# CALIBRATION ON MERTON JUMP DIFFUSION USING BAYESIAN MCMC METHOD

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ABSTRACT.

## 1. INTRODUCTION

Advances in computing powers and numerical methods have largely improve the capability of solving econometric and statistical models using computational intense methods, includes Markov Chain Monte Carlo (MCMC) method. Especially in dynamic asset pricing models, the MCMC method is widely utilized to extracting information about latent state variables (such as implied volatility), structural parameters and market prices of risk (volatility or jump risks) from observed prices or market quotes. The Bayesian inference is to obtain the distribution of parameter set, $\Theta$, and (optional) state variables, $X$, conditioning on the observed prices, $Y$. That is, the posterior distribution, $p(\Theta, X|Y)$ is vital to the parameters estimation and their statistical inference.

Consider a stochastic process $\{X_t\}$, where each $X_t$ assumes value in space $\Omega$. Then the process $\{X_t\}$ is a Markov process if given the value of $X_t$, the values of $X_{t+h}$, $h > 0$, do not depend on the values $X_s$, $s < t$. That is, $\{X_t\}$ is a Markov process if its conditioanl distribution function satisfies

$$\mathbb{P}(X_{t+h}|X_s, s \leq t) = \mathbb{P}(X_{t+h}|X_t), \quad h > 0.$$

In continuous-time asset pricing models, MCMC that explore their posterior distributions samples from high-dimensional and sophisticated distributions by generating Markon process over $(\Theta, X)$, $\{\Theta^{(g)}, X^{(g)}\}_{g=1}^{G}$. And the equilibrium distribution of $(\Theta, X)$ is $p(\Theta, X|Y)$. Then Monte Carlo methods use these samples for statistical inference on parameters and states.

However, $p(\Theta, X|Y)$ in continuous-time asset pricing models is usually not easy to obtain. Johannes and Polson [4] listed the reasons for this difficulty, which summarize as

(1) market prices are observed discretely (e.g. on daily basis) while the asset pricing models specify the prices and states to evolve continuously;
(2) the state variables are latent based on researcher's perspective but not observable on the market;
(3) $p(\Theta, X|Y)$ is usually in high dimension, causing common sampling method to fail;
(4) the transition distributions for prices and states of the asset pricing model are non-normal and non-standard, complication the standard estimation methods such as MLE and GMM;
(5) the parameters of the asset pricing models are usually nonlinear and non-analytic form as the implicit solution to a stochastic differential equations.

A typical application of MCMC technique in asset pricing model is Jacquier, Polson and Rossi [3], where a cyclic Metropolis algorithm is used to construct a Markov-chain simulation on stochastic volatility model.

## 2. MODEL SPECIFICATION

2.1. **Geometric Brownian Motion (Black-Scholes).** The baseline model selected for fitting the underlying stock returns is Black-Scholes model [1], where the stock price dynamic, $S_t$, follows

