Black-Scholes World

Introduction

In this section we will derive the seminal Black-Scholes equation(s) and find formulae for vanilla call and put options. This work is fundamental to pricing in the Black-Scholes environment.

Here we present the classical BSE model and derivation. By classical we mean in the sense that it is the original 1973 derivation and arguably the best. Best in terms of flexibility (easy to adapt to different situations, models and contracts).

The ideas discussed here keep returning time and time again in equity derivatives, exotics, fixed income and credit.

The assumptions used in the derivation are essentially incorrect, but despite this the Black-Scholes Model is robust. The fashion these days is to criticize this model.

When we talk about the Black-Scholes derivation framework, the following points spring to mind:

1. Model - set of assumptions. Ideal market conditions were considered for both stock and option.
2. Equation - classic Nobel prize winning PDE
3. Formulae - famous closed form pricing formulas for calls and puts expressed in terms of the standardized Normal cumulative distribution function $N(x)$.

Recall *Arbitrage* – taking advantage of the mispricing of two or more securities to make an instantaneous risk free profit. Quant Finance is based on the idea that you must take a risk to make a profit – "there's no such thing as a free lunch". Arbitrage opportunities are possible when

1. The same asset does not trade at the same price on all markets

2. Two assets with identical cash flows do not trade at the same price.

The reduction of risk is caled **hedging**. The perfect elimination of risk, by exploiting correlation between two instruments is generally called **Delta hedging**. This means that the perfect hedge must be continually rebalanced.

Notation

Consider an options contract

$$V(S, t; \sigma, \mu; E, T; r).$$

Semi-colons separate different types of variables and parameters.

- S and t are variables;
- σ and μ are parameters associated with the asset price;
- E and T are parameters associated with the particular contract;
- r is a parameter associated with the currency.

For the moment just use $V(S, t)$ to denote the option value.

The Black–Scholes assumptions

- The underlying stock price process follows a Geometric Brownian motion (this ensures the stock price will not be negative) with known volatility. In the original model it was constant. However we can extend to consider a time dependent volatility.

$$\sigma(t) = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)\, d\tau}$$

  This is simply the average over the time to expiry. So this GBM assumption ignores the fact that stocks can jump giving rise to jump diffusion models (Merton 1976).

- The short -term risk-free interest rate is a known constant through time but these can be extended to consider a function of time

$$r(t) = \frac{1}{T-t}\int_t^T r(\tau)\, d\tau$$

  Again, initially r was a constant, how ever we can incorporate the average of a time varying rate of interest.

- Borrowing and short-selling are allowed without restriction and with full proceeds available. The borrowing and lending rates are equal.

- There are no dividends on the underlying, nor does the stock pay any other cashflows during the life of the option

- Delta hedging is done continuously

- The market is frictionless/perfect liquidity, i.e. there are no transaction costs on the underlying, no taxes or limits to trading. When you delta hedge stock must be sold - which costs

- There are no arbitrage opportunities. (A portfolio consisting of an option and stock is constructed. Delta hedging eliminates risk hence it can only grow at the *risk-free* rate), the market is considered arbitrage free

- The option is 'European' in that it can only be exercised at the expiration date.

The resulting PDE is essentially the Binomial Model in a continuous time setting.

Constructing the portfolio

A call option will $\left\{ \begin{array}{c} \text{rise} \\ \text{fall} \end{array} \right.$ in value if the underlying asset $\left\{ \begin{array}{c} \text{rises} \\ \text{falls} \end{array} \right.$ – positive correlation

A put option will $\left\{ \begin{array}{c} \text{rise} \\ \text{fall} \end{array} \right.$ in value if the underlying asset $\left\{ \begin{array}{c} \text{falls} \\ \text{rises} \end{array} \right.$ – negative correlation

Set up the following portfolio special portfolio Π consisting of one long option position and a short position in some quantity Δ, **Delta**, of the underlying asset:

$$\Pi = V(S,t) - \Delta S.$$

This is **hedging**

The value of V is what we wish to find; we have a model for S; and Δ we can choose. So the asset evolves according to the SDE

$$dS = \mu S dt + \sigma S dW.$$

The obvious question we ask is how does the value of the portfolio change over one time-step dt? That is as $t \longrightarrow t + dt$:

$$d\Pi = dV - \Delta dS.$$

Δ is chosen at the start of the time-step. We hold Δ fixed during the time step and change when rehedging. Now use Itô for $V(S,t)$ which gives

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2.$$

and using the form for dS yields

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW.$$

Substituting in $d\Pi$ gives the following portfolio change

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \\ &\quad \sigma S \frac{\partial V}{\partial S} dW - \Delta \left(\mu S dt + \sigma S dW \right) \end{aligned}$$

So we note that the change contains risk which is present due to

$$\sigma S \frac{\partial V}{\partial S} dW - \Delta \left(\sigma S dW \right),$$

i.e. coefficients of dW. Ideally we want this expression to vanish,

$$\sigma S \frac{\partial V}{\partial S} dW - \Delta \left(\sigma S dW \right) = 0,$$

which gives

$$\Delta = \frac{\partial V}{\partial S}.$$

This choice of Δ renders the randomness zero, i.e. the change in the portfolio is deterministic. The beauty of this is we do no have to worry about things like the evaluation of risk or how much the market wants to be compensated for taking risk, etc.

More importantly we term the reduction of risk as **hedging**. The perfect elimination of risk, by exploiting correlation between two instruments (in this case an option and its underlying) is generally called **Delta hedging**.

Delta hedging is an example of a **dynamic hedging** strategy, because $\frac{\partial V}{\partial S}$ is always changing.

From one time step to the next the quantity $\frac{\partial V}{\partial S}$ changes, since it is, like V a function of the ever-changing variables S and t.

This means that the perfect hedge must be continually rebalanced.

After choosing the quantity Δ, i.e. the number of shares we have to sell, as suggested above, we hold a portfolio whose value changes by the amount

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

This change is completely *riskless.*

So having used dynamic delta hedging to eliminate risk we now appeal to the idea of **no arbitrage**. Is there another such portfolio?

Suppose we put some money Π in a bank for a time period dt at an interest rate r. This grows by an amount $r\Pi dt$.

So - if we have a completely risk-free change $d\Pi$ in the value Π then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi\, dt.$$

This is an example of the **no arbitrage** principle. That is, either

1. put money in the bank and get $r\Pi\, dt$, or
2. buy an option, short some stock and get

   $$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

   which is riskless. Therefore a portfolio, which is locally risk neutral has been created by taking an amount of stock and one security derivative:

Then both portfolios should give exactly the same return, else there would be arbitrage.

Hence we find that

$$\begin{aligned} \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt &= r\,(V - \Delta S)\, dt \\ &= r\left(V - S\frac{\partial V}{\partial S} \right) dt, \end{aligned}$$

i.e. the change in the hedged option portfolio equals the risk-free return on the same portfolio.

On dividing by dt (which is actually taking limits) and rearranging we get the **Black–Scholes equation (BSE)** for the price of an option,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The Black-Scholes equation is a **linear parabolic partial differential equation.** This means that

- if V_1 and V_2 are solutions of the BSE then so is $V_1 + V_2$ and
- if V is a solution of the BSE and k is any constant then kV is also a solution

In words: two options are worth twice as much as one option, and a portfolio consisting of two different options has value equal to the sum of the individual options.

Two simple solutions of the BSE are

1. Asset $V(S,t) = S$
2. Cash $V(S,t) = S_0 e^{rt}$

These are easily verifiable by substituition.

Final and Boundary conditions

The equation is *backwards* parabolic. To solve the Black-Scholes PDE we need to impose suitable boundary and final conditions. Until we do so the BSE knows nothing about what kind of option we are pricing.

If we remind ourselves of the structure of this equation, i.e. first order in time and second order in asset price - this tells us that we need one time condition and two boundary conditions.

1. **Final Condition** provides information on t. This is called the *Payoff.*
2. **Boundary Condition** tells us something about the underlying for two values of S. In this case we choose $S = 0$ and $S \longrightarrow \infty$ (i.e. when the underlying becomes large).

Recall that in the absence of such conditions we obtain a general solution. PDEs (unlike ODEs) are generally solved for particular solutions, as most equations are obtained from physical situations hence we have some information about their behaviour. This is dealt with by the **final condition**. We must specify the option value V as a function of the underlying at the expiry date T. That is, we must prescribe $V(S,T)$, the payoff.

Options on dividend-paying equities

Now generalise the Black-Scholes model to include dividends. Normally a dividend D is paid discretely. The types of dividend structure are

- Discrete payment of D at time t_d; assume value of D is known in advance
- *Dividend yield* d_y paid discretely: payment of $d_y S$ made at time t_d.
- *Continuous dividend yield.* So a small percentage of the stock is paid out in dividends continuously, this keeps the model nice and simple and we get a closed form solution. In one time step dt the asset receives an amount DS (assume $D \propto$ stock). Hence in each interval $[t, t+dt]$ the payment is $DSdt$.

