Journal of Financial Economics 3 (1976) 167-179. © North-Holland Publishing Company

THE PRICING OF COMMODITY CONTRACTS*

Fischer BLACK

Sloan School of Management, M.I.T., Cambridge, Mass. 02139, U.S.A.

Received July 1975, revised version received July 1975

The contract price on a forward contract stays fixed for the life of the contract, while a futures contract is rewritten every day. The value of a futures contract is zero at the start of each day. The expected change in the futures price satisfies a formula like the capital asset pricing model. If changes in the futures price are independent of the return on the market, the futures price is the expected spot price. The futures market is not unique in its ability to shift risk, since corporations can do that too. The futures market is unique in the guidance it provides for producers, distributors, and users of commodities. Using assumptions like those used in deriving the original option formula, we find formulas for the values of forward contracts and commodity options in terms of the futures price and other variables.

1. Introduction

The market for contracts related to commodities is not widely understood. Futures contracts and forward contracts are often thought to be identical, and many people don't know about the existence of commodity options. One of the aims of this paper is to clarify the meaning of each of these contracts.[1]

The spot price of a commodity is the price at which it can be bought or sold for immediate delivery. We will write p for the spot price, or $p(t)$ for the spot price at time t.

The spot price of an agricultural commodity tends to have a seasonal pattern: it is high just before a harvest, and low just after a harvest. The spot price of a commodity such as gold, however, fluctuates more randomly.

Predictable patterns in the movement of the spot price do not generally imply profit opportunities. The spot price can rise steadily at any rate lower than the storage cost for the commodity (including interest) without giving rise to a profit opportunity for those with empty storage facilities. The spot price can fall during a harvest period without giving rise to a profit opportunity for growers, so long as it is costly to accelerate the harvest.

*I am grateful for comments on earlier drafts by Michael Jensen, Myron Scholes, Edward Thorp, and Joseph Williams. This work was supported in part by the Center for Research in Security Prices (sponsored by Merrill Lynch, Pierce, Fenner & Smith Inc.) at the Graduate School of Business, University of Chicago.

[1]For an introduction to commodity markets, see Chicago Board of Trade (1973).

The futures price of a commodity is the price at which one can agree to buy or sell it at a given time in the future without putting up any money now. We will write x for the futures price, or $x(t, t^*)$ for the futures price at time t for a transaction that will occur at time t^*.

For example, suppose that it is possible today to enter into a contract to buy gold six months from now at $160 an ounce, without either party to the contract being compensated by the other. Both parties may put up collateral to guarantee their ability to fulfill the contract, but if the futures price remains at $160 an ounce for the next six months, the collateral will not be touched. If the contract is left unchanged for six months, then the gold and the money will change hands at that time. In this situation, we say that the six month futures price of gold is $160 an ounce.

The futures price is very much like the odds on a sports bet. If the odds on a particular baseball game between Boston and Chicago are 2:1 in favor of Boston, and if we ignore the bookie's profit, then a person who bets on Chicago wins $2 or loses $1. No money changes hands until after the game. The odds adjust to balance the demand for bets on Chicago and the demand for bets on Boston. At 2:1, balance occurs if twice as many bets are placed on Boston as on Chicago.

Similarly, the futures price adjusts to balance demand to buy the commodity in the future with demand to sell the commodity in the future. Whenever a contract is opened, there is someone on each side. The person who agrees to buy is long the commodity, and the person who agrees to sell is short. This means that when we add up all positions in contracts of this kind, and count short positions as negative, we always come out with zero. The total long interest in commodity contracts of any type must equal the total short interest.

When the two times that specify a futures price are equal, the futures price must equal the spot price,

$$x(t, t) \equiv p(t). \tag{1}$$

Expression (1) holds for all times t. For example, it says that the May futures price will be equal to the May spot price in May, and the September futures price will be equal to the September spot price in September.

