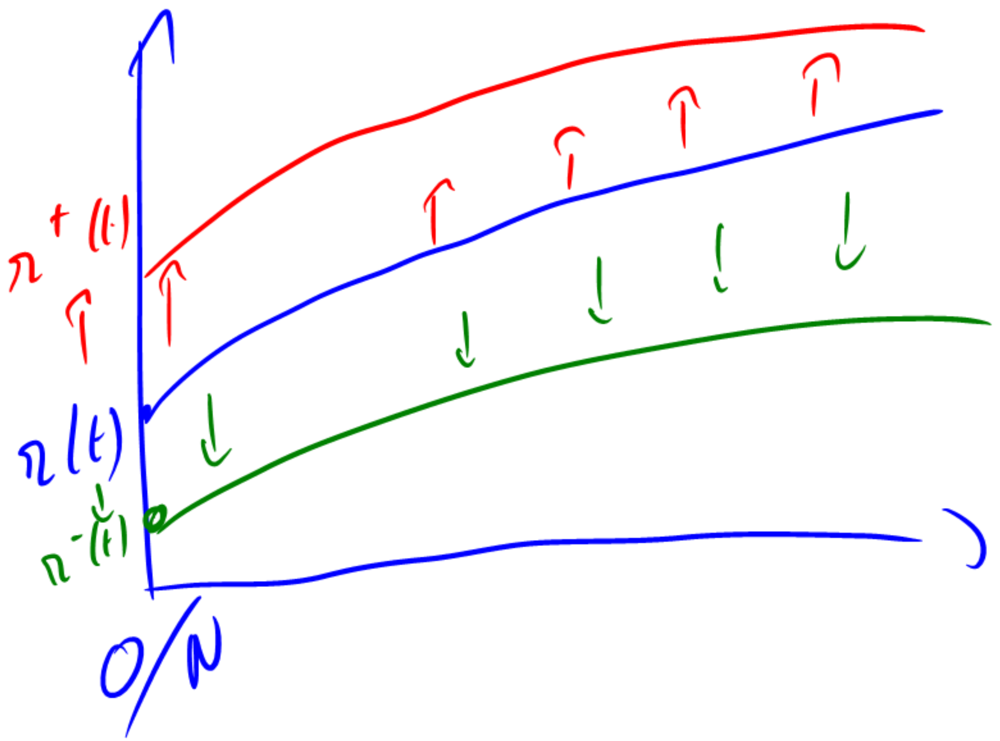
$$\boxed{dS_t = \mu S_t\,dt + \cancel{\sigma}\sqrt{v_t}\, S_t\, dX_1(t)} + \gamma S_t\, dJ(t) + \int_{\mathbb{R}\setminus\{0\}} \xi(z)\, S_t\, N(dt,dz)$$