What happens to the asset when a dividend is paid? By arbitrage,

$$S\,(\text{before}) = S\,(\text{after}) + \text{dividend}.$$

So for discrete yield d_y paid on t_d,

$$S\left(t_d^-\right) = S\left(t_d^+\right) + d_y S\left(t_d^-\right).$$

To build a continuous dividend yield into the derivation of the equation, start with the hedged portfolio

$$\begin{aligned} \Pi &= V - \Delta S \\ d\Pi &= dV - \Delta dS - \Delta DSdt, \end{aligned}$$

because we are short the stock. Using the earlier hedging argument gives

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t}+\mu S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\right)dt+\sigma S\frac{\partial V}{\partial S}dW \\ &\quad -\Delta\left(\mu Sdt+\sigma SdW\right)-\Delta DSdt \end{aligned}$$

from which the BSE is obtained

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\left(r-D\right)S\frac{\partial V}{\partial S}-rV=0.$$

We can also write down the SDE for a dividend paying stock as

$$dS=\left(\mu-D\right)Sdt+\sigma SdW.$$

For simulations we use the risk-neutral version

$$dS=\left(r-D\right)Sdt+\sigma SdW,$$

more on this later.

Currency Options

Consider an option on a foreign currency S. Holding this gives us interest at the foreign rate r_f. In one time step the currency receives an amount r_fSdt. So the effect is the same as receiving a continuous dividend yield. The BSE is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\left(r-r_f\right)S\frac{\partial V}{\partial S}-rV=0,$$

where r and r_f are the domestic and foreign rates of interest in turn. from which the BSE is obtained.

We can also write down the SDE for a foreign currency as

$$dS=\left(r-r_f\right)Sdt+\sigma SdW.$$

Commodity Options

Commodities have an associated **cost of carry.** Physical storage of assets such as grains, oil and metals is not without cost - we have to pay to hold the commodity.

Suppose q is the fraction of the commodity S which goes towards payment of cost of carry, i.e. $q\propto S$.

Then in one time step dt an amount $qSdt$ will be required to finance the holding, hence

$$d\Pi=dV-\Delta dS+\Delta qSdt.$$

The resulting BSE is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\left(r+q\right)S\frac{\partial V}{\partial S}-rV=0.$$

As before we can write down the SDE for simulation purposes,

$$dS=\left(\mu+q\right)Sdt+\sigma SdW.$$

The risk-neutral version is

$$dS=\left(r+q\right)Sdt+\sigma SdW.$$

Calls and Puts

For a **call option** we use the following:

Payoff:

$$V(S,T) = \max(S - E, 0).$$

Boundary Conditions:

$$S = 0 \Longrightarrow V(S,t) = 0$$

If we put $S = 0$ in $dS = \mu S dt + \sigma S dW$ then the change will be zero.

$$S \longrightarrow \infty \Longrightarrow V(S,t) \sim S$$

As S becomes very large if we look at $\max(S - E, 0)$ then we find that $S >> E$, hence V is approximately similar to S.

For a **put option** we use the following:

Payoff:

$$V(S,T) = \max(E - S, 0).$$

Boundary Conditions:

$$S = 0 \Longrightarrow V(S,t) = Ee^{-r(T-t)}$$

This is obtained from the **put call parity:**

$$C(S,t) - P(S,t) = S - Ee^{-r(T-t)}.$$

where C and P represent a call and put in turn. We know when $S = 0$, $C = 0$.

$$S \longrightarrow \infty \Longrightarrow V(S,t) \sim 0$$

This is all the information we need to solve the BSE.

Solving the Equation

The Black–Scholes equation is now solved for plain vanilla calls and puts. Starting with

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

The three main steps are:

- Turn the BSE into a one dimensional heat equation by a series of transformations.
- Use a known solution of the heat equation called the *fundamental solution,* or *source solution.*
- Reverse the transformations.

Step 1

Recalling that the payoff is received at time T but that we are valuing the option at time t this suggests that we write

$$V(S,t)=e^{-r(T-t)}U(S,t)$$

$$\begin{aligned}
\frac{\partial V}{\partial t} &= re^{-r(T-t)}U+e^{-r(T-t)}\frac{\partial U}{\partial t}.\\
\frac{\partial V}{\partial S} &= e^{-r(T-t)}\frac{\partial U}{\partial S}\longrightarrow\frac{\partial^2 V}{\partial S^2}=e^{-r(T-t)}\frac{\partial^2 U}{\partial S^2}
\end{aligned}$$

This takes our differential equation to

$$\frac{\partial U}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2}+rS\frac{\partial U}{\partial S}=0.$$

For a more mathematically intuitive argument, write the Black-Scholes equation as

$$rV=\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}$$

and argue the point that we wish to remove the term on the left-hand side. So look for a solution of the form

$$V(S,t)=F(t)\,\mathbf{V}(S,t).$$

Differentiating with respect to t

$$\frac{\partial V}{\partial t}=\mathbf{V}\frac{dF}{dt}+F\frac{\partial \mathbf{V}}{\partial t},$$

and substituting this in to the BSE

$$rF\mathbf{V}=\mathbf{V}\frac{dF}{dt}+F\frac{\partial \mathbf{V}}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 \mathbf{V}}{\partial S^2}F+rS\frac{\partial \mathbf{V}}{\partial S}F.$$

Comparing $\mathbf{V}$ terms gives

$$\frac{dF}{dt}=rF$$

$$\begin{aligned}
\int_t^T\frac{dF}{F} &= r\int_t^T d\tau\\
F(t) &= F(T)\,e^{-r(T-t)}
\end{aligned}$$

Step 2

As we are solving a backward equation we can write

$$\tau' = T - t.$$

The time to expiry is more useful in an option's value than simply the time. We can use the chain rule to rewrite the equation in the new time variable τ

$$\begin{aligned}\frac{\partial}{\partial t} &\equiv \frac{\partial \tau'}{\partial t}\frac{\partial}{\partial \tau'}\\ &= -\frac{\partial}{\partial \tau'}\end{aligned}$$

Under the new time variable

$$\tau' = 0 \Longrightarrow t = T \ \text{(expiry)}$$

so that now τ will be increasing from zero. So as $t' \uparrow \tau \downarrow$.

The BSE becomes

$$\frac{\partial U}{\partial \tau'} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S},$$

which is simply the Kolmogorov equation. So $V(S,t)$ is the discounted solution of the Kolmogorov equation.

Step 3

We now wish to cancel out the variable coefficients S and S^2. When we first started modelling equity prices we used intuition about the asset price *return* and the idea of a lognormal random walk. Let's write

$$\xi = \log S.$$

Again use the chain rule to write the stock in terms of ξ. With this as the new variable, we find that this is equivalent to $S = e^{\xi}$

$$\frac{\partial}{\partial S} = \frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi} = \frac{1}{S}\frac{\partial}{\partial \xi}$$

$$\begin{aligned}\frac{\partial^2}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial}{\partial \xi}\right) = \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial}{\partial \xi}\right) - \frac{1}{S^2}\frac{\partial}{\partial \xi}\\ &= \frac{1}{S}\frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\right) - \frac{1}{S^2}\frac{\partial}{\partial \xi}\\ &= \frac{1}{S^2}\frac{\partial^2}{\partial \xi^2} - \frac{1}{S^2}\frac{\partial}{\partial \xi} = \frac{1}{S^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right)\end{aligned}$$

Now the Black–Scholes equation can be written under this transformation as

$$\frac{\partial U}{\partial \tau'} = \frac{1}{2}\sigma^2 S^2 \frac{1}{S^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right)U + rS\frac{1}{S}\frac{\partial}{\partial \xi}U$$

which simplifies to

$$\frac{\partial U}{\partial \tau'} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi}$$

We need to eliminate the first order derivative term in ξ.

Final Step

Perform a translation of the co-ordinate system

$$x = \xi + \left(r - \frac{1}{2}\sigma^2\right)\tau'; \quad \tau = \tau'$$

So we are transforming from (ξ, τ') to (x, τ). So apply chain rule I

$$\begin{aligned}
\frac{\partial}{\partial \tau'} &= \frac{\partial x}{\partial \tau'}\frac{\partial}{\partial x} + \frac{\partial \tau}{\partial \tau'}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} \\
\frac{\partial}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial \tau}{\partial \xi}\frac{\partial}{\partial \tau} = 1.\frac{\partial}{\partial x} \Longrightarrow \frac{\partial^2}{\partial \xi^2} = \frac{\partial^2}{\partial x^2}
\end{aligned}$$

So $U(\xi, \tau') = W(x, \tau)$.

$$\frac{\partial U}{\partial \tau'} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi}$$

becomes

$$\left(\frac{\partial}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x}\right)W = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial W}{\partial x}$$

After this change of variables the BSE becomes

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2}. \tag{1}$$

To summarize the steps taken to get this 1D heat equation:

$$\begin{aligned}
V(S,t) &= \\
e^{-r(T-t)}U(S,t) &= e^{-r\tau}U(S, T-\tau) = e^{-r\tau}U(e^{\xi}, T-\tau)
\end{aligned}$$

$$= e^{-r\tau}U\left(e^{x-\left(r-\frac{1}{2}\sigma^2\right)\tau}, T-\tau\right) = e^{-r\tau}W(x,\tau).$$

So we will start by solving for $W(x,\tau)$. The equation for this function is solved using the similarity reduction method, for the *fundamental solution* $W_f(x,\tau;x')$. This is all familiar methodology. We define

$$W_f(x,\tau;x') = \tau^{\alpha} f\left(\frac{(x-x')}{\tau^{\beta}}\right),$$

where x' is an arbitrary constant, and the parameters α and β are constant, to be chosen shortly. We choose $\frac{(x-x')}{\tau^{\beta}}$ because it is a constant coefficient problem.

