

IMPLIED VOLATILITY SURFACE AND MODEL OF VIX

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ABSTRACT.

1. INTRODUCTION

Black [2] first introduced the European futures options pricing based on Black-Scholes framework [3]. The common assumption for the process followed by future price F in risk-neutral framework is

$$dF = \sigma F dW, \quad (1)$$

where σ is a constant and W is the Wiener process. Similar to a non-dividend-paying stock, the differential equation satisfied by a derivative dependent on a futures price is

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} \sigma^2 F^2 = r f.$$

Then the European call price c and the European put price p for the futures option are given by following equations,

$$c = e^{-rT} [F_0 N(d_1) - K N(d_2)], \quad (2)$$

$$p = e^{-rT} [K N(-d_2) - F_0 N(-d_1)], \quad (3)$$

where

$$d_1 = \frac{\log(F_0/K) + \sigma^2 T/2}{\sigma \sqrt{T}},$$

$$d_2 = \frac{\log(F_0/K) - \sigma^2 T/2}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}.$$

Following the Black-Scholes framework, more recent researches come up with stochastic volatility (SV) model [5, 1, 4], where the volatility surface (w.r.t. different terms and strikes) are fitted better to the empirical observations. The SV modelings are motivated by the intermittency and clustering feature of volatility from underlying dynamics. A common stochastic volatility process satisfies the SDE:

$$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dW_1, dv_t = \alpha(S_t, v_t, t) dt + \eta \beta(S_t, v_t, t) \sqrt{v_t} dW_2,$$

where $\mathbb{E}[dW_1 dW_2] = \rho dt$. μ_t is the (deterministic) instantaneous drift of stock price returns, η is the volatility of volatility and ρ is the correlation between random stock price returns and changes in v_t . W_1 and W_2 are Wiener processes.

The volatility drift and vol of vol term are in most generic functional forms and can determine by time (t), instantaneous volatility level (v_t) and the spot price (S_t). Note that the SV setup ensures the standard time-dependent volatility version of the Black-Scholes formula may be retrieved in the limit $\eta \rightarrow 0$.

Particularly, S&P500 index returns, shown in Figure 1, reinforce the stochastic volatility and volatility mean-reverting characteristics. More importantly, the “volatility” of the S&P500 index

is tradable. The VIX index represents the volatility of SPX in a precise way, and CBOE has listed futures on VIX index since 2004 and VIX option in 2006.