$dX_2(t)$ $\quad dX_1\, dX_2 \to \rho dt$

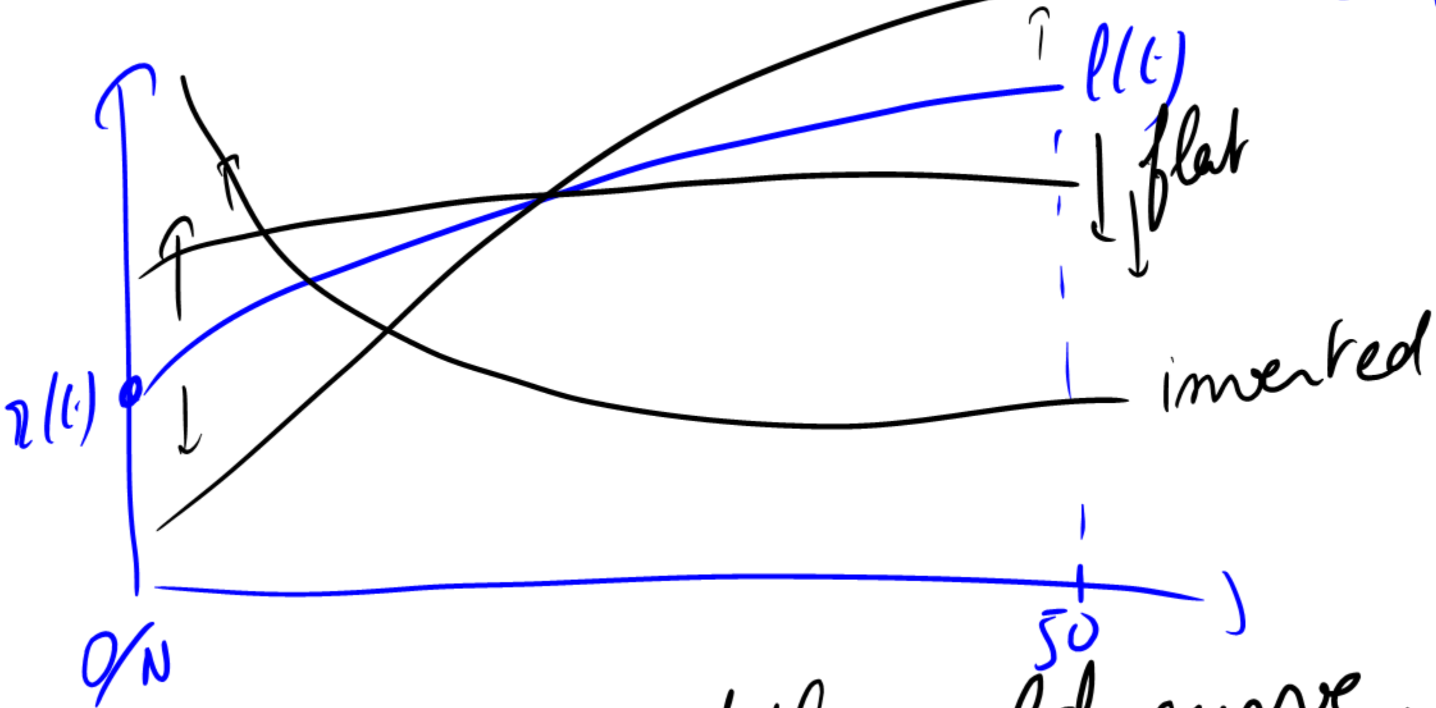
$dv_t = \ldots$

A rough guide to interest rate models:

1. Single factor models → 1 rate : the short rate

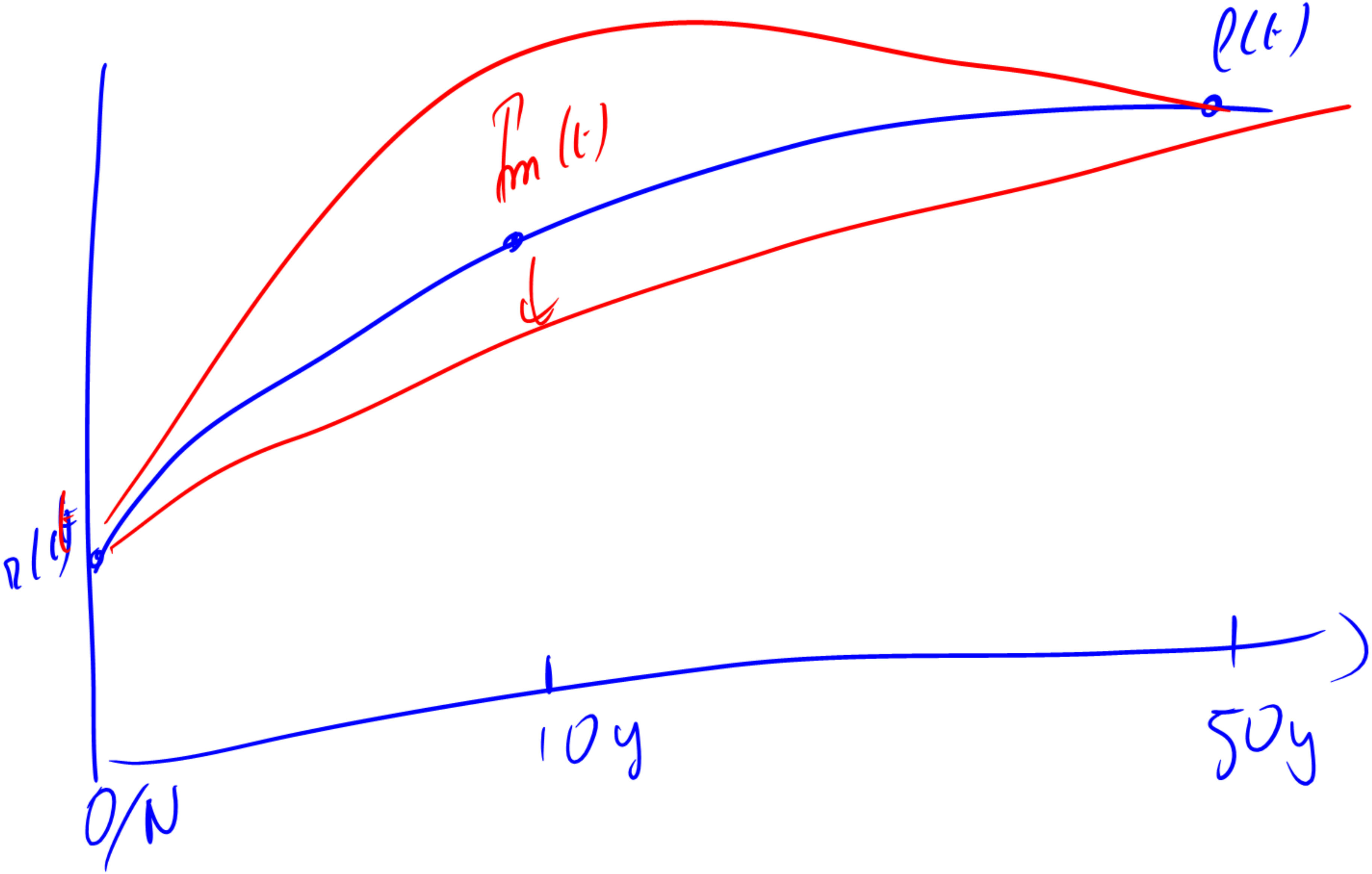


② 2 factor model → 2 rates → short rate O/N
→ long rate 50y.



→ change the slope of the yield curve.

