

Problem Set 1 (Lectures 1 and 2)

1. (The σ -algebra generated by a collection of sets, **Optional Problem**) Let Ω be an arbitrary non-empty set and let A be a collection of elements of 2^Ω . Define

$$F^*(A) \equiv \{\mathcal{F} \mid \mathcal{F} \text{ is a } \sigma\text{-algebra of } \Omega \text{ containing } A\}$$

- i) Show that $F^*(A)$ is non-empty.
- ii) Let $\sigma(A)$ denote the intersection over all the σ -algebras contained in $F^*(A)$. Show that $\sigma(A)$ is a σ -algebra.

COMMENT: Skip this if it does not sound interesting to you.

OPTIONAL: It turns out that there is no other σ -algebra \mathcal{F} such that $A \subseteq \mathcal{F}$ and $\mathcal{F} \subset \sigma(A)$ (you can also show this if you are interested). The set $\sigma(A)$ is the (unique) smallest σ -algebra containing A .

2. (**Optional Problem**) Show that σ -algebra is “closed” under countable intersections. That is, if $F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ then $\cap_{n=1}^\infty F_n \in \mathcal{F}$.
3. (Properties of a probability measure) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that:
- (a) $\mathbb{P}(F_1 \cup F_2) = \mathbb{P}(F_1) + \mathbb{P}(F_2) - \mathbb{P}(F_1 \cap F_2)$ for any $F_1, F_2 \in \mathcal{F}$
 - (b) $\mathbb{P}(\cup_{n \in \mathbb{N}} F_n) \leq \sum_{n \in \mathbb{N}} \mathbb{P}(F_n)$ for any countable collection $\{F_n\}$

COMMENT: These are useful properties implied by the definition of probability measure. We will use some of them throughout the course.

4. Proof the following Proposition: If F_X is the c.d.f. of a random variable $X : \Omega \rightarrow \mathbb{R}$ then
- (a) F_X is non-decreasing.
 - (b) $\lim_{x \uparrow \infty} F_X(x) = 1$
 - (c) $\lim_{x \downarrow -\infty} F_X(x) = 0$
 - (d) $\lim_{h \rightarrow 0^+} F(x+h) = F(x)$

COMMENT: Combined with the optional part below, this gives a full characterization of how c.d.f.s for real-valued random variables can look.

(OPTIONAL) Furthermore, if F is a function satisfying 1,2,3,4, then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X : \Omega \rightarrow \mathbb{R}$ such that F coincides with F_X .

5. (Moments of some common distributions) Use the definitions of expectations provided in class to solve the following problems:

- (a) Show that if X is a Bernoulli random variable with parameter p , then $\mathbb{E}_F[X] = p$ and $\mathbb{E}_F[(X - p)^2] = p(1 - p)$
- (b) Show that if X is a Normal Random variable with parameters μ and σ^2 then $\mathbb{E}_F[X] = \mu$ and $\mathbb{E}_F[(X - \mu)^2] = \sigma^2$.¹
- (c) Show that if X is a Pareto distribution with parameters (x_m, α) , then for any $n \geq \alpha$, $\mathbb{E}_F[X^n] = \infty$.
- (d) Show that the moment generating function of a Normal random variable with parameters μ and σ^2 is given by:

$$\mu_X(t) = \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right)$$

- (e) Show that if $X \sim \mathcal{N}(0, 1)$, then the random variable $Y : \mathbb{R} \rightarrow \mathbb{R}$ given by $\mu + \sigma X$ has the c.d.f. of Normal random variable with parameters (μ, σ^2) .

6. (95% of the mass within 1.96 standard deviations) Let $X \sim \mathcal{N}(0, 1)$. Convince yourself that

$$\mathbb{P}_X(|X| < 1.96) = \int_{-1.96}^{1.96} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx$$

and convince yourself that it is hard to give an analytic solution for this integral.

- a Go to matlab and use the command `randn(10000,1)` to get a vector with 10,000 realizations of a standard normal random variable. Report the share of the realizations that fall in the interval $[-1.96, 1.96]$.
- b Now, suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$. Use the result in a) to give an approximation for the value of

$$\mathbb{P}_X(|X - \mu| < 1.96\sigma)$$

¹Here you can use the fact that $\int_{-\infty}^{\infty} u(1/\sqrt{2\pi})e^{-u^2/2} du = 0$ and $\int_{-\infty}^{\infty} u^2(1/\sqrt{2\pi})e^{-u^2/2} du = 1$.

References