The Heath, Jarrow and Morton Model

Forward Curve. Factorisation with Principal Component Analysis

In this lecture. . .

- the short rate process, bond price and forward rates
- evolving the forward curve – the Heath, Jarrow & Morton model
- interest rate derivatives pricing by Monte-Carlo
- a multi-factor model and its calibration with PCA
- yield curve data analysis: revealing the internal structure of factors for yield curve movement

By the end of this lecture you will:

- understand forward rates and their bootstrapping
- have an introduction to modelling a yield curve as a whole
- be able to analyse the yield curve changes data to obtain historic volatilities and their principal components
- understand the HJM framework and its calibration issues
- be able to price simple rate derivatives by Monte-Carlo

Introduction

The Heath, Jarrow & Morton approach to modeling of the forward curve was a major breakthrough that improved pricing and risk management in fixed income.

They built a framework encompassing all of the models we have seen so far (and many techniques that we haven't).

One-factor models, such as Vasicek, CIR, Ho & Lee, Hull & White, evolve the short rate $r(t)$.

- $r(t) = f(t, t)$ represents only one point on the yield curve, which is a multi-dimensional object.
- One-factor models are calibrated using short-term instruments, $Z_M(t^*, T)$.
- Empirically, at **the short end** the yield curve changes independently from its other sections.

$$\mathbb{C}orr[\Delta r_t, \Delta f_j] \approx 0.$$

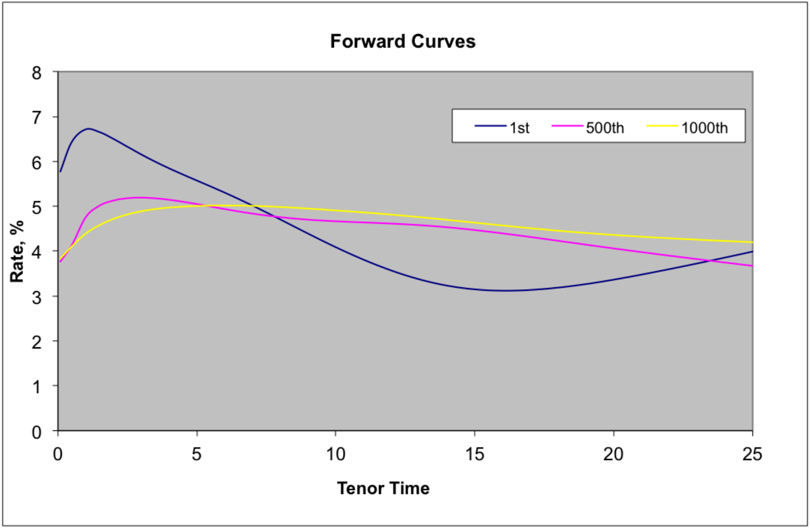
A calibrated $r(t)$ process is unable to produce full curves that match what is observed in markets.

- Many forward rate linked instruments became available and liquid since after one-factor models were introduced.

HJM evolves the whole forward curve. We operate with a system of the alike SDEs df_j, each evolves rate at a fixed, constant tenor $\tau_j = T_j - t$.

- Fitting to a market yield curve occurs naturally within the HJM model. It does not appear as an afterthought.

Forward Curve



It makes sense to model a sequence of forward rates $f(0; t_i, t_{i+1})$

rather than overlapping and always-decreasing discount factors $Z(0, T_1), Z(0, T_2), Z(0, T_3)$ ... where $T_1 < T_2 < T_3 <$ ....

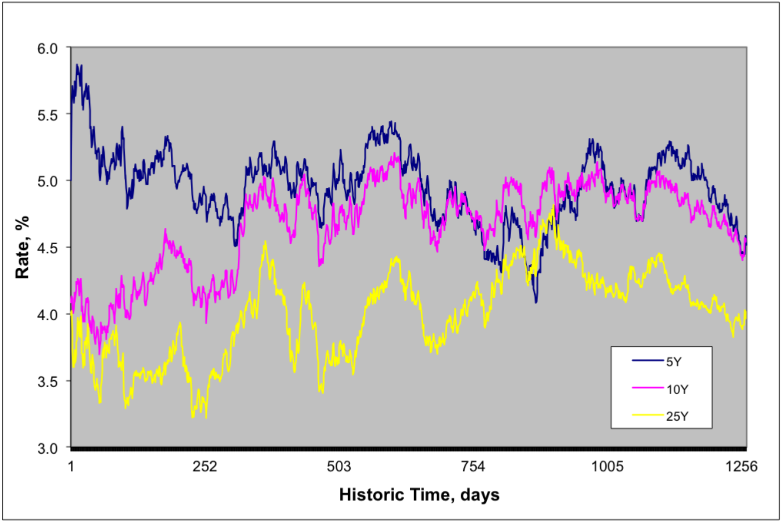
$$\begin{aligned} Z(0, T_1) \times Z(0; T_1, T_2) &= Z(0, T_2) \\ Z(0; T_1, T_2) &= \exp\left[-f(0; t_1, t_2)(t_2 - t_1)\right] \\ f_2 &= -\frac{\ln Z_2 - \ln Z_1}{t_2 - t_1} \\ &\quad \text{etc.} \end{aligned}$$

