1 Review of Probability

1.1 Probability

- *Sample space* Ω is the collection of all possible outcomes.
- An *event* is a subset of Ω.
- σ-field $\mathcal{F}$ is a collection of events such that: (i) $\emptyset \in \mathcal{F}$; (ii) if an event $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$; (iii) if $\{A_i \in \mathcal{F} : i \in \mathbb{N}\}$, then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$.
- Measure space $(\Omega, \mathcal{F})$.
- Probability $P : \mathcal{F} \mapsto [0,1]$ is a function such that: (i) $P(A) \geq 0$ for all $A \in \mathcal{F}$; (ii) $P(\Omega) = 1$; (iii) if $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, are mutually disjoint, then $P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i)$.
- Probability space $(\Omega, \mathcal{F}, P)$.
- Measurable function: a function X is called $(\Omega, \mathcal{F}) / (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ measurable if $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$ for any $B \in \mathcal{B}_{\mathbb{R}}$.
- Random variable: $X : \Omega \mapsto \mathbb{R}$ is a measurable function from $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.
- Cumulative distribution function (CDF)
- Probability density function (PDF)
  - Univariate random variable
  - Bivariate random variable
  - Multivariate random variable
- Conditional distribution

1.2 Expectation

- *Expectation* is an average of a random variable. Expectation is nothing but an integration, where the integrand is the random variable, and the measure is the probability measure. We write $E[X]$, instead of $\int X \mathrm{d} P_X$, just for a concise notation when the underlying probability measure is clear.
- Conditional expectation
- Law of iterated expectation: $E[E[Y|X]] = E[Y]$.
- Properties of conditional expectations
  1. $E[E[Y|X_1, X_2]|X_1] = E[Y|X_1]$
  2. $E[E[Y|X_1]|X_1, X_2] = E[Y|X_1]$
  3. $E[h(X)Y|X] = h(X)E[Y|X]$