PRICING KERNEL AND IMPLICATIONS OF BANSEL YARON MODEL

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ABSTRACT.

1. INTRODUCTION

Define the utility recursively with the time aggregator as in [3] and [4],

$$U_t = \left[(1-\beta)c_t^{\rho} + \beta\mu_t(U_{t+1})^{\rho}\right]^{1/\rho}. \tag{1}$$

With this utility function, the pricing kernel yields

$$m_{t,t+t} = \beta g_{t+1}^{\rho-1}\left[g_{t+1}u_{t+1}/\mu_t(g_{t+1}u_{t+1})\right]^{\alpha-\rho}. \tag{2}$$

With the recursive utility, a loglinear approximation is commonly used to derive the pricing kernel,

$$\log u_t \approx b_0 + b_1 \log\mu_t(g_{t+1}u_{t+1}). \tag{3}$$

In derivation of pricing kernel, the cumulant generating functions are useful. Let $k_t(s;y) = \log E_t(e^{sy_t+1})$. The utility recursively defined in 1, the certainty equivalent function,

$$\mu_t(U_{t+1}) = [E_t(U_{t+1}^{\alpha})]^{1/\alpha}. \tag{4}$$

Then the log of 4 of $e^{a_t+b_t y_{t+1}}$ is

$$\log\mu_t(e^{a_t+b_t y_{t+1}}) = a_t + k_t(\alpha b_t)/\alpha.$$

The cumulant generating function for the standard normals is $k_t(s;\omega_{t+1}) = s^2/2$. The cumulant generating function for the jump component is $k_t(s;z_{t+1}) = (e^{s\theta+(s\delta)^2/2} - 1)h$. We will use the results in pricing kernel in section 2.1 and 3.1.

2. BANSEL AND YARON MODEL WITH FLUCTUATING ECONOMIC UNCERTAINTY

2.1. Pricing Kernel.

In [2], Bansel and Yaron presents a consumption growth (g_t) model incorporated with fluctuating economic uncertainty (or stochastic variance). The state-space representation of the dynamic is,

$$\begin{aligned}
x_{t+1} &= \rho x_t + \gamma_1 v_t^{1/2}\omega_{x,t+1} \\
g_{t+1} &= g + x_t + v_t^{1/2}\omega_{g,t+1} \\
v_{t+1} &= v + \nu(v_t^2 - v) + v_{\sigma}^{1/2}\omega_{v,t+1} \\
&\quad\ \omega_{x,t+1}, \omega_{g,t+1}, \omega_{v,t+1} \sim N.i.i.d.(0,1),
\end{aligned} \tag{5}$$

where σ_{t+1}^2 represents the time-varying economic uncertainty incorporated in consumption growth rate and σ^2 is its unconditional mean.

More concisely, we express the model with the state variable x_t embedded in the consumption growth dynamic, where

$$\begin{aligned}
\log g_t &= \log g + \gamma(B) v_{t-1}^{1/2}\omega_{g,t}, \\
v_t &= v + \nu(B)\omega v, t,
\end{aligned} \tag{6}$$

where $\omega_{g,t}$ and $\omega v, t$ are independent iid standard normal random variables. B is the lag operator, where $Bx_j = x_{j-1}$ for $j \geq 1$. Also,

$$\gamma(B) = \sum_{j=0}^{\infty} \gamma_j B^j, and \quad \gamma(B)\omega_t = \sum_{j=0}^{\infty} \gamma_j \omega_{t-j}.$$

The value function should have the form with innovative terms ω_t's in consumption growth and stochastic variance. And we guess

$$\log u_t = \log u + c_g(B) v_{t-1}^{1/2} \omega_{g,t} + c_v(B)\omega_{v,t}$$

with parameter set $\Theta = (u, c_g, c_v)$ to be determined.

Then the corresponding certainty equivalent is computed, given the initial guess of the value function. Using a trick $c_v(B)\omega_{v,t+1} = (c_v(B) - c_{v,0})\omega_{v,t+1} + c_{v,0}\omega_{v,t+1}$, we have,

$$\begin{aligned}
\log(g_{t+1} u_{t+1}) &= \log g + \log u + [\gamma(B) + c_g(B)] v_t^{1/2} \omega_{g,t+1} + c_v(B)\omega_{v,t+1} \\
&= \log(gu) + [\gamma(B) + c_g(B) - (\gamma_0 + c_{g,0})] v_t^{1/2} \omega_{g,t+1} \\
&\quad + [c_v(B) - c_{v,0}]\omega_{v,t+1} + (\gamma_0 + c_{g,0}) v_t^{1/2} \omega_{g,t+1} + c_{v,0}\omega_{v,t+1}
\end{aligned}$$

