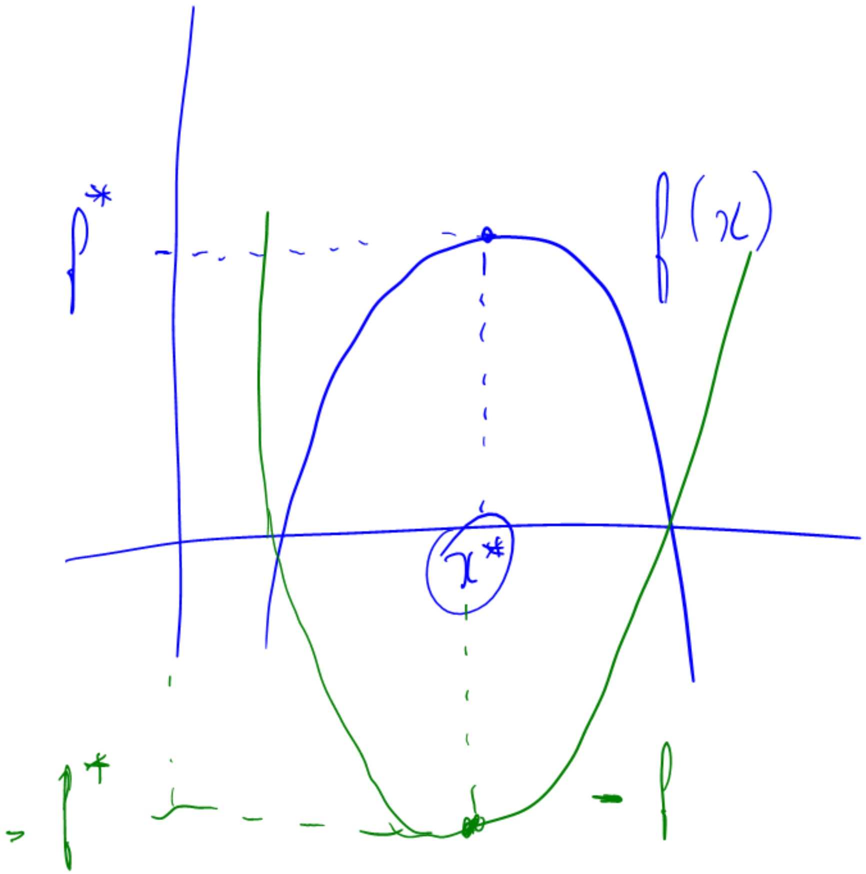
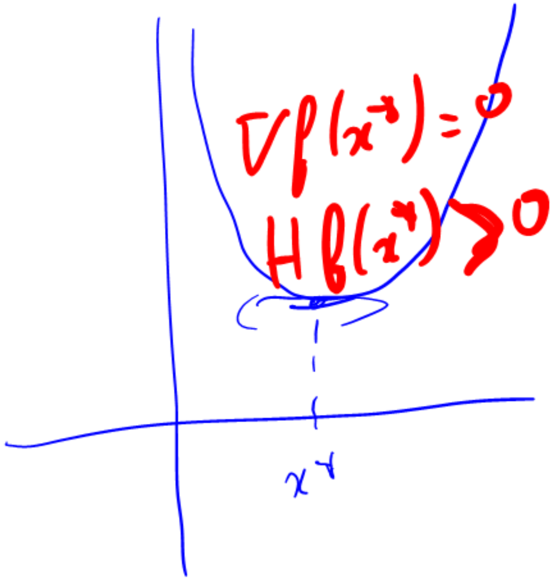