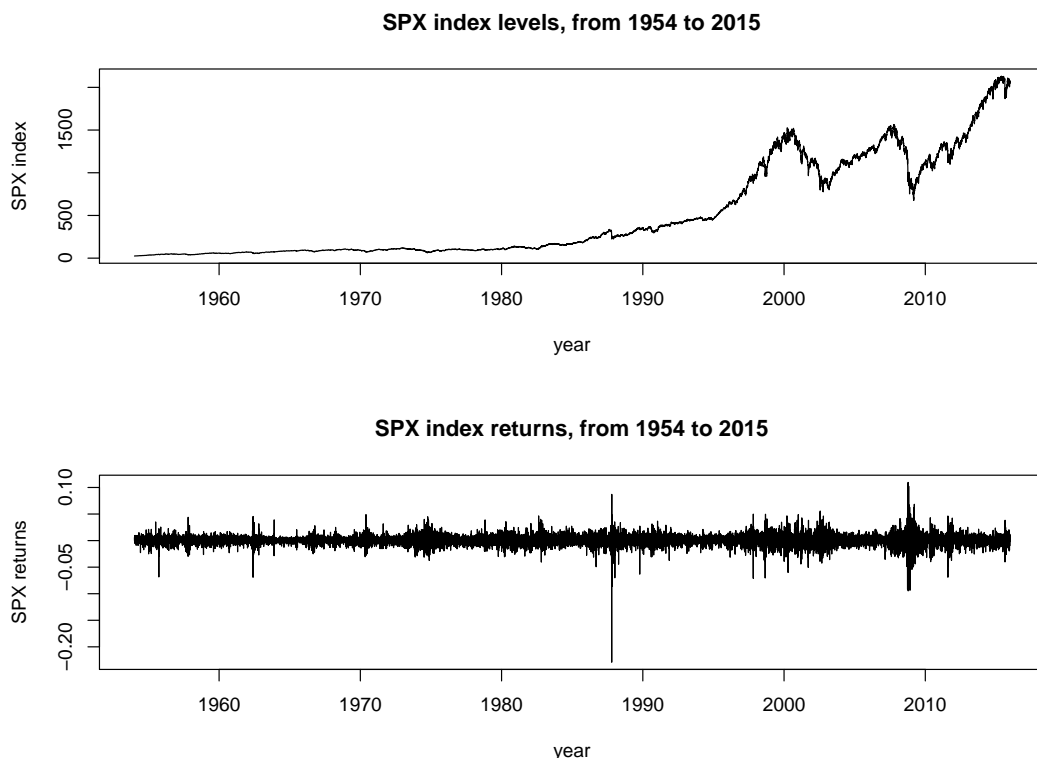


Figure 1 The SPX index levels and return on daily basis. Time period is from 1954 to 2015.

The rest of the paper lays out as following: Section 2 presents the observations and findings on implied volatility from VIX options. Section 3 conjectures a two-factor VIX model and discusses the joint VIX model with SPX dynamics.

2. VIX IMPLIED VOLATILITY

The option data from OptionMetrics contains daily standardized VIX option bid/offer price, Delta of the option and other information including expiration and strike. The forward price, here the underlying future price, is interpolated from OptionMetrics' standard volatility surface using log-linear interpolation. The reasoning of selecting log-linear interpolation is that the future price is positively correlated with discount factor, and we assume a flat-forward interpolation keeps most of properties from term structure. The zero yield curve is also from OptionMetrics database.

The VIX future has multiplier of 1000, so the strike price of options are in magnitude of 1000 times VIX. To back-out the implied volatility of VIX options, we use the Black model as specified in 2. The market premium is calculated from the average of best bid and best offer price. The underlying future price is interpolated log-linearly from standard option volatility, and the interest rate is interpolated linearly from the term structure of zero yield.

According call-put parity, the implied volatility from call option and put option with same strike will be the same. Figure 2 plots the averaged implied volatility over the terms. The graph on the top shows the implied volatility dynamics of VIX option with strike price ranging from 5%



Figure 2 The averaged VIX option implied volatility over different terms. The graph plots the implied volatility dynamic on various strikes. The blue line is the fitted implied volatility smile by lowess regression. The graph on the top has strikes (of VIX) from 5 to 130. The graph on the bottom has strikes from 5 to 22.

to 130%. Unlike the volatility skew shape as in equity option, the VIX option has higher implied vol (SPX's vol of vol). The strike-zoom-in graph on the bottom from Figure 2 show the volatility smile at relatively low VIX strike levels, ranging from 5% to 23%.

Figure 3 separates the average on implied volatility when the short term and long term are high/low respectively. From the graphs, it is obvious that the implied volatility levels have different shapes, in terms of both magnitude and the slope. That is, the implied volatility smile of VIX options are not only driven by the VIX level, but also term structure and skew. Therefore, empirical observations conflict one-factor stochastic volatility model, which captures only the level of volatility but not sufficient to fit term structure additionally.

Then the idea of two-factor stochastic volatility modeling becomes natural. In Section 3, we will discuss the rationale of the double-stochastic-variance (DSV) model.

3. MODELING VIX AND SPX JOINTLY

Before proceeding to the conjecture of the VIX model, we firstly show that VIX represents the volatility of SPX in a precise way. Therefore one can model VIX using variance swap instead of the implied volatilities. And also the relationship between the prices of options on SPX and options on VIX can be established. In particular, the fair values of the variance swaps can be expressed in terms of the market prices of European options, which provides an model-independent way of linking SPX and VIX.