Using the cumulant generating function of standard normals, the certainty equivalent is

$$\begin{aligned}
\log \mu_t(g_{t+1} u_{t+1}) &= \log(gu) + [\gamma(B) + c_g(B) - (\gamma_0 + c_{g,0})] v_t^{1/2} \omega_{g,t+1} \\
&\quad + [c_v(B) - c_{v,0}]\omega_{v,t+1} + (\alpha/2)(\gamma_0 + c_{g,0})^2 v_t + (\alpha/2) c_{v,0}^2 \\
&= \log(gu) + [\gamma(B) + c_g(B) - (\gamma_0 + c_{g,0})] v_t^{1/2} \omega_{g,t+1} \\
&\quad + [c_v(B) - c_{v,0}]\omega_{v,t+1} + (\alpha/2)(\gamma_0 + c_{g,0})^2 [v + \nu(B)\omega_{v,t}] + (\alpha/2) c_{v,0}^2
\end{aligned}$$

Substitute the certainty equivalent into 3 and line up with the parameters with the terms,

$$\begin{aligned}
\log u &= b_0 + b_1[\log(gu) + (\alpha/2) c_{v,0^2} + (\alpha/2)(\gamma_0 + c_{g,0})^2 v] \\
c_g(B)B &= b_1[\gamma(B) + c_g(B) - (\gamma_0 + c_{g,0})] \\
c_v(B)B &= b_1[c_v(B) + c_{v,0} + (\alpha/2)(\gamma_0 + c_{g,0})^2 \nu(B)B]
\end{aligned}$$

Apply the same setting as in [1], choose $B = b_1$. We have

$$\begin{aligned}
\gamma_0 + c_{g,0} &= \gamma(b_1), \\
c_{v,0} &= (\alpha/2)\gamma(b_1)^2 b_1 \nu(b_1),
\end{aligned}$$

Construct the pricing kernel from 2, we have

$$\begin{aligned}
\log(g_{t+1} u_{t+1}) &- \log \mu_t(g_{t+1} u_{t+1}) = \\
&= -(\alpha/2)\gamma(b_1)^2 v + \gamma(b_1) v_t^{1/2} \omega_{g,t+1} + (\alpha/2)\gamma(b_1)^2 [b_1 \nu(b_1) - \nu(B)B]\omega_{v,t+1}
\end{aligned}$$

Finally, the stochastic variance Bansel-Yaron pricing kernel is

$$
\begin{aligned}
\log m_{t,t+1} &= \log\beta + (\rho-1)\log g \\
&\quad -(\alpha-\rho)\left\{(\alpha/2)\gamma(b_1)^2 v + [(\alpha/2)\gamma(b_1)^2 b_v \nu(b_1)]^2\right\} \\
&\quad +[(\rho-1)\gamma(B) + (\alpha-\rho)\gamma(b_1)]\, v_t^{1/2}\omega_{g,t+1} \\
&\quad +(\alpha-\rho)(\alpha/2)\gamma(b_1)^2[b_1\nu(b_1) - \nu(B)B]\omega_{v,t+1} \\
&= \text{constant} + \\
&\quad +[(\rho-1)\gamma(B) + (\alpha-\rho)\gamma(b_1)]\, v_t^{1/2}\omega_{g,t+1} \\
&\quad +(\alpha-\rho)(\alpha/2)\gamma(b_1)^2[b_1\nu(b_1) - \nu(B)B]\omega_{v,t+1}
\end{aligned}
$$

which is a (complex) constant plus the innovative terms (labelled in red above) with certain coefficients.

2.2. **Data Implication.**

3. Bansel and Yaron Model with Stochastic Variance and Jumps

3.1. **Pricing Kernel.** The model specification of consumption growth with stochastic variance and jumps is

$$
\begin{aligned}
\log g_t &= \log g' + \gamma(B) v_{t-1}^{1/2}\omega_{g,t} + \psi(B) z_{g,t}, \\
v_t &= v + \nu(B)\omega v, t, \\
h_t &= h + \eta(B)\omega_{h,t},
\end{aligned}
\tag{7}
$$

where the innovative terms $\omega_{g,t}, \omega_{v,t}, \omega_{h,t}$ are independent and standard-normal distributed. Also, $\log g = \log g' - \psi(h) h\theta$. The jump component $z_{g,t}$ is Poisson distributed, conditionally on j number of jumps whose mean and variance are $j\theta$ and $j\delta^2$ respectively.

The value function should have the form with innovative terms ω_t's in consumption growth, variance, jump number and jump size. And we guess

$$
\log u_t = \log u + c_g(B) v_{t-1}^{1/2}\omega_{g,t} + c_z(B) z_{g,t} + c_v(B)\omega_{v,t} + c_h(B)\omega_{h,t}
$$

with parameter set $\Theta = (u, c_g, c_z, c_v, c_h)$ to be determined.

