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1 Panel Data

A panel dataset tracks the same individuals across time $t = 1, \dots, T$. The potential endogeneity of the regressors motivates the panel data models. We assume the observations are i.i.d. across $i = 1, \dots, n$, while we allow some form of dependence within a group across $t = 1, \dots, T$ for the same i . We maintain the linear equation

$$y_{it} = \beta_1 + x_{it}\beta_2 + u_{it}, \quad i = 1, \dots, n; \quad t = 1, \dots, T \quad (1)$$

where $u_{it} = \alpha_i + \epsilon_{it}$ is called the *composite error*. Note that α_i is the time-invariant unobserved heterogeneity, while ϵ_{it} varies across individuals and time periods.

2 Fixed Effect

If $\text{cov}(\alpha_i, x_{it}) = 0$, OLS is consistent for (1); otherwise the consistency breaks down. The fixed effect model allows α_i and x_{it} to be arbitrarily correlated. The trick to regain consistency is to eliminate $\alpha_i, i = 1, \dots, n$. The rest of this section develops the consistency and asymptotic distribution of the *within estimator*, the default fixed-effect (FE) estimator. The within estimator transforms the data by subtracting all the observable variables by the corresponding group means. Averaging the T equations in (1) for the same i , we have

$$\bar{y}_i = \beta_1 + \bar{x}_i\beta_2 + \bar{u}_{it} = \beta_1 + \bar{x}_i\beta_2 + \alpha_i + \bar{\epsilon}_{it}. \quad (2)$$

where $\bar{y}_i = \frac{1}{T} \sum_{t=1}^T y_{it}$. Subtracting (2) from (1) gives

$$\tilde{y}_{it} = \tilde{x}_{it}\beta_2 + \tilde{\epsilon}_{it} \quad (3)$$

where $\tilde{y}_{it} = y_{it} - \bar{y}_i$. We then run OLS with the demeaned data, and obtain the within estimator

$$\hat{\beta}_2^{FE} = \left(\tilde{X}' \tilde{X} \right)^{-1} \tilde{X}' \tilde{y},$$

where $\tilde{y} = (y_{it})_{i,t}$ stacks all the nT observations into a vector, and similarly defined is \tilde{X} as an $nT \times K$ matrix, where K is the dimension of β_2 .

We know that OLS in (3) would be consistent if $\mathbb{E}[\tilde{\epsilon}_{it}|\tilde{x}_{it}] = 0$. Below we provide a sufficient condition, which is often called *strict exogeneity*.

Assumption (FE.1). $\mathbb{E}[\epsilon_{it}|\alpha_i, \mathbf{x}_i] = 0$ where $\mathbf{x}_i = (x_{i1}, \dots, x_{iT})$.

Its strictness is relative to the contemporary exogeneity $\mathbb{E}[\epsilon_{it}|x_{it}] = 0$. FE.1 is more restrictive as it assumes that the error ϵ_{it} is mean independent of the past, present and future explanatory variables.

When we talk about the consistency in panel data, typically we are considering $n \rightarrow \infty$ while T stays fixed. This asymptotic framework is appropriate for panel datasets with many individuals but only a few time periods.

Lemma (FE consistency). *If FE.1 is satisfied, then $\hat{\beta}_2^{FE}$ is consistent.*

The variance estimation for the FE estimator is a little bit tricky. We assume a homoskedasticity condition to simplify the calculation. Violation of this assumption changes the form of the asymptotic variance, but does not jeopardize the asymptotic normality.

Assumption (FE.2). $\text{var}(\epsilon_i|\mathbf{x}_i) = \sigma_\epsilon^2 I_T$.

Under FE.1 and FE.2, $\hat{\sigma}_\epsilon^2 = \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=1}^T \hat{\epsilon}_{it}^2$ is a consistent estimator of σ_ϵ^2 , where $\hat{\epsilon} = \tilde{y}_{it} - \tilde{x}_{it} \hat{\beta}_2^{FE}$. Note that the denominator is $n(T-1)$, not nT .