3 factor model -> 3 rates
-> short
-> long
-> mid



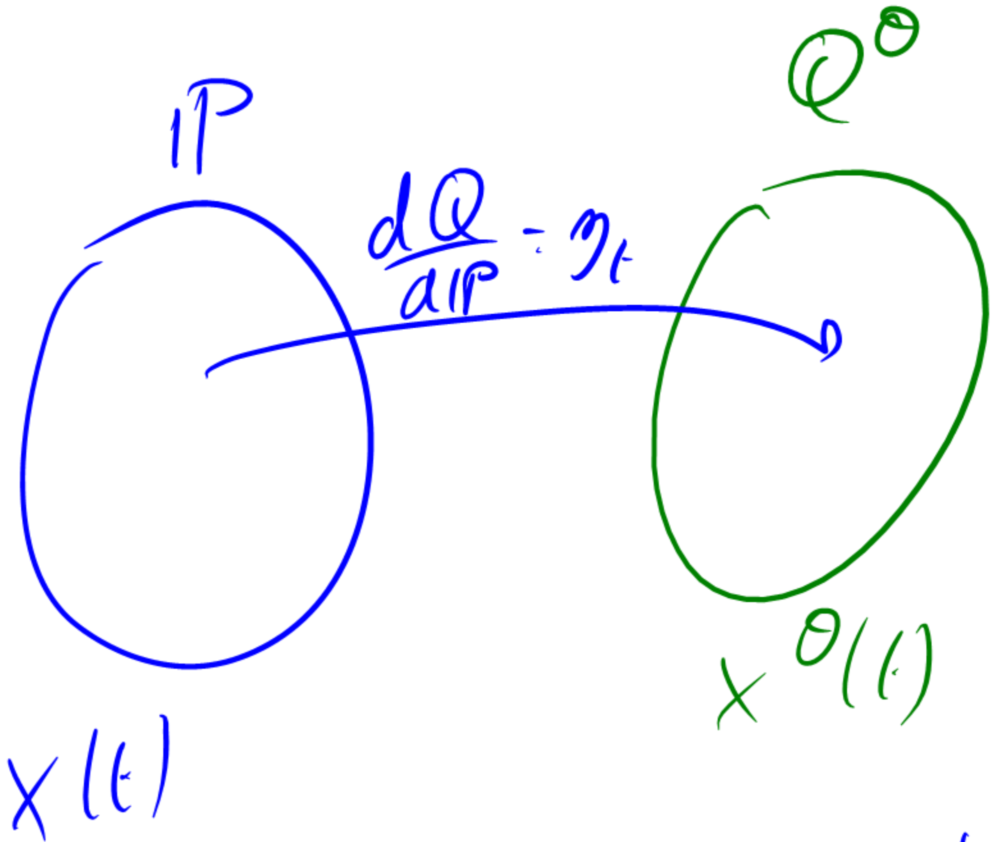
→ "convertible bond" → 1 factor model.

→ PCA / variance decomposition

85% is explained by a parallel shift
9% is explained by a change in slope
4% is explained by a change of curvature.

3 - 5 factors

→ 1 factor model



$$X^\theta(t) = X(t) + \int_0^t \theta(s)\,ds$$

$$\iff X(t) = X^\theta(t) - \int_0^t \theta(s)\,ds$$

$$dX^\theta(t) = dX(t) + \theta(t)\,dt$$

$$dX(t) = dX^\theta(t) - \theta(t)\,dt$$

Under the measure $\mathbb{P}$

$$dr(t) = \mu_t dt + \sigma_t \underbrace{dx(t)}$$

$dx(t)$
$= dx^{\theta}(t) - \theta(t)dt$

Under $\mathbb{Q}^{\theta}$

$$dr(t) = \mu_t dt + \sigma_t \left(dx^{\theta}(t) - \theta(t)dt \right)$$

$$\boxed{dr(t) = \left(\mu_t - \sigma_t \theta(t) \right)dt + \sigma_t dx^{\theta}(t)}$$

$Z^*(t,T)$ is a martingale w.r.t $\mathbb{Q}^\theta$ and $\Pi(t)$ is a mar with respect to $\mathbb{P}$.

By the Martingale Representation Theorem, there exists a process γ such that

$$\Pi(t) = \Pi(0) + \int_0^t \gamma(s)\, dX(t)$$

$$\Rightarrow \boxed{d\Pi(t) = \gamma(t)\, dX(t)}$$

$\mathbb{P}$

$dX(t) = dX^\theta(t) - \theta(t)\,dt$

$\mathbb{Q}^\theta$

$$d\Pi(t) = \gamma(t)\left(-\theta(t)\,dt + dX^\theta(t)\right)$$

$$\begin{tikzcd}
\mathbb{P} \arrow[r, bend left, "\eta_t = \frac{d\mathbb{Q}}{d\mathbb{P}}"] & \mathbb{Q} \arrow[l, bend left, "\frac{d\mathbb{P}}{d\mathbb{Q}} = \eta_t^{-1}"]
\end{tikzcd}$$