$$\max f(x) = -\min(-f(x))$$

$$\min g(x) = -\max(-g(x))$$

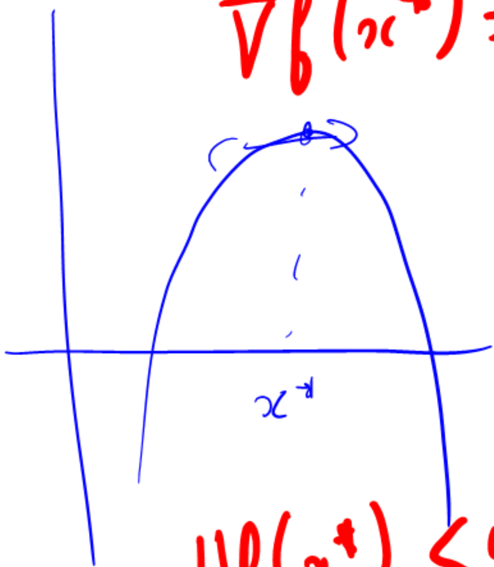
Three types of situations



$\nabla f(x^*) = 0$

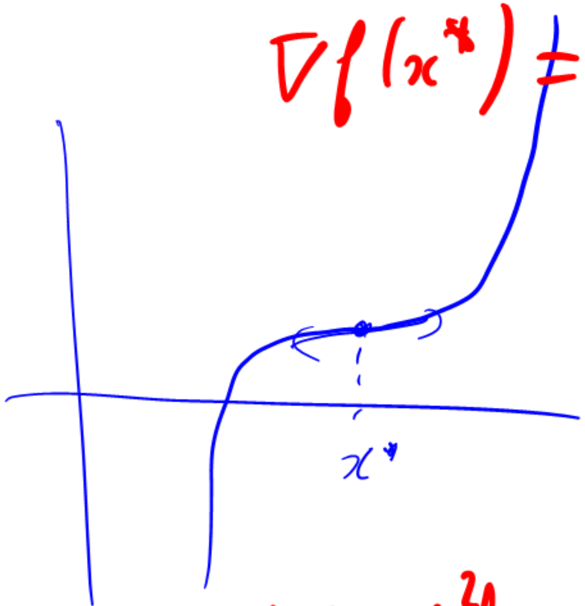
$Hf(x^*) > 0$

$\nabla f(x^*) = 0$



$Hf(x^*) < 0$

$\nabla f(x^*) = 0$



in 1d, $\dfrac{d^2 f}{dx^2} = 0$

in nd, depends!

1^{st} order derivative : gradient

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

n - element vector

Df

→ 1^{st} order (necessary) condition

→ identify candidate points

2nd order derivative

$$Hf = D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & & \\ \vdots & & \ddots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & & & & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

-> 2nd order (sufficient) condition.

