Empirical Asset Pricing

Part 3: Option Pricing

UCLA | Fall 2016

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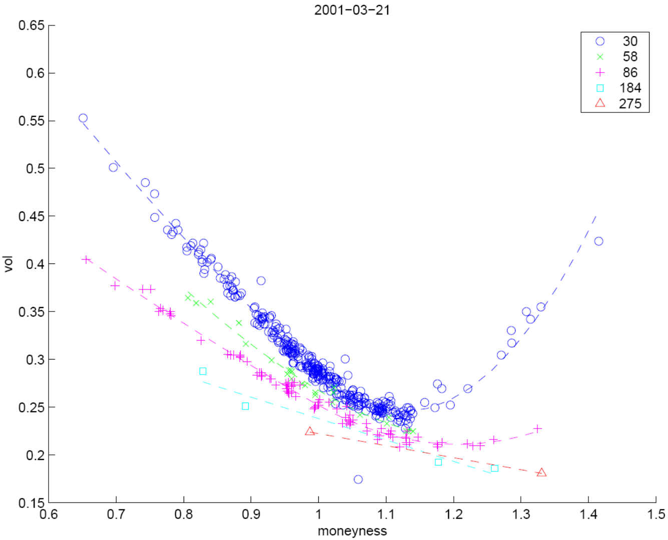
This version: October 29, 2016

15. Option Puzzles

Option Puzzle I: Implied volatility smile



Source: Broadie, Chernov, and Johannes (2007)

Option Puzzle II: The gap between at-the-money implied and realized volatility

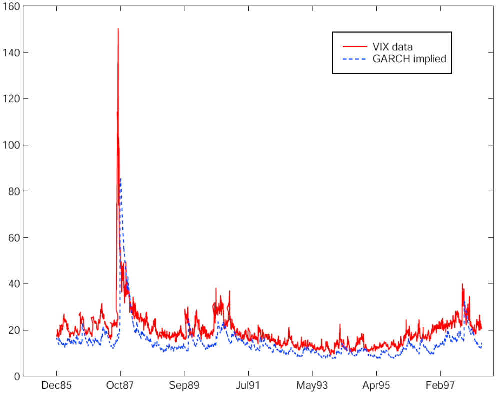


Figure 7.2: Time Series of *VIX* and *GARCH*-implied volatilities for the *S&P*500 index.

Source: Singleton (2006)

Option Puzzle III: High returns to writing out-of-the-money puts

Moneyness	0.94	0.96	0.98	1.00	1.02
08/1987 to 06/2005	−56.8	−52.3	−44.7	−29.9	−19.0
Standard error	14.2	12.3	10.6	8.8	7.1
t-stat	−3.9	−4.2	−4.2	−3.3	−2.6
p-value, %	0.0	0.0	0.0	0.0	0.4
Skew	5.5	4.5	3.6	2.5	1.8
Kurt	34.2	25.1	16.7	10.5	7.1
Subsamples					
01/1988 to 06/2005	−65.2	−60.6	−51.5	−34.1	−21.6
01/1995 to 09/2000	−85.5	−71.6	−63.5	−50.5	−37.5
10/2000 to 02/2003	+67.2	+54.3	+44.5	+48.2	+40.4
08/1987 to 01/2000	−83.9	−63.2	−55.7	−39.5	−25.5

Table 2: Average put option returns. The first panel contains the full sample (215 monthly returns) results, with standard errors, t-statistics, p-values, and skewness and kurtosis statistics. The second panel analyzes subsamples. All relevant statistics are in percentages per month.

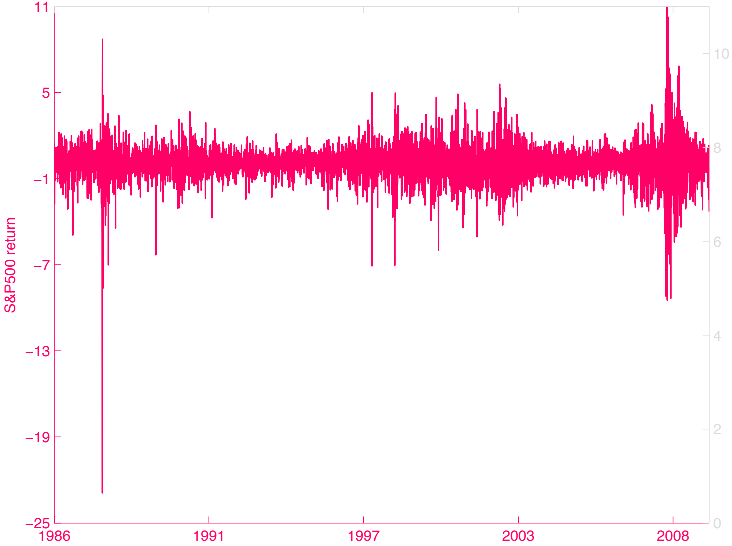
Moneyness	0.94	0.96	0.98	1.00	1.02
CAPM α, %	−48.3	−44.1	−36.8	−22.5	−12.5
Std.err., %	11.6	9.3	7.1	4.8	2.9
t-stat	−4.1	−4.7	−5.1	−4.6	−4.2
p-value, %	0.0	0.0	0.0	0.0	0.0
Sharpe ratio	−0.27	−0.29	−0.29	−0.23	−0.18

Table 3: Risk-corrected measures of average put option returns. The first panel provides CAPM α's with standard errors and the second panel provides put option Sharpe ratios. These quantities were computed based on the full sample of 215 monthly returns from August 1987 to June 2005. All relevant statistics except for the Sharpe ratios are in percentages per month. Sharpe ratios are monthly. The p-values are computed under the (incorrect) null hypothesis that average option returns are equal to zero.

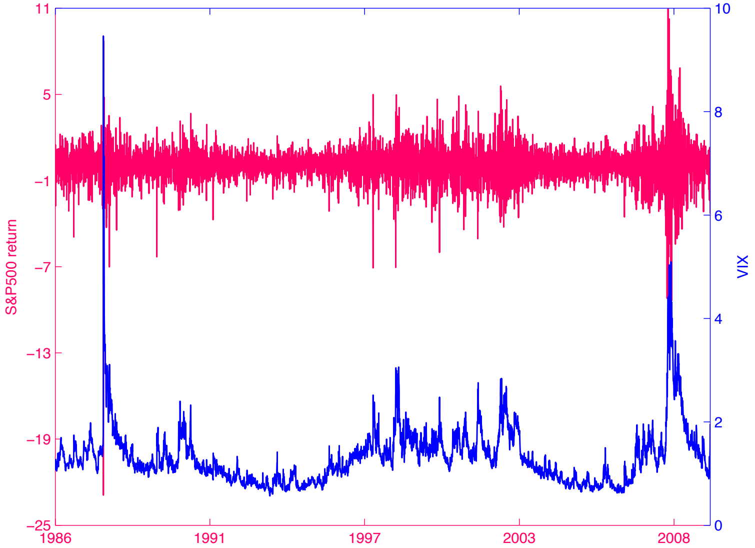
Source: Broadie, Chernov, and Johannes (2009)

16. Properties of index returns

The target



The target



11
5
−1
−7
−13
−19
−25
S&P500 return
10
8
6
4
2
0
VIX
1986
1991
1997
2003
2008

Conditional Moments

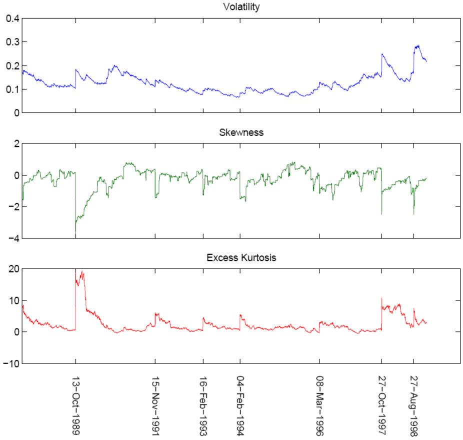


Figure 7.1: Rolling Sample Moments of $S\&P500$ returns.

Source: Singleton (2006)

Starting point

- See Das and Sundaram (1999) for an early analysis along these lines
- All the models that we consider are extensions of the Black-Scholes-Merton framework

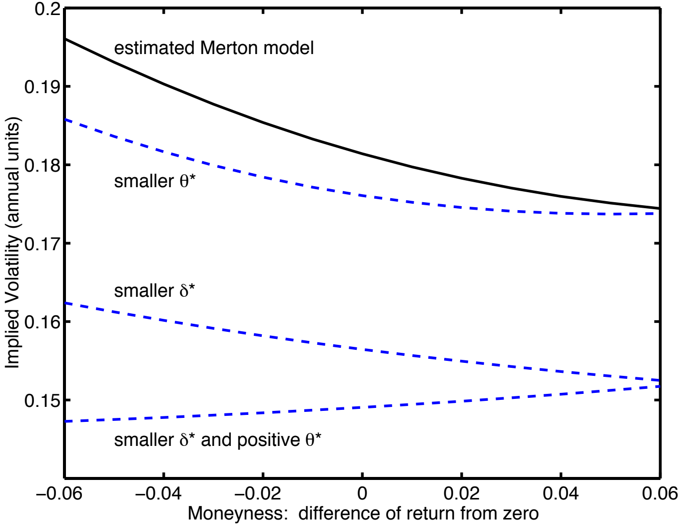
$$\begin{aligned} r_{t+1} &= \mu + v^{1/2}\varepsilon_{t+1} + z_{t+1}\xi_{t+1}, \\ \text{Prob}(z_t = j) &= e^{-\omega}\omega^j/j!, \\ \xi_t|j &\sim \mathcal{N}(j\theta, j\delta^2) \end{aligned}$$

- Cumulants:

$$\begin{aligned} \kappa_1 &= \mu + \omega\theta,\ \kappa_2 = v + \omega(\theta^2 + \delta^2), \\ \kappa_3 &= \omega\theta(\theta^2 + 3\delta^2),\ \kappa_4 = \omega(\theta^4 + 6\theta^2\delta^2 + 3\delta^4) \end{aligned}$$

- Note: mean=κ_1, vol=$\kappa_2^{1/2}$, skew=$\kappa_3/\kappa_2^{3/2}$, kurt=κ_4/κ_2^2

Options



Source: Backus, Chernov, and Martin (2011)

Stochastic volatility (SV)

- For a long time SV was viewed as more attractive than the Merton
- Can generate varying volatility as is evident in the data
- Can generate smiles via the leverage effect
- We have to digress on the SV models to understand some of the issues involved

SV models

- Consider an affine continuous-time model:

  $$v_{t+1} = \kappa\theta + (1-\kappa)v_t + \sigma_v v_t^{1/2}\varepsilon_{t+1}^v$$

  This is the familiar Feller process (known is CIR for interest rates, Heston for vol)

- Can discretize it directly or use the Poisson mixture of Gammas

- Some alternatives (see Chernov, Gallant, Ghysels, and Tauchen, 2003 for more):

  $$\begin{aligned} \log v_{t+1}^{1/2} &= \kappa\theta + (1-\kappa)\log v_t^{1/2} + \sigma_v \varepsilon_{t+1}^v \text{ (SVOL)} \\ v_{t+1} &= \kappa\theta + (1-\kappa)v_t + \sigma_v v_t \varepsilon_{t+1}^v \text{ (GARCH diffusion)} \end{aligned}$$

- In all cases the shock to vol is correlated with the shock to returns – the leverage effect

Discrete-time vs GARCH

- The literature is peppered with confusion regarding the differences between SV and GARCH models.
- GARCH is
  1. often advertised as a discrete-time counterpart to SV models
  2. shown to converge to SV models as time interval converges to zero
- We know that one can easily specify an SV model in discrete time
- The difference is in the shock structure

$$\begin{aligned} \sigma_t^2 &= \lambda + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2\varepsilon_t^2 \text{ (GARCH)}, \\ \sigma_t^2 &= \lambda + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2\varepsilon_t^v \text{ (GARCH diffusion)} \end{aligned}$$

- Observe:
  - The same shocks affect returns and vol in GARCH
  - More rigid (quadratic) shocks structure in GARCH
- Corradi (2000) shows that convergence results are highly sensitive w.r.t. assumptions

Feller SV

- Consider an SV model with leverage:

