Intermediate Financial Economics
Lecture IV: Asset Pricing I

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Today's Lecture:

- I. General consumption-based asset pricing
  - A. A model with a representative agent

A. A Model with A Representative Agent

- Environment:
  - Exchange economy: no productions; consumers receive exogenous endowment of wealth and exchange with each other in the asset market.
  - Goods market: one perishable consumption goods.
  - Numeraire: money doesn't exist in this world. All wealth is dominated in terms of the consumption goods.
  - Asset market: there are n assets indexed by 1, …, n, each with random future payoff x_i (in units of consumption). The net supply of all assets are zero.

Environment Cont.

- People: there is one representative consumer.
  - You can take this assumption literally: there is just one guy, Robinson Crusoe.
  - Or you can think of a world populated with many identical individuals.
- The utility function of the rep. consumer is u(·), which is a well behaved function:
  - It is smooth.
  - u' >0. The more the merrier.
  - u"<0. Risk averse. Implies that marginal utility u' is decreasing; The more one eats, the less additional pleasure he would have from each additional bite.
  - Time separable and state independent.
- Time: start with two periods and generalize to multiple periods.

The Representative Consumer

- She maximizes her expected utility:

(1)
$$\max_{\{h_i\}} u(c_0) + E[\beta u(c_1)] \ s.t.$$

$$c_0 + p_1 h_1 + \cdots + p_n h_n = e_0$$

$$c_1 = e_1 + h_1 x_1 + \cdots + h_n x_n$$

where c_0 and c_1 are consumption in today and tomorrow, e_0 and e_1 are her endowment of consumption today and tomorrow, h_i is the asset holding for asset i, β denotes her time impatience, and x_i is the random payoff of the asset tomorrow.

- Time impatience measures how patient she is about deferring consumption to the future. The less the β, one unit of utility means less to her now, the less patient she is.
- She decides how much to invest and consume.

First Order Condition

- Plugging the two budget equations into the objective function, and taking derivative with respect to h_i , we have

(2) $$p_i u'(c_0) = E[\beta u'(c_1) x_i]$$

- This is the fundamental equation for asset pricing, a.k.a. the Euler equation.
  - $p_i u'(c_0)$ represents the loss in marginal utility when p_i dollars are used for investment in asset i instead of consumption today.
  - $E[bu'(c_1)x_i]$ is the expected gain in marginal utility when asset i pays x_i dollars in the future.
  - These two have to equalize in equilibrium for each asset i.
  - What would happen if they don't?

A1. STOCHASTIC DISCOUNT FACTOR (SDF)

- Equation (2) can be rearranged as

(3) $$p_i = E[\beta \frac{u'(c_1)}{u'(c_0)} x_i]$$

- Denote $m = \beta \frac{u'(c_1)}{u'(c_0)}$
  - The marginal rates of substitution (MRS).
  - In general, m is called the stochastic discount factor, or the pricing kernel. The MRS is just a special case.
- We have

(4) $$p_i = E[m x_i]$$

  - This is the most general formula for asset pricing.
  - Different asset pricing models simply play around with m.

Riskless Bond

- Suppose the asset is a riskless bond that pays one unit of consumption at date 1 risk free.
  - So x=1, no longer random.
- We can derive an expression for the gross rate of return of the bond $R_f = 1/p_f$.
- When x=1, then the price of the riskless bond is

(5)
$$P_f = E[\beta \frac{u'(c_1)}{u'(c_0)}] = E[m]$$

Therefore the return is payoff over price:

(6)
$$R_f = \frac{1}{E[\beta \frac{u'(c_1)}{u'(c_0)}]} = \frac{1}{E[m]}$$

- It answers the question where the interest rate comes from in Lecture 1.

EXAMPLE 1

- $u(c)=\ln c$, $\beta=1$. Suppose there are two states tomorrow with equal probabilities. The stock's payoff is either 4 or -1, denoted by $x=(4,-1)$. The agent's consumption today is 1, and her consumption tomorrow is either 3 or 1, denoted by (3,1).
- The pricing kernel is $m=c_0/c_1=(1/3,1)$. The price of the riskless bond is $P_f = 2/3$, and the gross return is $R_f = 3/2$.
- The price of the stock is $p=E[mx]=(4*1/3 - 1)/2 = 1/6$. The expected return of the stock is $R=E[x]/p=9$.
- Note: we didn't specify where the consumption comes from. Will discuss this soon.

