Asian Option Pricing using Monte Carlo Simulation

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1 Stock Price Simulation

1.1 Milstein Scheme Simulation

The underlying stock price follows the Geometric Brownian Motion (GBM), whose dynamic is

$$dS_t = r_t S_t dt + \sigma_t dW_t \tag{1}$$

where r_t is the short rate at time t, and σ_t is the implied volatility at time t. W_t is the Wiener process.

The Forward Euler-Maruyama methods for the GBM gives

$$S_{t+\Delta t} - S_t = r_t S_t \Delta t + \sigma_t \phi \sqrt{\Delta t}$$

where Δt refers to as the time step in discrete time. And ϕ is a standard normal random number. That is, $\phi \sim N(0,1)$.

The Milstein method corrects the Forward Euler-Maruyama with a term on error level of $O(\delta t)$. Given a stochastic process Y_t with

$$dY_t = A(Y_t, t)dt + B(Y_t, t)dW_t$$

the discretization using Milstein scheme is

$$Y_{t+\Delta t} - Y_t = A\Delta t + B\phi\sqrt{\Delta t} + \frac{1}{2} B \frac{\partial B}{\partial Y_t}(\phi^2 - 1)\Delta t$$

where $\frac{1}{2}(\phi^2 - 1)\Delta t$ is the Milstein correction term. For GBM, the Milstein scheme yields

$$S_{t+\Delta t} - S_t = r_t S_t \Delta t + \sigma_t S_t \phi \sqrt{\Delta t} + \frac{1}{2} S_t \sigma^2 (\phi^2 - 1)\Delta t$$

1.2 Antithetic Variance Reduction

The antithetic variable technique attempts to reduce the variance by introducing negatively correlated random numbers between pair of observations. While simulating the GBM, one set of standard normal random numbers is generated, labeled as $\phi^n \sim N(0,1)$. Then $-\phi^n$ also has a standard normal distribution. Regular normal random number generating methods include Box-Muller method and Polar-Marsaglia method.

The pairs $\{(\phi^n, -\phi^n)\}$ are distributed more desirable than $2n$ independent samples, since the samples with antithetic variable have mean 0 and negative correlation.

In this report, the input parameters for the simulation are set as $S0 = 100$, $E = 100$, $T - t = 1$, $\Delta t = 1/365$, $\sigma = 20\%$, $r = 5\%$ and $\omega = 1000$. Therefore, each time step represents a time increment of one calendar day, and there will be 2000 scenarios/paths of underlying stock price using antithetic variable.

1.3 Simulation Testing

Before being used into the Asian option pricing, the stock price simulations need to be tested to ensure their mathematical or statistical properties. The main testings focus on the normal random number generation and martingale property of the underlying stock price.

Figure 1 presents the main testing results on the simulated stock price dynamics. The left graph plots one particular (randomly selected) stock price path over the simulation period. The stock price is simulated using Milstein scheme, where there is a correction term on the order of $O(\Delta t)$. The middle graph testifies the distribution of the random numbers that simulate the Wiener process W_t. The blue mark draws the empirical QQ plot of the random numbers used in the stock price simulation, which is compared to the standard normal distribution QQ plot (red line). The blue labels are within a reasonable range around the red line, which indicates that the random numbers approximately follow the standard normal distribution.

The right graph plots the averaged discounted stock price over each simulation periods. The theoretical base of this testing is the martingale property of the risk neutral pricing, that is

$$\mathbb{E}\left[e^{-\int_0^t r_s ds} S_t\right] = S_0$$

Hence the expected value of the discounted stock price should be S_0. In the right graph of Figure 1, the blue line is then (equally) averaged discounted stock price at each time period, and the red line is the reference line at S_0 level. The comparison graph shows that the simulated stock prices are within the range of $[99.8, 100.2]$, which is ± 0.2 around S_0. The simulations pass the martingale test.

2 Asian Option Pricing

2.1 Payoff Scheme

In general, the expected value of the discounted payoff under the risk neutral density $\mathbb{Q}$ is

$$V(S(T), T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_\tau d\tau} \textbf{Payoff } (S(T))\right]$$

The payoff of the Asian option varies on option type (Call vs. Put), averaging method (Arithmetic vs. Geometric sampling) and strike scheme (Fixed vs. Floating strike) and so on. This report mainly discuss the option pricing differences on averaging methods and strike schemes.

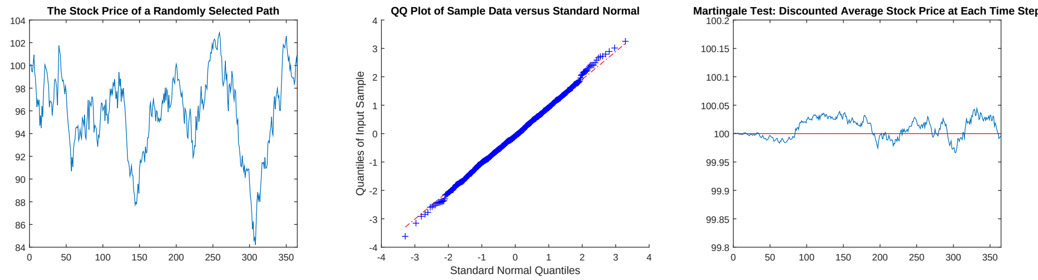

Figure 1: The testings on the stock price simulation, including plot of a randomly selected stock price path (left), QQ plot of the random numbers for the same scenario (middle) and discounted stock price over the simulation period (right).

2.1.1 Averaging Method

Consider a regular fixed strike Asian option. An Asian call option pays out

$$C(T) = \max(A(0,T) - K, 0)$$

where $A(0,T)$ is the averaged underlying price over the period $[0,T]$.

