CQF Exercises The Black Scholes Model

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$
\begin{aligned}
d_1 &= \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\log(S/E) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\phi^{\ 2}/2)\, d\phi
\end{aligned}
$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$

the expiry date, $r \geq 0$ the interest rate, D is the dividend yield and σ is the volatility of S.

1. Note:

$$
\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} \quad \text{and} \quad \frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}
$$

So

$$
\begin{aligned}
\Delta &= \frac{\partial C}{\partial S} \\
&= e^{(-D(T-t))} N(d_1) + S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1{}^2}{2}\right)} \frac{\partial d_1}{\partial S} - E \exp(-r(T-t)) \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_2{}^2}{2}\right)} \frac{\partial d_2}{\partial S} \\
&= e^{(-D(T-t))} N(d_1) + \frac{1}{\sqrt{2\pi}} \frac{\partial d_1}{\partial S} \underbrace{\left(S e^{(-D(T-t))} e^{\left(-\frac{d_1{}^2}{2}\right)} - E e^{(-r(T-t))} e^{\left(-\frac{d_2{}^2}{2}\right)} \right)}_{=0} \\
&= e^{(-D(T-t))} N(d_1) \quad \text{because the term in the bracket above is zero.}
\end{aligned}
$$

$$
\begin{aligned}
\upsilon &= \frac{\partial C}{\partial \sigma} \\
&= S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1{}^2}{2}\right)} \frac{\partial d_1}{\partial \sigma} - E e^{(-r(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_2{}^2}{2}\right)} \frac{\partial d_2}{\partial \sigma} \\
&= S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1{}^2}{2}\right)} \left(\frac{\partial d_2}{\partial \sigma} + \sqrt{T-t} \right) - \frac{1}{\sqrt{2\pi}} E e^{(-r(T-t))} e^{\left(-\frac{d_2{}^2}{2}\right)} \frac{\partial d_2}{\partial \sigma} \\
&= \sqrt{\frac{T-t}{2\pi}} S e^{(-D(T-t))} e^{\left(-\frac{d_1{}^2}{2}\right)} + \frac{\partial d_2}{\partial \sigma} \frac{1}{\sqrt{2\pi}} \left[\underbrace{S e^{(-D(T-t))} e^{\left(-\frac{d_1{}^2}{2}\right)} - E e^{(-r(T-t))} e^{\left(-\frac{d_2{}^2}{2}\right)}}_{=0} \right] \\
&= \sqrt{\frac{T-t}{2\pi}} S e^{(-D(T-t))} e^{\left(-\frac{d_1{}^2}{2}\right)} \quad \left(= \sqrt{\frac{T-t}{2\pi}} E e^{(-r(T-t))} e^{\left(-\frac{d_2{}^2}{2}\right)} \right)
\end{aligned}
$$

2. The Black–Scholes formula for a European call option $C(S,t)$ is

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2)$$

From this expression, find the Black–Scholes value of the call option in the following limits:

(a) (time tends to expiry) $t \to T^{-}$, $\sigma > 0 : \exp(-r(T-t))$, $\exp(-D(T-t)) \to 1$

1.
$$d_{1\ 2} \to \frac{\log(S/E)}{\sigma\sqrt{T-t}} + O\left(\sqrt{T-t}\right) \to \begin{cases} \infty & S > E \\ 0 & S = E \\ -\infty & S < E \end{cases} \qquad \text{so} \quad C \to \begin{cases} S-E & S > E \\ 0 & S = E \\ 0 & S < E \end{cases}$$

(b) (volatility tends to zero) $\sigma \to 0^{+}$, $t < T$;

$$\begin{aligned} d_{1\ 2} &\to \frac{\log(S/E) + (r-D)(T-t)}{\sigma\sqrt{T-t}} + O(\sigma) = \frac{\log(S\exp(-D(T-t))/E\exp(-r(T-t)))}{\sigma\sqrt{T-t}} + O(\sigma) \\ &\to \begin{cases} \infty & Se^{(-D(T-t))} > Ee^{(-r(T-t))} \\ 0 & Se^{(-D(T-t))} = Ee^{(-r(T-t))} \\ -\infty & Se^{(-D(T-t))} < Ee^{(-r(T-t))} \end{cases} \quad \text{so} \quad C \to \max\left[Se^{(-D(T-t))} - Ee^{(-r(T-t))}, 0\right] \end{aligned}$$

(c) (volatility tends to infinity) $\sigma \to \infty$, $t < T$;

$$\begin{aligned} d_{1\ 2} &\to \pm\frac{1}{2}\sigma\sqrt{T-t} + O\left(\frac{1}{\sigma}\right) \to \pm\infty \\ C &\to Se^{(-D(T-t))}N(\infty) - Ee^{(-r(T-t))}N(-\infty) = Se^{(-D(T-t))} \end{aligned}$$

3. Start with delta hedged portfolio

$$\Pi = V(S,t) - \Delta S.$$

with

$$dS = \mu S dt + \sigma S dX$$

over one time-step dt, where Δ is fixed from t to $t + dt$

$$d\Pi = \underbrace{dV - \Delta dS}_{\text{changes in } V \text{ and } S} + \underbrace{S^2 dt}_{\text{cash flow from holding } V}$$

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \\ &\quad \frac{\partial V}{\partial S} dS - \Delta dS + S^2 dt \end{aligned}$$

Only source of risk is in dS, so choose $\Delta = \frac{\partial V}{\partial S}$ to eliminate it.

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S^2\right) dt \\ &= r\Pi\, dt \\ &= r\left(V - \Delta S\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt \end{aligned}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S^2 = rV - rS\frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -S^2$$

At expiry we have $V(S,T) = S^2$. Look for a solution of the form

$$V(S,t) = \phi(t)\, S^2$$

then

$$\begin{aligned} \frac{\partial V}{\partial t} &= \dot{\phi}(t)\, S^2, \ \frac{\partial V}{\partial S} = 2S\phi(t), \ \frac{\partial^2 V}{\partial S^2} = 2\phi(t) \\ V(S,T) &= \phi(T)\, S^2 = S^2 \Longrightarrow \phi(T) = 1 \end{aligned}$$

$$\dot{\phi}(t)\, S^2 + \sigma^2 S^2 \phi(t) + 2rS^2\phi(t) - rS^2\phi(t) = -S^2 \longrightarrow$$

$$\dot{\phi}(t) + \left(\sigma^2 + r\right)\phi(t) = -1, \quad \phi(T) = 1$$

use integrating factor

$$e^{\left(\sigma^2+r\right)t}$$

$$\begin{aligned} \frac{d}{dt}\left(\phi(t)\, e^{\left(\sigma^2+r\right)t}\right) &= -e^{\left(\sigma^2+r\right)t} \\ \int d\left(\phi(t)\, e^{\left(\sigma^2+r\right)t}\right) &= -\int e^{\left(\sigma^2+r\right)t} dt \\ \phi(t)\, e^{\left(\sigma^2+r\right)t} &= -\frac{e^{\left(\sigma^2+r\right)t}}{\left(\sigma^2+r\right)} + A \\ \phi(t) &= -\frac{1}{\left(\sigma^2+r\right)} + Ae^{-\left(\sigma^2+r\right)t} \end{aligned}$$

we know $\phi(T) = 1$ so $A = \left(1 + \frac{1}{(\sigma^2+r)}\right) e^{\left(\sigma^2+r\right)t}$.

Hence

$$\phi(t) = \frac{1}{\sigma^2+r}\left(\left(\sigma^2+r+1\right) e^{\left(\sigma^2+r\right)(T-t)} - 1\right)$$