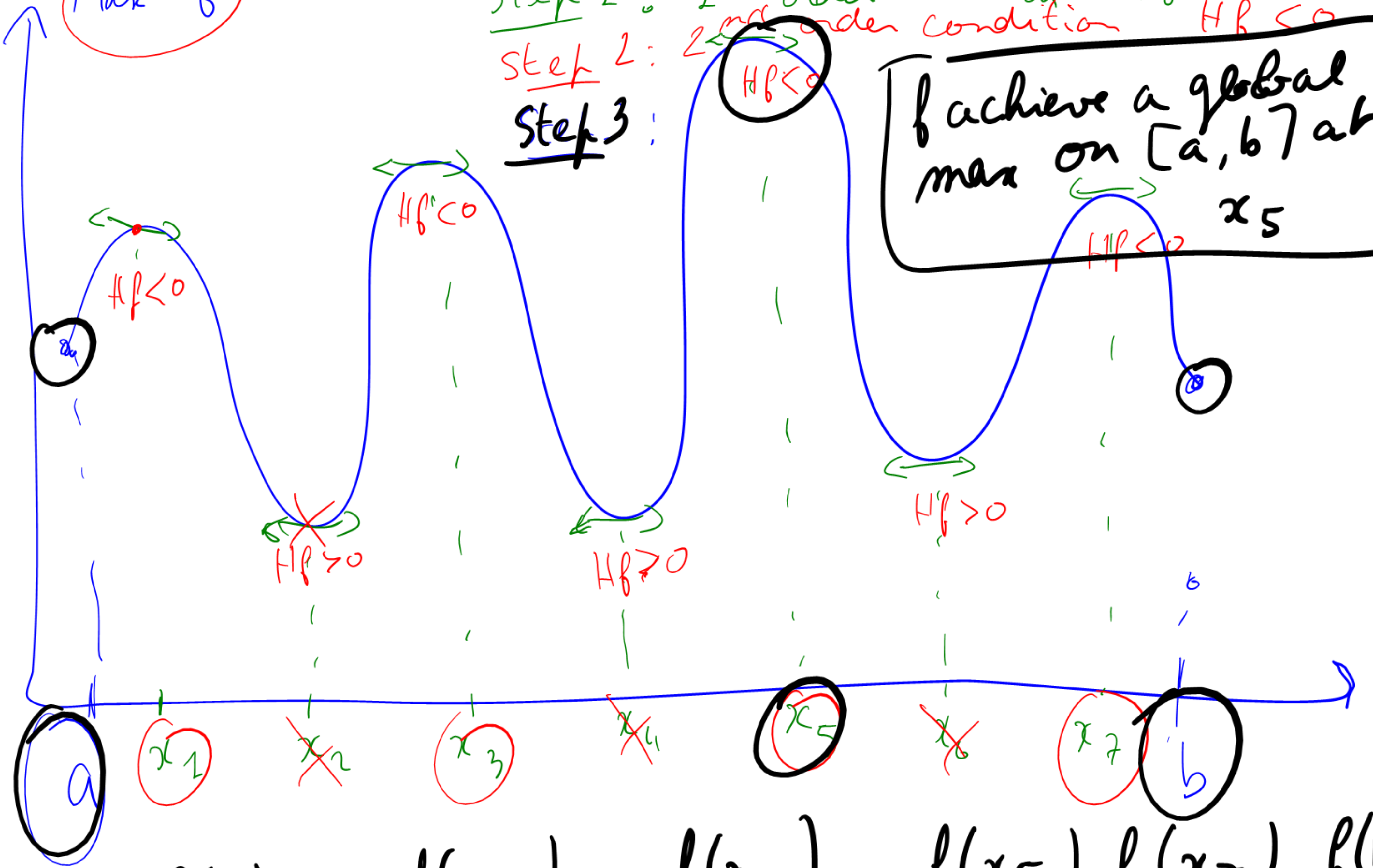
Max f

Step 1: 1^{st} order condition $\nabla f = 0$

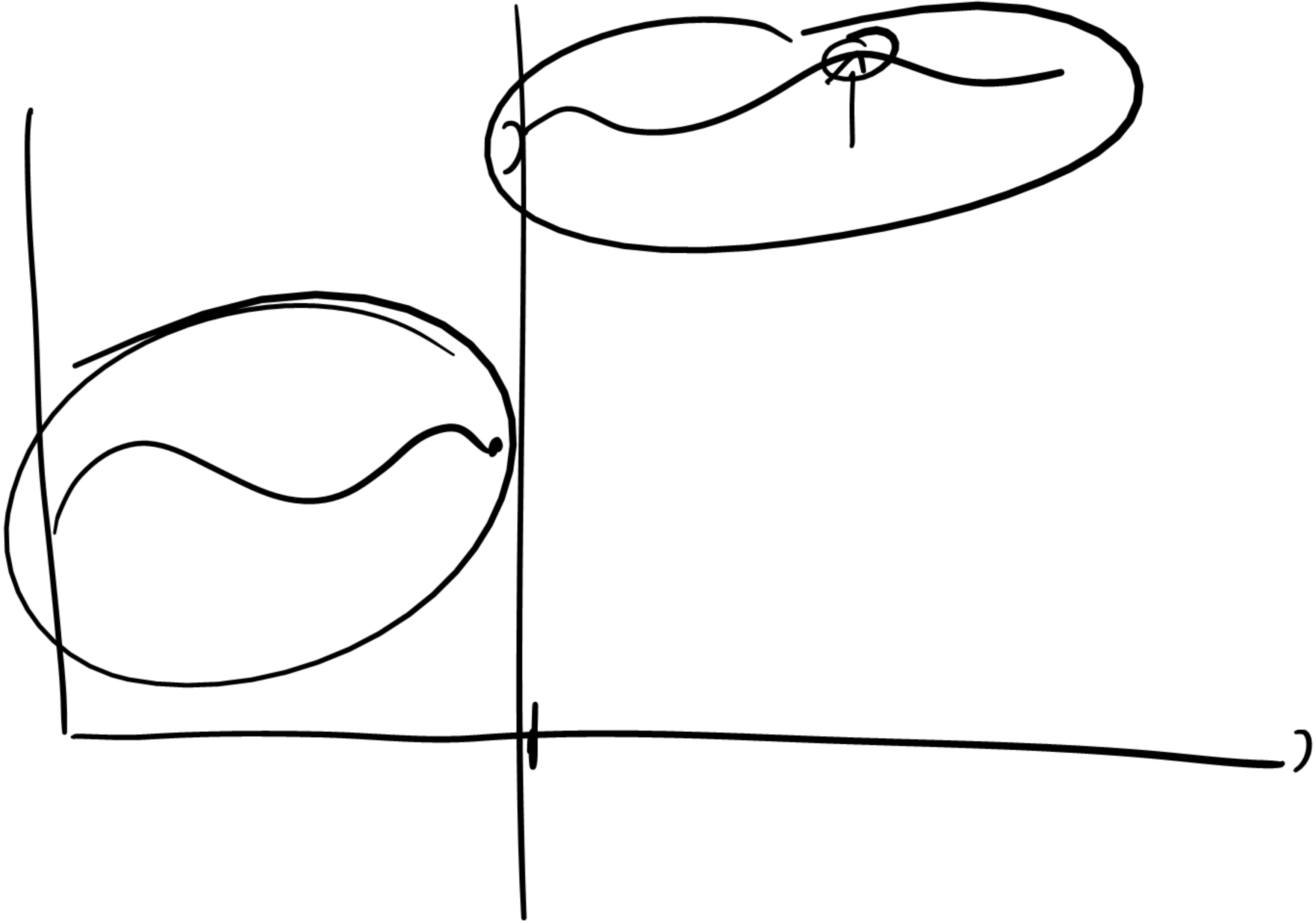
Step 2: 2^{nd} order condition $Hf < 0$

Step 3:

f achieve a global max on $[a, b]$ at x_5



$f(a), \quad f(x_1), \quad f(x_3), \quad f(x_5), f(x_7), f(b)$



1. $\min_{w} \ \sigma_{\pi}^2(w)$

Subject to

* $\mu_{\pi}(w) = 6\%$
* Budget

2

2. $\max_{w} \ \mu_{\pi}(w)$

s.t.

* $\sigma_{\pi}^2(w) = \omega$
* Budget

3. $\max \ \mu_{\pi}(w) - \frac{\lambda}{2} \sigma^2(w)$

s.t. Budget

$$\max_{w} \mu_\pi(w) - \frac{\lambda}{2}\sigma_\pi^2(w)$$

$$= \max_{w} \left(R + w'(\mu - R\mathbb{1})\right) - \frac{\lambda}{2} w'\Sigma^{-1} w$$

$$= R + \max_{w} \underbrace{\left[w'(\mu - R\mathbb{1}) - \frac{\lambda}{2} w'\Sigma^{-1} w \right]}_{f(w)}$$

1st order condition

$\hat{w} \to$ candidate point

$$\nabla f\Big|_{\hat{w}} = \frac{\partial}{\partial w}\left(\cancel{w'}(\mu - r\mathbb{1})\right) - \frac{\lambda}{2}\frac{\partial}{\partial w}\left(\cancel{w'}\Sigma^{-1}\hat{w}\right)$$

$\cancel{2 w'\Sigma^{-1}}$ or $2\Sigma^{-1} w$

$$= \mu - r\mathbb{1} - \cancel{\frac{\lambda}{2}} \times 2\Sigma^{-1}\hat{w}$$