$F(t, T, T + \Delta t)$ expires at time T and applies over an instant $\Delta t \to 0$, so we call it **an instantaneous forward rate** and denote as $f(t, T)$.

$$\begin{aligned} f(t,T) &= -\lim_{\Delta t \to 0} \frac{lnZ(0;T+\Delta t) - \ln Z(0;T)}{\Delta t} \\ &= -\frac{\partial}{\partial T} \ln Z(0;T) \qquad (1) \\ f(\tau) &= \frac{\partial}{\partial \tau} r(\tau)\tau \qquad \text{(Hagan \& West 2005)} \end{aligned}$$

Instantaneous forward rate does not apply to a cashflow. It is a number from a continuous process.

Historic evolution of forward rates



Bootstrapping fwds

A bootstrapping scheme for each discrete tenor *j* follows from the instantaneous forward rate maths (1)

$$f_0 = -\frac{\ln Z_1 - \ln 1}{T_1 - 0} \qquad f_1 = -\frac{\ln(Z_2/Z_1)}{T_2 - T_1}$$

Tenor, T	**0.08**	**0.17**	**0.25**	**0.33**
Spot	0.5052%	0.5295%	0.5500%	0.5682%
Z(0, T)	0.9996	0.9991	0.9986	0.9981
Forward	0.5063%	0.5552%	0.5927%	0.6244%

Bootstrapping fwds (cont.)

No arbitrage relationship between a sequence of forward rates and ZCB yield at time n (**spot rate**) is given by a geometric average

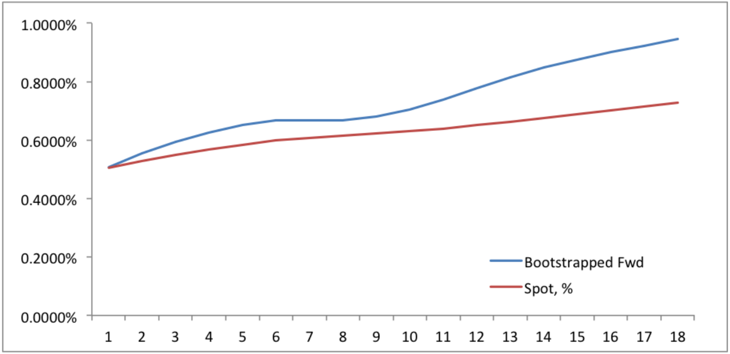
$$(1 + rs_n)^n = (1 + f_0) \times (1 + f_1) \times ... \times (1 + f_n).$$

where for the first period $f_0 = rs_0$.

You might know this no relationship well, and it leads to *a recursive* bootstrapping procedure *from* a set of spot rates.

Please examine *Yield Curve.xlsm* spreadsheet and its VBA coding.

Forward vs. Spot Curve



Data: Bank Liability Curve as of 30 January 2015 (Bank of England)

Big Picture

Let's draw a summary of relationship among **a.** the short rate process, **b.** spot curve and **c.** forward curve.

Bond price

The relationship that links the short rate $r(t)$ and bond price $Z(t;T)$ is

$$Z(r,t;T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right]$$

$$dr(t) = u(r)\,dt + \nu\, r^{\beta}\, dX \qquad \text{Calibration Lecture}$$

$$dr(t) = u(r,t)\,dt + \omega(r,t)\,dX \qquad \text{General Case}$$

- $r(t)$ is *any* stochastic process.
- $dr(t)$ is a one-factor model for it has only **one** term $dX = \phi\sqrt{\delta t}$ representing random movement in the factor.

Future-starting bond

If our $t \neq 0$ then we are pricing **a future-starting bond** using the corresponding forward rate $f(0, t_1, t_2)$

$$Z(t; T_1, T_2) = \exp\left(-\int_{T_1}^{T_2} f(s)ds\right)$$

$$f(t, T) = -\frac{\partial}{\partial T} \ln Z(t, T) \qquad \text{instantaneous rates}$$

Remember that in the instantaneous notation $f(t, T) = F(t, T, T + \Delta)$.

If we link the forward rate to a specific, constant tenor time $\tau_j = T_j - t$ then we can simply to f_j where j is a counter for tenor 0.5Y, 1Y, etc.

But a forward rate f_j is **a traded quantity** with assistance of the Forward Rate Agreements. Modelling such f_j will take us to the direction of and LIBOR Market Model...

Modeling bond price

The log-normal SDE must be a familiar model:

$$\frac{dZ}{Z} = \mu(t, T)\, dt + \sigma(t, T)\, dX \tag{2}$$

where bond price evolves with t but the maturity date T is fixed.