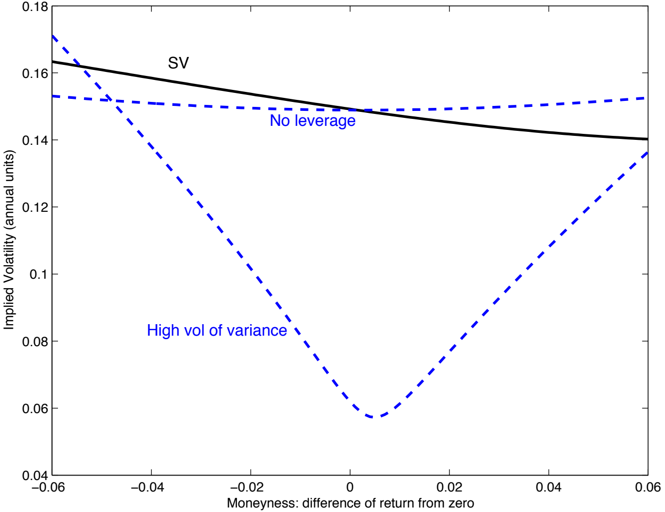
$$
\begin{aligned}
v_{t+1} &= \kappa\theta + (1-\kappa)v_t + \sigma(\kappa\theta/2 + (1-\kappa)v_t)^{1/2}\eta_{t+1} \\
r_{t+1} &= \mu + (1-\rho^2)^{1/2}(\kappa\theta/2 + (1-\kappa)v_t)^{1/2}\varepsilon_{t+1} \\
&+ \rho(\kappa\theta/2 + (1-\kappa)v_t)^{1/2}\eta_{t+1} \\
&= \mu - \rho\kappa\theta/\sigma - \rho(1-\kappa)v_t/\sigma + \rho v_{t+1}/\sigma \\
&+ (1-\rho^2)^{1/2}(\kappa\theta/2 + (1-\kappa)v_t)^{1/2}\varepsilon_{t+1}
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
& k_t((s_1, s_2); (r_{t+1}, v_{t+1}) = \mu s_1 + (1-\rho^2)(\kappa\theta/2 + (1-\kappa)v_t)s_1/2 \\
- \; & \rho\kappa\theta s_1/\sigma - \rho(1-\kappa)s_1 v_t/\sigma \\
+ \; & \frac{(s_2 + \rho s_1/\sigma)(1-\kappa)}{1 - (s_2 + \rho s_1/\sigma)\sigma^2/2} v_t - 2\kappa\theta \log(1 - (s_2 + \rho s_1/\sigma))/\sigma^2
\end{aligned}
$$

- The cumulants of r_{t+1} will be determined by the derivatives of $k_t((s_1, 0); (r_{t+1}, v_{t+1})$

Options

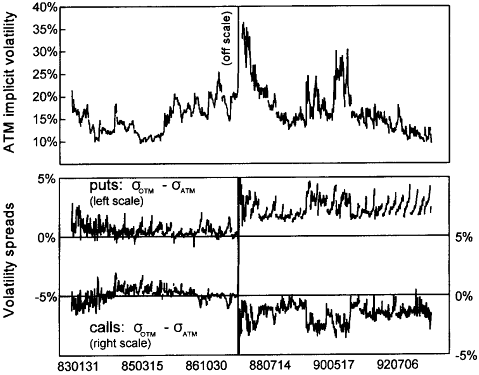


0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
Implied Volatility (annual units)
–0.06
–0.04
–0.02
0
0.02
0.04
0.06
Moneyness: difference of return from zero
SV
No leverage
High vol of variance

17. Studying the S&P 500 dynamics

Early evidence

- Bates (2000) asks: What explains the skewness observed post '87 crash: volatility or jumps?



Source: Bates (2000)

The model

- The model:

$$
\begin{aligned}
\log S_{t+1}/S_t &= \text{drift} + v_{1t}^{1/2} w_{1t+1} + v_{2t}^{1/2} w_{2t+1} + j_{t+1}\xi_j \\
v_{it+1} &= \alpha_i^* + (1-\beta_i^*) v_{it} + \sigma_{vi} v_{it}^{1/2} u_{it+1} \\
& \quad \text{cov}_t(w_{it+1}, u_{it+1}) = \rho_i \\
j_{t+1} &\sim \mathcal{P}(\lambda_t^*),\ \lambda_t = \lambda_0^* + \lambda_1^* v_{1t} + \lambda_2^* v_{2t} \\
o_t &= O_t/S_t = O^M(1, v_t, T_{i,t}, K_{i,t}/S_t, \Theta) + e_{i,t} \\
e_{i,t} &= \varepsilon_{l,t} + \sigma_l \eta_{i,t},\ \varepsilon_{l,t} = \rho_l \varepsilon_{l,t-1} + \nu_{l,t}
\end{aligned}
$$

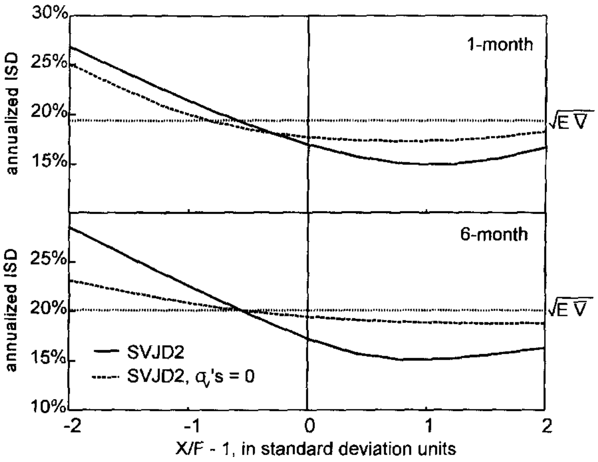
- Objective function: $\max_{\{v_t\},\Theta}$

$$
\log L_{\text{options}} = -0.5 \sum_t \log|\Omega_{t|t-1}| + (e_t - E_{t-1}e_t)'\Omega_{t|t-1}^{-1}(e_t - E_{t-1}e_t)
$$

- Incorporate the dynamics of state: $\max_{\{v_t\},\Theta,\beta_1,\beta_2}$

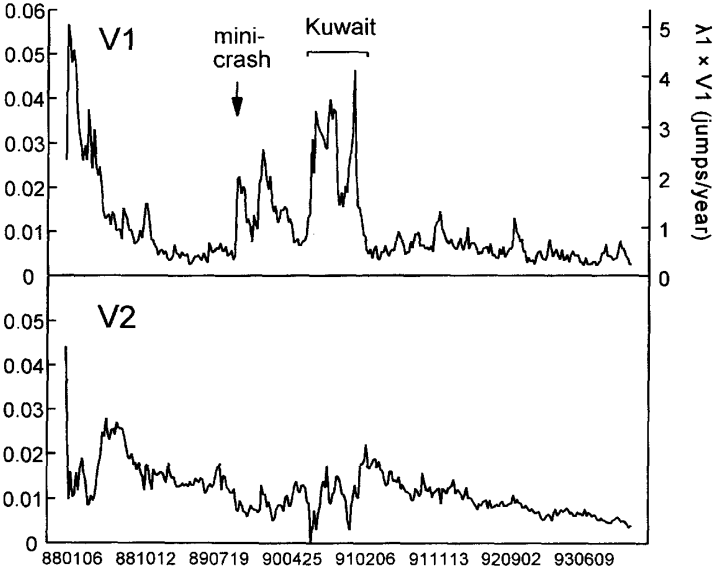
$$
\log L(\{v_t\}, \Theta, \beta_1, \beta_2) = \log L_{\text{options}} + \log L_{v_1} + \log L_{v_2}
$$

Sources of skewness



Source: Bates (2000)

Evidence of misspecification



Next stage

- Development of estimation methods lead to a flurry of papers focusing on:
  - Modelling of equity using long data spans (Andersen, Benzoni, and Lund, 2002; Chernov, Gallant, Ghysels, and Tauchen, 2003; Eraker, Johannes, and Polson, 2003)
  - Joint modelling of equity and options (Chernov and Ghysels, 2000; Eraker, 2004; Jones, 2003; Pan, 2002)

- Key questions that are addressed:
  - Functional form? Affine vs other specifications
  - How many vol factors and where?
  - Do we need jumps? Where?
  - Are these risk factors priced by the option market?

Preferred models of index only

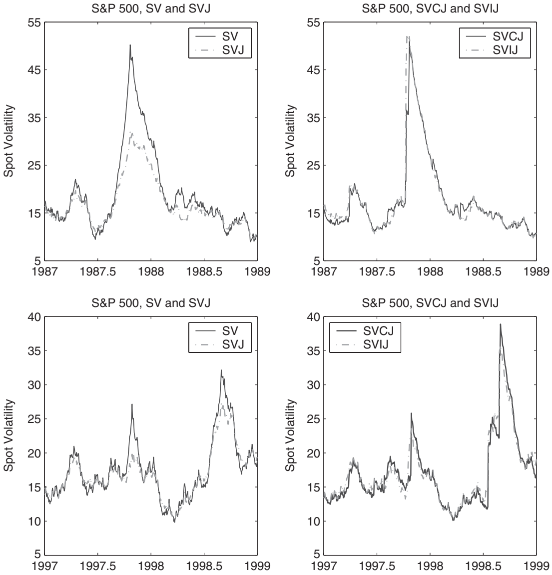
- ABL prefer SVOL with jumps in prices (constant intensity, zero mean)
- EJP prefer a model with jumps in prices and in variance (SVCJ):

$$\begin{aligned} v_{t+1} &= \kappa\theta + (1-\kappa)v_t + \sigma_v v_t^{1/2} w_{t+1}^v + j_{t+1}\xi_j^v \\ j_{t+1} &\sim \mathcal{P}(\lambda),\ \xi^s \sim \mathcal{N}(\mu_s, \sigma_s^2),\ \xi^v \sim \mathcal{E}(\mu_v) \end{aligned}$$

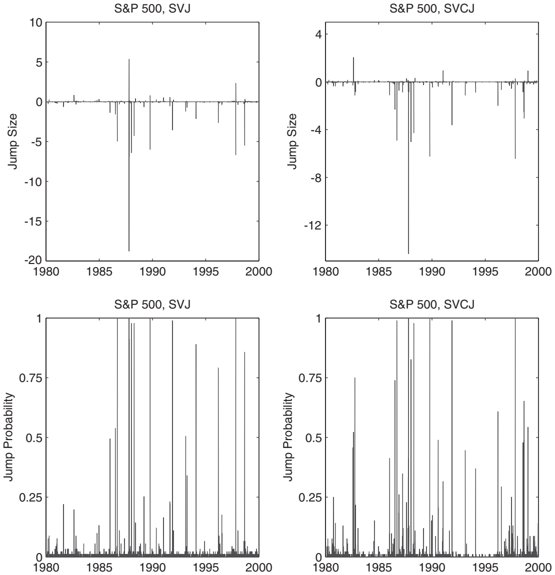
- CGGT prefer SVJ to SVCJ in the affine world, but the best is:

$$\begin{aligned} \log S_{t+1}/S_t &= \text{drift} + e^{\beta_0 + \beta_1 v_{1t} + \beta_2 v_{2t}} \\ &\times [\sqrt{1-\rho_1^2-\rho_2^2} w_{t+1}^s + \rho_1 w_{1t+1} + \rho_2 w_{2t+1}] \\ v_{1t+1} &= \kappa_1\theta_1 + (1-\kappa_1)v_{1t} + w_{1t+1} \\ v_{2t+1} &= \kappa_2\theta_2 + (1-\kappa_2)v_{2t} + (\alpha_0 + \alpha_1 v_{2t}) w_{2t+1} \end{aligned}$$

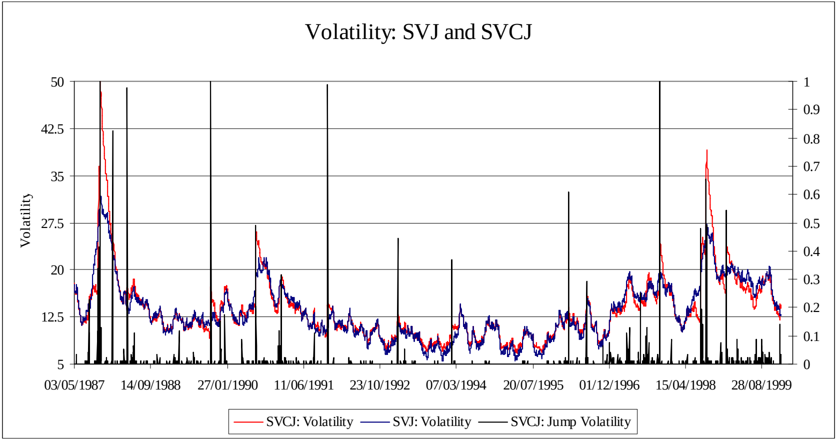
EJP states: two years



EJP jumps: full sample



EJP jumps in vol



Source: Eraker, Johannes and Polson (2003)

Affine Parameter Estimates

Objective Measure Parameter Estimates

Objective measure parameters estimated by Eraker, Johannes, and Polson (2003), Andersen, Benzoni, and Lund (2002), Chernov et al. (2003), and Eraker (2004). The parameter values correspond to daily percentage returns. These values could be easily converted to annual decimals—another common measure—by scaling some of the parameters: for example, multiplying κ_v, and λ 252, $\sqrt{252\theta_v}/100$ gives the mean volatility, and $\sqrt{252\mu_v}/100$ gives the mean jump in volatility. In the SVCJ model, in the column labeled μ_s we report $\hat{\mu}_s = \mu_s + \rho_s\mu_v$, which is the expected jump size.

