

# Applied Stochastic Calculus

In this lecture

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Popular models / asset price SDE

The evolution of financial assets is random and depends on time. They are examples of *stochastic processes* which are random variables indexed (parameterized) with time.

If the movement of an asset is discrete it is called a *random walk*. A continuous movement is called a *diffusion process*. We will consider the asset price dynamics to exhibit continuous behaviour and each random path traced out is called a *realization*.

We need a definition and set of properties for the randomness observed in an asset price realization, which will be *Brownian Motion*.

# 1 Construction of Brownian Motion

Brownian Motion can be constructed by careful scaling of a simple symmetric random walk. Consider the coin tossing experiment

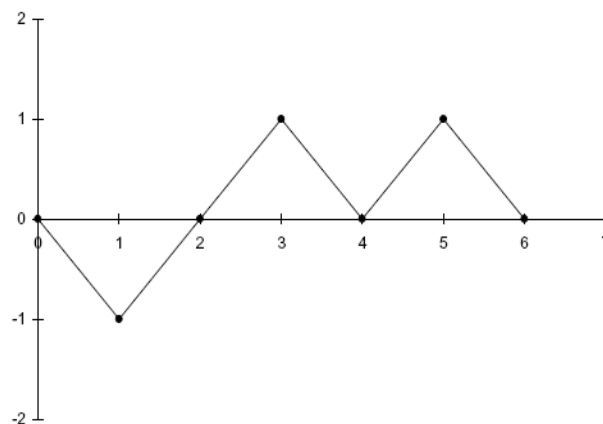


Figure 1: After 6 coin tosses

where we define the random variable

$$R_i = \begin{cases} +1 & \text{if } H \\ -1 & \text{if } T \end{cases}$$

and examine the statistical properties of  $R_i$ .

Firstly the mean

$$\mathbb{E}[R_i] = (+1)\frac{1}{2} + (-1)\frac{1}{2} = 0$$

and secondly the variance

$$\begin{aligned}\mathbb{V}[R_i] &= \mathbb{E}[R_i^2] - \underbrace{\mathbb{E}^2[R_i]}_{=0} \\ &= \mathbb{E}[R_i^2] = 1\end{aligned}$$

Suppose we now wish to keep a score of our winnings after the  $n^{\text{th}}$  toss  
- we introduce a new random variable

$$W_n = \sum_{i=1}^n R_i$$

This allows us to keep a track of our total winnings. This represents the position of a marker that starts off at the origin (no winnings). So starting with no money means

$$W_0 = R_0.$$

Now we can calculate expectations of  $W_n$

$$\mathbb{E}[W_n] = \mathbb{E}\left[\sum_{i=1}^n R_i\right] = \sum_{i=1}^n \mathbb{E}[R_i] = 0$$

$$\mathbb{E}[W_n^2] =$$

$$\begin{aligned} & \mathbb{E}[R_1^2 + R_2^2 + \dots + R_n^2 + 2R_1R_2 + \dots + 2R_{n-1}R_n] \\ &= \mathbb{E}\left[\sum_{i=1}^n R_i^2\right] + 2\mathbb{E}\left[\sum_{\substack{i=1 \\ j \neq i}}^n R_i R_j\right] = \sum_{i=1}^n \mathbb{E}[R_i^2] + 2 \underbrace{\sum_{i=1}^n \mathbb{E}[R_i] \mathbb{E}[R_j]}_{j \neq i} \\ &= n \cdot 1 + 2 \cdot 0 \cdot 0 = n \end{aligned}$$

$V[X] = \mathbb{E}[X^2]$  if  $\mu = 0$

$\sum_{i=1}^n \sum_{j \neq i}^n$

## A Note on Variations

Consider a function  $f_t$  where  $t_i = i \frac{t}{n}$ , we can define different measures of how much  $f_t$  varies over time as

$$V^N = \sum_{i=1}^n |f_{t_i} - f_{t_{i-1}}|^N$$

The cases  $N = 1, 2$  are important.

$$V = \sum_{i=1}^n |f_{t_i} - f_{t_{i-1}}|$$

*variation of trajectory - sum of absolute changes*

$$V^2 = \sum_{i=1}^n |f_{t_i} - f_{t_{i-1}}|^2$$

*quadratic variation - sum of squared changes*

Now look at the *quadratic variation* of the random walk.

After each toss, we have won or lost \$1. That is

$$W_n - W_{n-1} = \pm 1 \implies |W_n - W_{n-1}| = 1$$

Hence

$$\sum_{i=1}^n \underbrace{(W_i - W_{i-1})^2}_{=1} = n$$

Let's now extend this by introducing time dependence. Perform six tosses of a coin in a time  $t$ . So each toss must be performed in time  $t/6$ , and a bet size of  $\sqrt{t/6}$  (and not \$1), i.e. we win or lose  $\sqrt{t/6}$  depending on the outcome.

Let's examine the quadratic variation for this experiment

$$\begin{aligned}
 V^2 &= \sum_{i=1}^6 (W_i - W_{i-1})^2 \\
 &= \sum_{i=1}^6 \left( \pm \sqrt{t/6} \right)^2 \\
 &= 6 \times \frac{t}{6} = t
 \end{aligned}$$

$$\pm \sqrt{\frac{t}{6}}$$

Now speed up the game. So we perform  $n$  tosses within time  $t$  with each bet being  $\sqrt{t/n}$ . Time for each toss is  $t/n$ .

$$W_i - W_{i-1} = \pm \sqrt{t/n}$$

The quadratic variation is

$$\begin{aligned}
 V^2 &= \sum_{i=1}^n (W_i - W_{i-1})^2 = n \times \left( \pm \sqrt{t/n} \right)^2 \\
 &= t
 \end{aligned}$$



