* Stochastic calculus
* martingales

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* Portfolio selection → the big picture
→ optimisation

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* Call option on a stock →
* Martingale & PDE

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* Bonds & Call on Bonds

→ 8h :)

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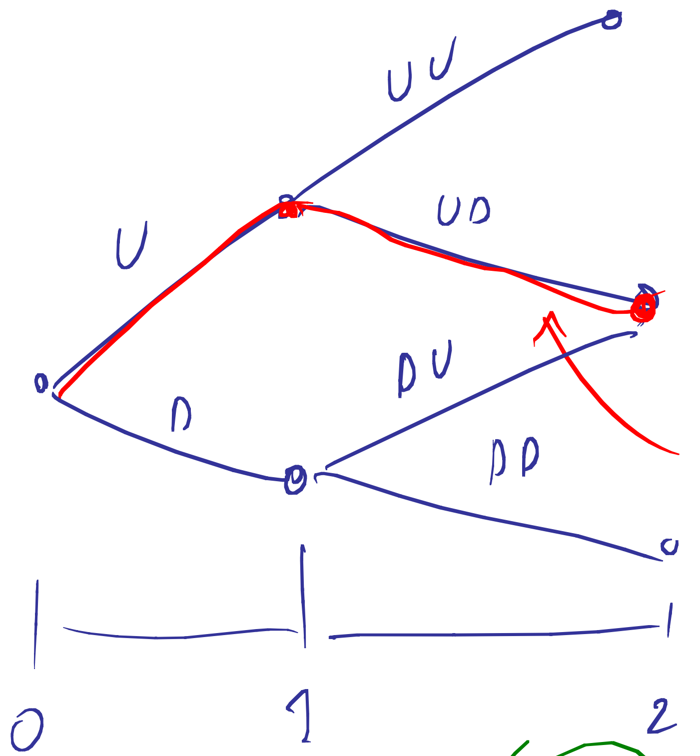
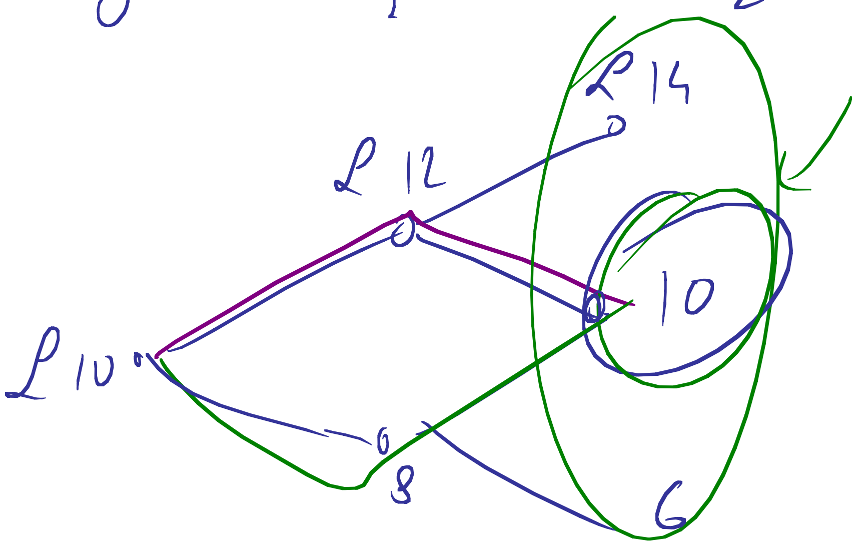
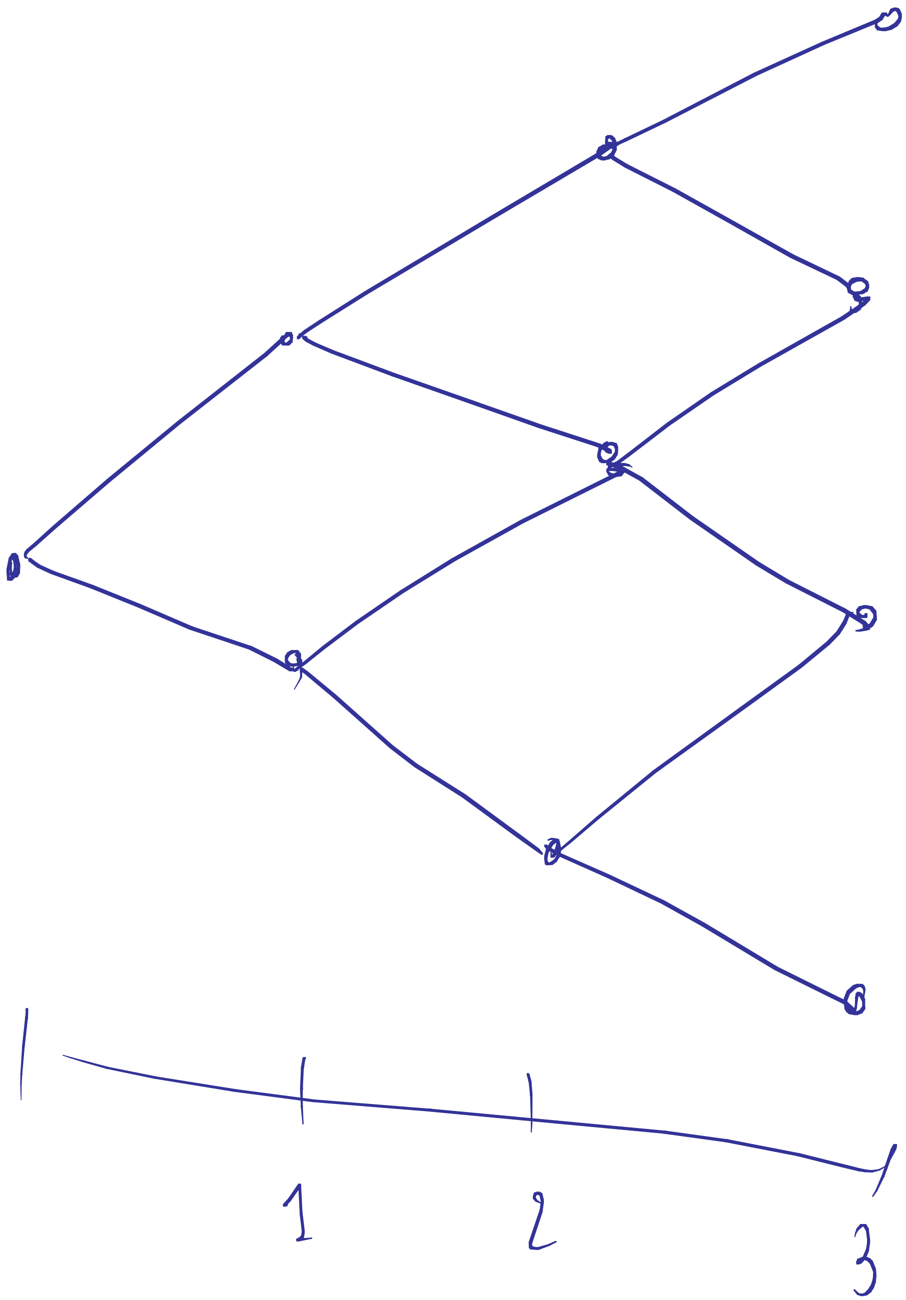


