

ECONOMETRICS I

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Introduction to Probability and Statistics for Economists (Ph.D. in Economics, 1st year)

Lectures 1 and 2

TWO MAIN BLOCKS

1. PROBABILITY THEORY
2. MATHEMATICAL STATISTICS

Probability Theory

How to model 'randomness'?

THINGS WE WILL LEARN IN LECTURE 1:

- ★ What is a probability space?
 1. What is measurable space?
 2. What is a probability space?
- ★ What is a random variable?
- ★ What is the 'distribution' or 'law' of a random variable?

What is a probability space?

$$(\Omega, \mathcal{F}, \mathbb{P})$$

PROBABILITY SPACE: TWO COMPONENTS

$$(\Omega, \mathcal{F}, \mathbb{P})$$

1. (Ω, \mathcal{F}) measurable space.

Ω : States of the world, \mathcal{F} : Set of events ($\subseteq \Omega$).

2. $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$. Probability Measure

‘How likely is an event in \mathcal{F} ’

MEASURABLE SPACE

See notes.

PROBABILITY MEASURES

★ $\mathbb{P}(\phi) = 0, \mathbb{P}(\Omega) = 1$ (Normalization)

★ For any finite collection A_1, A_2, \dots, A_m such that $A_i \cap A_j = \emptyset$

$$\mathbb{P}\left(\cup_{i=1}^m A_i\right) = \sum_{i=1}^m \mathbb{P}(A_i)$$

This property is called **additivity**.

★ If you replace finite by *countably infinite*, Property 2 is called **σ -additivity**.

IMPORTANT

Normalization and σ -additivity define a probability measure

What is a random variable?

$$X : \Omega \rightarrow S$$

RANDOM VARIABLE

$$X : \Omega \rightarrow S$$

- ★ Ω : Set of states of the world.
- ★ S : Image Space
- ★ X : Random Variable

What is the distribution or law of a random variable?

‘INDUCED’ PROBABILITY OF A RANDOM VARIABLE

- ★ The probability \mathbb{P} on Ω induces a probability on subsets of S :

$$\mathbb{P}_X[F] \equiv \mathbb{P}[\{\omega \mid X(\omega) \in F\}], \quad F \subseteq S$$

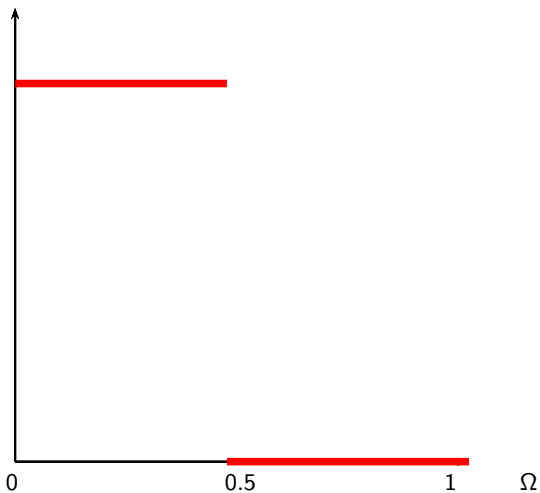
- ★ How likely are the states of the world in which F occurs?

The induced probability of a random variable is usually called its
DISTRIBUTION OR LAW

Different random variables can induce the same probability on S .

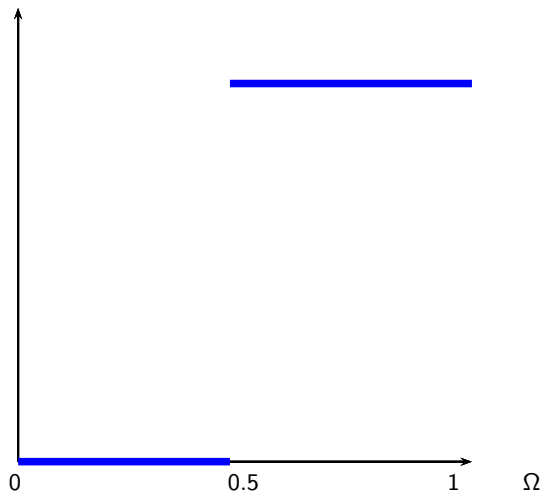
$$P_{X_2}(1) = .5$$

$$S = \{0, 1\}$$



$$P_{X_2}(1) = .5$$

$$S = \{0, 1\}$$



The Cumulative Distribution Function of Real-valued Random Variables

C.D.F OF AN \mathbb{R} -VALUED RANDOM VARIABLE

- ★ How likely is a realization of the random variable X below x ?
- ★ The c.d.f. summarizes this information