Then the corresponding certainty equivalent is computed, given the initial guess of the value function. Using a trick $c_v(B)\omega_{v,t+1} = (c_v(B) - c_{v,0})\omega_{v,t+1} + c_{v,0}\omega_{v,t+1}$. And we have,

$$
\begin{aligned}
\log(g_{t+1}u_{t+1}) &= \log g' + \log u + [\gamma(B) + c_g(B)]\, v_t^{1/2}\omega_{g,t+1} + [\psi(B) + c_z(B)] z_{g,t+1} \\
&\quad + c_v(B)\omega_{v,t+1} + c_h(B)\omega_{h,t+1} \\
&= \log(g'u) + [\gamma(B) + c_g(B) - (\gamma_0 + c_{g,0})]\, v_t^{1/2}\omega_{g,t+1} \\
&\quad + [\psi(B) + c_z(B) - (\psi_0 + c_{z,0})] z_{g,t+1} + [c_v(B) - c_{v,0}]\omega_{v,t+1} \\
&\quad + [c_h(B) - c_{h,0}]\omega_{h,t+1} + (\gamma_0 + c_{g,0})\, v_t^{1/2}\omega_{g,t+1} \\
&\quad + c_{v,0}\omega_{v,t+1} + c_{h,0}\omega_{h,t+1} + (\psi_0 + c_{z,0}) z_{g,t+1}
\end{aligned}
$$

where $g' =$

Using the cumulant generating function of standard normals and Poissons, the certainty equivalent is

$$
\begin{aligned}
\log\mu_t(g_{t+1}u_{t+1}) &= \log(g'u) + [\gamma(B)+c_g(B)-(\gamma_0+c_{g,0})]\,v_t^{1/2}\omega_{g,t+1} \\
&\quad +[\psi(B)+c_z(B)-(\psi_0+c_{z,0})]z_{g,t+1}+[c_v(B)-c_{v,0}]\omega_{v,t+1} \\
&\quad +(\alpha/2)(\gamma_0+c_{g,0})^2 v_t + (\alpha/2)(c_{v,0}^2+c_{h,0}^2) \\
&\quad +[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]h_t \\
&= \log(g'u) + [\gamma(B)+c_g(B)-(\gamma_0+c_{g,0})]\,v_t^{1/2}\omega_{g,t+1} \\
&\quad +[\psi(B)+c_z(B)-(\psi_0+c_{z,0})]z_{g,t+1}+[c_v(B)-c_{v,0}]\omega_{v,t+1} \\
&\quad +(\alpha/2)(\gamma_0+c_{g,0})^2[v+\nu(B)\omega_{v,t}] + (\alpha/2)(c_{v,0}^2+c_{h,0}^2) \\
&\quad +[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha][h+\eta(B)\omega_{h,t}]
\end{aligned}
$$

Substitute the certainty equivalent into 3 and line up with the parameters with the terms,

$$
\begin{aligned}
\log u &= b_0 + b_1[\log(g'u) + (\alpha/2)(c_{v,0^2}+c_{h,0}^2) + (\alpha/2)(\gamma_0+c_{g,0})^2 v] \\
&\quad +b_1[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2})/\alpha]h; \\
c_g(B)B &= b_1[\gamma(B)+c_g(B)-(\gamma_0+c_{g,0})] \\
c_z(B)B &= b_1[\psi(B)+c_z(B)-(\psi_0+c_{z,0})] \\
c_v(B)B &= b_1[c_v(B)+c_{v,0}+(\alpha/2)(\gamma_0+c_{g,0})^2\nu(B)B] \\
c_h(B)B &= b_1\left\{c_h(B)-c_{h,0}+[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]\eta(B)B\right\}
\end{aligned}
$$

Apply the same setting as in [1], choose $B = b_1$. We have

$$
\begin{aligned}
\gamma_0+c_{g,0} &= \gamma(b_1), \\
\psi_0+c_{z,0} &= \psi(b_1), \\
c_{v,0} &= (\alpha/2)\gamma(b_1)^2 b_1\nu(b_1), \\
c_{h,0} &= [(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]b_1\eta(b_1)
\end{aligned}
$$

Construct the pricing kernel from 2, we have

$$
\begin{aligned}
\log(g_{t+1}u_{t+1}) &- \log\mu_t(g_{t+1}u_{t+1}) = \\
&= -(\alpha/2)\gamma(b_1)^2 v - [(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]h \\
&+ \gamma(b_1)v_t^{1/2}\omega_{g,t+1}+\psi(b_1)z_{g,t+1}+(\alpha/2)\gamma(b_1)^2[b_1\nu(b_1)-\nu(B)B]\omega_{v,t+1} \\
&+ [(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha][b_1\eta(b_1)-\eta(B)B]\omega_{h,t+1}
\end{aligned}
$$