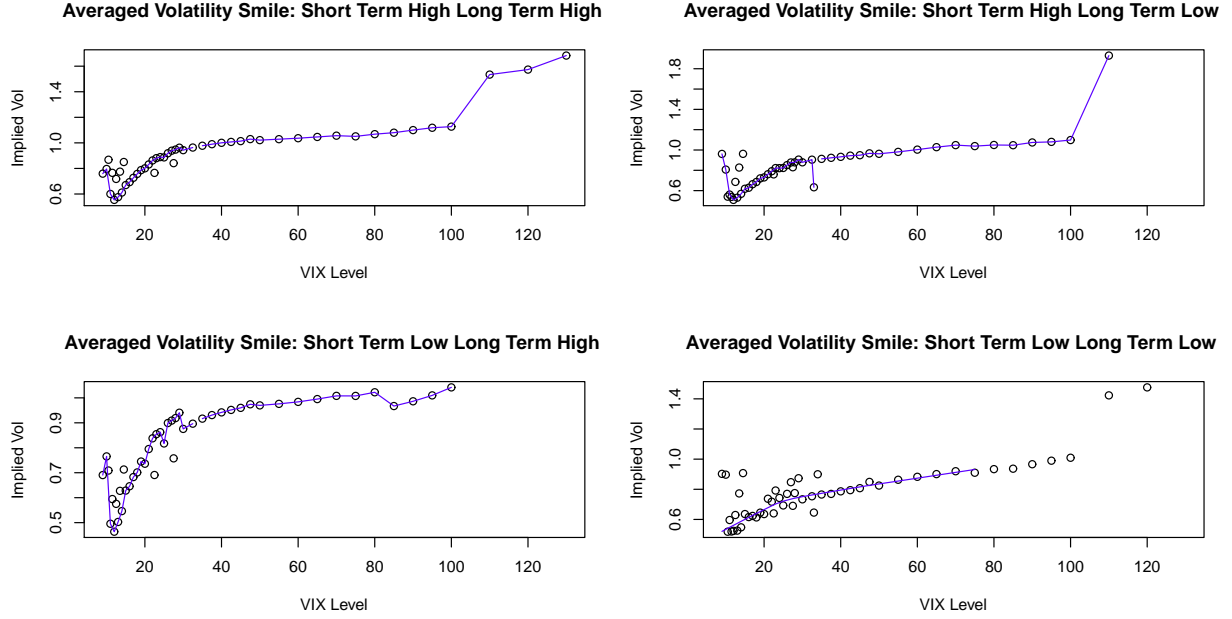


Figure 3 The averaged VIX option implied volatility over different terms, when the short/long term implied volatilities are high/low. Four graphs display the combination of the short- and long-term VIX. Each graph plots the implied volatility dynamic on various strikes.

Assume there are European options with all possible strikes K and maturity are traded on the market, then any twice-differentiable payoff at time T may be statically hedged using a portfolio of European options expiring at time T . To prove this, consider the value of a claim with payoff $g(S_T)$ at time T as

$$\begin{aligned} g(S_T) &= \int_0^\infty g(K) \delta(S_T - K) dK \\ &= \int_0^F g(K) \delta(S_T - K) dK + \int_F^\infty g(K) \delta(S_T - K) dK. \end{aligned}$$

Follow the derivation in ??, and integrate by parts twice, we have

$$\begin{aligned} g(S_T) &= g(F) - \int_0^F g'(K)(K - S_T)^+ dK \\ &\quad + \int_F^\infty g'(K)(S_T - K)^+ dK, \\ g(S_T) &= \int_0^F g''(K)(K - S_T)^+ dK + \int_F^\infty g''(K)(S_T - K)^+ dK \\ &\quad + g(F) + g'(F)[(F - S_T)^+ - (S_T - F)^+] \\ &= \int_0^F g''(K)(K - S_T)^+ dK + \int_F^\infty g''(K)(S_T - K)^+ dK + g(F) + g'(F)(F - S_T) \\ &\quad + g(F) + g'(F)(F - S_T). \end{aligned} \tag{4}$$

With $F = \mathbb{E}[S_T]$,

$$\mathbb{E}[g(S_T)] = g(F) + \int_0^F dK \tilde{P}(K) g''(K) + \int_F^\infty dK \tilde{C}(K) g''(K).$$

The Equation 4 shows that any European type twice-differentiable payoff can be replicated by a portfolio of European options (puts and calls) with strike from 0 to ∞ .