sample path

$$\Omega_2 = \{UU, UD, DU, DD\}$$



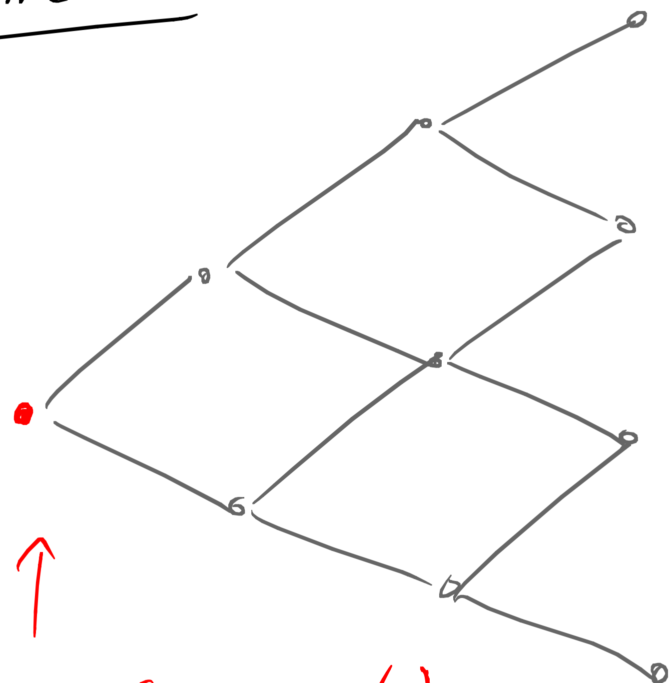
$$\Omega_2' = \{6, 10, 14\}$$



$$\Omega_3 = \{UUU, UDU, UUD, UDD, DUU, DUD, DDU, DDD\}$$

Periods	size of Ω
1	2
2	$2^2 = 4$
3	$2^3 = 8$
⋮	
10	$2^{10} = 1024$
⋮	
100	$2^{100} = 1.27 \times 10^{30}$