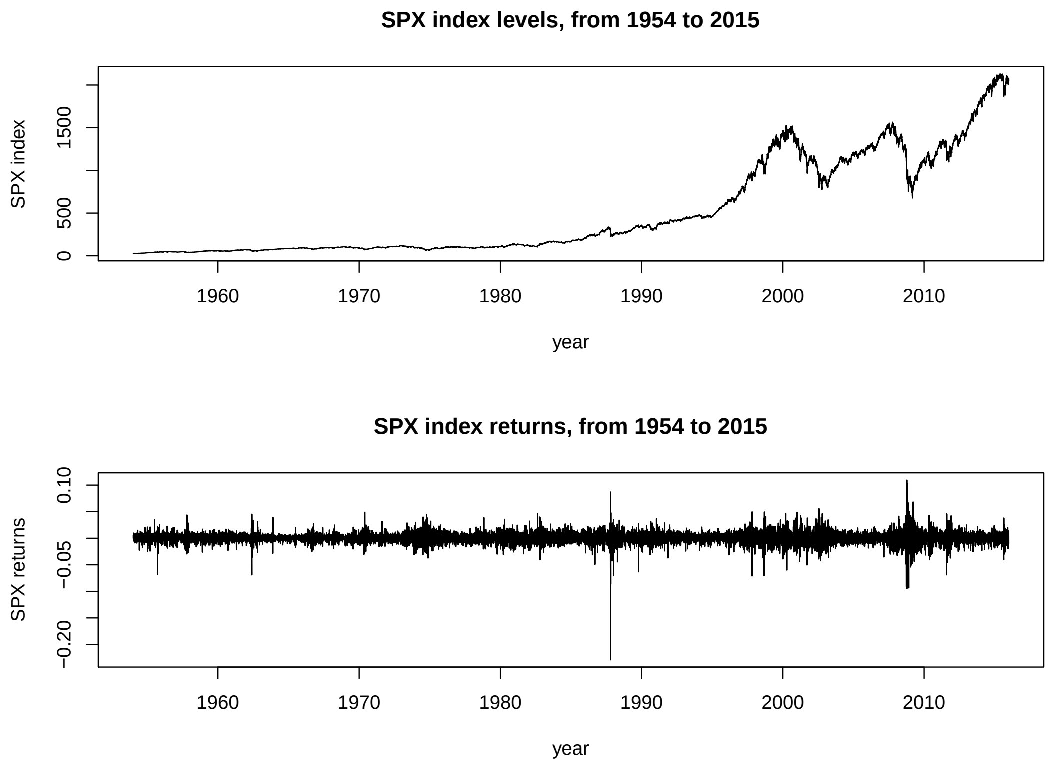


**Figure 1** The SPX index levels and return on daily basis. Time period is from 1954 to 2015.

Geometric Brownian Motion

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t$$

where $\mu$ is the drift term and $\sigma$ is the volatility. $W_t$ is the Wiener process. This model assumes the stock returns follow a random walk. In reality, the S&P500 index level and returns on daily basis are plotted in Figure 1.

In discrete time equally space, the model has close-form solution for the return

$$Y_t = \log(S_t/S_{t-1}) = \mu + \sigma\epsilon_t$$

where $\epsilon_t \sim N(0,1)$. We have $\Theta = (\mu, \sigma^2)$. There is no latent variable, which implies the posterior to be $p(\Theta|Y) = p(\mu, \sigma|Y)$.

Using Hammersley-Clifford theorem [2], $p(\mu|\sigma^2, Y)$ and $p(\sigma^2|\mu, Y)$ are complete conditionals to the posterior. Assuming independent priors on $\mu$ and $\sigma^2$, Bayes rule implies that

$$\begin{aligned} p(\mu|\sigma^2, Y) &\propto p(Y|\mu, \sigma^2)(\mu) \\ p(\sigma^2|\mu, Y) &\propto p(Y|\mu, \sigma^2)(\sigma^2) \\ p(Y|\mu, \sigma^2) &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^T \exp\left(-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{Y_t - \mu}{\sigma}\right)^2\right) \end{aligned}$$

where $T$ is the sample size. $p(\mu)$ and $p(\sigma^2)$ are priors. Here we choose the standard conjugate priors on $\mu$ and $\sigma^2$. First select the inverse gamma distribution as the prior for $\sigma^2$. The inverse gamma distribution relies on two parameters $\alpha$ and $\beta$. The density is

$$f(\sigma^2|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\alpha-1}\exp(-\beta/\sigma^2)$$

Therefore, the marginal density $p(\sigma^2)$ combines the prior $p(\sigma^2)$ and density $p(Y|\mu,\sigma^2)$, which yields

$$\begin{aligned}
p(\sigma^2|\mu, Y) &\propto p(Y|\mu,\sigma^2)\times p(\sigma^2)\\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^T \exp\left(-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{Y_t-\mu}{\sigma}\right)^2\right)\times\frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\alpha-1}\exp(-\beta/\sigma^2)\\
&\propto (\sigma^2)^{-T/2-\alpha-1}\exp\left(-\left[\frac{1}{2}\sum_{t=1}^{T}(Y_t-\mu)^2+\beta\right]/\sigma^2\right)\\
&\propto IG\left(\alpha+\frac{T}{2},\beta+\frac{1}{2}\sum_{t=1}^{T}(Y_t-\mu)^2\right)
\end{aligned}$$