Theorem (FE asymptotic normality). *If FE.1 and FE.2 are satisfied, then*

$$\frac{\left(\tilde{X}' \tilde{X} \right)^{1/2}}{\hat{\sigma}_\epsilon} \left(\hat{\beta}_2^{FE} - \beta_2^0 \right) \Rightarrow N(0, I_K).$$

Remark. We implicitly assume some regularity conditions that allow us to invoke a law of large numbers and a central limit theorem. We ignore those technical details here.

3 Random Effect

The random effect estimator pursues efficiency at a knife-edge special case $\text{cov}(\alpha_i, x_{it}) = 0$. As mentioned above, FE is consistent when α_i and x_{it} are uncorrelated. However, an inspection of the covariance matrix reveals that OLS is inefficient.

The model is again (1), while we assume

Assumption (RE.1). $\mathbb{E}[\epsilon_{it}|\alpha_i, \mathbf{x}_i] = 0$ and $\mathbb{E}[\alpha_i|\mathbf{x}_i] = 0$.

RE.1 obviously implies $\text{cov}(\alpha_i, x_{it}) = 0$, so

$$S = \text{var}(u_i|\mathbf{x}_i) = \sigma_\alpha^2 \mathbf{1}_T \mathbf{1}_T' + \sigma_\epsilon^2 I_T, \text{ for all } i = 1, \dots, n.$$

Because the covariance matrix is not a scalar multiplication of the identity matrix, OLS is inefficient.

As mentioned before, FE estimation kills all time-invariant regressors. In contrast, RE allows time-invariant explanatory variables. Let us rewrite (1) as

$$y_{it} = w_{it}\boldsymbol{\beta} + u_{it},$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ and $w_{it} = (1, x_{it})$ are $K+1$ vectors, i.e., $\boldsymbol{\beta}$ is the parameter including the intercept, and w_{it} is the explanatory variables including the constant. Had we known S , the GLS estimator would be

$$\hat{\boldsymbol{\beta}}^{RE} = \left(\sum_{i=1}^n \mathbf{w}_i' S^{-1} \mathbf{w}_i \right)^{-1} \sum_{i=1}^n \mathbf{w}_i' S^{-1} \mathbf{y}_i = (W' \mathbf{S}^{-1} W)^{-1} W' \mathbf{S}^{-1} \mathbf{y}$$

where $\mathbf{S} = I_T \otimes S$. (“ \otimes ” denotes the Kronecker product.) In practice, σ_α^2 and σ_ϵ^2 in S are unknown, so we seek consistent estimators. Again, we impose a simplifying assumption parallel to FE.2.

Assumption (RE.2). $\text{var}(\epsilon_i|\mathbf{x}_i, \alpha_i) = \sigma_\epsilon^2 I_T$ and $\text{var}(\alpha_i|\mathbf{x}_i) = \sigma_\alpha^2$.

Under this assumption, we can consistently estimate the variances from

the residuals $\hat{u}_{it} = y_{it} - x_{it}'\hat{\beta}^{RE}$. That is

$$\begin{aligned}\hat{\sigma}_u^2 &= \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \hat{u}_{it}^2 \\ \hat{\sigma}_\alpha^2 &= \frac{1}{n} \sum_{i=1}^n \frac{1}{T(T-1)} \sum_{t=1}^T \sum_{r=1}^T \sum_{r \neq t} \hat{u}_{it} \hat{u}_{ir}.\end{aligned}$$

Again, we claim the asymptotic normality.

Theorem (RE asymptotic normality). *If RE.1 and RE.2 are satisfied, then*

$$\left(W' \hat{\mathbf{S}}^{-1} W\right)^{1/2} \left(\hat{\beta}^{RE} - \beta_0\right) \Rightarrow N(0, I_{K+1})$$

where $\hat{\mathbf{S}}$ is a consistent estimator of \mathbf{S} .