CQF Module Two

EXAMINATION SOLUTIONS

June 2019 Cohort

Instructions

Please review these Solutions. Complete mathematical workings and computation had to be provided to obtain maximum credit. Mathematical techniques appropriate to the given task should have been used.

Solutions provided in the form of correct numerical answers and plots. No Excel or code released with this document.

As a quant, one cannot provide a computational result without considering if numbers make sense. Sensibility/sanity checks are fundamental to model validation. For example, optimal allocations could be confirmed with Solver/numerical optimiser. VaR backtesting should compare $N_{breaches}$ to 1% confidence.

Optimal Portfolio Allocation

Portfolio Optimisation Workings

Covariance $\boldsymbol{\Sigma} = diag(\boldsymbol{\sigma})\, \boldsymbol{R}\, diag(\boldsymbol{\sigma})$, where $diag(\boldsymbol{\sigma})$ is a diagonal matrix of standard deviations.

Question 1. Efficient Frontier Portfolio: to solve the min variance optimization problem for a target return m formulate the Lagrange function with two multipliers λ_1, λ_2,

$$L(\boldsymbol{w}, \lambda_1, \lambda_2) = \frac{1}{2}\boldsymbol{w'\Sigma w} + \lambda_1(\boldsymbol{w'\mu} - m) + \lambda_2(\boldsymbol{w'1} - 1)$$

$$\begin{aligned}
\frac{\partial L}{\partial \boldsymbol{w}} &= \boldsymbol{\Sigma w} + \lambda_1 \boldsymbol{\mu} - \lambda_2 \boldsymbol{1} = 0 \\
&\quad \text{gives} \quad \boldsymbol{w} = \boldsymbol{\Sigma}^{-1}(\lambda_1 \boldsymbol{\mu} + \lambda_2 \boldsymbol{1}) \\
\frac{\partial L}{\partial \lambda_1} &= \boldsymbol{w'\mu} - m = 0 \\
\frac{\partial L}{\partial \lambda_2} &= \boldsymbol{w'1} - 1 = 0
\end{aligned}$$

The required part ends. Three partial derivatives equated to zero give a system of linear equations as follows:

$$\begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\mu} & \boldsymbol{1} \\ \boldsymbol{\mu}' & 0 & 0 \\ \boldsymbol{1}' & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \boldsymbol{w} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ m \\ 1 \end{pmatrix}$$

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$$

Solution is obtained using

$$\boldsymbol{x} = \boldsymbol{A^{-1}}\,\boldsymbol{b}.$$

Inserting the result for allocations into the equations for the multipliers λ_1, λ_2 gives,

$$\begin{pmatrix} \boldsymbol{\mu'\Sigma^{-1}\mu} & \boldsymbol{\mu'\Sigma^{-1}1} \\ \boldsymbol{\mu'\Sigma^{-1}1} & \boldsymbol{1'\Sigma^{-1}1} \end{pmatrix} \times \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} m \\ 1 \end{pmatrix}$$

Define $A = \boldsymbol{1'\Sigma^{-1}1}$, $B = \boldsymbol{\mu'\Sigma^{-1}1} = \boldsymbol{1'\Sigma^{-1}\mu}$, and $C = \boldsymbol{\mu'\Sigma^{-1}\mu}$, the same as in the Portfolio Optimisation CQF Lecture.

$$\begin{cases} \lambda_1 = \frac{Am-B}{AC-B^2} \\ \lambda_2 = \frac{C-Bm}{AC-B^2} \end{cases}$$

Question 2. **Tangency Portfolio is one entirely invested in risky assets** but does so on the risk-adjusted basis. The original Markowitz thinking did not explicitly consider 'a bank', the ability to borrow and purchase risky assets.

$$\underset{\boldsymbol{w}}{\text{argmax}} \frac{\boldsymbol{w'\mu} - r_f}{\sqrt{\boldsymbol{w'\Sigma w}}} = \frac{\mu_\Pi - r_f}{\sigma_\Pi}$$
$$\text{s.t.} \;\; \boldsymbol{w'1} = 1$$

