ECONOMETRICS I

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Introduction to Probability and Statistics for Economists
(Ph.D. in Economics, 1st year)

Lectures 1 and 2

Two Main Blocks

1. Probability Theory

2. Mathematical Statistics

Probability Theory

How to model 'randomness'?

Things we will learn in Lecture 1:

⋆ What is a probability space?

1. What is measurable space?
2. What is a probability space?

⋆ What is a random variable?

⋆ What is the ‘distribution’ or ‘law’ of a random variable?

What is a probability space?

$$(\Omega, \mathcal{F}, \mathbb{P})$$

Probability Space: Two components

$$(\Omega, \mathcal{F}, \mathbb{P})$$

1. $(\Omega, \mathcal{F})$ measurable space.

$$\mathcal{F}: \text{ Set of events } (\subseteq \Omega).$$

2. $\mathbb{P} : \mathcal{F} \to [0, 1]$. Probability Measure

'How likely is an event in $\mathcal{F}$'

Measurable Space

See notes.

Probability Measures

- $\star$ $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$ (Normalization)

- $\star$ For any finite collection $A_1, A_2, \ldots A_m$ such that $A_i \cap A_j = \emptyset$

$$\mathbb{P}\Big(\cup_{i=1}^{m} A_i \Big) = \sum_{i=1}^{m} \mathbb{P}(A_i)$$

  This property is called additivity.

- $\star$ If you replace finite by *countably infinite*, Property 2 is called σ-additivity.

Important

Normalization and σ-additivity define a probability measure

What is a random variable?

$$X : \Omega \to S$$

Random Variable

$$X : \Omega \to S$$

- $\star$ Ω : Set of states of the world.
- $\star$ S : Image Space
- $\star$ X : Random Variable

What is the distribution or law of a random variable?

'Induced' Probability of a Random Variable

- ⋆ The probability $\mathbb{P}$ on Ω induces a probability on subsets of S:

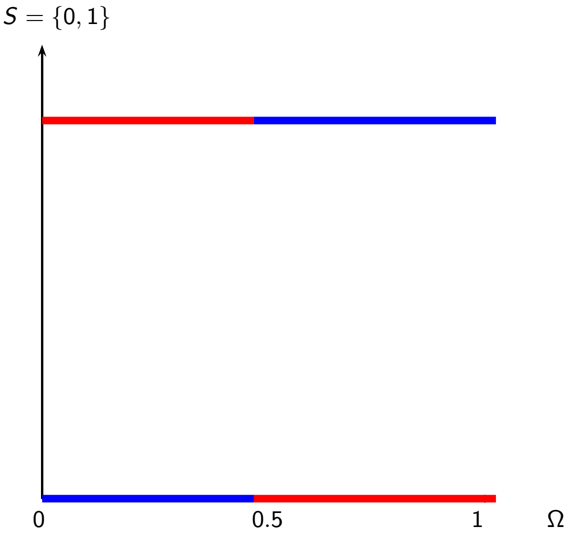
$$\mathbb{P}_X[F] \equiv \mathbb{P}[\,\{\omega \mid X(\omega) \in F\}\,], \quad F \subseteq S$$

- ⋆ How likely are the states of the world in which F occurs?

The induced probability of a random variable is usually called its
DISTRIBUTION OR LAW

Different random variables can induce the same probability on S.

$$P_{X_2}(1) = .5$$



The Cumulative Distribution Function of Real-valued Random Variables

c.d.f of an $\mathbb{R}$-valued random variable

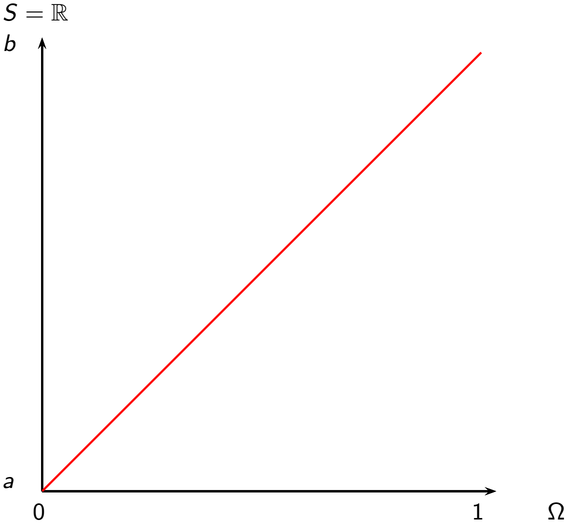
- ⋆ How likely is a realization of the random variable X below x?
- ⋆ The c.d.f. summarizes this information

$$F_X : \mathbb{R} \to [0, 1]$$

$$F_X(x) \equiv \mathbb{P}\Big\{\omega \in \Omega \mid X(\omega) \leq x\Big\}$$

Examples of c.d.f.