$$F_X : \mathbb{R} \rightarrow [0, 1]$$

$$F_X(x) \equiv \mathbb{P}\{\omega \in \Omega \mid X(\omega) \leq x\}$$

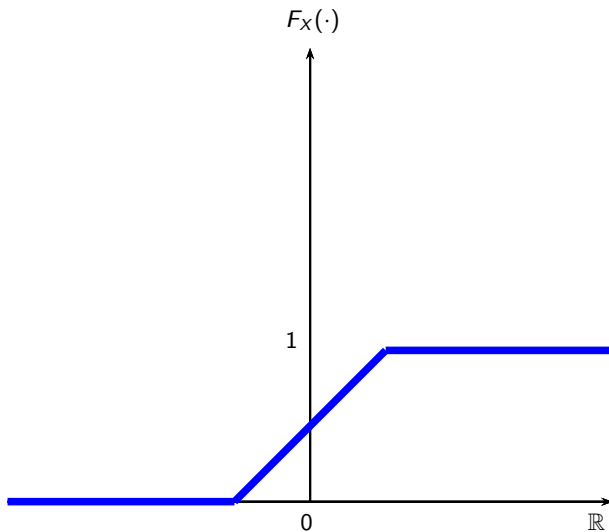
Examples of c.d.f.

► Uniform cdf and r.v. with Uniform distribution

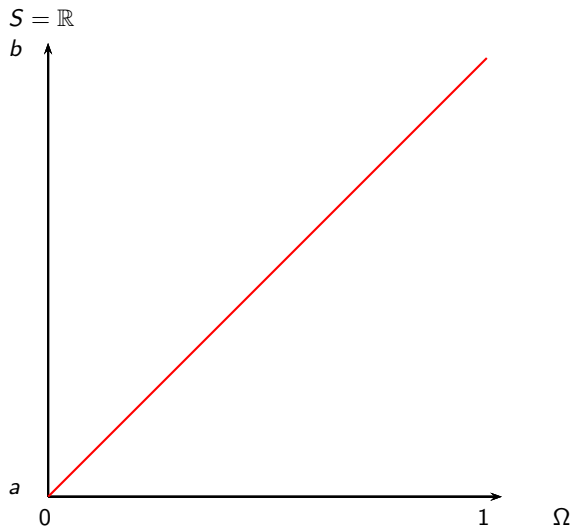
► Bernoulli cdf and r.v. with Bernoulli distribution

► Skip to Properties

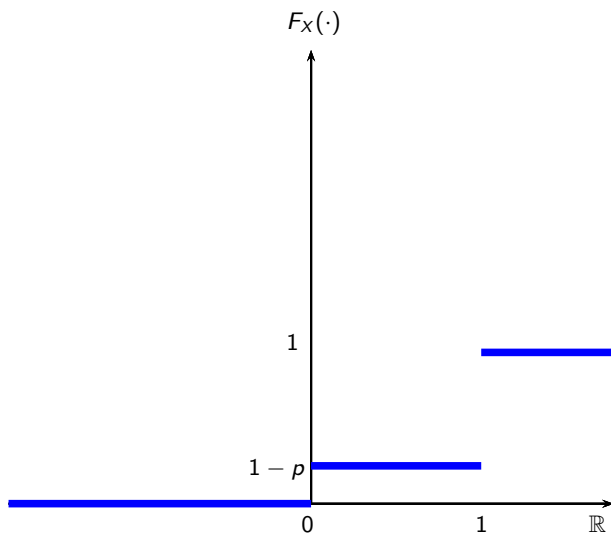
$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ (x - a)/b - a & \text{if } x \in [a, b) \\ 1 & \text{if } x \geq b \end{cases}$$