η_t is a $\mathbb{P}$-mart

η_t is a $\mathbb{Q}$-mart

$$dZ^*(t,T) = d(\Pi_t \eta_t^{-1})$$

under Q^θ

$$d\Pi_t = r'(t)\left[-\theta(t)dt + dx^\theta(t)\right]$$

$$d\eta^{-1}(t) = \theta(t)\eta^{-1}(t)dx^\theta(t)$$

$$= d\Pi_t \cdot \eta_t^{-1} + \Pi_t d\eta_t^{-1} + (r(t)\theta(t)\eta^{-1}(t))dt$$

$$= \left(\cancel{-\theta(t)r(t)\eta_t^{-1}dt} + r(t)\eta_t^{-1}dx^\theta(t)\right)$$

$$+ \theta(t)\eta^{-1}(t)\Pi(t)dx^\theta(t)$$

$$+ \cancel{r(t)\theta(t)\eta^{-1}(t)dt}$$

$$= \eta_t^{-1}(r_t + \Pi(t)\theta(t))dx^\theta(t)$$

$$\eta_t^{-1}\Pi_t = Z^*(t,T)$$

$$dZ^*(t,T) = \left[\eta_t^{-1}r_t + Z^*(t,T)\theta(t)\right]dx^\theta(t)$$

$$dB(t,T) = d\left(Z^*(t,T)\cdot A(t)\right)$$

$$= dZ^*(t,T)\cdot A(t) + Z^*(t,T)\,dA(t)$$

$$= \underbrace{\left[\eta_t^{-1} r(t) + Z^*(t,T)\theta(t)\right] A(t)\, dx^{\theta}(t)}_{\text{diffusion}}$$

$$\underbrace{+ r(t) Z^*(t) A(t)\, dt}_{\text{drift}}$$

$dA(t) = r(t) A(t)\, dt$

$dZ^*(t,T) = (\eta_t^{-1} r(t) + Z^*(t,T)\theta(t))\, dx^{\theta}(t)$

$Z^*(t) A(t) = B(t,T)$

$$= r(t) B(t,T)\, dt + \left[\eta_t^{-1} r_t A(t) + \underbrace{Z^*(t,T)\,\theta(t)\, A(t)}_{B(t,T)}\right] dx^{\theta}(t)$$

$$= r(t) B(t,T)\, dt + \left[\eta_t^{-1} r_t \underbrace{\frac{Z^*(t,T) A(t)}{Z^*(t,T)}}_{B(t,T)} + \theta(t) B(t,T)\right] dx^{\theta}$$

$$dB(t,T) = B(t,T)\left[r(t)\,dt + \left(\frac{\eta_t^{-1} r_t}{Z^*(t,T)} + \theta(t)\right) dx^{\theta}(t)\right]$$