Empirical observation: if we construct a constant maturity bond price, the changes in the yield give us **the model invariant** (an *iid* process or close to it)

$$\ln Z(t_2; T) - \ln Z(t_1; T) \sim IN(\mu, \sigma)$$

$$\ln Z(t; T) \propto f(t, T)$$

HJM Calibration

Data and Computational Preview

1. Take the data of *changes* in forward rates about constant tenors Δf_j and compute $j \times j$ covariance matrix $\boldsymbol{\Sigma}$.

$$\boldsymbol{\Sigma} = \frac{1}{N}\boldsymbol{X}\boldsymbol{X}'$$

2. Calibrate factor**s** from PCA of the covariance matrix $\boldsymbol{\Sigma}$.
   The factors provide linear decomposition of changes on average.

3. Fit volatility functions and add no-arbitrage condition, eg, HJM models' drift.

4. Pricing of interest rate derivatives is done by Monte-Carlo.

Start with the forward rate equation $f(t, T) = -\frac{\partial}{\partial T} \ln Z(t, T)$ and use the bond price 'GBM' SDE to derive the evolution of forward rates:

$$df(t, T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t, T) - \mu(t, T)\right] dt - \frac{\partial}{\partial T}\sigma(t, T)dX \qquad (3)$$

The SDE carries the drift of the bond price $\mu(t, T)$.

When we come to pricing, such drift terms are replaced by the risk-free interest rate $r(t)$.

Risk to risk-neutral

In the real world, our model for he evolution of the forward curve (3) was derived as

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T) - \underline{\mu(t,T)}\right]dt - \frac{\partial}{\partial T}\sigma(t,T)dX$$

In the risk-neutral world, under measure $\mathbb{Q}$, the model becomes

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T) - \underline{r(t)}\right]dt - \frac{\partial}{\partial T}\sigma(t,T)dX^{\mathbb{Q}} \qquad (4)$$

But $r(t)$ is not a function of T, so $\dfrac{\partial}{\partial T}r(t) = 0$

Change of measure

Replacing from $\mu(t, T)$ with $r(t)$ is a change of measure. According to the Girsanov theorem.

$$X_t^{\mathbb{Q}} = X_t + \int_t^T \theta_s ds$$

We also know the market price of risk result as $\theta = \frac{\mu - r}{\sigma}$.

$$\frac{dZ}{Z} = \mu(t, T)dt + \sigma dX_t$$

$$\frac{dZ}{Z} = r(t)dt + \sigma dX_t^{\mathbb{Q}}$$

In the risk-neutral economy, the expected return on any traded investment (a bond) is simply $r(t)$.

- By setting up a hedged portfolio $\Pi = Z(t, T_1) - \Delta Z(t, T_2)$ we found that to cancel the drift we require

$$\frac{\mu(t, T_1) - r(t)}{\sigma(t, T_1)} = \frac{\mu(t, T_2) - r(t)}{\sigma(t, T_2)}$$

- Only possible if both sides are equal to some parameter, which is independent of maturity dates T_1, T_2

$$\mu(t, T) = r(t) + \lambda(r, t)\sigma(t, T)$$

  where $\lambda(r, t)$ is the market price of risk (MPOR), a unifying global parameter, not a constant!

More in the CQF Extra *The Market Price of Risk: Fear and Greed...*

Forward rate drift

Apply chain rule to differentiate with respect to ∂T, and the model simplifies to

$$df(t,T) = \underbrace{\sigma(t,T)\frac{\partial}{\partial T}\sigma(t,T)dt}_{} - \frac{\partial}{\partial T}\sigma(t,T)dX^{\mathbb{Q}} \tag{5}$$

We expressed the risk-neutral forward rate drift as a function of volatility of interest rates.

Forward rate volatility

Converting into volatility of forward rates $\nu(t, T)$ adds simplicity,

$$\nu(t, T) = -\frac{\partial}{\partial T}\sigma(t, T)$$

The model becomes

$$df(t, T) = -\sigma(t, T)\nu(t, T)dt + \nu(t, T)dX^{\mathbb{Q}}$$

We want to express the full equation in terms of $\nu(t, T)$,

$$df(t, T) = \left[\nu(t, T)\int_t^T \nu(t, s)ds\right] dt + \nu(t, T)dX^{\mathbb{Q}} \tag{6}$$

From now on, we operate in the risk-neutral world, so we drop $\mathbb{Q}$.

Risk-neutral HJM dynamics

HJM SDE for an instantaneous forward rate at any point on the curve,

$$df(t,T) = m(t,T)dt + \nu(t,T)dX \tag{7}$$

The risk-neutral drift is function of volatility. It is **a no arbitrage condition**:

$$m(t,T) = \nu(t,T)\int_t^T \nu(t,s)ds \tag{8}$$

All we need is a volatility function. It will be estimated directly from a covariance matrix of forward rate changes.

HJM SDE applies at any point f_j of the forward curve, where j just refers to a fixed tenor $\tau_j = T_j - t$.

Therefore, we have **a system of SDEs** – each with its own drift $m_j(t, T)$ and factorised volatility $\nu_j(t, T) = \sum_{i=1}^{k} \nu_i(t, T_j) dX_i$.