UNIFORM DISTRIBUTION ON $[a, b]$ 

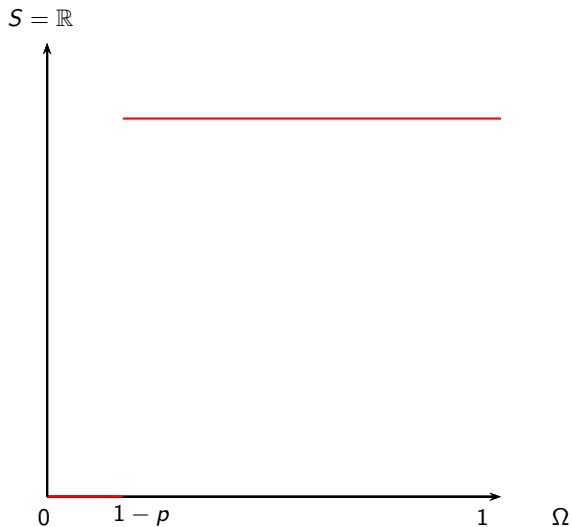
$$X(\omega) = a + \omega[b - a]$$



$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & \text{if } x \in [0, 1) \\ 1 & \text{if } x \geq 1 \end{cases}$$

BERNOULLI DISTRIBUTION WITH PARAMETER p 

$$X(\omega) = \mathbf{1}[\omega \geq 1 - p]$$



What are the common properties of c.d.f?

CHARACTERIZATION

1. F_X is non-decreasing
2. $\lim_{x \rightarrow \infty} F_X(x) = 1$
3. $\lim_{x \rightarrow -\infty} F_X(x) = 0$
4. $\lim_{h \rightarrow 0^+} F_X(x + h) = F_X(x)$

In fact, these 4 properties characterize the induced c.d.f. of a real-valued random variable!

Discrete and Continuous Type of Real-Valued Random Variables

DISCRETE DISTRIBUTIONS / DISCRETE R.V.S

DISTRIBUTION: Distributions for which \exists a countable set

$$\text{Supp} = \{x_1, x_2, \dots\}, x_i \in \mathbb{R},$$

such that

$$\text{a) } \mathbb{P}_X(X = x_i) > 0 \quad \forall \quad x_i \in \text{Supp}$$

$$\text{b) } \sum_{x_i \in \text{Supp}} \mathbb{P}_X(X = x_i) = 1$$

are called **discrete**.

P.M.F.

We will identify discrete distributions/r.v.s by its support and its
p.m.f.

(ABSOLUTELY) CONTINUOUS DISTRIBUTIONS

Random Variables for which

$$F_X(x) = \int_{-\infty}^x f(z) dz$$

for some $f(z) \geq 0 \forall z \in \mathbb{R}$ are called:

(Absolutely) Continuous

The function $f(z)$ is called

Probability Density Function (p.d.f.)

We will identify continuous distributions/r.v.s by its p.d.f.

The set

$$\{z \in \mathbb{R} \mid f(z) > 0\} \subseteq \mathbb{R}$$

is called the **support** of the continuous r.v.

FROM P.D.F. TO PROBABILITIES

Let X be a real-valued random variable with p.d.f. $f(z)$

$$\star \mathbb{P}_X[X \leq a] = \int_{-\infty}^a f(z) dz$$

$$\star \mathbb{P}_X[a \leq X \leq b] = \int_a^b f(z) dz$$

$$\star \mathbb{P}_X[X > a] = \int_a^{\infty} f(z) dz$$

Examples of (Univariate) Discrete Distributions

[▶ Bernoulli](#)[▶ Binomial](#)[▶ Skip to Continuous Distributions](#)

BERNOULLI DISTRIBUTION (p), $p \in (0, 1)$

★ The Bernoulli distribution with parameters p has support:

$$\text{Supp} = \{0, 1\}$$

and p.m.f. given by:

$$\mathbb{P}_X(X = x) = p^x(1 - p)^{1-x} \quad x \in \{0, 1\}$$

HOW DO WE KNOW IT IS A P.M.F.?

Two parts:

a) $\mathbb{P}_X(X = x) > 0 \quad \forall x \in \{0, 1\}$. Easy to verify:

$$p^x(1-p)^{1-x} > 0$$

b)

$$\sum_{x \in \{0,1\}} p^x(1-p)^{1-x} = (1-p) + p$$

◀ Examples

BINOMIAL DISTRIBUTION (n, p) , $n \in \mathbb{N}, p \in (0, 1)$

★ The binomial distribution with parameters (n, p) has support:

$$\text{Supp} = \{0, 1, 2, \dots, n\}$$

and p.m.f. given by:

$$\mathbb{P}_X(X = x) \equiv \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \quad x \in \text{Supp}$$

HOW DO WE KNOW IT IS A P.M.F.?