$$X(\omega) = a + \omega[b - a]$$



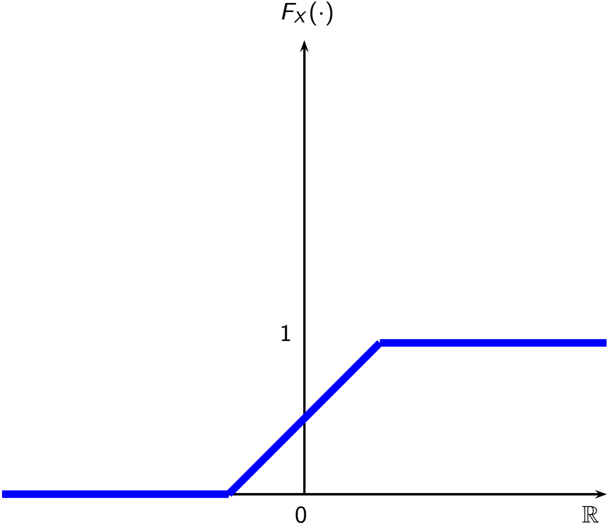
If $x \in [a, b]$:

$$\begin{aligned}
\mathbb{P}\{\omega \in \Omega \mid X(\omega) \leq x\} &= \mathbb{P}\{\omega \in \Omega \mid a + \omega(b-a) \leq x\} \\
&= \mathbb{P}\{\omega \in \Omega \mid \omega(b-a) \leq x - a\} \\
&= \mathbb{P}\{\,[0, x - a/(b-a)]\,\} \\
&= x - a/(b-a)
\end{aligned}$$

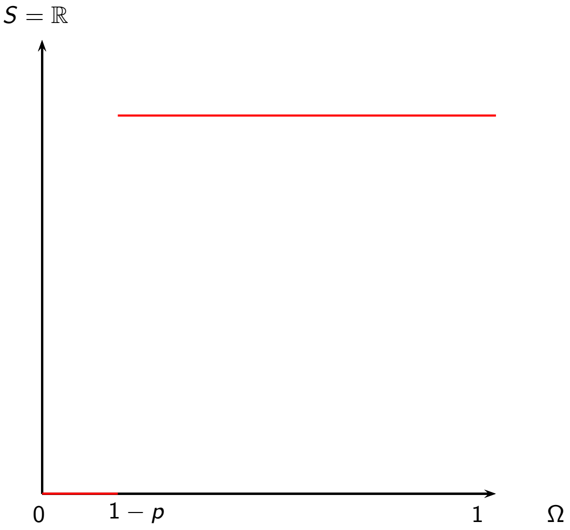
Hence,

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < a \\ (x-a)/b-a & \text{if} \quad x \in [a, b) \\ 1 & \text{if} \quad x \geq b \end{cases}$$

UNIFORM DISTRIBUTION ON $[a, b]$

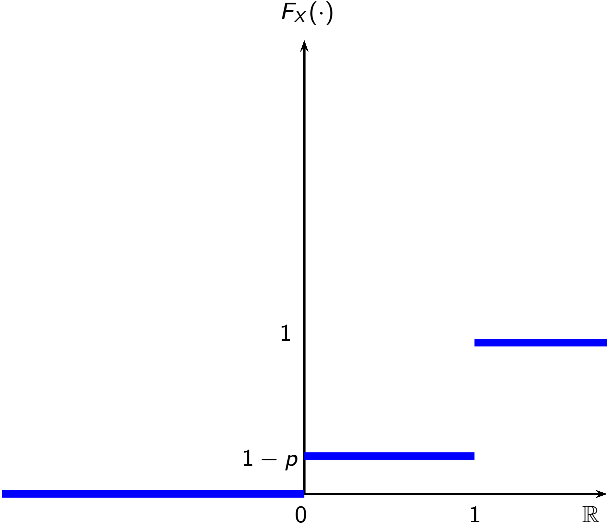


$$X(\omega) = \mathbf{1}[\omega \geq 1 - p]$$

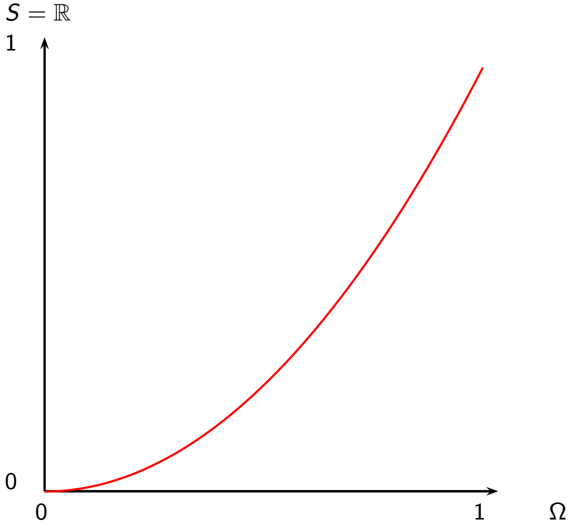


$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 - p & \text{if} \quad x \in [0, 1) \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

