

# Generalized Method of Moments

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## 1 Introduction

*Generalized method of moments* (GMM) is an estimation principle that extends *method of moments*. It seeks the parameter value that minimizes a quadratic form of the moments. It is particularly useful in estimating structural models in which moment conditions can be derived from economic theory. GMM emerges as one of the most popular estimators in modern econometrics, and it includes conventional methods like the two-stage least squares (2SLS) and the three-stage least square as special cases.

## 2 GMM in Linear Model

In this section we discuss GMM in a linear single structural equation. A structural equation is a model of economic interest. Consider the following linear structural model

$$y_i = x'_{1i}\beta_1 + z'_{1i}\beta_2 + \epsilon_i, \quad (1)$$

where  $x_{1i}$  is a  $k_1$ -dimensional endogenous explanatory variables,  $z_{1i}$  is a  $k_2$ -dimensional exogenous explanatory variables with the intercept included. In addition, we have  $z_{2i}$ , a  $k_3$ -dimensional excluded exogenous variables. Let  $K = k_1 + k_2$  and  $L = k_2 + k_3$ . Denote  $x_i = (x'_{1i}, z'_{1i})'$  as a  $K$ -dimensional explanatory variable, and  $z_i = (z'_{1i}, z'_{2i})'$  as an  $L$ -dimensional exogenous vector.

In the context of endogeneity, we can call the exogenous variable *instrument variables*, or simply *instruments*. Let  $\beta = (\beta'_1, \beta'_2)'$  be a  $K$ -dimensional parameter of interest. From now on, we rewrite (1) as

$$y_i = x'_i\beta + \epsilon_i, \quad (2)$$

and we have a vector of instruments  $z_i$ .

Before estimating any structural econometric model, we must check identification. A model is *identified* if there is a one-to-one mapping between the distribution of the observed variables and the parameters. In other words, in an identified model any two parameter values  $\beta$  and  $\tilde{\beta}$ ,  $\beta \neq \tilde{\beta}$ , cannot generate the same distribution for the observable data. In the context of (2), identification requires that the true value  $\beta_0$  is the only value on the parameters space that satisfies the moment condition

$$\mathbb{E} [z_i (y_i - x'_i\beta)] = 0_L. \quad (3)$$

The rank condition is sufficient and necessary for identification.

**Assumption** (Rank condition).  $\text{rank} (\mathbb{E} [z_i x'_i]) = K$ .

Note that  $\mathbb{E} [x_i' z_i]$  is a  $K \times L$  matrix. The rank condition implies the *order condition*  $L \geq K$ , which says that the number of excluded instruments must be no fewer than the number of endogenous variables.

**Theorem.** *The parameter in (3) is identified if and only if the rank condition holds.*

*Proof.* (The “if” direction). For any  $\tilde{\beta}$  such that  $\tilde{\beta} \neq \beta_0$ ,

$$\begin{aligned} \mathbb{E} [z_i (y_i - x_i' \tilde{\beta})] &= \mathbb{E} [z_i (y_i - x_i' \beta_0)] + \mathbb{E} [z_i x_i'] (\beta_0 - \tilde{\beta}) \\ &= 0_L + \mathbb{E} [z_i x_i'] (\beta_0 - \tilde{\beta}). \end{aligned}$$

Because  $\text{rank} (\mathbb{E} [z_i x_i']) = K$ , we would have  $\mathbb{E} [z_i x_i'] (\beta_0 - \tilde{\beta}) = 0_L$  if and only if  $\beta_0 - \tilde{\beta} = 0_K$ , which violates  $\tilde{\beta} \neq \beta_0$ . Therefore  $\beta_0$  is the unique value that satisfies (3).

(The “only if” direction is left as an exercise. Hint: By contraposition, if the rank condition fails, then the model is not identified. We can easily prove the claim by making an example.)  $\square$

Because identification is a prerequisite for structural estimation, from now on we always assume that the model is identified. When it is just-identified ( $L = K$ ), by (3) we can express the parameter as

$$\beta = (\mathbb{E} [z_i x_i'])^{-1} \mathbb{E} [z_i y_i]. \quad (4)$$

It follows by the principle of method of moments that

$$\hat{\beta} = \left( \frac{Z'X}{n} \right)^{-1} \frac{Z'y}{n} = (Z'X)^{-1} Z'y,$$

which is exactly the 2SLS when  $L = K$ .