$\Lambda(t) = \eta_t Z^*(t,T)$

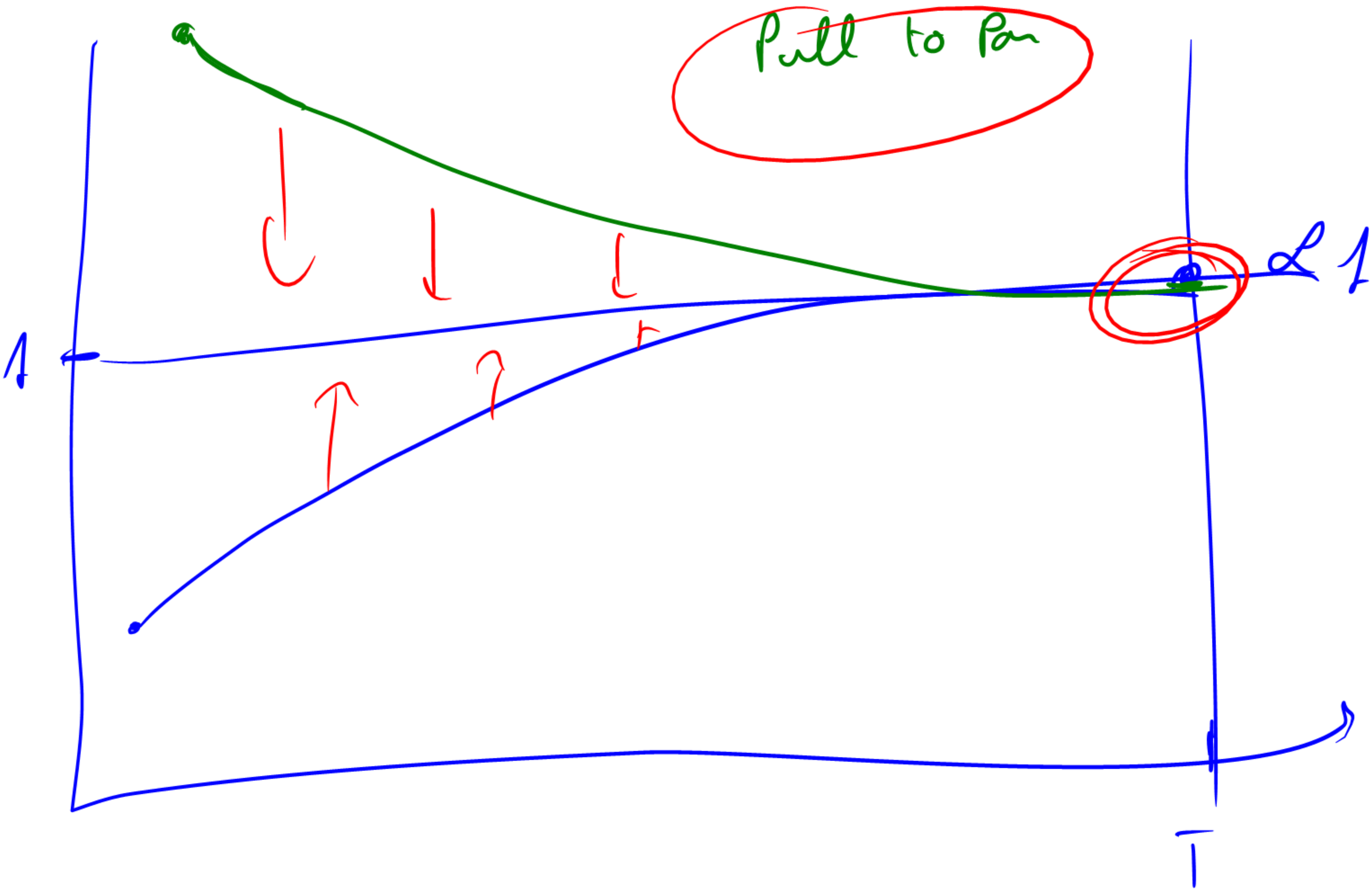
$$dB(t,T) = B(t,T)\left[r(t)\,dt + \left(\frac{\gamma_t}{\pi(t)} + \theta_t \right) dx^{\theta}(t) \right]$$

"Bond volatility"

$$b^{\theta}(t,T)$$

$$dB(t,T) = B(t,T)\left[r(t)\,dt + b^{\theta}(t,T)\, dx^{\theta}(t) \right]$$

$$B(t,T) = B(0,T)\,A(t)\,\underbrace{\exp\left\{ -\frac{1}{2}\int_t^T \left(b^{\theta}(0,T)\right)^2 ds + \int_t^T b^{\theta}(0,T)\, dx^{\theta}(s) \right\}}_{\text{Exp Mart}}$$



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Market price of risk

$\mathbb{Q}^\theta$ dynamics of $B(t,T)$

bond volatility

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt + b^\theta(t,T)\,dX^\theta(t) \quad , \qquad B(T,T) = 1$$