Now let us define the three kinds of commodity contracts: forward contracts, futures contracts, and option contracts. Roughly speaking, a forward contract is a contract to buy or sell at a price that stays fixed for the life of the contract; a futures contract is settled every day and rewritten at the new futures price; and an option contract can be exercised by the holder when it matures, if it has not been closed out earlier.

We will write v for the value of a forward contract, u for the value of a futures contract, and w for the value of an option contract. Each of these values will depend on the current futures price $x(t, t^*)$ with the same transaction time t^* as the contract, and on the current time t, as well as on other variables. So we will write $v(x, t)$, $u(x, t)$, and $w(x, t)$. The value of the short side of any contract will be just the negative of the value of the long side. So we will treat v, u, and w as the values of a forward contract to buy, a long futures contract, and an option to buy.

The value of a forward contract depends also on the price c at which the commodity will be bought, and the time t^* at which the transaction will take place. We will sometimes write $v(x, t, c, t^*)$ for the value of a long forward contract. From the discussion above, we know that the futures price is that price at which a forward contract has a current value of zero. We can write this condition as

$$v(x, t, x, t^*) \equiv 0. \tag{2}$$

In effect, eq. (2) says that the value of a forward contract when it is initiated is always zero. When it is initiated, the contract price c is always equal to the current futures price $x(t, t^*)$.

Increasing the futures price increases the value of a long forward contract, and decreasing the futures price decreases the value of the contract. Thus we have

$$\begin{aligned} v(x, t, c, t^*) &> 0, \qquad x > c, \\ v(x, t, c, t^*) &< 0, \qquad x < c. \end{aligned} \tag{3}$$

The value of a forward contract may be either positive or negative.

When the time comes for the transaction to take place, the value of the forward contract will be equal to the spot price minus the contract price. But by eq. (1), the futures price $x(t, t^*)$ will be equal to the spot price at that time. Thus the value of the forward contract will be the futures price minus the spot price,

$$v(x, t^*, c, t^*) = x - c. \tag{4}$$

Later we will use eq. (4) as the main boundary condition for a differential equation describing the value of a forward contract.

The difference between a futures contract and a forward contract is that the futures contract is rewritten every day with a new contract price equal to the corresponding futures price. A futures contract is like a series of forward contracts. Each day, yesterday's contract is settled, and today's contract is written with a contract price equal to the futures price with the same maturity as the futures contract.

Eq. (2) shows that the value of a forward contract with a contract price equal to the futures price is zero. Thus the value of a futures contract is reset to zero every day. If the investor has made money, he will be given his gains immediately. If he has lost money, he will have to pay his losses immediately. Thus we have

$$u(x, t) \equiv 0. \tag{5}$$

Technically, eq. (5) applies only to the end of the day, after the futures contract has been rewritten. During the day, the futures contract may have a positive or negative value, and its value will be equal to the value of the corresponding forward contract.

Note that the futures price and the value of a futures contract are not at all the same thing. The futures price refers to a transaction at times t^* and is never zero. The value of a futures contract refers to time t and is always zero (at the end of the day).

In the organized U.S. futures markets, both parties to a futures contract must post collateral with a broker. This helps to ensure that the losing party each day will have funds available to pay the winning party. The amount of collateral required varies from broker to broker.

The form in which the collateral can be posted also varies from broker to broker. Most brokers allow the collateral to take the form of Treasury Bills or marginable securities if the amount exceeds a certain minimum. The brokers encourage cash collateral, however, because they earn the interest on customers' cash balances.

The value of a futures customer's account with a broker is entirely the value of his collateral (at the end of the day). The value of his futures contracts is zero. The value of the collateral posted to ensure performance of a futures contract is not the value of the contract.

As futures contracts are settled each day, the value of each customer's collateral is adjusted. When the futures price goes up, those with long positions have money added to their collateral, and those with short positions have money taken away from their cellateral. If a customer at any time has more collateral than his broker requires, he may withdraw the excess. If he has less than his broker requires, he will have to put up additional collateral immediately.

