1 Probability

1.1 Probability Space

- *Sample space* Ω is the collection of all possible outcomes.
- An *event* A is a subset of Ω.
- A σ-field, denoted by $\mathcal{F}$, is a collection of events such that: (i) $\emptyset \in \mathcal{F}$; (ii) if an event $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$; (iii) if $A_i \in \mathcal{F}$ for $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$.
- $(\Omega, \mathcal{F})$ is called a *measure space*.
- A function $\mu : \mathcal{F} \mapsto [0, \infty]$ is called a *measure* if it satisfies (i) $\mu(A) \geq 0$ for all $A \in \mathcal{F}$; (ii) if $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, are mutually disjoint, then $\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i)$
- If $\mu(\Omega) = 1$, we call μ a *probability measure*. A probability measure is often denoted as P.
- $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

1.2 Random Variable

- A function $X : \Omega \mapsto \mathbb{R}$ is $(\Omega, \mathcal{F}) \setminus (\mathbb{R}, \mathcal{R})$ *measurable* if $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$ for any $B \in \mathcal{R}$, where $\mathcal{R}$ is the Borel σ-field on the real line. *Random variable* is an alternative name for a measurable function.
- Discrete random variable: the set $\{X(\omega) : \omega \in \Omega\}$ is finite or countable.
- Continuous random variable: the set $\{X(\omega) : \omega \in \Omega\}$ is uncountable.
- $P_X : \mathcal{R} \mapsto [0, 1]$ is also a probability measure if defined as $P_X(B) = P\left(X^{-1}(B)\right)$ for any $B \in \mathcal{R}$. This P_X is called the probability measure *induced* by the measurable function X .

1.3 Distribution Function

- (Cumulative) distribution function

$$F(x) = P(X \leq x) = P(\{\omega \in \Omega : X(\omega) \leq x\}).$$

- Properties of CDF: $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$, non-decreasing, and right-continuous

$$\lim_{y \to x^+} F(y) = F(x).$$

- Probability density function (PDF): if there exists a function f such that for all x,

$$F\left(x\right)=\int_{-\infty}^{x}f\left(y\right)dy,$$

then f is called the PDF of X.

- Properties: $f\left(x\right)\geq0$. $\int_{a}^{b}f\left(x\right)dx=F\left(b\right)-F\left(a\right)$

1.4 Examples

- Binary, Poisson, uniform, normal, χ^{2}, t, F.
- Parametric distribution verses nonparametric distribution.
- Implementation in `R`: `d` for density, `p` for probability, `q` for quantile, and `r` for random variable. For instance, **`dnorm`**, **`pnorm`**, **`qnorm`**, and **`rnorm`**.