We have a vector stochastic process $\boldsymbol{f}(t,T) = \bar{\boldsymbol{f}}(t,\tau)$ modelled by a system of SDEs, where index $j = 1 \ldots N$ refers to tenor 0.5Y, 1Y, etc.

$$
\begin{aligned}
d\bar{f}_1(t,\tau) &= \bar{m}(t,\tau_1)dt + \sum_{i=1}^{k} \bar{\nu}_i(t,\tau_1)dX_i \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\
d\bar{f}_j(t,\tau) &= \bar{m}(t,\tau_j)dt + \sum_{i=1}^{k} \bar{\nu}_i(t,\tau_j)dX_i
\end{aligned}
$$

Through calibration by PCA we will obtain **the fitted** volatility functions $\bar{\nu}_i(t,\tau)$ and drift function $\bar{m}(t,\tau)$.

Both are functions of **tenor time** τ.

It is a logical step to express this system of SDEs for specific, discrete tenors *in matrix form*

$$d\boldsymbol{f}(t,T) = \boldsymbol{M}(t,T)dt + \boldsymbol{V}\boldsymbol{\Lambda}^{\frac{1}{2}}\, d\boldsymbol{X} \tag{9}$$

where $d\boldsymbol{X}$ is a multi-dimensional Brownian Motion representing k independent factors.

Independence is achieved by decomposition of covariance matrix

$$\boldsymbol{\Sigma} = \mathbf{V}\boldsymbol{\Lambda}^{\frac{1}{2}}\left(\mathbf{V}\boldsymbol{\Lambda}^{\frac{1}{2}}\right)' = \boldsymbol{A}\boldsymbol{A}' \quad \text{Cholesky decomposition}$$

The covariance matrix is estimated from <u>**changes in forward rates**</u>

$$\boldsymbol{\Sigma} = \mathbb{C}ov[\Delta f(\tau_j),\, \Delta f(\tau_{j+m})].$$

We will have to re-visit HJM multi-factor calibration and introduce the spectral decomposition of covariance matrix...

That will give us a solution, in numerical sense.

Let's talk about pricing.

The analytical solution (PDE approach) is not feasible under the HJM. That leaves us with two choices:

1. **Monte-Carlo method** – estimation of an expectation by simulating the evolution of forward rates.

   Implemented under risk-neutral HJM dynamics and measure $\mathbb{Q}$.

2. The other is to build up **a tree** structure and formalise it into a Finite Difference grid.

Pricing by Monte-Carlo

Simulated Output of HJM

1. Simulation Simulate an evolution of the whole risk-neutral curve for the necessary length of time, from today t^* to T^*.

- For 'a risk-neutral forward curve' we use GLC data (UK Gilts).
- Realizations of the curve $\boldsymbol{f}(t, T)$ over time steps $dt = 0.01$ are **in rows**.
- The paths of forward rates for discretised tenors τ_j are **in columns**.

2. Discounting factors Obtain ZCB values for all required tenors up to T^*. However, discounting factors can come from the outside (e.g., OIS curve) creating a problem of how to match expectations.

3. Calculate the value of cashflows of interest and apply discounting.
Example: consider $(L - K)^+$ payoff and discount

(Discounting is assumed within the same risk-neutral expectation.)

5. Return to Step 1 to perform another round of simulation.

- Keep track of the running average of simulated prices, i.e., 1st, 1st + 2nd, 1st + 2nd + 3rd, etc. .
- It must demonstrate convergence/reduction in its variance.

Risk-neutral discount factor

Pricing a Zero Coupon Bond with any stochastic short rate $r(t)$

$$Z(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right]$$

Here is a case when we do not need the entire simulated curve.

By looking at the HJM simulation output, we already know that we only an evolution of the first point on the yield curve $f(t,t) = r(t)$.

$$\begin{aligned} Z(t,T) &= \exp\left(-\sum r_t \, \Delta t\right) \qquad \text{under MC} \qquad (10) \\ &= \exp\left(-\text{SUM(COLUMN)} \times 0.01\right) \end{aligned}$$

Monte-Carlo simulation is always discretised over time step dt. Therefore, integration becomes summation.

HJM and the short rate r(t)

Given by the forward rate for a maturity equal to the current date, i.e.

$$r(t) = f(t, t)$$

If the forward curve today is $f(t^*, T)$ then the short rate for *any* time t in the future is

$$r(t) = f(t^*, t) + \int_{t^*}^{t} df(s, t)$$

In terms of HJM output, $df(s, ...)$ means evolving a rate **in column**.

Check with $f(t^*, t) + f(t, t) - f(t^*, t) = f(t, t)$.

The HJM SDE in terms of spot rate is jammed into

$$dr(t) = \left[\frac{\partial f(t^*, t)}{\partial t} - \frac{\partial \mu(t,s)}{\partial s}\bigg|_{s=t} \right.$$

$$+ \int_{t^*}^{t} \left(\sigma(s,t) \frac{\partial^2 \sigma(s,t)}{\partial t^2} + \left(\frac{\partial \sigma(s,t)}{\partial t} \right)^2 - \frac{\partial^2 \mu(s,t)}{\partial t^2} \right) ds$$

$$\left. - \underline{\int_{t^*}^{t} \frac{\partial^2 \sigma(s,t)}{\partial t^2} dX(s)} \right] dt - \frac{\partial \sigma(t,s)}{\partial s}\bigg|_{s=t} dX$$

This SDE is odd because **the drift depends on the history of** σ from the date t^* to the future date t and the stochastic increments dX.