Note that the unknown function depends on only *one* variable

$$\eta = (x - x')/\tau^{\beta}$$

Again we use a combination of product and chain rule to write the PDE in terms of an ODE:

$$W_f(x,\tau;x') = \tau^{\alpha} f(\eta); \quad \eta = (x-x')/\tau^{\beta}$$

So

$$\frac{d\eta}{d\tau} = -\beta\tau^{-\beta-1}(x-x'); \quad \frac{d\eta}{dx} = \tau^{-\beta}$$

$$
\begin{aligned}
\frac{\partial W}{\partial \tau} &= \tau^{\alpha}\frac{\partial}{d\tau}f(\eta)+\alpha\tau^{\alpha-1}f(\eta)\\
&= \tau^{\alpha}\frac{df}{d\eta}\frac{d\eta}{d\tau}+\alpha\tau^{\alpha-1}f(\eta)\\
&= -\tau^{\alpha-1-\beta}\beta\frac{df}{d\eta}.(x-x')+\alpha\tau^{\alpha-1}f(\eta)\\
&= \tau^{\alpha-1}\left(-\beta\eta\frac{df}{d\eta}+\alpha f(\eta)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial W}{\partial x} &= \tau^{\alpha}\frac{\partial}{dx}f(\eta)\\
&= \tau^{\alpha}\frac{df}{d\eta}\frac{d\eta}{dx}=\tau^{\alpha}\tau^{-\beta}\frac{df}{d\eta}\\
&= \tau^{\alpha-\beta}\frac{df}{d\eta}\\
\frac{\partial^2 W}{\partial x^2} &= \tau^{\alpha-\beta}\frac{\partial}{dx}\left(\frac{df}{d\eta}\right)\\
&= \tau^{\alpha-\beta}\frac{d}{d\eta}\frac{d\eta}{dx}\left(\frac{df}{d\eta}\right)=\tau^{\alpha-\beta}\tau^{-\beta}\frac{d^2f}{d\eta^2}\\
&= \tau^{\alpha-2\beta}\frac{d^2f}{d\eta^2}
\end{aligned}
$$

So

$$\frac{\partial W}{\partial \tau}=\tau^{\alpha-1}\left(-\beta\eta\frac{df}{d\eta}+\alpha f(\eta)\right) \tag{2}$$

$$\frac{\partial^2 W}{\partial x^2}=\tau^{\alpha-2\beta}\frac{d^2f}{d\eta^2} \tag{3}$$

Substituting (2), (3) into (1) gives the 2nd order equation

$$\tau^{\alpha-1}\left(\alpha f-\beta\eta\frac{df}{d\eta}\right)=\frac{1}{2}\sigma^2\tau^{\alpha-2\beta}\frac{d^2f}{d\eta^2} \tag{4}$$

We still have a τ term in (4) and for similarity reduction we need to reduce the dimension of the problem. This implies

$$\alpha-1=\alpha-2\beta \quad\Longrightarrow\quad \beta=\frac{1}{2},$$

to give

$$\left(\alpha f-\frac{1}{2}\eta\frac{df}{d\eta}\right)=\frac{1}{2}\sigma^2\frac{d^2f}{d\eta^2}$$

With the correct choice of $\alpha,\ \beta$ we want

$$\int_{-\infty}^{\infty}W_f(x,\tau;x')dx=1\ \forall\tau$$

So

$$\int_{-\infty}^{\infty}W_f(x,\tau;x')dx=\tau^{\alpha}\int_{\mathbb{R}}f\left(\tfrac{x-x'}{\sqrt{\tau}}\right)dx$$

$$\begin{aligned} \eta &= \tfrac{x-x'}{\sqrt{\tau}} \\ \sqrt{\tau} d\eta &= dx \end{aligned}$$

So the integral becomes

$$\tau^{\alpha} \int_{\mathbb{R}} f(\eta) \sqrt{\tau} d\eta = \tau^{\alpha+1/2} \int_{\mathbb{R}} f(\eta) d\eta = 1$$

This implies that $\tau^{\alpha+1/2}$ should equal one, in order for the solution to be normalised regardless of time. Therefore $\alpha = -1/2$.

$f(\eta)$ becomes our PDF. The function f now satisfies

$$-\frac{1}{2}\left(f + \eta \frac{df}{d\eta}\right) = \frac{1}{2}\sigma^2 \frac{d^2 f}{d\eta^2}.$$

where the left hand side can be expressed as an exact derivative

$$-\frac{d}{d\eta}(\eta f) = \sigma^2 \frac{d^2 f}{d\eta^2}.$$

This can be integrated

$$\eta f + \sigma^2 \frac{df}{d\eta} = A$$

where the constant $A = 0$ because as η becomes large, both $f(\eta)$ and $f\prime(\eta)$ tend to zero.

This is variable separable

$$\begin{aligned} \eta f &= -\sigma^2 \frac{df}{d\eta} \\ \int \frac{df}{f} &= -\frac{1}{\sigma^2} \int \eta d\eta \\ \ln f &= -\frac{1}{2\sigma^2}\eta^2 + K \end{aligned}$$

Taking exponentials of both sides gives

$$f(\xi) = C \exp\left(-\frac{\eta^2}{2\sigma^2}\right)$$

C is a normalising constant such that

$$C \int_{\mathbb{R}} \exp\left(-\tfrac{\eta^2}{2\sigma^2}\right) d\eta = 1.$$

Easy to solve by substituting $u = \dfrac{\eta}{\sqrt{2}\sigma} \longrightarrow \sqrt{2}\sigma du = d\eta$, and converts the integral to

$$\begin{aligned} C\sqrt{2}\sigma \int_{\mathbb{R}} e^{-u^2} du &= 1 \\ C\sqrt{2}\sigma\sqrt{\pi} &= 1 \\ C &= \frac{1}{\sqrt{2\pi}\sigma} \end{aligned}$$

$$f(\eta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\eta^2}{2\sigma^2}}.$$

Replacing η gives us the fundamental solution :

$$W_f(x,\tau;x') = \frac{1}{\sqrt{2\pi\tau}\,\sigma} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}. \tag{5}$$

This is the probability density function for a Normal random variable x having mean of x' and standard deviation $\sigma\sqrt{\tau}$. For $\tau \neq 0$, W_f represents a series of Gaussian curves. (5) allows us to find the solution of the BSE at different points (e.g. $x' = 2;\ x' = -17$, etc).

Properties of The Solution

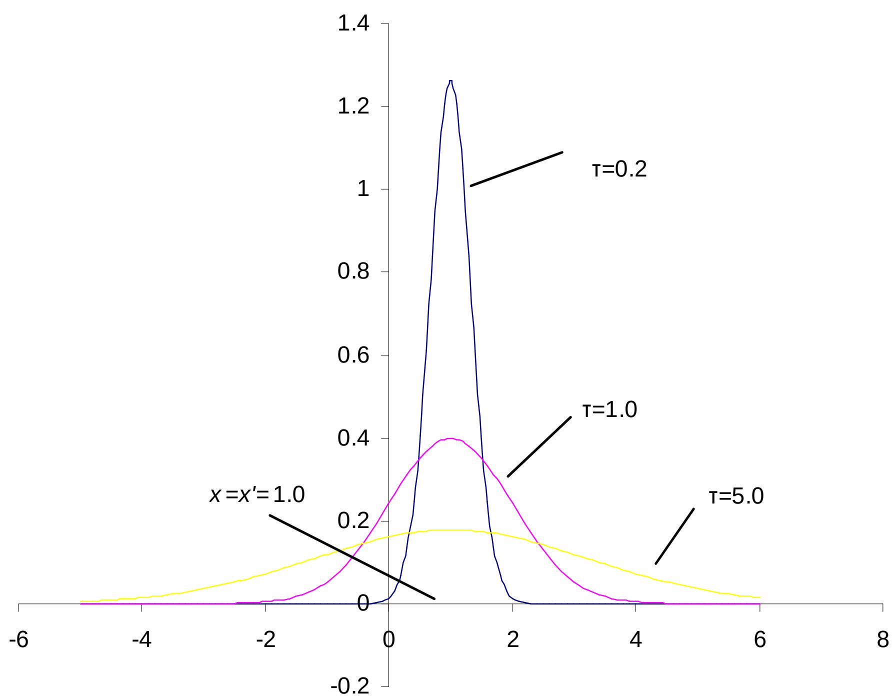
We have made sure from our solution method that

$$\int_{\mathbb{R}} W_f dx = 1$$

this has been fixed. At $x' = x$ ($\exp 0 = 1$)

$$W_f(x,\tau;x') = \frac{1}{\sqrt{2\pi\tau}\,\sigma}.$$

Then as $\tau \to 0$ (close to expiration), $W_f \longrightarrow \infty$, the Gaussian curve becomes taller but the area is confined to unity therefore it becomes slimmer to compensate. As x moves away from $x', \exp(-\infty) \longrightarrow 0$. $W_f(x,\tau;x')$ is plotted below for different values of τ. If τ is large then W_f is flat, as τ gets smaller W_f is increasingly peaked, and focused on x'.



