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1 Panel Data

A panel dataset tracks the same individuals across time $t = 1, \ldots, T$. The potential endogeneity of the regressors motivates the panel data models. We assume the observations are i.i.d. across $i = 1, \ldots, n$, while we allow some form of dependence within a group across $t = 1, \ldots, T$ for the same i. We maintain the linear equation

$$y_{it} = \beta_1 + x_{it}\beta_2 + u_{it},\ i = 1, \ldots, n;\ t = 1, \ldots, T \tag{1}$$

where $u_{it} = \alpha_i + \epsilon_{it}$ is called the *composite error*. Note that α_i is the time-invariant unobserved heterogeneity, while ϵ_{it} varies across individuals and time periods.

2 Fixed Effect

If $\mathrm{cov}(\alpha_i, x_{it}) = 0$, OLS is consistent for (1); otherwise the consistency breaks down. The fixed effect model allows α_i and x_{it} to be arbitrarily correlated. The trick to regain consistency is to eliminate $\alpha_i, i = 1, \ldots, n$. The rest of this section develops the consistency and asymptotic distribution of the *within estimator*, the default fixed-effect (FE) estimator. The within estimator transforms the data by subtracting all the observable variables by the corresponding group means. Averaging the T equations in (1) for the same i, we have

$$\overline{y}_i = \beta_1 + \overline{x}_i\beta_2 + \bar{u}_{it} = \beta_1 + \overline{x}_i\beta_2 + \alpha_i + \bar{\epsilon}_{it}. \tag{2}$$

where $\overline{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}$. Subtracting (2) from (1) gives

$$\tilde{y}_{it} = \tilde{x}_{it}\beta_2 + \tilde{\epsilon}_{it} \tag{3}$$

where $\tilde{y}_{it} = y_{it} - \overline{y}_i$. Running OLS with the demeaned data, we obtain the within estimator

$$\widehat{\beta}_2^{FE} = \left(\tilde{X}'\tilde{X}\right)^{-1}\tilde{X}'\tilde{y},$$

where $\tilde{y} = (y_{it})_{i,t}$ stacks all the nT observations into a vector, and similarly defined is $\tilde{X}$ as an $nT \times K$ matrix, where K is the dimension of β_2.

We know that OLS in (3) would be consistent if $\mathbb{E}\left[\tilde{\epsilon}_{it}|\tilde{x}_{it}\right] = 0$. Below we provide a sufficient condition, which is often called *strict exogeneity.*

Assumption (FE.1)**.** $\mathbb{E}\left[\epsilon_{it}|\alpha_i, \mathbf{x}_i\right] = 0$ *where* $\mathbf{x}_i = (x_{i1}, \ldots, x_{iT})$.

Its strictness is relative to the contemporary exogeneity $\mathbb{E}\left[\epsilon_{it}|x_{it}\right] = 0$. FE.1 is more restrictive as it assumes that the error ϵ_{it} is mean independent of the past, present and future explanatory variables.

When we talk about the consistency in panel data, typically we are considering $n \to \infty$ while T stays fixed. This asymptotic framework is appropriate for panel datasets with many individuals but only a few time periods.

Lemma (FE consistency)**.** *If FE.1 is satisfied, then* $\widehat{\beta}_2^{FE}$ *is consistent.*

The variance estimation for the FE estimator is a little bit tricky. We assume a homoskedasitcity condition to simplify the calculation. Violation of this assumption changes the form of the asymptotic variance, but does not jeopardize the asymptotic normality.

Assumption (FE.2)**.** $\mathrm{var}\left(\epsilon_i|\mathbf{x}_i\right) = \sigma_\epsilon^2 I_T$.

Under FE.1 and FE.2, $\widehat{\sigma}_\epsilon^2 = \frac{1}{n(T-1)}\sum_{i=1}^n \sum_{t=1}^T \widehat{\tilde{\epsilon}}_{it}^2$ is a consistent estimator of σ_ϵ^2, where $\widehat{\tilde{\epsilon}} = \tilde{y}_{it} - \tilde{x}_{it}\widehat{\beta}_2^{FE}$. Note that the denominator is $n(T-1)$, not nT.

