Empirical Asset Pricing

Part 2: Modelling the Yield Curve

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10. Term Structure of Interest Rates

Importance of the yield curve

- Conditional expectation of the pricing kernel is a yield:

$$y_t^n = -n^{-1} \log E_t e^{m_{t,t+n}} = -n^{-1} \sum_{j=1}^{\infty} \kappa_{jt}(m_{t,t+n})/j! \tag{1}$$

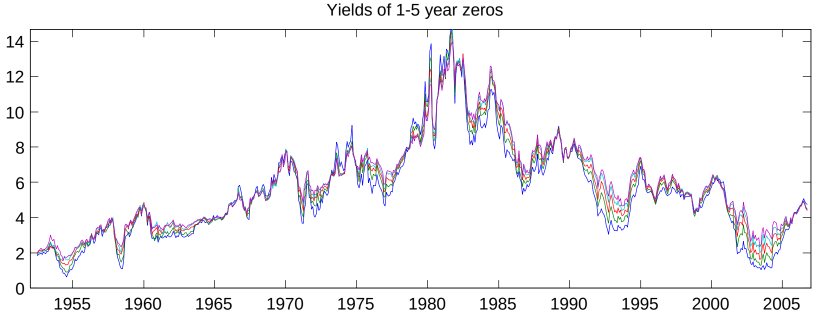
$$r_{t,f} = y_t^1 = -E_t m_{t,t+1} - var_t m_{t,t+1}/2 - \sum_{j=3}^{\infty} \kappa_{jt}(m_{t,t+1})/j! \tag{2}$$

- The equity premium puzzle is connected to the risk-free rate puzzle

- The yield can potentially tell us how the risk premium (variance of the pricing kernel) changes with horizon

- So, it is important to understand the properties of the yield curve

How many factors drive the term structure?



A principal components approach

- Eigenvalue decomposition (Litterman and Scheinkman, 1991)
- A (non-zero) vector q of dimension N is an eigenvector of a square ($N \times N$) matrix Σ iff

  $$\Sigma q = \lambda q$$

  where the scalar λ is the eigenvalue corresponding to q
- Construct matrix Q so that its i-th column is the eigenvector q_i of $\Sigma = cov(y)$
- Construct diagonal matrix Λ so that its diagonal elements are the corresponding eigenvalues, i.e. $\Lambda_{ii} = \lambda_i$.
- Then

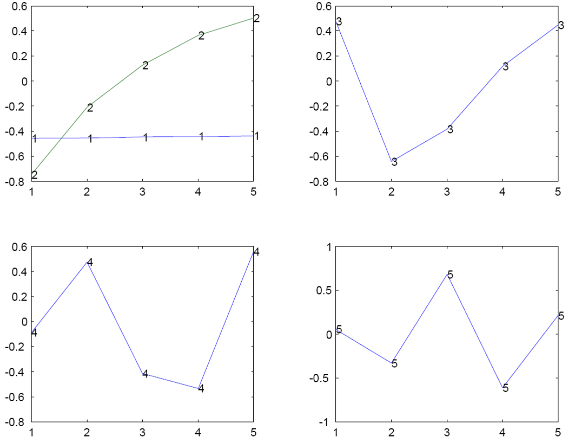
  $$\begin{aligned} \Sigma &= Q\Lambda Q' \\ x_t &= Q^{-1} y_t \\ cov(x) &= \Lambda \end{aligned}$$

- As a result we have a factor model of the yield curve:
  - Λ gives us the variances of the factors
  - $\lambda_i / \sum \lambda_i$ = fraction of variance explained by the ith factor
  - Columns of Q tell us how y loads on x

A principal components result

Standard results of PC decomposition of default-free yields

x	1	2	3	4	5
$\sigma(x)$	5.75	0.56	0.10	0.08	0.06

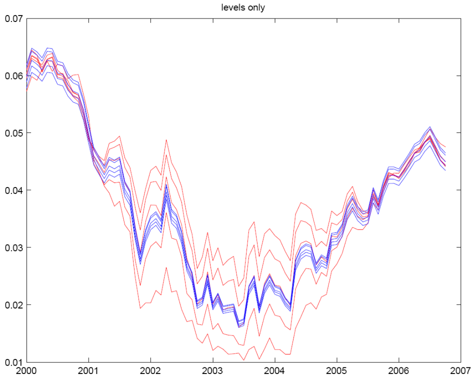


Factor models

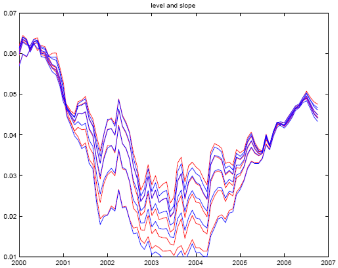
- Factor models come from dropping the small eigenvalues, then a larger number of series are driven by a smaller number of factors
- For example, what if we drop 4 and 5?

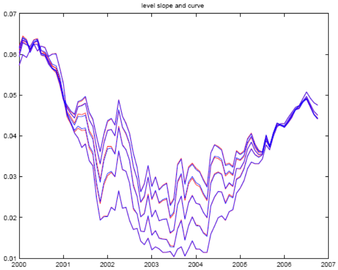
$$\begin{bmatrix} y_{t,1} \\ y_{t,2} \\ y_{t,3} \\ y_{t,4} \\ y_{t,5} \end{bmatrix} \approx q_1 \text{level}_t + q_2 \text{slope}_t + q_3 \text{curve}_t$$

- The factors are uncorrelated with each other by construction, so the equation above is a legitimate regression.



Factor models, cont'd





The persistence of interest rates

- The hypothesis that yields contain a unit root cannot be rejected.
- Some results from Pagan, Hall, and Martin (1995) that use an augmented Dickey-Fuller test (5% critical value is -2.87)
  - One-month yield. $ADF(12) = -2.02$, $\rho_1 = 0.98$, ρ_1 of changes $=$ 0.02
  - Three-month yield. $ADF(12) = -1.89$, $\rho_1 = 0.98$, ρ_1 of changes = 0.11
  - Six-month yield. $ADF(12) = -1.91$, $\rho_1 = 0.99$, ρ_1 of changes = 0.15
  - 10-year yield. $ADF(12) = -1.53$, $\rho_1 = 0.99$, ρ_1 of changes = 0.07
- Cross-sectional differences in yields (measures of slope) are stationary
- Cointegrating econometric results: Yields contain a single cointegrating vector

Estimation bias

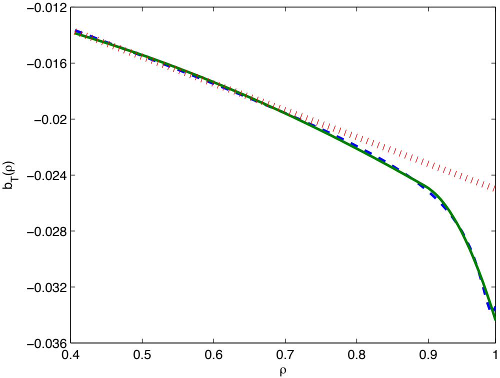


Figure 1:
Bias of the OLS estimator $\widehat{\rho}_T$ (sample size $T = 160$): exact (green solid line), spline approximation (blue dashes) and Kendall's approximation (red dots).

Source: Jardet, Monfort, and Pegoraro (2012)

Forecasting bias

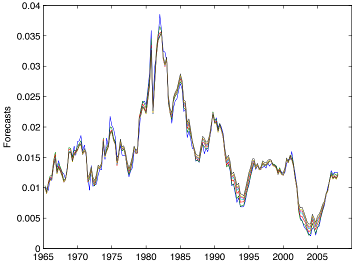


Figure 3:
q-step ahead short rate forecasts from the CVAR(3) model; $q = 1, 4, 8, 12, 16, 20, 40$ quarters.

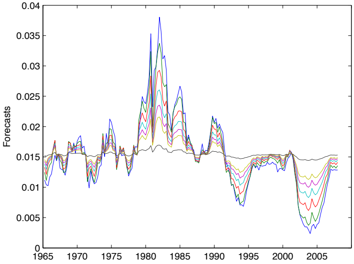


Figure 4:
q-step ahead short rate forecasts from the VAR(3) model; $q = 1, 4, 8, 12, 16, 20, 40$ quarters.

Source: Jardet, Monfort, and Pegoraro (2012)

Bond risk premiums

- Assuming log-normality and $m_t = 0$, equations (1) and (2) imply

$$
\begin{aligned}
y_t^n &= n^{-1}\left(-\sum_{i=1}^{n} E_t \Delta m_{t+i} - var_t\left(\sum_{i=1}^{n} \Delta m_{t+i}\right)/2\right) \\
r_{f,t} &= -E_t \Delta m_{t+1} - var_t \Delta m_{t+1}/2 \\
E_t r_{f,t+i} &= -E_t \Delta m_{t+i} - E_t\left(var_{t+i-1}(\Delta m_{t+i})\right)/2 \\
&= -E_t \Delta m_{t+i} - var_t \Delta m_{t+i}/2 + var_t\left(E_{t+i-1} \Delta m_{t+i}\right)/2 \\
y_t^n &= n^{-1} \sum_{i=0}^{n-1} E_t r_{f,t+i} + \ldots
\end{aligned}
$$

- The expectations hypothesis (EH) of interest rates
  - Strong form: Yields are averages of expected future short rates

$$
y_t^n = n^{-1} \sum_{i=0}^{n-1} E_t r_{t+i}
$$

  - Investors are risk-neutral with respect to bond risk
  - Weak form: Yields are a constant plus an average of expected future short rates – constant risk premium