$\mathbb{P}$-dynamics of $B(t,T)$

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt + b^\theta(t,T)\big(dX(t) + \theta(t)\,dt\big)$$

$$= \boxed{\Big[\underbrace{r(t)}_{\text{short term rate}} + \underbrace{b^\theta(t,T)}_{\#\text{ of units of risk taken}}\,\underbrace{\theta(t)}_{\text{price of 1 unit of risk}}\Big]}\,dt + b^\theta(t,T)\,dX(t)$$

price of 1 unit of risk = Mkt price of Risk !!!

equity

$$\boxed{\theta = \frac{\mu - \sigma}{r}}$$

FAPF $\rightarrow$

$$C(t) = A(t)\,\mathbb{E}^{Q}\left[\frac{(B(T,U)-K)^{+}}{A(T)}\,\middle|\,\mathcal{F}_t\right] \qquad \frac{\mathbb{1}_{\{B(T,U)>K\}}}{A(T)}$$

$$= \underbrace{A(t)\,\mathbb{E}^{Q}\left[\frac{B(T,U)\,\mathbb{1}_{\{B(T,U)>K\}}}{A(T)}\,\middle|\,\mathcal{F}_t\right]}_{1\ \text{"difficult"}} - \underbrace{A(t)\,K\,\mathbb{E}^{Q}\left[\frac{\mathbb{1}_{\{B>K\}}}{A(T)}\,\middle|\,\mathcal{F}_t\right]}_{\text{"easy"}}$$

$$① \quad A(t)\, \mathbb{E}^{Q}\left[\frac{B(T,U)\, \mathbb{1}_{\{B(T,U)>K\}}}{A(T)} \,\middle|\, \mathcal{F}_t \right]$$

$$= A(t)\, \mathbb{E}^{Q}\left[\frac{1}{\cancel{A(T)}}\, \cancel{A(T)}\, \mathbb{E}^{Q}\left[\frac{1}{A(U)} \,\middle|\, \mathcal{F}_T \right] \mathbb{1}_{\{B(T,U)>K\}} \,\middle|\, \mathcal{F}_t \right]$$

$$= A(t)\, \mathbb{E}^{Q}\left[\cancel{\mathbb{E}^{Q}}\left[\frac{\mathbb{1}_{\{B(T,U)>K\}}}{A(T)} \,\cancel{\middle|\, \mathcal{F}_T} \right] \,\middle|\, \mathcal{F}_t \right]$$

Tower Property

$$= A(t)\, \mathbb{E}^{Q}\left[\frac{\mathbb{1}_{\{B(T,U)>K\}}}{A(T)} \,\middle|\, \mathcal{F}_t \right]$$

FAPF: "Reboot"

$$\underbrace{\text{Discounting}}_{B(t,T)} \times \ \mathbb{E}^{\text{New Measure}}\left[CF_s \mid \mathcal{F}_t\right]$$

Futures and Forwards ... again!!!

Buying today

$Y(t)$

Payoff $= Y(T) - Y(t)\, e^{\int_0^t r(s)\,ds}$

T

Borrowing

$Y(t)$

Loan Repayment

$Y(t)\, e^{\int_0^t r(s)\,ds}$

Forward (no cost)

