CQF Module 4, Session 5: Martingales and Fixed Income Valuation

CQF

In this lecture...

... we will apply probabilistic and martingale methods to the pricing of bonds and derivatives struck on bonds using short-term rate models. We will see:

- the pricing of interest rate products in a probabilistic setting;
- the equivalent martingale measure**s**;
- the fundamental asset pricing formula for bonds;
- application for popular interest rates models;
- the dynamics of bond prices;
- the forward measure;
- the fundamental asset pricing formula for derivatives on bonds;
- rights and wrongs of short-term interest rate models;

Introduction

1. A Model for the Short-Term Rate

In the world of equity, a single class of models (the Geometric Borwnian Motion and its extensions) dominates the landscape. The situation is markedly different in the world of interest rates, where several classes of models coexist:

1. *Single factor short-term rate models*, such as the Merton, Vasicek, Cox-Ingersoll-Ross, Ho-Lee or Hull-White models;
2. *Multifactor models* such as the Brennan-Schwartz, Longstaff-Schwartz and Fong-Vasicek models;
3. *Forward rate models* such as the Heath-Jarrow-Morton model and its modification by Brace, Gatarek and Musiela.

Our focus today will be on single factor short-term rate models whose dynamics under the physical measure $\mathbb{P}$ is of the form

$$dr(t) = \mu(t, r_t)dt + \sigma(t, r_t)dX(t), \qquad r(0) = r \tag{1}$$

where $r(t)$ is the short-term rate at time t and $X(t)$ is a Brownian motion.

To simplify the notation, we will write this equation as

$$dr(t) = \mu_t dt + \sigma_t dX(t), \qquad r(0) = r \tag{2}$$

$\mu_t = \mu(t, r_t)$

The specification (2) is general enough to cover both

1. *Equilibrium models* such as the Vasicek or Cox-Ingersoll-Ross models;
2. *No arbitrage models* such as Ho-Lee and Hull-White;

$dr(t) = \mu_t dt + \sigma_t dX(t)$

Equipped with a short-term interest rate model, we can define a money market or "money-in-the-bank" process $A(t)$ as:

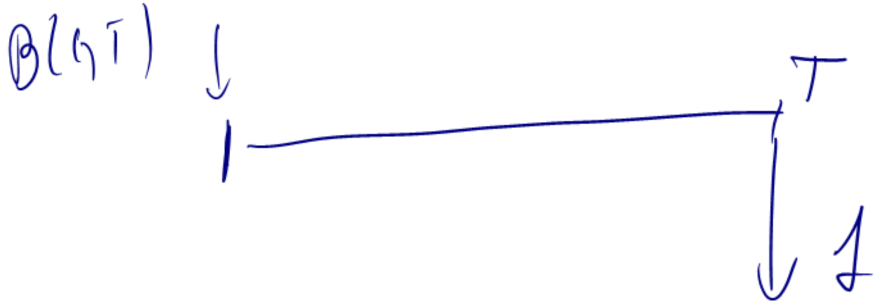
$$A(t) = e^{\int_0^t r(s)ds} \tag{3}$$

$dA(t) = r(t)A(t)dt$

$\Rightarrow \boxed{A(t) = A(0)\, e^{\int_0^t r(s)ds}}$ $\qquad A(0) = 1$

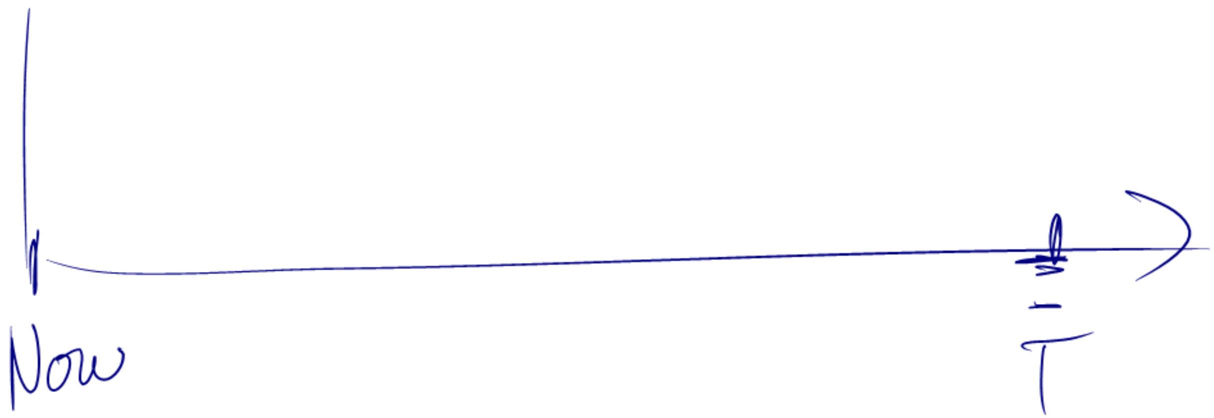
$A^{-1}(t) = e^{-\int_0^t r(s)ds} \Rightarrow dA^{-1}(t) = -r(t)A^{-1}(t)dt$

2. The Zero-Coupon Bond Market



A *zero-coupon bond* $B(t, T)$ is a bond which pays 1 at maturity time T.

We will define the *zero-coupon bond market* as the set of all the zero-coupon bonds $B(t, T)$ for all $t \leq T \leq \overline{T}$.