Testing the EH

- Bonds with maturities n (long) and 1 (short)

$$y_t^n = n^{-1} \sum_{i=0}^{n-1} E_t y_{t+i}^1 + c$$

- Obvious method: Switch around previous equation, replace expectations with realizations, add residual term

$$n^{-1} \sum_{i=0}^{n-1} y_{t+i}^1 = -c + 1 \times y_t^n + e$$

- Problem: Nonstationarity

Campbell and Shiller (1991) regressions

- Alternative: explain changes in the yield on the longer-maturity yield
- Step expectation equation ahead by 1

$$y_{t+1}^{n-1} = (n-1)^{-1} \sum_{i=1}^{n-1} E_{t+1} y_{t+i}^{1} + c^*$$

- Multiply by leading fraction, subtract original expectations equation

$$(n-1)y_{t+1}^{n-1} - ny_t^n = (c^* - c) + (E_{t+1} - E_t)\left(\sum_{i=1}^{n-1} y_{t+i}^{1}\right) - E_t y_t^1$$

- Add y_t^n to both sides, turn into a regression equation by denoting change in expectations by unforecastable residual ε_{t+1}

$$(n-1)y_{t+1}^{n-1} - (n-1)y_t^n = (c^* - c) + y_t^n - y_t^1 + \varepsilon_{t+1}$$

- Denote cross-sectional difference in yields by S_t (slope), divide by $n-1$

$$y_{t+1}^{n-1} - y_t^n = (c^* - c) + \frac{1}{n-1} S_{t,1,n} + \varepsilon_{t+1}$$

Campbell and Shiller (1991) results

Table 1

Campbell–Shiller long-rate regression. The slope coefficients ϕ_{nT} are estimated from the indicated linear projections using the smoothed Fama and Bliss (1987) data set. The maturities n are given in months and s.e. is the estimated standard error of ϕ_{nT}.

$$R_{t+1}^{(n-1)} - R_t^n = \text{constant} + \phi_{nT}\,(R_t^n - r_t)/(n-1) + \text{residual}$$

Maturity	3	6	9	12	24	36	48	60	84	120
ϕ_{nT}	−0.428	−0.883	−1.228	−1.425	−1.705	−1.190	−2.147	−2.433	−3.096	−4.173
s.e.	(0.481)	(0.640)	(0.738)	(0.825)	(1.120)	(1.295)	(1.418)	(1.519)	(1.705)	(1.985)

Source: Dai and Singleton (2002)

- Weak form of expectations hypothesis is overwhelmingly rejected
- Regressions have a finite-sample bias in favor of the expectations hypothesis, so this evidence understates strength of rejection
- A major challenge in building no-arbitrage term structure models is fitting these patterns

Further evidence: Excess returns

- Let $rx^n_{t,t+1}$ be the excess one year return on the n period zero-coupon Treasury bond.
- Using "in n for 1" forward rates $f^n_t = q^n_t - q^{n+1}_t$, regress

$$rx^n_{t,t+1} = \alpha^{(n)} + \beta^{(n)}_1 y^1_t + \beta^{(n)}_2 f^2_t + \ldots + \beta^{(n)}_5 f^5_t + \varepsilon^n_{t+1}$$

- Cochrane and Piazzesi (2005) find that, to a close approximation, we can write

$$\begin{aligned} rx^n_{t,t+1} &= b_n \left[\gamma_0 + \gamma_1 y^1_t + \gamma_2 f^2_t + \ldots + \gamma_5 f^5_t\right] + \varepsilon^n_{t+1} \\ &= b_n \left[\gamma' f_t\right] + \varepsilon^n_{t+1}. \end{aligned}$$

- That is, a single "factor" or linear combination of forward rates (or yields) $\gamma' f_t$ captures expected bond returns at *all* maturities.
- This factor explains a large fraction of the average excess returns across all maturities (very high R^2)
- The risk premium $E_t r^n_{t,t+1} = b_n [\gamma' f_t]$ is time-varying

Regression results

A. Estimates of the return-forecasting factor, $\overline{rx}_{t+1} = \boldsymbol{\gamma}^\top \mathbf{f}_t + \bar{\varepsilon}_{t+1}$

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	R^2	$\chi^2(5)$
OLS estimates	−3.24	−2.14	0.81	3.00	0.80	−2.08	0.35	
Asymptotic (Large T) distributions								
HH, 12 lags	(1.45)	(0.36)	(0.74)	(0.50)	(0.45)	(0.34)		811.3
NW, 18 lags	(1.31)	(0.34)	(0.69)	(0.55)	(0.46)	(0.41)		105.5
Simplified HH	(1.80)	(0.59)	(1.04)	(0.78)	(0.62)	(0.55)		42.4
No overlap	(1.83)	(0.84)	(1.69)	(1.69)	(1.21)	(1.06)		22.6
Small-sample (Small T) distributions								
12 lag VAR	(1.72)	(0.60)	(1.00)	(0.80)	(0.60)	(0.58)	[0.22, 0.56]	40.2
Cointegrated VAR	(1.88)	(0.63)	(1.05)	(0.80)	(0.60)	(0.58)	[0.18, 0.51]	38.1
Exp. Hypo.							[0.00, 0.17]	

B. Individual-bond regressions

	Restricted, $rx_{t+1}^{(n)} = b_n(\boldsymbol{\gamma}^\top \mathbf{f}_t) + \varepsilon_{t+1}^{(n)}$					Unrestricted, $rx_{t+1}^{(n)} = \boldsymbol{\beta}_n \mathbf{f}_t + \varepsilon_{t+1}^{(n)}$			
n	b_n	Large T	Small T	R^2	Small T	R^2	EH	Level R^2	$\chi^2(5)$
2	0.47	(0.03)	(0.02)	0.31	[0.18, 0.52]	0.32	[0, 0.17]	0.36	121.8
3	0.87	(0.02)	(0.02)	0.34	[0.21, 0.54]	0.34	[0, 0.17]	0.36	113.8
4	1.24	(0.01)	(0.02)	0.37	[0.24, 0.57]	0.37	[0, 0.17]	0.39	115.7
5	1.43	(0.04)	(0.03)	0.34	[0.21, 0.55]	0.35	[0, 0.17]	0.36	88.2

Notes: The 10-percent, 5-percent and 1-percent critical values for a $\chi^2(5)$ are 9.2, 11.1, and 15.1 respectively. All p-values are less than 0.005. Standard errors in parentheses "( )", 95-percent confidence intervals for R^2 in square brackets "[ ]". Monthly observations of annual returns, 1964–2003.

Factor loadings

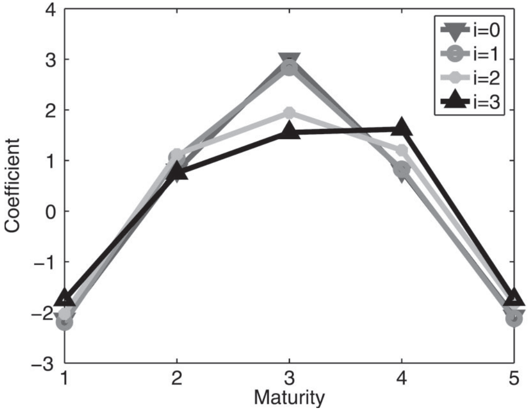


FIGURE 3. COEFFICIENTS IN REGRESSIONS OF AVERAGE (ACROSS MATURITY) EXCESS RETURNS ON LAGGED FORWARD RATES, $\overline{rx}_{t+1} = \boldsymbol{\gamma}^{\top}\mathbf{f}_{t-i/12} + \varepsilon_{t+1}$

The source of variation

- C&P show that the common component in bond risk premia is linked to the business cycle fluctuations of the 70's, 80's and 90's.

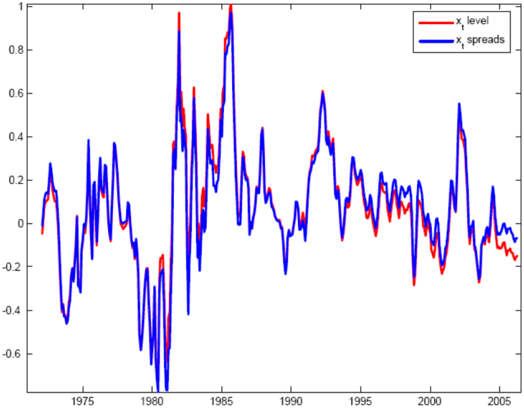


Figure 5: Return-forecasting factor $x_t = \gamma' f_t$ constructed from all forward rates (level) and using only forward spreads (spreads).

Example: Vasicek model

- Pricing kernel

$$\begin{aligned} m_{t+1} &= m + a(B)w_{t+1} \\ &= m + a_0 w_{t+1} + a_1 w_t + a_2 w_{t-1} + \cdots \\ w &\sim \text{NID}(0,1) \end{aligned}$$

- Interest rate

$$y_t^1 = -\log E_t(e^{m_{t+1}}) = -m - a_0^2/2 - a_1 w_t - a_2 w_{t-1} - \cdots$$

- ARMA(1,1) for m_t is AR(1) [Vasicek] for the interest rate

$$\begin{aligned} a_{j+1} &= \varphi a_j,\ j \geq 1, \\ y_t^1 &= (1-\varphi)(-m - a_0^2/2) + \varphi y_{t-1}^1 - a_1 w_t \end{aligned}$$

- Define partial sums

$$A_n = a_0 + a_1 + a_2 + \cdots + a_n$$

Bond pricing

- The LOOP and LIE:

$$Q_t^{n+1} = E_t M_{t,t+n+1} = E_t(M_{t,t+1} E_{t+1} M_{t+1,t+n+1}) = E_t(M_{t,t+1} Q_{t+1}^n)$$

- Guess: $q_{t+1}^n = q^n + \gamma^n(B) w_{t+1}$
- Then $m_{t,t+1} + q_{t+1}^n = m + q^n + (a(B) + \gamma^n(B)) w_{t+1}$
- Evaluating the expectation and lining up terms gives us

$$\begin{aligned} q^{n+1} &= m + q^n + (a_0 + \gamma_0^n)^2/2 \\ \gamma_j^{n+1} &= a_{j+1} + \gamma_{j+1}^n \end{aligned}$$

