Module 1 Further Exercises in SDEs

Throughout this problem sheet, you may assume that W_t is a Brownian Motion (Weiner Process) and dW_t is its increment. You may assume$W_0 = 0$. SDE(s) refers to Stochastic Differential Equation(s).

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$. Calculate $\mathbb{E}[\psi]$ and $\mathbb{V}[\psi]$ where $\psi = \sqrt{dt}\phi$. dt is a small time-step. **Note: No integration is required.**
   $\mathbb{E}[\psi] = \mathbb{E}\left[\sqrt{dt}\phi\right] = \sqrt{dt}\mathbb{E}[\phi]$, because dt is not a RV and we also know that $\mathbb{E}[\phi]$, therefore $\mathbb{E}[\psi] = 0$.
   $\mathbb{V}[\psi] = \mathbb{E}\left[\psi^2\right] - \mathbb{E}^2[\psi] \to \mathbb{E}\left[\phi^2 dt\right] - \mathbb{E}^2[\psi] \Rightarrow \mathbb{V}[\psi] = dt.$

2. Consider the following examples of Stochastic Differential Equations (SDE); Write these in standard form, i.e. $dG = A(G,t)dt + B(G,t)dW_t$. Give the drift and diffusion for each case.

   (a) $df + dW_t - dt + 2\mu t f dt + 2\sqrt{f}dW_t = 0$; where $f = f(W_t, t)$
   $df = (1 - 2\mu t f)\, dt + \left(-1 - 2\sqrt{f}\right) dW_t$
   drift $= 1 - 2\mu t f$ volatility $= -1 - 2\sqrt{f}$

   (b) $\dfrac{dy}{y} = (A + By)\, dt + (Cy)\, dW_t$ where $y = y(W_t, t)$
   $dy = (Ay + By^2)\, dt + (Cy^2)\, dW_t$
   drift $= Ay + By^2$ volatility $= Cy^2$

   (c) $dS = (\nu - \mu S)dt + \sigma dW_t + 4dS$
   $dS = \frac{1}{3}(-\nu + \mu S)dt + \left(-\frac{\sigma}{3}\right) dW_t$
   drift $= \frac{1}{3}(-\nu + \mu S)$ volatility $= -\frac{\sigma}{3}$

3. Show that
$$\int_0^1 (1-t)\cos W_t dW_t = \int_0^1 (a + bt)\sin W_t dt,$$
and determine the values of a and b.
$$\frac{\partial F}{\partial W_t} = (1-t)\cos W_t \Longrightarrow F(W_t, t) = (1-t)\sin W_t$$
from which we can obtain the other derivative terms:
$$\frac{\partial F}{\partial t} = -\sin W_t, \quad \frac{\partial^2 F}{\partial W_t^2} = -(1-t)\sin W_t$$
and
$$F(W_1, 1) = 0, \; F(W_0, 0) = 0$$
to give
$$\begin{aligned}\int_0^1 (1-t)\cos W_t dW_t &= -\int_0^1 \left(-\sin W_t - \frac{1}{2}(1-t)\sin W_t\right) dt \\ &= \int_0^1 \left(\sin W_t + \frac{1}{2}(1-t)\sin W_t\right) dt \\ &= \int_0^1 \left(\tfrac{3}{2} - \tfrac{t}{2}\right)\sin W_t dt \equiv \int_0^1 (a+bt)\sin W_t dt\end{aligned}$$
hence
$$a = \tfrac{3}{2} \text{ and } b = -\tfrac{1}{2}$$

4. The function $V(S,t) = \log(tS)$, where S evolves according to the SDE $dS = \mu S dt + \sigma S dW_t$; show that

$$dV = \left(\frac{1}{t} + \mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\sigma S \frac{\partial V}{\partial S}\right) dW_t$$

$$\begin{aligned} \text{if } V(S,t) &= \log(tS) \rightarrow \\ V_t &= \frac{1}{t}; \quad V_S = \frac{1}{S} \quad \text{and } V_{SS} = -\frac{1}{S^2} \end{aligned}$$