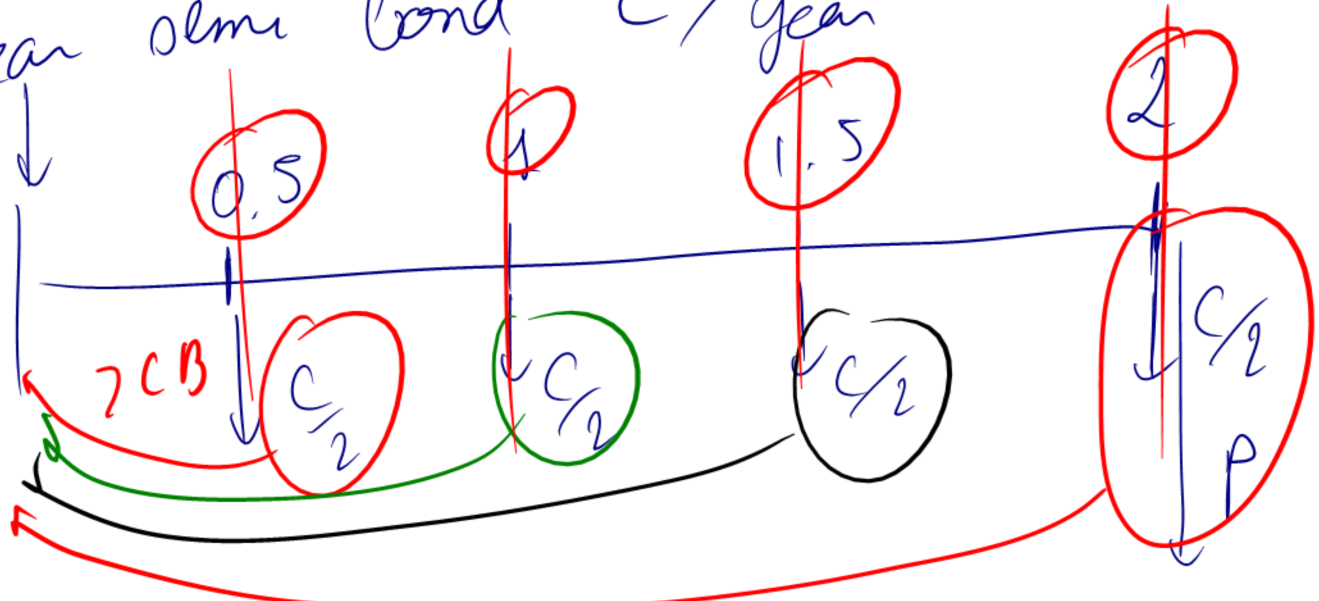
Our definition of the zero-coupon bond market is clearly unrealistic:

- in reality, zero-coupon bonds are scarce and illiquid investments;
- we cannot expect to find a zero-coupon bond maturing at any time $t \leq T \leq \overline{T}$.

However, we will find this representation convenient since zero-coupon bonds are the building block of the fixed income market:

- any bond can be decomposed as a sequence of coupons;
- each coupon can be modelled as a zero-coupon bond.

2-year semi bond C/year



Another reason for defining the entire bond market rather than a single security stems from the observation in Lecture 4.2 that since the short-term rate cannot be traded directly, the only way to hedge a bond is by trading another bond.

With this idea, we have made our way from the so-called *complete markets* of Modules 2 and 3 to *incomplete markets*.

3. Pricing Zero-Coupon Bonds

As in lecture 4.2, our starting point will be the valuation of zero-coupon bonds.

3.1. Equivalent Martingale Measure for the Zero-Coupon Bond Market

Definition (Equivalent Martingale Measure for the Zero-Coupon Bond Market)

An equivalent martingale measure for the zero-coupon bond market is a measure $\mathbb{Q}$

1. equivalent to the physical measure $\mathbb{P}$;
2. such that for any maturity $T \in [0, \overline{T}]$ the process

discounted ZCB price

$$Z^*(t,T) = \frac{B(t,T)}{A(t)} \tag{4}$$

ZCB

discounted using the money market account

is a martingale for all $t \in [0, T]$.

3.1.1 Equivalent Measure or Equivalent Measures?

First, Definition is consistent with the way we defined the equivalent martingale measure for a stock in Lecture 3.3.

But the key difference in the zero-coupon bond case is that the equivalent martingale measure does not apply to a single security, but to a continuum of securities.

As a result, we should not expect to find a unique martingale measure to fit Definition 14. This is a trademark of incomplete markets: in some instances no martingale measure will exist while in others a multiplicity (or even an infinite number) of equivalent martingale measures may exist.

In our case, the latter applies: we can find multiple equivalent martingale measures.

3.1.2 Equivalent Martingale Measure and No-Arbitrage

As shown in Lecture 3.6, the existence of (at least) one equivalent martingale measure implies an absence of arbitrage opportunity.

Our zero-coupon bond market is therefore arbitrage-free.

3.2. Behaviour of the Short-Term Rate under the Equivalent Martingale Measure

We consider an equivalent martingale measure $\mathbb{Q}^{\theta}$ for which there exists a process $\theta(t)$

- satisfying the Novikov condition

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^{\bar{T}}\theta_s^2 ds\right)\right] < \infty$$

- and which defines the Radon Nicodým derivative:

$$\begin{aligned} \frac{d\mathbb{Q}^{\theta}}{d\mathbb{P}} &= \eta_t \\ &= \underbrace{\exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dX_s\right)}_{\text{exponential martingale}}, \quad t \in [0, \bar{T}] \end{aligned}$$

Standard BM w.r.t to $\mathbb{P}$

By Girsanov's theorem, the process X^θ defined as

Standard BM w.r.t $\mathbb{Q}^\theta$

$$X_t^\theta = X_t + \int_0^t \theta(s)ds, \quad t \in [0, \overline{T}] \tag{5}$$

is a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{Q}^\theta)$.

As a result, the dynamics of $r(t)$ under $\mathbb{Q}^\theta$ is:

$$dr(t) = (\mu_t - \sigma_t\theta_t)\, dt + \sigma_t dX^\theta(t), \qquad r(0) = r \tag{6}$$

3.3. The Fundamental Asset Pricing Formula for Zero-Coupon Bonds

$$Y(t) = \mathbb{E}^{Q}\left[\text{Discounted Cash Flows} \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}^{Q}\left[\frac{A(t)}{A(T)} Y(T) \,\middle|\, \mathcal{F}_t\right] = A(t)\,\mathbb{E}^{Q}\left[\frac{Y(T)}{A(T)} \,\middle|\, \mathcal{F}_t\right]$$

Applying the fundamental asset pricing formula (derived in Lecture 3.3) to the zero-coupon bond, we get

w.r.t Q^{θ}

$$A(t) = e^{\int_0^t r(s)ds}$$

$$B(t, T) = A(t)\mathbf{E}^{\theta}\left[\frac{B(T, T)}{A(T)}\middle|\mathcal{F}_t\right] \tag{7}$$