- The second equation implies: $\gamma_j^n = \sum_{i=1}^n a_{j+i} \equiv A_{n+j} - A_j$
- The first equation implies: $q^n = nm + \sum_{j=1}^n A_{j-1}^2/2$
- And, the unconditional slope is

$$E(y_t^n - y_t^1) = -n^{-1} q^n + q^1 = n^{-1} \sum_{j=1}^n (A_0^2 - A_{j-1}^2)/2$$

Excess returns

- Excess returns are, of course, a direct way to look at risk premiums
- Returns

$$
\begin{aligned}
r^n_{t,t+1} &= q^{n-1}_{t+1} - q^n_t = q^{n-1} - q^n + \gamma^{n-1}_0 w_{t+1} + (\gamma^{n-1}_1 - \gamma^n_0) w_t + \dots \\
&= -m - A^2_{n-1}/2 + (A_{n-1} - A_0) w_{t+1} - a_1 w_t - a_2 w_{t-1} - \dots
\end{aligned}
$$

- Expected excess returns

$$
\begin{aligned}
E_t(rx^n_{t+1}) &= -m - A^2_{n-1}/2 - a_1 w_t - a_2 w_{t-1} - \dots \\
&+ \quad m + A^2_0/2 + a_1 w_t + a_2 w_{t-1} + \dots = (A^2_0 - A^2_{n-1})/2
\end{aligned}
$$

- Risk premiums are constant
- We see that risk premiums are controlled by the gap between a_0 and higher order terms in the MA representation of m
- In the case of ARMA(1,1), these terms are $a_1 \varphi^j$
- Both a_1 and φ are determined by the dynamic properties of r_f, so a_0 controls risk premiums

Expectations hypothesis?

- The expected future short rate is:

$$E_t y^1_{t+i} = -m - a_0^2/2 - a_{i+1} w_t - a_{i+2} w_{t-1} - \dots$$

- Therefore

$$\begin{aligned} n^{-1} \sum_{i=0}^{n-1} E_t y^1_{t+i} &= -m - a_0^2/2 - n^{-1} \sum_{i=0}^{n-1} a_{i+1} w_t - n^{-1} \sum_{i=0}^{n-1} a_{i+2} w_{t-1} - \dots \\ &= -m - a_0^2/2 - n^{-1}(A_n - A_0) w_t - n^{-1}(A_{n+1} - A_1) w_{t-1} \\ &= -m - a_0^2/2 + n^{-1} q^n + y^n_t \end{aligned}$$

- Weak-form expectation hypothesis holds, so not a very good model
- Note that $a_0 = 0$ does not imply strong-from EH

Varying risk premiums

- We have to extend the basic Vasicek model to account for multi-factor structure of yields and for variation in risk premiums
- Multifactor Vasicek model

$$
\begin{aligned}
X_{t+1} &= \mu + \phi X_t + v_{t+1}, \quad v_{t+1} \sim N(0, I) \\
m_{t+1} &= -\delta_0 - \delta_1' X_t - \lambda_t' \lambda_t / 2 - \lambda_t' v_{t+1} \\
\lambda_t &= \lambda_0 + \lambda_1 X_t
\end{aligned}
$$

- Pricing:

$$
\begin{aligned}
Q_t^1 &= E_t\left(M_{t+1}\right) = e^{-\delta_0 - \delta_1' X_t} \\
y_t^1 &= \delta_0 + \delta_1' X_t
\end{aligned}
$$

- For longer maturities, guess:

$$
\begin{aligned}
q_t^n &= A_n - B_n' X_t \\
B_n' &= -\delta_1'(I + \phi^* + \phi^{*2} + \ldots + \phi^{*n-1}) = -\delta_1'(I - \phi^{*n})(I - \phi^*)^{-1} \\
\phi^* &= \phi - \lambda_1
\end{aligned}
$$

- Excess returns are time-varying

$$
E_t r x_{t,t+1}^n = B_{n-1}' \left(\lambda_0 - \frac{1}{2} B_{n-1} \right) + \left(B_{n-1}' \lambda_1 \right) X_t
$$

11. Structural models of real yields

Economic properties of entropy (II)

- Dynamics: horizon dependence, $I(n) = n^{-1} EL_t(M_{t,t+n})$ – avg over n periods

$$H(n) \quad \equiv \quad I(n) - I(1) = -E(y_t^n - y_t^1)$$

Derivation of the horizon dependence

- First, note that if M_{t+1} is iid, then $H(n) = 0$
- In a stationary environment, conditional entropy over n periods is

$$
\begin{aligned}
L_t(M_{t,t+n}) &= \log E_t M_{t,t+n} - E_t \log M_{t,t+n} \\
&= q_t^n - E_t \sum_{j=1}^{n} m_{t+j-1,t+j}.
\end{aligned}
$$

  Entropy is therefore

$$
I(n) = n^{-1} E q_t^n - E m_{t,t+1}.
$$

- Bond yields are related to prices by $y_t^n = -n^{-1} q_t^n$. Therefore horizon dependence is related to mean yield spreads by

$$
H(n) = -E(y_t^n - y_t^1).
$$

- Horizon dependence is negative if the mean yield curve is increasing, positive if it is decreasing, and zero if it is flat.

Example: Vasicek model

- Pricing kernel

$$
\begin{aligned}
m_{t+1} &= m + a(B) w_{t+1} \\
&= m + \underbrace{a_0 w_{t+1}}_{\text{entropy}} + \underbrace{a_1 w_t + a_2 w_{t-1} + \cdots}_{\text{horizon dependence}} \\
w &\sim \text{NID}(0,1)
\end{aligned}
$$

- Entropy

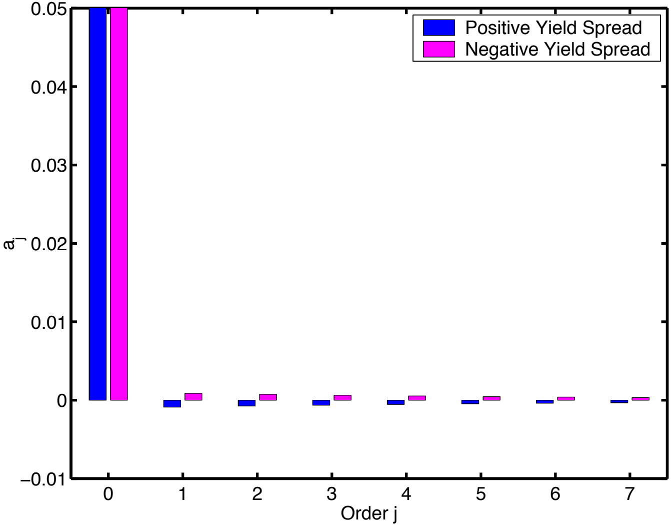
$$
EL_t(M_{t+1}) = a_0^2/2 = A_0^2/2 \Rightarrow a_0 \text{ “big”}
$$

- Horizon dependence

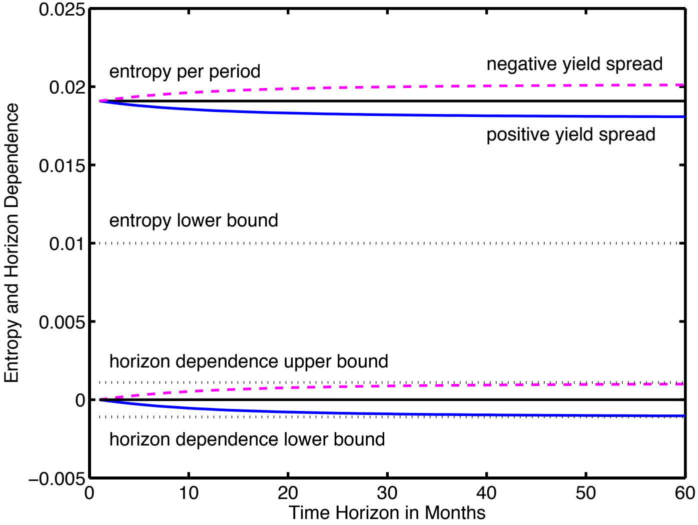
$$
H(n) = n^{-1} \sum_{j=1}^{n} (A_{j-1}^2 - A_0^2)/2 \Rightarrow a_j \text{ “small”}
$$

- Calibrate the ARMA(1,1) version:
  - φ and a_1 to match the serial correlation and variance of y_t^1
  - a_0 to match the mean 10-year term spread