Two parts:

a) $\mathbb{P}_X(X = x) > 0 \quad \forall x \in \{0, 1, 2, \dots, n\}$. Easy to verify:

$$\frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} > 0$$

b)

$$\sum_{x \in \{0, 1, \dots, n\}} \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = 1?$$

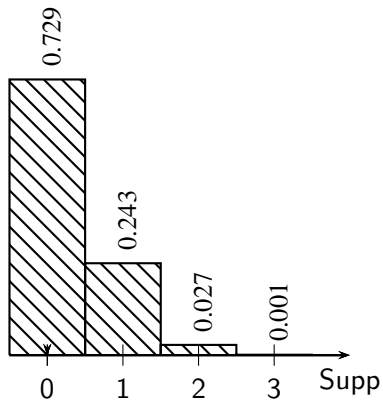
HOW DO WE KNOW IT IS A P.M.F.?

Use the Binomial Theorem

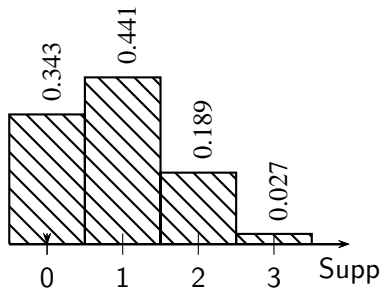
$$(a + b)^n = \sum_{x \in \{0, 1, \dots, n\}} \frac{n!}{(n-x)!x!} a^x b^{n-x} = 1$$

$$a = p, \quad b = 1 - p$$

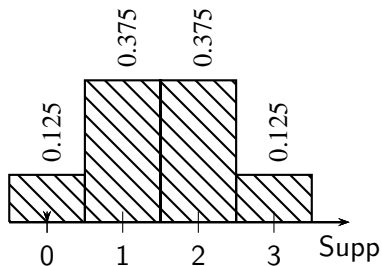
BINOMIAL (3, .1)



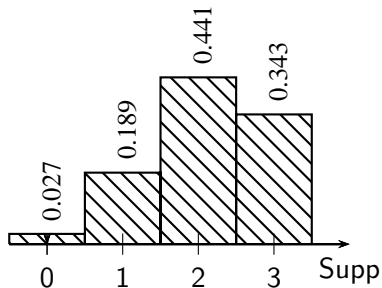
BINOMIAL (3, .3)



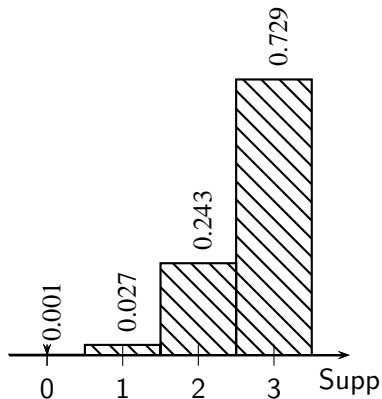
BINOMIAL (3, .5)



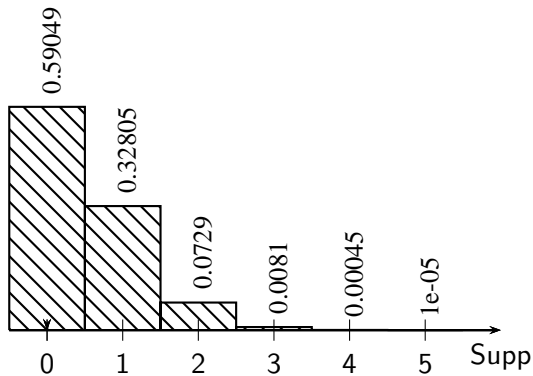
BINOMIAL (3, .7)



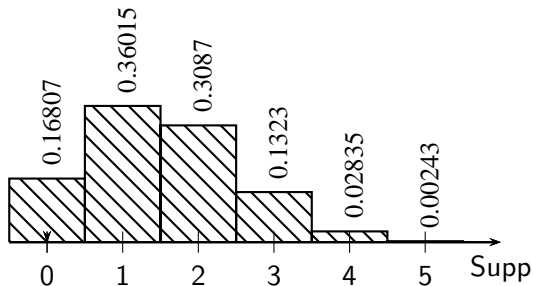
BINOMIAL (3, .9)