		κ_v	θ_v	σ_v	ρ	λ	μ_s (%)	σ_s (%)	μ_v
SV	EJP	0.023	0.90	0.14	−0.40	·	·	·	·
	ABL	0.016	0.66	0.08	−0.38	·	·	·	·
	CGGT	0.013	0.59	0.06	−0.27	·	·	·	·
	Eraker	0.017	0.88	0.11	−0.37	·	·	·	·
SVJ	EJP	0.013	0.81	0.10	−0.47	0.006	−2.59	4.07	·
	ABL	0.013	0.66	0.07	−0.32	0.020	0 (fixed)	1.95	·
	CGGT	0.011	0.62	0.04	−0.43	0.007	−3.01	0.70	·
	Eraker	0.012	0.83	0.08	−0.47	0.003	−3.66	6.63	·
SVCJ	EJP	0.026	0.54	0.08	−0.48	0.006	−2.63	2.89	1.48
	CGGT	0.014	0.61	0.07	−0.46	0.007	−1.52	1.73	0.72
	Eraker	0.016	0.57	0.06	−0.46	0.004	−2.84	4.91	1.25

18. Solving the S&P 500 option puzzles

Methodology

- The discussion is based, primarily, on Broadie, Chernov, and Johannes (2007), (2009)

- The methodology is very simple
  - Treat the $\mathbb{P}$ parameters as known (estimated by MCMC)
  - Estimate the $\mathbb{Q}$ parameters via

$$\widehat{\Theta}^{\mathbb{Q}} = \arg\min \sum_{t=1}^{T} \sum_{n=1}^{N_t} \left[IV_t\left(K_n, \tau_n, S_t\right) - IV\left(\Theta^{\mathbb{Q}} | \Theta^{\mathbb{P}}, K_n, \tau_n, S_t, v_t\right)\right]^2,$$

where T is the number of days in our sample, N_t is the number of cross sectional option prices observed on date t

Notes

- The objective of minimizing squared deviations between model and market option prices places heavy weight on expensive in-the-money and long maturity options.

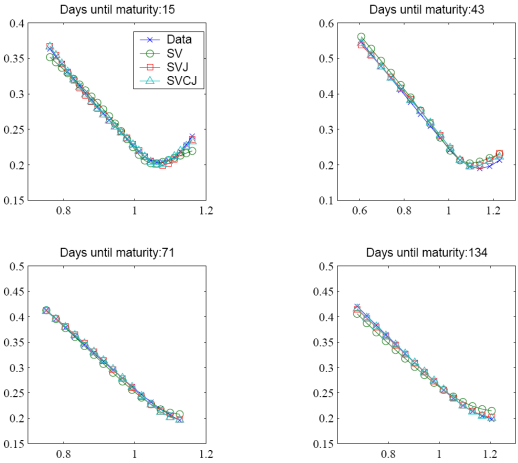
- The BCJ objective function has a computational disadvantage because one has to compute model-based implied volatilities – a numerical procedure – at each trial value of parameters $\Theta^{\mathbb{Q}}$.

$$\widehat{\Theta}^{\mathbb{Q}} = \arg\min \sum_{t=1}^{T} \sum_{n=1}^{N_t} \left[\frac{O_t\left(K_n, \tau_n, S_t\right) - O\left(\Theta^{\mathbb{Q}} | \Theta^{\mathbb{P}}, K_n, \tau_n, S_t, v_t\right)}{\mathcal{V}\left(K_n, \tau_n, S_t\right)} \right]^2,$$

where and $\mathcal{V}\left(K_n, \tau_n, S_t\right)$ is the Black-Scholes vega computed on the basis of the observed option price.

Attacking Puzzle I with SV

- Leverage effect should generate the smile
- SV model calibrated to the cross-section of options (+SVJ, SVCJ)



Source: Broadie, Chernov, and Johannes (2007)

Attacking Puzzle II with SV

- Volatility risk premium explains the gap between implied and realized variance
- $\kappa^{\mathbb{Q}} < \kappa^{\mathbb{P}}$, so vol is more persistent under $\mathbb{Q}$
- Consider a "theoretical" variance swap: you receive future variance $\tau^{-1}\sum_{i=1}^{\tau} V_{t+i}$ in exchange for the payment of $\tau^{-1}E_t^{\mathbb{Q}}\sum_{i=1}^{\tau} V_{t+i}$
- The risk premium is (e.g., Chernov, 2007)

$$VRP_t(\tau) = \frac{1}{\tau}E_t^{\mathbb{P}}\sum_{i=1}^{\tau} V_{t+i} - \frac{1}{\tau}E_t^{\mathbb{Q}}\sum_{i=1}^{\tau} V_{t+i}$$

$$\frac{1}{\tau}E_t^{\mathbb{M}}\sum_{i=1}^{\tau} V_{t+i} = A^{\mathbb{M}}(\tau)V_t + B^{\mathbb{M}}(\tau)$$

$$A^{\mathbb{M}}(\tau) = \frac{1-\kappa^{\mathbb{M}}}{\kappa^{\mathbb{M}}\tau}\left(1-(1-\kappa^{\mathbb{M}})^{\tau}\right)$$

$$B^{\mathbb{M}}(\tau) = \theta^{\mathbb{M}}(1-A^{\mathbb{M}}(\tau))$$

- $VRP_t(\tau) < 0 \rightarrow$ volatility is a hedge to equity

Attacking Puzzle II with SV: A Bonus

- Does implied variance forecast future realized variance?
- Run a regression

$$RV_{t+\tau,\tau} = a + b \cdot IV^2_{t,\tau} + c \cdot RV_{t,\tau} + \varepsilon_{t+\tau}$$

- Null hypotheses:
  - IV is unbiased: $a = 0, b = 1.$ In practice, $a \neq 0, b < 1$
  - IV is informationally efficient: $c = 0.$ In practice, $c \neq 0.$
- Explanation:

$$\begin{aligned} IV^2_{t,\tau} &\approx E^{\mathbb{Q}}_t\left(RV_{t+\tau,\tau}\right) \\ &= E^{\mathbb{P}}_t\left(RV_{t+\tau,\tau}\right) + \left[E^{\mathbb{Q}}_t\left(RV_{t+\tau,\tau}\right) - E^{\mathbb{P}}_t\left(RV_{t+\tau,\tau}\right)\right] \\ &= \text{unbiased expectation - vol risk premium} \end{aligned}$$

- Vol risk premium is negative, hence the bias (Chernov, 2007)

Attacking Puzzle III with SV

- SV seems to be able to explain option returns

	Moneyness		0.94	0.96	0.98	1.00
Average returns	Data, %		−56.8	−52.3	−44.7	−29.9
	BS	$E^{\mathbb{P}}$,%	−20.6	−17.6	−14.6	−12.0
		p-value,%	8.1	1.7	0.4	2.2
	SV	$E^{\mathbb{P}}$,%	−25.8	−21.5	−17.5	−13.7
		p-value,%	24.1	9.3	3.0	7.3
CAPM αs	Data, %		−48.3	−44.1	−36.8	−22.5
	BS	$E^{\mathbb{P}}$,%	−17.9	−15.3	−12.7	−10.4
		p-value,%	12.6	2.7	0.3	1.2
	SV	$E^{\mathbb{P}}$,%	−23.6	−19.5	−15.8	−12.4
		p-value,%	39.1	14.1	3.4	8.7
Sharpe ratios	Data		−0.27	−0.29	−0.29	−0.23
	BS	$E^{\mathbb{P}}$	−0.05	−0.07	−0.08	−0.09
		p-value,%	4.9	1.9	1.2	4.0
	SV	$E^{\mathbb{P}}$	−0.04	−0.07	−0.09	−0.10
		p-value,%	21.5	12.0	7.7	14.3

Table 5: This table reports population expected option returns, CAPM α's, and Sharpe ratios and finite sample distribution p-values for the Black-Scholes (BS) and stochastic volatility (SV) models. We assume that all risk premia (except for the equity premium) are equal to zero. The p-values are computed using the finite sample distributions of the respective statistics. The distributions were constructed from 25,000 simulated paths, each with 215 monthly observations.

Source: Broadie, Chernov, and Johannes (2009)

How did we get this?

- Buy-and-hold put returns are defined as

$$r^p_{t,T} = \frac{(K - S_{t+T})^+}{P_{t,T}(K, S_t)} - 1$$

- Expected put option returns are given by

$$E^{\mathbb{P}}_t\left(r^p_{t,T}\right) = \frac{E^{\mathbb{P}}_t\left[(K - S_{t+T})^+\right]}{P_{t,T}(S_t, K)} - 1 = \frac{E^{\mathbb{P}}_t\left[(K - S_{t+T})^+\right]}{E^{\mathbb{Q}}_t\left[e^{-rT}(K - S_{t+T})^+\right]} - 1$$

- Simulated put returns are for a given moneyness $\kappa = K/S$

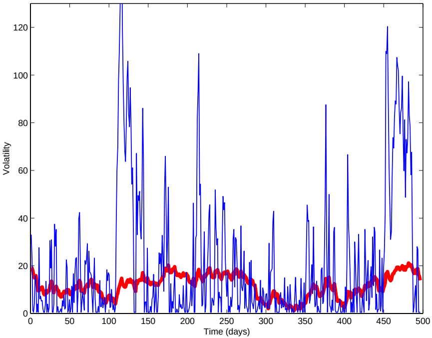
$$r^{p,(g)}_{t,T} = \frac{\left(\kappa - R^{(g)}_{t,T}\right)^+}{e^{-rT} E^{\mathbb{Q}}_t\left[(\kappa - R_{t,T})^+\right]} - 1,$$

  $t = 1, \ldots, N = 215$ and $g = 1, \ldots, G = 25000$.
- Average option returns for simulation g using N months of data are

$$\bar{r}^{p,(g)}_T = \frac{1}{N} \sum_{t=1}^{N} r^{p,(g)}_{t,T}.$$

So Why Worry About Jumps?

- An SV model calibrated to match the IV smile implies crazy time-series behaviour of volatility

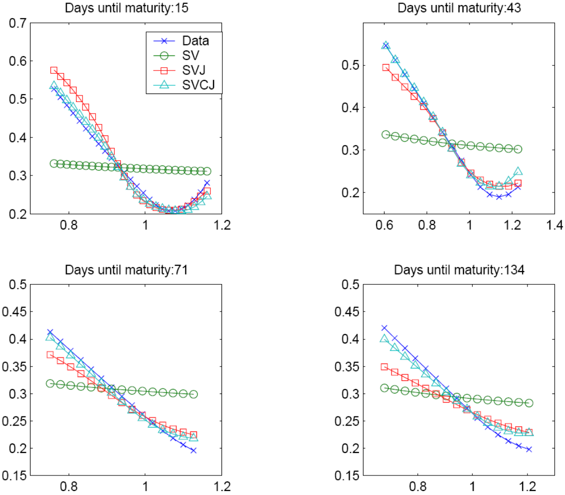


Source: Broadie, Chernov, and Johannes (2007)

- Smile-calibrated: $\sigma_v = 2.82$, $\rho = -0.70$
- Time-series-estimated: $\sigma_v = 0.14$, $\rho = -0.40$

So Why Worry About Jumps?