Solved in the usual way with the Lagrangian method (derivation was not required). **We have provided partial derivatives in the relevant Webex section.**

$$L(w, \lambda) = (\boldsymbol{w'\mu} - r_f)(\boldsymbol{w'}\Sigma\boldsymbol{w})^{-\frac{1}{2}} + \lambda(1 - \boldsymbol{w'1})$$

$$\boldsymbol{w_T} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{1})}{\boldsymbol{1'\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{1})} \iff \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{1})}{B - Ar}$$

Efficient Frontier for the optimisation problem with the risk-free asset (tangency portfolio) is a line, called Capital Market Line. The slope is also a Sharpe Ratio.

$$\text{SR} = \frac{\mu_\Pi - r_f}{\sigma_\Pi} \quad \Rightarrow \quad \mu_\Pi = r_f + \text{SR} \times \sigma_\Pi \quad \text{is the equation for CML}$$

Computational Summary (June 2019)

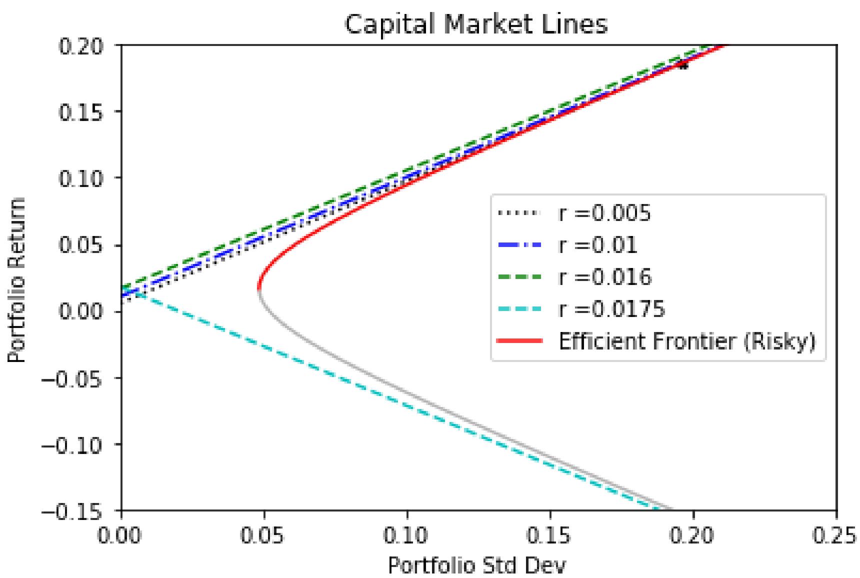
		Stressed x1.25	Stressed x1.5
w_A	0.7851	0.8182	0.8762
w_B	0.0539	-0.0094	-0.1461
w_C	0.1336	0.179	0.3257
w_D	0.0275	0.0122	-0.0557
σ_Π	**0.0584**	**0.0607**	**0.0636**

While scaling correlation matrix, control for diagonal elements to stay at "1" because they represent correlation of asset to itself.

Increase in correlation is associated with 'risk off' mode of today's markets. Observe that increased correlation leads to less-balanced allocations – even as changes to portfolio risk are minuscule.

Table 1: Tangency Portfolio (the last column gives negative risk premium *rhs*)