$B(t,T)$ $\quad$ $B(T,T) = 1$

$$B(t) = A(t)\,\mathbb{E}^{\theta}\left[\frac{1}{A(T)} \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}^{\theta}\left[\frac{A(t)}{A(T)} \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}^{\theta}\left[e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{F}_t\right]$$

Since

$$B(T,T) = 1$$

and

$$A(T) = e^{\int_0^T r(s)ds}$$

then formula (7) simplifies to

$$B(t,T) \quad = \quad \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right]$$

Q^{θ} ;

$$dr(t) = \left(\mu_t - \sigma_t\,\theta(t)\right)dt + \sigma_t\,dX^{\theta}(t)$$

← drift $\frac{\partial B}{\partial r}$

$\frac{1}{2}\sigma_t^2\,\frac{\partial^2 B}{\partial r^2}$

Key Fact (The Fundamental Asset Pricing Formula for Zero-Coupon Bonds)

The price of a zero-coupon bond is given by the expectation:

$$B(t,T) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds}\,\middle|\,\mathcal{F}_t\right] \tag{8}$$

← $-r\,B(t,T)$

1

F.K (PDE)

$$\boxed{\frac{\partial B}{\partial t} + \left(\mu_t - \sigma_t\,\theta(t)\right)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma_t^2\,\frac{\partial^2 B}{\partial r^2} - rB = 0 \qquad \text{T.C.}\quad B(T,T) = 1}$$

3.4. Bond Pricing in Practice: Analytical Solutions

Analytical solutions exist for the most popular models such as the Vacicek model or the CIR model.

In general, the best way to derive them is to apply Feynman-Kač to the fundamental asset pricing formula to obtain a PDE, and then solve that PDE[1].

[1] Although in the case of the Vaiscek model, there is an elegant and insightful analytical solution through probabilities.

By Feynman-Kač, the PDE associated with expectation (8) and the $\mathbb{Q}^{\theta}$-process $r(t)$ given by equation (6) is:

$$
\begin{aligned}
\frac{\partial B}{\partial t}(t,r) + (\mu_t - \sigma_t\theta_t)\,\frac{\partial B}{\partial r}(t,r(t)) + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 B}{\partial r^2}(t,r(t)) & \\
-r(t)B(t,r) &= 0 \\
B(T,r(T)) &= 1
\end{aligned}
$$

We are now in familiar territory as this PDE was derived and then solved in a few cases earlier in this module.

3.5. Bond Pricing in Practice: Numerical Solutions

Given

- a short-rate model with dynamics

$$dr(t) = (\mu_t - \sigma_t\theta_t)\,dt + \sigma_t dX^{\theta}(t), \qquad r(0) = r \qquad (9)$$

  under the equivalent martingale measure $\mathbb{Q}^{\theta}$;
- the fundamental asset pricing formula applied to zero-coupon bonds (equation (8))

$$B(t,T) \quad = \quad \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right]$$

it is not difficult to obtain a numerical solution using **Monte Carlo methods**.

The Monte-Carlo "algorithm" would look like this:

- $r(0) = r$
- For Simulation $j = 1$ to N,
  - For TimeStep t_i, $i = 1$ to M
    - Simulate a $N(0,1)$ random variable Z_i;
    - Simulate the short-term interest rate as

$$\begin{aligned} r(t_i) &= r(t_{i-1}) + \left[\mu(t_{i-1}, r_{t_{i-1}}) \right. \\ &\left. -\sigma(t_{i-1}, r_{t_{i-1}})\theta(t_{i-1})\right](t_i - t_{i-1}) \\ &+\sigma(t_{i-1}, r_{t_{i-1}})Z_i\sqrt{t_i - t_{i-1}} \end{aligned}$$

    - Compute the value of the integral term $\int_t^T r(s)ds$ recursively as:

$$Int = Int + r(t_{i-1})(t_i - t_{i-1})$$

  - Next TimeStep

- ...
  - Compute the value $B(j)$ of the zero-coupon bond under simulation j:

$$B(j) = exp\left\{-Int\right\}$$

- Next Simulation
- Compute the value of the bond as

$$B = \frac{1}{M}\sum_{j=1}^{M} B(j)$$

Done!

4. Dynamics of the Zero-Coupon Bond Price

Later in this session, we will consider the pricing of derivatives struck on a zero-coupon bond. To price such derivatives, we will need to understand no only the dynamics of the short-term rate but also the dynamics of the zero-coupon bond prices.

Our starting point will be somewhat unusual. We will look at the process

discounted ZCB

$$M(t) = Z^*(t,T)\eta_t = \frac{B(t,T)}{A(t)}\eta_t$$

exp mart $\equiv \frac{d\mathbb{Q}^\theta}{d\mathbb{P}}$

where η_t is the Radon Nikodým derivative.

$$Z^*(t,T) = \frac{B(t,T)}{A(t)}$$

The reason for considering this unusual process $M(t)$ is that

Fact

If a process $Y(t)$ is a martingale under $\mathbb{Q}^\theta$ and $\eta_t = \frac{d\mathbb{Q}^\theta}{d\mathbb{P}}$, then the process $\boxed{M(t) = Y(t)\eta_t}$ is a martingale under $\mathbb{P}$.

$\frac{B(t,T)}{A(t)} = Z^*(t,T) \rightarrow \mathbb{Q}^\theta$ - martingale

$M(t) := Z^*(t)\,\eta_t \rightarrow \mathbb{P}$ - martingale

Since $Z^*(t,T)$ is a martingale under $\mathbb{Q}^\theta$ (by definition of $\mathbb{Q}^\theta$), $M(t) = Z^*(t,T)\eta_t$ is a martingale under $\mathbb{P}$. Hence, by the *Martingale Representation Theorem*, there exists a process γ such that $M(t)$ can be represented as

$$\begin{aligned} M(t) &= M(0) + \int_0^t \gamma(s)dX(t) \\ &= Z^*(0,T) + \int_0^t \gamma(s)dX(t) \end{aligned}$$

and hence

$$dM(t) = \gamma(t)dX(t)$$

However, we still do not know how $Z^*(t,T)$ behaves.
Recalling that $M(t) = Z^*(t,T)\eta_t$, we can express $Z^*(t,T)$ as

$$Z^*(t,T) = M(t)\eta_t^{-1} = \frac{M(t)}{\eta(t)}$$

and in particular,

differential

$$\boxed{dZ^*(t,T) = d(M(t)\eta_t^{-1})} \qquad (10)$$

This calls for the Itô product rule !!!