- Time-series consistent calibration:



Source: Broadie, Chernov, and Johannes (2007)

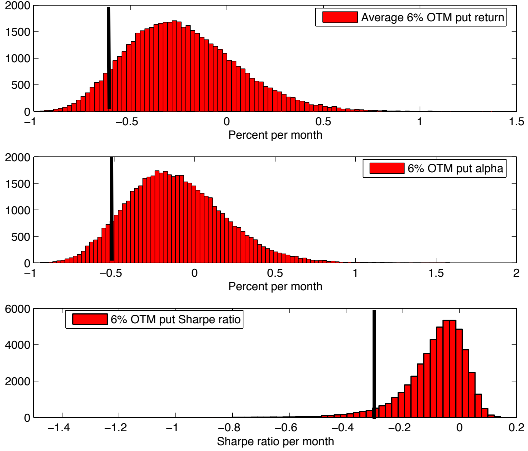
A clarification

- Time-series consistency and no-arbitrage:
  - Equivalent $\mathbb{P}$ and $\mathbb{Q}$ is the same as no-arb
  - In continuous time this implies, in particular, that $\sigma_v^{\mathbb{P}} = \sigma_v^{\mathbb{Q}}$, $\rho^{\mathbb{P}} = \rho^{\mathbb{Q}}$

- What about discrete time?
  - Everything can change
  - But everything is a jump

So Why Worry About Jumps?

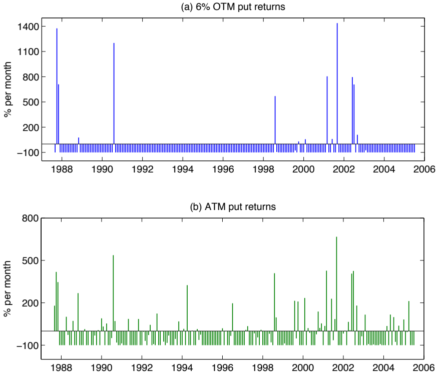
- Put returns are not informative. Black-Scholes world:



Source: Broadie, Chernov, and Johannes (2009)

So Why Worry About Jumps?

- Put returns are not informative. Real world:



Source: Broadie, Chernov, and Johannes (2009)

- Straddle returns are informative : -15.7% in the data; 1.4% in the SV model

Plan B: SV model with jumps in returns

- The Bates (1996)-Scott (1997) SVJ model under a measure $\mathbb{M}$ ($\mathbb{P}$ or $\mathbb{Q}$)

$$\log(S_{t+1}/S_t) = \left(\mu^{\mathbb{M}} - \delta\right) + \sqrt{V_t}\varepsilon^s_{t+1}(\mathbb{M}) + Z^s_{t+1}(\mathbb{M}) - \lambda^{\mathbb{M}}\overline{\mu}^{\mathbb{M}}$$
$$V_{t+1} = \kappa_v^{\mathbb{M}}\theta_v^{\mathbb{M}} + (1-\kappa_v^{\mathbb{M}})V_t + \sigma_v\sqrt{V_t}\varepsilon^v_{t+1}(\mathbb{M})$$

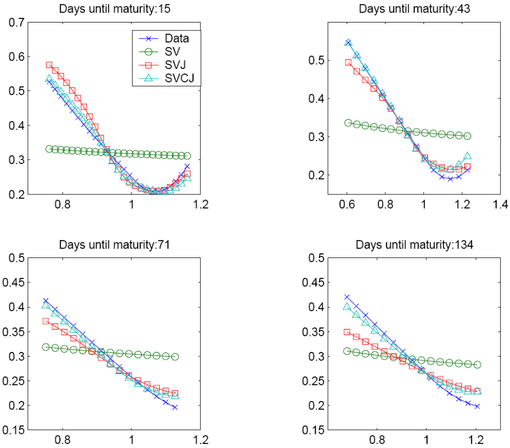
  where the new components are jumps arriving with a probability $e^{-\lambda^{\mathbb{M}}}\lambda^{\mathbb{M}j}/j!$ with a jump size $Z^s(\mathbb{M})\,|j \sim \mathcal{N}\left(\mu_z^{\mathbb{M}}j, (\sigma_z^{\mathbb{M}})^2 j\right)$.

- Empirical evidence is that $\kappa_v^{\mathbb{Q}} \approx \kappa_v^{\mathbb{P}}$

- Note the versatility of the jump risk premia

Attacking Puzzle I with SVJ

- Jump assymetry should generate the smile, 50% fit improvement
- SVJ model calibrated to the cross-section of options (+SV, SVCJ)



Source: Broadie, Chernov, and Johannes (2007)

Attacking Puzzles II & III with SVJ

- Consider total variance (*quadratic variation*) now:

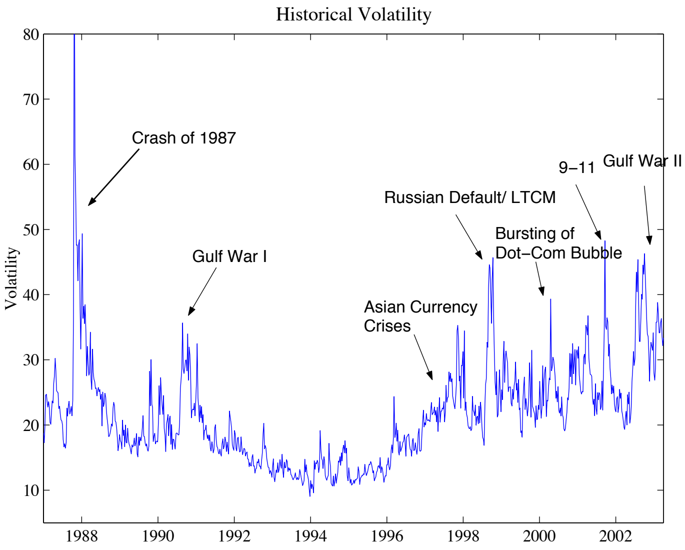
$$\overline{QV}_t(\tau) = 1/\tau \sum_{i=1}^{\tau} \left[V_{t+i} + Z_{t+i}^{s2} \Delta N_{t+i} \right]$$

- Total volatility risk premium (assuming $\kappa_v^{\mathbb{Q}} = \kappa_v^{\mathbb{P}}$):

$$\begin{aligned} -VRP_t(\tau) &= E_t^{\mathbb{Q}} \overline{QV}_t(\tau) - E_t^{\mathbb{P}} \overline{QV}_t(\tau) \\ &= \left(\theta_v^{\mathbb{Q}} - \theta_v^{\mathbb{P}}\right) \left(1 + \frac{1-\kappa_v}{\kappa_v \tau}((1-\kappa_v)^{\tau} - 1)\right) \\ &+ \lambda^{\mathbb{Q}} \left(\left(\mu_s^{\mathbb{Q}}\right)^2 + \left(\sigma_s^{\mathbb{Q}}\right)^2 \right) - \lambda \left(\left(\mu_s^{\mathbb{P}}\right)^2 + \left(\sigma_s^{\mathbb{P}}\right)^2 \right) \end{aligned}$$

- Total volatility risk premium explains the gap between implied and realized variance via the jump risk premium and straddle returns (-11%, $p-$value=12%)

Are We Done?



Source: Broadie, Chernov, and Johannes (2007)

SVJ Cannot Match Volatility

- Define

$$\begin{aligned} V_{skew} &= skew\left(\frac{V_{t+1}-V_t}{\sqrt{V_t}}\right) \text{ and } V_{kurt} = kurt\left(\frac{V_{t+1}-V_t}{\sqrt{V_t}}\right) \\ R_{skew} &= skew\left(R_{t+1}/\sqrt{V_t}\right) \text{ and } R_{kurt} = kurt\left(R_{t+1}/\sqrt{V_t}\right) \end{aligned}$$

Model/Data	Period	V_{kurt}	V_{skew}	R_{kurt}	R_{skew}
SVJ Model	1987 to 2003	850.41	21.16	17.75	-1.20
	1988 to 2003	15.66	1.37	6.10	-0.42
SVJ Model (RP)	1987 to 2003	1048.51	24.21	15.91	-1.06
	1988 to 2003	16.01	1.40	7.09	-0.42
	Quantile	V_{kurt}	V_{skew}	R_{kurt}	R_{skew}
	0.50	3.05	0.15	22.05	-1.48
SVJ simulations	0.95	3.23	0.22	106.05	-5.07
	0.99	3.34	0.26	226.77	-8.66

Source: Broadie, Chernov, and Johannes (2007)

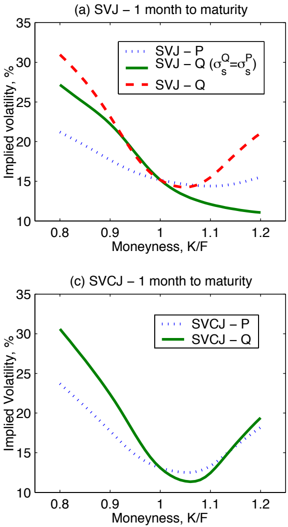
The Big Gun: SVJ + jumps in variance

- The Duffie-Pan-Singleton (2000)-Matytsin (2000) SVCJ model under a measure $\mathbb{M}$ ($\mathbb{P}$ or $\mathbb{Q}$)

$$\log(S_{t+1}/S_t) = \left(\mu^{\mathbb{M}} - \delta\right) + \sqrt{V_t}\varepsilon^s_{t+1}(\mathbb{M}) + Z^s_{t+1}(\mathbb{M}) - \lambda^{\mathbb{M}}\overline{\mu}^{\mathbb{M}}$$
$$V_{t+1} = \kappa_v^{\mathbb{M}}\theta_v^{\mathbb{M}} + (1-\kappa_v^{\mathbb{M}})V_t + \sigma_v\sqrt{V_t}\varepsilon^v_{t+1}(\mathbb{M}) + Z^v_{t+1}(\mathbb{M})$$

  where the new component is the volatility jump size $Z^v(\mathbb{M})|j \sim \mathcal{G}amma\left(j, \mu_v^{\mathbb{M}}\right)$.
- We have seen that the fit to the smile is not much different from SVJ
- However, ...

Jump Risk Premia



Source: Broadie, Chernov, and Johannes (2007)

Jump Risk Premia

Risk-Neutral Parameter Estimates

For each parameter and model, the table gives the point estimate, computed as the average parameter value across 50 bootstrapped samples, and the bootstrapped standard error. For the SVJ and SVCJ models, an entry of σ_s in the $\sigma_s^{\mathbb{Q}}$ column indicates that we impose the constraint $\sigma_s = \sigma_s^{\mathbb{Q}}$.