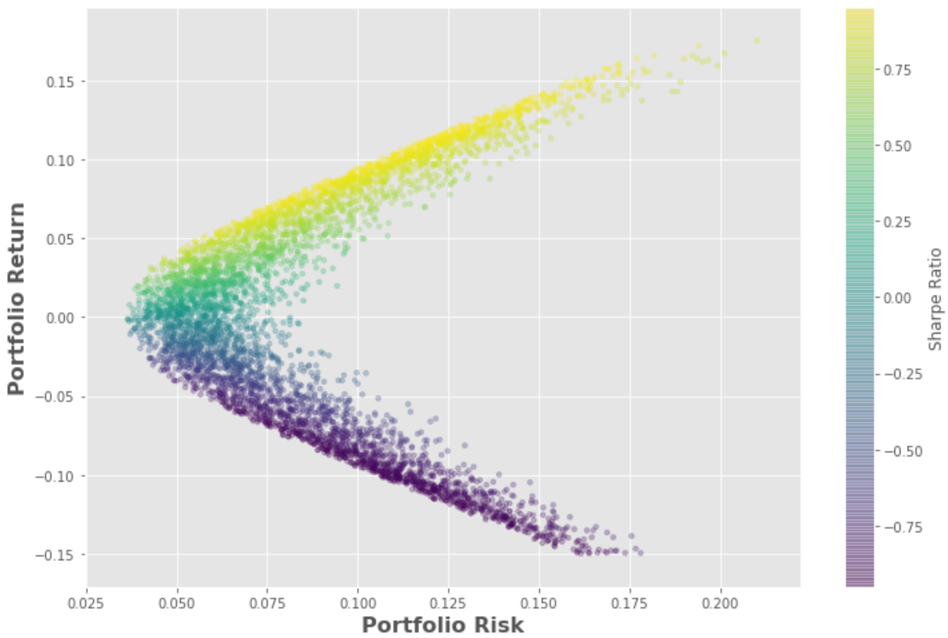
	50bps	75bps	100bps	150bps	175bps
w_A	0.0168	-0.2533	-0.746	-8.645	8.104
w_B	-0.2294	-0.3289	-0.511	-3.423	2.752
w_C	0.8143	1.0537	1.490	8.490	-6.351
w_D	0.3982	0.5285	0.766	4.578	3.504
μ_Π	0.186	0.236	0.326	1.777	-1.299
σ_Π	0.197	0.251	0.351	1.972	1.474



In the presence of bank account/bond asset, the actionable Efficient Frontier is not the concave envelope itself (Markowitz Bullet) but a set of tangent lines (CMLs). When the risk-free rate is above the return on Global Min Variance portfolio, Sharpe Ratio turns negative.

$$r_f > \mu_G \qquad \mu_G \approx 0.01607$$

When risk premium $(\mu_\Pi - r_f)$ is negative the Capital Market Line is tangent to the parabola from the bottom (negative slope), and the portfolio is non-feasible. For each additional unit of risk taken, the investor reduces their excess return!



Quesiton 1 also required simulation of Markowitz Bullet, which transpires if randomly generated allocations $\boldsymbol{w}$ are inserted into analytical solution for Min Variance portfolio. It is difficult to generate portfolios that have risk $> 20\%$, which was the maximum asset risk given (Asset D).

B. Backtesting Value at Risk $\boxed{\text{Solutions:}}$

Question 3

- For historical standard deviation method (rolling window of 22 returns), there were 24 breaches out of 1,218 eligible observations to compare VaR_t and r_{10D} forward return. Percentage of VaR breaches 1.97%. There were 14 consecutive breaches, and so conditional probability of the next breach $14/22 = 63.6\%$.

- For EWMA smoothed variance method, there were 32 breaches out of 1,218 eligible observations. The percentage of VaR breaches was, therefore, 2.63%. There were 17 consecutive breaches, and so conditional probability of the next breach $17/32 = 53.1\%$.

Some implementations began EWMA computation for the initial 20 days, just as r_{t-1} became available. Those likely to have resulted in 36 breaches and 19 consecutive breaches. That way of computation **incorrectly** assumed $\sigma_{t-1} = 0$. The past volatility is not zero but an unknown quantity due to sample cut.

Better way was to substitute a reasonable value for σ_{t-1}, as many have done for Day 22 to make eligible observations count the same for SD and EWMA methods.

Most implementations applied r_{t-1} but where r_{t-2} was used due to how EWMA equation was given in exam task paper – that was acknowledged.