A2. Relation to PV

- Suppose there is no uncertainty, meaning that all x's are constants.
  - All assets are effectively bonds that pay deterministic coupons.
- Then equation (3) reduces to $p=x/R_f$, which coincides with the PV formula discussed in the first lecture.
  - $1/R_f$ is the discount factor.
- If there is uncertainty, investors will typically demand higher returns for holding the risky assets with the same expected payoff.
- Then we have $p_i = E[x_i]/R_i$, where $1/R_i$ is the asset-specific risk-adjusted discount factor for asset i.
  - This is the PV formula for pricing stocks or risky projects.
- The pricing equation $p_i = E[mx_i]$ in (4) is more general.
  - The discount factor is no longer asset-specific and deterministic like $1/R_i$, but a universal and stochastic one that applies to all assets.

Risk Neutrality

- When all investors are risk neutral, asset prices are the PV of future expected payoffs with riskfree discount rates, even when there is uncertainty.
- Recall that risk neutrality means that u(·) is linear.
- So the MRS is 1 and $m=\beta$. So $p = E[mx]= \beta E[x]$.
- The riskfree return is $R^f=1/E[m]=1/\beta$.
- Then $p=\beta E[x]=E[x]/R^f$.
- Therefore the PV formula holds when all investors are risk neutral.
  - This makes sense, because a world with uncertainty is equivalent to a world with certainty to risk-neutral investors.
- We'll show later that risk neutrality is still relevant even in a model with risk averse investors.

Multiple Periods

- Equation (3) and (4) also holds in a multi-period model under some conditions.

(3') $$p_t^i = E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}^i]$$

(4') $$p_t^i = E_t[m_{t+1} x_{t+1}^i]$$

PV model of Stocks

- For a stock, $x_{t+1} = p_{t+1} + d_{t+1}$. (Let's drop i.) So (4') becomes

(7) $$p_t = E_t[m_{t+1}(p_{t+1} + d_{t+1})]$$

- The same equation holds for time t+1

(8) $$p_{t+1} = E_{t+1}[m_{t+2}(p_{t+2} + d_{t+2})]$$

- (8) into (7), we have

$$p_t = E_t[m_{t+1}d_{t+1}] + E_t[m_{t+1,t+2}d_{t+2}] \\ +E_t[m_{t+1,t+2}p_{t+2}]$$

- where $m_{t+1,t+2} = m_{t+1}\ m_{t+2}$ and where we have used a result from probability theory that

$$E_t\ [E_{t+1}\ [x]]=E_t\ [x]$$

- Repeating this for T times, we will have

$$p_t = E_t[\Sigma_{j=1}^{T} m_{t+1,t+j} d_{t+j}] + E_t[m_{t+1,t+T} p_{t+T}]$$

- Taking the limit as $T \to \infty$, and assuming no bubbles exist (the second term on the RHS vanishes), we have

(9) $$p_t = E_t[\Sigma_{j=1}^{\infty} m_{t+1,t+j} d_{t+j}]$$

- In summary, (3) and (4) are generalizations of the PV formulas.
  - There exists a single series of stochastic discount factors $\{m_{t+1}\}$, the same for all assets.
  - m_{t+1} is random.

A3. Risk Corrections

- Let's still work with Equation (3) and (4).
- By the definition of covariances,

$$cov(m, x_i) = E[mx_i] - E[m]E[x_i]$$

- We have

$$p_i = E[m]E[x_i] + cov(m, x_i)$$

or

(10) $$p_i = \frac{E[x_i]}{R_f} + cov(m, x_i)$$

- Asset price = PV of future expected payoff discounted by the riskless rate + risk correction
- The risk correction is determined by the covariance between the stochastic discount factor and the payoff.

Understanding Covariance

- Covariance describes the statistical association between two random variables.
- Positive covariance: when the value of one variable rises over its mean, the value of the other variable (usually) rises over its mean as well.
- Negative covariance: when the value of one variable rises over its mean, the value of the other variable (usually) falls below its mean.
- Definition: $\mathrm{Cov}(X, Y) = E[(X - EX)*(Y - EY)]$
- Variance is a special case of covariance.
- The covariance of an RV WITH ITSELF is just the variance:

  $\mathrm{Cov}(X, X)=E[(X-EX)(X-EX)]=E[(X - EX)^2]=\mathrm{Var}(X)$

  $=p_1(X_1 - EX)^2 + \cdots + p_s(X_s - EX)^2$

Correlation Coefficient ($\rho_{A,B}$)

- Correlation coefficient between two RVs = covariance divided by the product of the SDs of the 2 RVs.

$$Corr(r_a, r_b) = \rho_{a,b} = \frac{Cov(r_a, r_b)}{\sigma(r_a)\sigma(r_b)}$$

- Also called in short the *correlation*.
- Varies from -1 when there's a perfect linear relation with negative coefficient
- To zero when there's no association
- To +1 when there's a perfect linear relation with positive coefficient

Example 2

- Consider a gamble where you will \$2 if heads and lose \$1 if tails. Let the random payoff of this gamble be X. And let Y be the random payoff of another gamble, in which you will win \$5 if heads or lose \$5 if tails.