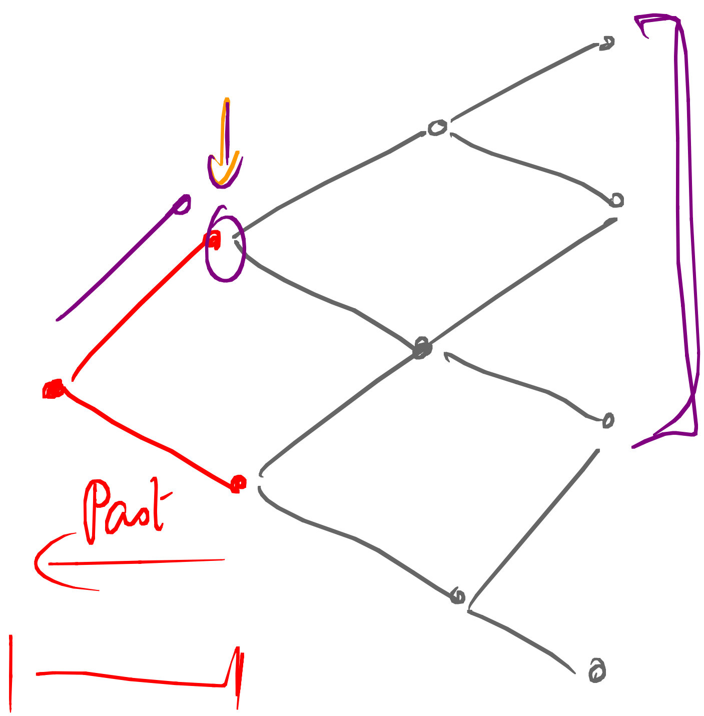
At time 0 :



$\mathcal{F}_0 = \{ \Omega, \emptyset \}$

$\Omega = \{UUU, UUD, UDU, UDD, DUU, DUD, DDU, DDD\}$

At time 1 .

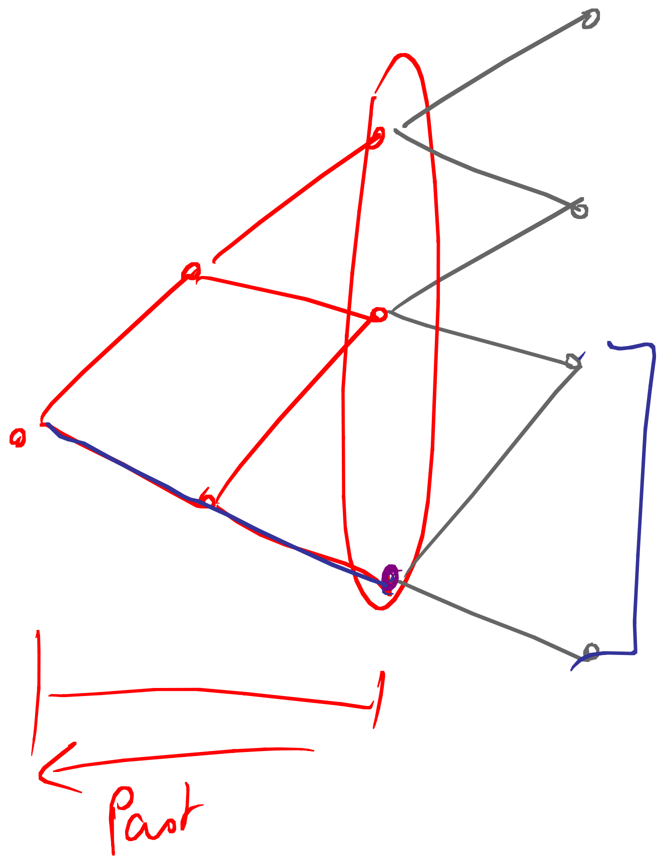


$u = \{UUU, UUD, UDU, UDD\}$

$d = \{DDD, DDU, DUD, DDD\}$

$F_1 = \{\Omega, \emptyset, u, d\}$

At time 2 :