This behaviour of decay away from one point x', unbounded growth at that point and constant area means that W_f has turned in to a **Dirac delta function** $\delta(x' - x)$ as $\tau \to 0$.

Dirac delta function

This is written $\delta(x-x') = \lim_{\tau \longrightarrow 0} \delta(x-x')$, such that

$$\delta(x-x') = \begin{cases} \infty & x = x' \\ 0 & x \neq x' \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-x')\,dx = 1$$

or $\int_0^{\infty} \delta(x-x')\,dx = 1$

If $g(x)$ is a continuous function then

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x')\,dx = g(x')$$

So if we take a delta function and multiply it by any other function - and calculate the area under this product - this is simply the function $g(x)$ evaluated at the point $x = x'$. What is happening here?

The delta function picks out the value of the function at which it is singular (in this case x'). All other points are irrelevant because we are multiplying by zero.

In the limit as $\tau \to 0$ the function W_f becomes a delta function at $x = x'$. This means that

$$\lim_{\tau \to 0} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x'-x)^2}{2\sigma^2\tau}} g(x')dx' = g(x).$$

Here we have swapped x and x' - it makes no difference due to the $(x'-x)^2$ term hence either can be the spatial variable.

So

$$\frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x'-x)^2}{2\sigma^2\tau}}$$

is a delta function and $g(x')$ will be replaced by the payoff function. The term above is also an example of a *Green's function*, which allows us to write down the general solution of the BSE in integral form.

So as we get closer to expiration, i.e. $\tau \to 0$, the delta function picks out the value of $g(x')$ at which $x' = x$

Now introduce the payoff at $t = T$ $(\tau = 0)$:

$$V(S,T) = \text{Payoff}(S).$$

Recall $x = \xi + \left(r - \frac{1}{2}\sigma^2\right)\tau$, so at expiry $\tau = 0 \Longrightarrow x = \xi = \log S$. Hence $S = e^x$ to give

$$W(x,0) = \text{Payoff}(e^x).$$

The solution of this for $\tau > 0$ is

$$W(x,\tau) = \int_{-\infty}^{\infty} W_f(x,\tau;x')\text{Payoff}(e^{x'})\,dx'.$$

We have converted the backward BSE to the Forward Equation. Look at

$$\text{Payoff}(e^{x'})\,dx'.$$

We know

$$\begin{aligned} x' &= \log S' \Longrightarrow dx' = \frac{dS'}{S'} \text{ and} \\ e^{x'} &= S' \end{aligned}$$

therefore Payoff$(e^{x'}).dx'$ becomes

$$\text{Payoff}(S').\frac{dS'}{S'}$$

This result is important. As $\log S$ does not exist in the negative plane the integral goes from 0 to infinity, with the lower limit acting as an asymptote.

Let's start unravelling some of the early steps and transformations. Returning to our Green's function

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{(x'-x)^2}{2\sigma^2(T-t)}} =$$

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{\left(-\frac{1}{2\sigma^2(T-t)} \cdot \left(\underbrace{\log S + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}_{x} - \underbrace{\log S'}_{x'}\right)^2\right)} =$$

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{\left(-\frac{1}{2\sigma^2(T-t)} \cdot \left(\log \frac{S}{S'} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2\right)}$$

So putting this together with the Payoff function as an integrand we have

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty e^{\left(-\frac{1}{2\sigma^2(T-t)} \cdot \left(\left(\log \frac{S}{S'} + \left(r - \frac{1}{2}\sigma^2\right)\right)(T-t)\right)^2\right)} \text{Payoff}(S').\frac{dS'}{S'}$$

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \times$$

$$\int_0^\infty e^{-\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2 / 2\sigma^2(T-t)} \text{Payoff}(S') \frac{dS'}{S'}. \tag{6}$$

This expression works because the equation is linear - so we just need to specify the payoff condition. It can be applied to any European option on a single lognormal underlying asset.

Equation (6) gives us the risk-neutral valuation. $e^{-r(T-t)}$ present values to today time t. The integral is the expected value of the payoff with respect to the lognormal transition pdf. The future state is (S', T) and today is (S, t). So it represents $\mathbb{P}\left[(S,t) \longrightarrow (S',T)\right]$.

Also note the presence of the risk-free IR r in the pdf. So the expected payoff is as if the underlying evolves according to the *risk-neutral* random walk

$$\frac{dS}{S} = r dt + \sigma dX.$$

The real world drift μ is now replaced by the risk-free return r. The delta hedging has eliminated all the associated risk. This means that if two investors agree on the volatility they will also agree on the price of the derivatives even if they disagree on the drift.

This brings us on to the idea of *risk-neutrality.*

So we can think of the option as discounted expectation of the payoff under the assumption that S follows the risk neutral random walk

$$V(S,t) = e^{-r(T-t)} \int_0^\infty \widetilde{p}\,(S,t;S',T)\,V\,(S',T)\,dS'$$

where $p\,(S,t;S',T)$ represents the transition density and gives the probability of going from (S,t) to (S',T) under $\frac{dS}{S} = rdt + \sigma dW$.

So clearly we have a definition for $\widetilde{p}$, i.e. the lognormal density given by

$$\widetilde{p}\,(S,t;S',T) = \frac{1}{\sigma S'\sqrt{2\pi(T-t)}} e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}.$$

Two important points

- $\widetilde{p}\,(S,t;S',T)$ is a Green's for the BSE. As the PDE is linear we can write the solution down as the integrand consisting of this function and the final condition.
- The BSE is essentially the backward Kolmogorov equation whose solution is the transition density $\widetilde{p}\,(S,t;S',T)$ with (S',T) fixed and varying (S,t); but with the discounting factor.

Formula for a call

The call option has the payoff function

$$\text{Payoff}(S) = \max(S-E,0).$$

When $S < E$, $\max(S-E,0) = 0$ therefore

$$\int_0^\infty \equiv \int_0^E + \int_E^\infty = \int_E^\infty \because \int_0^E 0 = 0$$

Expression (6) can then be written as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}(S'-E)\frac{dS'}{S'}.$$

Return to the variable $x' = \log S' \Longrightarrow -x' = \log 1/S'$ so we can write the above integral as

$$\begin{aligned}
&\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^\infty e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\left(e^{x'}-E\right)dx' \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^\infty e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} e^{x'}\,dx' \\
&\quad -E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^\infty e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,dx'.
\end{aligned}$$

Just a couple more steps are required to simplify these messy looking integrals. Let's look at the second integral

$$-E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^\infty e^{-\frac{1}{2}\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/\sigma^2(T-t)}\,dx'$$

use the substitution

$$
\begin{aligned}
u &= \frac{\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}} \\
du &= \frac{-1}{\sigma\sqrt{(T-t)}}dx' \longrightarrow -\sigma\sqrt{(T-t)}du = dx'
\end{aligned}
$$

and the limits:

$$
\begin{aligned}
x' &= \infty \longrightarrow u = -\infty \\
x' &= \log E \longrightarrow u = \frac{\left(-\log E+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}
\end{aligned}
$$

$$
\begin{aligned}
& -E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\frac{\left(\log S/E+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}}^{-\infty} e^{-\frac{1}{2}u^2}. -\sigma\sqrt{(T-t)}du \\
=& -E\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\frac{\left(\log S/E+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}}^{-\infty} e^{-\frac{1}{2}u^2}. -du \\
=& -E\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\left(\log S/E+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}} e^{-\frac{1}{2}u^2}du \\
=& -Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{1}{2}u^2}du \\
=& -Ee^{-r(T-t)}N\left(d_2\right)
\end{aligned}
$$

The first integral requires similar treatment however before we do that we complete the square on the exponent. The integrand is

$$
= e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}e^{x'-r(T-t)}
$$

Now just work on the exponent, and put $\tau = T-t$ temporarily to simplify working

$$
\begin{aligned}
& \frac{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)\tau\right)^2}{2\sigma^2\tau} + x' - r\tau \\
=& \frac{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)\tau\right)^2+2(x'-r\tau)\sigma^2\tau}{2\sigma^2\tau} \\
=& \frac{-\left(\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)\tau\right)^2-2(x'-r\tau)\sigma^2\tau\right)}{2\sigma^2\tau} \\
=& -\frac{1}{2}\frac{\left(\left(-x'+\log S+r\tau-\frac{1}{2}\sigma^2\tau\right)^2-2(x'-r\tau)\sigma^2\tau\right)}{\sigma^2\tau}
\end{aligned}
$$