As  $n$  increases, time between subsequent tosses decreases and the bet sizes become smaller. The time and bet size decrease in turn like

$$\text{time decrease} \sim O\left(\frac{1}{n}\right)$$

$$\text{bet size} \sim O\left(\frac{1}{\sqrt{n}}\right)$$

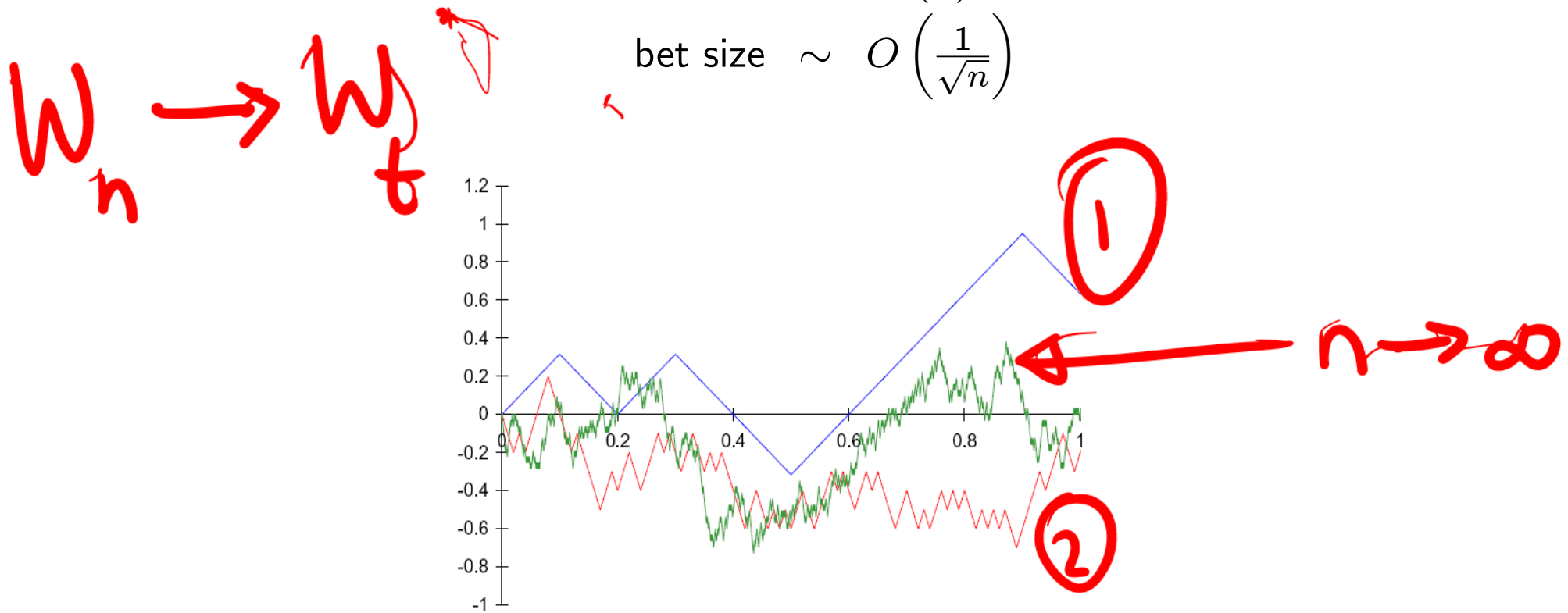


Figure 2: A series of coin tossing experiments

The scaling we have used has been chosen carefully to both keep the random walk finite and also not becoming zero. i.e. In the limit  $n \rightarrow \infty$ , the random walk stays finite. It has an expectation conditional on a starting value of zero, of

$$\begin{aligned}\mathbb{E}[W_t] &= \mathbb{E}\left[\lim_{n \rightarrow \infty} \sum_{i=1}^n R_i\right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[R_i] = n \cdot 0\end{aligned}$$

$$\text{Mean of } W_t = 0$$

$$\begin{aligned}\mathbb{E}[W_t^2] &= \mathbb{E}\left[\lim_{n \rightarrow \infty} \sum_{i=1}^n R_i^2\right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[R_i^2] = \lim_{n \rightarrow \infty} n \cdot \left(\sqrt{t/n}\right)^2 \\ \mathbb{V}[W_t] &= \mathbb{E}[W_t^2] = t\end{aligned}$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

This limiting process as  $dt$  tends to zero is called Brownian Motion and denoted  $W_t$ .

Alternative notation for Brownian motion/Wiener process is  $X_t$  or  $B_t$ .

## Properties of a Wiener Process

A stochastic process  $\{W(t) : t \in \mathbb{R}_+\}$  is defined to be Brownian motion (or a Wiener process) if

- $W(0) = 0$  (with probability one)

a.s.

- Continuity - paths of  $W(t)$  are continuous (no jumps). Differentiable nowhere.

$$W_t \sim N(0, t)$$

$$-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}$$

- for each  $t > 0$  and  $s > 0$ ,  $W(t) - W(s)$  is normal with mean 0 and variance  $|t - s|$ ,

i.e.  $(W(t) - W(s)) \sim N(0, |t - s|)$ . Coin tosses are Binomial, but due to a large number and the C.L.T we have a distribution that is normal. That is  $W(t) - W(s)$  has a pdf given by

$$p(x) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2|t-s|}\right)$$

So Brownian motion has independent Gaussian increments.

- $W(t+s) - W(t)$  is independent of  $W(t)$ .  
This means  $dW_1 = W(t_1) - W(t_0)$  is independent of  $dW_2 = W(t_2) - W(t_1)$ , is independent of  $dW_3 = W(t_3) - W(t_2), \dots, dW_n = W(t_n) - W(t_{n-1})$ .

Also called *standard Brownian motion*.

If we want to be a little more pedantic then we can write some of the properties above as

$$W_t \sim N^{\mathbb{P}}(0, t)$$

i.e.  $W_t$  is normally distributed under the probability measure  $\mathbb{P}$ .

The *covariance function* for a Brownian motion can be calculated as follows. If  $t > s$ ,

$$\begin{aligned} \mathbb{E}[W_t W_s] &= \mathbb{E}[(W_t - W_s) W_s + W_s^2] \\ &= \underbrace{\mathbb{E}[W_t - W_s]}_{N(0, |t-s|)} \mathbb{E}[W_s] + \mathbb{E}[W_s^2] \\ &= (0) \cdot 0 + \mathbb{E}[W_s^2] \\ &= s \end{aligned}$$

The first term on the second line follows from independence of increments. Similarly, if  $s > t$ ; then  $\mathbb{E}[W_t W_s] = t$  and it follows that

$$\mathbb{E}[W_t W_s] = \min\{t, s\}.$$

Brownian motion is a *martingale*.