Commodity options have a bad image in the U.S., because they were recently used to defraud investors of many millions of dollars. There are no organized commodity options markets in this country. In the U.K., however, commodity options have a long and relatively respectable history.

A commodity option is an option to buy a fixed quantity of a specified commodity at a fixed time in the future and at a specified price. It differs from a security option in that it can't be exercised before the fixed future date. Thus it is a 'European option' rather than an 'American option'.

A commodity option differs from a forward contract because the holder of the option can choose whether or not he wants to buy the commodity at the specified price. With a forward contract, he has no choice: he must buy it, even if the spot price at the time of the transaction is lower than the price he pays.

At maturity, the value of a commodity option is the spot price minus the contract price, if that is positive, or zero. Writing c^* for the exercise price of the option, and noting that the futures price equals the spot price at maturity, we have

$$\begin{aligned} w(x, t^*) &= x - c^*, \qquad x \geqq c^*, \\ &= 0, \qquad x < c^*. \end{aligned} \tag{6}$$

Expression (6) looks like the expression for the value of a security option at maturity as a function of the security price.

2. The behavior of the futures price

Changes in the futures price for a given commodity at a given maturity give rise to gains and losses for investors with long or short positions in the corresponding futures contracts. An investor with a position in the futures market is bearing risk even though the value of his position at the end of each day is zero. His position may also have a positive or negative expected dollar return, even though his investment in the position is zero.

Since his investment is zero, it is not possible to talk about the percentage or fractional return on the investor's position in the futures market. Both his risk and his expected return must be defined in dollar terms.

In deriving expressions for the behavior of the futures price, we will assume that taxes are zero. However, tax factors will generally affect the behavior of the futures price. There are two peculiarities in the tax laws that make them important.

First, the IRS assumes that a gain or loss on a futures contract is realized only when the contract is closed out. The IRS does not recognize, for tax purposes, the fact that a futures contract is effectively settled and rewritten every day. This makes possible strategies for deferring the taxation of capital gains. For example, the investor can open a number of different contracts, both long and short. The contracts that develop losses are closed out early, and are replaced with different contracts so that the long and short positions stay balanced. The contracts that develop gains are allowed to run unrealized into the next tax year. In the next year, the process can be repeated. Whether this process is likely to be profitable depends on the special factors affecting each investor, including the size of the transaction costs he pays.

Second, the IRS treats a gain or loss on a long futures position that is closed out more than six months after it is opened as a long-term capital gain or loss, while it treats a gain or loss on a short futures position as a short-term capital gain or loss no matter how long the position is left open. Thus if the investor opens both long and short contracts, and if he realizes losses on the short contracts and gains on the long contracts, he can convert short-term gains (from other transactions) into long-term gains. Again, whether this makes sense for a particular investor will depend on his transaction costs and other factors.

However, we will assume that both taxes and transaction costs are zero. We will further assume that the capital asset pricing model applies at each instant of time.[2] This means that investors will be compensated only for bearing risk that cannot be diversified away. If the risk in a futures contract is independent of the risk of changes in value of all assets taken together, then investors will not have to be paid for taking that risk. In effect, they don't have to take the risk because they can diversify it away.

The usual capital asset pricing formula is

$$E(\tilde{R}_i)-R = \beta_i[E(\tilde{R}_m)-R]. \tag{7}$$

In this expression, $\tilde{R}_i$ is the return on asset i, expressed as a fraction of its initial value; R is the return on short-term interest-bearing securities; and $\tilde{R}_m$ is the return on the market portfolio of all assets taken together. The coefficient β_i is a measure of the extent to which the risk of asset i cannot be diversified away. It is defined by

$$\beta_i = \operatorname{cov}(\tilde{R}_i, \tilde{R}_m)/\operatorname{var}(\tilde{R}_m). \tag{8}$$

The market portfolio referred to above includes corporate securities, personal assets such as real estate, and assets held by non-corporate businesses. To the extent that stocks of commodities are held by corporations, they are implicitly included in the market portfolio. To the extent that they are held by individuals and non-corporate businesses, they are explicitly included in the market portfolio. This market portfolio cannot be observed, of course. It is a theoretical construct.

