CQF Module 2 Examination

January 2016 Cohort

Instructions

Answers to all questions are required. Complete mathematical and computational workings must be provided to obtain maximum credit. Submission must include Excel file(s) and code if used. Books and lecture notes may be referred to. Questions to Richard.Diamond@fitchlearning.com.

Portfolio computational tasks are best solved by matrix manipulation on a spreadsheet. Use Excel functions *MMULT()*, *MINV() and TRANSPOSE()*. If familiar, use Python, Matlab or R.

A. Optimal Portfolio Allocations [48%]

Consider an investment universe composed of the following risky assets:

Asset	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$
A	0.04	0.07
B	0.08	0.12
C	0.12	0.18
D	0.15	0.26

with a correlation structure

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

1. The global minimum variance portfolio is obtained by optimising as follows *s.t.* the budget constraint

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2}\boldsymbol{w'}\,\boldsymbol{\Sigma}\,\boldsymbol{w} \qquad \text{s.t. } \boldsymbol{w'1} = 1$$

   - Obtain analytical solution for optimal allocations $\mathbf{w}^*$. Provide full workings, including derivation and analytical result for the Lagrangian multiplier.

2. Consider the following optimization task for a targeted return $m = 10\%$, for which the net of allocations invested (borrowed) in a risk-free asset:

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \qquad \text{s.t. } r + (\boldsymbol{\mu} - r\mathbf{1})'\mathbf{w} = 0.1$$

   - Obtain analytical solution for optimal allocations $\mathbf{w}^*$. Provide full workings, including derivation and analytical result for the Lagrangian multiplier.

- For the target return of 10% and risk-free rate of 3% calculate optimal allocations $\mathbf{w}^*$ and portfolio standard deviation $\sigma_\Pi = \sqrt{\mathbf{w}^{*\prime}\boldsymbol{\Sigma}\mathbf{w}^*}$ using asset data and correlation above.
- Provide the definition of *a tangency portfolio* and calculate μ_T, σ_T for the same level of risk-free rate. Calculate the slope of Capital Market Line and briefly explain its role in evaluating investments.

3. Compute tangency portfolio (allocations $\mathbf{w_T}$) Analytical VaR with $c = 99\%$ confidence,

$$\text{VaR}(X) = \mathbf{w}'_\mathbf{T}\boldsymbol{\mu} + \text{Factor} \times \sqrt{\mathbf{w}'_\mathbf{T}\boldsymbol{\Sigma}\mathbf{w_T}}$$

where the Factor is a standardised percentile drawn from the inverse **CDF** function for

- the Normal distribution Factor $= \Phi^{-1}(1-c)$.
- the Student's t distribution Factor $= T_\nu^{-1}(1-c)$ with $\nu = 30$.
  Student's t distribution allows more realistic modelling of 'fatter tails' of returns distribution. The distribution changes its shape depending on the degrees of freedom parameter ν.

Excel function *TINV()* provides a value for the two-tail Factor, while VaR is one-tail.

B. Value at Risk on FTSE 100 [20%]

Imagine that each morning you come and calculate 99%/10day VaR based on available data. Once ten days pass you compare that VaR number to the realised index return and check if your prediction about the worst loss was breached. You are given a dataset of FTSE100 index returns, continue in Excel.

B.1 Calculate the rolling 99%/10day Value at Risk for an investment in the market index using a sample standard deviation, as follows:

- The rolling standard deviation for a sample of 21 days is computed for data for days 1-21, 2-22, etc. Starting on Day 21, you have a time series of σ_t.
- Re-scale standard deviation to reflect a ten days move $\sigma_{10D,t} = \sqrt{10 \times \sigma_t^2}$ (we can add variances) and scale an average daily return as $\mu_{10D} = \mu \times 10$ where μ is a mean of all data.
- Calculate Value at Risk for each day t (starting on Day 21) as follows:

$$\text{VaR} = \mu_{10D} + \text{Factor} \times \sigma_{10D} \quad \dagger$$

  where Factor is a percentile of the Standard Normal Distribution that 'cuts' 1% on the tail.

In Excel, you will have two final columns, for $\sigma_{10D,t}$ and VaR_t.

B.2 Calculate two numbers: the percentage of VaR breaches and the conditional probability of breach in VaR for the next period, given that a breach was observed for the previous period.

- VaR is fixed at time t and compared to the realised return at time $t+10$. A breach occurs when a realised 10-day return $\ln(S_{t+10}/S_t)$ is below the VaR quantity (negative scale).
- 20/08/2009 is the first day on which VaR_{SD} computation is available. Number of breaches divided by number of observations will give the percentage of breaches.
- Plot time series of VaR_t and indicate breaches. Briefly discuss, are the breaches independent?

In Excel, you will add columns, for $r_{10D,t}$ and $0, 1$ indicator, where 1 means a breach.

C. Stochastic Calculus [32%, 8% each]

1. Consider a basket of N assets each following the Geometric Brownian Motion SDE

$$dS_i = S_i\mu_i dt + S_i\sigma_i dX_i \qquad \text{for } 1 \le i \le N$$

The price changes are correlated as measured by the linear correlation coefficients ρ_{ij}.
Invoke the multi-dimensional Itô Lemma to write down the SDE for $F(S_1, S_2, \dots, S_N)$ *in the most compact form possible* but with clear drift and diffusion terms. Apply $dX_i dX_j \to \rho_{ij} dt$.

2. Construct an SDE for the process $Y(t) = e^{\sigma X(t) - \frac{1}{2}\sigma^2 t}$ and show it is, in fact, an Exponential Martingale of the form $dY(t) = Z(t)\, g(t)\, dX(t)$. Identify the terms $g(t)$ and $Z(t)$.

3. What about the process $Y(t) = \sqrt{t}X(t) - \int_0^t \frac{X(s)}{2\sqrt{s}} ds$, is it a martingale? This <u>must not</u> be solved by differentiation. Instead, think which function to apply the Itô Lemma to (from $Y(t)$ expression), and show that Y(t) is equal to an Itô Integral.

4. Covariance matrix can be decomposed as $\boldsymbol{\Sigma} = \boldsymbol{AA'}$ (Cholesky decomposition). The result, lower triangular matrix $\boldsymbol{A}$, is used for imposing correlation on a vector of random Normal variables $\boldsymbol{X}$.

$$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

- For the two-variate case, show analytically what $\boldsymbol{\Sigma} = \boldsymbol{AA'}$ is equal to.
- Write down the results for $Y_1(t)$ and $Y_2(t)$ which are correlated via $\boldsymbol{Y} = \boldsymbol{AX}$.
- Briefly discuss, does $Y_2(t)$ keep the properties of the Brownian Motion if $X_1(t), X_2(t)$ are random Normal? **Note:** consider distribution/variance of the increment of $Y_2(t)$.