$$= (\mu - r\mathbb{1}) - \lambda\Sigma^{-1}\hat{w} = 0$$

$$\Leftrightarrow \quad \lambda\Sigma^{-1}\hat{w} = \mu - r\mathbb{1} \quad \Leftarrow\Rightarrow$$

$$\hat{w} = \frac{1}{\lambda} \Sigma^{-1} (\mu - r\mathbb{1})$$

Check the 2nd order condition

$$Hf(\hat{w}) = \frac{\partial}{\partial w}\left(\nabla f(\hat{w})\right)$$

$$= \frac{\partial}{\partial w}\left(\cancel{(\mu - r\mathbb{1})} - \lambda \Sigma \cancel{\hat{w}}\right)$$

$$= \underbrace{-\lambda \Sigma}_{\text{neg. definite matrix}} < 0$$

$\lambda > 0$

$\Sigma > 0$ positive definite

x

$x' M x > 0$ $\quad$ $M \to$ pos. def.

$x' M x < 0$ $\quad$ $M \to$ neg def.

$$L(w,\lambda,\gamma) = \frac{1}{2} w' \Sigma w + \lambda (m - w'\mu) + \gamma (1 - w'1)$$

1st order condition

$$\frac{\partial L}{\partial w}\Big|_{\hat{w},\hat{\lambda},\hat{\gamma}} := 0$$

$$\frac{\partial L}{\partial w}\Big|_{\hat{w},\hat{\lambda},\hat{\gamma}} = \cancel{\frac{1}{2} \times 2} \Sigma \hat{w} - \hat{\lambda}\hat{\mu} - \hat{\gamma} 1 = 0$$

$$\Leftrightarrow \Sigma \hat{w} = \hat{\lambda}\hat{\mu} + \hat{\gamma} 1$$

$$\Leftrightarrow \boxed{\hat{w} = \Sigma^{-1} (\hat{\lambda}\hat{\mu} + \hat{\gamma} 1)}$$