Commodity contracts, however, are not included in the market portfolio. Commodity contracts are pure bets, in that there is a short position for every long position. So when we are taking all assets together, futures contracts, forward contracts, and commodity options all net out to zero.

Eq. (7) cannot be applied directly to a futures contract, because the initial value of the contract is zero. So we will rewrite the equation so that it applies to dollar returns rather than percentage returns.

Let us assume that asset i has no dividends or other distributions over the period. Then its fractional return is its end-of-period price minus its start-of-period price, divided by its start-of-period price. Writing P_{i0} for the start-of-period price of asset i, writing $\tilde{P}_{i1}$ for its end-of-period price, and substituting from eq. (8), we can rewrite eq. (7) as

$$\begin{aligned} E\{(\tilde{P}_{i1}-P_{i0})/P_{i0}\}-R = {} & [\operatorname{cov}\{(\tilde{P}_{i1}-P_{i0})/P_{i0}, \tilde{R}_m\}/\operatorname{var}(\tilde{R}_m)] \\ & \times [E(\tilde{R}_m)-R]. \end{aligned} \tag{9}$$

[2]For an introduction to the capital asset pricing model, see Jensen (1972). The behavior of futures prices in a model of capital market equilibrium was first discussed by Dusak (1973).

Multiplying through by P_{i0}, we get an expression for the expected dollar return on an asset,

$$E(\tilde{P}_{i1}-P_{i0})-RP_{i0} = [\mathrm{cov}\,(\tilde{P}_{i1}-P_{i0},\tilde{R}_m)/\mathrm{var}\,(\tilde{R}_m)][E(\tilde{R}_m)-R]. \tag{10}$$

The start-of-period value of a futures contract is zero, so we set P_{i0} equal to zero. The end-of-period value of a futures contract, before the contract is rewritten and its value set to zero, is the change in the futures price over the period. In practice, commodity exchanges set daily limits which constrain the reported change in the futures price and the daily gains and losses of traders. We will assume that these limits do not exist. So we set $\tilde{P}_{i1}$ equal to $\Delta\tilde{P}$, the change in the futures price over the period,

$$E(\Delta\tilde{P}) = [\mathrm{cov}\,(\Delta\tilde{P},\tilde{R}_m)/\mathrm{var}\,(\tilde{R}_m)][E(\tilde{R}_m)-R]. \tag{11}$$

In effect, we have applied expression (10) to a futures contract, and have come up with expression (11), which refers to the change in the futures price. For the rest of this section, we can forget about futures contracts and work only with the futures price.

Writing β^* for the first factor on the right-hand side of eq. (11), we have

$$E(\Delta\tilde{P}) = \beta^*[E(\tilde{R}_m)-R]. \tag{12}$$

Expression (12) says that the expected change in the futures price is proportional to the 'dollar beta' of the futures price. If the covariance of the change in the futures price with the return on the market portfolio is zero, then the expected change in the futures price will be zero,[3]

$$E(\Delta\tilde{P}) = 0, \quad \text{when} \quad \mathrm{cov}\,(\Delta\tilde{P},\tilde{R}_m) = 0. \tag{13}$$

Expressions (12) and (13) say that the expected change in the futures price can be positive, zero, or negative. It would be very surprising if the β^* of a futures price were exactly zero, but it may be approximately zero for many commodities. For these commodities, neither those with long futures positions nor those with short futures positions have significantly positive expected dollar returns.

3. Futures prices and spot prices

When eq. (13) holds at all points in time, the expected change in the futures price will always be zero. This means that the expected futures price at any time t' in the future, where t' is between the current time t and the transaction time

[3]In the data she analyzed on wheat, corn, and soybean futures, Dusak (1973) found covariances that were close to zero.

t^*, will be equal to the current futures price. The mean of the distribution of possible futures prices at time t' will be the current futures price.[4]

But the futures price at time t^* is the spot price at time t^*, from expression (1). So the mean of the distribution of possible spot prices at time t^* will be the current futures price, when eq. (13) always holds.