The next step is to apply the Itô product rule to (10).

Since

$$
\begin{aligned}
d\eta_t &= -\theta_t \eta_t dX(t) \quad \longleftarrow \quad \mathbb{P}\text{-mart} \\
&= -\theta_t \eta_t \left(dX^{\theta}(t) - \theta_t dt \right)
\end{aligned}
$$

then

$$
\begin{aligned}
d\eta_t^{-1} &= \theta_t \eta_t^{-1} dX^{\theta}(t) \quad \longleftarrow \quad \mathbb{Q}^{\theta}\text{-mart} \\
&= \theta_t \eta_t^{-1} \left(dX(t) + \theta_t dt \right)
\end{aligned}
$$

$$\eta_t^{-1} = \frac{1}{\eta_t} = \exp\left\{ \frac{1}{2}\int_0^t \theta^2(s)\,ds + \int_0^t \theta(s)\,dX(s) \right\}$$

... and we can finally derive the dynamics of $Z^*(t,T)$ under $\mathbb{Q}^\theta$:

$$\begin{aligned} dZ^*(t,T) &= d(M(t)\eta_t^{-1}) \\ &= M(t)d\eta_t^{-1} + dM(t)\eta_t^{-1} + \gamma_t\eta_t^{-1}\theta_t dt \end{aligned}$$

...

And therefore,

$$
\begin{aligned}
dZ^*(t,T) &= M(t)\left(\eta_t^{-1}\theta_t dX^\theta(t)\right) + \gamma_t\eta_t^{-1}\left(dX^\theta(t) - \theta_t dt\right) \\
&\quad +\gamma_t\eta_t^{-1}\theta_t dt \\
&= \eta_t^{-1}\left(\gamma_t + M(t)\theta_t\right) dX^\theta(t)
\end{aligned}
$$

$Z^*(t,T) = \frac{B(t,T)}{A(t)} \iff B(t,T) = Z^*(t,T)\,A(t)$

$dB(t,T) = d(Z^*(t,T).A(t))$

Recalling that $Z^*(t, T) = \frac{B(t,T)}{A(t)}$, we can finally derive the dynamics of $B(t, T)$ under $\mathbb{Q}^\theta$:

$dA(t) = r(t)A(t)\,dt + 0$

$$
\begin{aligned}
dB(t, T) &= d(Z^*(t, T)A(t)) \\
&= Z^*(t, T)dA(t) + dZ^*(t, T)A(t) \\
&= rZ^*(t, T)A(t)dt + \left(\eta_t^{-1}\gamma_t + Z^*(t, T)\theta_t\right) A(t)dX^\theta(t)
\end{aligned}
$$

...

Thus,

$$
\begin{aligned}
dB(t,T) &= rB(t,T)dt + \left(\eta_t^{-1}\gamma_t A(t) + B(t,T)\theta_t\right) dX^{\theta}(t) \\
&= rB(t,T)dt + \left(\frac{\gamma_t}{Z^*(t,T)\eta_t} B(t,T) + B(t,T)\theta_t\right) dX^{\theta}(t) \\
&= rB(t,T)dt + \left(\frac{\gamma_t}{M(t)} + \theta_t\right) B(t,T) dX^{\theta}(t)
\end{aligned}
$$

...

And hence,

$$\begin{aligned} dB(t,T) &= rB(t,T)dt + b^{\theta}(t,T)B(t,T)dX^{\theta}(t) \\ &= B(t,T)\left[rdt + b^{\theta}(t,T)dX^{\theta}(t)\right] \end{aligned}$$

where

$$b^{\theta}(t,T) = \frac{\gamma_t}{M(t)} + \theta_t$$

To conclude,

Key Fact (Dynamics of the Zero-Coupon Bond Price under $\mathbb{Q}^\theta$)

The dynamics of a zero-coupon bond $B(t,T)$ under the equivalent martingale measure $\mathbb{Q}^\theta$ is given by

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + b^\theta(t,T)dX^\theta(t), \qquad B(T,T) = 1 \qquad (11)$$

where b^θ is the volatility of the zero-coupon bond. Therefore

$$\begin{aligned} B(t,T) \;=\; & B(0,T)A(t)\exp\left(-\frac{1}{2}\int_0^t \left(b^\theta(s,T)\right)^2 ds \right. \\ & \left. + \int_0^t b^\theta(s,T)dX^\theta(s)\right) \end{aligned} \qquad (12)$$

5. The Market Price of Interest Risk

Under an equivalent martingale measure Q^θ, the dynamics of a zero-coupon bond is given by

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + b^\theta(t,T)dX^\theta, \qquad B(T,T) = 1$$

As a result, the dynamics of a zero-coupon bond under the actual probability measure $\mathbb{P}$ is

$$\frac{dB(t,T)}{B(t,T)} = \Big(r(t) + \theta(t)b^\theta(t,T)\Big)\, dt + b^\theta(t,T)dX(t), \quad B(T,T) = 1$$

The process $\theta(t)$ through which we defined the relation between the physical measure $\mathbb{P}$ and the equivalent martingale measure $\mathbb{Q}^{\theta}$ has an economic meaning: it is the *market price of risk*.

The market price of risk represents the compensation paid by the market to an investor per unit of risk.

In our framework, the risk is represented by the volatility of the zero-coupon bond, $b(t, T)$, and the total compensation for risk that an investor should expect is therefore equal to $\theta(t)b^{\theta}(t, T)$.

This observation is consistent with the complete market case we explored in Lecture 3.3. In the stock option case, the market price of risk was also the process $\theta = \frac{\mu - r}{\sigma}$.

The key difference between the complete stock market and the incomplete bond market is that while in the stock market the process θ (and hence the market price of risk) was uniquely determined, the process θ is not unique in the bond market.