Bernoulli Distribution with parameter p



$$X(\omega) = \omega^2$$



If $x \in [0, 1]$:

$$\begin{aligned}
\mathbb{P}\{\omega \in \Omega \mid X(\omega) \leq x\} &= \mathbb{P}\{\omega \in \Omega \mid \omega^2 \leq x\} \\
&= \mathbb{P}\{\omega \in \Omega \mid \omega \leq \sqrt{x}\} \\
&= \mathbb{P}\{\, [0, \sqrt{x}] \,\} \\
&= \sqrt{x}
\end{aligned}$$

Hence,

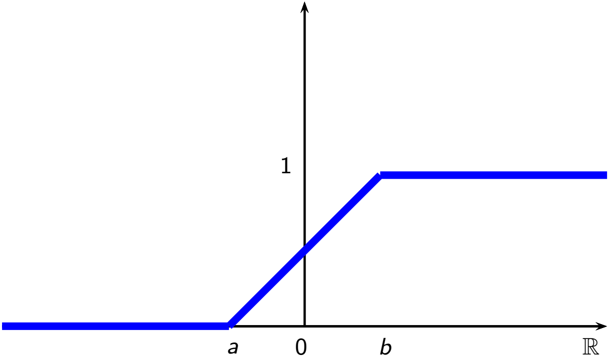
$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \sqrt{x} & \text{if} \quad x \in [0, 1) \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

Discrete and Continuous Type of Real-Valued Random Variables

Let's take a look to the c.d.f.s we have computed

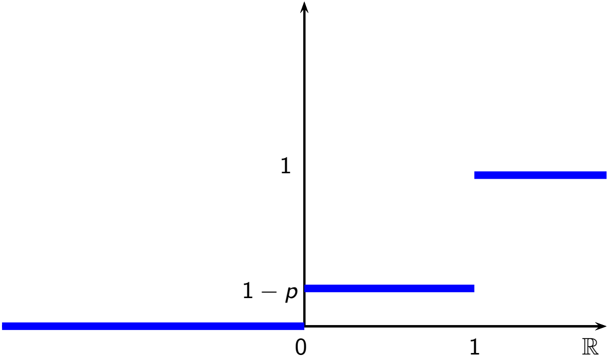
Uniform Distribution on $[a, b]$

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < a \\ (x-a)/b-a & \text{if} \quad x \in [a, b) \\ 1 & \text{if} \quad x \geq b \end{cases}$$



Bernoulli Distribution with parameter p

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1-p & \text{if} \quad x \in [0,1) \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$



What are the common properties?

1. F_X is non-decreasing
2. $\lim_{x\to\infty} F_X(x) = 1$
3. $\lim_{x\to-\infty} F_X(x) = 0$
4. $\lim_{h\to 0^+} F_X(x+h) = F_X(x)$

In fact, these 4 properties characterize the induced c.d.f. of a real-valued random variable!

Discrete Distributions / Discrete r.v.s

DISTRIBUTION: Disributions for which $\exists$ a countable set

$$\text{Supp} = \{x_1, x_2, \ldots\}, x_i \in \mathbb{R},$$

such that

a) $\mathbb{P}_X(X = x_i) > 0 \quad \forall \quad x_i \in \text{Supp}$

b) $\sum_{x_i \in \text{Supp}} \mathbb{P}_X(X = x_i) = 1$

are called discrete.

P.M.F.

We will identify discrete distributions/r.v.s by its support and its p.m.f.

Uniform[a,b]

- ⋆ Note that the $U[a, b]$ is not discrete. Why?
- ⋆ $P_X(X = x) = 0 \quad \forall \quad x \in \mathbb{R}$.
- ⋆ However, $U[a, b]$ has a special property as well!

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{b-a}\mathbf{1}\{z \in [a, b]\}dz$$

(Absolutely) Continuous Distributions

Random Variables for which

$$F_X(x) = \int_{-\infty}^{x} f(z)dz$$

for some $f(z) \geq 0 \ \forall z \in \mathbb{R}$ are called:

(Absolutely) Continuous

The function $f(z)$ is called

Probability Density Function (p.d.f.)

We will identify continuous distributions/r.v.s by its p.d.f.

Examples of (Univariate) Discrete Distributions

Discrete Finite Support

Bernoulli Distribution $(p),\ p \in (0,1)$

$\star$ The Bernoulli distribution with parameters p has support:

$$\text{Supp} = \{0, 1\}$$

and p.m.f. given by:

$$\mathbb{P}_X(X = x) = p^x (1-p)^{1-x} \quad x \in \{0, 1\}$$

How do we know it is a p.m.f.?