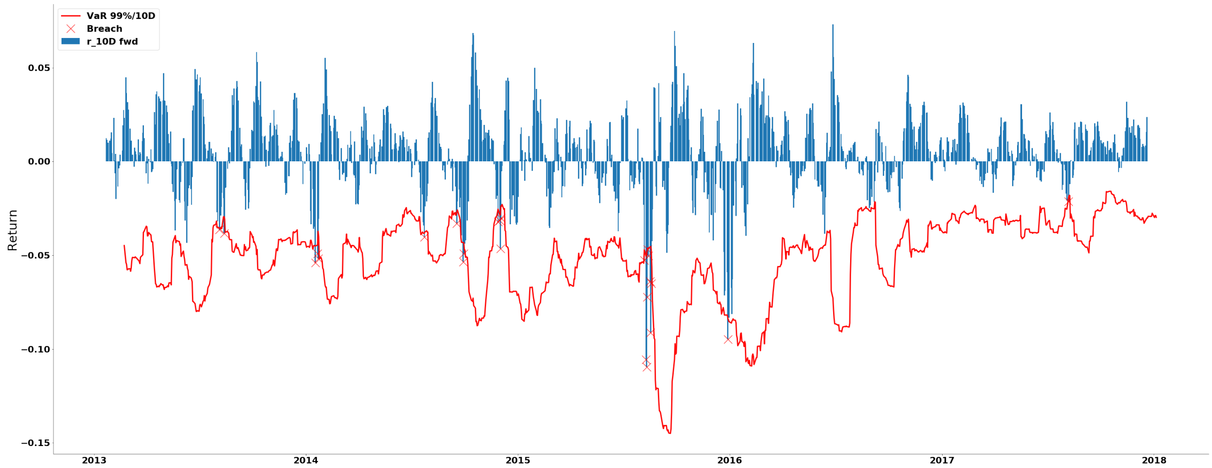


Figure 1: VaR Backtesting (22D rolling sample). Python *matplotlib* used.

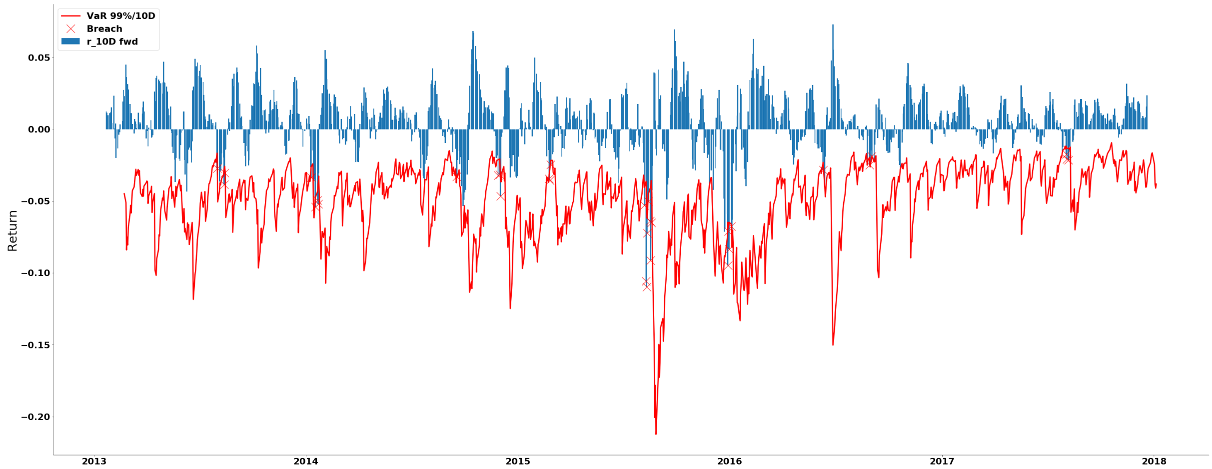


Figure 2: VaR Backtesting (22D EWMA recursive volatility).

Question 4 EVT VaR for the dataset is computed as follows:

$$\begin{aligned} VaR_{95\%} &= -0.012617 + \frac{0.00002}{0.4}\left[\left(\frac{1249}{62}(1-0.95)\right)^{-0.4} - 1\right] \\ &= -0.01261714443 \end{aligned}$$

when choosing suitable threshold and fitting ξ, β parameters I found that β came up to be near zero in most cases (while ξ was in predicted range of $0.2 - 0.4$ for financial data by Hull). Potential cause was that we operated on returns, rather than dollar losses.

Question 5

$$\begin{aligned}\text{LVaR} &= \text{VaR} + \Delta_{Liquidity} \\ &= \text{Portfolio Value} \times \left[\underbrace{-\mu + \text{Factor} \times \sigma}_{} + \frac{1}{2}(\mu_{Spread} + \text{Factor} \times \sigma_{Spread})\right]\end{aligned}$$

where braced expression is our analytical VaR formula (percentage), and $\Delta_{Liquidity}$ is effectively half VaR of the spread itself.