A stochastic process  $M_t$  is called a  $\mathbb{P}$ –martingale if  $\mathbb{E}_t^{\mathbb{P}} |M_T| < \infty$  for  $t < T$  and

$$\mathbb{E}_t^{\mathbb{P}} [M_T] = M_t$$

That is, it's a conditional expectation and we write formally as

$$\mathbb{E}_t^{\mathbb{P}} [M_T | \mathcal{F}_t] = M_t; \quad t < T$$

$\mathcal{F}_t$  here is an information set called a *filtration*. It is the flow of information associated with a stochastic process.

Taking expectations of both sides gives

$$\mathbb{E}_t [M_T] = \mathbb{E}_t [M_t]; \quad t < T$$

so martingales have constant mean.

A process  $M_t$  which has

$$\mathbb{E}_t^{\mathbb{P}} [M_T | \mathcal{F}_t] \geq M_t$$

is called a *submartingale* and if it has

$$\mathbb{E}_t^{\mathbb{P}} [M_T | \mathcal{F}_t] \leq M_t$$

is called a *supermartingale*.

Using the earlier betting game as an example (where probability of a win or a loss was  $\frac{1}{2}$ )

submartingale - gambler wins money on average  $\mathbb{P}(H) > \frac{1}{2}$

supermartingale- gambler loses money on average  $\mathbb{P}(H) < \frac{1}{2}$

The above definitions tell us that every martingale is also a submartingale and a supermartingale. The converse is also true.

For a Brownian motion, again where  $t < T$

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}} [W_T] &= \mathbb{E}_t^{\mathbb{P}} [W_T - W_t + W_t] \\ &= \underbrace{\mathbb{E}_t^{\mathbb{P}} [W_T - W_t]}_{N(0, |T-t|)} + \mathbb{E}_t^{\mathbb{P}} [W_t]\end{aligned}$$

The next step is important - and requires a little subtlety

The first term is zero. We are taking expectations at time  $t$ — hence  $W_t$  is known, i.e.  $\mathbb{E}_t^{\mathbb{P}} [W_t] = W_t$ . So

$$\mathbb{E}_t^{\mathbb{P}} [W_T] = W_t.$$



Another important property of Brownian motion is that of a *Markov process*. That is if you observe the path of the B.M from 0 to  $t$  and want to estimate  $W_T$  where  $T > t$  then the only relevant information for predicting future dynamics is the value of  $W_t$ . That is, the past history is fully reflected in the present value. So the conditional distribution of  $W_t$  given up to  $t < T$  depends only on what we know at  $t$  (latest information).

Markov is also called memoryless as it is a stochastic process in which the distribution of future states depends only on the present state and not on how it arrived there.

# Mean Square Convergence

Consider a function  $F(X)$ . If

$$\mathbb{E} \left[ (F(X) - l)^2 \right] \longrightarrow 0$$

then we say that  $F(X) = l$  in the *mean square limit*, also called *mean square convergence*. We present a full derivation of the mean square limit. Starting with the quantity:

$$\mathbb{E} \left[ \left( \sum_{j=1}^n (W(t_j) - W(t_{j-1}))^2 - t \right)^2 \right]$$

where  $t_j = \frac{jt}{n} = j\Delta t$ .

Hence we are saying that *up to mean square convergence*,

$$(dW)^2$$

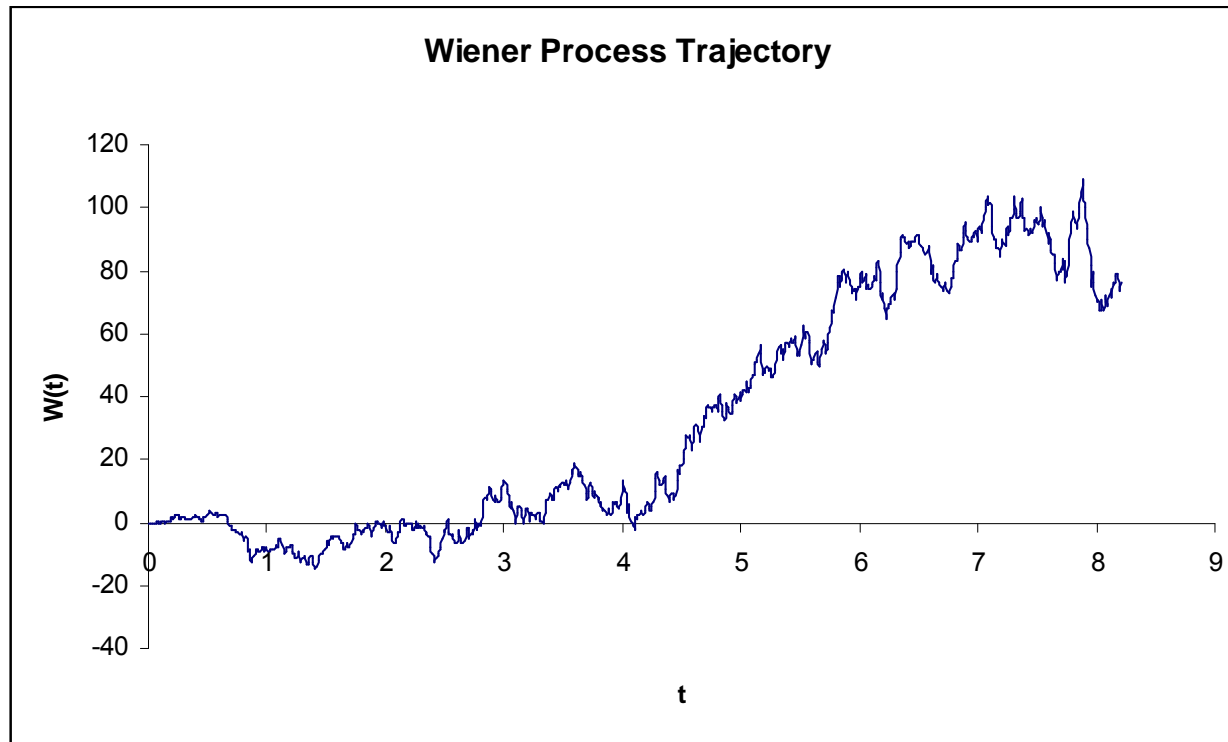
$$dW^2 = dt.$$

This is the symbolic way of writing this property of a Wiener process, as the partitions  $\Delta t$  become smaller and smaller.