	η_v	$\mu_s^{\mathbb{Q}}$ (%)	$\sigma_s^{\mathbb{Q}}$ (%)	$\mu_v^{\mathbb{Q}}$	RMSE (%)
SV	0.005 (0.07)	—	—	—	7.18
SVJ	0.010 (0.03)	−9.97 (0.51)	σ_s	—	4.08
SVJ	0.006 (0.02)	−4.91 (0.36)	9.94 (0.41)	—	3.48
SVJ	0	−9.69 (0.58)	σ_s	—	4.09
SVJ	0	−4.82 (0.33)	9.81 (0.58)	—	3.50
SVCJ	0.030 (0.21)	−6.58 (0.53)	σ_s	10.81 (0.45)	3.36
SVCJ	0.031 (0.18)	−5.39 (0.40)	5.78 (0.70)	8.78 (0.42)	3.31
SVCJ	0	−7.25 (0.50)	σ_s	5.29 (0.18)	3.58
SVCJ	0	−5.01 (0.38)	7.51 (0.83)	3.71 (0.22)	3.39

Source: Broadie, Chernov, and Johannes (2007)

Volatility Dynamics

Model/Data	Period	V_{kurt}	V_{skew}	R_{kurt}	R_{skew}
SVCJ	1987 to 2003	1015.13	23.62	16.62	-1.12
	1988 to 2003	15.16	1.34	6.31	-0.40
SVCJ (RP)	1987 to 2003	546.52	16.08	15.96	-1.02
	1988 to 2003	13.44	1.38	6.77	-0.35
	Quantile	V_{kurt}	V_{skew}	R_{kurt}	R_{skew}
	0.01	3.28	0.20	3.02	0.06
SVCJ simul	0.05	13.21	1.01	3.21	-0.02
	0.50	217.70	8.98	7.13	-0.46
	0.95	1150.76	27.53	37.92	-2.20
	0.99	2012.62	39.34	94.16	-4.11

Source: Broadie, Chernov, and Johannes (2007)

19. “Model-free” implied variance of S&P 500

BSIV and MFIV

- Intuitively, Black-Scholes IV produces market (or $\mathbb{Q}$) expectations of the future realized variance
    - A very useful object as it gives the price of variance, forecast of variance, or variance risk premium depending on how you prefer to think about it (similar to forward and futures in other markets)
    - The theoretical link between BSIV and future variance is a bit messy (Chernov, 2007)
    - Moreover, many researchers believe (erroneously) that using BSIV is wrong if the true data-generating process has SV and/or jumps
- An alternative is Model-Free IV that intuitively serves the same purpose
    - It has emerged from the practitioner literature concerned with trading volatility (Carr and Madan, 1998)
    - MFIV is inferred from options as well, but using a portfolio of options as opposed to a misspecified model (Black - Scholes)
    - Many researchers believe (erroneously) that this measure is model-free, hence the name
    - Finally, there are some important implementation issues

Variance Swap

- The starting point for our discussion is variance swap
  - One leg of the swap receives realized variance

$$(\log S_\Delta/S_0)^2 + (\log S_{2\Delta}/S_\Delta)^2 + \ldots + (\log S_T/S_{T-\Delta})^2$$

  - The other leg pays $\tilde{V}$ that is selected in such a way that there is no exchange of payments at time 0:

$$\tilde{V} = E_0^{\mathbb{Q}}[(\log S_\Delta/S_0)^2 + (\log S_{2\Delta}/S_\Delta)^2 + \ldots + (\log S_T/S_{T-\Delta})^2]$$

- Questions:
  - What is $\tilde{V}$?
  - How does one hedge this contract, that is, can we replicate $\tilde{V}$ via prices of known assets?
- The main result is derived assuming $\Delta \to 0$ **and** one has access to options with a continuum of strikes between 0 and ∞
- Issues:
  - What is the assumption about the dynamics of S_t to get this result?
  - What happens when we only have limited number of options?
  - What happens when Δ is fixed?

Valuation of the variance swap

- Assume $dS_t = rS_t dt + \sigma_t S_t dW_t$ under $\mathbb{Q}$ (no jumps), dividend yield = 0
- Take the limit of the expression for $\tilde{V}$ as $\Delta \to 0$ to get:

$$\tilde{V} = E_0^{\mathbb{Q}}\left[\int_0^T (d\log S_t)^2\right] = E_0^{\mathbb{Q}}\left[\int_0^T \sigma_t^2 dt\right] = 2E_0^{\mathbb{Q}}\left[\int_0^T \frac{dS_t}{S_t} - \int_0^T d\log S_t\right] = 2rT - 2E_0^{\mathbb{Q}} \log \frac{S_T}{S_0}$$

- Apply the remarkable result:

$$\begin{aligned} f(S_T) &= f(k) + f'(k)[(S_T - k)^+ - (k - S_T)^+] \\ &+ \int_0^k f''(K)(K - S_T)^+ dK + \int_k^\infty f''(K)(S_T - K)^+ dK \end{aligned}$$

  to $f(S) = \log S$, $k =$ forward price of S_t to get

$$\tilde{V} = 2e^{rT}\left[\int_0^{F_T} P(T,K)/K^2 dK + \int_{F_T}^\infty C(T,K)/K^2 dK\right]$$

- CBOE defines, $VIX^2 = \tilde{V}/T$ and computes it as:

$$VIX^2 = \frac{2e^{rT}}{T}\sum_{i=1}^n \frac{Q(T,K_i)}{K_i^2}\Delta K_i - \frac{1}{T}\left(\frac{F_T}{K_f} - 1\right)^2,$$

  where K_i are the available strikes, $\Delta K_i = K_i - K_{i-1}$, K_f denotes the first strike available below F_T; the last term corrects for $F_T \neq K_f$.
- Martin (2011) challenges that theoretical VIX recovers $\mathrm{var}_0^{\mathbb{Q}}(S_T/S_0)$ in the presence of jumps

VIX and Entropy

- Use the fact that $E_0^{\mathbb{Q}} S_T/S_0 = \exp(rT)$ and

$$e^{-rT} E_0^{\mathbb{Q}} \log S_t/S_0 = rTe^{-rT} - Te^{-rT}/2 \cdot VIX^2$$

- We get:

$$VIX^2 = 2/T(\log E_0^{\mathbb{Q}} S_T/S_0 - E_0^{\mathbb{Q}} \log S_T/S_0) \equiv 2/T \cdot L_0^{\mathbb{Q}}(S_T/S_0)$$

- Here L is (risk-adjusted) entropy of returns

$$\begin{aligned} L_0^{\mathbb{Q}}(S_T/S_0) &= k_0^{\mathbb{Q}}(1; \log S_T/S_0) - E_0^{\mathbb{Q}} \log S_T/S_0 \\ &= \kappa_2^{\mathbb{Q}}/2! + \kappa_3^{\mathbb{Q}}/3! + \kappa_4^{\mathbb{Q}}/4! + \ldots \end{aligned}$$

- So,

$$T/2 \cdot VIX^2 = \mathrm{var}_0^{\mathbb{Q}} \log(S_T/S_0) = \kappa_2^{\mathbb{Q}}$$

when $\log S_T/S_0$ is normal

Simple Variance Swap

- Martin (2011) proposes an alternative swap that recovers $\mathrm{var}_0^{\mathbb{Q}}(S_T/S_0)$
- One leg of the swap receives realized variance

$$((S_\Delta - S_0)/S_0)^2 + ((S_{2\Delta} - S_\Delta)/F_\Delta)^2 + \ldots + ((S_T - S_{T-\Delta})/F_{T-\Delta})^2$$

- The other leg pays V that is selected in such a way that there is no exchange of payments at time 0:

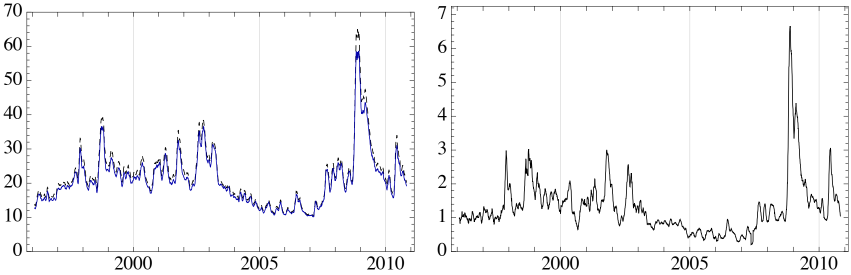
$$\begin{aligned} V &= E_0^{\mathbb{Q}}[((S_\Delta - S_0)/S_0)^2 + ((S_{2\Delta} - S_\Delta)/F_\Delta)^2 + \ldots \\ &+ ((S_T - S_{T-\Delta})/F_{T-\Delta})^2] \end{aligned}$$

- As $\Delta \to 0$

$$V = \frac{2e^{-rT}}{S_0^2}\left[\int_0^{F_T} P(T,K)dK + \int_{F_T}^{\infty} C(T,K)dK\right]$$

- Define, by analogy with VIX, $SVIX^2 = 2e^{rT}/T \cdot V$
- Martin (2011) shows that $T \cdot SVIX^2 = \mathrm{var}_0^{\mathbb{Q}}(S_T/S_0)$

VIX and SVIX



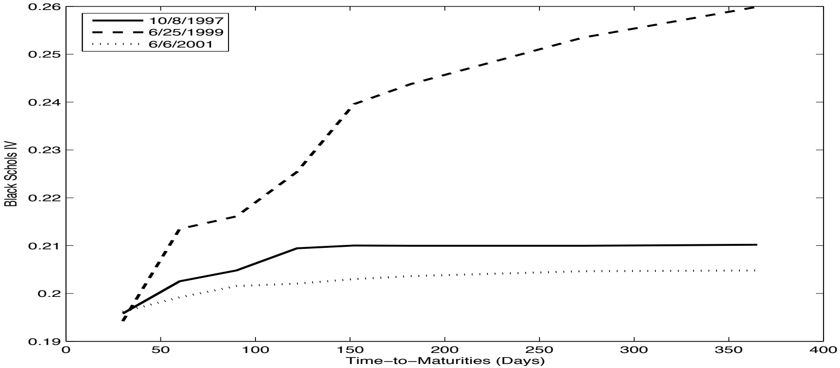
(a) VIX and SVIX (b) VIX minus SVIX

Source: Martin (2011)

- In the log-normal world, $VIX - SVIX < 0$

20. Time-varying central tendency of variance

The term structure of IV



Source: Cheung (2008)

Time-varying central tendency

- On the basis of related observations Amengual (2009); Cheung (2008) propose:

$$\log(S_{t+1}/S_t) = (\mu - \delta) + \sqrt{V_t}\varepsilon^s_{t+1} + Z^s_{t+1} - \lambda\bar{\mu}$$
$$V_{t+1} = \kappa_v\theta_t + (1-\kappa_v)V_t + \sigma_v\sqrt{V_t}\varepsilon^v_{t+1} + Z^v_{t+1}$$
$$\theta_{t+1} = \kappa_\theta\bar{\theta} + (1-\kappa_\theta)\theta_t + \sigma_\theta\sqrt{\theta_t}\varepsilon^\theta_{t+1}$$

- Jumps: both assume constant jump intensity λ
  - Cheung: $\xi^v_t \sim \mathcal{E}(\mu_v)$, $\xi^s_t \sim \mathcal{N}(\mu_s, \sigma^2_s)$
  - Amengual: $\xi^v_t \sim \mathcal{E}(\beta_1 V_t)$, $\xi^s_t | \xi^v_t \sim \mathcal{N}(\beta_2 + \beta_3\xi^v_t, \beta_4)$

- Risk premia:
  - Cheung: κ_v, κ_θ, λ, μ_v, μ_s, σ_s are different under $\mathbb{Q}$
  - Amengual: κ_v, κ_θ are different under $\mathbb{Q}$

Estimation approach

- Both use some form of variance-related derivative as additional signal about V_t and θ_t
  - Cheung uses data on 1-month and 1-year ATM IV's

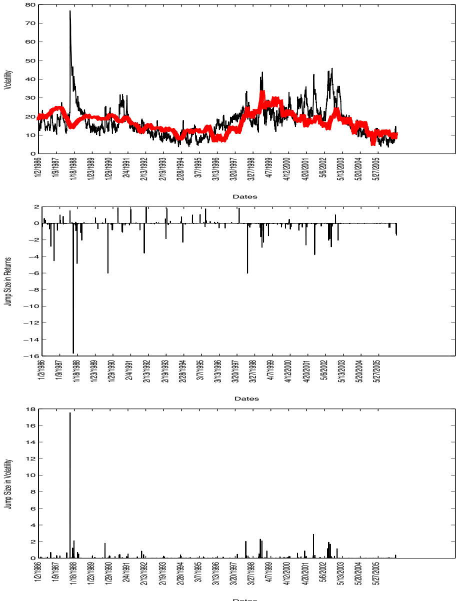
$$IV_t(\tau) = E_t^{\mathbb{Q}} \overline{QV}_t(\tau) + \text{ fat-tailed error}$$

  - Amengual uses data on 1, 2, 3, 6, 12 and 24-month variance swaps

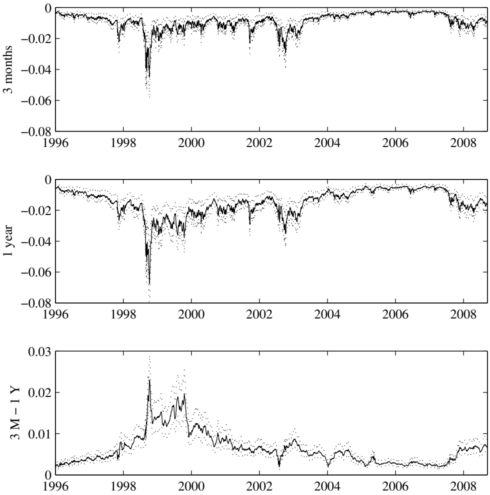
$$\log VS_t(\tau) = \log E_t^{\mathbb{Q}} \tau^{-1} \sum_{i=1}^{\tau} V_{t+i} + \text{ normal error}$$