To formalize,

$\theta \rightarrow \frac{d\mathbb{Q}^\theta}{d\mathbb{P}}$

Key Fact

The relationship between the physical measure $\mathbb{P}$ and an equivalent martingale $\mathbb{Q}^\theta$ measure is established by the market price of risk which acts as the change of measure process.

In complete markets the equivalent martingale measure is unique and so is the market price of risk.

In incomplete markets, we may have several equivalent martingale measures, each with its own market price of risk.

6. Pricing Bond Derivatives

Fixed income markets present a wide diversity of instruments: bonds, of course, but also forwards, options, caps and floors, numerous swaps and swaptions, structure notes... and this is without mentioning the interconnection between fixed income markets and credit market or between fixed income products and inflation-linked products.

All this to say that pricing zero-coupon bond is simply the starting point of any serious attempt to price fixed income products. In this section, we will start to expand our horizons by considering the pricing of derivatives on zero-coupon bonds.

To alleviate the notation, we will drop the θ superscript and simply refer to the $\mathbb{Q}$-measure to indicate one of the equivalent martingale measures. Consequently, we will denote by $X^{\mathbb{Q}}(t)$ the $\mathbb{Q}$-standard Brownian motion and express the bond price dynamics as

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + b(t,T)dX^{\mathbb{Q}}(t), \qquad B(T,T) = 1$$

and

$$\begin{aligned} B(t,T) &= B(0,T)A(t)\exp\left(-\frac{1}{2}\int_0^t (b(s,T))^2\,ds \right. \\ &\qquad \left. + \int_0^t b(s,T)dX^{\mathbb{Q}}(s)\right) \end{aligned}$$

6.1. Applying the Fundamental Asset Pricing Formula

The fundamental asset pricing formula tells us that the time t price of a contingent claim paying some (random) amount Y at time T is given by

$$V(t) = A(t)\mathbf{E}^{\mathbb{Q}}\left[\frac{Y}{A(T)}\middle|\mathcal{F}_t\right]$$

In particular, the value of a zero-coupon bond maturing at time T is given by

$$B(t,T) = A(t)\mathbf{E}^{\mathbb{Q}}\left[\frac{1}{A(T)}\middle|\mathcal{F}_t\right]$$

Now, what would happen if we wanted to price a call option $C(t)$ on a zero-coupon bond maturing at time U? The call option has strike K and expiry $T < U$.

NOW

expiring for an option

maturity

t $\qquad T \qquad U$

$\max[B(T,U) - K, 0]$ $\qquad$ £1

Applying the fundamental asset pricing formula, we would get

$$
\begin{aligned}
C(t) &= A(t)\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{(B(T,U)-K)^{+}}{A(T)}\right|\mathcal{F}_t\right] \\
&= A(t)\left(\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{B(T,U)\mathbf{1}_{\{B(T,U)>K\}}}{A(T)}\right|\mathcal{F}_t\right]\right. \\
&\qquad \left. -K\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{\mathbf{1}_{\{B(T,U)>K\}}}{A(T)}\right|\mathcal{F}_t\right]\right)
\end{aligned}
$$

...

$$
\begin{aligned}
C(t) &= A(t)\left(\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{A(T)\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(U)}\right|\mathcal{F}_T\right]\mathbf{1}_{\{B(T,U)>K\}}}{A(T)}\right|\mathcal{F}_t\right]\right.\\
&\qquad\left.-K\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{\mathbf{1}_{\{B(T,U)>K\}}}{A(T)}\right|\mathcal{F}_t\right]\right)\\
&= A(t)\left(\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{\mathbf{1}_{\{B(T,U)>K\}}}{A(U)}\right|\mathcal{F}_t\right]-K\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{\mathbf{1}_{\{B(T,U)>K\}}}{A(T)}\right|\mathcal{F}_t\right]\right)
\end{aligned}
$$

And that's it. We cannot go any further.

To go any further, we would need to know at time t the joint distribution of $B(T, U)$, $A(U)$ and $A(T)$. This is unlikely, unless we make very constraining assumptions.

This approach appears to lead to a dead end.

One way out of this situation would be to look for a measure $\mathbb{P}_T$ such that the expectation in the fundamental asset pricing formula would be a sole function of the derivative payoff $(B(T,U) - K)^+$.

This idea implies that rather than having the "classic" formula

$$V(t) \quad = \quad A(t)\mathbf{E}^{\mathbb{Q}}\left[\frac{(B(T,U)-K)^+}{A(T)}\middle|\mathcal{F}_t\right]$$

we would have the "modified" formula

$$V(t) \quad = \quad B(t,T)\mathbf{E}^{\mathbb{P}_T}\left[(B(T,U)-K)^+|\mathcal{F}_t\right] \tag{13}$$

$B(T,U)$

T U

To be in a position to use this "modified" formula, we must answer 2 questions:

1. We do not know what $\mathbb{P}_T$ is. In fact, we do not even know if $\mathbb{P}_T$ exists.
2. Given information up to time t, what would $B(T, U)$ be equal to? $\rightarrow$ Forward Price !!!

Let's start with the second, and easiest, question. If we are at time t and we would like to know the price at some future time T of a bond maturing at time U, we would use the forward price for that bond.

This leads us to a possible answer to our first question. To define $\mathbb{P}_T$, we could look for a "forward martingale measure", that is, an equivalent martingale measure defined with respect to forward prices.

Hence, to use the "modified" fundamental asset pricing formula, we need to know a little bit about forwards.

6.2. Forward Contracts and Forward Prices

Forward contracts are OTC derivatives securities in which the long party has the obligation to buy an agreed upon quantity of an underlying asset (securities, commodities or others) at an agreed upon time and at an agreed upon price called the forward price.

Forward contracts are symmetrical contracts. Therefore, the obligations of the short party mirror those of the long party. The contract is settled at maturity and typically no cash flow is exchanged in the meantime. As they are OTC derivatives, forward contracts are subject to counterparty risk.

How do we price a forward contract?

Let's say that we want to enter into a (long) forward contract on a financial instrument (stock, bond, currency...) whose value at time t is $Y(t)$. The forward matures at time T.