Vasicek model: moving average coefficients



Vasicek model: horizon dependence



Endowment economies revisited

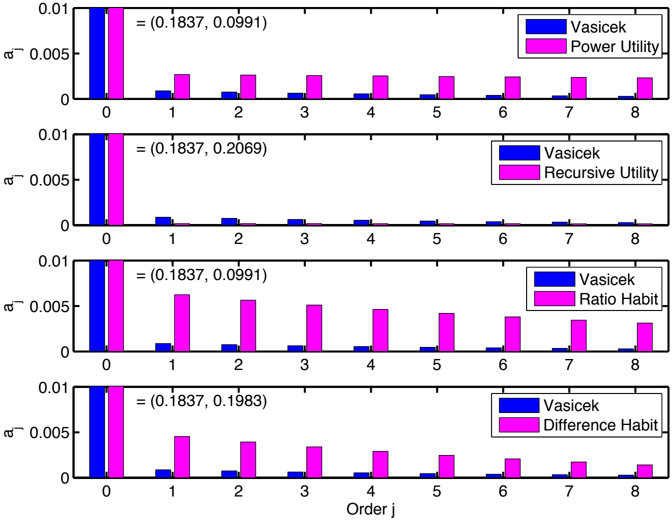
- Consumption growth

$$g_t = g + \gamma(B) v^{1/2} w_t$$

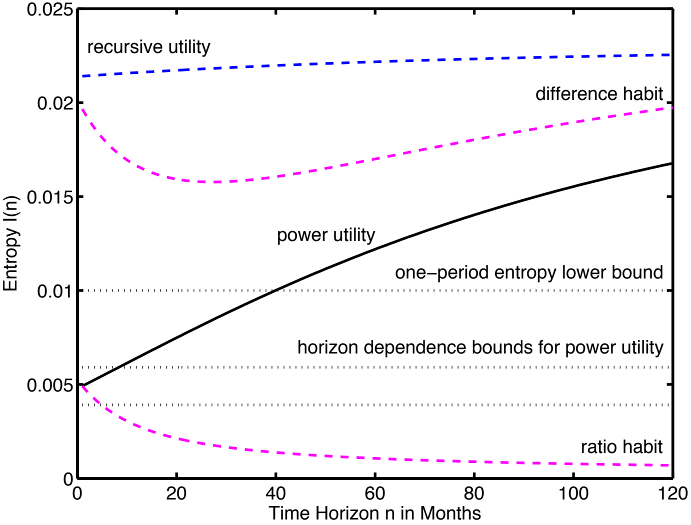
- Pricing kernel (recursive, ratio habit, diff habit)

$$\begin{aligned}
m_{t+1}^{REC} &= \text{const} + \underbrace{[(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)]v^{1/2}}_{a_0} w_{t+1} \\
&\quad + \underbrace{(\rho-1)\gamma_1 v^{1/2}}_{a_1} w_t + \underbrace{(\rho-1)\gamma_2 v^{1/2}}_{a_2} w_{t-1} + \cdots \\
m_{t+1}^{RH} &= \text{const} + \underbrace{(\rho-1)\gamma_0 v^{1/2}}_{a_0} w_{t+1} \\
&\quad + \underbrace{[(\rho-1)\gamma_1 - \rho\chi_0\gamma_0]v^{1/2}}_{a_1} w_t + \underbrace{[(\rho-1)\gamma_2 - \rho\chi_1\gamma_1]v^{1/2}}_{a_2} w_{t-1} + \cdots \\
m_{t+1}^{DH} &= \text{const} + \underbrace{(\rho-1)(1/s)\gamma_0 v^{1/2}}_{a_0} w_{t+1} \\
&\quad + \underbrace{(\rho-1)(1/s)[\gamma_1 - (1-s)\chi_0\gamma_0]v^{1/2}}_{a_1} w_t + \underbrace{(\rho-1)(1/s)[\gamma_2 - (1-s)\chi_1\gamma_1]v^{1/2}}_{a_2} w_{t-1} + \cdots
\end{aligned}$$

Calibrated MA coefficients



Term structure of entropy



Stochastic volatility?

- In CC habits, the term structure is flat when $b = 0$
- Recursive: Consumption growth

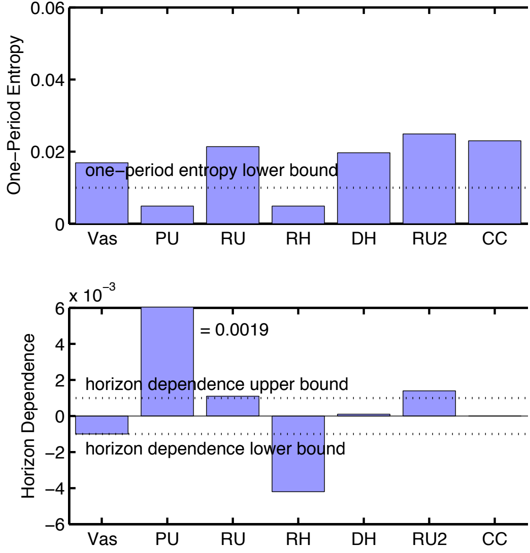
$$
\begin{aligned}
g_t &= g + \gamma(B) v_{t-1}^{1/2} w_{gt}, \ \gamma_{j+1} = \varphi_g \gamma_j, \ j \geq 1 \\
v_t &= v + \nu(B) w_{vt}, \ \nu_{j+1} = \varphi_v \nu_j, \ j \geq 0
\end{aligned}
$$

- Pricing kernel

$$
\begin{aligned}
m_{t+1} = \ & \text{const} + [(\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1)] v_t^{1/2} w_{gt+1} \\
& + (\alpha - \rho)(\alpha/2)\gamma(b_1)^2 b_1 \nu(b_1) w_{vt+1} + (\rho - 1)[\gamma(B)/B]_+ v_{t-1}^{1/2} w_{gt} \\
& - (\alpha - \rho)(\alpha/2)\gamma(b_1)^2 \nu(B) w_{vt}
\end{aligned}
$$

- w_v lead to negative horizon dependence, the result of the different signs of the initial and subsequent MA coefficients. The overall impact on $H(n)$ depends on the relative magnitudes of the consumption and variance effects and the nonlinear interaction between the consumption growth and conditional variance processes.
- Relative to the constant variance case, $H(120)$ rises from 0.0011 to 0.0012. This suggests that H is dominated, with these parameter values, by the dynamics of consumption growth. The increase in $H(120)$ indicates that nonlinear interactions between the two processes are quantitatively significant.

Summary ... so far



12. Long-Run Properties of the Structural Models

What do the models imply at infinite horizons?

- Can we select (or reject) models on the basis of their long-run properties?
- Suppose you could link properties of the pricing kernel to long-dated assets ... bonds are a natural choice
- The discussion is based on the work of Bansal and Lehmann (1997); Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Hansen (2012)

Multiplicative decomposition

- Consider an eigenvalue problem:

$$E_t(M_{t,t+1}e_{t+1}) = \lambda e_t$$

- The Perron-Frobenius theorem asserts that the largest real eigenvalue corresponds to an eigenvector with strictly positive components (uniqueness is guaranteed for matrixes, that is, Markov-switching process)
- PFT suggests a way of searching for the largest eigenvalue: guess e_t as an exponential of shocks (or states) in the model (but beware the uniqueness issue!)
- The Hansen-Scheinkman decomposition

$$M_{t,t+1} = \lambda \cdot \tilde{M}_{t,t+1} \cdot e_t / e_{t+1},$$

where $\tilde{M}_{t,t+1}$ is defined by the decomposition and is a multiplicative martingale, $E_t(\tilde{M}_{t,t+1}) = 1$

- The Alvarez-Jermann decomposition

$$M_{t,t+1} = M^P_{t,t+1} \cdot M^T_{t,t+1},$$

where $M^P_{t,t+1} = \tilde{M}_{t,t+1}$ is labeled as a “permanent” component, and $M^T_{t,t+1} = \lambda e_t / e_{t+1}$ is labelled as a “transient” component

Connection to bonds

- Bond returns:

$$\begin{aligned} R^{\infty}_{t,t+1} &= 1/M^{T}_{t,t+1} \\ Er^{\infty}_{t,t+1} &= -\log\lambda \end{aligned}$$

- Proof:

$$\frac{Q^{k-1}_{t+1}}{Q^{k}_{t}} = \frac{E_{t+1}(\tilde{M}_{t+1,t+k}\lambda^{k-1}e_{t+1}/e_{t+k})}{E_t(\tilde{M}_{t,t+k}\lambda^{k}e_t/e_{t+k})} = \frac{e_{t+1}}{\lambda e_t}\frac{\tilde{E}_{t+1}(1/e_{t+k})}{\tilde{E}_t(1/e_{t+k})}$$

- Therefore, the long bond is the growth optimal asset if $M^P = \tilde{M}$ is constant.
  - This happens when Λ_{t+1} in $M_{t,t+1} = \Lambda_{t+1}/\Lambda_t$ is iid
  - If there is a utility-maximizing investor then Λ_t is marginal utility; it is iid if, e.g., C_t is iid (not C_t/C_{t-1}!) in the PU case

Connection to entropy

- Similarly,

$$
\begin{aligned}
n^{-1}\log E_t M_{t,t+n} &= \log\lambda + n^{-1}\tilde{E}_t e_t / e_{t+n} \to \log\lambda \\
I(\infty) &= \log\lambda - m
\end{aligned}
$$

- The approach results in a potentially useful characterization of the models we discussed so far

Long-run properties of pricing kernels

- The CRRA PK: $M_{t,t+1} = \beta G_{t+1}^{\rho-1}$

- Consider models that can be expressed as:

$$M'_{t,t+1} = \beta G_{t+1}^{\varepsilon} d_{t+1}/d_t,$$

  where d_t is stationary and ε is an exponent TBD.

- Then for M' the $I(\infty)$ is the same as that of CRRA with $\rho = \varepsilon + 1$

- Proof. The PK $M'_{t,t+1} = M_{t,t+1} d_{t+1}/d_t$ has the same eigenvalue λ and eigenfunction $e'_t = e_t/d_t$. Therefore, $n^{-1} \log E_t M'_{t,t+n} \to \log \lambda$. Because d_t is stationary, $Em'_{t,t+1} = Em_{t,t+1}$.