$$\lim_{dt \rightarrow 0} dW^2 \rightarrow dt$$

Mean Square Limit

Mean Square Logic



A realisation of a Wiener process, with  $\Delta t = 0.0001$

$$\phi \sim N(0, 1) \quad \sqrt{\delta t} \phi$$

Numerical Scheme:

Start:  $t_0, W_0 = 0$ ; define  $\Delta t = T/n$

loop  $i = 1, 2, \dots, n$  :

$$t_i = t_{i-1} + \Delta t$$

draw  $\phi \sim N(0, 1)$

$$W_i = W_{i-1} + \phi \sqrt{\Delta t}$$

## Taylor Series and Itô

$$dW^2 \rightarrow dt$$

If we were to do a naive Taylor series expansion of  $F$ , completely disregarding the nature of  $W$ , and treating  $dW$  as a small increment in  $W$ , we would get

$$F(W + dW) = F(W) + \frac{dF}{dW}dW + \frac{1}{2} \frac{d^2F}{dW^2}dW^2,$$

ignoring higher-order terms.

$dF$

$\rightarrow dt$

We could argue that  $F(W + dW) - F(W)$  was just the 'change in'  $F$  and so

$$dF = \frac{dF}{dW}dW + \frac{1}{2} \frac{d^2F}{dW^2}dW^2.$$

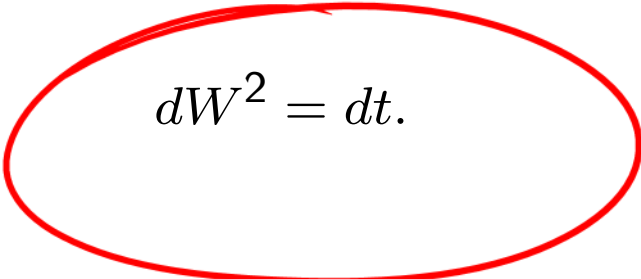
This is *almost* correct.

Because of the way that we have defined Brownian motion, and have seen how the quadratic variation behaves, it turns out that the  $dW^2$  term isn't really random at all.

The  $dW^2$  term becomes (as all time steps become smaller and smaller) the same as its average value,  $dt$ .

Taylor series and the 'proper' Itô are very similar. The only difference being that the correct Itô's lemma has a  $dt$  instead of a  $dW^2$ .

You can, with little risk of error, use Taylor series with the 'rule of thumb'


$$dW^2 = dt.$$

- and in practice you will get the right result.

We can now answer the question, “If  $F = W^2$  what is  $dF$ ?” In this example

$$\frac{dF}{dW} = 2W \text{ and } \frac{d^2 F}{dW^2} = 2.$$

Therefore Itô’s lemma tells us that

$$dF = dt + 2W dW.$$

This is an example of a **stochastic differential equation (SDE)**.



$$t + e^W$$

$$t^2 \sin W$$

$$te^{tW}$$

Now consider a slight extension. A function of a Wiener Process  $f = f(t, W(t))$ , so we can allow both  $t$  and  $W(t)$  to change, i.e.

$$\begin{aligned} t &\longrightarrow t + dt \\ W &\longrightarrow W + dW. \end{aligned}$$

Using Taylor as before

$$\begin{aligned} f(t + dt, W + dW) &= f(t, W) + \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W} dW + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} dW^2 + \dots \\ df &= f(t + dt, W + dW) - f(t, W) = \\ &\quad \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW \end{aligned}$$

This gives another form of Itô:

$$df = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW \quad (*)$$

This is also a SDE.

drift diffusion

## Examples:

1. Obtain a SDE for  $f = te^{W(t)}$ . We need  $\frac{\partial f}{\partial t} = e^{W(t)}$ ;  $\frac{\partial f}{\partial W} = te^{W(t)} = \frac{\partial^2 f}{\partial W^2}$ , then substituting in (\*)

$$df = \left( e^{W(t)} + \frac{1}{2}te^{W(t)} \right) dt + te^{W(t)}dW.$$

We can factor out  $te^{W(t)}$  and rewrite the above as

$$\frac{df}{f} = \left( \frac{1}{t} + \frac{1}{2} \right) dt + dW.$$

2. Consider the function of a stochastic variable  $f = t^2W^n(t)$

$$\frac{\partial f}{\partial t} = 2tW^n; \quad \frac{\partial f}{\partial W} = nt^2W^{n-1}; \quad \frac{\partial^2 f}{\partial W^2} = n(n-1)t^2W^{n-2},$$

in (\*) gives

$$df = \left( 2tW^n + \frac{1}{2}n(n-1)t^2W^{n-2} \right) dt + nt^2W^{n-1}dW.$$

## A Formula for Stochastic Integration

If we take the 2D form of Itô given by (\*), rearrange and integrate over  $[0, t]$ , we obtain a very nice formula for integrating functions of the form  $f(t, W(t))$ :

$$\int_0^t \frac{\partial f}{\partial W} dW = f(t, W(t)) - f(0, W(0)) - \int_0^t \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) d\tau$$

**Example:** Show that

$$\int_0^t (t + e^W) dW = tW + e^W - 1 - \int_0^t (W_\tau + \frac{1}{2} e^{W_\tau}) d\tau.$$