Now expand the bracket in the numerator

$$
\begin{aligned}
& \Big(x'^2+\log^2 S+r^2\tau^2+\frac{1}{4}\sigma^4\tau^2-2x'\log S-2x'r\tau+x'\sigma^2\tau+2r\tau\log S \\
& -\sigma^2\tau\log S-r\sigma^2\tau^2\Big)-2x'\sigma^2\tau+2r\sigma^2\tau^2 \\
& \Big(x'^2+\log^2 S+r^2\tau^2+\frac{1}{4}\sigma^4\tau^2-2x'\log S-2x'r\tau-x'\sigma^2\tau+2r\tau\log S \\
& -\sigma^2\tau\log S+r\sigma^2\tau^2\Big) \quad \text{now complete the square} \\
=& \left(-x'+\log S+r\tau+\frac{1}{2}\sigma^2\tau\right)^2-2\sigma^2\tau\log S \\
=& \left(-x'+\log S+\left(r+\frac{1}{2}\sigma^2\right)\tau\right)^2-2\sigma^2\tau\log S
\end{aligned}
$$

Let's return to the integral

$$
\begin{aligned}
& \frac{1}{\sigma\sqrt{2\pi\tau)}} \int_{\log E}^{\infty} e^{-\frac{1}{2\sigma^2\tau}\left(\left(-x'+\log S+\left(r+\frac{1}{2}\sigma^2\right)\tau\right)^2-2\sigma^2\tau\log S\right)} dx' \\
= & \frac{S}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^{\infty} e^{-\frac{1}{2\sigma^2(T-t)}\left(\left(-x'+\log S+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)^2\right)} dx'
\end{aligned}
$$

and as before use a similar substitution

$$
\begin{aligned}
v &= \frac{\left(-x'+\log S+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}} \\
dv &= \frac{-1}{\sigma\sqrt{(T-t)}} dx' \longrightarrow -\sigma\sqrt{(T-t)}dv = dx'
\end{aligned}
$$

and the limits as before:

$$
\begin{aligned}
x' &= \infty \longrightarrow v = -\infty \\
x' &= \log E \longrightarrow v = \frac{\left(-\log E+\log S+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}.
\end{aligned}
$$

Following the earlier working reduces this to

$$
\begin{aligned}
& S\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\left(\log S/E+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}} e^{-\frac{1}{2}v^2} dv \\
= & S\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}v^2} dv \\
= & SN\left(d_1\right)
\end{aligned}
$$

Thus the option price can be written as two separate terms involving the cumulative distribution function for a Normal distribution:

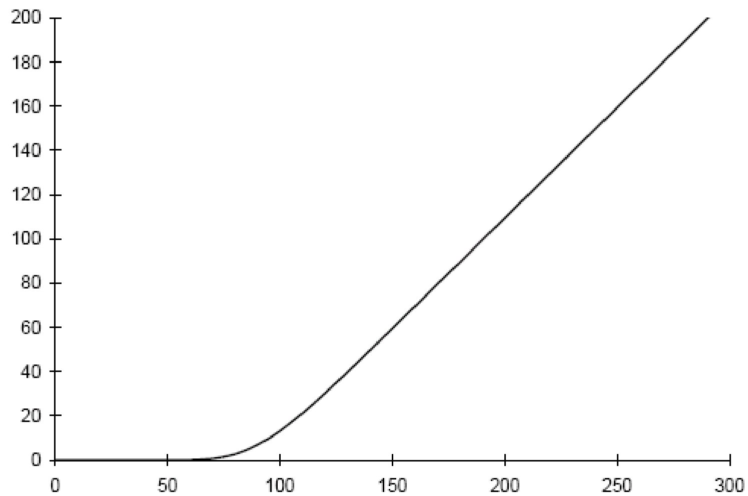
$$
V\left(S,t\right) = SN(d_1) - Ee^{-r(T-t)}N(d_2)
$$

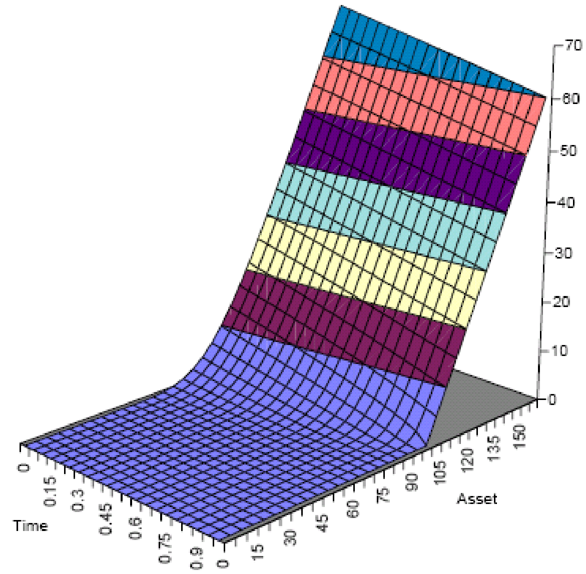
where

$$
d_1 = \frac{\log(S/E)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and}
$$

$$
d_2 = \frac{\log(S/E)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.
$$

$$
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi.
$$





The diagrams above show

a) The value of a call option as a function of the underlying at a fixed time prior to expiry

b) The value of a call option as a function of asset and time

Observations:

- Call values decrease as the strike increases
- Call prices decrease as we get closer to expiry $(T - t) \longrightarrow 0$.
- Call prices increase with volatility
- Call prices increase with interest rates.

These can be verified by studying the pricing formulas.

When there is a continuous dividend yield D or the option is on a currency which receives interest at the foreign rate (replace D by r_f), then the call option simply becomes

$$C(S,t) = Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \text{ and}$$

$$d_2 = \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

At-The-Money-Forward-Options: A nice approximation

Option price having an exercise equal to its forward price. Within the FX world At-The-Money-Forward (ATMF) options are the most heavily traded. When an option is struck ATMF, it means that the strike $E = Se^{(r-D)\tau}$, where we use the earlier definition of $\tau = T - t$. So the strike equals the forward price.

There exists a very nice approximation for ATMF options near expiry.

Begin by writing the call option formula

$$\begin{aligned} C(S,t) &= Se^{-D\tau}N(d_1) - Se^{(r-D)\tau}e^{-r\tau}N(d_2) \\ &= Se^{-D\tau}\left(N(d_1) - N(d_2)\right) \end{aligned}$$

Now simplify d_1 and d_2

$$\begin{aligned} d_1 &= \frac{\log\left(S/Se^{(r-D)\tau}\right) + \left(r - D + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\log e^{-(r-D)\tau} + \left(r - D + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ &= \frac{-(r-D)\tau + \left(r - D + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = \frac{\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \\ &= \frac{1}{2}\sigma\sqrt{\tau} \end{aligned}$$

Similar working shows $d_2 = -\frac{1}{2}\sigma\sqrt{\tau}$.

Returning to the earlier definition of the option price

$$C(S,t) = Se^{-D\tau}\left(N\left(\frac{1}{2}\sigma\sqrt{\tau}\right) - N\left(-\frac{1}{2}\sigma\sqrt{\tau}\right)\right)$$

Consider the CDF for a variable x, i.e. $N(x)$. This can be approximated due to Kendall and Stuart (1943)

$$N(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(x - \frac{x^3}{6} + \frac{x^5}{40} + O\left(x^7\right)\right).$$

If x is small then to leading order this becomes $N(x) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x$.

So if we are close to expiry, then $\tau\ (= T - t)$ is small hence

$$N\left(\pm\frac{1}{2}\sigma\sqrt{\tau}\right) \approx \frac{1}{2} \pm \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}\sigma\sqrt{\tau}\right)$$

$$\left(N\left(\frac{1}{2}\sigma\sqrt{\tau}\right)-N\left(-\frac{1}{2}\sigma\sqrt{\tau}\right)\right)=\frac{1}{\sqrt{2\pi}}\sigma\sqrt{\tau}\approx 0.4\sigma\sqrt{\tau}$$

Putting this altogether gives

$$C(S,t)\approx 0.4Se^{-D(T-t)}\sigma\sqrt{T-t}$$

Formula for a put

The put option has payoff

$$\text{Payoff}(S)=\max(E-S,0).$$

A similar working as in the case of a call yields

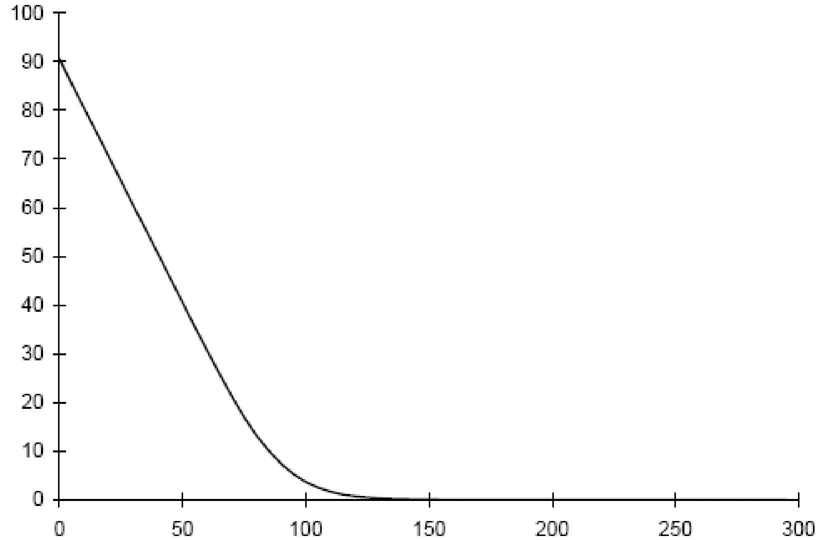
$$V(S,t)=-SN(-d_1)+Ee^{-r(T-t)}N(-d_2),$$