In the rest of this section, we focus on the over-identified case ( $L > K$ ). When  $L > K$ , (3) involves more equations than the number of parameters, so that directly taking the inverse as in (4) is inapplicable.

In order to express  $\beta$  explicitly, we define a criterion function

$$Q(\beta) = \mathbb{E} [z_i (y_i - x_i \beta)]' W \mathbb{E} [z_i (y_i - x_i \beta)],$$

where  $W$  is an arbitrary  $L \times L$  positive-definite symmetric matrix. Because of the quadratic form,  $Q(\beta) \geq 0$  for all  $\beta$ . Identification indicates that  $Q(\beta) = 0$  if and only if  $\beta = \beta_0$ . Therefore we conclude

$$\beta_0 = \arg \min_{\beta} Q(\beta).$$

Since  $Q(\beta)$  is a smooth function of  $\beta$ , the minimizer  $\beta_0$  can be characterized by the first-order condition

$$0_K = \frac{\partial}{\partial \beta} Q(\beta_0) = -\mathbb{E} [x_i z_i'] W \mathbb{E} [z_i (y_i - x_i \beta_0)]$$

Rearranging the above equation, we have

$$\mathbb{E} [x_i z_i'] W \mathbb{E} [z_i x_i'] \beta_0 = \mathbb{E} [x_i z_i'] W \mathbb{E} [z_i y_i].$$

Denote  $\Sigma = \mathbb{E} [z_i x_i']$ . Under the rank condition,  $\Sigma' W \Sigma$  is invertible so that we can solve

$$\beta_0 = (\Sigma' W \Sigma)^{-1} \Sigma' W \mathbb{E} [z_i y_i].$$

In practice, we use the sample moments to replace the corresponding population moments. The GMM estimator mimics its population formula.

$$\begin{aligned}\hat{\beta} &= \left( \frac{1}{n} \sum x_i z_i' W \frac{1}{n} \sum z_i x_i' \right)^{-1} \frac{1}{n} \sum x_i z_i' W \frac{1}{n} \sum z_i y_i \\ &= \left( \frac{X'Z}{n} W \frac{Z'X}{n} \right)^{-1} \frac{X'Z}{n} W \frac{Z'y}{n} \\ &= (X'ZWZ'X)^{-1} X'ZWZ'y.\end{aligned}$$

**Exercise.** The same GMM estimator  $\hat{\beta}$  can be obtained by minimizing

$$\left[ \frac{1}{n} \sum_{i=1}^n z_i (y_i - x_i \beta) \right]' W \left[ \frac{1}{n} \sum_{i=1}^n z_i (y_i - x_i \beta) \right] = \frac{(y - X\beta)' Z}{n} W \frac{Z' (y - X\beta)}{n},$$

or more concisely,

$$\hat{\beta} = \arg \min_{\beta} (y - X\beta)' Z W Z' (y - X\beta).$$

Now we check the asymptotic properties of  $\hat{\beta}$ . A few assumptions are in order.

**Assumption (A.1).**  $Z'X/n \xrightarrow{P} \Sigma$  and  $Z'\epsilon/n \xrightarrow{P} 0_L$ .

A.1 assumes that we can apply a law of large numbers, so that that the sample moments  $Z'X/n$  and  $Z'\epsilon/n$  converge in probability to their population counterparts.

**Theorem.** Under A.1,  $\hat{\beta}$  is consistent.

*Proof.* The step is similar to the consistency proof of OLS.

$$\begin{aligned}\hat{\beta} &= (X'ZWZ'X)^{-1} X'ZWZ' (X'\beta_0 + \epsilon) \\ &= \beta_0 + \left( \frac{X'Z}{n} W \frac{Z'X}{n} \right)^{-1} \frac{X'Z}{n} W \frac{Z'\epsilon}{n} \\ &\xrightarrow{P} \beta_0 + (\Sigma'W\Sigma)^{-1} \Sigma'W0 = \beta_0.\end{aligned}$$

□

To check asymptotic normality, we assume that a central limit theorem can be applied.

**Assumption (A.2).**  $\frac{1}{\sqrt{n}} \sum_{i=1}^n z_i' \epsilon_i \Rightarrow N(0_L, \Omega)$ , where  $\Omega = \mathbb{E}[z_i' z_i \epsilon_i^2]$ .