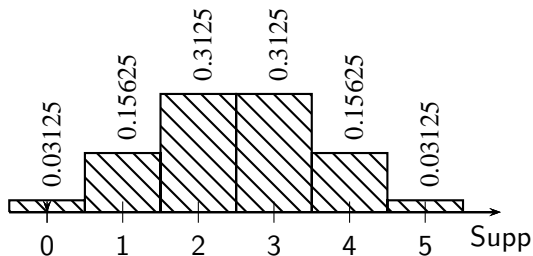
BINOMIAL (5, .1)



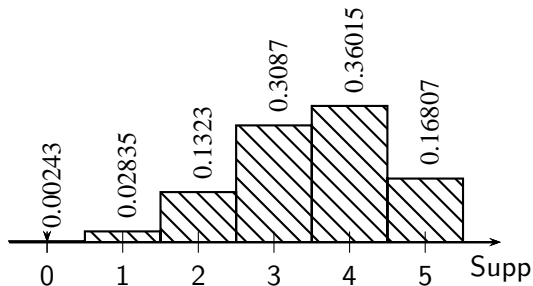
BINOMIAL (5, .3)



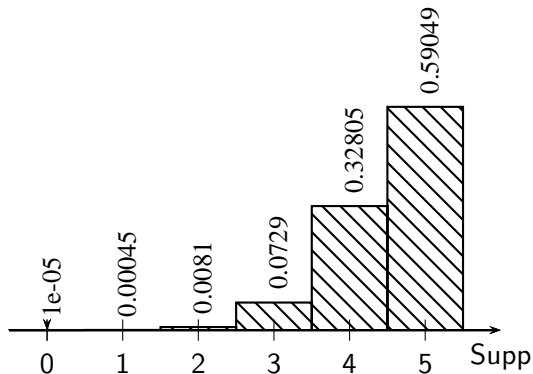
BINOMIAL (5, .5)



BINOMIAL (5, .7)



BINOMIAL (5, .9)



[◀ Examples](#)

Examples of (Univariate) Continuous Distributions

[▶ Uniform](#)[▶ Normal](#)[▶ Skip to Moments](#)

UNIFORM $[A, B]$

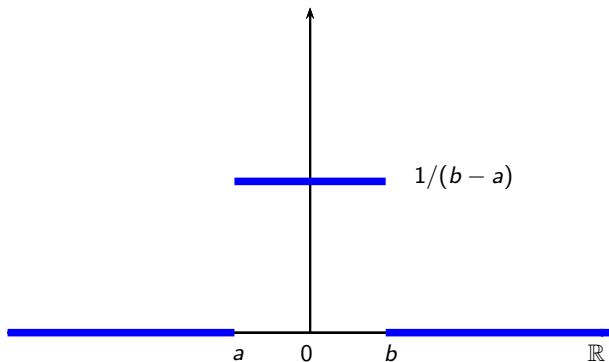
- ★ The uniform distribution with parameters $[a, b]$ has p.d.f.

$$f(z) = \frac{1}{b-a} \mathbf{1}_{\{z \in [a, b]\}}$$

- ★ and support $[a, b]$

P.D.F. OF THE UNIFORM DISTRIBUTION ON $[a, b]$

$$f(z) = \begin{cases} 0 & \text{if } z < a \\ 1/(b-a) & \text{if } z \in [a, b] \\ 0 & \text{if } z > b \end{cases}$$



NORMAL DISTRIBUTION (μ, σ^2)

- ★ The normal distribution with parameters (μ, σ^2) has p.d.f.

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(z - \mu)^2\right)$$

- ★ and support \mathbb{R}

NORMAL DISTRIBUTION (μ, σ^2)

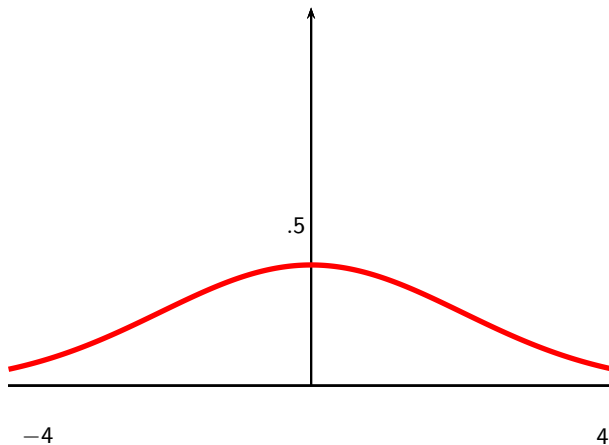
★ $f(z) > 0$ ✓

★ How do we know that:

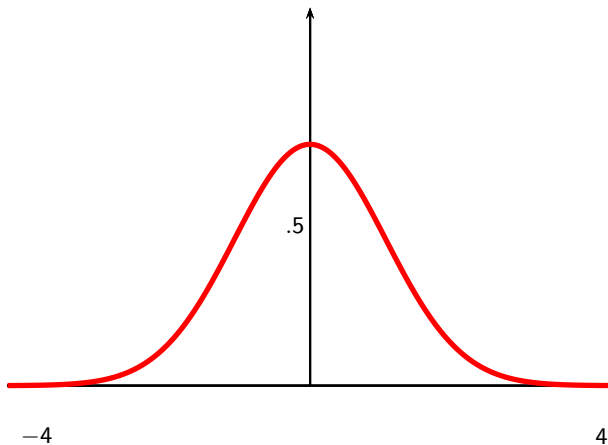
$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(z-\mu)^2\right) dz = 1?$$

Euler-Poisson Integral/Gaussian Integral

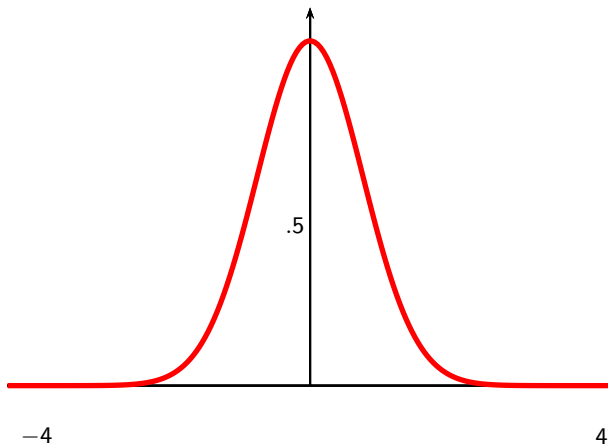
$$N(0, 1)$$



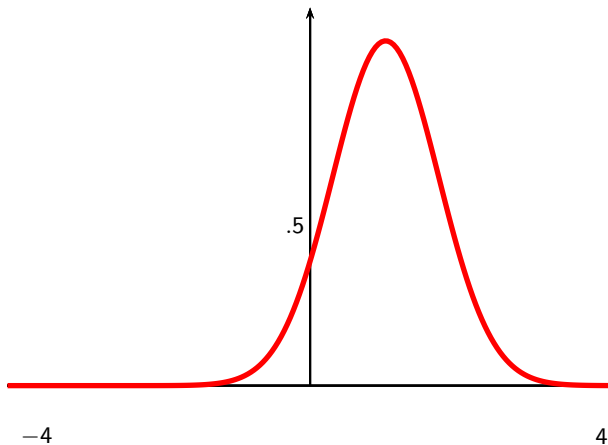
$$N(0, .5)$$



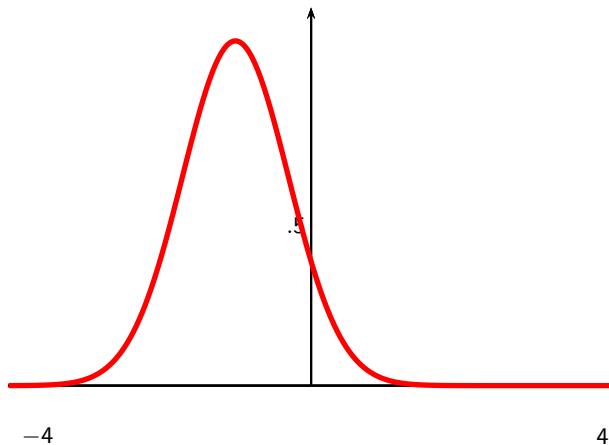
$$N(0, .35)$$



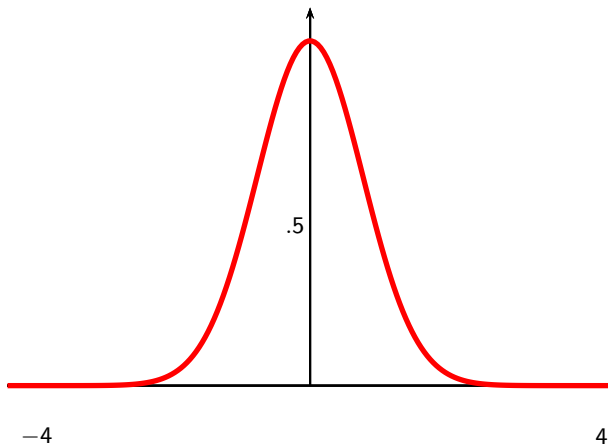
$$N(.5, .35)$$

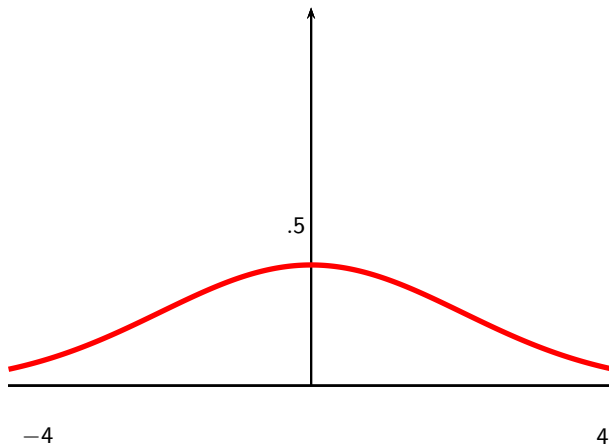


$$N(-5, .35)$$



$$N(0, .35)$$



$N(0,1)$ 

Moments

MEAN OF A DISCRETE R.V.

- ★ Let X be a discrete r.v. with support S and p.m.f. \mathbb{P}_X
- ★ The mean or expected value of X is defined as:

$$\mathbb{E}_{\mathbb{P}_X}[X] \equiv \mu \equiv \sum_{x_n \in S} x_n \mathbb{P}_X[X = x_n]$$

- ★ The variance of X is defined as:

$$\mathbb{E}_{\mathbb{P}_X}[(X - \mu)^2] \equiv \sigma^2 \equiv \sum_{x_n \in S} (x_n - \mu)^2 \mathbb{P}_X[X = x_n]$$

MEAN OF A CONTINUOUS TYPE R.V.

★ For a continuous-type random variable $g(X)$, where $X \sim f_X$:

$$\mathbb{E}_{f_X}[X] \equiv \int_{-\infty}^{\infty} g(z)f_X(z)dz$$

★ Therefore, the mean and the variance of X are given by:

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} zf_X(z)dz \\ \sigma^2 &= \int_{-\infty}^{\infty} (z - \mu)^2 f_X(z)dz\end{aligned}$$

Examples

► Mean of a Uniform $[a,b]$

► Second Moment of a Uniform $[a,b]$