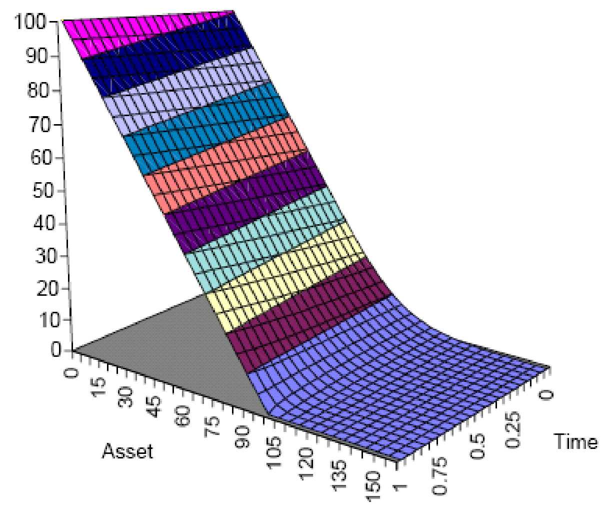
with the same d_1 and d_2. Naturally the more sensible approach is to exploit the put-call parity. If the price of a call and put are denoted in turn by $C(S,t)$ and $P(S,t)$

$$C-P=S-Ee^{-r(T-t)}$$

hence rearranging, using the formula for a call together with $N(x)+N(-x)=1$, gives

$$\begin{aligned}
P &= C-S+Ee^{-r(T-t)} \\
&= SN(d_1)-Ee^{-r(T-t)}N(d_2)-S+Ee^{-r(T-t)} \\
&= S\underbrace{(N(d_1)-1)}_{=-N(-d_1)}+Ee^{-r(T-t)}\underbrace{(1-N(d_2))}_{=N(-d_2)} \\
&= -SN(-d_1)+Ee^{-r(T-t)}N(-d_2)
\end{aligned}$$





The diagrams above show

a) The value of a put option as a function of the underlying at a fixed time prior to expiry

b) The value of a put option as a function of asset and time

Observations:

- Put values increase as the strike increases
- Put prices increase as we get closer to expiry $(T-t) \longrightarrow 0$.
- Put prices increase with volatility
- Put prices increase as interest rates decrease.

Binary Options

Also known as *digital* options. Two general types: *cash-or-nothing* or *asset-or-nothing* options.

In the first type, a fixed amount of cash is paid at expiry if option is in-the-money, whilst the second pays out the value of the underlying asset. The payoff is defined in terms of the Heaviside 'unit step' function

$$\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Some definitions have

$$\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$

and

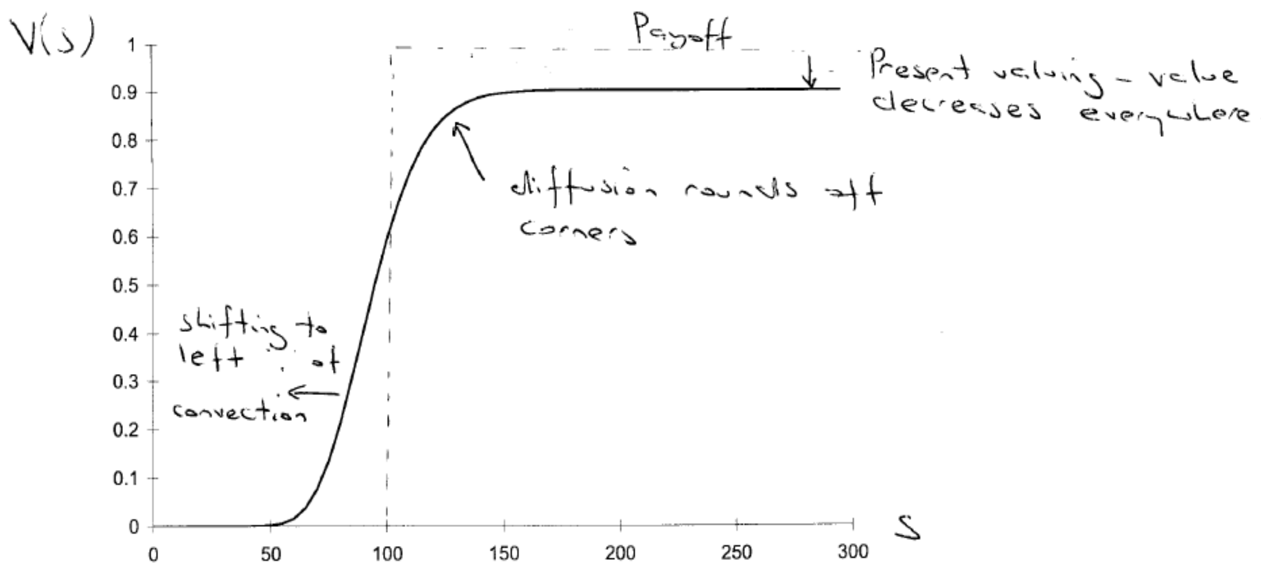
$$\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

For a binary call option, becomes

$$\text{Payoff}(S) = \mathcal{H}(S) = \begin{cases} 1 & S(T) > E \\ 0 & \text{otherwise} \end{cases}$$

So $\mathcal{H}$ takes the value one when it has a non-negative argument and zero otherwise.

The diagram shows the value of a binary call sometime before expiration.



What is happening here?

Each part of the BSE plays a role here.

$-rV$: Present valuing, has the affect of discounting

$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$: diffusion rounds off the corners

$+rS\frac{\partial V}{\partial S}$: convection shifts the profile to the left

Pricing Formulas:

Cash-or-nothing (example for FX where r_d is the domestic interest rate).

$$
\begin{aligned}
C &= e^{-r_d \tau} N(d_2) \text{ FOR call/DOM put} \\
P &= e^{-r_d \tau} N(-d_2) \text{ FOR put/DOM call}
\end{aligned}
$$

Asset-or-nothing

$$
\begin{aligned}
C &= S e^{-r_f \tau} N(d_1) \text{ FOR call/DOM put} \\
P &= S e^{-r_f \tau} N(-d_1) \text{ FOR put/DOM call}
\end{aligned}
$$

(example for FX where r_f is the foreign currency interest rate)

Note

$$
\begin{aligned}
C + P &= e^{-r_d \tau} N(d_2) + e^{-r_d \tau} N(-d_2) \\
&= e^{-r_d \tau} \left(N(d_2) + N(-d_2) \right) = e^{-r_d \tau}
\end{aligned}
$$

So the total worth of a binary call and binary put (on same underlying) is the discounted value of a unit of cash.

The greeks

We now examine the sensitivity of an option price to the input variables/parameters.

The Greeks are forms of measurement on options that express the change of the option price when some parameter changes given every other parameter stays the same. This is an essential form of risk management carried out by all option traders. The next table defines some of the basic greeks.

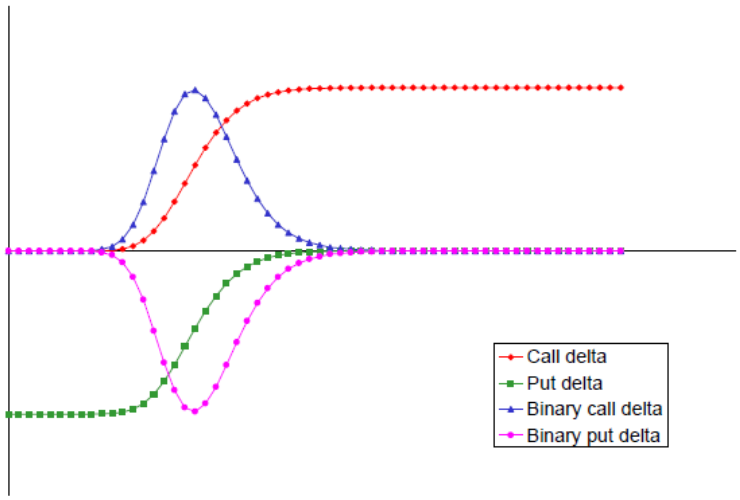
greek	**symbol**	**Measures change in**
delta	$\Delta = \frac{\partial V}{\partial S}$	option price change when underlying price increases by 1
theta	$\Theta = \frac{\partial V}{\partial t}$	option price when time to expiry decreases by 1 day
gamma	$\Gamma = \frac{\partial \Delta}{\partial S}$	delta when the stock price increases by 1
vega	$\frac{\partial V}{\partial \sigma}$	option price when volatility increases by 1% (100 basis points)
rho	$\rho = \frac{\partial V}{\partial r}$	option price when interest rate increases by 1% (100 basis points)
psi	$\Psi = \frac{\partial V}{\partial D}$	option price when dividend yield increases by 1%

The greeks above are all first order derivatives with the exception of gamma which is $\frac{\partial^2 V}{\partial S^2}$ and from the list is the only sensitivity that does not measure a change in the option price change, but rather, it measures the change in delta. Theta is the only Greek that is in the negative domain as it measures decreases in time. There is no greek letter assigned to vega, although occasionally υ is used, whilst some academics refer to it by κ. vega, ρ and Ψ all measure one percent increases (100 basis points), such as the risk-free rate increasing from 3.5% to 4.6% (an example of ρ).