Based on our definition of forward contracts, the payoff $G(T, Y_T)$ of the contract is

$$G(T, Y_T) = Y(T) - F_Y(t, T)$$

where $F_Y(t, T)$ is the forward price of Y determined at time t for delivery at time T

Plugging this into the fundamental asset pricing formula, we see that the time $t \leq \tau \leq T$ value $V(\tau)$ of a forward contract entered into at time t is equal to

$$V(\tau) \quad = \quad A(\tau)\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{Y(T) - F_Y(t,T)}{A(T)}\right|\mathcal{F}_\tau\right]$$

This formula can be simplified by noting that $F_Y(t,T)$ is a constant:

$$V(\tau) \quad = \quad A(\tau)\left(\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{Y(T)}{A(T)}\right|\mathcal{F}_\tau\right] - F_Y(t,T)\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(T)}\right|\mathcal{F}_\tau\right]\right)$$

and now we know the value of a forward contract for any time $t \leq \tau \leq T$.

How do we know the forward price?

The forward price $F_Y(t,T)$ was originally set at time t so that the value of the forward contract at time t is 0. Hence,

$$V(t) = A(t)\left(\mathbf{E}^{\mathbb{Q}}\left[\frac{Y(T)}{A(T)}\middle|\mathcal{F}_t\right] - F_Y(t,T)\mathbf{E}^{\mathbb{Q}}\left[\frac{1}{A(T)}\middle|\mathcal{F}_t\right]\right) = 0$$

Rearranging,

$$F_Y(t,T) = \frac{\mathbf{E}^{\mathbb{Q}}\left[\frac{Y(T)}{A(T)}\middle|\mathcal{F}_t\right]}{\mathbf{E}^{\mathbb{Q}}\left[\frac{1}{A(T)}\middle|\mathcal{F}_t\right]} =: \frac{W(t)}{B(t,T)} \tag{14}$$

where

$$W(t) = A(t)\mathbf{E}^{\mathbb{Q}}\left[\frac{Y(T)}{A(T)}\Big|\mathcal{F}_t\right] \tag{15}$$

that is $W(t)$ is the value of a claim paying $Y(T)$ at time T.

6.3. The Forward Martingale Measure

We will define the *forward martingale measure*, or simply *forward measure*, via the Radon-Nikodým derivative λ_t defined as

$$\lambda_t = \frac{d\mathbb{P}_T}{d\mathbb{Q}}$$

$$= \mathbf{E}^{\mathbb{Q}}\left[\left.\frac{A(0)B(T,T)}{A(T)B(0,T)}\right|\mathcal{F}_t\right]$$

$$= \frac{A(0)}{B(0,T)} \mathbb{E}\left[\left.\frac{B(T,T)}{A(T)}\right|\mathcal{F}_t\right]$$

Spot = "Instantaneous Forward"

$$= \frac{B(t,T)}{B(0,T)}$$

$$\lambda_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{\cancelto{1}{A(0)}\,\cancelto{1}{B(T,T)}}{A(T)\,B(0,T)} \,\middle|\, \mathcal{F}_t \right]$$

Thus,

$$\lambda_t = \frac{1}{B(0,T)} \mathbf{E}^{\mathbb{Q}}\left[\frac{1}{A(T)} \middle| \mathcal{F}_t \right] \times \frac{A(t)}{A(t)}$$

$$= \frac{1}{A(t)B(0,T)} \left(\underbrace{A(t)\mathbf{E}^{\mathbb{Q}}\left[\frac{1}{A(T)} \middle| \mathcal{F}_t \right]}_{B(t,T)} \right)$$

$$\lambda_t = \frac{A^{-1}(t)\,B(t,T)}{B(0,T)}$$

And hence,

$$\lambda_t = \frac{A^{-1}(t)B(t,T)}{B(0,T)}$$

The strange formula

$$\lambda_t = \frac{A^{-1}(t)B(t, T)}{B(0, T)}$$

can be interpreted as the ratio of two trades.

The denominator represents

- The purchase at time 0 of a bond maturing at time T. This trade costs $B(0, T)$ at time 0 and pays 1 at time T;

The numerator is

- the purchase at time t of a zero-coupon bond maturing at time T, for a price $B(t, T)$. This bond pays 1 at time T;
- the time 0 value of this trade is therefore equal to $\frac{1}{A(t)}B(t, T)$;

Since both trade will end up paying 1 at time T, then to prevent arbitrage, they should have *on average* the same value at time 0. Restating this observation in more mathematical terms:

To prevent arbitrage, $\lambda_t = \frac{A^{-1}(t)B(t,T)}{B(0,T)}$ **must be a martingale.**

We must emphasize the *on average*. Indeed, the short-term interest rate $r(t)$ is stochastic and therefore, both

- $A(t)$, and
- $B(t, T)$

are not know with certainty at time 0.

Let's check that this is indeed a measure by verifying that the Radon-Nikodým derivative λ_t is indeed an exponential martingale.

We know that the time t value of a zero-coupon bond is given by formula (12)

$$B(t, T) = B(0, T)A(t)\underbrace{\exp\left(-\frac{1}{2}\int_0^t (b(s, T))^2\, ds + \int_0^t b(s, T)dX_s^{\mathbb{Q}}\right)}$$

and

$$A(t) = e^{\int_0^t r(s)ds}$$

$$\lambda_t = \frac{A^{-1}(t)}{B(0,T)} \times B(t,T)$$

$$= \frac{\cancel{A^{-1}(t)}}{\cancel{B(0,T)}} \times \cancel{B(0,T)}\,\cancel{A(t)}\,\boxed{\exp\left\{-\frac{1}{2}\int_0^t b^2(s,T)ds + \int_0^t b(s,T)dX_s^{\mathbb{Q}}\right\}}$$

Therefore,

$$
\begin{aligned}
\lambda_t &= \frac{A^{-1}(t)B(t,T)}{B(0,T)} \\
&= \exp\left\{-\frac{1}{2}\int_0^t (b(s,T))^2\, ds + \int_0^t b(s,T) dX_s^{\mathbb{Q}}\right\}
\end{aligned}
$$

As long as the bond volatility $b(s, T)$ remains finite, λ_t is indeed an exponential martingale.

As a result, the forward measure $\mathbb{P}_T$ is well-defined.