► Skip to Moment Generating Function

$U[A,B]: \mu$

★ $\mu = \frac{b+a}{2}$

★ $f(z) = \frac{1}{b-a} \mathbf{1}\{z \in [a, b]\}$

★ Derivation:

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} z f(z) dz = \frac{1}{b-a} \int_a^b z dz \\ &= \frac{1}{2} \frac{1}{b-a} z^2 \Big|_a^b \\ &= \frac{1}{2} \frac{1}{b-a} (b^2 - a^2) \\ &= \frac{b+a}{2} \end{aligned}$$

$$U[A,B]: \mathbb{E}[X^2]$$

$$\star \mathbb{E}[X^2] = \frac{b^2+ab+a^2}{3}$$

★ Derivation:

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} z^2 f(z) dz = \frac{1}{b-a} \int_a^b z^2 dz \\ &= \frac{1}{3} \frac{1}{b-a} z^3 \Big|_a^b \\ &= \frac{1}{3} \frac{1}{b-a} (b^3 - a^3) \\ &= \frac{b^2 + ab + a^2}{3}\end{aligned}$$

Hence, σ^2 is given by:

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} = \frac{(b-a)^2}{12}$$

Moment Generating Function

MGF

- ★ The \mathbb{R} -valued r.v. X has moment generating function $m_X(\cdot)$ if

$$m_X(t) \equiv E_F \left[\exp(tX) \right] < \infty, \quad \forall t \in (-\epsilon, \epsilon).$$

- ★ The k -th moment of X is the k -th derivative of $m_X(t)$ at $t = 0$.

EXAMPLE

★ $X \sim \text{Bernoulli}(p)$

$$\begin{aligned}m_X(t) &= E_F\left[\exp(tX)\right] \\ &= p \exp(t) + (1-p) \exp(0)\end{aligned}$$

★ Moments:

$$E_F[X] = p$$

$$E_F[X^2] = p$$

$$\vdots$$

$$E_F[X^k] = p$$