Delta

The most important greek is delta quantity as it tells us how much of the stock we need to sell in order to hedge.

Not just a theoretical device or line in some complex mathematical derivation - it is a practical thing because you calculate delta and you delta hedge. Beacuse it's a function of the stock and time you have to rehedge. Problem with Black-Scholes is hedging has to be done continuoulsy. In practice we can't - we may do it daily, this is *discrete hedging.* This means we are no longer in the Black-Scholes world!



The graph illustrates the behaviour of both call and put option deltas for Europeans and Binaries as they shift from being OTM to ATM and finally ITM. Note that calls and puts have opposite deltas - call options are positive and put options are negative. The binary deltas have been rescaled so they can be observed on the same plot.

Here V can be the value of a single contract or of a whole portfolio of contracts. The delta of a portfolio of options is just the sum of the deltas of all the individual positions.

Option delta is represented as the price change given a 1 point move in the underlying asset and is usually displayed as a decimal value.

Delta values range between 0 and 1 for call options and -1 to 0 for put options (which means decimal notation). Some traders refer to the delta as a whole number between 0 to 100 for call options and -100 to 0 for put options.

For example consider a call that has $\Delta = 0.4$, we would expect the option price to vary by 40% of the change in the underlying. So if the underlying stock rose 1 point, the option value should rise approximately $\frac{4}{10}$ of one point.

The options delta is used to measure the anticipated percentage of change in the price in relation to a change in the price of the underling security.

We can write

$$\Delta = \frac{V(S + \delta S, t) - V(S, t)}{\delta S}$$

where δS is the unit move. Since $\Delta = \Delta(S, t)$, this means that the number of assets held must be continuously changed to maintain a delta neutral position, i.e. $\Delta = 0$. This procedure is called *dynamic hedging*.

Changing the number of assets held requires the continual purchase and/or sale of the stock. This is called rehedging or rebalancing the portfolio.

Consider a Call $C(S,t)$, where

$$C(S+\delta S,t) = C(S,t) + \frac{\partial C}{\partial S}\delta S + \tfrac{1}{2!}\frac{\partial^2 C}{\partial S^2}\delta S^2 + \tfrac{1}{3!}\frac{\partial^3 C}{\partial S^3}\delta S^3 + ...$$

For any small change in the underlying S, we can use the quantity Δ as a guide to determine how much of the underlying to buy or sell to create a hedged portfolio. This means the second order term can be neglected.

When the change in the value of the underlier is not small, the second-order term Γ, cannot be ignored. In practice, maintaining a delta neutral portfolio requires continual recalculation of the position's Greeks and rebalancing of the underlier's position. Typically, this rebalancing is performed daily or weekly.

Recall the Put-Call parity

$$C - P = S - Ee^{-r(T-t)}$$

which differentiated with respect to S gives

$$\Delta_C - \Delta_P = 1,$$

i.e. relationship between a call and put delta.

Gamma

Once you've done your delta hedging, you are left with curvature. You hedge with delta to get rid of linear risk, Γ is an important measure of what risk is left over. When the change in the value of the underlying is not small, the second-order term in the Taylor series expansion above, called *Gamma*, denoted Γ cannot be ignored. Delta and gamma are arguably the two most important sensitivities as they are partial derivatives with respect to the underlying stock. Γ also tells us about transaction costs. Because gamma is the change in delta, if there are transaction costs, it tells us the size of these, because you hold delta, delta changes, you have to buy yourself more stock so gamma is change in delta and tells you how much to buy and sell. Large gamma means greater losses due to transaction costs.

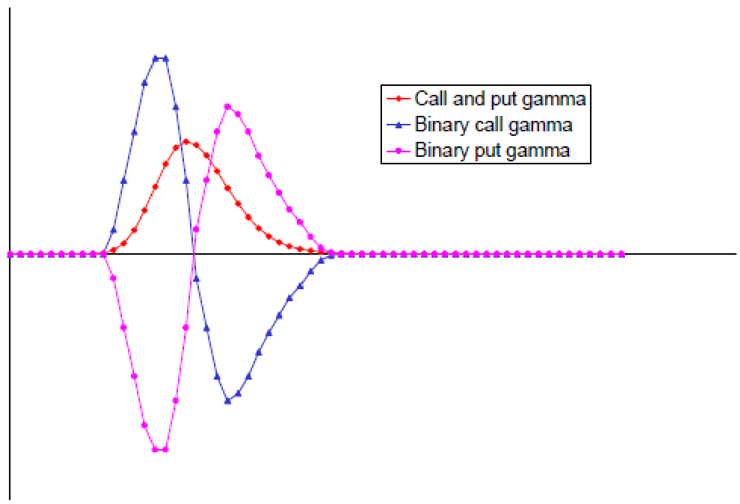
Gamma also important in volatility arbitrage - find an option that is cheap, you buy it and make lots of money - how much money you make depends on Γ.

The change in option price is at the greatest percentage of the option price when the option is close to a payoff of zero. This is when Gamma is at its highest values. It can be thought of as the acceleration of the option when the stock changes. This information can be used to predict how much can be made or lost based on the movement of the underlying position. Since gamma is the sensitivity of the delta to the underlying it is a measure of by how much or how often a position must be rehedged in order to maintain a delta-neutral position.

We know from earlier that $\Delta_C - \Delta_P = 1$. Hence further differentiation with respect to S shows

$$\Gamma_C = \Gamma_P,$$

i.e. the gamma for a call is the same as gamma for a put.



gamma attains its maximum at the money and is lowest deep in/out the money.

Theta

By writing the Black Scholes equation in terms of the greeks

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$$

we can see how theta is related to the option's delta and gamma. In a delta-hedged portfolio it contributes to ensuring that the portfolio earns at the risk-free rate. Because $rS\Delta - rV$ are small, we can say $\Theta \approx -\frac{1}{2}\sigma^2 S^2 \Gamma$.

Here we use basic differentiation techniques to demonstrate the simplicity in obtaining (for example) the delta of a **European Put**. The idea is to show how straightforward it actually is to produce complex looking formulae. We have just written the price of a put

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - Se^{-D(T-t)}N(-d_1)$$

So we want $\Delta = \frac{\partial P}{\partial S}$.

Useful results:

If $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-\phi^{\,2}/2)d\phi$ then $\frac{dN}{dx} = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$: Leibniz Rule

$$\left.\begin{array}{c} d_1 = \dfrac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{array}\right\} \Longrightarrow \frac{\partial(d_1)}{\partial S} = \frac{\partial(d_2)}{\partial S}$$

Another result of importance (messy to prove)

$$Se^{-D(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-d_1^{\,2}/2) = Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-d_2^{\,2}/2) \tag{A}$$

Write

$$Ee^{-r(T-t)}N(-d_2) \tag{a}$$

$$Se^{-D(T-t)}N(-d_1) \tag{b}$$

and

$$\frac{\partial}{\partial S}(\text{a}) = Ee^{-r(T-t)}\frac{\partial}{\partial S}N(-d_2)$$

now use chain rule

$$\begin{aligned} & Ee^{-r(T-t)}\frac{\partial}{\partial d_2}N(-d_2)\frac{\partial(-d_2)}{\partial S} \\ = & -Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-d_2^{\,2}/2)\frac{\partial(d_2)}{\partial S} \end{aligned}$$

$$\frac{\partial}{\partial S}(\text{b}) = e^{-D(T-t)}\frac{\partial}{\partial S}SN(-d_1)$$

use product rule then chain rule

$$\begin{aligned} & e^{-D(T-t)}\left(N(-d_1) + S\frac{\partial}{\partial S}N(-d_1)\right) \\ = & e^{-D(T-t)}\left(N(-d_1) + S\frac{\partial}{\partial d_1}N(-d_1)\frac{\partial(-d_1)}{\partial S}\right) \\ = & e^{-D(T-t)}\left(N(-d_1) - S\frac{1}{\sqrt{2\pi}}\exp(-d_1^{\,2}/2)\frac{\partial(d_1)}{\partial S}\right) \end{aligned}$$

