x_1)) - (1 - F(x_2)) + \left((1 - F(x_1))^{-\frac{1}{\alpha}} + (1 - F(x_2))^{-\frac{1}{\alpha}} - 1\right)^{-\alpha} \\ &= F(x_1) + F(x_2) - 1 + \left((1 - F(x_1))^{-\frac{1}{\alpha}} + (1 - F(x_2))^{-\frac{1}{\alpha}} - 1\right)^{-\alpha} \end{aligned}$$

(b) We can simply replace $F(x_i) = u_i$ in order to obtain the associated copula function for this bivariate Pareto copula:

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left((1 - u_1)^{-\frac{1}{\alpha}} + (1 - u_2)^{-\frac{1}{\alpha}} - 1\right)^{-\alpha}.$$

It is also known as the Clayton copula of the Archimedean family that allows to model the left tail dependence.

Extending Exercises 3 and 4, several important observations are to be made:

- The solution showed that a joint distribution function (cdf) can be expressed as a copula function. While it is possible to find ready analytical solutions for the common Archimedean copulae, this property is not guaranteed.
- Elliptical copulae of consequence (Gaussian and Student's t) **do not** have explicit copula functions $C(u_1, u_2)$–notice the capital C. They are expressed as below, **without further solution**:

$$C(u_1, u_2, \ldots, u_n) = \Phi_n\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n); \Sigma\right)$$

  where Φ_n is a *multivariate* standard Normal distribution and Σ is a *correlation* matrix. On the other hand, the copula probability density function (*pdf*) $c(u_1, u_2)$—notice the lower case c—is known for both, Gaussian and Student's t (see Lecture Slides).
- Copula functions of the Archimedean family allow the representation as a sum of u_i, as seen in the previous exercise, and in general form:

$$C(u_1, u_2, \ldots, u_n) = \phi^{-1}\left(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_n)\right)$$

  where ϕ is *a copula generator* of some known functional form.

  Some sources invert $\psi(u) = \phi^{-1}(u)$ and express Archimedean copulae as

$$C(u_1, u_2, \ldots, u_n) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \ldots + \psi^{-1}(u_n)\right)$$

  These representations are equivalent to each other.

- In general, the copula of a multivariate random variable $\mathbf{X}$ is a joint distribution of the uniform *grades* $\mathbf{U} \equiv F(x)$, where $\mathbf{U} \sim \mathrm{U}[0,1]$ and $F(x) = \Pr(X \leq x)$.

  One can think of the copula as 'a pure joint distribution'—that is, a standardised distribution that describes the joint features of a multivariate random variable, i.e., co-dependence.

4. Consider **a copula function** of the Archimedean family

$$C(u_1, u_2, \ldots, u_n) = \phi^{-1}\left(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_n)\right)$$

Given the copula generator

$$\phi(u) = -\ln\left(\frac{e^{-\alpha u} - 1}{e^{-\alpha} - 1}\right)$$

show that the copula function can be expressed explicitly as

$$C(u_1, u_2, \ldots, u_n) = -\frac{1}{\alpha}\ln\left[1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right]$$

Identify this copula function by name. What does parameter α represent?

$\boxed{\textbf{Solution:}}$

The first step is to find the function for u, the inverse copula generator $\phi^{-1}(u)$, because it determines the form of the copula function.

$$\begin{aligned} e^{-\phi} &= \frac{e^{-\alpha u} - 1}{e^{-\alpha} - 1} \\ e^{-\alpha u} &= 1 + e^{-\phi}(e^{-\alpha} - 1) \\ u &= -\frac{1}{\alpha}\ln[1 + (e^{-\alpha} - 1)e^{-\phi}] \end{aligned}$$

The copula function is

$$C(u_1, u_2, \ldots, u_n) = \phi^{-1}\left(\sum_{i=1}^{n}\phi(u_i)\right)$$

Noting that $\displaystyle\sum_{i=1}^{n}\phi(u_i) = -\sum_{i=1}^{n}\ln\left(\frac{e^{-\alpha u_i} - 1}{e^{-\alpha} - 1}\right) = -\ln\prod_{i=1}^{n}\left(\frac{e^{-\alpha u_i} - 1}{e^{-\alpha} - 1}\right)$

$$\begin{aligned} C(u_1, u_2, \ldots, u_n) &= -\frac{1}{\alpha}\ln\left[1 + (e^{-\alpha} - 1)\prod_{i=1}^{n}\frac{(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)}\right] \\ &= -\frac{1}{\alpha}\ln\left[1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right] \end{aligned}$$

The Frank Copula is invariant to permutations of its arguments u_i, therefore, correlation does not compound. Dependence (association) parameter α is a proxy to correlation and converted to the Kendall's tau as $\rho_K = 1 - \frac{4}{\alpha}[1 - D_1(\alpha)]$ using the Debye function

$$D_1(\alpha) = \frac{1}{\alpha}\int_0^{\alpha}\frac{x}{e^x - 1}dx$$

This parametrisation of α will give positive values for positive correlation, but notice that the copula function uses $-\alpha$ throughout.

5. **Copula function can price multi-asset options.** For example, a bivariate European digital **put** pays one unit of currency if two assets are both below the strike. Because of the correlation (modelled via association parameter α), we cannot simply multiply the probabilities of each event. Consider a simplified scenario for assets 1 and 2:

$$T = 1, r = 0, K_1 = K_2 = 100, \sigma_1 = \sigma_2 = 20\%, S_1 = S_2 = 102.02$$

Use Frank Copula with $\alpha = 5$ to price a bivariate digital put.

$$C(u_1, u_2, \cdots, u_n) = -\frac{1}{\alpha} \ln \left[1 + \frac{\prod_{i=1}^{n} (e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}} \right]$$

Hint: the risk-neutral probability of a European call option being *in the money* at maturity is $u = N(d_2)$.

$\boxed{\textbf{Solution:}}$

Price of a single-asset digital (binary) option is equal to the probability that it will be in the money at maturity. For a put option,

$$B(S, t) = e^{-r(T-t)}(1 - N(d_2)) = u \qquad \text{(will use as copula input)}$$

$$d_2 = \frac{\ln(S/K) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$

This probability is risk-neutral and the option value is discounted, all according to the the Black-Scholes approach to option pricing.

$$d_2 = \frac{\ln(102.02/100) + (0 - 0.5 \times 0.2^2) \times 1}{0.2 \times \sqrt{1}} \approx 0$$

Therefore, the value each (of two) single-asset binary put is $u_1 = u_2 = 1 - N(0) = 0.5$.

The joint probability of both assets being below their respective strikes at maturity is

$$\begin{aligned} C(u_1, u_2) &= -\frac{1}{5} \ln \left[1 + \frac{\left(e^{-5\times0.5} - 1\right)\left(e^{-5\times0.5} - 1\right)}{(e^{-5} - 1)} \right] \\ &= -\frac{1}{5} \ln [0.151716] \\ &= 0.3771. \end{aligned}$$

Copula function is equivalent to the joint cumulative probability $C(u_1, u_2) \equiv F(x_1, x_2)$ by Sklar theorem.

Archimedean copulae give the benefit of an analytical solution to the joint cumulative probability (copula function) but model co-dependence of entities using the same association parameter. An assumption is made that the correlation is homogeneous among all assets.