Finally, the pricing kernel is

$$\begin{aligned}
\log m_{t,t+1} &= \log\beta + (\rho-1)\log g \\
&\quad -(\alpha-\rho)\left\{(\alpha/2)\gamma(b_1)^2 v - [(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]h\right\} \\
&\quad -(\alpha-\rho)(\alpha/2)\left\{[(\alpha/2)\gamma(b_1)^2 b_v \nu(b_1)]^2 + [[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]b_1\eta(b_1)]^2\right\} \\
&\quad +[(\rho-1)\gamma(B)+(\alpha-\rho)\gamma(b_1)]v_t^{1/2}\omega_{g,t+1} + [(\rho-1)\psi(B)+(\alpha-\rho)\psi(b_1)]z_{g,t+1} \\
&\quad +(\alpha-\rho)(\alpha/2)\gamma(b_1)^2[b_1\nu(b_1)-\nu(B)B]\omega_{v,t+1} \\
&\quad +(\alpha-\rho)[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha][b_1\eta(b_1)-\eta(B)B]\omega_{h,t+1} \\
&= \text{constant} + \\
&\quad +[(\rho-1)\gamma(B)+(\alpha-\rho)\gamma(b_1)]v_t^{1/2}\omega_{g,t+1} \\
&\quad +[(\rho-1)\psi(B)+(\alpha-\rho)\psi(b_1)]z_{g,t+1} \\
&\quad +(\alpha-\rho)(\alpha/2)\gamma(b_1)^2[b_1\nu(b_1)-\nu(B)B]\omega_{v,t+1} \\
&\quad +(\alpha-\rho)[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha][b_1\eta(b_1)-\eta(B)B]\omega_{h,t+1}
\end{aligned}$$

which is a (complex) constant plus the innovative terms (labelled in red above) with certain coefficients. This format makes it easier to calculate the entropy.

3.2. **Entropy and Horizon Dependence.** The definition of entropy is

$$L(x) \equiv \log E(x) - E(\log x) \geq 0,$$

for $x > 0$. In this case, we apply the cumulant generating functions, where $k_t(s;\omega_{t+1}) = s^2/2$ and $k_t(s;z_{t+1}) = (e^{s\theta+(s\delta)^2/2}-1)h$,

$$\begin{aligned}
E(\log m_{t,t+1}) &= \text{constant}' \\
\log E(m_{t,t+1}) &= \text{constant}' \\
&\quad +[(\rho-1)\gamma(B)+(\alpha-\rho)\gamma(b_1)]^2 v/2 \\
&\quad +\left\{\left(e^{(\alpha^*-1)\theta+[(\alpha^*-1)\delta]^2}-1\right)-\left(\alpha^*-1\right)\theta\right\}h_t \\
&\quad +[(\alpha-\rho)(\alpha/2)\gamma(b_1)^2 b_1\nu(b_1)]^2/2 \\
&\quad +\left\{(\alpha-\rho)[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]b_1\eta(b_1)\right\}^2/2 \\
L(m_{t,t+1}) &= \log E(m_{t,t+1}) - E(\log m_{t,t+1}) \\
&= [(\rho-1)\gamma(B)+(\alpha-\rho)\gamma(b_1)]^2 v/2 \\
&\quad +\left\{\left(e^{(\alpha^*-1)\theta+[(\alpha^*-1)\delta]^2}-1\right)-\left(\alpha^*-1\right)\theta\right\}h_t \\
&\quad +[(\alpha-\rho)(\alpha/2)\gamma(b_1)^2 b_1\nu(b_1)]^2/2 \\
&\quad +\left\{(\alpha-\rho)[(e^{\alpha(\psi_0+c_{z,0})\theta+(\alpha(\psi_0+c_{z,0})\delta)^2/2}-1)/\alpha]b_1\eta(b_1)\right\}^2/2
\end{aligned}$$

where $(\alpha^*-1) = (\rho-1)\psi_0 + (\alpha-\rho)\psi(b_1)$. Note that $constant'$ in above equation does not equal to the $constant$ in the pricing kernel. But $constant'$s are the same in $E(\log m_{t,t+1})$ and in $\log E(m_{t,t+1})$.

Analogous to the derivation on Appendix F in []

3.3. **Sensitivity of Parameters.**

3.4. **Hansen-Scheinkman Decomposition.**

3.5. **Expected Return on Console Bond.**

3.6. **Expected Return in the Economy.**

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Appendix: Code for Model Implications