So now

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial S}(\mathrm{a}) - \frac{\partial}{\partial S}(\mathrm{b}) \\
&= -Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-{d_2}^2/2)\frac{\partial(d_2)}{\partial S} - \\
&\quad e^{-D(T-t)}\left(N(-d_1) - S\frac{1}{\sqrt{2\pi}}\exp(-{d_1}^2/2)\frac{\partial(d_1)}{\partial S}\right) \\
&= -e^{-D(T-t)}N(-d_1) + \\
&\quad \frac{\partial(d_1)}{\partial S}\left(Se^{-D(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-{d_1}^2/2) - Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-{d_2}^2/2)\right) \\
&= -e^{-D(T-t)}N(-d_1) + \frac{\partial(d_1)}{\partial S}(0)
\end{aligned}$$

Using

$$N(x) + N(-x) = 1 \Longrightarrow N(-x) = 1 - N(x)$$

$$\begin{aligned}
\Delta &= -e^{-D(T-t)}(1 - N(d_1)) \\
&= e^{-D(T-t)}(N(d_1) - 1)
\end{aligned}$$

This is probably a good point at which to prove the earlier (messy) result (A)

$$Se^{(-d_1^2/2)} = Ee^{-r(T-t)}e^{-d_2^2/2}.$$

Let's start with the left hand side $Se^{(-d_1^2/2)} =$

$$e^{\ln S}e^{-\frac{1}{2}\left(\frac{\log(S/E)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)^2}$$

Look at exponent

$$\frac{\sigma^2(T-t)\ln S - \frac{1}{2}(\ln(S/E))^2 - (r+\sigma^2/2)(T-t)\ln(S/E) - \frac{1}{2}(r+\sigma^2/2)^2(T-t)^2}{\sigma^2(T-t)}$$

$=$

$$\frac{-\frac{1}{2}\left[(\ln S/E)^2+\left\{\sigma\ln S+r\ln E+\left(\sigma^2/2\right)\ln E-r\ln S-\left(\sigma^2/2\right)\ln S\right\}(T-t)-\frac{1}{2}\left(r-\sigma^2/2\right)^2(T-t)^2-r\sigma^2(T-t)^2\right]}{\sigma^2(T-t)}$$

$$\frac{-\frac{1}{2}(\ln S/E)^2+\left\{\left(\sigma^2/2\right)\ln S-\left(\sigma^2/2\right)\ln E-r\ln S+r\ln E\right\}(T-t)-\frac{1}{2}\left(r-\sigma^2/2\right)^2(T-t)^2}{\sigma^2(T-t)} + \ln E - r(T-t)$$

$$\begin{aligned}
&= e^{\ln E-r(T-t)-\frac{1}{2}\left(\frac{\log(S/E)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)^2} \\
&= Ee^{-r(T-t)}e^{-d_1^2/2}.
\end{aligned}$$

The *Speed* of an option is the sensitivity of its gamma to changes in the underlying stock price S and written

$$\text{Speed} = \frac{\partial \Gamma}{\partial S} = \frac{\partial^3 V}{\partial S^3},$$

where V is the option price. Given that for a European call option $C(S,t) = Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$, gamma is

$$\Gamma = \frac{e^{-D(T-t)}}{\sigma S\sqrt{T-t}} N'(d_1),$$

by suitable differentiation of Γ, we show that

$$\text{Speed} = -\frac{\Gamma}{S}\left(1 + \frac{d_1}{\sigma\sqrt{T-t}}\right).$$

Start by writing $\tau = T - t$

$$\frac{\partial \Gamma}{\partial S} = \frac{\partial}{\partial S}\left(\frac{e^{-D\tau}}{\sigma S\sqrt{\tau}} N'(d_1)\right) = \frac{e^{-D\tau}}{\sigma\sqrt{\tau}}\frac{\partial}{\partial S}\left(\frac{1}{S}N'(d_1)\right)$$

Concentrate on $\displaystyle \frac{\partial}{\partial S}\left(\frac{1}{S}N'(d_1)\right) = -\frac{1}{S^2}N'(d_1) + \frac{1}{S}\frac{dN'}{dd_1}\frac{\partial d_1}{\partial S}$

$$\begin{aligned}
&= -\frac{1}{S^2}N'(d_1) - \frac{1}{S}\left(\frac{d_1}{\sigma S\sqrt{\tau}}\underbrace{\frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}}_{=N'(d_1)}\right) \\
&= -\frac{N'(d_1)}{S^2}\left(1 + \frac{d_1}{\sigma\sqrt{\tau}}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial \Gamma}{\partial S} &= -\frac{N'(d_1)}{S^2}\frac{e^{-D\tau}}{\sigma\sqrt{\tau}}\left(1 + \frac{d_1}{\sigma\sqrt{\tau}}\right) \\
&= -\underbrace{\frac{e^{-D(T-t)}}{\sigma S\sqrt{T-t}}N'(d_1)}_{\Gamma}\frac{1}{S}\left(1 + \frac{d_1}{\sigma\sqrt{T-t}}\right) \\
&= -\frac{\Gamma}{S}\left(1 + \frac{d_1}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

Deltas

$$\begin{aligned}
\text{Call } \Delta_C &= e^{-D(T-t)} N(d_1) \\
\text{Put } \Delta_P &= e^{-D(T-t)} \left(N(d_1) - 1\right) \\
\text{Binary Call } \Delta_{BC} &= \frac{e^{-r(T-t)}}{\sigma S \sqrt{T-t}} N'(d_2) \\
\text{Binary Put } \Delta_{BP} &= -\frac{e^{-r(T-t)}}{\sigma S \sqrt{T-t}} N'(d_2)
\end{aligned}$$

Gammas

$$\begin{aligned}
\text{Call } \Gamma_C &= \frac{e^{-D(T-t)}}{\sigma S \sqrt{T-t}} N'(d_1) \\
\text{Put } \Gamma_P &= \frac{e^{-D(T-t)}}{\sigma S \sqrt{T-t}} N'(d_1) \\
\text{Binary Call } \Gamma_{BC} &= -\frac{e^{-r(T-t)} d_1}{\sigma^2 S^2 \sqrt{T-t}} N'(d_2) \\
\text{Binary Put } \Gamma_{BP} &= \frac{e^{-r(T-t)} d_1}{\sigma^2 S^2 \sqrt{T-t}} N'(d_2)
\end{aligned}$$

Thetas

$$\begin{aligned}
\text{Call } \Theta_C &= -\frac{\sigma S e^{-D(T-t)}}{2\sqrt{T-t}} N'(d_1) + DSN(d_1) e^{-D(T-t)} - rEe^{-r(T-t)} N(d_2) \\
\text{Put } \Theta_P &= -\frac{\sigma S e^{-D(T-t)}}{2\sqrt{T-t}} N'(-d_1) - DSN(-d_1) e^{-D(T-t)} + rEe^{-r(T-t)} N(-d_2) \\
\text{Binary Call } \Theta_{BC} &= re^{-r(T-t)} N(d_2) + e^{-r(T-t)} N'(d_2) \left(\frac{d_1}{2\,(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}} \right) \\
\text{Binary Put } \Theta_{BP} &= re^{-r(T-t)} \left(1 - N(d_2)\right) - e^{-r(T-t)} N'(d_2) \left(\frac{d_1}{2\,(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}} \right)
\end{aligned}$$

Vegas

$$\begin{aligned}
\text{Call } vega &= S\sqrt{T-t}\, e^{-D(T-t)} N'(d_1) \\
\text{Put } vega &= S\sqrt{T-t}\, e^{-D(T-t)} N'(d_1) \\
\text{Binary Call } vega &= -e^{-r(T-t)} N'(d_2) \left(\sqrt{T-t} + \frac{d_2}{\sigma} \right) \\
\text{Binary Put } vega &= e^{-r(T-t)} N'(d_2) \left(\sqrt{T-t} + \frac{d_2}{\sigma} \right)
\end{aligned}$$

rhos

$$\begin{aligned}
\text{Call } \rho_C &= E\,(T-t)\, e^{-r(T-t)} N(d_2) \\
\text{Put } \rho_P &= -E\,(T-t)\, e^{-r(T-t)} N(-d_2) \\
\text{Binary Call } \rho_{BC} &= -\,(T-t)\, e^{-r(T-t)} N(d_2) + \frac{\sqrt{T-t}}{\sigma} e^{-r(T-t)} N'(d_2) \\
\text{Binary Put } \rho_{BP} &= -\,(T-t)\, e^{-r(T-t)} \left(1 - N(d_2)\right) - \frac{\sqrt{T-t}}{\sigma} e^{-r(T-t)} N'(d_2)
\end{aligned}$$

Psi

$$
\begin{aligned}
\text{Call } \Psi_C &= -(T-t)\, S e^{-D(T-t)} N(d_1) \\
\text{Put } \Psi_P &= (T-t)\, S e^{-D(T-t)} N(-d_1) \\
\text{Binary Call } \Psi_{BC} &= -\frac{\sqrt{T-t}}{\sigma} e^{-r(T-t)} N'(d_2) \\
\text{Binary Put } \Psi_{BP} &= \frac{\sqrt{T-t}}{\sigma} e^{-r(T-t)} N'(d_2)
\end{aligned}
$$