CQF Lecture 5.6 CDO and Copula Models

Exercises

1. A **synthetic CDO**, based on balance sheet information, is structured as follows:

Assets:	125 single-name CDS
Principal:	0.8 million (per name)
Maturity:	5 years
CDS spread:	200 bps
Payments:	Act/360 quarterly in arrears

Tranche	Attachment point	Expected Loss	Fair Spread	Rating
Senior	7%-10%	0.002%	L+45	AAA
Class A	5%-7%	0.1%	L+70	AA-
Class B	2%-5%	2.3%	L+20	BBB-
Class C	0%-2%	26.27%	Excess spread	NR

Table 1: CDO Capital Structure

   (a) Holders of which trache are long correlation and why? Which tranche is the most sensitive to changes in default correlation?

   (b) What about exposure of mezzanine noteholders to default correlation?

   (c) Assuming 0% recovery, how many defaults should occur before Senior tranche experiences a capital loss? If we assume 40% recovery how much more protection does that afford to Senior noteholder?

   (d) A downgrade is triggered when the entire Equity tranche is lost. Assuming 0% recovery, how many defaults should occur before the implied ratings are downgraded.

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2. Consider a random variable X that provides information about default time, conditional on intensity parameter θ. X follows the exponential distribution with *cdf*:

$$\Pr(X \leq x|\theta) \equiv F(x|\theta) = 1 - e^{-\theta x}$$

Assuming that intensity follows Gamma distribution, i.e., $\theta \sim \Gamma(\alpha, \beta)$ with *pdf*:

$$g(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$$

Use the result below to show that the unconditional **marginal cdf** of X follows Pareto distribution–that is,

$$\Pr(X \leq x) \equiv F(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}$$

Hint: In order to recover a *cdf* for the unconditional distribution, we need to integrate over the conditional distribution as follows:

$$F(x) = \int_0^\infty F(x|\theta)\, g(\theta)\, d\theta.$$

3. Consider two identically distributed variables X_1 and X_2. When conditioned on θ, same as in the previous exercise, they are independent. Their unconditional joint *cdf* is

$$\begin{aligned} F(x_1, x_2) &\equiv \Pr(X_1 \leq x_1, X_2 \leq x_2) \\ &= 1 - \Pr(X_1 > x_1) - \Pr(X_2 > x_2) + \Pr(X_1 > x_1, X_2 > x_2) \end{aligned}$$

In a practical context, X_1 and X_2 represent default times τ_1 and τ_2 respectively, so that $F(t_1, t_2) = \Pr(\tau_1 \leq t_1, \tau_2 \leq t_2)$ but let's continue working in 'random variable X' notation.

(a) Express the **joint *cdf*** $F(x_1, x_2)$ as a function of the isolated marginal *cdf*s $F(x_1)$ and $F(x_2)$ (also called 'marginals').

Hint: We can spot $F(x) = 1 - \Pr(X > x)$ but the unconditional term is unknown: $\Pr(X_1 > x_1, X_2 > x_2)$? We need to calculate it by integration over the product of $\Pr(X_1 > x_1\,|\theta)\Pr(X_2 > x_2\,|\theta)\,g(\theta)$, treating *conditional* X_1 and X_2 as independent.

(b) By substituting uniform variables u_1, u_2 instead of marginals $F(x_1)$ and $F(x_2)$ show that the associated **copula function** is

$$\begin{aligned} C(u_1, u_2) &\equiv \Pr(U_1 \leq u_1, U_2 \leq u_2) \\ &= u_1 + u_2 - 1 + \left((1-u_1)^{-\frac{1}{\alpha}} + (1-u_2)^{-\frac{1}{\alpha}} - 1\right)^{-\alpha} \end{aligned}$$

4. Consider **a copula function** of the Archimedean family

$$C(u_1, u_2, \ldots, u_n) = \phi^{-1}\left(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_n)\right)$$

Given the copula generator

$$\phi(u) = -\ln\left(\frac{e^{-\alpha u} - 1}{e^{-\alpha} - 1}\right)$$

show that the copula function can be expressed explicitly as

$$C(u_1, u_2, \ldots, u_n) = -\frac{1}{\alpha}\ln\left[1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right]$$

Identify this copula function by name. What does parameter α represent?

5. **Copula functions can price multi-asset options.** There is movement co-dependence among assets that can be modelled by a suitable copula. Copula function is equivalent to joint probability $C(u_1, u_2) \equiv F(x_1, x_2)$ by Sklar theorem. Because of correlation, one cannot use multiplication rule of simple probabilities to find out the Pr of the joint event.

We need the risk-neutral probability of an option being *in the money* at maturity, which is $u = N(d_2)$ for a European call. This risk-neutral probability can also be obtained by Monte Carlo.

For example, bi-variate European digital **put** pays one unit of currency if two underlying assets are both below a pair of strikes at maturity. Consider a simplified scenario of two identical assets and the same strikes:

$$T = 1, r = 0, K_{1,2} = 100, \sigma_{1,2} = 20\%, S_{1,2} = S_0 e^{\frac{1}{2}\sigma^2}, S_0 = 100,$$

Use Frank Copula function with $\alpha = 5$ to calculate the price of the bi-variate digital put.

$$C(u_1, u_2, \cdots, u_n) = -\frac{1}{\alpha}\ln\left[1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right]$$