Forward Price

$F(t,T) = Y(t)\, e^{\int_0^t r(s)\,ds}$

$1 \cdot e^{-\int_0^t r(s)\,ds} = ZCB$

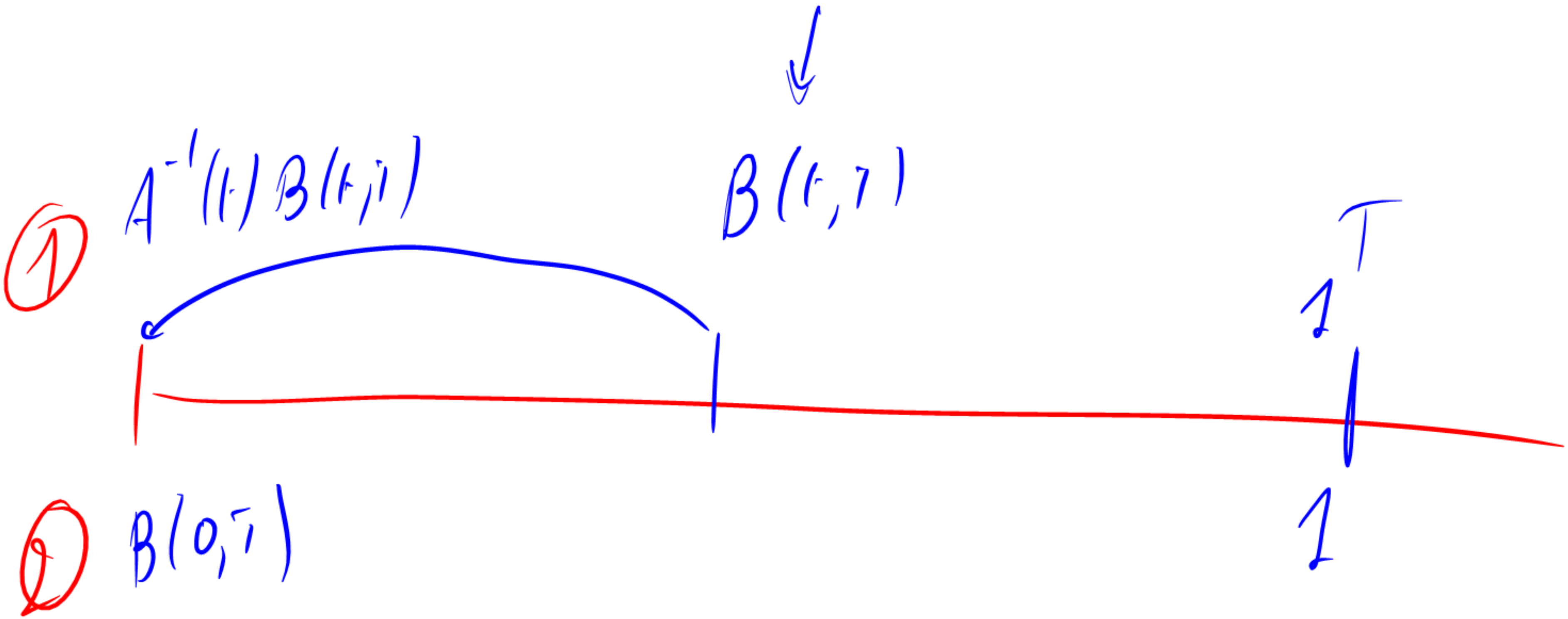
$= \dfrac{Y(t)}{e^{-\int_0^t r(s)\,ds}} = \dfrac{Y(t)}{B(t,T)}$

When interest rates are stochastic:

$F_Y(t,T) = \dfrac{Y(t)}{B(t,T)}$

$$\boxed{F_Y(t,T) = \frac{A(t)\, \mathbb{E}^Q\left[\frac{Y_T}{A(T)} \,\middle|\, \mathcal{F}_t\right]}{B(t,T)}}$$

$$\lambda_t = \boxed{\frac{A^{-1}(t)\, B(t,T)}{B(0,T)}} \text{ must be a martingale.}$$



Start from the FAPF :

$$V(t) = A(t)\, \mathbb{E}^{Q}\left[\frac{Y_T}{A(T)} \,\Big|\, F_t \right]$$

Note that $\lambda_T = \dfrac{A^{-1}(T)\, B(T,T)}{B(0,T)} = \dfrac{1}{A(T)\, B(0,T)}$

Then

$$V(t) = A(t)\, \mathbb{E}^{Q}\left[\frac{B(0,T)}{B(0,T)} \times \underbrace{\frac{Y_T}{A(T)}}_{\lambda_T} \,\Big|\, F_t \right]$$

$$\boxed{V(t) = A(t)\, B(0,T)\, \mathbb{E}^{Q}\left[Y_T\, \lambda_T \,\big|\, F_t \right]}$$