(a) Technology company example (exact factor value was used, rounded up for presentation)

$$\begin{aligned}\text{LVaR}_{99\%/1D} &= 16,000,000 \times [-0.01 + 2.32635 \times 0.03 + \frac{1}{2}(0.0035 + 2.32634787 \times 0.0150)] \\ &= \$1,263,809\end{aligned}$$

within that computation, classical VaR contribution 75.7%, and liquidity adjustment contribution 24.3%.

(b) UK Gilts (government bonds) portfolio example (exact factor value was used)

$$\begin{aligned}LVaR_{99\%/1D} &= 40,000,000 \times [2.32635 \times 0.03 + \frac{1}{2}(0.0015)] \\ &= \pounds 2,821,617\end{aligned}$$

within that computation, classical VaR contribution 98.9%, and liquidity adjustment contribution 1.1%.

(c) if the spread increases to from 15bps 125bps,

$$\begin{aligned}LVaR_{99\%/1D} &= 40,000,000 \times [2.32635 \times 0.03 + \frac{1}{2}(0.0125)] \\ &= \pounds 3,041,614\end{aligned}$$

within that computation, classical VaR contribution 91.8%, and liquidity adjustment contribution increases to 8.2%.

Extra on independence of breaches.

	Date	Closing Price	LogReturn	SD	VAR10D
135	2013-08-05	1707.140015	-0.001481	0.004827	-0.035512
141	2013-08-13	1694.160034	0.002772	0.004000	-0.029430
250	2014-01-17	1838.699951	-0.003903	0.006135	-0.045130
251	2014-01-21	1843.800049	0.002770	0.006061	-0.044586
252	2014-01-22	1844.859985	0.000575	0.005060	-0.037227
379	2014-07-24	1987.979980	0.000488	0.005203	-0.038276
418	2014-09-18	2011.359985	0.004879	0.003625	-0.026665
425	2014-09-29	1977.800049	-0.002550	0.005876	-0.043229
426	2014-09-30	1972.290039	-0.002790	0.005891	-0.043340
468	2014-11-28	2067.560059	-0.002546	0.003456	-0.025425
469	2014-12-01	2053.439941	-0.006853	0.003871	-0.028481
470	2014-12-02	2066.550049	0.006364	0.003880	-0.028546
471	2014-12-03	2074.330078	0.003758	0.003214	-0.023642
641	2015-08-07	2077.570068	-0.002879	0.006396	-0.047051
642	2015-08-10	2104.179932	0.012727	0.006889	-0.050681
643	2015-08-11	2084.070068	-0.009603	0.006787	-0.049930
644	2015-08-12	2086.050049	0.000950	0.006346	-0.046688
645	2015-08-13	2083.389893	-0.001276	0.006260	-0.046050
646	2015-08-14	2091.540039	0.003904	0.006331	-0.046575
647	2015-08-17	2102.439941	0.005198	0.006183	-0.045484
648	2015-08-18	2096.919922	-0.002629	0.006189	-0.045528
649	2015-08-19	2079.610107	-0.008289	0.006389	-0.046998
740	2015-12-29	2078.360107	0.010574	0.011386	-0.083762
1144	2017-08-07	2480.909912	0.001646	0.002595	-0.019094

Figure 3: Breaches list (22D rolling sample)