Buidlding a tree

If we attempt to use an evolution path for a forward rate $f(t,t) = r(t)$ to build a tree

- Then we'll find ourselves with an unfortunate result: an up move followed by a down move will **not** end up in the same state.
- Our tree structure becomes 'bushy'. The number of branches grows *exponentially* with the addition of new time steps.

This is a feature of a non-Markov model. Equivalence of paths is what makes the pricing by Binomial Method so efficient.

Non-Markovian nature of HJM

The highly path-dependent drift of dr(t) makes the movement of the short rate **non-Markov**.

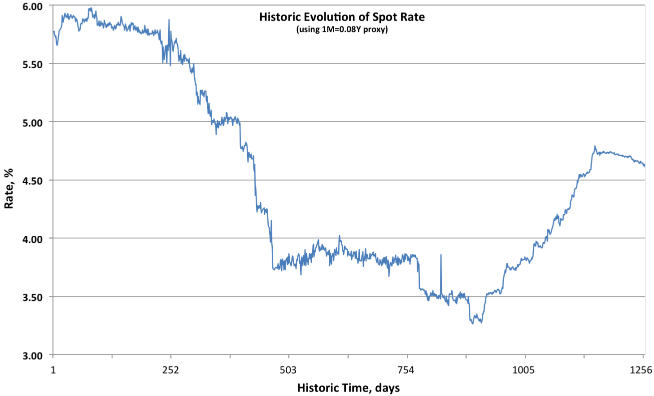
$$r + dr$$

is not recombinable.

In a **Markov chain** only the present state of a variable determines the possible future (albeit random) state.

Markov process is a stochastic process without a **memory**. But let's look at the empirical $r(t)$.

Memory in r(t)



The **long memory** for stationary time series means that decay in autocorrelation is slower than exponential $\mathrm{Corr}[r_t, r_s] = \beta^{t-s}$.

Working in tenor time τ

In terms of the HJM SDE simulation it is useful to model the forward rate $f(t, T)$ and its volatility $\nu(t, T)$ at each tenor point, **not** at each *maturity date* T, 1 Jan 2010, 1 June 2010, 1 Jan 2011, etc.

- $df(t, T)$ means evolving a forward rate as a process.
- However, as t changes, the time distance $\tau = T - t$ changes too. So we are modelling a different rate $T_{\tau-0.01}, T_{\tau-0.02}, T_{\tau-0.03}, \ldots$
- Eventually the front end of the curve expires, and new section of the curve appears at the long end.

We need to model in tenor time τ. This is called **Musiela Parametrisation**.

$$f(t, T) \quad \Rightarrow \quad \bar{f}(t, T - t) = \bar{f}(t, \tau)$$

What will happen to our HJM SDE?

$$\boxed{d\bar{f}(t, \tau) = \bar{m}(t, \tau)dt + \bar{\nu}(t, \tau)dX}$$

The volatility function keeps its form:

$$\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T) \quad \Rightarrow \quad -\frac{\partial}{\partial \tau}\bar{\sigma}(t,\tau)\,\cancel{\frac{\partial \tau}{\partial T}} = \bar{\nu}(t,\tau)$$

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial T}\frac{\partial \tau}{\partial \tau} = \frac{\partial}{\partial \tau} \qquad \text{since } \frac{\partial \tau}{\partial T} = 1$$

The drift gains an extra forward-looking term:

$$\bar{m}(t,\tau) = \bar{\nu}(t,\tau)\int_0^{\tau}\bar{\nu}(t,s)ds + \frac{\partial}{\partial \tau}\bar{f}(t,\tau)$$

Musiela Parametrisation relies on Itô and the chain rule

$$3.\ \boxed{d\bar{f} \equiv df + \frac{\partial f}{\partial T}dt}$$

$$2.\ \frac{d\bar{f}}{dt} \equiv \frac{df}{dt} + \frac{\partial f}{\partial T} \qquad \text{and} \qquad 1.\ \frac{d\bar{f}}{dt} \equiv \frac{\partial f}{\cancel{\partial T}}\frac{\cancel{\partial T}}{\partial t} + \frac{\partial f}{\cancel{\partial t}}\frac{\cancel{\partial t}}{\partial T}$$

$$\begin{aligned} d\bar{f}(t,\tau) &= \underbrace{df(t,T)} + \frac{\partial f(t,T)}{\partial T}dt \\ &= \underbrace{\left(\nu(t,T)\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX} + \frac{\partial f(t,T)}{\partial T}dt \\ &= \left(\bar{\nu}(t,\tau)\int_0^\tau \bar{\nu}(t,s)ds + \frac{\partial \bar{f}(t,\tau)}{\partial \tau}\right)dt + \bar{\nu}(t,\tau)dX \end{aligned}$$

The extra forward derivative term $\frac{\partial \bar{f}(t,\tau)}{\partial \tau}$ is a slope of the yield curve.