$F_2 = \{\Omega, \phi, u, d, \dots, uu, ud, du, dd, \dots, uu \cup ud\}$

u

$uu = \{UUU, UUD\}$

$dd = \{DDD, DDU\}$

$du = \{DUD, DUU\}$

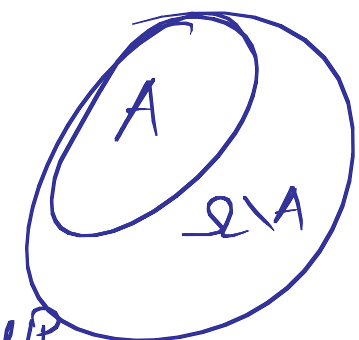
$ud = \{UDD, UDU\}$

A nice trick:

$$\mathbb{E}[h(X)] = \int_\Omega h(X(\omega))\, d\mathbb{P}(\omega)$$

indicator function $\mathbb{1}_{X\in A} = \begin{cases} 1 & \text{if } X \in A \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{E}[\mathbb{1}_{X\in A}] = \int_\Omega \mathbb{1}_{X\in A}\, d\mathbb{P}$$

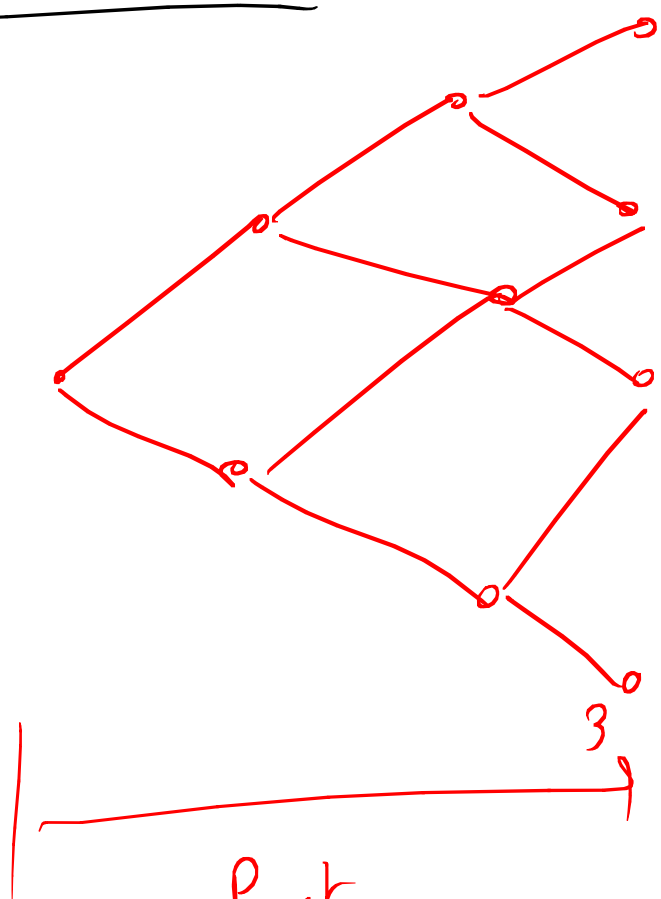


$$= \int_A 1\, d\mathbb{P} + \underbrace{\int_{\Omega\setminus A} 0\, d\mathbb{P}}_{0}$$

$$= \cancel{\int}_A \cancel{d}\mathbb{P}$$

$$= \mathbb{P}(A)$$ ← Probability that your outcome is in set A!