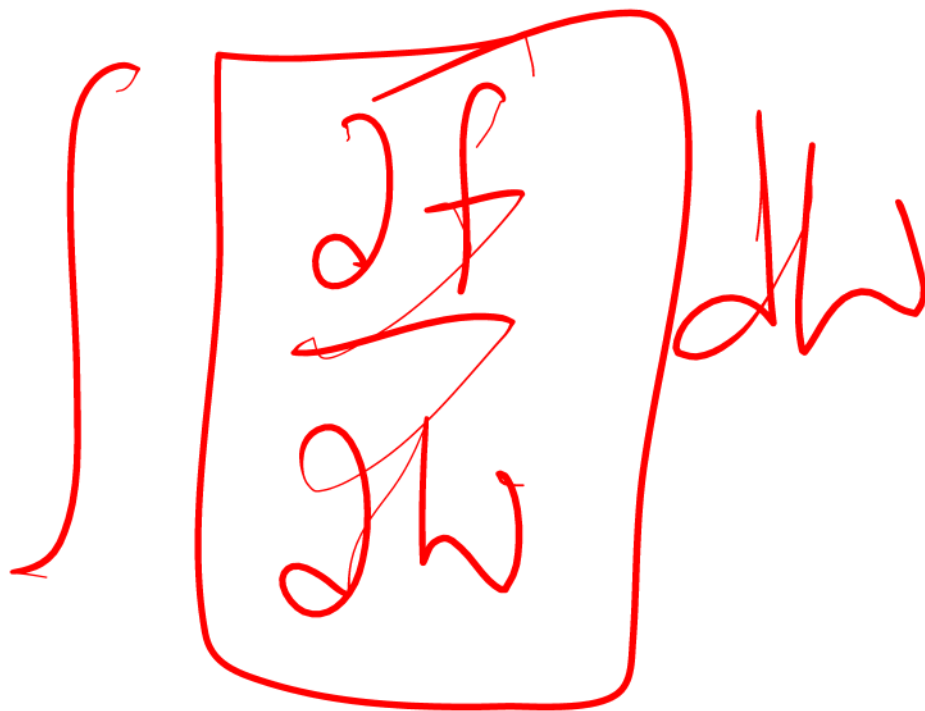
Comparing this to the stochastic integral formula above, we see that  $\frac{\partial f}{\partial W} \equiv t + e^W \implies f = tW + e^W$ . Also

$$\frac{\partial^2 f}{\partial W^2} = e^{W_t}, \quad \frac{\partial f}{\partial t} = W_t.$$

Substituting all these terms in to the formula and noting that  $f(0, W(0)) = 1$  verifies the result.

Naturally if  $f = f(W(t))$  then the integral formula simply collapses to

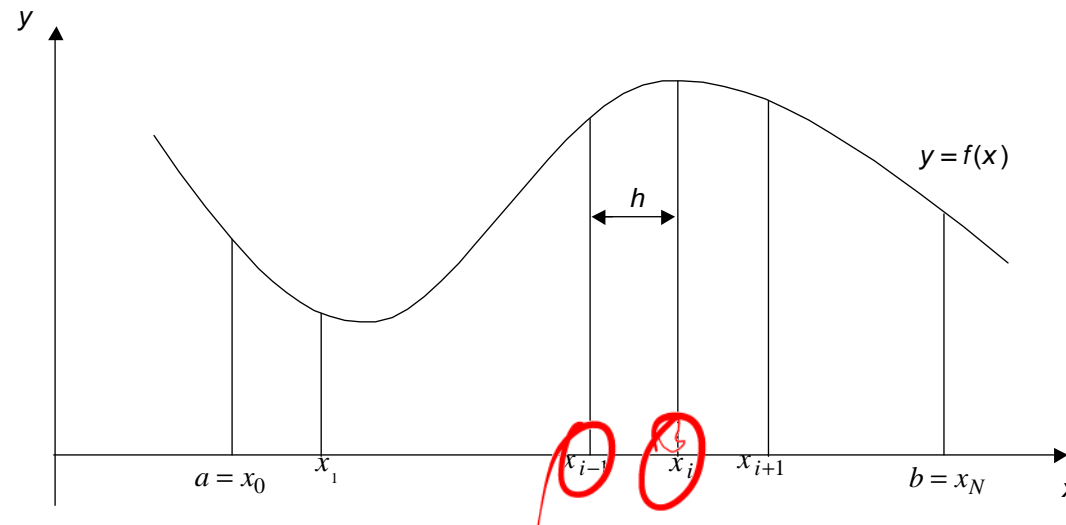
$$\int_0^t \frac{df}{dW} dW = f(W(t)) - f(W(0)) - \frac{1}{2} \int_0^t \frac{d^2 f}{dW^2} d\tau$$



# Itô Integral

Recall the usual Riemann definition of a definite integral

$$\int_a^b f(x) dx$$



which represents the area under the curve between  $x = a$  and  $x = b$ , where the curve is the graph of  $f(x)$  plotted against  $x$ .

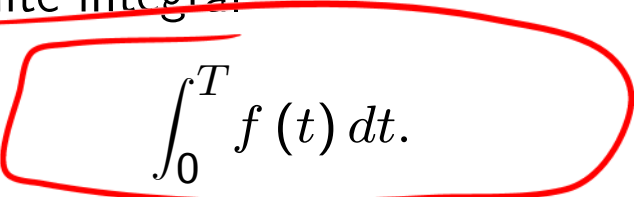
Assuming  $f$  is a "well behaved" function on  $[a, b]$ , there are many different ways (which all lead to the same value for the definite integral).

Start by partitioning  $[a, b]$  into  $N$  intervals with end points  $x_0 = a < x_1 < x_2 < \dots < x_{N-1} < x_N = b$ , where the length of an interval  $dx = x_i - x_{i+1}$  tends to zero as  $N \rightarrow \infty$ . So there are  $N$  intervals and  $N + 1$  points  $x_i$ .

Discretising  $x$  gives

$$x_i = a + idx$$

Now consider the definite integral


$$\int_0^T f(t) dt.$$

With Riemann integration there are a number of ways we can approximate this, all leading to the same calculation e.g.