Finally, the double stochastic variance model has dynamics of

$$\begin{aligned} dS_t &= \sqrt{v_t} S_t dW_t^1, \\ dv_t &= \kappa_1(v'_t - v_t)dt + \xi_1 v_t dW_t^2, \end{aligned} \tag{5}$$

$$dv'_t = \kappa_2(\theta - v'_t)dt + \xi_2 v'_t dW_t^3, \tag{6}$$

where the Brownian motions W_i could all be correlated with $\mathbb{E}[dW_t^i dW_t^j] = \rho_{ij}dt$.

REFERENCES

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- [3] Black, Fischer and Scholes, Myron. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.
- [4] Duffie, Darrell, Jun Pun, and Kenneth Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68:1343–1376, 2000.
- [5] Heston, Steven L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

APPENDIX: CODE

```

1 setwd('C:\\Users\\ranzhao\\Documents\\Empirical-Asset-Pricing\\Assignment 7')
2 setwd('D:\\PhD FE\\Empirical-Asset-Pricing\\Assignment 7')
3 setwd('D:\\Empirical-Asset-Pricing\\Assignment 7')

5 spx_index_values = read.csv('spx_index_values.csv', header = TRUE)
6 par(mfrow=c(2,1))
7 plot(as.Date(as.character(spx_index_values$Date), "%m/%d/%Y"), spx_index_values$SPX.
   Index, type='l',
   main='SPX index levels, from 1954 to 2015',
9   xlab='year', ylab='SPX index')

11 # calculate the return series
12 spx_index_values$Return = rep(0, dim(spx_index_values)[1])
13 spx_index_values$Return[2:length(spx_index_values$Return)] =
   log(spx_index_values$SPX.Index[2:length(spx_index_values$SPX.Index)] /
15   spx_index_values$SPX.Index[1:(length(spx_index_values$SPX.Index)-1)])
16 data.length = length(spx_index_values$Return)

17 plot(as.Date(as.character(spx_index_values$Date), "%m/%d/%Y"), spx_index_values$Return
   , type='l',
19   main='SPX index returns, from 1954 to 2015',
   xlab='year', ylab='SPX returns')

21 require(data.table)
22 option.data = fread('VIXoptions.csv', header = T, sep = ',')
23 implied.data = fread('VIXoptionsStd.csv', header = T, sep = ',')
24 ir.data = fread('zeroyieldcurve.csv', header = T)

27 option.data = as.data.frame(option.data)
28 implied.data = as.data.frame(implied.data)
29 ir.data = as.data.frame(ir.data)

31 option.data = option.data[,c("date", "exdate", "cp_flag", "strike_price", "best_bid", "best
   _offer", "delta")]
32 option.data$days = as.numeric(as.Date(as.character(option.data$exdate), format="%Y/%m/%d
   ") - as.Date(as.character(option.data$date), format="%Y/%m/%d"))
33 option.data$T = option.data$days/360

35 option.data = option.data[!is.na(option.data$delta),]
36 implied.data = implied.data[implied.data$cp_flag=="C", c("date", "days", "forward_price")
   ]

37 # interpolation future prices and interest rate
38 option.data$F0 = 0
39 option.data$r = 0

41 all.dates = unique(option.data$date)
42 for (i in all.dates){ # slow
43   all.days = unique(option.data$days[option.data$date==i])
44   for (j in all.days){
45     ix0 = implied.data$days[implied.data$date==i]
46     iy0 = implied.data$forward_price[implied.data$date==i]
47     option.data$F0[option.data$date==i&option.data$days==j] = linear.inter(ix0, iy0, j
       , "log")
48   }
49   # handle missing data
50   ir.ix0 = ir.data$days[ir.data$date==i]