At time 3:



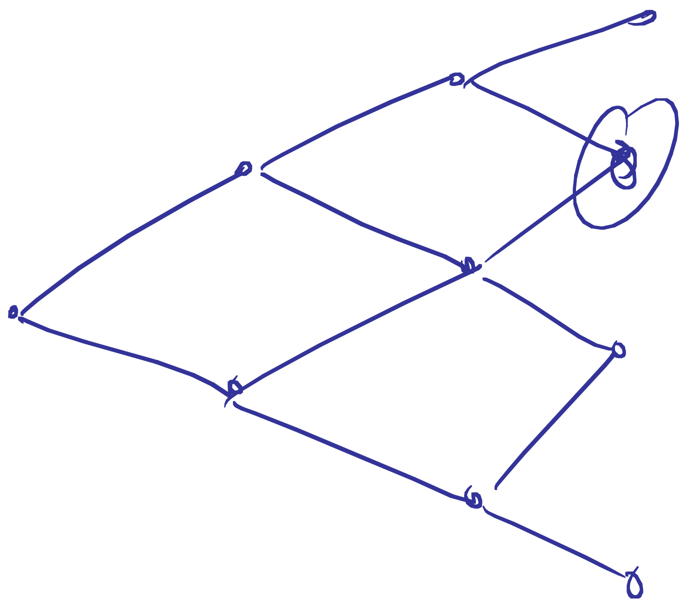
Past

$\mathcal{F}_3 = \{ \Omega, \phi, UUU, UDU, UUD, \ldots$

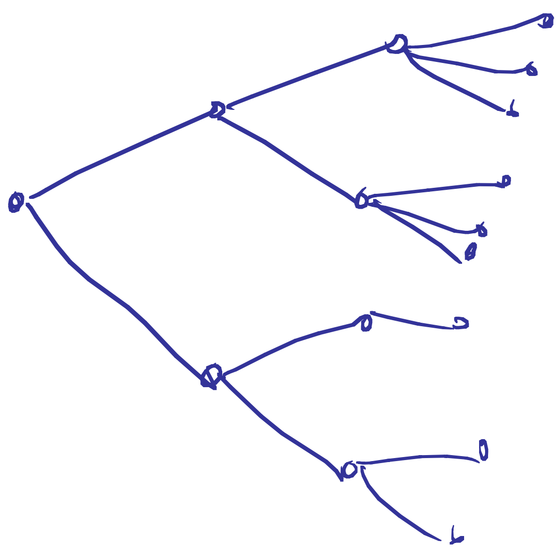
$U = \{UUU, UUD, UDD, UDU\}$

$UU = \{UUU, UUD\}$

$UUU =$



Pricing



Portfolio selection

stochastic prgm.

* $\mathbb{E}[h(X)] = \int_{-\infty}^{+\infty} h(x) \, \boxed{p(x) \, dx} = \int_{-\infty}^{+\infty} h(x) \, dP(x)$

~~Probability~~ Density Function (PDF)

CDF

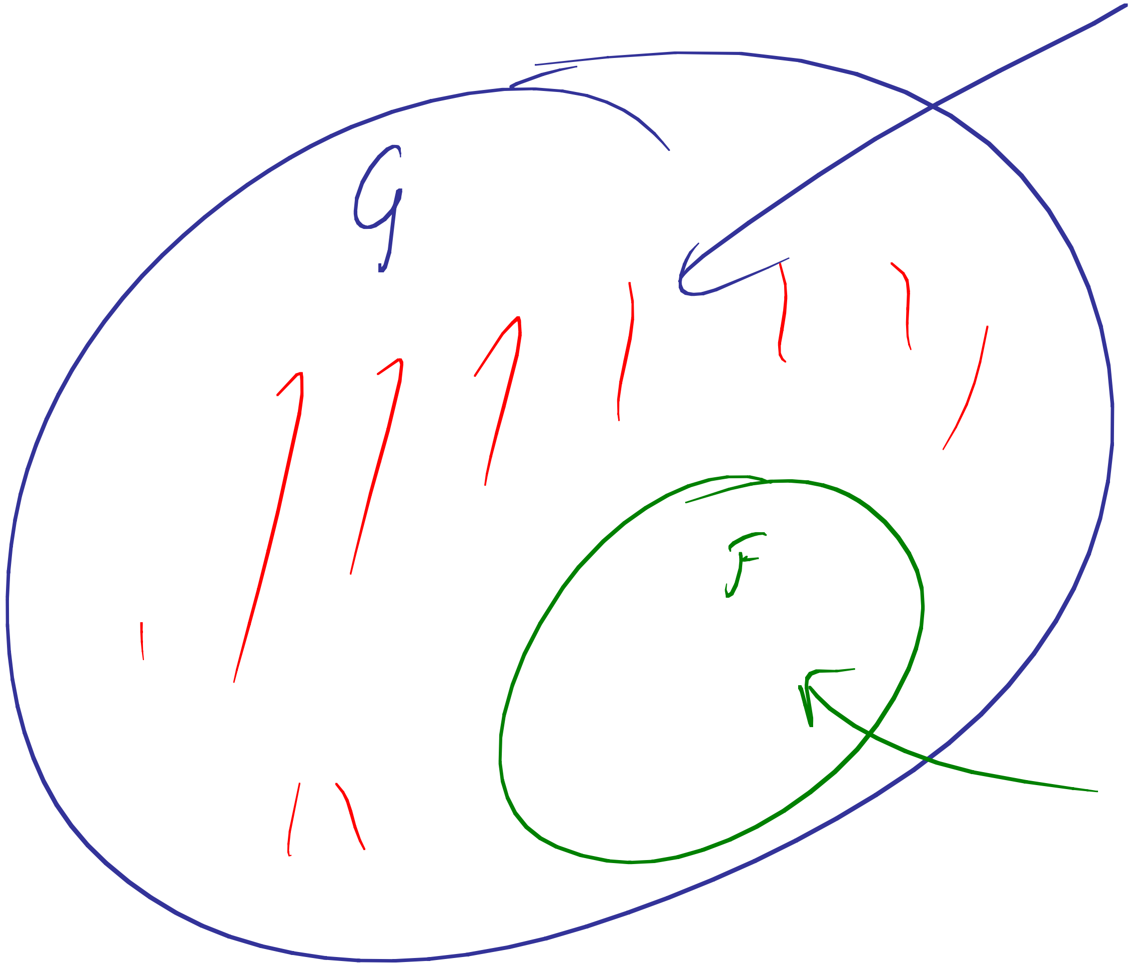
$P(X \le x) \leftarrow$ Cumulative Density Function (CDF)