Examples

- Difference habit

$$M'_{t,t+1} = \beta G_{t+1}^{\rho-1}(S_{t+1}/S_t)^{\rho-1}, \quad \varepsilon = \rho - 1, \quad d_t = S_t^{\rho-1}$$

- Ratio habit

$$\begin{aligned} M'_{t,t+1} &= \beta G_{t+1}^{\rho-1}(X_{t+1}/X_t)^{-\rho} \\ &= \beta G_{t+1}^{-1}[(X_{t+1}/C_{t+1})/(X_t/C_t)]^{-\rho}, \quad \varepsilon = -1, \quad d_t = (X_t/C_t)^{-\rho} \end{aligned}$$

- Recursive
  - Pricing kernel

$$\begin{aligned} M'_{t,t+1} &= \beta G_{t+1}^{\rho-1}(G_{t+1}u_{t+1}/\mu_t(G_{t+1}u_{t+1}))^{\alpha-\rho} \\ &= \beta G_{t+1}^{\alpha-1}(u_{t+1}/\mu_t(G_{t+1}u_{t+1}))^{\alpha-\rho} \end{aligned}$$

  - Hansen (2012) considers an "interesting limit" (corresponding to $b_0 = 0$ and $b_1 = 1$) where

$$M'_{t,t+1} = \beta' G_{t+1}^{\alpha-1}(u_{t+1}/u_t)^{\alpha-\rho}, \quad \varepsilon = \alpha - 1, \quad d_t = u_t^{\alpha-\rho}$$

Issues with interpretation

- The permanent/transient language is confusing, however
- Write $M_{t,t+1} = e^{-r_t} L_{t,t+1}$. What's wrong with $L_{t,t+1} = M^P_{t,t+1}$ and $e^{-r_t} = M^T_{t,t+1}$?
- The two components are correlated with each other, so cannot separate their impact. Consider a basic example:

$$m_{t+1} = m + a_0 w_{t+1} + a_1 w_t$$

Then

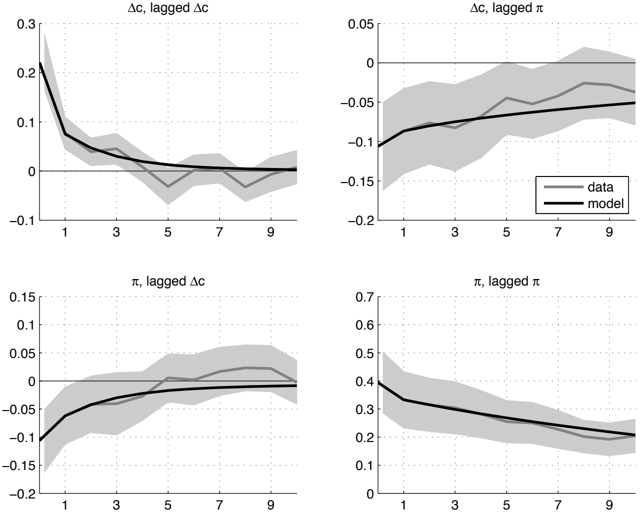
$$\begin{aligned} M^P_{t+1} &= e^{-(a_0+a_1)^2/2+(a_0+a_1)w_{t+1}} \\ M^T_{t+1} &= e^{m+(a_0+a_1)^2/2-a_1 w_{t+1}+a_1 w_t} \end{aligned}$$

13. Structural Models of Nominal Yields

From real to nominal rates

- Most models that we've considered imply negatively sloping real yield curve
- Reasonable people disagree whether this is a correct pattern
- Regardless, the general view is that we know that the nominal slope is positive on average
- We have an extra lever to "fix" the real curve: inflation
- Recall that $m^{\$}_{t,t+1} = m_{t,t+1} - \pi_{t,t+1}$

Consumption growth and inflation



Source: Piazzesi and Schneider (2006)

CC habits

- Wachter (2006) considers the case of $b > 0$ implying a positive real slope

- Estimate a model (restricted VAR):

$$\begin{aligned} g_{t+1} &= (1-\varphi_g)g + \varphi_g g_t + v_g^{1/2} w_{gt+1} + \theta_g v_g^{1/2} w_{gt} \\ \pi_{t+1} &= (1-\varphi_\pi)\pi + \varphi_\pi \pi_t + v_\pi^{1/2} w_{\pi t+1} + \theta_\pi v_\pi^{1/2} w_{\pi t} \end{aligned}$$

  where shocks are correlated, $\hat{\rho} = -0.2$

- Discard the equation for g (it is different in the CC model), and use the estimated π

- The model seems to match the Campbell-Shiller pattern. What about time series?

Short rate

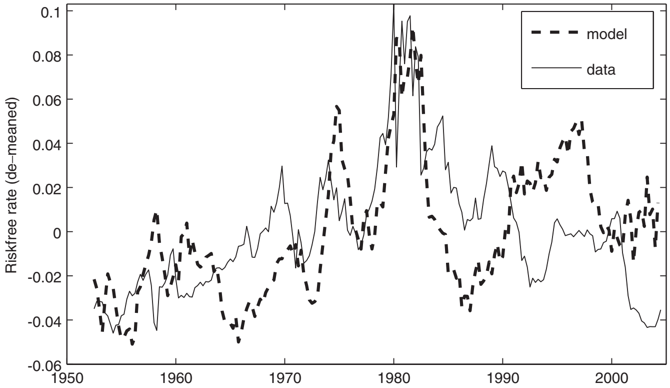


Fig. 7. Time series of the three-month yield in the data and predicted by the model. The solid line shows the time series of the nominal three-month yield in quarterly data. The dashed line shows the implied time series when quarterly data on consumption and the price level are fed into the model. Expected inflation is taken to be its mean conditional on past inflation data, given the maximum likelihood estimates in Table 1. Using Eq. (2), surplus consumption is generated from actual consumption. Both series are de-meaned. Data are quarterly, begin in the second quarter of 1952, and end in the second quarter of 2004.

Term spread (5 years)

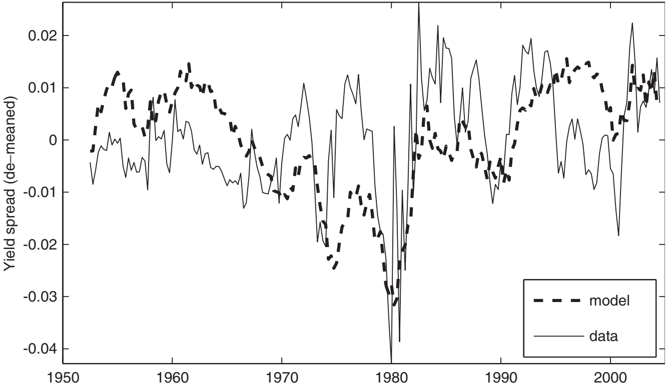


Fig. 8. Time series of the yield spread in the data and predicted by the model. The yield spread is the difference in yields between the five-year nominal bond and the three-month bond. The solid line shows the time series of the yield spread between bonds in the data. The dashed line shows the implied time series when quarterly data on consumption and the price level are fed into the model. Expected inflation is taken to be its mean conditional on past inflation data, given the maximum likelihood estimates in Table 1. Using Eq. (2), surplus consumption is generated from actual consumption. Both series are de-meaned. Data are quarterly, begin in the second quarter of 1952, and end in the second quarter of 2004.

Recursive preferences

- Several authors considered this framework, e.g., Bansal and Shaliastovich (2013); Piazzesi and Schneider (2006)
- The BS model:

$$
\begin{aligned}
g_{t+1} &= g + x_{gt} + v_g^{1/2}\omega_{gt+1} \\
\pi_{t+1} &= \pi + x_{\pi t} + v_\pi^{1/2}\omega_{\pi t+1} \\
x_{gt+1} &= \varphi_g x_{gt} + \varphi_{g\pi} x_{\pi t} + v_{gt}^{1/2} e_{gt+1} \\
x_{\pi t+1} &= \varphi_\pi x_{\pi t} + v_{\pi t}^{1/2} e_{\pi t+1} \\
v_{gt+1} &= (1-\nu_g) v_{g0} + \nu_g v_{gt} + v_{wg}^{1/2} w_{gt+1} \\
v_{\pi t+1} &= (1-\nu_\pi) v_{\pi 0} + \nu_\pi v_{\pi t} + v_{w\pi}^{1/2} w_{\pi t+1}
\end{aligned}
$$

- Recall that if ω's and e's are perfectly correlated, we can re-write this process as ARMA(1,1), so similar to the one in Wachter
- PS consider homoscedastic shocks to g and π, and focus on the implications of learning about the consumption and inflation dynamics for bond yields.

Estimation

- Assume that nominal 1-, 3-, and 5-year yields, and expectations of inflation are observed without an error. Invert x_{gt}, v_{gt}, and $v_{\pi t}$ from the yields

- Some estimates: $\hat{\varphi}_{g\pi} = -0.047$, $1-\hat{\alpha} = 21$, $1/(1-\hat{\rho}) = 1.8$

- Results:

Consumption and inflation

Consumption growth and inflation rate: U.S. data and model

	Data		Model
Consumption:			
Mean	1.94	(0.20)	1.94
Std. Dev.	1.03	(0.08)	1.03
AR(1)	0.29	(0.12)	0.22
Inflation:			
Mean	3.61	(0.50)	3.61
Std. Dev.	1.76	(0.21)	1.76
AR(1)	0.56	(0.11)	0.62
Corr($\pi, \Delta c$)	-0.11	(0.10)	-0.25

Properties of consumption growth and inflation rate in the data and the estimated model. Data are quarterly observations of consumption and inflation in the United States from 1947 to 2010; population values for the model. Standard errors are Newey-West adjusted.

Nominal bonds

Bond markets: U.S. data and model

	1y	2y	3y	4y	5y
Data:					
Yield Level	6.09	6.33	6.52	6.68	6.79
Std. Dev.	3.09	2.97	2.87	2.78	2.70
EH Slope:		−0.41	−0.78	−1.14	−1.15
		(0.44)	(0.56)	(0.63)	(0.67)
CP Slope:		0.44	0.85	1.28	1.43
		(0.11)	(0.23)	(0.33)	(0.42)
CP R^2		0.15	0.17	0.20	0.17
		(0.10)	(0.10)	(0.11)	(0.12)
Model:					
Yield Level:	6.10	6.29	6.50	6.73	6.97
Std. Dev.	2.37	2.29	2.24	2.20	2.17
EH Slope:		−0.45	−0.51	−0.57	−0.61
CP Slope		0.44	0.83	1.19	1.53
CP R^2		0.16	0.17	0.17	0.16

Nominal term structure, slopes in the expectations hypothesis regressions, and slopes and R^2s in Cochrane and Piazzesi (2005) single-factor bond premium regressions. Data are second-month-of-the-quarter observations of quarterly yields from 1969 to 2010; model output is based on population values.