Multi-factor Musiela HJM SDE

The multi-factor SDE that we are going to use for simulating forward curve is (*HJM MC spreadsheet*)

$$d\bar{f}(t,\tau) = \left(\sum_{i=1}^{k} \bar{\nu}_i(t,\tau) \int_0^{\tau} \bar{\nu}_i(t,s)ds + \frac{\partial \bar{f}(t,\tau)}{\partial \tau} \right) dt + \sum_{i=1}^{k} \bar{\nu}_i(t,\tau) dX_i \tag{11}$$

The uncertainty around the $k \ll N$ linearly independent factors is simulated by the uncorrelated Brownian Motions dX_i times volatility.

- Diffusion term, such as $\bar{\nu}(t,\tau)\, dX$ represents uncertainty about curve movement coming from one factor.

Case for a multi-factor model

A single-factor model for the short rate $r(t)$ cannot hope to capture the richness of yield curve movements.

An example of instrument sensitive to more than one factor is *a spread option*. Its payoff is the difference between rates at two different tenors, e.g., $(L_{6M} - L_{3M})$.

- **If movements of two rates are not correlated (going up/down in sync), there is an extra source of risk**, ie, another factor.

  Cannot hedge with a single bond.

Bucket risks vs. systematic factors

Take another example of two ‘natural’ factors, some short rate $f_{0.25Y}$ and a very long-term rate f_{25Y}.

They represent **just one systematic factor**, one kind of movement:

- Steepening or flattening of the curve.

1) Changes in the rate at certain tenor point can be particularly sensitive to a systematic factor so, traders and rate analysts assign causality, e.g., “long-end moves the curve”.

2) Interest rate *changes* are correlated across all tenors, but variance of changes can be explained with a few independent factors.

For the multi-factor Musiela parameterisation of HJM SDE (11),

Instead of relying on any particular analytical solution for

$$\bar{\nu}_i(t,\tau) = \ldots$$

we will estimate the volatility structure that matches the data.

Principal Component Analysis is the method for such calibration.

It implements a *linear decomposition* of changes in interest rates at each tenor.

- PCA provides systematic factors that describe movement of a curve as a whole.
- Factor attribution is well-established for yield curve analysis.

Data Preparation (technical)

If we have forward rate time series going back a few years, we can calculate covariances between **the changes** in the rates.

- Instantaneous forward rates data is provided by the BOE.
- $6M$ increment up to $25Y$ maturity gives 50 columns.
- The spreadsheet uses Government Liability Curve (GLC), pre-January 2007 regime.

GLC bootstrapped from repo agreements, spot bonds (gilts) and bond futures.

More suitable for pricing IR derivatives would be HJM calibration obtained from Bank Liability Curve (BLC) built using FRAs and short sterling futures.

Covariance Matrix Estimation (technical)

To estimate the covariance matrix $\boldsymbol{\Sigma}$,

1. Obtain **daily differences** in interest rate (for each tenor, columnwise) and subtract the mean if necessary.
2. $\boldsymbol{\Sigma}$ would then be 50×50 symmetric matrix with the variances along the diagonal and the covariances between tenors off the diagonal.
3. Since we used daily changes and percentages, we have to re-annualised covariance with $\times \frac{252}{100 \times 100}$.

Matrix Decomposition

$\boldsymbol{\Sigma}$ is the covariance matrix of forward rate changes. Any symmetric matrix can be decomposed according to *the spectral theorem*:

$$\boldsymbol{\Sigma} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}'$$

- $\boldsymbol{\Lambda}$ is a diagonal matrix with eigenvalues $\lambda_1 > \cdots > \lambda_n > 0$ positive and usually ranked in software output (Matlab, R).

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

- $\mathbf{V}$ is a vectorised matrix of eigenvectors $vec(\mathbf{e}^{(1)}\ \mathbf{e}^{(2)}\ \ldots\ \mathbf{e}^{(n)})$.

$$\Delta f(\tau_j) = \sqrt{\lambda_1}\boldsymbol{e}^{(1)}_{\tau_j} + \sqrt{\lambda_2}\boldsymbol{e}^{(2)}_{\tau_j} + \sqrt{\lambda_3}\boldsymbol{e}^{(3)}_{\tau_j} + \ldots \quad \text{in rows}$$

Eigenvectors

Volatility functions of τ

- For each *column* eigenvector $\mathbf{e}^{(\mathbf{i})}$, the first entry is the movement of one-month rate ($\tau = 0.08$), the second entry is of the six-month rate ($\tau = 0.5$) and so on.

$$\overline{\nu}_i(t^*, \tau) = \sqrt{\lambda_i}\, \boldsymbol{e}^{(\boldsymbol{i})}_{\boldsymbol{\tau}} \tag{12}$$

  To obtain a volatility function, it is naturally convenient to fit a column eigenvector to tenor τ. Eigenvector $\boldsymbol{e}^{(i)}_{\tau}$ has values at

$$\tau = 0.08Y,\ 0.5Y,\ 1Y,\ \dots,\ 25Y$$

- Instead of picking numbers from the matrix of eigenvectors **V**, **we use the fitted volatility functions**.