$$p(x) = \frac{dP(x)}{dx} \iff p(x)\,dx = \boxed{dP(x)}$$

$\mathbb{P}$ is a little bit like P, the CDF

* Expectation with measure

$$\boxed{\mathbb{E}[h(X)] = \int_{\Omega} h(X(\omega)) \, d\mathbb{P}(\omega)} \leftarrow$$



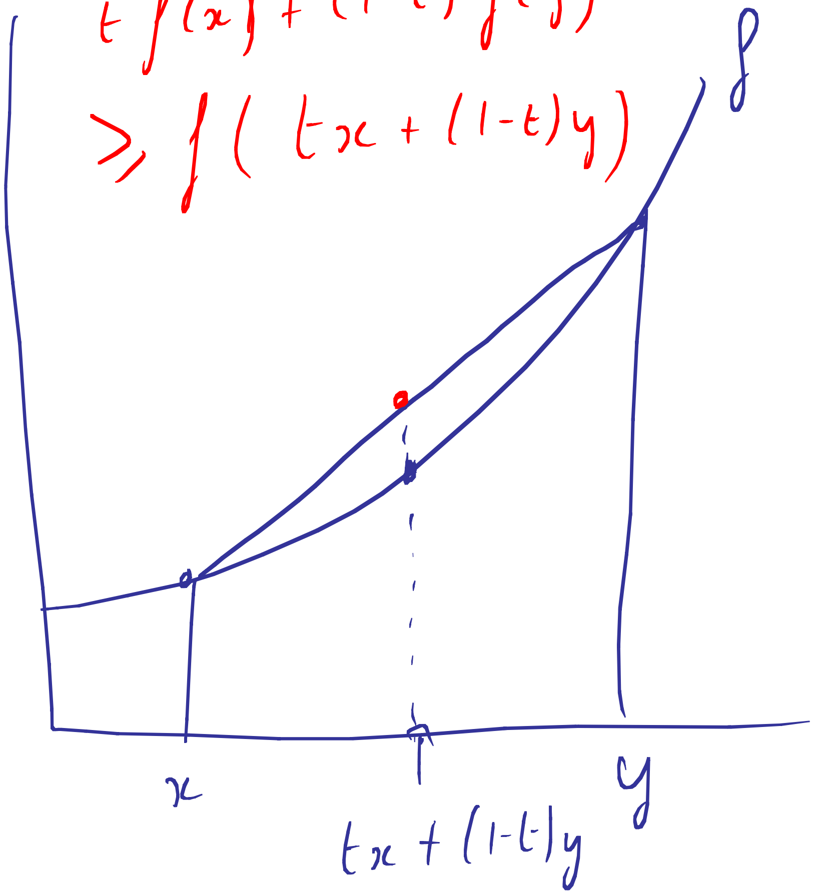
$$Y = \mathbb{E}[X \mid \mathcal{G}]$$

stock price

$$Z = \mathbb{E}[Y \mid \mathcal{F}] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{F}] = \mathbb{E}[X \mid \mathcal{F}]$$