- Both use Bayesian MCMC

Estimated states

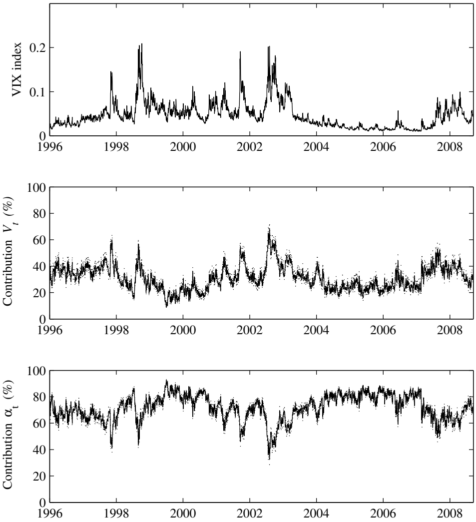


Dynamics of risk premia



Source: Amengual (2009)

Decomposition of risk premia



Source: Amengual (2009)

21. Options and structural models

Risk Aversion

- The Radon-Nikodym derivative is connected to preferences via the pricing kernel:

$$e^{-r_t} L_{t+1} = M_{t+1} = \frac{U'_{t+1}}{U'_t}$$

- Example: CRRA preferences

$$\begin{aligned} U_t &= (1-\beta) C_t^{\rho} / \rho + \beta E_t U_{t+1}, \\ m_{t+1} &= \log \beta - (1-\rho) \log C_{t+1}/C_t \equiv \log \beta - (1-\rho) g_{t+1} \end{aligned}$$

- Here, $1-\rho$ is the relative risk aversion

$$1-\rho = -\frac{C_{t+1} U''_{t+1}}{U'_{t+1}} = -\frac{\partial m_{t+1}}{\partial c_{t+1}} \equiv RRA_{t+1}$$

- We can also define absolute risk aversion, $ARA_{t+1} = RRA_{t+1}/C_{t+1}$

Inferring Risk Aversion

- A major part of this literature is using option-implied $\mathbb{Q}$ to infer risk aversion
- If we know $\mathbb{P}$ and $\mathbb{Q}$ distribution, we can recover risk aversion, at least in theory
- Extra assumptions: $c_t = W_t$ (wealth) and $W_t \approx S_t$ (S&P 500)
- Therefore,

$$\begin{aligned} ARA_{t+1} &= \frac{\partial p^{\mathbb{P}}(S_{t+1}|S_t)/\partial S_{t+1}}{p^{\mathbb{P}}(S_{t+1}|S_t)} - \frac{\partial p^{\mathbb{Q}}(S_{t+1}|S_t)/\partial S_{t+1}}{p^{\mathbb{Q}}(S_{t+1}|S_t)} \\ RRA_{t+1} &= \frac{\partial \log p^{\mathbb{P}}(S_{t+1}|S_t)}{\partial \log S_{t+1}} - \frac{\partial \log p^{\mathbb{Q}}(S_{t+1}|S_t)}{\partial \log S_{t+1}} \end{aligned}$$

Extracting PDFs

- We can use results of parametric estimation

$$p(Y|\Theta) = \int p(Y,X|\Theta)dX \approx \sum \frac{1}{G}\sum_{g=1}^{G} P(Y,X^{(g)}|\Theta^{(g)})$$

- However, very few papers deliver $p^{\mathbb{Q}}$ via Bayesian MCMC
- Difficult to compute this using the frequentists approach (can use transform methods)
- Historically, such analysis started before all the developments in jump-diffusion modelling
- A non-parametric alternative is to use the Breeden-Litzenberger (1978) formula:

$$p^{\mathbb{Q}}(S_{t+\tau} = K|S_t) = e^{r\tau}\frac{\partial^2 O(S_t, K, \tau)}{\partial K^2}$$

Computing PDFs

- The BL formula suggests:

$$\frac{\partial^2 O(S_t, K, \tau)}{\partial K^2} \approx \frac{O(S_t, K+h, \tau) - 2O(S_t, K, \tau) + O(S_t, K-h, \tau)}{h^2}$$

  which is the scaled price of a butterfly centered at K

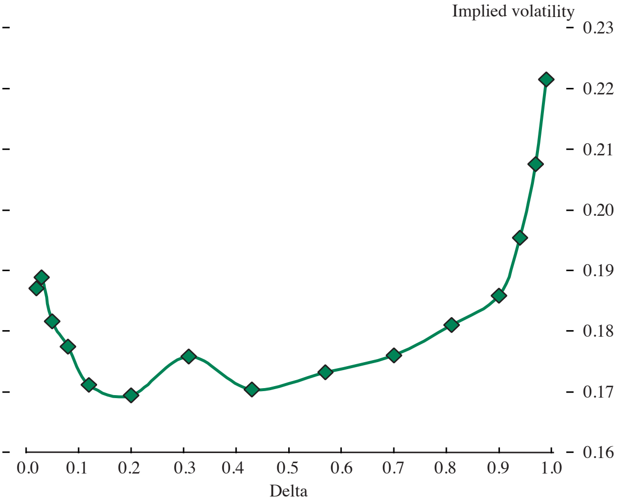
- Cannot choose arbitrary h, do not have the full range of K in practice

- The general idea is to keep $p^{\mathbb{Q}}(S_{t+\tau}|S_t)$ from BS:

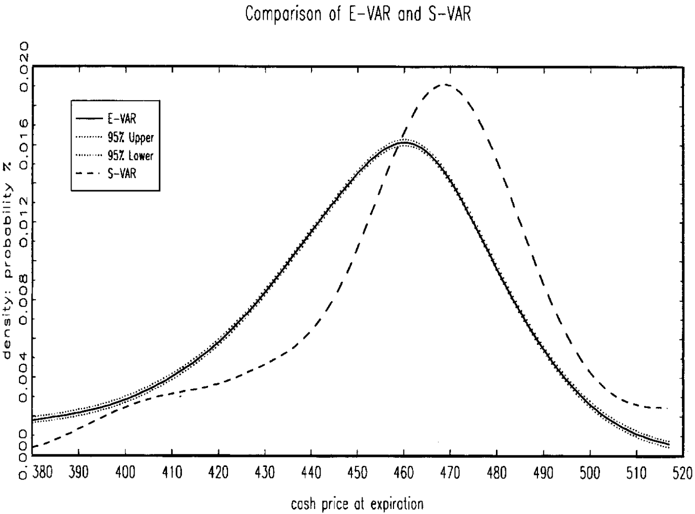
$$p^{\mathbb{Q}}(S_{t+\tau}|S_t) = \frac{1}{S_{t+\tau}\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{[\log S_{t+\tau}/S_t - (r_t - \sigma^2/2)\tau]^2}{2\sigma^2\tau}\right),$$

  construct BSIVs from each available strike, then smooth them using flexible functional form (kernel regression, spline, etc), and then plug back instead of σ

Example of smoothed IVs

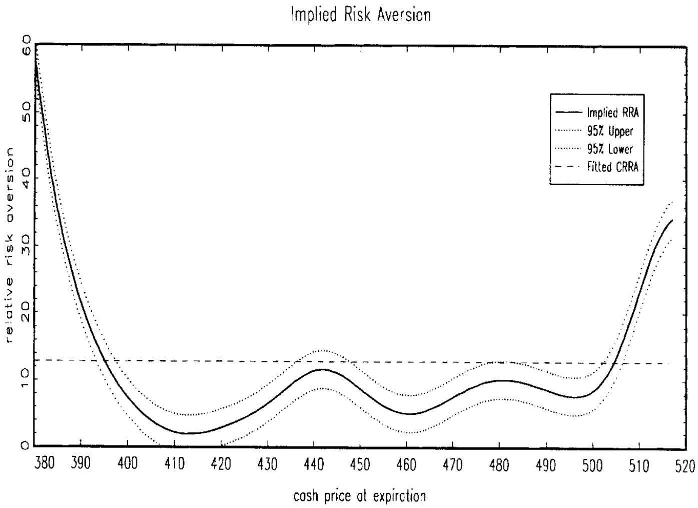


Estimated PDFs



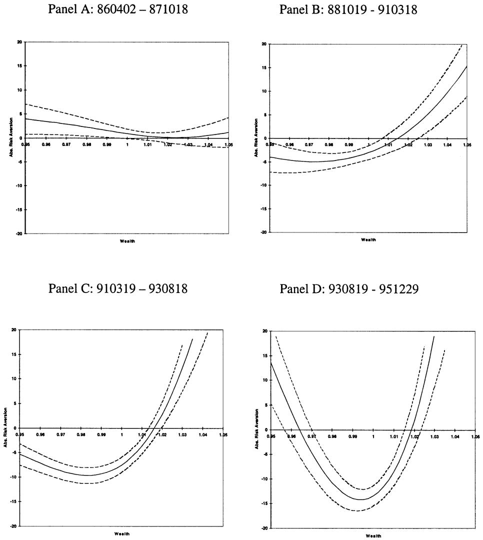
Source: Aït-Sahalia and Lo (2000)

Estimated RRA



Source: Aït-Sahalia and Lo (2000)

Estimated ARA



Source: Jackwerth (2000)

A new puzzle?

- How robust are these findings to assumptions being made about the underlying economic environment and the dimensionality of the state vector?

- These studies presume that stock prices follow a univariate process and, in particular, rule out stochastic volatility.

- Preferences are also defined over this single state variable, wealth.

- Pricing kernels implied by a large class of preferences can be represented as:

$$M_{t+1} = \beta(G_{t+1})^{\rho-1} f(x_{t+1})/f(x_t),$$

  where x_t is an extra state variable

State Dependence and RA smiles

- Chabi-Yo, Garcia, and Renault (2008) argue that the findings can be explained by the lack of conditioning on state variables other than wealth (S&P 500 index)

- Consider $\mathbb{P}$ $p^{\mathbb{P}}(r_{t+1} = r) = S_t e^r p^{\mathbb{P}}(S_{t+1} = S_t e^r | S_t)$

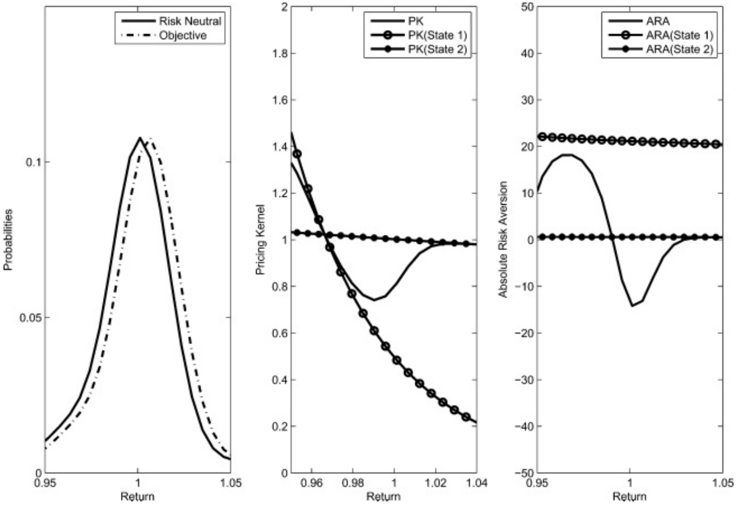
$$p^{\mathbb{P}}(r_{t+1} = r) = E(p^{\mathbb{P}}(r_{t+1} = r | x^{t+1}))$$