Even when (13) doesn't hold, we may still be able to use eq. (12) to estimate the mean of the distribution of possible spot prices at time t^*. To use (12), though, we need to know β^* at each point in time between t and t^*, and we need to know $E(\tilde{R}_m)-R$.

A farmer may not want to know the mean of the distribution of possible spot prices at time t^*. He may be interested in the discounted value of the distribution of possible spot prices. In fact, it seems plausible that he can make his investment decisions as if β^* were zero, even if it is not zero. He can assume that the β^* is zero, and that the futures price is the expected spot price.

To see why this is so, note that he can hedge his investments by taking a short position in the futures market. By taking the right position in the futures market, he can make the β of his overall position zero. Assuming that the farmer is not concerned about risk that can be diversified away, he should make the same investment decisions whether or not he actually takes offsetting positions in the futures market.

In fact, futures prices provide a wealth of valuable information for those who produce, store, and use commodities. Looking at futures prices for various transaction months, participants in this market can decide on the best times to plant, harvest, buy for storage, sell from storage, or process the commodity. A change in a futures price at time t is related to changes in the anticipated distribution of spot prices at time t^*. It is not directly related to changes in the spot price at time t. In practice, however, changes in spot prices and changes in futures prices will often be highly correlated.

Both spot prices and futures prices are affected by general shifts in the cost of producing the commodity, and by general shifts in the demand for the commodity. These are probably the most important factors affecting commodity prices. But an event like the arrival of a prime producing season for the commodity will cause the spot price to fall, without having any predictable effect on the futures price.

Changes in commodity prices are also affected by such factors as the interest rate, the cost of storing the commodity, and the β of the commodity itself.[5] These factors may affect both the spot price and the futures price, but in different ways.

Commodity holdings are assets that form part of investors' portfolios, either directly or indirectly. The returns on such assets must be defined to include

[4] The question of the relation between the futures price and the expected spot price is discussed under somewhat different assumptions by Cootner (1960a, 1960b) and Telser (1960).

[5] Some of the factors affecting changes in the spot price are discussed by Brennan (1958) and Telser (1958).

such things as the saving to a user of commodities from not running out in the middle of a production run, or the benefit to anyone storing the commodity of having stocks on hand when there is an unusual surge in demand. The returns on commodity holdings must be defined net of all storage costs, including deterioration, theft, and insurance premiums. When the returns on commodity holdings are defined in this way, they should obey the capital asset pricing model, as expressed by eq. (7), like any other asset. If the β of the commodity is zero, as given in eq. (7), then we would expect the β^* of a futures contract to be approximately zero too, as given in eq. (12). And vice versa.

The notion that commodity holdings are priced like other assets means that investors who own commodities are able to diversify away that part of the risk that can be diversified away. One way this can happen is through futures markets: those who own commodities can take short positions, and those who hold diversified portfolios of assets can include long positions in commodity contracts.

But there are other ways that the risk in commodity holdings can be largely diversified away. The most common way for risk to be spread is through a corporation. The risk of a corporation's business or assets is passed on to the holders of the corporation's liabilities, especially its stockholders. The stockholders have, or could have, well diversified portfolios of which this stock is only a small part.

Thus if stocks of a commodity are held by a corporation, there will normally be no need for the risk to be spread through the futures market. (There are special cases, however, such as where the corporation has lots of debt outstanding and the lenders insist that the commodity risk be hedged through the futures market.) There are corporations at every stage in a commodity's life cycle: production, distribution, and processing. Even agricultural commodities are generally produced by corporations these days, though the stock may be closely held. Any of these corporate entities can take title to the stocks of commodities, no matter where they are located, and thus spread the risk to those who are in the best position to bear it. For example, canners of tomatoes often buy a farmer's crop before the vines are planted. They may even supply the vines.