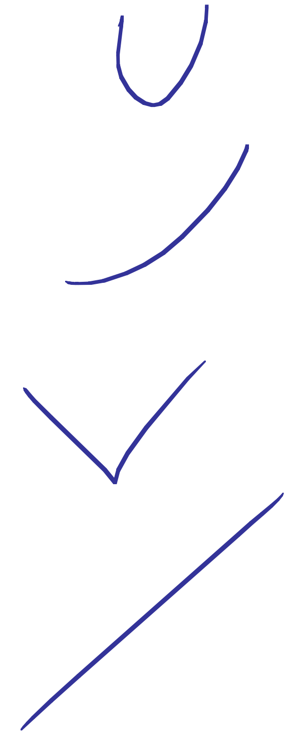
Convex functions:

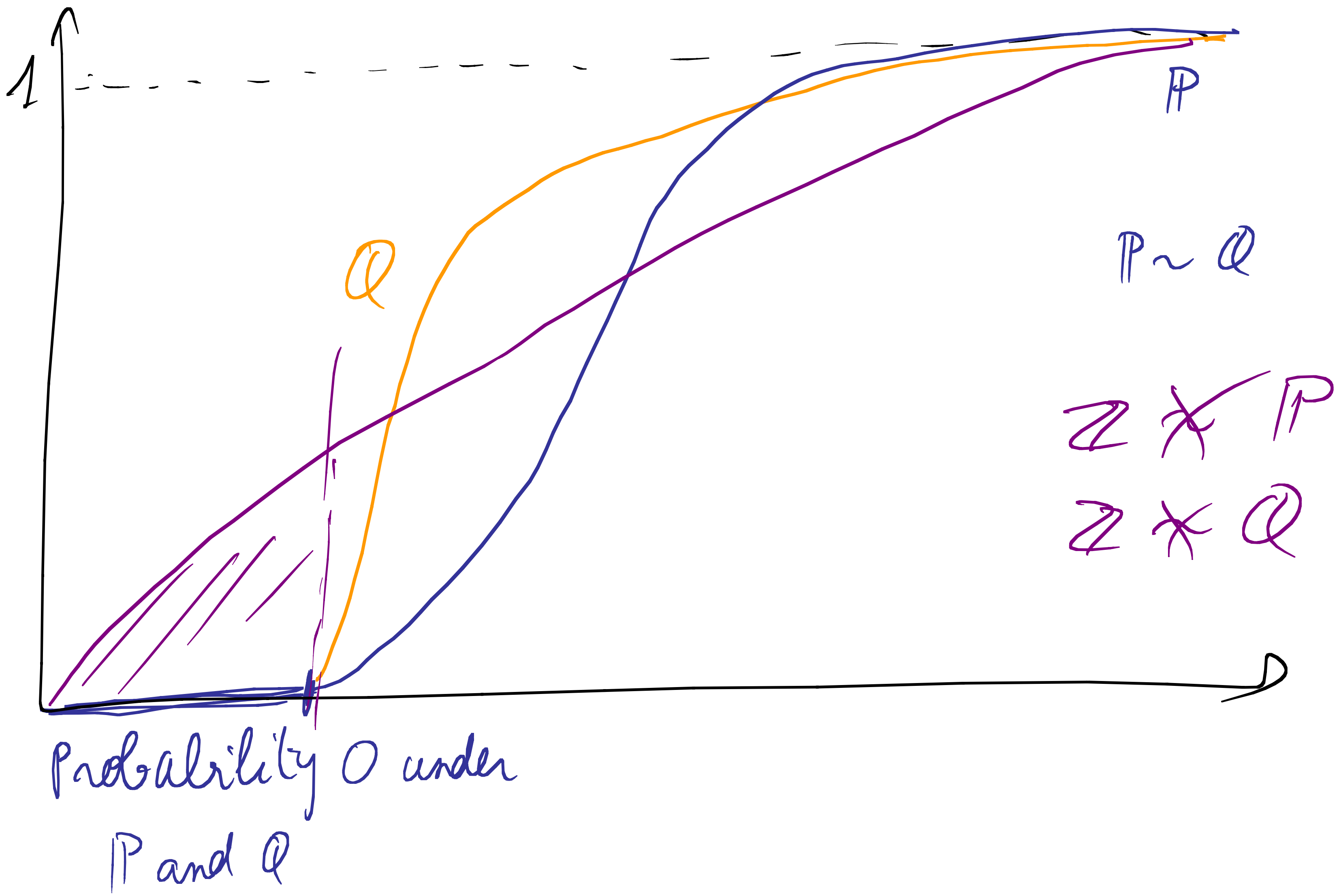
$$t f(x) + (1-t) f(y) \geq f(tx + (1-t)y)$$