Moreover, by Girsanov's theorem, the process X^T defined as

$$\boxed{X_t^T = X_t^{\mathbb{Q}} - \int_0^t b(s,T)ds, \quad t \in [0, \overline{T}]}$$

is a standard Brownian Motion under the forward measure $\mathbb{P}_T$.

X^T is called the *forward Brownian motion*.

6.4. Pricing a Derivative Under the Forward Measure

We now have a candidate measure $\mathbb{P}_T$. But before we can use the forward asset pricing formula

$$V(t) = B(t,T)\mathbf{E}^{\mathbb{P}_T}[Y|\mathcal{F}_t]$$

we need to make sure that it will give the same result as the "classic" fundamental asset pricing formula

$$V(t) = A(t)\mathbf{E}^{\mathbb{Q}}\left[\frac{Y}{A(T)}\middle|\mathcal{F}_t\right]$$

which we know to be correct.

We will start with what we know: the fundamental asset pricing formula

$$V(t) = A(t)\mathbf{E}^{\mathbb{Q}}\left[\frac{Y}{A(T)}\middle|\mathcal{F}_t\right]$$

Noting that

$$\lambda_T = \frac{A^{-1}(T)B(T,T)}{B(0,T)} = \frac{A^{-1}(T)}{B(0,T)} \tag{16}$$

the asset pricing formula can be rewritten as

$$\begin{aligned} V(t) &= A(t)\mathbf{E}^{\mathbb{Q}}\left[B(0,T)A(T)\frac{Y\lambda_T}{A(T)} \middle| \mathcal{F}_t \right] \\ &= A(t)B(0,T)\mathbf{E}^{\mathbb{Q}}\left[Y\lambda_T \middle| \mathcal{F}_t \right] \end{aligned}$$

$$\lambda_T = \frac{d\mathbb{P}^T}{d\mathbb{Q}}$$

It turns out that we can write [2]

$$\boxed{\mathbf{E}^{\mathbb{Q}}\left[Y\lambda_T|\mathcal{F}_t\right] =} \mathbf{E}^T\left[Y|\mathcal{F}_t\right]\mathbf{E}^{\mathbb{Q}}\left[\lambda_T|\mathcal{F}_t\right]$$

Therefore,

$$V(t) \quad = \quad A(t)B(0,T)\mathbf{E}^T\left[Y|\mathcal{F}_t\right]\mathbf{E}^{\mathbb{Q}}\left[\lambda_T|\mathcal{F}_t\right]$$

λ_T is a mart under $\mathbb{Q}$! $\quad \mathbb{E}^{\mathbb{Q}}\left[\lambda_T|\mathcal{F}_t\right] = \lambda_t = \frac{B(t,T)}{A(t)B(0,T)}$

$$V(t) = \cancel{A(t)}\cancel{B(0,T)}\,\mathbb{E}^T\left[Y|\mathcal{F}_t\right] \times \frac{B(t,T)}{\cancel{A(t)}\cancel{B(0,T)}} = B(t,T)\,\mathbb{E}^T\left[Y|\mathcal{F}_t\right]$$

[2] This is due to a rather advanced stochastic analysis result which is an extension of Bayes' formula dealing with change of measure and conditional expectations.

Now, λ_T is a martingale under $\mathbb{Q}$. Hence

$$\begin{aligned}\mathbf{E}^{\mathbb{Q}}[\lambda_T|\mathcal{F}_t] &= \lambda(t) \\ &= \frac{A^{-1}(t)B(t,T)}{B(0,T)}\end{aligned}$$

...

... and we can finally conclude that

$$V(t) \;=\; B(t,T)\mathbf{E}^{\mathbb{P}_T}\left[Y|\mathcal{F}_t\right]$$

Key Fact

We want to price a European derivative expiring at time T on a zero-coupon bond maturing at time U, with $T < U$. The payoff of the derivative is $G(B(T,U))$.

Under the forward (martingale) measure $\mathbb{P}_T$, the value of this derivative is given by the forward asset pricing formula

$$V(t) \quad = \quad B(t,T)\mathbf{E}^{\mathbb{P}_T}\left[G(B(T,U))|\mathcal{F}_t\right] \tag{17}$$

The forward (martingale) measure $\mathbb{P}_T$ is defined in terms of the equivalent martingale measure $\mathbb{Q}$ via the Radon-Nikodým derivative

$$\lambda_t \quad = \quad \frac{d\mathbb{P}^T}{d\mathbb{Q}} = \frac{A^{-1}(t)B(t,T)}{B(0,T)} \tag{18}$$

6.5 Pricing a Call on a Zero-Coupon Bond

In the case of a call on a zero-coupon bond, a Black-Scholes-type formula exists.

To see this, we will need to express the call payoff at time T, not in terms of the zero-coupon bond, but in terms of a forward on the zero-coupon bond[3] as

$$(B(T,U) - K)^{+} = (F_B(T,T,U) - K)^{+} \tag{19}$$

where $F_B(t,T,U)$ is the forward price at time t for settlement at time T of a zero-coupon bond maturing at time $U > T$. Note that $F(T,T,U)$ is simply the "instantaneous forward price" at time T, which is equal to the spot price $B(T,U)$.

[3]After all, this should be more consistent with the forward measure-based pricing framework we have developed.

