Complex Variables

1. (a) Verify that the Cauchy-Riemann equations are satisfied for the function $f(z) = \sinh 4z$.

   (b) Show that the function $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic and hence find the conjugate harmonic function v to express $u + iv$ as an analytic function of z.

2. Use Cauchy's Integral Formula to evaluate

$$\oint_C \frac{e^{iz}}{z^3} dz$$

   where C is the circle $|z| = 2$.

3. By considering a suitable contour integral show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4x + 5)^2} = \frac{\pi}{2}$$

4. Using the Residue Theorem show that

$$\int_C \frac{e^{imz}}{z^2 + 1} dz = \pi e^{-m}, \quad m > 0$$

   where C is the usual closed contour in the upper half plane, from $x = -R$ to $x = R$ (as $R \longrightarrow \infty$) together with the semi-circular arc $\left\{z = Re^{i\theta} : 0 \leq \theta \leq \pi\right\}$.
   Hence show that

$$\int_0^{\infty} \frac{\cos mx}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}.$$