	Date	Closing Price	LogReturn	SD	SDEWMA	VAR10D
133	2013-08-01	1706.869995	0.012463	0.004805	0.002255	-0.016585
141	2013-08-13	1694.160034	0.002772	0.004000	0.003753	-0.027609
142	2013-08-14	1685.390015	-0.005190	0.004164	0.003506	-0.025793
245	2014-01-10	1842.369995	0.002304	0.005717	0.003560	-0.026187
250	2014-01-17	1838.699951	-0.003903	0.006135	0.006501	-0.047822
251	2014-01-21	1843.800049	0.002770	0.006061	0.005890	-0.043329
252	2014-01-22	1844.859985	0.000575	0.005060	0.005208	-0.038314
418	2014-09-18	2011.359985	0.004879	0.003625	0.004352	-0.032012
468	2014-11-28	2067.560059	-0.002546	0.003456	0.002893	-0.021280
469	2014-12-01	2053.439941	-0.006853	0.003871	0.002800	-0.020597
470	2014-12-02	2066.550049	0.006364	0.003880	0.004335	-0.031891
526	2015-02-24	2115.479980	0.002755	0.008250	0.004262	-0.031351
527	2015-02-25	2113.860107	-0.000766	0.008129	0.003899	-0.028682
529	2015-02-27	2104.500000	-0.002961	0.007531	0.002934	-0.021586
641	2015-08-07	2077.570068	-0.002879	0.006396	0.005519	-0.040598
642	2015-08-10	2104.179932	0.012727	0.006889	0.004924	-0.036226
643	2015-08-11	2084.070068	-0.009603	0.006787	0.007925	-0.058303
644	2015-08-12	2086.050049	0.000950	0.006346	0.008429	-0.062007
646	2015-08-14	2091.540039	0.003904	0.006331	0.006121	-0.045030
647	2015-08-17	2102.439941	0.005198	0.006183	0.005590	-0.041121
648	2015-08-18	2096.919922	-0.002629	0.006189	0.005483	-0.040335
649	2015-08-19	2079.610107	-0.008289	0.006389	0.004856	-0.035722
740	2015-12-29	2078.360107	0.010574	0.011386	0.008802	-0.064752
741	2015-12-30	2063.360107	-0.007243	0.011481	0.009332	-0.068652
742	2015-12-31	2043.939941	-0.009456	0.011607	0.008797	-0.064718
743	2016-01-04	2012.660034	-0.015422	0.011714	0.008987	-0.066112
853	2016-06-10	2096.070068	-0.009218	0.006159	0.003125	-0.022990
908	2016-08-29	2180.379883	0.005214	0.003592	0.002655	-0.019530
912	2016-09-02	2179.979980	0.004192	0.003393	0.002536	-0.018657
1142	2017-08-03	2472.159912	-0.002186	0.003452	0.001612	-0.011857
1143	2017-08-04	2476.830078	0.001887	0.003458	0.001791	-0.013177
1144	2017-08-07	2480.909912	0.001646	0.002595	0.001819	-0.013379

Figure 4: Breaches list (22D EWMA recursive volatility).

VaR breaches here **not** independent of the level of volatility: underestimation of volatility is not corrected fast, and use of EWMA does not alleviate the problem.

Statistical tests for Normality and autocorrelation are typically applied to such data as regression residuals – D'Agostino K-squared and Shapiro-Wilk are good choices. However, applying them to *returns that caused VaR breaches is non-informative*: the tests suggest non-Normality because of a slight deviation from Normal skewness and kurtosis. The same applies to Q-Q plots. Empirical asset returns will always show as non-normal, whether they have sequences of out-sized asset returns or not.

There is a simple *iid* check with a lag plot: if returns were *iid*, then all observations would remain within a circle.

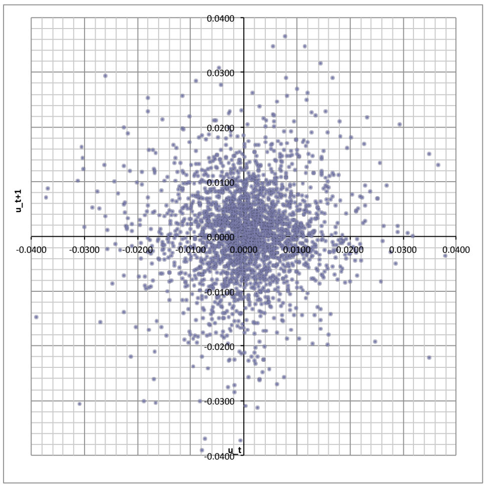


Figure 5: Examining location-dispersion ellipsoid (u_t plotted vs. u_{t-1}) gives a robust check for *iid*-ness.

However, lag plots and conditional probability of consecutive breach – all are simple tools. Independence of exceedances (VaR breaches) can be analysed using Christoffersen's 1998 Exceedance Independence test and Loss-Quantile Independence test:

- Christoffersen's test relies on classification of breach following non-breach $I_t = 1, I_{t-1} = 0$, and non-breach following breach $I_t = 0, I_{t-1} = 1$ (as well as two other obvious situations). Likelihood Ratio is computed and compared agaist a critical value from χ^2 distribution. For VaR backtesting with historical standard deviation, it was computed,

$$\text{LR} = 87.825, \qquad \text{critical value } \chi^2_{95\%} = 3.84$$

  the null hypothesis of breach independence rejected strongly.