Under arithmetic averaging scheme, the continuous time formula of the average is

$$A(0,T) = \frac{1}{T}\int_0^T S(t)dt$$

In discrete time, the arithmetic average is

$$A(0,T) = \frac{1}{N+1}\sum_{i=0}^{N} S(t_i) \qquad t_i = i \cdot \frac{T}{N}$$

Here the ending (time) point of the stock price is included into averaging calculation. Though this ending point is excluded from the averaging in some literature, the pricing impact of the exclusion will be trivial.

Under geometric averaging scheme, the continuous time formula of the average is

$$A(0,T) = \exp\left(\frac{1}{T}\int_0^T \ln(S(t))dt\right)$$

In discrete time, the geometric average is

$$A(0,T) = \left(\prod_{i=0}^{N+1} S(t_i)\right)^{\frac{1}{N+1}} \qquad t_i = i \cdot \frac{T}{N}$$

For Asian put option, the payoff function is

$$P(T) = \max(K - A(0,T), 0)$$

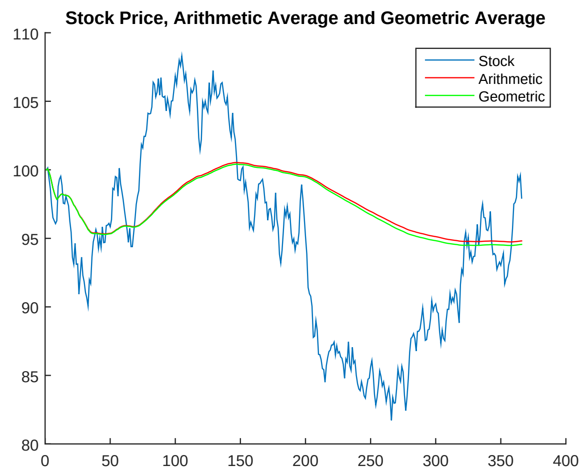

Figure 2: The stock price dynamics and its arithmetic and geometric averages.

and the formulae for the geometric and arithmetic averaging are the same as that of the call option.

Figure 2 shows the stock price simulation and its arithmetic (red) and geometric (green) averages. The Asian option has the feature that its final payoff is related to these averages.

Table 1 presents the Asian option pricing results using arithmetic or geometric averaging methods. The call and put option prices with different number of paths are calculation given the simulation results. Approximately there is $\pm 0.4\%$ difference between arithmetic and geometric results. The results are not very sensitive to the number of scenarios once it passes 10,000.

Fixed Strike	Call		Put	
Averaging	Arithmetic	Geometric	Arithmetic	Geometric
1,000	5.6991	5.4916	3.3107	3.4248
10,000	5.7328	5.5181	3.3238	3.4415
100,000	5.7424	5.5262	3.3285	3.4465

Table 1: The Asian pricing results on call and put option type with arithmetic or geometric averaging methods. The pricing results with 1000, 10000, 100000 paths are compared.

2.1.2 Strike Scheme

Beside the averaging method, the strike scheme also affects the payoff of the Asian option. The fixed strike Asian call option has the payoff

$$C(T) = \max(A(0,T) - K, 0)$$

For Asian put option, the fixed strike payoff is

$$P(T) = \max(K - A(0,T), 0)$$

And the floating strike (floating rate) Asian call option has the payoff

$$C(T) = \max(S(T) - kA(0,T), 0)$$

where $k = 1$ in this report. For Asian put option, the floating strike payoff is

$$P(T) = \max(kA(0,T) - S(T), 0)$$

Table 2 presents the Asian option pricing results using arithmetic or geometric averaging methods under the floating strike scheme. The call and put option prices with different number of paths are calculation given the simulation results. Approximately there is ±0.4% difference between arithmetic and geometric results. The results are not very sensitive to the number of scenarios once it passes 10,000. Comparing to the fixed strike scheme pricing, the floating strike scheme consistently has higher option price except for the Geometric averaging method of put option pricing.

Floating Strike	Call		Put	
Averaging	Arithmetic	Geometric	Arithmetic	Geometric
1,000	5.7635	5.9615	3.3585	3.2349
10,000	5.8262	6.0333	3.3782	3.2528
100,000	5.8489	6.0574	3.395	3.2692

Table 2: The Asian pricing results on call and put option type with arithmetic or geometric averaging methods. The pricing results with 1000, 10000, 100000 paths are compared.

2.2 Comparison with Theoretical Price

Given the GBM dynamic for the underlying stock price movement and S_{avg} as the geometric average of the stock price, S_{avg} is also lognormally distributed. Under the risk neutral assumption, assumption, the geometric average price option can be treated like a regular option with the volatility σ_{adj} set equal to $\sigma\sqrt{3}$ and the adjust the dividend yield equal to

$$q_{adj} = r - \frac{1}{2}(r - q - \frac{\sigma^2}{6}) = \frac{1}{2}(r + q + \frac{\sigma^2}{6})$$

Using the Black-Scholes-Merton formula, we price geometric average price Asian options as

$$\begin{aligned} C(0) &= S_0 e^{-q_{adj}T} N(d_1) - Ke^{-rT} N(d_2) \\ P(0) &= Ke^{-rT} N(-d_2) - S_0 e^{-q_{adj}T} N(-d_1) \end{aligned}$$

where

$$
\begin{aligned}
\sigma_{adj} &= \sigma\sqrt{3} \\
q_{adj} &= \frac{1}{2}(r + q + \frac{\sigma^2}{6}) \\
d_1 &= \frac{1}{\sigma_{adj}\sqrt{T}}\left(\log(\frac{S_0}{K}) + (r - q_{adj} + \frac{1}{2}\sigma_{adj}^2)T\right) \\
d_2 &= d_1 - \sigma_{adj}\sqrt{T}
\end{aligned}
$$

3 Extensions