That is, given $\mu$ and $Y_t$, we are able to generate the $\sigma^2$ according to the marginal density $p(\sigma^2|\mu, Y)$. Similarly, select normal distribution as prior for $\mu$. The density is

$$f(\mu|\theta,\delta) = \frac{1}{\sqrt{2\pi\delta^2}}\exp\left(-frac12\left(\frac{\mu-\theta}{\delta}\right)^2\right)$$

and the marginal density is

$$\begin{aligned}
p(\mu|\sigma^2, Y) &\propto p(Y|\mu,\sigma^2)\times p(\mu)\\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^T \exp\left(-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{Y_t-\mu}{\sigma}\right)^2\right)\times\frac{1}{\sqrt{2\pi\delta^2}}\exp\left(-frac12\left(\frac{\mu-\theta}{\delta}\right)^2\right)
\end{aligned}$$

To deal with $Y_t-\mu$, denote $\hat{\mu} = \left(\sum_{t=1}^{T} Y_t\right)/T$, and

$$\begin{aligned}
\sum_{t=1}^{T}(Y_t-\mu)^2 &= \sum_{t=1}^{T}(Y_t-\hat{\mu}+\hat{\mu}-\mu)^2\\
&= \sum_{t=1}^{T}(Y_t-\hat{\mu})^2+2(\hat{\mu}-\mu)\sum_{t=1}^{T}(Y_t-\hat{\mu})+\sum_{t=1}^{T}(\hat{\mu}-\mu)^2\\
&= \sum_{t=1}^{T}(Y_t-\hat{\mu})^2+T(\hat{\mu}-\mu)^2
\end{aligned}$$

Continuing on the marginal density, we have

$$\begin{aligned}
p(\mu|\sigma^2, Y) &\propto \exp\left(-\frac{T}{2\sigma^2}(\mu-\mu)^2-\frac{1}{2\delta^2}(\mu-\theta)^2\right)\\
&\propto \exp\left(-\frac{T}{2\sigma^2}(-2\hat{\mu}\mu+\mu^2)-\frac{1}{2\delta^2}(\mu^2-2\mu\theta)\right)\\
&\propto \exp\left(-\frac{1}{2\delta^{*2}}\left(\mu-\left(\frac{T\hat{\mu}}{\sigma^2}+\frac{\theta}{\delta^2}\right)\delta^{*2}\right)^2\right)\\
&\propto N\left(\left(\sum_{t=1}^{T}Y_t/\sigma^2+\theta/\delta^2\right)\delta^{*2},\delta^{*2}\right)
\end{aligned}$$

where $\delta^{*2} = (T/\sigma^2+1/\delta^2)^{-1}$.

Given the prior distributions, the complete MCMC method to conduct parameter estimation and statistical inference is

(1) initialize the parameters $\mu^{(0)}$ and $(\sigma^2)^0$;
(2) specify the parameters of the prior $\alpha, \beta, \theta, \delta$;
(3) draw $\mu^{g+1} \sim p(\mu|(\sigma^2)^g, Y)$;
(4) draw $(\sigma^2)^{g+1} \sim p(\sigma^2|()^g, Y)$;
(5) estimate parameters in $\{\mu^g, (\sigma^2)^g\}_{g=1}^G$

### 2.2. **Merton Jump Diffusion Model.**

## 3. DATA AND EMPIRICAL RESULTS

## 4. CONCLUSION

## References

[1] Black F., Scholes M. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[2] John Hammersley and Peter Clifford. Markov fields on finite graphs and lattices. *Unpublished Manuscipt*, 1970.

[3] Jacquier, Eric, Nicholas G. Polson, and Peter Rossi. Bayesian analysis of stochastic volatility models. *Journal of Business and Economic Statistics*, 12:69–87, 1994.

[4] Johannes, Michael, and Nicholas G. Polson. Mcmc methods in financial econometrics, in yacine aït-sahalia, and lars hansen. *Handbook of Financial Econometrics (Elsevier: Oxford)*, 2002.