1. left hand rectangle rule;

$$\int_0^T f(t) dt = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(t_i) (t_{i+1} - t_i)$$

or

2. right hand rectangle rule;

$$\int_0^T f(t) dt = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(t_{i+1}) (t_{i+1} - t_i)$$

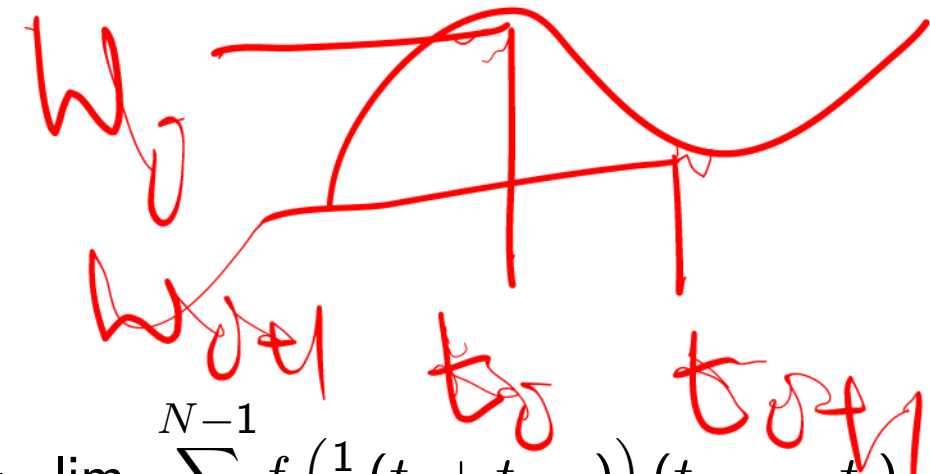
or

3. trapezium rule;

$$\int_0^T f(t) dt = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \frac{1}{2} (f(t_i) + f(t_{i+1})) (t_{i+1} - t_i)$$

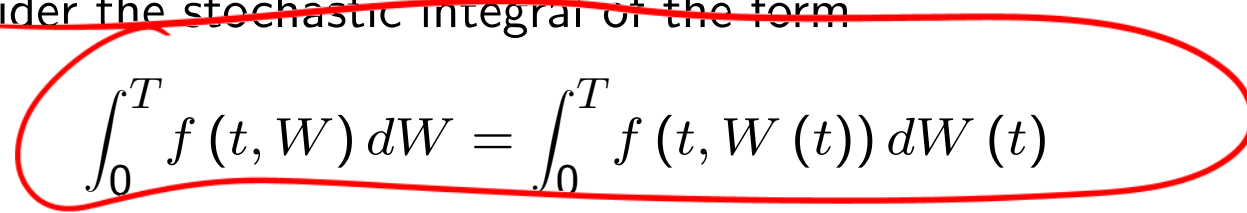
or

#### 4. midpoint rule



$$\int_0^T f(t) dt = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f\left(\frac{1}{2}(t_i + t_{i+1})\right) (t_{i+1} - t_i)$$

In the limit  $N \rightarrow \infty$ ,  $f(t)$  we get the same value for each definition of the definite integral, provided the function is integrable.

Now consider the ~~stochastic integral of the form~~


$$\int_0^T f(t, W) dW = \int_0^T f(t, W(t)) dW(t)$$

where  $W(t)$  is a Brownian motion. We can define this integral as


$$\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(t_i, W_i) (W_{i+1} - W_i),$$



where  $W_i = W(t_i)$ , or as

$$\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(t_{i+1}, W_{i+1}) (W_{i+1} - W_i),$$

or as

$$\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, W_{i+\frac{1}{2}}\right) (W_{i+1} - W_i),$$

where  $t_{i+\frac{1}{2}} = \frac{1}{2}(t_i + t_{i+1})$  and  $W_{i+\frac{1}{2}} = W\left(t_{i+\frac{1}{2}}\right)$  or in many other ways. So clearly drawing parallels with the above Riemann form.

**Very Important:** In the case of a stochastic variable  $dW(t)$  the value of the stochastic integral **does** depend on which definition we choose.

In the case of a stochastic integral, the definition

$$I = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(t_i, W_i) (W_{i+1} - W_i)$$

Non  
anticipatory

is special. This definition results in the **Itô Integral**.

It is special because it is **non-anticipatory**; given that we are at time  $t_i$  we know  $W_i = W(t_i)$  and therefore we know  $f(t_i, W_i)$ . The only uncertainty is in the  $W_{i+1} - W_i$  term.

Compare this to a definition such as

$$\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(t_{i+1}, W_{i+1}) (W_{i+1} - W_i),$$

right

Anticipatory

which is **anticipatory**; given that at time  $t_i$  we know  $W_i$  but are uncertain about the future value of  $W_{i+1}$ . Thus we are uncertain about *both* the value of

$$f(t_{i+1}, W_{i+1})$$

and the value of  $(W_{i+1} - W_i)$  — there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to anticipate the future value of  $W_{i+1}$  so that we may evaluate  $f(t_{i+1}, W_{i+1})$ .

The main thing to note about Itô integrals is that  $I$  is a random variable (unlike the deterministic case). Additionally, since  $I$  is essentially the limit of a sum of normal random variables, then by the CLT  $I$  is also normally distributed, and can be characterized by its mean and variance.

Example: Show that Itô's lemma implies that

$$3 \int_0^T W^2 dW = W(T)^3 - W(0)^3 - 3 \int_0^T W(t) dt.$$

Show that the result also can be found by writing the integral

$$3 \int_0^T W^2 dW = \lim_{N \rightarrow \infty} 3 \sum_{i=0}^{N-1} W_i^2 (W_{i+1} - W_i)$$

Hint: use  $3b^2(a - b) = a^3 - b^3 - 3b(a - b)^2 - (a - b)^3$ .

The Itô integral here is defined as

$$\int_0^T 3W^2(t) dW(t) = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} 3W_i^2 (W_{i+1} - W_i)$$

Now note the hint:

$$3b^2(a - b) = a^3 - b^3 - 3b(a - b)^2 - (a - b)^3$$

hence

$$\begin{aligned} &\equiv 3W_i^2 (W_{i+1} - W_i) \\ &= W_{i+1}^3 - W_i^3 - 3W_i (W_{i+1} - W_i)^2 - (W_{i+1} - W_i)^3, \end{aligned}$$

so that

$$\sum_{i=0}^{N-1} 3W_i^2 (W_{i+1} - W_i) =$$

$$\begin{aligned} & \sum_{i=0}^{N-1} W_{i+1}^3 - \sum_{i=0}^{N-1} W_i^3 - \sum_{i=0}^{N-1} 3W_i (W_{i+1} - W_i)^2 \\ & - \sum_{i=0}^{N-1} (W_{i+1} - W_i)^3 \end{aligned}$$

Now the first two expressions above give

$$\begin{aligned} \sum_{i=0}^{N-1} W_{i+1}^3 - \sum_{i=0}^{N-1} W_i^3 &= W_N^3 - W_0^3 \\ &= W(T)^3 - W(0)^3. \end{aligned}$$

In the limit  $N \rightarrow \infty$ , i.e.  $dt \rightarrow 0$ ,  $(W_{i+1} - W_i)^2 \rightarrow dt$ , so

$$\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} 3W_i (W_{i+1} - W_i)^2 = \int_0^T 3W(t) dt$$

Finally  $(W_{i+1} - W_i)^3 = (W_{i+1} - W_i)^2 \cdot (W_{i+1} - W_i)$  which when  $N \rightarrow \infty$  behaves like  $dW^2 dW \sim O(dt^{3/2}) \rightarrow 0$ .