Endogenous inflation via the Taylor rule

- Monetary policy, as proxied by the Taylor rule, determines inflation (Gallmeyer, Hollifield, Palomino, and Zin, 2007, 2009)
- Assume a standard endowment setup, which implies a pricing kernel M_t. Therefore, the real and nominal interest rates are:

$$
\begin{aligned}
r_t &= -\log E_t M_{t+1} \\
M^{\$}_{t+1} &= M_{t+1} \cdot P_t / P_{t+1} \\
r^{\$}_t &= -\log E_t M^{\$}_{t+1} = -\log E_t \left(M_{t+1} \cdot e^{-\pi_{t+1}} \right)
\end{aligned}
$$

- Under conditional log-normality:

$$
\begin{aligned}
r^{\$}_t &= -E_t(m_{t+1} - \pi_{t+1}) - var_t(m_{t+1} - \pi_{t+1})/2 \\
&= r_t + E_t(\pi_{t+1}) + cov_t(m_{t+1}, \pi_{t+1}) - var_t \pi_{t+1}/2
\end{aligned}
$$

- Now impose a Taylor rule:

$$
\begin{aligned}
r^{\$}_t &= \tau_0 + \tau_g g_t + \tau_\pi \pi_t + s_t && \text{(the rule)} \\
s_t &= \varphi_s s_{t-1} + v_s^{1/2} w_{st} && \text{(the MP shock)}
\end{aligned}
$$

- One equation in one unknown: inflation

Equilibrium Inflation: "Guess and Verify"

$$\overbrace{\tau_0 + \tau_g g_t + \tau_\pi \underbrace{(\pi_0 + \pi_g g_t + \pi_v v_t + \pi_s s_t)}_{\text{guess for } \pi_t} + s_t}^{r_t^\$}$$

$$= -\log E_t[\exp\{\overbrace{m_{t+1} - \underbrace{(\pi_0 + \pi_g g_{t+1} + \pi_v v_{t+1} + \pi_s s_{t+1})}_{\text{guess for } \pi_{t+1}}}^{m_{t+1}^\$}\}]$$

$$\begin{aligned}
\pi_g &= \frac{(1-\rho)\varphi_g - \tau_g}{\tau_\pi - \varphi_g} \\
\pi_v &= \frac{\alpha(\alpha-\rho)(1-b_1\varphi_g)^{-2} - (\lambda_g + \pi_g)^2}{2(\tau_\pi - \varphi_v)}, \quad \lambda_g = (1-\alpha) - (\alpha-\rho)\frac{b_1\varphi_g}{1-b_1\varphi_g} \\
\pi_s &= -\frac{1}{\tau_\pi - \varphi_g}
\end{aligned}$$

Calibration

- Endowment Growth (Piazzesi-Schneider):

  $\varphi_g = 0.36$, $\theta_g = 0.006$ [assume AR(1) for g_t]

- Exogenous inflation (Piazzesi-Schneider):

  $\varphi_\pi = 0.8471$, $\theta_\pi = 0.0093$, $v_\pi = 0.0063(1-\varphi_\pi^2)$

- Stochastic volatility (Bansal-Yaron, quarterly adj.):

  $\varphi_v = 0.973$, $\theta_v = 0.0001825$, $\sigma_v = 0.9884 \times 10^{-5}$

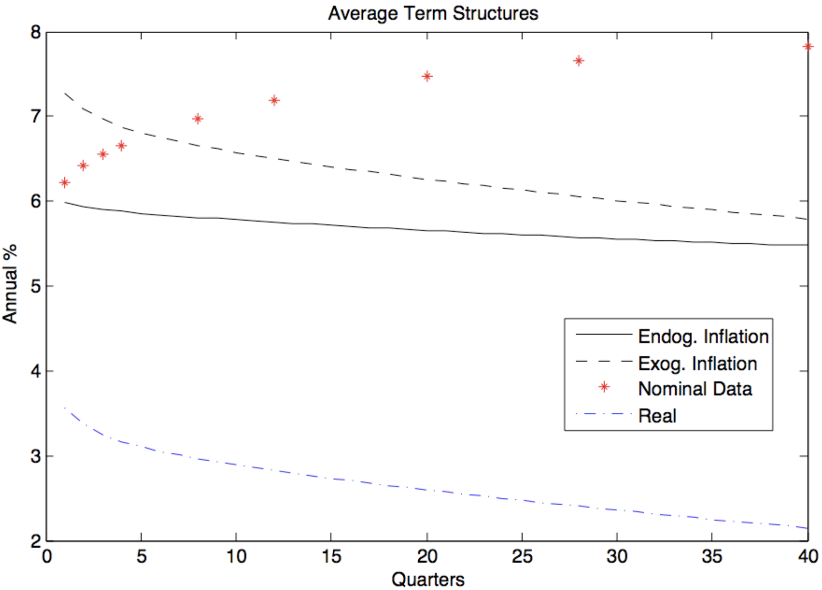
- Policy Shock (Ang-Dong-Piazzesi):

  $\varphi_s = 0.922$, $v_s = (0.023 \times 10^{-4})$.

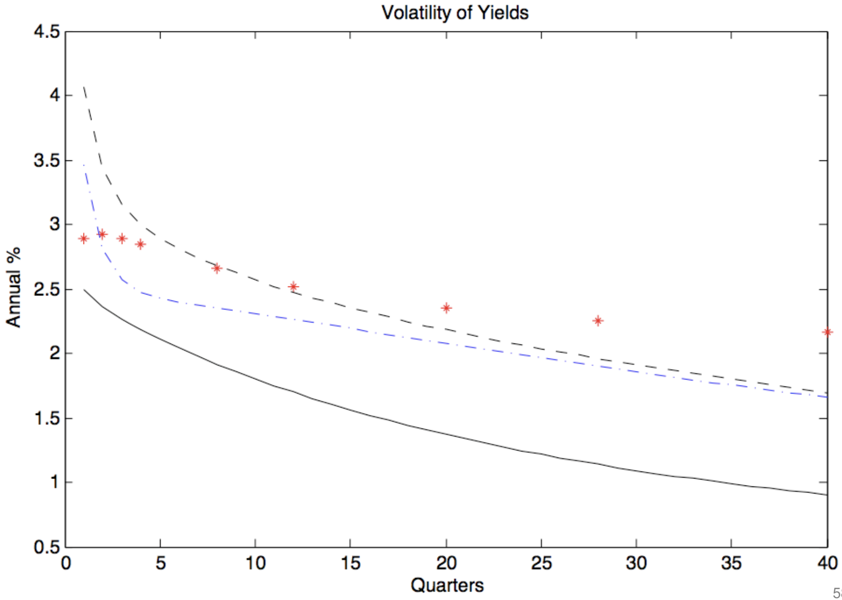
Calibration II

- Fix a level of ρ for each plot
- Pick α and β to minimize the distance between the average nominal yields and yield volatilities in the data and those in the exogenous inflation economy
- Pick τ_0, τ_g, and τ_π to minimize the distance between the average nominal yields and yield volatilities in the data and those in the endogenous inflation economy

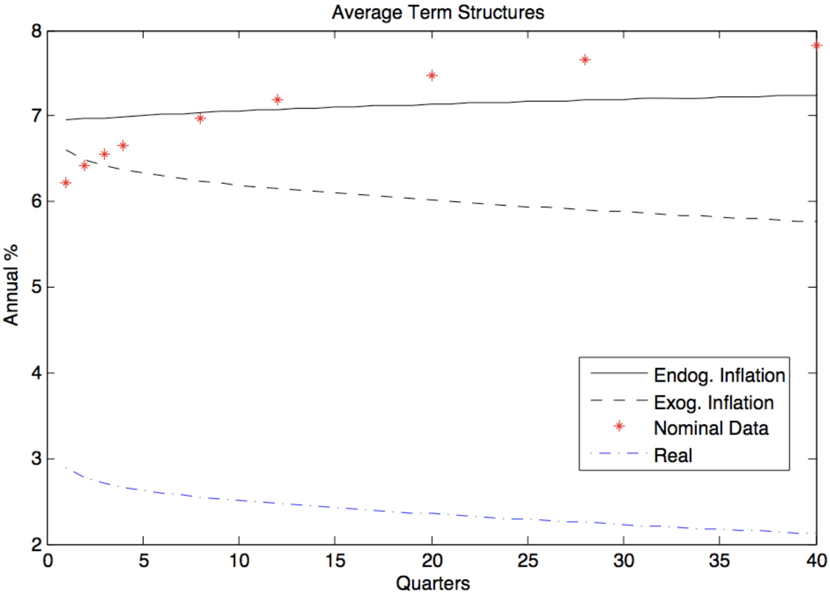
$$\rho = -0.5,\ \alpha = -4.835,\ \beta = 0.999$$
$$\tau_0 = 0.003,\ \tau_g = 1.2475,\ \tau_\pi = 1.000$$



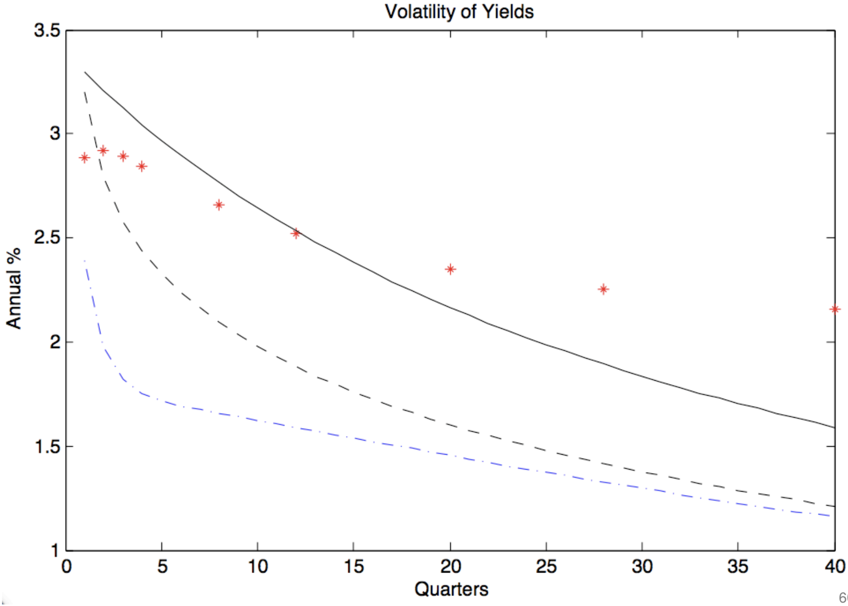
$$\rho = -0.5,\ \alpha = -4.835,\ \beta = 0.999$$
$$\tau_0 = 0.003,\ \tau_g = 1.2475,\ \tau_\pi = 1.000$$