Two parts:

a) $\mathbb{P}_X(X = x) > 0 \quad \forall\, x \in \{0, 1\}$. Easy to verify:

$$p^x(1-p)^{1-x} > 0$$

b)

$$\sum_{x \in \{0,1\}} p^x(1-p)^{1-x} = (1-p) + p$$

Binomial Distribution (n, p), $n \in \mathbb{N}, p \in (0, 1)$

⋆ The binomial distribution with parameters (n, p) has support:

$$\text{Supp} = \{0, 1, 2, \ldots n\}$$

and p.m.f. given by:

$$\mathbb{P}_X(X = x) \equiv \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \quad x \in \text{Supp}$$

How do we know it is a p.m.f.?

Two parts:

a) $\mathbb{P}_X(X = x) > 0 \quad \forall\, x \in \{0, 1, 2, \dots n\}$. Easy to verify:

$$\frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} > 0$$

b)

$$\sum_{x \in \{0,1,\dots n\}} \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = 1?$$

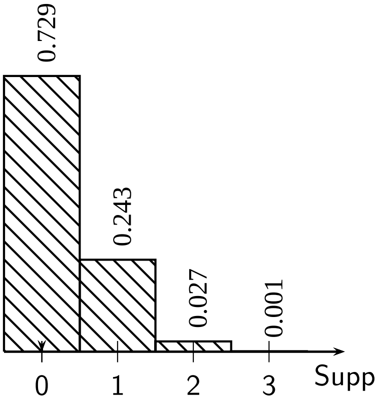
How do we know it is a p.m.f.?

Use the Binomial Theorem

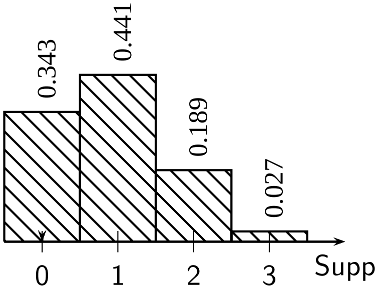
$$(a+b)^n = \sum_{x\in\{0,1,\ldots n\}} \frac{n!}{(n-x)!x!} a^x b^{n-x} = 1$$

$$a = p, \quad b = 1-p$$

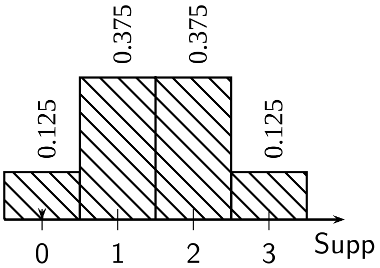
BINOMIAL (3, .1)



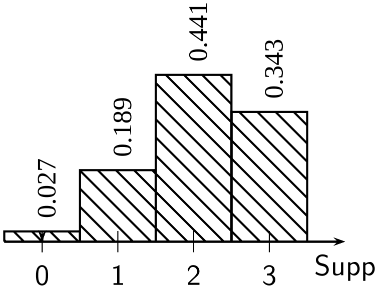
BINOMIAL (3, .3)



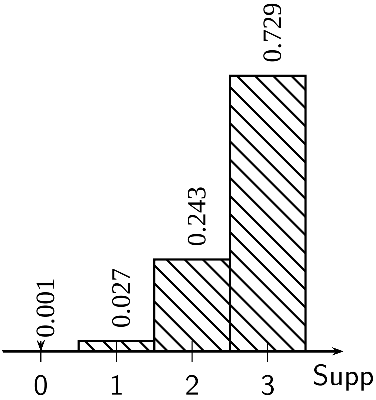
BINOMIAL (3, .5)



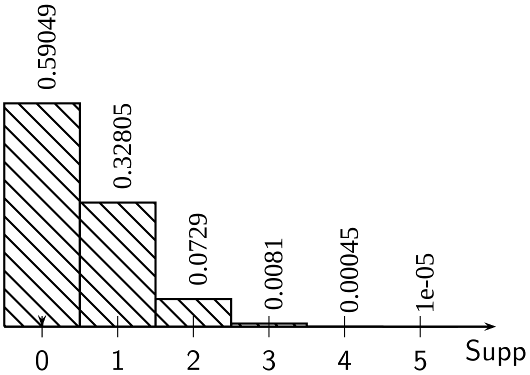
Binomial (3, .7)



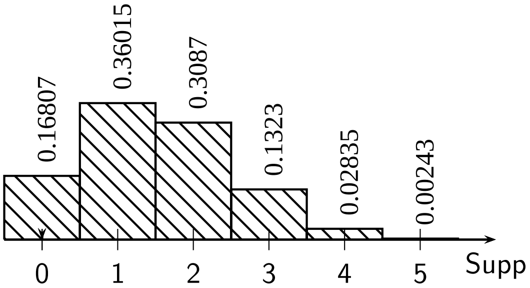
Binomial (3, .9)



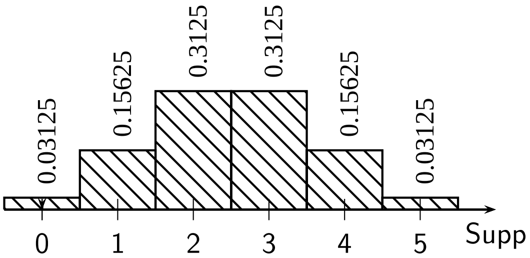
BINOMIAL (5, .1)