- As an example, CGR consider regimes shifts:

$$g_{t+1} = \mu_{s_{t+1}} + \sigma_{s_{t+1}} \varepsilon_{t+1}$$

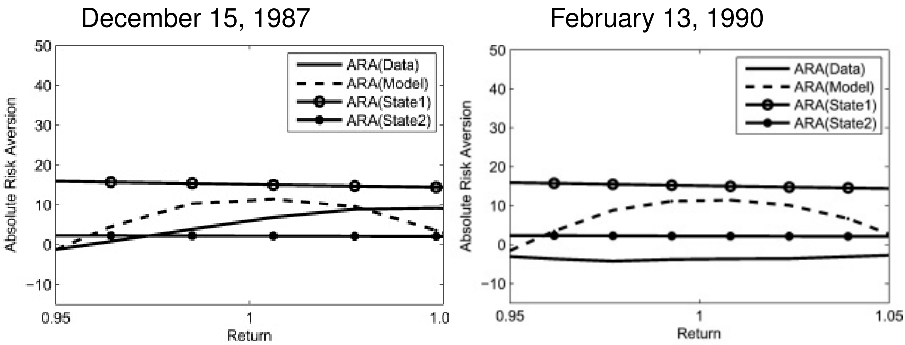
s_{t+1} is a two-state Markov chain

ARA with recursive preferences



Source: Chabi-Yo, Garcia, and Renault (2008)

Recursive ARA and data



Source: Chabi-Yo, Garcia, and Renault (2008)

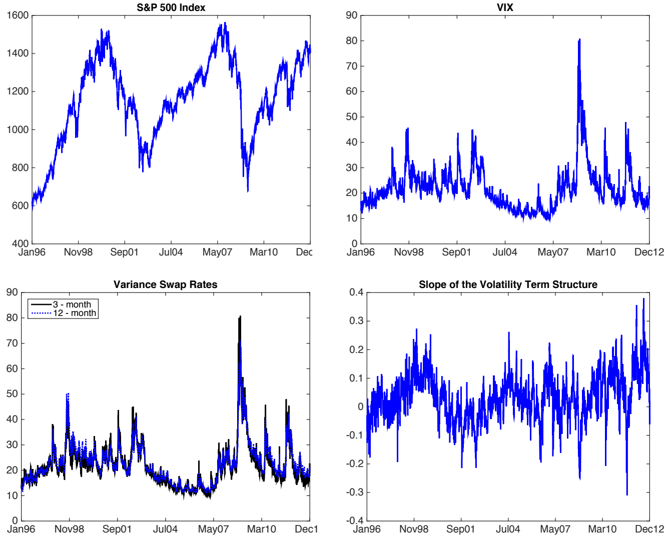
Further evidence on state dependence

- Song and Xiu (2015) bring information from VIX options
- First, revisit the Breeden-Litzenberger (1978) formula with explicit conditioning on state V_t

$$
\begin{aligned}
p^{\mathbb{Q}}(S_{t+\tau} &= K_S | S_t, V_t) = e^{r\tau} \frac{\partial^2 O^{\text{SPX}}(S_t, V_t, K_S, \tau)}{\partial K^2} \\
p^{\mathbb{Q}}(V_{t+\tau} &= K_V | S_t, V_t) = e^{r\tau} \frac{\partial^2 O^{\text{VIX}}(S_t, V_t, K_V, \tau)}{\partial K^2}
\end{aligned}
$$

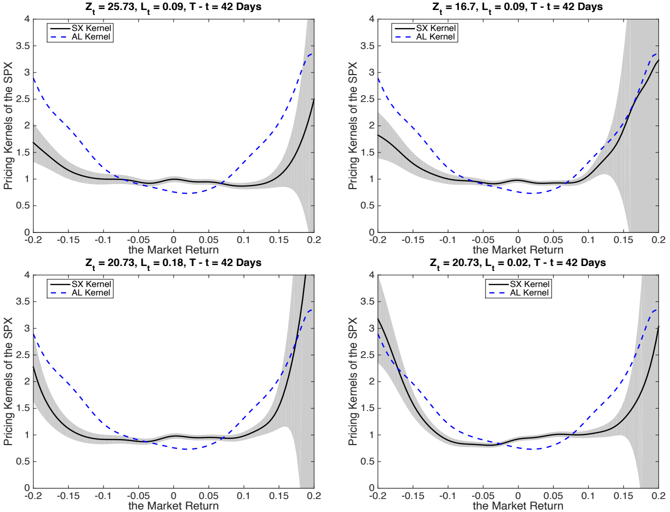
- Observations:
  - Cross-section of options is informative about *marginal* distributions only
  - V_t is not observable so have to develop inference over $Z_t = VIX_t$(3-month)
  - Variance has multivariate structure, so also use $L_t = VIX_t(12) - VIX_t(3)$ – the slope of the variance curve
- Estimate $p^{\mathbb{Q}}$ and p non-parametrically, and then get the pricing kernel as a ratio
- Data: daily close
  - SPX options: 01/1996 to 12/2012
  - VIX options: 06/2007 to 12/2012
  - VIX: 01/1996 to 12/2012

Data



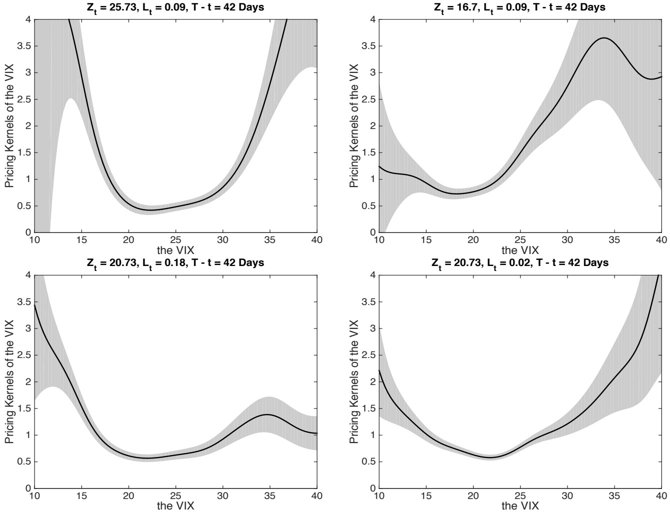
Source: Song and Xiu (2015)

Pricing kernel of returns



Source: Song and Xiu (2015)

Pricing kernel of variance



Source: Song and Xiu (2015)

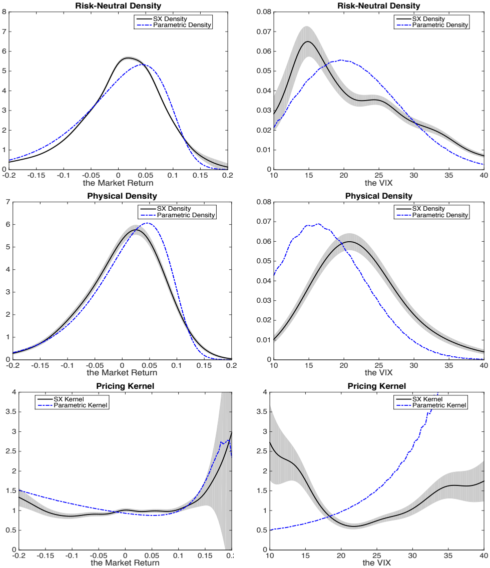
Consistent with an affine model?

- Consider a model that is similar to Amengual/Chung:

$$\begin{aligned}\log(S_{t+1}/S_t) &= (\mu_0 + \mu_1 V_t) + \sqrt{V_t}\varepsilon^s_{t+1} + Z^s_{t+1} - \lambda\bar{\mu} \\ V_{t+1} &= \eta + \kappa_v\theta_t + (1-\kappa_v)V_t + \sigma_v\sqrt{V_t}\varepsilon^v_{t+1} + Z^v_{t+1} \\ \theta_{t+1} &= \kappa_\theta\bar{\theta} + (1-\kappa_\theta)\theta_t + \sigma_\theta\sqrt{\theta_t}\varepsilon^\theta_{t+1}\end{aligned}$$

- Jumps: intensity is λ and jump sizes are: $\xi^v_t \sim \mathcal{E}(\beta)$, $\xi^s_t \sim \mathcal{E}(\beta_+)$ with prob p and $\mathcal{E}(\beta_-)$ with prob $1-p$

Comparison with the nonpara evidence



The Recovery Theorem

- Ross (2015) considers state-space that follows a Markov chain with N states and transition probabilities p_{ij}
- a_{ij} denotes price of Arrow-Debrew security that pays \$1 if we transition from i to j
  - Option prices allows us estimating a_{ij} because $a_{ij} = e^{-r_i} p_{ij}^{\mathbb{Q}}$
  - Can't use option prices to establish probabilities and preferences: $a_{ij} = M_{ij} p_{ij}$ are N^2 equations in $N^2 + N^2 - N$ unknowns
- Consider the Perron-Frobenius problem, that is, find the largest λ that solves the eigenvalue problem for $A : Ae = \lambda e$, or $\sum_j a_{ij} e_j = \lambda e_i$
  - Hint: e that is associated with the largest λ is positive
  - Decompose $A = \lambda D \tilde{P} D^{-1}$, $diag(D) = e$, $\tilde{P} = \lambda^{-1} D^{-1} AD$, or

$$a_{ij} = \lambda \tilde{p}_{ij} e_i / e_j$$

  - N^2 equations in N^2 unknowns [$\tilde{P}$ is $N^2 - N$, e is $N - 1$ (up to scale), λ is 1]

What is recovered?

- We recover probabilities in the sense that $\sum_j \tilde{p}_{ij} = 1$. Are they equal to p_{ij}?
- In order to link to the previous results, note that

$$M_{ij} p_{ij} = a_{ij} = \lambda \tilde{p}_{ij} e_i / e_j \text{ implies } M_{ij} = \lambda \tilde{M}_{ij} e_i / e_j$$

- This is equivalent to the Hansen-Scheinkman decomposition

$$M_{t+1} = \lambda \tilde{M}_{t+1} e_t / e_{t+1}, \; E_t(\tilde{M}_{t+1}) = 1$$

- So, $\tilde{p}_{ij} = p_{ij}$, or $\tilde{p}_t = p_t$ is equivalent to $\tilde{M}_t = 1$ (the long bond is the growth optimal asset)
- Ross assumes a "transition-independent" pricing kernel, that is

$$M_{t+1} = \delta f_{t+1} / f_t$$

  with stationary f_t
- The HS decomposition implies $\delta = \lambda$, $f_t = 1/e_t$, and $\tilde{M}_t = 1$
- See Borovicka, Hansen, and Scheinkman (2014)

Properties of the endowment: Disasters

- Consumption growth iid

$$\begin{aligned} \log g_{t+1} &= w_{t+1} + z_{t+1}\xi_{t+1}, \\ \text{Prob}(z_t = j) &= e^{-\omega}\omega^j/j!, \\ \xi_t|j &\sim \mathcal{N}(j\theta, j\delta^2) \end{aligned}$$

- Average number of disasters ($\omega = 0.01$), mean ($\theta = -0.3$) and variance ($\delta^2 = 0.15^2$) (similar to Barro (2006))
- Equity premium is

$$\begin{aligned} E(r_{t+1} - r^f) &= \lambda E(\log g_{t+1}) + k(\rho - 1; \log g_{t+1}) \\ &- k(\lambda + \rho - 1; \log g_{t+1}), \\ k(s; \log g_{t+1}) &= s\mu + (s\sigma^2)/2 + \omega[e^{s\theta + (s\delta)^2/2} - 1] \end{aligned}$$

- The kicker is (for $\rho = -4$, $\lambda = 5$)
$\lambda\omega\theta + \omega e^{(\rho-1)\theta + ((\rho-1)\delta)^2/2}[1 - e^{\lambda\theta + (\lambda\delta)^2/2 + \lambda(\rho-1)\delta^2}] \approx \omega(5 + \lambda\theta)$

Is this a realistic model of consumption?