Cubic Spline

- The fitting is done by a single **cubic spline** *wrt* tenor τ

$$\overline{\nu}(t,\tau) = \beta_0 + \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 \quad \forall\,\tau_j$$

A spline is a piecewise-defined smooth polynomial function.
It balances between exactness and flexibility.

Despite using *LINEST()* to calculate β, here we are **not** conducting any regression analysis.

In our PCA application to Pound Sterling curve, three factors explain 93.33% of movement (variation) in the yield curve.

Tenor	λ	Cum. R^2
1Y	0.002027	71.31%
25Y	0.000463	87.58%
6Y	0.000164	93.33%

But how did we choose these $k = 3$ eigenvectors to be our volatility functions?

$$\boldsymbol{e}^{(1Y)}, \ \boldsymbol{e}^{(6Y)}, \ \boldsymbol{e}^{(25Y)}$$

By the largest corresponding eigenvalue.

Factor Significance

Eigenvalue λ_i **is variance** of the movements of a curve in each eigendirection. For example, the first factor explains

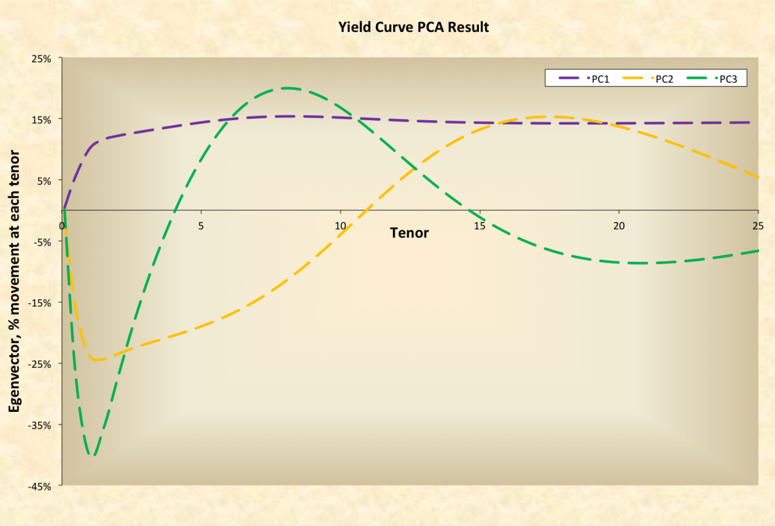
$$\frac{\lambda_1}{\lambda_1 + \lambda_2 + \cdots + \lambda_N}$$

The cumulative goodness of fit statistic for the k-factor model is

$$\text{Cum. } R^2 = \frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{N} \lambda_i}$$

By choosing the largest-impact factors we reduce an N-dimensional model to the three-factor model. Each factor represents systemic movement by the curve.

PCA Result: Three Largest Factors



Factor Attribution

- **Parallel shift in overall level of rates** is the largest principal component of forward curve movement, common to all tenors.
- **Steepening/flattening of the curve** is the second important component (i.e., change of *skew* across the term structure)

  Inverted curve (backwardation for commodities term structure) would have a different shape for the PC2.
- **Bending about specific maturity points** is the third component to curve movement that mostly affects *curvature* (convexity).

Disclaimer. Factor attribution is subject to the stable regime in interest rates. During a period of rapid rates policy changes, PCA attribution might not hold.

SDE for simulation

We simulate the HJM

$$d\bar{f}(t+\Delta t,\tau)=\bar{f}(t,\tau)+d\bar{f}(t,\tau)$$

The multi-factor SDE (summations) that we use for simulating a forward curve (*HJM MC spreadsheet*), coming from (11)

$$d\bar{f}(t,\tau)=\bar{m}(\tau)dt+\sum_{i=1}^{k}\bar{\nu}_i(t,\tau)dX_i+\frac{\bar{f}(t,\tau)}{\partial\tau}dt \tag{13}$$

where

$$\bar{\nu}_i(t,\tau)\,dX_i=\sqrt{\lambda_i}\,\boldsymbol{e}_{\boldsymbol{\tau}}^{(\boldsymbol{i})}\,\phi_i\sqrt{\delta t}$$

The uncertainty around the $k \ll N$ linearly independent factors is simulated by the uncorrelated Brownian Motions dX_i times volatility.