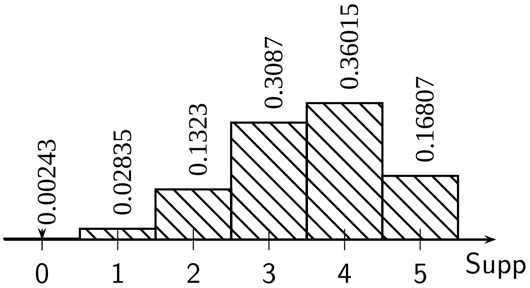
Binomial (5, .3)



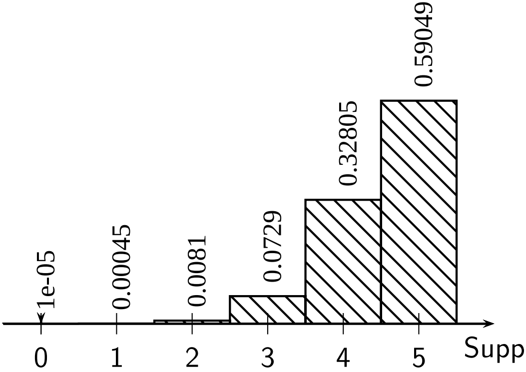
BINOMIAL (5, .5)



Binomial (5, .7)



Binomial (5, .9)



Examples of (Univariate) Continuous Distributions

(Absolutely) Continuous Real-valued r.v.s

Random Variables for which

$$F_X(x) = \int_{-\infty}^{x} f(z)dz$$

for some $f(z) \geq 0 \ \forall z \in \mathbb{R}$ are called:

(Absolutely) Continuous

The function $f(z)$ is called

Probability Density Function (p.d.f.)

Properties of a p.d.f.

⋆ A function $f : \mathbb{R} \to \mathbb{R}$ is a p.d.f. if

a) $f(z) \geq 0$

b) $\int_{\{z \in \mathbb{R} \mid f(z) > 0\}} f(z) dz = 1$

⋆ a), b) $\implies$ $F(x) = \int_{-\infty}^{x} f(z) dz$ is a c.d.f.

⋆ The set

$$\{z \in \mathbb{R} \mid f(z) > 0\} \subseteq \mathbb{R}$$

is called the support of the continuous r.v.

From p.d.f. to probabilities

Let X be a real-valued random variable with p.d.f. $f(z)$

- ⋆ $\mathbb{P}_X[X \leq a] = \int_{-\infty}^{a} f(z)dz$
- ⋆ $\mathbb{P}_X[a \leq X \leq b] = \int_{a}^{b} f(z)dz$
- ⋆ $\mathbb{P}_X[X > a] = \int_{a}^{\infty} f(z)dz$

Examples of Absolutely Continuous Distributions

Uniform [a,b]

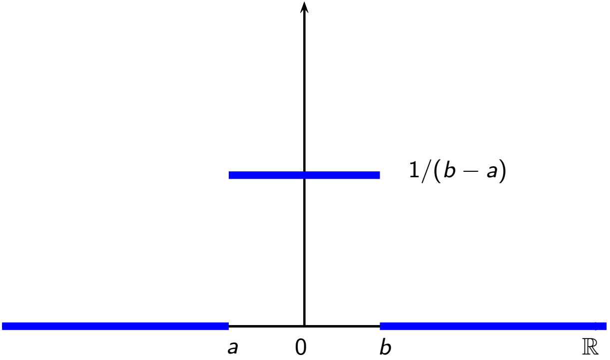
⋆ The uniform distribution with parameters $[a, b]$ has p.d.f.

$$f(z) = \frac{1}{b-a}\mathbf{1}\{z \in [a, b]\}$$

⋆ and support $[a, b]$

p.d.f. of the Uniform Distribution on $[a, b]$

$$f(z) = \begin{cases} 0 & \text{if} \quad z < a \\ 1/(b-a) & \text{if} \quad z \in [a, b] \\ 0 & \text{if} \quad z > b \end{cases}$$



Normal Distribution (μ, σ^2)

- ⋆ The normal distribution with parameters (μ, σ^2) has p.d.f.

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(- \frac{1}{2\sigma^2}(z-\mu)^2\Big)$$

- ⋆ and support $\mathbb{R}$

Normal Distribution (μ, σ^2)

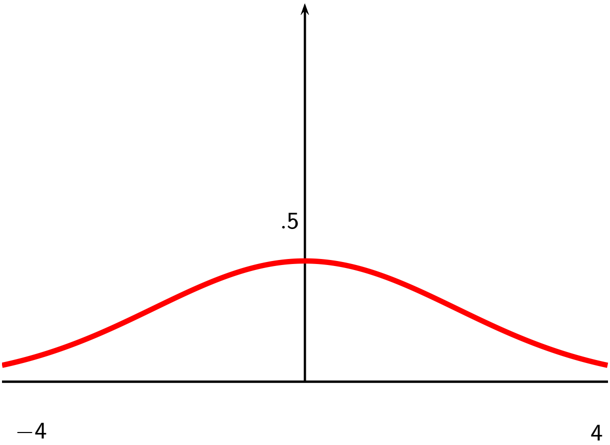
- $\star$ $f(z) > 0 \checkmark$

- $\star$ How do we know that:

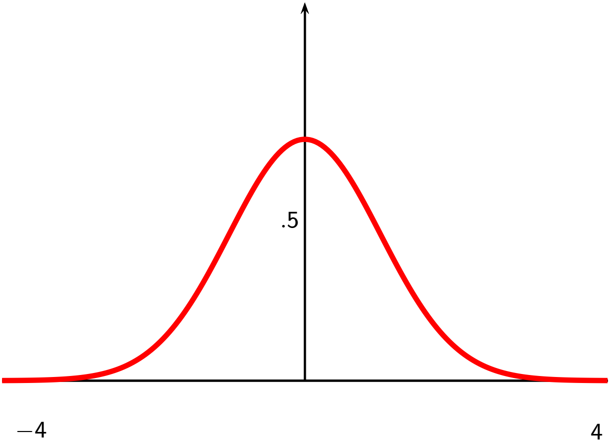
$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{1}{2\sigma^2}(z-\mu)^2 \Big) = 1?$$

Euler-Poisson Integral/Gaussian Integral

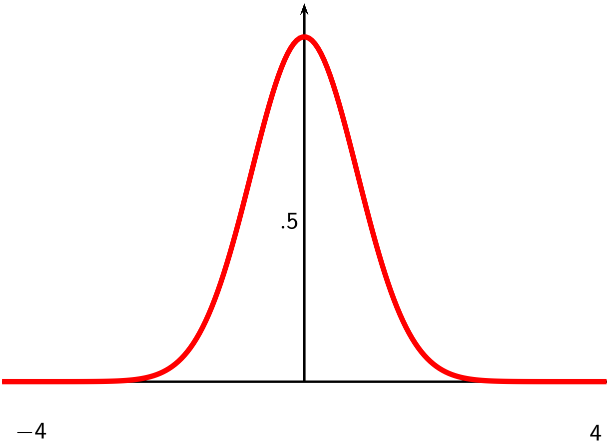
$N(0,1)$



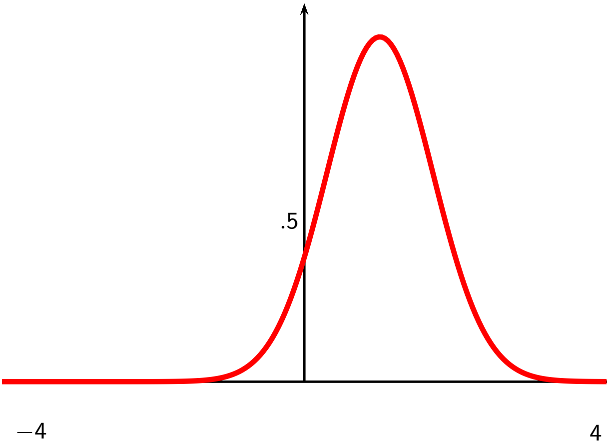
$N(0, .5)$



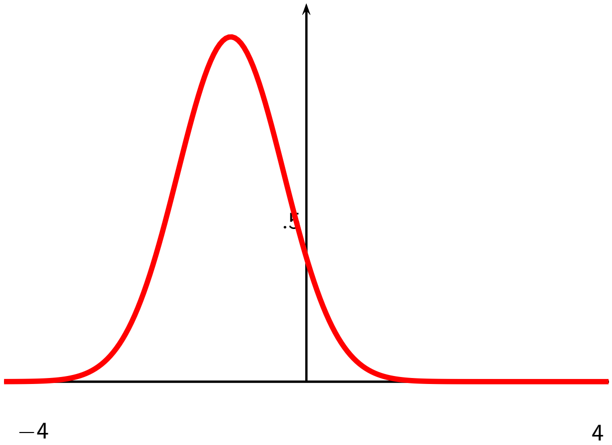
$N(0, .35)$



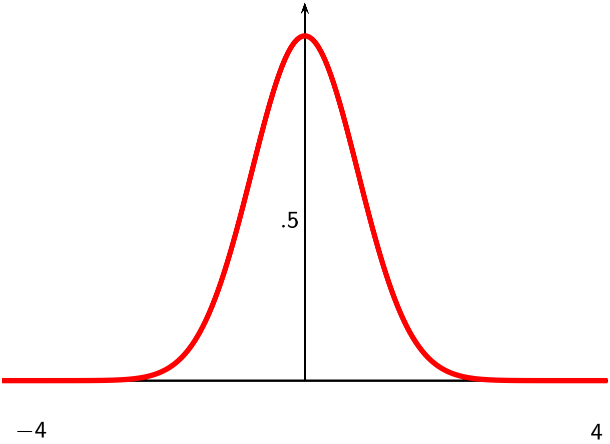
$N(.5, .35)$



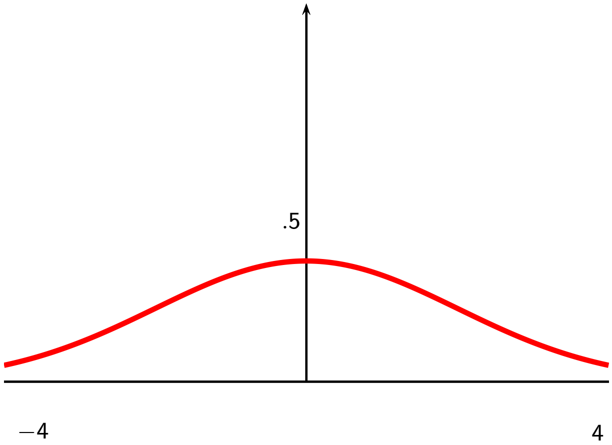
$N(-5, .35)$



$N(0, .35)$



N(0,1)



Mean (μ) and variance (σ^2)

Definiton: Mean of a discrete r.v.

- ⋆ Let X be a discrete r.v. with $(S, \mathbb{P}_X)$
- ⋆ The mean or expected value of X is defined as:

$$\mathbb{E}_{\mathbb{P}_X}[X] \equiv \mu \equiv \sum_{x_n \in S} x_n \, \mathbb{P}_X[X = x_n]$$

- ⋆ The variance of X is defined as:

$$\mathbb{E}_{\mathbb{P}_X}[(X-\mu)^2] \equiv \sigma^2 \equiv \sum_{x_n \in S} (x_n - \mu)^2 \, \mathbb{P}_X[X = x_n]$$

Let's compute the μ and σ^2 for some of the discrete distributions we have introduced