**Theorem (Asymptotic Normality).** Under A.1 and A.2,

$$\sqrt{n} (\hat{\beta} - \beta_0) \Rightarrow N(0_K, (\Sigma'W\Sigma)^{-1} \Sigma'W\Omega W\Sigma (\Sigma'W\Sigma)^{-1}). \quad (5)$$

*Proof.* Multiply  $\hat{\beta} - \beta_0$  by the scaling factor  $\sqrt{n}$ ,

$$\begin{aligned}\sqrt{n} (\hat{\beta} - \beta_0) &= \left( \frac{X'Z}{n} W \frac{Z'X}{n} \right)^{-1} \frac{X'Z}{n} W \frac{Z'\epsilon}{\sqrt{n}} \\ &= \left( \frac{X'Z}{n} W \frac{Z'X}{n} \right)^{-1} \frac{X'Z}{n} W \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i' \epsilon_i.\end{aligned}$$

The conclusion follows as

$$\frac{X'Z}{n}W\frac{Z'X}{n} \xrightarrow{P} \Sigma'W\Sigma$$

and

$$\frac{X'Z}{n}W\frac{1}{\sqrt{n}}\sum z'_i\epsilon_i \Rightarrow \Sigma'W \times N(0, \Omega).$$

□

It is clear from (5) that the GMM estimator's asymptotic variance depends on the choice of  $W$ . A natural question follows: can we optimally choose a  $W$  to make the asymptotic variance as small as possible? Here we claim the result without a proof.

*Claim.* The choice  $W = \Omega^{-1}$  makes  $\hat{\beta}$  an asymptotically efficient estimator, under which the asymptotic variance is

$$\left(\Sigma'\Omega^{-1}\Sigma\right)^{-1}\Sigma'\Omega^{-1}\Omega\Omega^{-1}\Sigma\left(\Sigma'\Omega^{-1}\Sigma\right)^{-1} = \left(\Sigma'\Omega^{-1}\Sigma\right)^{-1}.$$

In practice,  $\Omega$  is unknown but can be estimated. Hansen (1982) suggests the following procedure, which is known as the *two-step GMM*.

1. Choose any valid  $W$ , say  $W = I_L$ , to get a consistent (but inefficient in general) estimator  $\hat{\beta}$ . Save the residual  $\hat{\epsilon}_i = y_i - x'_i\hat{\beta}$  and estimate the variance matrix  $\hat{\Omega} = \frac{1}{n}\sum z_i z'_i \hat{\epsilon}_i^2$ .
2. Set  $W = \hat{\Omega}^{-1}$  and obtain a second estimator

$$\hat{\beta} = \left(X'Z\hat{\Omega}^{-1}Z'X\right)^{-1}X'Z\hat{\Omega}^{-1}Z'y.$$

This second estimator is asymptotic efficient.

If we further assume conditional homoskedasticity, then  $\Omega = \mathbb{E}[z_i z'_i \epsilon_i^2] = \mathbb{E}[z_i z'_i \mathbb{E}[\epsilon_i^2 | z_i]] = \sigma^2 \mathbb{E}[z_i z'_i]$ . Therefore in the first-step of the two-step GMM we can estimate the variance of the error term by  $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_i^2$  and the variance matrix by  $\hat{\Omega} = \hat{\sigma}^2 \frac{1}{n}\sum_{i=1}^n z_i z'_i = \hat{\sigma}^2 Z'Z/n$ . When we plug this  $W = \hat{\Omega}^{-1}$  into the GMM estimator,

$$\begin{aligned}\hat{\beta} &= \left(X'Z\left(\hat{\sigma}^2\frac{Z'Z}{n}\right)^{-1}Z'X\right)^{-1}X'Z\left(\hat{\sigma}^2\frac{Z'Z}{n}\right)^{-1}Z'y \\ &= \left(X'Z(Z'Z)^{-1}Z'X\right)^{-1}X'Z(Z'Z)^{-1}Z'y.\end{aligned}$$

This is exactly the same expression of 2SLS for  $L > K$ . Therefore, 2SLS can be viewed as a special case of GMM with  $W = (Z'Z/n)^{-1}$ . Under conditional homoskedasticity, 2SLS is the efficient estimator; otherwise 2SLS is inefficient.

*Remark.* 2SLS gets its name because it can be obtained using two steps: first regress  $X$  on all instruments  $Z$ , and then regress  $y$  on the fitted value along with the included exogenous variables. However, 2SLS can actually be obtained by one step using the above equation. It is a special case of GMM.