$$0 < t < 1$$

- $f(x) = \boxed{x^2}$
- $f(x) = \boxed{e^x}$
- $f(x) = |x|$
- $f(x) = x$




1
Q
P
P ~ Q
Z ✗ P
Z ✗ Q
Probability 0 under
P and Q

* A function, some stochastic process(es)

$$V(S_1(t), S_2(t)) = \boxed{S_1(t) \times S_2(t)}$$

$$dS_1(t) = f_1(t, S_1(t), S_2(t))\,dt + g_1(t, S_1(t), S_2(t))\,dX_1(t)$$

$$dS_2(t) = f_2(.)\,dt + g_2(.)\,dX_2(t)$$

$$dX_1(t)\,dX_2(t) \to \boxed{\rho\,dt}$$

ρ is your correlation coefficient

$$dV(t) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_1} f_1 + \frac{\partial V}{\partial S_2} f_2 + \frac{1}{2}\frac{\partial^2 V}{\partial S_1^2} g_1^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S_2^2} g_2^2 + \frac{\partial^2 V}{\partial S_1 \partial S_2} \times \rho g_1 g_2 \right) dt + \frac{\partial V}{\partial S_1} g_1 \, dX_1(t) + \frac{\partial V}{\partial S_2} g_2 \, dX_2(t)$$

$$V = S_1 \times S_2$$

$$\frac{\partial V}{\partial t} = 0$$

$$\frac{\partial V}{\partial S_1} = S_2$$

$$\frac{\partial V}{\partial S_2} = S_1$$

$$\boxed{\frac{\partial^2 V}{\partial S_1^2} = 0}$$

$$\boxed{\frac{\partial^2 V}{\partial S_2^2} = 0}$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = 1$$

Ordinary Calculus : $\frac{d}{dx}(f \times g) = \frac{df}{dx} \times g + f \times \frac{dg}{dx}$

Stochastic (Itô) Calculus : $dX^2(t) \to dt$

$dX_1 dX_2 \to \rho dt$

$$dV = \underbrace{dS_1 \times S_2 + S_1 \times dS_2}_{\text{Ordinary calculus}} + \boxed{dS_1 dS_2}$$

Cross variation adjustment

adjustment for the quadratic variation property

$$dV = \underbrace{S_1(t)\, dS_2(t)} + \underbrace{S_2(t)\, dS_1(t)} + \boxed{dS_1(t)\, dS_2(t)}$$

$$S_1\left(f_2\, dt + g_2\, dX_2(t)\right) + S_2(t)\left(f_1\, dt + g_1\, dX_1(t) + \rho g_1 g_2\, dt\right)$$

$$\boxed{dS_1(t)\, dS_2(t)} = \left(f_1\, dt + g_1\, dX_1(t)\right)\left(f_2\, dt + g_2\, dX_2(t)\right)$$

$$= \cancel{f_1 f_2\, dt^2} + \cancel{f_1 g_2\, dt\, dX_2(t)} + \cancel{g_1 f_2\, dt\, dX_1(t)} + g_1 g_2\, \underbrace{dX_1\, dX_2(t)}_{\rho\, dt}$$

$(dt)^{3/2}$ $(dt)^{3/2}$

$g_1 g_2\, \rho\, dt$

$$dV = \left(f_1 S_2 + f_2 S_1 + \rho g_1 g_2 \right) dt + S_1 g_2 dx_2 + S_2 g_1 dx_1$$

$\Lambda x -$

$$dS_1(t) = (a_1 + b_1 X(t))dt + \sigma_1 dW_1$$

$$dS_2(t) = (a_2 + b_2 X(t))dt + \sigma_2 dW_2$$

$$dX(t) = - - -$$