```

```

51   ir.iy0 = ir.data$rate[ir.data$date==i]/100
    if (length(ir.ix0)==0){
53       ir.ix0 = ir.data$days[ir.data$date==i.last]
       ir.iy0 = ir.data$rate[ir.data$date==i.last]/100
55   }
    option.data$r[option.data$date==i&option.data$days==j] = linear.inter(ir.ix0, ir.
        iy0, j, "linear")
57 }
    i.last = i
59 }

61 # calculate the BS implied volatility
option.data$BS_iv = 0
63 for (k in 1:length(option.data$date)){ # very slow
    option.data$BS_iv[k] = bs.iv(option.data$F0[k],
65                                option.data$strike_price[k]/1000,
                                option.data$T[k],
67                                option.data$r[k],
                                0.5*(option.data$best_bid[k]+option.data$best_offer[k])
69                                ,
                                option.data$cp_flag[k])
}
71 # option.data$BS_iv = lapply(cbind(option.data$F0, option.data$strike_price/1000,
    option.data$T, option.data$r, 0.5*(option.data$best_bid+option.data$best_offer),
    option.data$cp_flag), bs.iv)

73 option.data = option.data[!is.na(option.data$BS_iv),]
all.strikes = sort(unique(option.data$strike_price))
75 vix.vol.smile = rep(0, length(all.strikes))
for (s in 1:length(all.strikes)){
77     vix.vol.smile[s] = mean(option.data$BS_iv[option.data$strike_price==all.strikes[s]])
}
79

par(mfrow=c(2,1))
81 plot(all.strikes/1000, vix.vol.smile, type='p', main='Averaged Volatility Smile on VIX
    ', xlab='VIX Level', ylab='Implied Vol')
lines(lowess(all.strikes/1000, vix.vol.smile), col="blue") # lowess line (x,y)
83

plot(all.strikes[1:20]/1000, vix.vol.smile[1:20], type='p', main='Averaged Volatility
    Smile on VIX', xlab='VIX Level', ylab='Implied Vol')
85 lines(lowess(all.strikes[1:20]/1000, vix.vol.smile[1:20]), col="blue") # lowess line (x
    ,y)

87 short.term.vol.thres = mean(option.data$BS_iv[option.data$days<=60])
long.term.vol.thres = mean(option.data$BS_iv[option.data$days>=120])
89

short.term.high.long.term.high = c()
91 short.term.high.long.term.low = c()
short.term.low.long.term.high = c()
93 short.term.low.long.term.low = c()

95 for (i in all.dates){
    short.term.vol = mean(option.data$BS_iv[option.data$date == i & option.data$days
        <=60])
97     long.term.vol = mean(option.data$BS_iv[option.data$date == i & option.data$days
        >=120])
        if (short.term.vol >= short.term.vol.thres & long.term.vol >= long.term.vol.thres){

```



```

99     short.term.high.long.term.high = rbind(short.term.high.long.term.high, option.data
      [option.data$date == i,])
    } else if (short.term.vol >= short.term.vol.thres & long.term.vol < long.term.vol.
      thres){
101     short.term.high.long.term.low = rbind(short.term.high.long.term.low, option.data[
      option.data$date == i,])
    } else if (short.term.vol < short.term.vol.thres & long.term.vol >= long.term.vol.
      thres){
103     short.term.low.long.term.high = rbind(short.term.low.long.term.high, option.data[
      option.data$date == i,])
    } else {
105     short.term.low.long.term.low = rbind(short.term.low.long.term.low, option.data[
      option.data$date == i,])
    }
107 }

109 vix.vol.smile.short.term.high.long.term.high = rep(0, length(all.strikes))
vix.vol.smile.short.term.high.long.term.low = rep(0, length(all.strikes))
111 vix.vol.smile.short.term.low.long.term.high = rep(0, length(all.strikes))
vix.vol.smile.short.term.low.long.term.low = rep(0, length(all.strikes))
113
for (s in 1:length(all.strikes)){
115   vix.vol.smile.short.term.high.long.term.high[s] = mean(short.term.high.long.term.
      high$BS_iv[short.term.high.long.term.high$strike_price==all.strikes[s]])
   vix.vol.smile.short.term.high.long.term.low[s] = mean(short.term.high.long.term.low$
      BS_iv[short.term.high.long.term.low$strike_price==all.strikes[s]])
117   vix.vol.smile.short.term.low.long.term.high[s] = mean(short.term.low.long.term.high$
      BS_iv[short.term.low.long.term.high$strike_price==all.strikes[s]])
   vix.vol.smile.short.term.low.long.term.low[s] = mean(short.term.low.long.term.low$BS
      _iv[short.term.low.long.term.low$strike_price==all.strikes[s]])
119 }
par(mfrow=c(2,2))
121 plot(all.strikes/1000, vix.vol.smile.short.term.high.long.term.high, type='p', main='
      Averaged Volatility Smile: Short Term High Long Term High',xlab='VIX Level',ylab='
      Implied Vol')
lines(lowess(all.strikes/1000,vix.vol.smile.short.term.high.long.term.high), col="blue"
      ) # lowess line (x,y)
123
plot(all.strikes/1000, vix.vol.smile.short.term.high.long.term.low, type='p', main='
      Averaged Volatility Smile: Short Term High Long Term Low',xlab='VIX Level',ylab='
      Implied Vol')
125 lines(lowess(all.strikes/1000,vix.vol.smile.short.term.high.long.term.low), col="blue"
      ) # lowess line (x,y)

127 plot(all.strikes/1000, vix.vol.smile.short.term.low.long.term.high, type='p', main='
      Averaged Volatility Smile: Short Term Low Long Term High',xlab='VIX Level',ylab='
      Implied Vol')
lines(lowess(all.strikes/1000,vix.vol.smile.short.term.low.long.term.high), col="blue"
      ) # lowess line (x,y)
129
plot(all.strikes/1000, vix.vol.smile.short.term.low.long.term.low, type='p', main='
      Averaged Volatility Smile: Short Term Low Long Term Low',xlab='VIX Level',ylab='
      Implied Vol')
131 lines(lowess(all.strikes/1000,vix.vol.smile.short.term.low.long.term.low), col="blue"
      ) # lowess line (x,y)

133 linear.inter = function(ix0,iy0,ix,inter.method="log"){
# log-linear and linear interpolation function