Theorem (FE asymptotic normality)**.** *If FE.1 and FE.2 are satisfied, then*

$$\frac{\left(\tilde{X}'\tilde{X}\right)^{1/2}}{\widehat{\sigma}_\epsilon}\left(\widehat{\beta}_2^{FE} - \beta_2^0\right) \Rightarrow N\left(0, I_K\right).$$

Remark. We implicitly assume some regularity conditions that allow us to invoke a law of large numbers and a central limit theorem. We ignore those technical details here.

3 Random Effect

The random effect estimator pursues efficiency at a knife-edge special case $\mathrm{cov}\left(\alpha_i, x_{it}\right) = 0$. As mentioned above, FE is consistent when α_i and x_{it} are uncorrelated. However, an inspection of the covariance matrix reveals that OLS is inefficient.

The model is again (1), while we assume

Assumption (RE.1)**.** $\mathbb{E}\left[\epsilon_{it}|\alpha_i, \mathbf{x}_i\right] = 0$ *and* $\mathbb{E}\left[\alpha_i|\mathbf{x}_i\right] = 0$.

RE.1 obviously implies $\mathrm{cov}\left(\alpha_i, x_{it}\right) = 0$, so

$$S = \mathrm{var}\left(u_i|\mathbf{x}_i\right) = \sigma_\alpha^2 \mathbf{1}_T \mathbf{1}_T' + \sigma_\epsilon^2 I_T, \text{ for all } i = 1, \ldots, n.$$

Because the covariance matrix is not a scalar multiplication of the identity matrix, OLS is inefficient.

As mentioned before, FE estimation kills all time-invariant regressors. In contrast, RE allows time-invariant explanatory variables. Let us rewrite (1) as

$$y_{it} = w_{it}\boldsymbol{\beta} + u_{it},$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2')'$ and $w_{it} = (1, x_{it})$ are $K+1$ vectors, i.e., $\boldsymbol{\beta}$ is the parameter including the intercept, and w_{it} is the explanatory variables including the constant. Had we known S, the GLS estimator would be

$$\widehat{\boldsymbol{\beta}}^{RE} = \left(\sum_{i=1}^{n} \mathbf{w}_i' S^{-1} \mathbf{w}_i\right)^{-1} \sum_{i=1}^{n} \mathbf{w}_i' S^{-1} \mathbf{y}_i = \left(W' \mathbf{S}^{-1} W\right)^{-1} W' \mathbf{S}^{-1} y$$

where $\mathbf{S} = I_T \otimes S$. ("$\otimes$" denotes the Kronecker product.) In practice, σ_α^2 and σ_ϵ^2 in S are unknown, so we seek consistent estimators. Again, we impose a simplifying assumption parallel to FE.2.

Assumption (RE.2)**.** $\mathrm{var}\left(\epsilon_i|\mathbf{x}_i, \alpha_i\right) = \sigma_\epsilon^2 I_T$ *and* $\mathrm{var}\left(\alpha_i|\mathbf{x}_i\right) = \sigma_\alpha^2$.

Under this assumption, we can consistently estimate the variances from the residuals $\widehat{u}_{it} = y_{it} - x_{it}\widehat{\boldsymbol{\beta}}^{RE}$. That is

$$\begin{aligned}
\widehat{\sigma}_u^2 &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{u}_{it}^2 \\
\widehat{\sigma}_\alpha^2 &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{r\neq t}\widehat{u}_{it}\widehat{u}_{ir}.
\end{aligned}$$

Again, we claim the asymptotic normality.

Theorem (RE asymptotic normality)**.** *If RE.1 and RE.2 are satisfied, then*

$$\left(W'\widehat{\mathbf{S}}^{-1}W\right)^{1/2}\left(\widehat{\boldsymbol{\beta}}^{RE} - \boldsymbol{\beta}_0\right) \Rightarrow N\left(0, I_{K+1}\right)$$

where $\widehat{\mathbf{S}}$ *is a consistent estimator of* $\mathbf{S}$.