Bernoulli (p)

- ⋆ S={0,1}
- ⋆ $\mathbb{E}_{\mathbb{P}_X}[X] \equiv \mu = (0)(1-p) + 1(p) = p$
- ⋆ $\mathbb{E}_{\mathbb{P}_X}[(X-\mu)^2] \equiv \sigma^2 =$

$$\begin{aligned}(0-p)^2(1-p) + (1-p)^2 p &= p(1-p)[p + (1-p)] \\ &= p(1-p)\end{aligned}$$

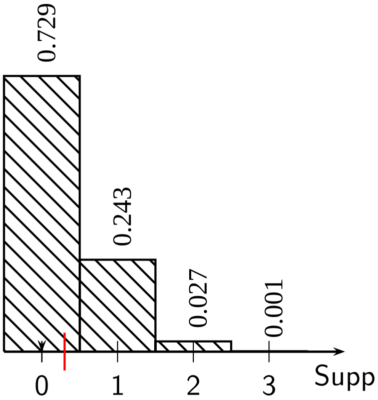
Binomial (n,p)

- $\star$ S=$\{0,1, \ldots n\}$
- $\star$ $\mathbb{E}_{\mathbb{P}_X}[X] \equiv \mu = \ldots$

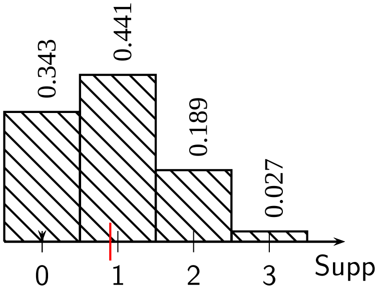
$$\begin{aligned}
\sum_{m=0}^{n} \frac{mn!p^m(1-p)^{n-m}}{(n-m)!m!} &= \sum_{m=1}^{n} \frac{n!p^m(1-p)^{n-m}}{(n-m)!m-1!} \\
&= np\sum_{m=1}^{n} \frac{n-1!p^{m-1}(1-p)^{n-m}}{(n-m)!m-1!} \\
&= np
\end{aligned}$$

- $\star$ $\mathbb{E}_{\mathbb{P}_X}[(X-\mu)^2] \equiv \sigma^2 = np(1-p)$

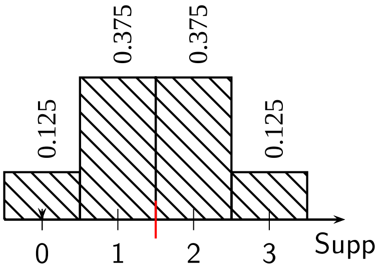
Binomial $(3, .1)$ $\mu = .3, \sigma^2 = .27$



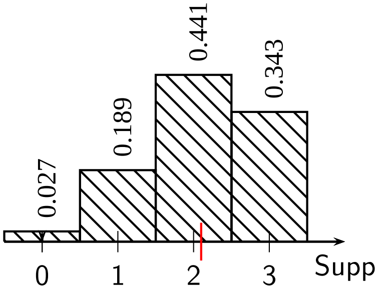
BINOMIAL $(3, .3)$, $\mu = .9, \sigma^2 = .63$