Substituting into expression for dV gives

$$\begin{aligned} dV &= \left(\frac{1}{t} + \mu S\left(\frac{1}{S}\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\right)\right) dt + \left(\sigma S . \frac{1}{S}\right) dW_t \\ dV &= \left(\frac{1}{t} + \mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \end{aligned}$$

5. Show that

$$G = \exp\left(t + ae^{W_t}\right)$$

is a solution of the stochastic differential equation

$$dG(t) = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right) dt + G(\ln G - t)\, dW_t$$

$$\frac{\partial G}{\partial t} = G, \quad \frac{\partial G}{\partial X} = aGe^X, \quad \frac{\partial^2 G}{\partial X^2} = ae^X G + ae^X \frac{\partial G}{\partial X} = ae^X G + a^2 e^{2X} G$$

In Itô, i.e.

$$\begin{aligned} dG &= \left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\right) dt + \frac{\partial G}{\partial X} dX \\ &= \left(G + \tfrac{1}{2}ae^X G + \tfrac{1}{2}a^2 e^{2X} G\right) dt + ae^X G dX \end{aligned}$$

From $G = \exp(t + a\exp(X(t)))$ we have

$$ae^X + t = \ln G \Longrightarrow ae^X = \ln G - t$$

so we can write the SDE in terms of the process G

$$dG = G\left(1 + \tfrac{1}{2}ae^X + \tfrac{1}{2}a^2 e^{2X}\right) dt + ae^X G dX$$

So

$$dG = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right) dt + G(\ln G - t)\, dX$$

6. Consider the stochastic differential equation

$$dG(t) = a(G,t)\, dt + b(G,t)\, dW_t.$$

Find $a(G,t)$ and $b(G,t)$ where

   (a) $G(t) = W_t^2$

$$dG = 2W_t dW_t + dt = 2\sqrt{G} dW_t + dt.$$

   Therefore

$$a(G,t) = 1 \text{ and } b(G,t) = 2\sqrt{G}$$

(b) $G(t) = 1 + t + e^{W_t}$

$$dG = e^{W_t} dW_t + \left(1 + \tfrac{1}{2} e^{W_t}\right) dt.$$

Rearranging the formula for $G(t)$ we have $\exp(W_t) = G(t) - 1 - t$, and so

$$dG = \underbrace{(G(t) - 1 - t)}_{b(G,t)} dW_t + \tfrac{1}{2} \underbrace{(1 + G(t) - t)}_{a(G,t)} dt.$$

(c) $G(t) = f_t W_t$, where f_t is a bounded and continuous function.

$$dG = f(t)\, dW_t + W_t(t) \frac{df}{dt} dt = f(t)\, dW_t + \frac{G(t)}{f(t)} \frac{df}{dt} dt$$

therefore

$$a(G,t) = \frac{G(t)}{f(t)} \frac{df}{dt} \text{ and } b(G,t) = f(t)$$

7. Use Itô's lemma to show that

$$d(\cos W_t) = \alpha (\cos W_t)\, dt + \beta (\sin W_t)\, dW_t$$

&

$$d(\sin W_t) = \alpha (\sin W_t)\, dt - \beta (\cos W_t)\, dW_t$$

and determine the constants α & β.

$$\left.\begin{array}{l} F = \cos(X(t)) \\ G = \sin(X(t)) \end{array}\right\} \Rightarrow \text{ Itô gives}$$

$$\left.\begin{array}{rl} dF & = \dfrac{\partial F}{\partial X} dX + \tfrac{1}{2} \dfrac{\partial^2 F}{\partial X^2} dt = -\sin(X) dX - \tfrac{1}{2} \cos(X)\, dt \\ dG & = \dfrac{\partial G}{\partial X} dX + \tfrac{1}{2} \dfrac{\partial^2 G}{\partial X^2} dt = \cos(X) dX - \tfrac{1}{2} \sin(X)\, dt \end{array}\right\}$$

comparing with earlier expressions gives

$$\alpha = -\tfrac{1}{2}; \quad \beta = -1$$