CQF Module 1.4 Exercises

Stochastic Differential Equations and Itô's Lemma

Throughout this problem sheet, you may assume that W_t or $W(t)$ is a Brownian Motion (Weiner Process) and dW_t (or $dW(t)$) is its increment. $W_0 = 0$.

1. The change in a share price $S(t)$ satisfies

$$dS = A(S,t)\, dW_t + B(S,t)\, dt,$$

for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + B \frac{\partial f}{\partial S} + \frac{1}{2} A^2 \frac{\partial^2 f}{\partial S^2} \right) dt + A \frac{\partial f}{\partial S} dW_t.$$

Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.

2. Show that $F(W_t) = \arcsin(2aW_t + \sin F_0)$ is a solution of the stochastic differential equation

$$dF = 2a^2 (\tan F) \left(\sec^2 F\right) dt + 2a (\sec F)\, dW_t,$$

where F_0 and a is a constant.

3. Show that

$$\int_0^t W(\tau) \left(1 - e^{-W^2(\tau)}\right) dW(\tau) = \overline{F}(W(t)) + \int_0^t G(W(t))\, d\tau.$$

where the functions $\overline{F}$ and G should be determined.

4. Consider a two factor model in which the stock price dynamics S_t, follows Geometric Brownian Motion and the stock variance v_t is itself stochastic and follows a square root process

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_1(t),$$

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta \sqrt{v_t} dW_2(t).$$

The two processes have a correlation coefficient ρ, i.e.

$$dW_1(t) dW_2(t) = \rho dt$$

The parameters μ, λ, $\bar{v}$ and η are all constant. Let $F = F(t, S_t, v_t)$. Using Itô, consider the SDE for dF and integrate over $[0,t]$ to obtain an expression for $F(t, S_t, v_t)$.