Module 1 Further Exercises in SDEs

Throughout this problem sheet, you may assume that W_t is a Brownian Motion (Weiner Process) and dW_t is its increment. You may assume$W_0 = 0$. SDE(s) refers to Stochastic Differential Equation(s).

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$. Calculate $\mathbb{E}[\psi]$ and $\mathbb{V}[\psi]$ where $\psi = \sqrt{dt}\phi$. dt is a small time-step. **Note: No integration is required.**

2. Consider the following examples of Stochastic Differential Equations (SDE); Write these in standard form, i.e. $dG = A(G,t)dt + B(G,t)dW_t$. Give the drift and diffusion for each case.

   (a) $df + dW_t - dt + 2\mu t f dt + 2\sqrt{f}dW_t = 0 \qquad$ where $f = f(W_t, t)$

   (b) $\dfrac{dy}{y} = (A + By)\,dt + (Cy)\,dW_t \qquad$ where $y = y(W_t, t)$

   (c) $dS = (\nu - \mu S)dt + \sigma dW_t + 4dS$

3. Show that
$$\int_0^1 (1-t)\cos W_t dW_t = \int_0^1 (a+bt)\sin W_t dt,$$
and determine the values of a and b.

4. The function $V(S,t) = \log(tS)$, where S evolves according to the SDE $dS = \mu S dt + \sigma S dW_t$; show that
$$dV = \left(\frac{1}{t} + \mu - \frac{1}{2}\sigma^2\right)dt + \sigma dX.$$

5. Show that
$$G = \exp\left(t + ae^{W_t}\right)$$
is a solution of the stochastic differential equation
$$dG(t) = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right)dt + G(\ln G - t)\,dW_t$$

6. Consider the stochastic differential equation
$$dG(t) = a(G,t)\,dt + b(G,t)\,dW_t.$$
Find $a(G,t)$ and $b(G,t)$ where

   (a) $G(t) = W_t^2$

   (b) $G(t) = 1 + t + e^{W_t}$

   (c) $G(t) = f_t W_t$, where f_t is a bounded and continuous function.

7. Use Itô's lemma to show that
$$d(\cos W_t) = \alpha(\cos W_t)\,dt + \beta(\sin W_t)\,dW_t$$
&
$$d(\sin W_t) = \alpha(\sin W_t)\,dt - \beta(\cos W_t)\,dW_t$$
and determine the constants α & β.