BINOMIAL $(3, .5)$, $\mu = 1.5, \sigma^2 = .75$



BINOMIAL $(3, .7),\ \mu = 2.1, \sigma^2 = .63$



Expected Value-Continuous Type

- ⋆ For a discrete random variable X we have defined the **expected value of a transformation:**

$$\sum_{i=1}^{n} u(x_i)\mathbb{P}_X(X = x_i)$$

- ⋆ The analogous definition for a continuous-type random variable X with p.d.f. f_X is:

$$\mathbb{E}_{f_X}[X] = \int_{-\infty}^{\infty} u(z) f_X(z) dz$$

μ and σ

⋆ Therefore, the mean and the variance of a continuous-type r.v. X are given by:

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} z f_X(z) dz \\
\sigma^2 &= \int_{-\infty}^{\infty} (z-\mu)^2 f_X(z) dz
\end{aligned}$$

Let's compute μ and σ^2 for the uniform $[a, b]$

U[A,B]: μ

- $\star$ $f(z) = \frac{1}{b-a}\mathbf{1}\{z \in [a,b]\}$
- $\star$ μ is given by:

$$\begin{aligned}\mu = \int_{-\infty}^{\infty} zf(z)dz &= \frac{1}{b-a}\int_a^b zdz \\ &= \frac{1}{2}\frac{1}{b-a}z^2\Big]_a^b \\ &= \frac{1}{2}\frac{1}{b-a}(b^2 - a^2) \\ &= \frac{b+a}{2}\end{aligned}$$

U[A,B]: σ^2

- Note that

$$\mathbb{E}_f[(X-\mu)^2] = \mathbb{E}_f[X^2] - 2\mu\mathbb{E}_f[X] + \mu^2$$

- σ^2 is given by:

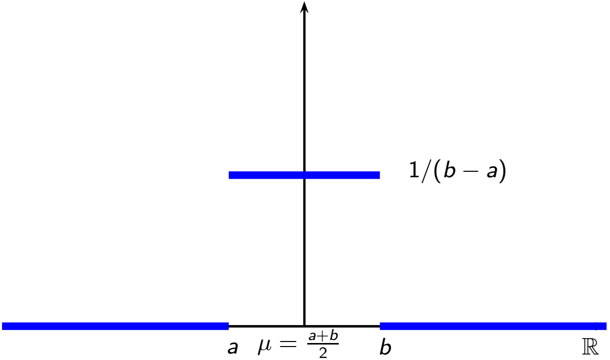
$$\begin{aligned}
\mathbb{E}[X^2] = \int_{-\infty}^{\infty} z^2 f(z)dz &= \frac{1}{b-a}\int_a^b z^2 dz \\
&= \frac{1}{3}\frac{1}{b-a}z^3\Big]_a^b \\
&= \frac{1}{3}\frac{1}{b-a}(b^3-a^3) \\
&= \frac{b^2+ab+a^2}{3}
\end{aligned}$$

Hence, σ^2 is given by:

$$\frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} = \frac{(b-a)^2}{12}$$

$\mathrm{U}[a,b] : \mu$

$$f(z) = \begin{cases} 0 & \text{if} \quad z < a \\ 1/(b-a) & \text{if} \quad z \in [a,b] \\ 0 & \text{if} \quad z > b \end{cases}$$



Let's now compute the expectation and variance of the normal distribution

$\mathcal{N}(\mu, \sigma^2)$

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(z-\mu)^2 \right)$$

Note first that:

$$\int_{-\infty}^{\infty} z \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2} z^2 \right) dz = 0 \ \text{(why?)}$$

$\mathcal{N}(\mu, \sigma^2)$

Let

$$u = z - \mu$$

Note that:

$$\begin{aligned}\int_{-\infty}^{\infty} z \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{1}{2\sigma^2}(z-\mu)^2\Big) dz &= \\ \int_{-\infty}^{\infty} (u+\mu) \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{1}{2\sigma^2}u^2\Big) du &= \mu\end{aligned}$$

Moment Generating Function

MGF

- ⋆ The real-valued random variable X is said to have a moment generating function $m_X : (-\epsilon, \epsilon) \to [0, \infty]$ if

$$E_F\left[\exp\left(tX\right)\right] < \infty, \quad \forall\, t \in (-\epsilon, \epsilon).$$

- ⋆ If the MGF is differentiable at $t = 0$ we can obtain the k-th moment of X as the k-th derivative of the MGF evaluated at $t = 0$.

Example

⋆ $X \sim \text{Bernoulli}(p)$

$$\begin{aligned} m_X(t) &= E_F\Big[\exp\Big(tX\Big)\Big] \\ &= p\exp(t) + (1-p)\exp(0) \end{aligned}$$

⋆ Moments:

$E_F[X] = p$

$E_F[X^2] = p$

$\vdots$

$E_F[X^k] = p$