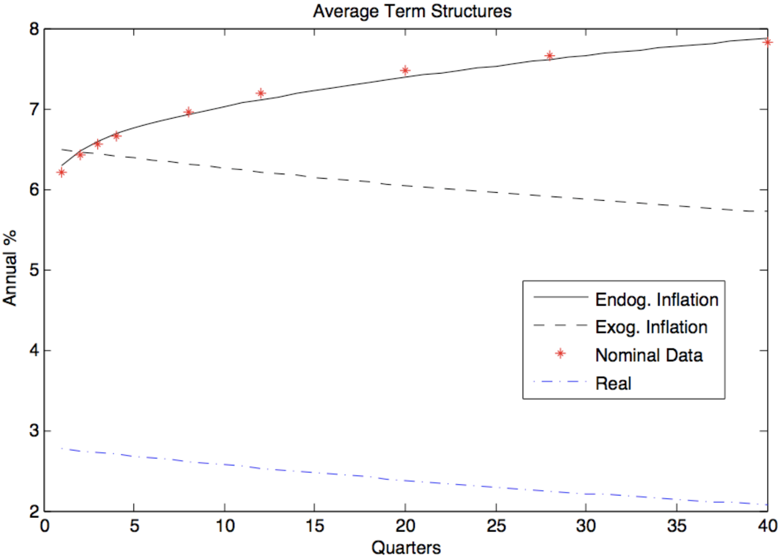
$\rho = 0.0$, $\alpha = -4.061$, $\beta = 0.998$
$\tau_0 = 0.003$, $\tau_g = 0.973$, $\tau_\pi = 0.973$



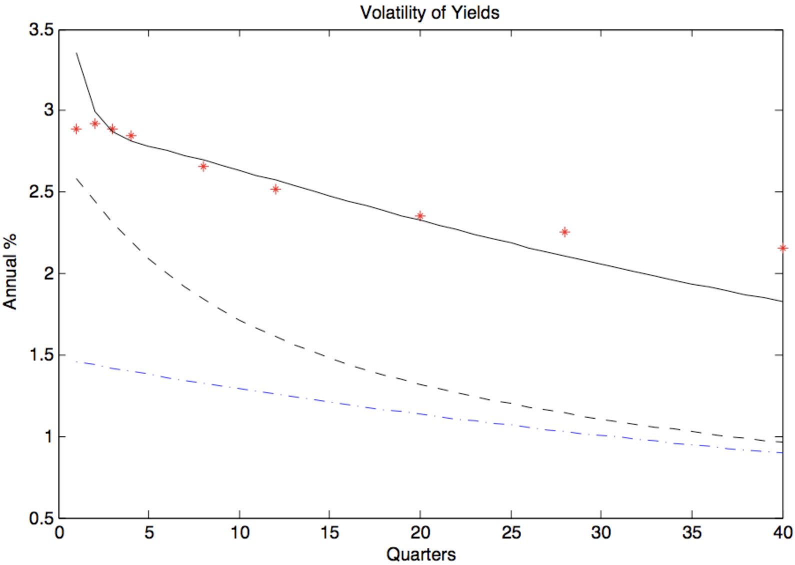
$$\rho = 0.0,\ \alpha = -4.061,\ \beta = 0.998$$
$$\tau_0 = 0.003,\ \tau_g = 0.973,\ \tau_\pi = 0.973$$



$$\rho = 1.0,\ \alpha = -6.079,\ \beta = 0.990$$
$$\tau_0 = -0.004,\ \tau_g = 1.534,\ \tau_\pi = 1.607$$



$$\rho = 1.0,\ \alpha = -6.079,\ \beta = 0.990$$
$$\tau_0 = -0.004,\ \tau_g = 1.534,\ \tau_\pi = 1.607$$



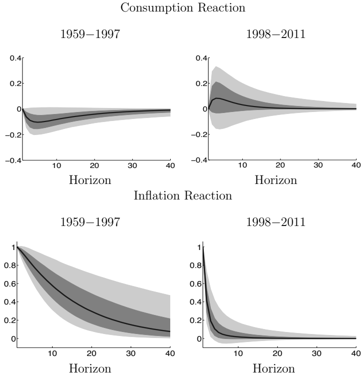
Discussion

- $\rho = 1$
  - As ρ increases, EIS increases, implying less demand for smoothing consumption over time
  - Representative agent's demand for long-term bonds for the purpose of intertemporal consumption smoothing declines
  - This leads to lower equilibrium prices and higher yields for real long-term bonds
  - The average real yield curve therefore is less downward-sloping
  - Also the sensitivity of long-term real yields to output growth is reduced leading to less volatile long-term yields

- Inflation and consumption:

$$cov_t(g_{t+1}, \pi_{t+1}) = \pi_g v_t = -\frac{\tau_g - (1-\rho)\varphi_g}{\tau_\pi - \varphi_g} v_t < 0$$

Subsample evidence



	Pre-1998	Post-1998	Full Sample
	Annualized Average Bond Yields		
Mean (y_{3m})	6.07	2.64	5.16
Mean (y_{1y})	6.51	2.88	5.55
Mean (y_{3y})	6.87	3.35	5.94
Mean (y_{5y})	7.05	3.78	6.19
Mean (y_{10y})	7.35	4.38	6.57

Source: Song (2014)

A regime-switching model

- Song (2014): Bansal-Shaliastovich meets GHPZ with regimes

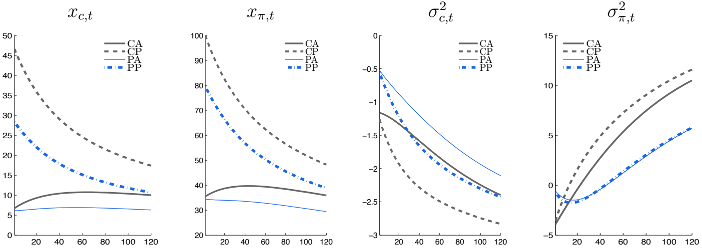
- Key elements:
  - S_t captures the state of the economy pertaining to the MP and correlation between variables
  - Equations:

$$
\begin{aligned}
r_t^{\$} &= \tau_0(S_t) + x_{\pi t} + \tau_g(S_t)(g_t - g) + \tau_\pi(S_t)(\pi_t - x_{\pi t}) + s_t \\
&\hspace{20em}\text{(the MP rule)} \\
x_{\pi t} &= \varphi_\pi(S_t) x_{\pi t-1} + v_{\pi t-1}^{1/2} e_{\pi t} \qquad \text{(inflation target)} \\
x_{gt} &= \varphi_g(S_t) x_{gt-1} + v_{gt-1}^{1/2} e_{gt} + \chi_{g\pi}(S_t) v_{\pi t-1}^{1/2} e_{\pi t} \\
&\hspace{14em}\text{(expected consumption growth)}
\end{aligned}
$$

- Also, see Campbell, Pflueger, and Viceira (2014)

Results

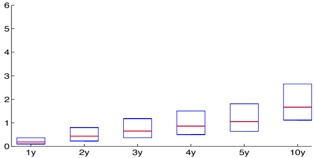
	Data	Model					
	$\mathrm{corr}(\Delta c_t, \pi_t)$	$\mathrm{corr}(\Delta c_t, \pi_t)$			$\mathrm{corr}(\mathbb{E}\Delta c_{t+1}, \mathbb{E}\pi_{t+1})$		
Regime	Estimate	Median	5%	95%	Median	5%	95%
CA	-0.24	-0.58	[-0.80,	-0.22]	-0.93	[-0.99,	-0.64]
CP	-0.09	-0.48	[-0.78,	0.02]	-0.74	[-0.95,	-0.15]
PA	0.01	0.17	[-0.13,	0.42]	0.59	[0.27,	0.80]
PP	0.03	0.19	[-0.14,	0.47]	0.27	[0.44,	0.84]



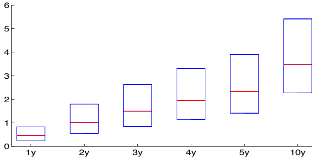
Source: Song (2014)

Slopes in the different (absorbing) regimes

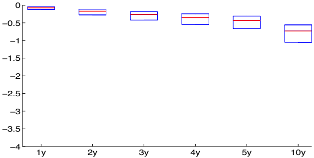
$corr(\pi, \Delta c) < 0$, Active MP



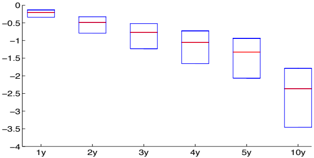
$corr(\pi, \Delta c) < 0$, Passive MP



$corr(\pi, \Delta c) > 0$, Active MP



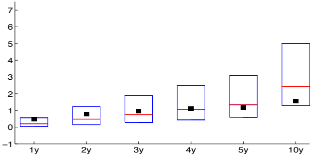
$corr(\pi, \Delta c) > 0$, Passive MP



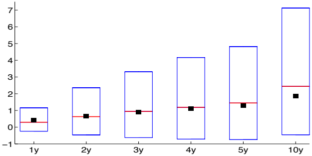
Source: Song (2014)

Slopes in the different regimes

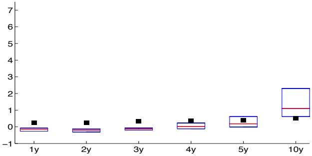
$corr(\pi, \Delta c) < 0$, Active MP



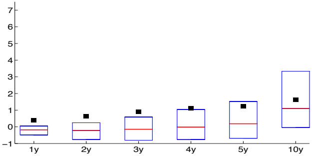
$corr(\pi, \Delta c) < 0$, Passive MP



$corr(\pi, \Delta c) > 0$, Active MP



$corr(\pi, \Delta c) > 0$, Passive MP



Source: Song (2014)

14. Affine Term Structure Models

Definition

- As we've seen already, the CGF should be linear in the factors. When it comes to term structure models, the requirement is for expectations under $\mathbb{Q}$
- This property implies that bond prices have the same functional structure and yields are linear in state variables
- We've seen lots of examples already.
- One needs the affine specification for pricing, thus, the model does not have to be affine under $\mathbb{P}$. Historically, researchers wanted to have affinity under $\mathbb{P}$ as well for estimation tractability. However, Le, Singleton, and Dai (2010) show that in discrete time the tractability of likelihood is not affected.