```

```

135     order = sort(ix0, index.return=TRUE)
136     x0 = order$x
137     y0 = iy0[order$ix]
138     n = length(ix0)
139     if (ix < min(ix0)){
140         ix1 = ix0[1]
141         ix2 = ix0[2]
142         iy1 = iy0[1]
143         iy2 = iy0[2]
144     } else if (ix > max(ix0)){
145         ix1 = ix0[n-1]
146         ix2 = ix0[n]
147         iy1 = iy0[n-1]
148         iy2 = iy0[n]
149     } else {
150         ix1 = ix0[max(which(ix>=ix0))]
151         ix2 = ix0[min(which(ix<=ix0))]
152         iy1 = iy0[max(which(ix>=ix0))]
153         iy2 = iy0[min(which(ix<=ix0))]
154     }
155     if (ix1 == ix2){
156         return(iy1)
157     }
158     else {
159         if (inter.method == "log"){
160             iy = (ix-ix1)/(ix2-ix1)*log(iy2) + (ix2-ix)/(ix2-ix1)*log(iy1)
161             return(exp(iy))
162         } else if (inter.method == "linear") {
163             iy = (ix-ix1)/(ix2-ix1)*iy2 + (ix2-ix)/(ix2-ix1)*iy1
164             return(iy)
165         }
166     }
167 }

169 bs.iv = function(S, K, T, r, market, type){
170 # calculate Black-Scholes implied volatility
171     sig <- 0.20
172     sig.up <- 2
173     sig.down <- 0.001
174     count <- 0
175     err <- BS(S, K, T, r, sig, type) - market
176
177     ## repeat until error is sufficiently small or counter hits 1000
178     while(abs(err) > 0.0001 && count<3000){
179         if(err < 0){
180             sig.down <- sig
181             sig <- (sig.up + sig)/2
182         } else{
183             sig.up <- sig
184             sig <- (sig.down + sig)/2
185         }
186         err <- BS(S, K, T, r, sig, type) - market
187         count <- count + 1
188     }
189
190     ## return NA if counter hit 1000
191     if(count==3000){

```

```

193     return(NA)
    } else {
195         return(sig)
    }
197 }

199 BS = function(S, K, T, r, sig, type="C"){
# calculation option price using Black-Scholes model
201     d1 <- (log(S/K) + (r + sig^2/2)*T) / (sig*sqrt(T))
    d2 <- d1 - sig*sqrt(T)
203     if(type=="C"){
        value <- S*exp(-r*T)*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
205     }
    if(type=="P"){
207         value <- K*exp(-r*T)*pnorm(-d2) - S*exp(-r*T)*pnorm(-d1)
    }
209     return(value)
}

```

assignment7.R