- Backus, Chernov, and Martin (2011) perform two exercises:
  1. Price options on the basis of

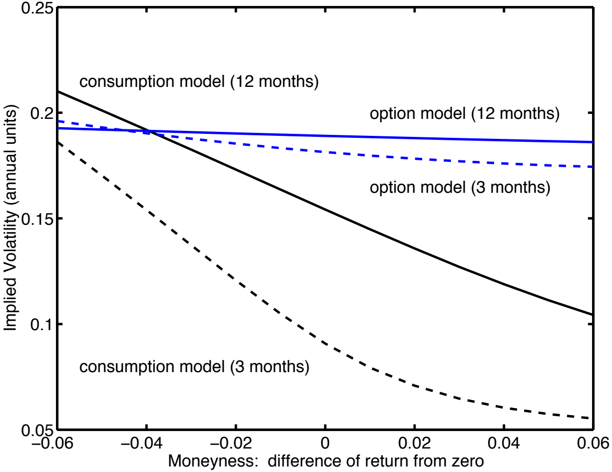
$$
\begin{aligned}
p^{\mathbb{Q}}(g) &= e^{r^f} p^{\mathbb{P}}(g) m(g), \; r_{t+1} = \lambda \log g_{t+1} + \text{const} \\
\omega^{\mathbb{Q}} &= \omega e^{(\rho-1)\theta + ((\rho-1)\delta)^2/2}, \; \lambda\theta^{\mathbb{Q}} = \lambda(\theta + (\rho-1)\delta^2), \lambda\delta^{\mathbb{Q}} = \lambda\delta
\end{aligned}
$$

  2. Infer distribution of g_{t+1} on the basis of:

$$
\begin{aligned}
p^{\mathbb{P}}(g) &= e^{-r^f} p^{\mathbb{Q}}(g)/m(g), \; r_{t+1} = \lambda \log g_{t+1} + \text{const} \\
\omega^{\mathbb{Q}} &= 1.5, \; \lambda\theta^{\mathbb{Q}} = -0.05, \; \lambda\delta^{\mathbb{Q}} = 0.098
\end{aligned}
$$

  (estimates are from Broadie, Chernov, and Johannes, 2007)

Comparing models: options implied by macro model



Source: Backus, Chernov, and Martin (2011)

Comparing models: consumption implied by options

	Calibration	Implied	
α	5.19	8.70	
ω	0.0100	1.3987	
θ	–0.3000	–0.0074	
δ	0.1500	0.0191	
Skew	–11.02	–0.28	– 0.35
Excess Kurt	145.06	0.48	1.10
Tail prob (≤ -3 st dev)	0.0090	0.0081	Great Depression
Tail prob (≤ -5 st dev)	0.0079	0.0001	

Source: Backus, Chernov, and Martin (2011)

Extensions with recursive preferences

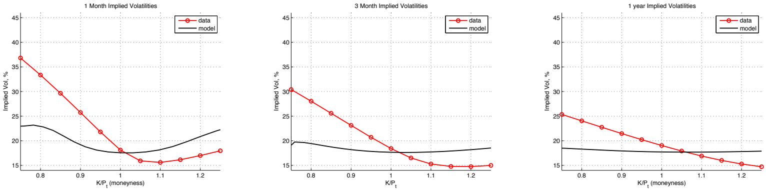
- Drechsler and Yaron (2011):

$$
\begin{aligned}
g_{t+1} &= g + x_t + \sqrt{V_t}\varepsilon^c_{t+1} \\
x_{t+1} &= \rho_x x_t + \phi_x \sqrt{V_t}\varepsilon^x_{t+1} + z_{t+1}\xi^x_{t+1} \\
V_{t+1} &= (1-\rho_v)\theta_t^2 + \rho_v V_t + \phi_v \sqrt{V_t}\varepsilon^v_{t+1} + z_{t+1}\xi^v_{t+1} \\
\theta_{t+1} &= (1-\rho_\theta)\bar{\theta} + \rho_\theta \theta_t + \phi_\theta \varepsilon^\theta_{t+1} \\
z_{t+1} &\sim \mathcal{P}(\lambda + \lambda_v V_t),\ \xi^x \sim \mathcal{N}(0,\sigma_x^2),\ \xi^v \sim \mathcal{E}(\mu_v)
\end{aligned}
$$

- Matches many facts about S&P 500 returns, variance risk premium, predictability of returns with VRP; cannot match the smile

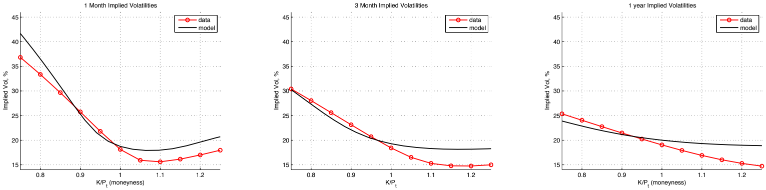
- Drechsler (2013) adds Knightian uncertainty to the framework to get the smile

Smiles: No Uncertainty



Source: Drechsler (2013)

Smiles: Kinightian Uncertainty



Source: Drechsler (2013)

More recursive examples

- Consumption growth

$$\begin{aligned} g_t &= g + \gamma(B)v^{1/2}w_{gt} + \psi(B)z_{gt} - \psi(1)h\theta, \\ h_t &= h + \eta(B)w_{ht} \end{aligned}$$

- Pricing kernel

$$\begin{aligned} m_{t,t+1} &= \text{constant} + [(\rho-1)\gamma(B) + (\alpha-\rho)\gamma(b_1)]v^{1/2}w_{gt+1} \\ &\quad + [(\rho-1)\psi(B) + (\alpha-\rho)\psi(b_1)]z_{gt+1} \\ &\quad + (\alpha-\rho)[(e^{\alpha\psi(b_1)\theta + (\alpha\psi(b_1)\delta)^2/2} - 1)/\alpha][b_1\eta(b_1) - \eta(B)B]w_{ht+1} \end{aligned}$$

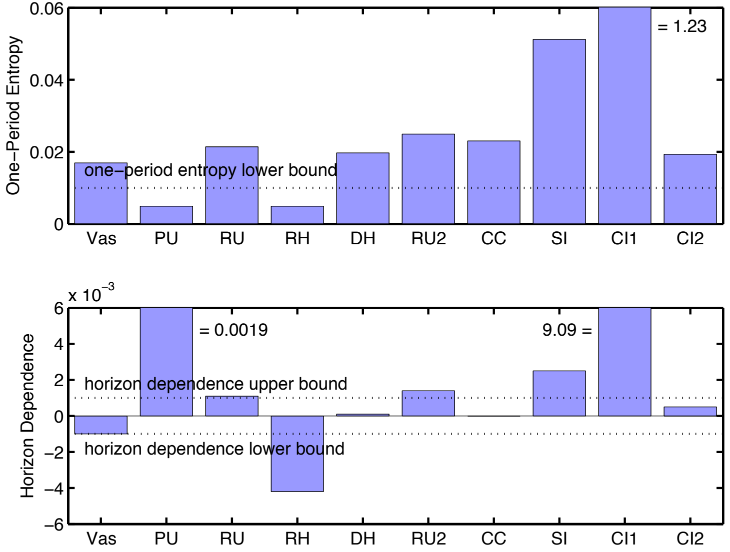
- Entropy

$$\begin{aligned} L_t(M_{t,t+1}) &= [(\rho-1)\gamma(B) + (\alpha-\rho)\gamma(b_1)]^2 v/2 \\ &\quad + \left\{\left(e^{(\alpha^*-1)\theta + [(\alpha^*-1)\delta]^2/2} - 1\right) - (\alpha^*-1)\theta\right\} h_t \\ &\quad + \left\{(\alpha-\rho)\left[(e^{\alpha\psi(b_1)\theta + [\alpha\psi(b_1)\delta]^2/2} - 1)/\alpha\right] b_1\eta(b_1)\right\}^2/2, \\ \alpha^* - 1 &= (\rho-1)\psi_0 + (\alpha-\rho)\psi(b_1) \end{aligned}$$

Nested models

Parameter or property	IID w/ Jumps (1)	Stochastic Intensity (2)	Constant Intensity 1 (3)	Constant Intensity 2 (4)
Preference parameters				
ρ	$1/3$	$1/3$	$1/3$	$1/3$
α	-9	-9	-9	-9
β	0.9980	0.9980	0.9980	0.9980
Consumption growth process				
$v^{1/2}$	0.0025	0.0025	0.0021	0.0079
h	0.0008	0.0008	0.0008	0.0008
θ	-0.3000	-0.3000	-0.3000	-0.1500
δ	0.1500	0.1500	0.1500	0.1500
η_0	0	0.0001	0	0
φ_h		0.9500		
γ_0	1	1	1	1
γ_1			0.0271	0.0281
φ_g			0.9790	0.9690
ψ_0	1	1	1	1
ψ_1			0.0271	
φ_z			0.9790	
Derived quantities				
b_1	0.9974	0.9973	0.9750	0.9979
$\gamma(b_1)$	1	1	1.5806	1.8481
$\psi(b_1)$	1	1	1.5806	1
$\eta(b_1)$	0	0.0016	0	0
Entropy and horizon dependence				
$I(1) = EL_t(m_{t,t+1})$	0.0485	0.0512	1.2299	0.0193
$I(\infty)$	0.0485	0.0542	15.730	0.0200
$H(120) = I(120) - I(1)$	0	0.0025	9.0900	0.0005
$H(\infty) = I(\infty) - I(1)$	0	0.0030	14.5000	0.0007

Summary



One−Period Entropy
0.06
0.04
0.02
0
= 1.23
one−period entropy lower bound
Vas
PU
RU
RH
DH
RU2
CC
SI
CI1
CI2
Horizon Dependence
x 10−3
6
4
2
0
−2
−4
−6
= 0.0019
9.09 =
horizon dependence upper bound
horizon dependence lower bound
Vas
PU
RU
RH
DH
RU2
CC
SI
CI1
CI2

Further restrictions from variance swaps

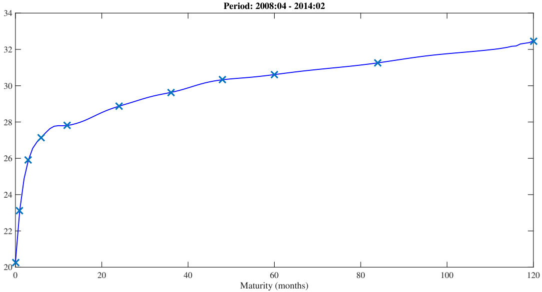
- Dew-Becker, Giglio, Le, and Rodriguez (2015) study pricing of variance risk using the term structure of variance swaps
- They construct "zero-coupon" swaps:

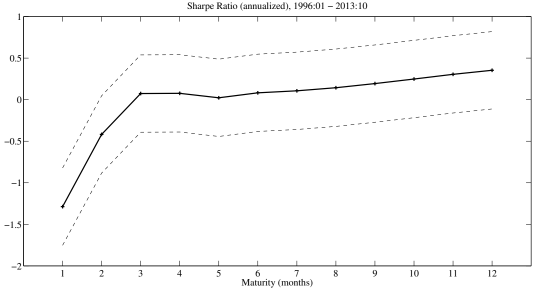
$$
\begin{aligned}
\tilde{V}_t^n &= E_t^{\mathbb{Q}} \sum_{i=1}^{n} (\log S_{t+i}/S_{t+i-1})^2 \\
Z_t^n &\equiv E_t^{\mathbb{Q}} (\log S_{t+n}/S_{t+n-1})^2 = \tilde{V}_t^n - \tilde{V}_t^{n-1}
\end{aligned}
$$

- Also, construct "returns" on these claims:

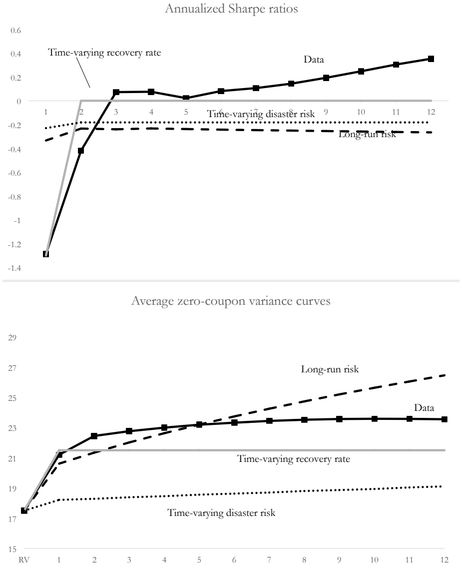
$$R_{t+1}^n = Z_{t+1}^{n-1}/Z_t^n - 1$$

Evidence


Period: 2008:04 - 2014:02
34
32
30
28
26
24
22
20
0
20
40
60
80
100
120
Maturity (months)


Sharpe Ratio (annualized), 1996:01 – 2013:10
1
0.5
0
−0.5
−1
−1.5
−2
1
2
3
4
5
6
7
8
9
10
11
12
Maturity (months)

Economic models



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