Hence putting together gives

$$W(T)^3 - W(0)^3 - \int_0^T 3W(t) dt$$

which is consistent with Itô's lemma.

# Diffusion Process

$G$  is called a diffusion process if

$$dG(t) = A(G, t)dt + B(G, t)dW(t) \quad (1)$$

This is also an example of a Stochastic Differential Equation (SDE) for the process  $G$ . It consists of two components:

1.  $A(G, t)dt$  is deterministic – coefficient of  $dt$  is known as the *drift* of the process.
2.  $B(G, t)dW$  is random – coefficient of  $dW$  is known as the *diffusion* or *volatility* of the process.

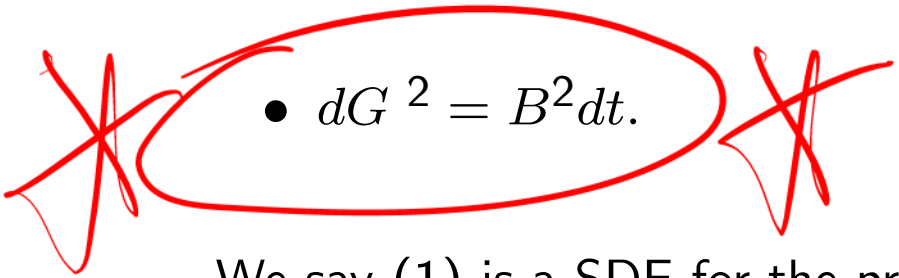
We say  $G$  evolves according to (or follows) this process.

For example

$$dG(t) = (G(t) + G(t-1))dt + dW(t)$$

is not a diffusion (although it is a SDE)

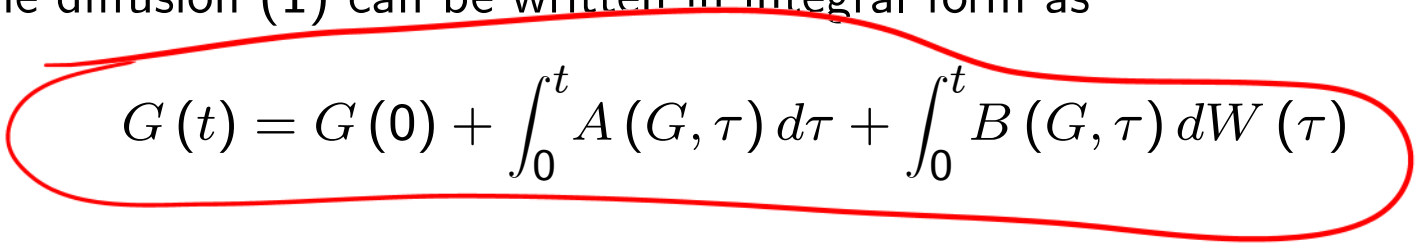
- $A \equiv 0$  and  $B \equiv 1$  reverts the process back to Brownian motion
- Called time-homogeneous if  $A$  and  $B$  are not dependent on  $t$ .



- $dG^2 = B^2 dt.$

We say (1) is a SDE for the process  $G$  or a *Random Walk* for  $dG$ .

The diffusion (1) can be written in integral form as


$$G(t) = G(0) + \int_0^t A(G, \tau) d\tau + \int_0^t B(G, \tau) dW(\tau)$$



Remark: A diffusion  $G$  is a *Markov* process if - once the present state  $G(t) = g$  is given, the past  $\{G(\tau), \tau < t\}$  is irrelevant to the future dynamics.

We have seen that Brownian motion can take on negative values so its direct use for modelling stock prices is unsuitable. Instead a non-negative variation of Brownian motion called *Geometric Brownian motion* (GBM) is used

If for example we have a diffusion  $G(t)$

$$dG = \mu G dt + \sigma G dW_t \quad (2)$$

then the drift is  $A(G, t) = \mu G$  and diffusion is  $B(G, t) = \sigma G$ .

The process (2) is also called Geometric Brownian Motion (GBM).

Brownian motion  $W(t)$  is used as a basis for a wide variety of models. Consider a pricing process  $\{S(t) : t \in \mathbb{R}_+\}$ : we can model its instantaneous change  $dS$  by a SDE

$$dS = a(S, t) dt + b(S, t) dW_t \quad (3)$$

By choosing different coefficients  $a$  and  $b$  we can have various properties for the diffusion process.

A very popular finance model for generating asset prices is the GBM model given by (2). The instantaneous return on a stock  $S(t)$  is a constant coefficient SDE

$$\frac{dS}{S} = \mu dt + \sigma dW_t \quad (4)$$

where  $\mu$  and  $\sigma$  are the return's drift and volatility, respectively.

$$\frac{S(t+1) - S(t)}{S(t)}$$

## Appendix

### Developing the terms inside the expectation

First, we will simplify the notation in order to deal more easily with the outer (right most) squaring. Let  $Y(t_j) = (W(t_j) - W(t_{j-1}))^2$ , then we can rewrite the expectation as:

$$\mathbb{E} \left[ \left( \sum_{j=1}^n Y(t_j) - t \right)^2 \right]$$

Expanding we have:

$$\mathbb{E} [(Y(t_1) + Y(t_2) + \dots + Y(t_n) - t) \times (Y(t_1) + Y(t_2) + \dots + Y(t_n) - t)]$$