Rotations

- A typical affine term-structure model has 3 to 5 factors, so one worries about identification of parameters and states
- As a result, the literature has arrived at the canonical representation which is not very intuitive but easy to identify. Example:

$$\begin{aligned}
r_t^{\$} &= \delta_0 + \delta_1 X_{1t} + \delta_2 X_{2t} \\
X_{1,t+1} &= \mu_1 + \phi_1 X_{1,t} + \sqrt{X_{1,t}}\varepsilon_{1,t+1} \\
X_{2,t+1} &= \phi_2 X_{2,t} + \phi_{21} X_{1,t} + \sqrt{1 + \beta X_{1,t}}\varepsilon_{2,t+1}
\end{aligned}$$

- What does it mean? Rewrite $(v_t, r_t^{\$})' = LX_t + \Upsilon$

$$L = \begin{pmatrix} \beta & 0 \\ \delta_1 & \delta_2 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} 0 \\ \delta_0 \end{pmatrix}$$

then

$$\begin{aligned}
v_{t+1} &= \widetilde{\mu}_1 + \phi_1 v_t + \sigma_v \sqrt{v_t}\varepsilon_{1,t+1} \\
r_{t+1}^{\$} &= \widetilde{\mu}_2 + \phi_2 r_t^{\$} + \phi_{21} v_t + \sigma_r \sqrt{1 + v_t}\varepsilon_{2,t+1}
\end{aligned}$$

Do latent factors have a structural interpretation?

- Observed panel of bond yields can be summarized by a set of parameters for an affine model, implied state vectors, and yield errors
  - Can use these to reproduce original set of data
- Any affine model can be rotated into an observationally equivalent model
$$X_t^* = LX_t + \Upsilon$$
  State vector, parameters of rotated model are different from original model
- Therefore nothing "fundamental" about a particular representation
- We are free to choose the rotation that is easiest to interpret

Affine models and the Taylor rule

- Starting with Ang and Piazzesi (2003), the literature incorporates the Taylor rule into affine models:

  $$r_t^{\$} = \tau_0 + \tau_g g_t + \tau_\pi \pi_t + s_t$$

  where g_t usually measures economic growth, e.g., GDP growth; s_t could be a vector and is referred to as "finance" factors, g_t and π_t are either added on to the state or represented as linear functions of the state.
- Compute yields, estimate the model. We have estimates of monetary policy and how it influences yields (expectations and risk premiums)
- Problem: one assumes the Taylor rule and inflation dynamics, while, as we have seen, inflation must be determined by the Taylor rule

Example

- Let's continue with the example from Chernov and Mueller (2012)
- Start with the real curve:

$$r_t = \delta_{x,0} + \delta_x' x_t.$$

The log-real pricing kernel is:

$$m_{t+1} = -r_t - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1},$$

where Λ_t is the price of risk. If we assume joint VAR for $z_t = (x_t', g_t, \pi_t)'$, then:

$$\begin{aligned} \pi_{t+1} &= e_t + \sigma\varepsilon^{\pi}_{t+1}, \\ e_t &= e_\ell' (\mu + \Phi z_t), \\ \sigma\varepsilon^{\pi}_t &= e_\ell' \Sigma \varepsilon_t. \end{aligned}$$

where e_ℓ is a vector of zeros with a one in the last position.

Example, cont'd

- The log-nominal pricing kernel is related to the log-real pricing kernel via:

$$m^{\$}_{t+1} = m_{t+1} - \pi_{t+1} = -r_t - e_t - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t\varepsilon_{t+1} - e_\ell'\Sigma\varepsilon_{t+1}.$$

- The nominal spot rate is:

$$\begin{aligned} r^{\$}_t &= -\log E_t(e^{m^{\$}_{t+1}}) = r_t + e_t - \frac{1}{2}e_\ell'\Sigma\Sigma' e_\ell - \left(e_\ell'\Sigma\right)\Lambda_t \\ &= \tilde{\delta}_{z,0} + \tilde{\delta}_z' z_t, \end{aligned}$$

where

$$\begin{aligned} \tilde{\delta}_{z,0} &= \delta_{x,0} + e_\ell'\mu - \frac{1}{2}e_\ell'\Sigma\Sigma' e_\ell - \left(e_\ell'\Sigma\right)\Lambda_0 \\ \tilde{\delta}_z' &= (\delta_x', 0, 0) + e_\ell'\Phi - \left(e_\ell'\Sigma\right)\Lambda_1. \end{aligned}$$

- Thus, the "macro-finance" specification does not necessarily tell us anything about the Taylor rule.

Predictability of yields

- Recall that Cochrane and Piazzesi (2005) established strong predictability of yields, a.k.a. variation in risk premiums

- As we've seen this aspect can be easily captured by affine models with state-dependent variance of the pricing-kernel

- CP further argue that their factor $\gamma' f_t$ is is not spanned by the three usual factors *level, slope, and curve*

- These are correlated with the "return factor", but cannot account for the full time-variation C&P document in expected excess bond returns

- The principal components we usually toss out (number 4, 5 etc) are important for the dynamic behavior of bond risk premia

A fourth factor?

	x_t	$level$	$slope$	$curve$	$factor4$	$factor5$	R^2
% of var(f)		97.7	2.2	0.10	0.01	0.00	
	0.24						0.46
	(9.00)						
		0.19					0.03
		(0.80)					
		0.21	-1.72				0.13
		(0.95)	(-2.51)				
		0.22	-1.74	4.79			0.25
		(1.11)	(-2.72)	(2.82)			
		0.20	-1.70	4.73	-0.28	-15.49	0.34
		(1.13)	(-2.60)	(2.80)	(-0.09)	(-2.87)	

Table 2. Regression coefficients, t-statistics and R^2 for forecasting the average (across maturity) excess return $\overline{rx}_{t+1}$ in GSW data, based on the return-forecast factor x_t and eigenvalue-decomposition factors of forward rates. The top row gives the fraction of variance explained, $100 \times \Lambda_i / \sum_{j=1}^{15} \Lambda_j$. Monthly observations of annual returns 1971-2006. Standard errors include a Hansen-Hodrick correction for serial correlation due to overlap.

An affine model with the CP result?

- General asset pricing point: Asset prices today depend on all information investors have about future cash flows

- Because we can think of time $t + 1$ prices as future cash flows (if we sell the bonds), this means today's bond prices contain all info about tomorrow's bond prices

- Corollary: nothing affects the cross-section of bond yields that does not affect the time series of bond yields, and vice versa.

- Markov latent variable structure of affine models incorporates this intuition
  - Given parameters, can invert time t term structure to get state vector
  - Parameters and state vector are complete description of dynamics

Hidden factors

- Duffee (2011) shows that this is not true

$$
\begin{aligned}
\begin{pmatrix} r_t \\ f_t \end{pmatrix} &= \mu + \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} r_{t-1} \\ f_{t-1} \end{pmatrix} + \Sigma \varepsilon_t \\
\begin{pmatrix} r_t \\ f_t \end{pmatrix} &= \mu^{\mathbb{Q}} + \begin{pmatrix} \phi_{11}^{\mathbb{Q}} & 0 \\ \phi_{21}^{\mathbb{Q}} & \phi_{22}^{\mathbb{Q}} \end{pmatrix} \begin{pmatrix} r_{t-1} \\ f_{t-1} \end{pmatrix} + \Sigma \varepsilon_t^{\mathbb{Q}}
\end{aligned}
$$

- So, under $\mathbb{Q}$ r_t is determined only by itself. It means that bond prices depend on r only.
- However, expected excess returns depend on f:

$$
\begin{aligned}
E_t\left(rx_{t+1}^{(n)}\right) &= ny_t^n - (n-1)E_t\left(y_{t+1}^{n-1}\right) - r_t \\
& \quad nA(n) + nB(n)r_t - (n-1)A(n-1) \\
&- \quad (n-1)B(n-1)E_t\left(r_{t+1}\right) - r_t \\
E_t\left(r_{t+1}\right) &= \mu_r + \phi_{11} r_t + \phi_{12} f_t
\end{aligned}
$$

Consistent with the CP evidence?

- Duffee (2011) estimates a 5-factor model with nominal yields, and finds that the 5th factor is hidden, but it does not replace the CP factor in predictive regressions

- Joslin, Priebsch, and Singleton (2014) consider nominal yields and 2 macro variables in 5-factor model and assume that inflation and GDP growth are hidden, but these macro factors do not drive out CP

- Chernov and Mueller (2012) use nominal and real yields, macro variables, and survey forecasts of inflation, and find that a factor driving inflation expectations is hidden. Yet, this factor co-exists with the CP one in predictive regressions.

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