This means that a futures market does not have a unique role in the allocation of risk. Corporations in the commodity business play the same role. Which kind of market is best for this role depends on the specifics of such things as transaction costs and taxes in each individual case. It seems clear that corporations do a better job for most commodities, because organized futures markets don't even exist for most commodities. Where they do exist, most of the risk is still transferred through corporations rather than through futures markets.

Thus there is no reason to believe that the existence of a futures market has any predictable effect on the path of the spot price over time. It is primarily the storage of a commodity that reduces fluctuations in its price over time.

Storage will occur whether or not there is any way of transferring risk. If there were no way to transfer risk, the price of a seasonal commodity might be somewhat higher before the prime production periods than it is now. But since there are good ways to transfer risk without using the futures market, even this benefit of futures markets is minimal.

I believe that futures markets exist because in some situations they provide an inexpensive way to transfer risk, and because many people both in the business and out like to gamble on commodity prices. Neither of these counts as a major benefit to society. The big benefit from futures markets is the side effect: the fact that participants in the futures markets can make production, storage, and processing decisions by looking at the pattern of futures prices, even if they don't take positions in that market.

This, of course, assumes that futures markets are efficient. It assumes that futures prices incorporate all available information about the future spot price of a commodity. It assumes that investors act quickly on any information they receive, so that the price reacts quickly to the arrival of the information. So quickly that individual traders find it very difficult to make money consistently by trading on information.

4. The pricing of forward contracts and commodity options

We have already discussed the pricing of futures contracts and the behavior of futures prices. In order to derive formulas for the other kinds of commodity contracts, we must make a few more assumptions.

First, let us assume that the fractional change in the futures price over any interval is distributed log-normally, with a known variance rate s^2. The derivations would go through with little change if we assumed that the variance rate is a known function of the time between t and t^*, but we will assume that the variance rate is constant.

Second, let us assume that all of the parameters of the capital asset pricing model, including the expected return on the market, the variance of the return on the market, and the short-term interest rate, are constant through time.

Third, let us continue to assume that taxes and transaction costs are zero.

Under these assumptions, it makes sense to write the value of a commodity contract only as a function of the corresponding futures price and time. If we did not assume the parameters of the capital asset pricing model were constant, then the value of a commodity contract might also depend on those parameters. Implicitly, of course, the value of the contract still depends on the transaction price and the transaction time.

Now let us use the same procedure that led to the formula for an option on a security.[6] We can create a riskless hedge by taking a long position in the

[6]The original option formula was derived by Black and Scholes (1973). Further results were obtained by Merton (1973).

option and a short position in the futures contract with the same transaction date. Since the value of a futures contract is always zero, the equity in this position is just the value of the option.

The size of the short position in the futures contract that makes the combined position riskless is the derivative of $w(x, t)$ with respect to x, which we will write w_1. Thus the change in the value of the hedged position over the time interval Δt is

$$\Delta w - w_1 \Delta x. \tag{14}$$

Expanding Δw, and noting that the return on the hedge must be at the instantaneous riskless rate r, we have the differential equation[7]

$$w_2 = rw - \tfrac{1}{2}s^2x^2w_{11}. \tag{15}$$

Note that this is like the differential equation for an option on a security, but with one term missing. The term is missing because the value of a futures contract is zero, while the value of a security is positive.

The main boundary condition for this equation is expression (6).[8] Using standard methods to solve eqs. (15) and (6), we obtain the following formula for the value of a commodity option:

$$w(x, t) = e^{r(t-t^*)}[xN(d_1) - c^*N(d_2)], \tag{16}$$

$$d_1 = \left[\ln \frac{x}{c^*} + \frac{s^2}{2}(t^* - t)\right] \Big/ s\sqrt{(t^* - t)},$$

$$d_2 = \left[\ln \frac{x}{c^*} - \frac{s^2}{2}(t^* - t)\right] \Big/ s\sqrt{(t^* - t)}.$$

This formula can be obtained from the original option formula by substituting $xe^{r(t-t^*)}$ for x everywhere in the original formula.[9] It is the same as the value of an option on a security that pays a continuous dividend at a rate equal to the stock price times the interest rate, when the option can only be exercised at maturity.[10] Again, this happens because the investment in a futures contract is zero, so an interest rate factor drops out of the formula.