Numerical Methods (reference)

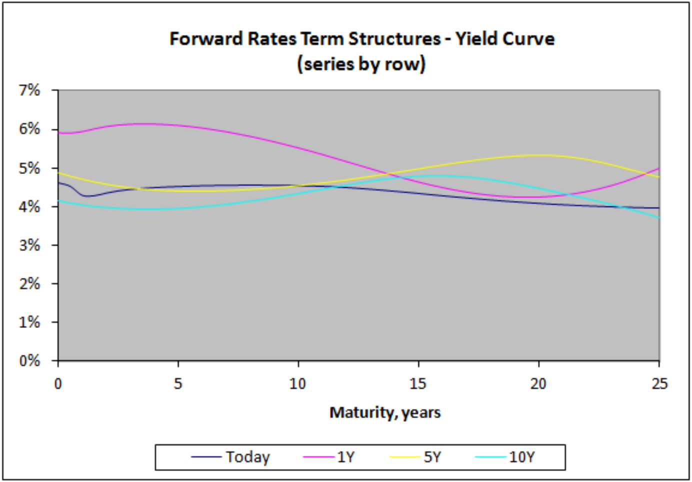
- Drift $\bar{m}(t, \tau)$ is calculated using *numerical integration* over the fitted volatility functions. Simple **trapezium rule** is used.

- Eigenvectors and eigenvalues are obtained orthogonal rotation of a matrix to eliminate the off-diagonal elements.
Jacobi Transformation preserves the order of original $\mathbf{\Sigma}$.

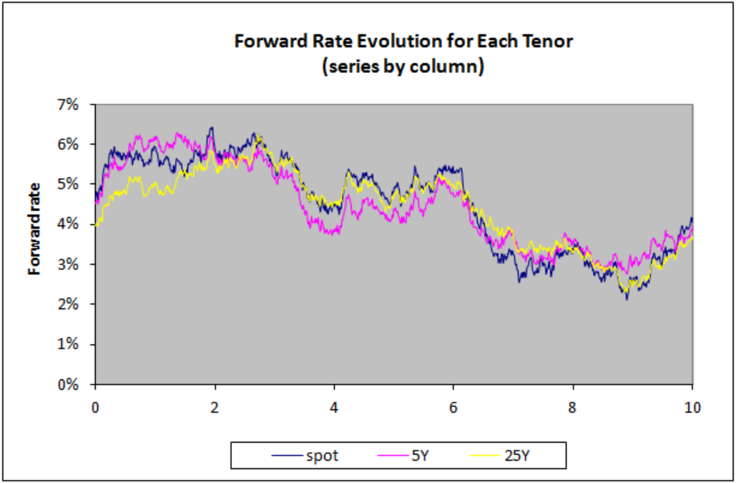
- To initialise the simulation, we use the last row (day) of known observations. **Euler scheme** is used $dX_i = \phi_i \sqrt{\delta t}$, where ϕ_i is generated using *RAND()*.

- The forward derivative $\frac{\partial \bar{f}(t,\tau)}{\partial \tau}$ is calculated using the previous row, so we simulate a realisation of the entire curve at each time step.

Simulated Forward Curves



Simulated Instantaneous Forward Rates



ZCB Pricing

There are **two approaches** for pricing a bond under HJM framework.

- Integrating over a current forward curve $\bar{f}(t^*, \tau_j)$ (in a row)

$$Z(t,T) = \exp\left(-\int_0^{T=\tau} \bar{f}(t^*, \tau) d\tau\right)$$

  This requires a no-arbitrage interpolation of the curve, followed by numerical integration. Things can get very technical!

- Using a simulated path of $r(t) = f(t,t)$ (from the first column)

$$Z(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s) ds\right)\right]$$

Risk-neutral discount factor

Using $r(t)$, the calculation becomes a summation over the first column of simulated data (HJM Model - MC Excel)

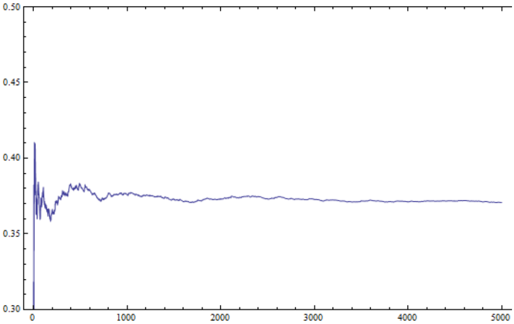
$$
\begin{aligned}
Z(t,T) &= \exp\left(-\sum r_t\,\Delta t\right) \qquad \text{under MC} \qquad (14)\\
&= \exp\left(-\mathrm{SUM(COLUMN)} \times 0.01\right)
\end{aligned}
$$

Solution: to price a half-year bond starting today Z(0; 0.5), we will carry out summation over 50 rows of the first column, $\Delta t = 0.01$.

Convenient!

To satisfy the risk-neutral expectation $\mathbb{E}^{\mathbb{Q}}$ we have to conduct the Monte-Carlo.

ZCB price convergence, $T > 10Y$



Monte-Carlo pricing means that we produce **a running average** of simulated prices, e.g., 1st, 1st + 2nd, 1st + 2nd + 3rd. The running average must demonstrate convergence/reduction in its variance.

Please take away the following important ideas:

- with the HJM model, we evolve the entire forward curve
- calibration means linear factorisation of forward rate volatility.
- Principal Component Analysis reveals the internal data structure:

  the key factors of curve movement are level, steepness/flatness and curvature
- pricing under the HJM is done using the Monte-Carlo