1 Exercise 2.1 Solutions

1. Denote by w_A^G and w_B^G the weights of the global minimum variance portfolio invested respectively in assets A and B.

   (a) The standard deviation of the portfolio return is given (see slide 22) by

   $$\sigma_\Pi(w_A, w_B) = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_A w_B \sigma_A \sigma_B} \tag{1}$$

   The budget equation $w_A + w_B = 1$ tells us that the investor's wealth must be fully invested in the portfolio. This equation implies that $w_B = 1 - w_A$. Substituting in equation (1), we can now express the standard deviation of the portfolio return as a sole function of w_A:

   $$\sigma_\Pi(w_A) = \sqrt{w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2\rho_{AB}w_A(1-w_A)\sigma_A\sigma_B}$$

   Developing and factoring, this expression yields

   $$\sigma_\Pi(w_A) = \sqrt{w_A^2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2w_A\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2}$$

   From this relation, we deduce an equation for the **variance** $\sigma_\Pi^2(w_A)$ of the portfolio return as a sole function of w_A:

   $$\sigma_\Pi^2(w_A) = w_A^2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2w_A\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2$$

   and we note that $\sigma_\Pi^2(w_A)$ is a quadratic function of w_A.

   (b) To derive the weight w_A^G of the global minimum variance portfolio invested in A, we need to solve the unconstrained optimization problem

   $$\min_{w_A} \sigma_\Pi^2(w_A) \tag{2}$$

   Differentiating the objective function $\sigma_\Pi^2(w_A)$ yields the first order (necessary) condition

   $$\left.\frac{d\sigma_\Pi^2(w_A)}{dw_A}\right|_{w_A^G} = 0$$

   which implies that

   $$2w_A^G(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2\sigma_B(\rho_{AB}\sigma_A - \sigma_B) = 0$$

   and in turns results in the candidate solution

   $$w_A^G = \frac{\sigma_B(\sigma_B - \rho_{AB}\sigma_A)}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B} \tag{3}$$

   Before concluding, we need to check that the candidate solution w_A^G defined in equation (3) actually yields a minimum point for the function σ_Π^2. By the second order (sufficient) condition we need to have

   $$\left.\frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\right|_{w_A^G} > 0$$

for a minimum to be reached at w_A^G

Here,

$$\frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\Big|_{w_A^G} = 2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)$$

Because $-1 \leq \rho_{AB} \leq 1$, we observe that

$$2(\sigma_A - \sigma_B)^2 \leq \frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\Big|_{w_A^G} \leq 2(\sigma_A + \sigma_B)^2$$

which implies that $\frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\Big|_{w_A^G} > 0$ as long as either

- $\sigma_A \neq \sigma_B$, or;
- $\rho_{AB} < 1$

2. Denote by w_A^t and w_B^t the weights of the tangency portfolio invested respectively in assets A and B.

(a) The slope S of the tangency line is equal to the Sharpe ratio:

$$S = \frac{\mu_t - r_f}{\sigma_t} \tag{4}$$

where r_f is the risk-free return and the return μ_t of the tangency portfolio and the standard deviation σ_t of the tangency portfolio are respectively given by

$$\mu_t = w_A^t\mu_A + w_B^t\mu_B$$

and

$$\sigma_t = \sqrt{(w_A^t)^2\sigma_A^2 + (w_B^t)^2\sigma_B^2 + 2\rho_{AB}(w_A^t)(w_B^t)\sigma_A\sigma_B}$$

Substituting into equation (5), we find a functional form $S(w_A^t, w_B^t)$ for the slope of the tangency line:

$$S(w_A^t, w_B^t) = \frac{w_A^t\mu_A + w_B^t\mu_B - r_f}{\sqrt{(w_A^t)^2\sigma_A^2 + (w_B^t)^2\sigma_B^2 + 2\rho_{AB}(w_A^t)(w_B^t)\sigma_A\sigma_B}} \tag{5}$$

(b) Because the tangency portfolio is fully invested in risky assets, the budget equation $w_A^t + w_B^t = 1$ applies. Substituting the budget equation into equation (5), we can express the slope of the tangency line as a sole function $S(w_A^t)$ of the weight w_A^t invested in asset A:

$$\begin{aligned} S(w_A^t) &= \frac{w_A^t\mu_A + (1 - w_A^t)\mu_B - r_f}{\sqrt{(w_A^t)^2\sigma_A^2 + (1 - w_A^t)^2\sigma_B^2 + 2\rho_{AB}(w_A^t)(1 - w_A^t)\sigma_A\sigma_B}} \\ &= \frac{w_A^t(\mu_A - \mu_B) + \mu_B - r_f}{\sqrt{(w_A^t)^2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2(w_A^t)\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2}} \end{aligned} \tag{6}$$

(c) As long as $\mu_B > r_f$ or $\mu_A > r_f$, the slope of the tangency line will be positive. In this case, rather than finding w_A^t as the maximizer of $S(w_A^t)$, we could instead find w_A^t as the maximizer of $S^2(w_A^t)$ by solving

$$\max_{w_A^t} S^2(w_A^t)$$

with

$$S^2(w_A^t) = \frac{\left(w_A^t(\mu_A - \mu_B) + \mu_B - r_f\right)^2}{(w_A^t)^2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2(w_A^t)\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2}$$

Denote by

$$f(w_A^t) := \left(w_A^t(\mu_A - \mu_B) + \mu_B - r_f\right)^2$$

and by

$$g(w_A^t) := (w_A^t)^2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2(w_A^t)\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2 = \sigma_t^2(w_A)$$

so that

$$S^2(w_A^t) = \frac{f(w_A)}{g(w_A)}$$

Considering the first order (necessary) condition associated with this optimization problem, we are looking for w_A^t such that

$$\frac{dS^2(w_A^t)}{dw_A^t} = 0$$

i.e. such that

$$\frac{dS^2(w_A^t)}{dw_A^t} = \frac{f'(w_A)g(w_A) - f(w_A)g'(w_A)}{g^2(w_A)} = 0$$

where

$$\begin{aligned} f'(w_A^t) &= \frac{df(w_A^t)}{dw_A^t} \\ &= 2(\mu_A - \mu_B)\left(w_A^t(\mu_A - \mu_B) + \mu_B - r_f\right) \end{aligned}$$

and

$$\begin{aligned} g'(w_A) &= \frac{dg(w_A^t)}{dw_A^t} \\ &= 2(w_A^t)(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2\sigma_B(\rho_{AB}\sigma_A - \sigma_B) \end{aligned}$$

After some rather tedious calculations (substituting, simplifying as much as possible and concentrating exclusively on the numerator since the denominator is positive), we finally get

$$w_A^t := \frac{\sigma_B\left((\mu_B - r_f)\rho_{AB}\sigma_A - (\mu_A - r_f)\sigma_B\right)}{-(\mu_a - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_A^2 + (\mu_A + \mu_B - 2r_f)\rho_{AB}\sigma_A\sigma_B} \tag{7}$$

Checking the second order (sufficient) condition for a maximization, that is

$$\frac{dS^2(w_A^t)}{d{w_A^t}^2} < 0$$

is equally unpleasant.

We will develop a much more efficient approach to this problem using matrix algebra in Lecture 2.2.