Applying formula (14), we deduce that the zero-coupon forward price $F_B(t, T, U)$ is given by

$$
\begin{aligned}
F_B(t, T, U) &= \frac{\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{B(T,U)}{A(T)}\right|\mathcal{F}_t\right]}{\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(T)}\right|\mathcal{F}_t\right]} \\
&= \frac{A(t)\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{B(T,U)}{A(T)}\right|\mathcal{F}_t\right]}{A(t)\mathbf{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(T)}\right|\mathcal{F}_t\right]} \\
&= \frac{B(t, U)}{B(t, T)} \qquad (20)
\end{aligned}
$$

As a result, the $\mathbb{Q}$-dynamics of the forward price is given by

$$\begin{aligned}
F_B(t,T,U) &= F_B(0,T,U)\exp\left(-\frac{1}{2}\int_0^t (b(s,U))^2 - (b(s,T))^2\,ds\right.\\
&\quad \left.+\int_0^t (b(s,U)-b(s,T))\,dX^{\mathbb{Q}}(s)\right)\\
&= F_B(0,T,U)\exp\left(-\int_0^t b(s,T)\,(b(s,U)-b(s,T))\,ds\right)\\
&\quad \times\exp\left(-\frac{1}{2}\int_0^t (b(s,U)-b(s,T))^2\,ds\right.\\
&\quad \left.+\int_0^t (b(s,U)-b(s,T))\,dX^{\mathbb{Q}}(s)\right) \qquad (21)
\end{aligned}$$

and hence

$$
\begin{aligned}
\frac{dF_B(t,T,U)}{F_B(t,T,U)} &= -b(s,T)\left(b(t,U)-b(t,T)\right)dt \\
&\quad + \left(b(t,U)-b(t,T)\right)dX^{\mathbb{Q}}(t) \qquad (22)
\end{aligned}
$$

Deriving the dynamics of $F_B(t, T, U)$ under the $\mathbb{Q}$-measure is a promising start. But it is not enough.

As we are going to price the call option using the forward asset pricing formula, we need to know the dynamics of the forward price $F_B(t, T, U)$ under the forward measure $\mathbb{P}_T$.

Recalling that

$$X_t^T \quad = \quad X_t^{\mathbb{Q}} - \int_0^t b(s, T)ds, \quad t \in [0, \overline{T}]$$

is a standard Brownian Motion under the forward measure $\mathbb{P}_T$, we immediately get

$$\frac{dF_B(t, T, U)}{F_B(t, T, U)} \quad = \quad (b(s, U) - b(s, T))\, dX^T(s) \tag{23}$$

and therefore

$$\begin{aligned} F_B(t, T, U) \quad = \quad & F_B(0, T, U) \exp\left\{ \int_0^T (b(s, U) - b(s, T))\, dX^T(s) \right. \\ & \left. - \frac{1}{2} \int_0^T (b(s, U) - b(s, T))^2\, ds \right\} \end{aligned} \tag{24}$$

which implies that the forward price is a martingale under the forward measure.

We now have all we need to solve the Call option pricing problem using the forward asset pricing formula

$$C(t) \quad = \quad B(t, T)\mathbf{E}^{\mathbb{P}_T}\left[(F_B(T, T, U) - K)^+ | \mathcal{F}_t\right] \qquad (25)$$

Key Fact

The time t price of a European Call expiring at time T and with strike K, written on a zero-coupon bond maturing at time $U > T$ is given by the following Black-Scholes type of formula:

Black 1976

$$
\begin{aligned}
C(t) &= B(t,U)N\left[d_1(B(t,U),t,T)\right] \\
&\quad -KB(t,T)N\left[d_2(B(t,U),t,T)\right] \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1(b,t,T) &= \frac{\ln\left(\frac{b}{K}\right) - \ln B(t,T) + \frac{1}{2}\upsilon_U^2(t,T)}{\upsilon_U(t,T)} \\
d_2(b,t,T) &= d_1 - \upsilon_U(t,T) \\
\upsilon_U^2(t,T) &= \int_t^T \left(b(s,U) - b(s,T)\right)^2 ds
\end{aligned}
$$

7. What Should We Make of this Entire Approach?

7.1. What's Unsatisfactory?

First, when we look at specific interest rate models such as the CIR or the Vasicek models, the probabilistic approach does not bring us any new insights. What's worse, we have to use Feynman-Kǎc to transform the problem into a PDE problem if we want to solve it analytically...

Second, if we want the bond price to have a lognormal-type behaviour in a short rate model, the bond volatility function $b(t, T)$ will unfortunately be a "fudge function".

This unpleasantness will however motivate us to turn our models around and specify a bond dynamics first, and then deduce a dynamics of interest rates. This approach forms the base of forward rate models such as the HJM class of models.

7.2. What is Satisfactory?

The forward measure.

The existence of the forward measure and the critical role played by forwards in the pricing of bond derivatives also provides a powerful motivation for looking at the term structure of forward (as opposed to spot) rates (see the HJM class of models).

Moreover, the interpretation of the Radon-Nikod'ym derivative $\lambda_t = \frac{A^{-1}(t)B(t,T)}{B(0,T)}$ as an intertemporal no-arbitrage condition is particularly intuitive and elegant.

Another aspect is worth noting, as long as the bond price follows a geometric dynamics, then irrespective of the specific interest rate model we chose, the value of a bond derivative will always be of the same form.

This raises an intriguing question: if we assume a geometric dynamics for the bond price, how many interest rate models do we have access to? Many, or few?

This question can be reformulated in a slightly more mathematical language as: what is the most general interest rate model we can find such that the bond price follows a geometric dynamics?

7.3. Where to Next? Forward Rate Models

The answer to this question, and next chapter in the development of interest rate models is the derivation of models of the forward rate dynamics. This critical step was achieved by Heath, Jarrow and Morton (1992) and then further developed by Brace, Gatarek and Musiela (1997).

The key attraction of forward rate models is

- they start from a (nice) geometric dynamics for the zero-coupon bond price and then deduce the behaviour of the term structure of forward rates;
- they are a "meta"-model which encompasses all existing interest rate models;
- as such, you can use them to price or manage the risk of anything, from vanilla derivatives (which typically only require a short-term interest rate model) to complex fixed income portfolios (which are heavily dependent on an accurate modelling of the term structure).

The key problems related to forward rate models:

- non-Markov. All the mathematics we have are based on Markov models. Hence, we need to choose our parameters carefully and make assumptions to ensure that the forward rate models we work with are indeed Markov;
- mathematically sophisticated, sometimes too sophisticated for the applications at hand (such as pricing vanilla derivatives);
- "meta"-model: no clear indication of which form to use and when to use it.

In this lecture, we have seen...

- the pricing of interest rate products in a probabilistic setting;
- the equivalent martingale measure**s**;
- the fundamental asset pricing formula for bonds;
- application for popular interest rates models;
- the dynamics of bond prices;
- the forward measure;
- the fundamental asset pricing formula for derivatives on bonds;
- right and wrongs of this approach to short-term interest rate models;