The term inside the Expectation is equal to

$$\begin{aligned}
& Y(t_1)^2 + Y(t_1)Y(t_2) + \dots + Y(t_1)Y(t_n) - Y(t_1)t \\
& + Y(t_2)^2 + Y(t_2)Y(t_1) + \dots + Y(t_2)Y(t_n) - Y(t_2)t \\
& \vdots \\
& + Y(t_n)^2 + Y(t_n)Y(t_1) + \dots + Y(t_n)Y(t_{n-1}) - Y(t_n)t \\
& - tY(t_1) - tY(t_2) - \dots - tY(t_n) + t^2
\end{aligned}$$

Rearranging

$$\begin{aligned}
& Y(t_1)^2 + Y(t_2)^2 + \dots + Y(t_n)^2 \\
& 2Y(t_1)Y(t_2) + 2Y(t_1)Y(t_3) + \dots + 2Y(t_{n-1})Y(t_n) \\
& - 2Y(t_1)t - 2Y(t_2)t - \dots - 2Y(t_n)t \\
& + t^2
\end{aligned}$$

We can now factorize to get

$$\sum_{j=1}^n Y(t_j)^2 + 2 \sum_{i=1}^n \sum_{j<i}^n Y(t_i)Y(t_j) - 2t \sum_{j=1}^n Y(t_j) + t^2$$

Substituting back  $Y(t_j) = (W(t_j) - W(t_{j-1}))^2$  and taking the expectation, we arrive at:

$$\begin{aligned} & \mathbb{E} \left[ \sum_{j=1}^n (W(t_j) - W(t_{j-1}))^4 \right. \\ & + 2 \sum_{i=1}^n \sum_{j < i} (X(t_i) - X(t_{i-1}))^2 (W(t_j) - W(t_{j-1}))^2 \\ & - 2t \sum_{j=1}^n (W(t_j) - W(t_{j-1}))^2 \\ & \left. + t^2 \right] \end{aligned}$$

# Computing the expectation

By linearity of the expectation operator, we can write the previous expression as:

$$\begin{aligned} & \sum_{j=1}^n \mathbb{E} \left[ \left( W(t_j) - W(t_{j-1}) \right)^4 \right] \\ & + 2 \sum_{i=1}^n \sum_{j < i} \mathbb{E} \left[ \left( W(t_i) - W(t_{i-1}) \right)^2 \left( W(t_j) - W(t_{j-1}) \right)^2 \right] \\ & - 2t \sum_{j=1}^n \mathbb{E} \left[ \left( W(t_j) - W(t_{j-1}) \right)^2 \right] \\ & + t^2 \end{aligned}$$

Now, since  $Z(t_j) = W(t_j) - W(t_{j-1})$  follows a Normal distribution with mean 0 and variance  $\frac{t}{n}$  ( $= dt$ ), it follows (standard result) that its fourth moment is equal to  $3\frac{t^2}{n^2}$ . We will show this shortly.

Firstly we know that  $Z(t_j) \sim N\left(0, \frac{t}{n}\right)$ , i.e.

$$\mathbb{E}\left[Z(t_j)\right] = 0, \quad \mathbb{V}\left[Z(t_j)\right] = \frac{t}{n}$$

therefore we can construct its PDF. For any random variable  $\psi \sim N(\mu, \sigma^2)$  its probability density is given by

$$p(\psi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{(\psi - \mu)^2}{\sigma^2}\right)$$

hence for  $Z(t_j)$  the PDF is

$$p(z) = \frac{1}{\sqrt{t/n}\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{z^2}{t/n}\right)$$

$$\begin{aligned} \mathbb{E}\left[\left(W(t_j) - W(t_{j-1})\right)^4\right] &= \mathbb{E}\left[Z^4\right] \\ &= 3 \frac{t^2}{n^2} \quad \text{for } j = 1, \dots, n \end{aligned}$$

So

$$\begin{aligned}\mathbb{E} [Z^4] &= \int_{\mathbb{R}} Z^4 p(z) dz \\ &= \sqrt{\frac{n}{2t\pi}} \int_{\mathbb{R}} Z^4 \exp\left(-\frac{1}{2} \frac{z^2}{t/n}\right) dz\end{aligned}$$

now put

$$u = \frac{z}{\sqrt{t/n}} \longrightarrow du = \sqrt{n/t} dz$$

Our integral becomes

$$\sqrt{\frac{n}{2t\pi}} \int_{\mathbb{R}} \left(\sqrt{\frac{t}{n}} u\right)^4 \exp\left(-\frac{1}{2} u^2\right) \sqrt{\frac{t}{n}} du$$



$$\begin{aligned}
&= \sqrt{\frac{1}{2\pi}} \frac{t^2}{n^2} \int_{\mathbb{R}} u^4 \exp\left(-\frac{1}{2}u^2\right) du \\
&= \frac{t^2}{n^2} \cdot \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}} u^4 \exp\left(-\frac{1}{2}u^2\right) du \\
&= \frac{t^2}{n^2} \cdot \mathbb{E}\left[u^4\right].
\end{aligned}$$

So the problem reduces to finding the fourth moment of a standard normal random variable. Here we do not have to explicitly calculate any integral. Two ways to do this.

Either use the Moment Generating Function to find the fourth moment to be three.

Or the other method is to make use of the fact that the kurtosis of the standardised normal distribution is 3.

That is

$$\mathbb{E} \left[ \frac{(\phi - \mu)^4}{\sigma^4} \right] = \mathbb{E} \left[ \frac{(\phi - 0)^4}{1^4} \right] = 3.$$

Hence  $\mathbb{E} [u^4] = 3$  and we can finally write  $3 \frac{t^2}{n^2}$ .

and

$$\mathbb{E} \left[ \left( W(t_j) - W(t_{j-1}) \right)^2 \right] = \frac{t}{n} \quad \text{for } j = 1, \dots, n$$

We can now conclude that the expectation is equal to:

$$\begin{aligned} & 3n \frac{t^2}{n^2} + n(n-1) \frac{t^2}{n^2} - 2tn \frac{t}{n} + t^2 \\ &= 3 \frac{t^2}{n} + t^2 - \frac{t^2}{n} - 2t^2 + t^2 = 2 \frac{t^2}{n} \\ &= O\left(\frac{1}{n}\right) \end{aligned}$$

So, as our partition becomes finer and finer and  $n$  tends to infinity, the quadratic variation will tend to  $t$  *in the mean square limit*.