[7] For the details of this expansion, see Black and Scholes (1973, p. 642 or p. 646).

[8] Another boundary condition and a regularity condition are needed to make the solution to (15) and (6) unique. The boundary condition is $w(0, t) = 0$. The need for these additional conditions was not noted in Black and Scholes (1973).

[9] Thorp (1973) obtains the same formula for a similar problem, related to the value of a security option when an investor who sells the underlying stock short does not receive interest on the proceeds of the short sale.

[10] Merton (1973) discusses the valuation of options on dividend-paying securities. The formula he obtains (f. 62) should be eq. (16), but he forgets to substitute $xe^{r(t-t^*)}$ for x in d_1 and d_2.

Eq. (16) applies to a 'European' commodity option, that can only be exercised at maturity. If the commodity option can be exercised before maturity, the problem of finding its value becomes much more complex.[11] Among other things, its value will depend on the spot price and on futures prices with various transaction dates before the option expires.

Eq. (16) also assumes that taxes are zero. But if commodity options are taxed like security options, then there will be substantial tax benefits for high tax bracket investors who write commodity options.[12] These benefits may be passed on in part or in full to buyers of commodity options in the form of lower prices. So taxes may reduce the values of commodity options.

Compared with the formula for a commodity option, the formula for the value of a forward contract is very simple. The differential equation it must satisfy is the same. Substituting $v(x, t)$ for $w(x, t)$ in eq. (15), we have

$$v_2 = rv - \tfrac{1}{2}s^2x^2v_{11}. \tag{17}$$

The main boundary condition is eq. (4), which we can rewrite as

$$v(x, t^*) = x - c. \tag{18}$$

The solution to (17) and (18) plus the implicit boundary conditions is

$$v(x, t) = (x - c)\,e^{r(t-t^*)}. \tag{19}$$

Expression (19) says that the value of a forward contract is the difference between the futures price and the forward contract price, discounted to the present at the short-term interest rate. It is independent of any measure of risk. It does not depend on the variance rate of the fractional change in the futures price or on the covariance rate between the change in the futures price and the return on the market.

[11]See Merton (1973) for a discussion of some of the complexities in finding a value for an option that can be exercised early.

[12]For a discussion of tax factors in the pricing of options, see Black (1975).

References

Black, F., 1975, Fact and fantasy in the use of options, Financial Analysts Journal 31, July/Aug.

Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81, May/June, 637-654.

Brennan, M.J., 1958, The supply of storage, American Economic Review 48, March, 50-72.

Chicago Board of Trade, 1973, Commodity trading manual (Board of Trade of the City of Chicago, Chicago, Ill.).

Cootner, P.H., 1960a, Returns to speculators: Telser versus Keynes, Journal of Political Economy 68, Aug., 396-404.

Cootner, P.H., 1960b, Rejoinder, Journal of Political Economy 68, Aug., 415-418.

Dusak, K., 1973, Futures trading and investor returns: An investigation of commodity market risk premiums, Journal of Political Economy 81, Nov./Dec., 1387- 1406.
Jensen, M.C., 1972, Capital markets: Theory and evidence, Bell Journal of Economics and Management Science 3, Autumn, 357–398.
Merton, R.C., 1973, The theory of rational option pricing, Bell Journal of Economics and Management Science 4, Spring, 141–183.
Telser, L., 1958, Futures trading and the storage of cotton and wheat, Journal of Political Economy 66, June, 233–255.
Telser, L., 1960, Returns to speculators: Telser versus Keynes, Reply, Journal of Political Economy 67, Aug., 404–415.
Thorp, E., 1973, Extensions of the Black-Scholes options model, Bulletin of the International Statistical Institute, Proceedings of the 39th Session, 522–529.