	X	Y
heads	\$2.0	\$5.0
tails	-\$1.0	-\$5.0

- What is the correlation coefficient between the payoffs of the two gambles? Does it depend on p, the probability of heads (winning)?
- Intuitively, whenever X pays, Y pays for sure; whenever X loses, Y loses for sure. This suggests a perfection positive correlation.

- Standard deviations of X and Y, and covariances and correlations for different values of p:

	X	Y				
heads	\$2.0	\$5.0				
tails	-\$1.0	-\$5.0				
p	E(X)	E(Y)	SD(X)	SD(Y)	Cov(X,Y)	Corr
0.5	\$0.5	\$0.0	\$1.50	\$5.00	7.50	1.00
0.6	\$0.8	\$1.0	\$1.47	\$4.90	7.20	1.00
0.7	\$1.1	\$2.0	\$1.37	\$4.58	6.30	1.00
0.8	\$1.4	\$3.0	\$1.20	\$4.00	4.80	1.00
0.9	\$1.7	\$4.0	\$0.90	\$3.00	2.70	1.00
1.0	\$2.0	\$5.0	\$0.00	\$0.00	0.00	undefined

- What if Y=$5 with tails?

	X	Y
heads	$2.0	-$5.0
tails	-$1.0	$5.0

- $\rho = -1$
- A random variable with two possible states, like X or Y, is called a binomial random variable.
- In fact, we can prove that the correlation between two binomial random variables is always 1 or -1, as long as $0<p<1$.

Example 3: Covariance (Four states with equal probabilities)

T's payoff	T - E(T)	S' payoff	S - E(S)	Product
-20	-37.5	5	-0.5	18.75
10	-7.5	20	14.5	-108.75
30	12.5	-12	-17.5	-218.75
50	32.5	9	3.5	113.75
				-195
E(T)=	17.5	E(S)=	5.5	
				-48.75

Cov(S, T)= - 48.75

Correlation= - 0.16

Interpreting Risk Corrections

- For the case of MRS, we have

$$p_i = \frac{E[x_i]}{R_f} + \frac{\beta}{u'(c_t)} cov(u'(c_{t+1}), x_i)$$

- The risk correction is positive if the asset's payoff is positively correlated with future marginal utility.
  - Suppose the asset's payoff is high (low). The positive covariance implies a high (low) marginal utility.
  - Since marginal utility is decreasing in consumption, it takes a high (low) value when one is hungry (full).
  - This is an asset that pays off well (badly) in the states that you need it the most (least). Such an asset is *valuable*.
- The risk correction is negative if the asset's payoff is negatively correlated with future marginal utility.
  - Suppose the asset's payoff is low (high). The negative covariance implies a high (low) marginal utility. The marginal utility is high (low) when one is hungry (full).
  - This is an asset that pays off badly (well) in the states that you need it the most (least). Such an asset is *risky*.

Covariance v.s. Variance

- (10) says that an asset's price is determined by the *covariance* of its payoffs with the SDF.
  - It implies that an asset with a large volatility is not necessarily risky.
  - As long as cov(m, x) = 0, the price will be $E[x]/R^f$, regardless of how large var(x) is.
- <u>Example 4</u>: Consider Asset A, B and the SDF m:

p	A	m	B
0. 33	4	0. 5	5
0. 33	3	0. 75	3
0. 33	2	0. 5	1

- Cov(A, m)=Cov(B,m)=0. Yet σ(A)=0.82 and σ(B)=1.63.
- P(A)=P(B)=1.75.

Idiosyncratic Risk v.s. Systematic Risk

- The part that is correlated with the SDF is the systematic risk, and the part that is uncorrelated with the SDF is the idiosyncratic risk.
- Mathematically, any payoff x can be decomposed into two parts: $x = b*m + \varepsilon$, where ε and m are uncorrelated.
  - This is just a linear regression of x on m without an intercept.
  - The best estimator of b that minimizes the mean squared error is $$b = \frac{E[mx]}{E[m^2]}$$
  - Systematic risk is bm, and the idiosyncratic risk is ε.
- The price of the systematic risk is $p(bm)=E[bm^2]=E[mx]=p(x)$, so the idiosyncratic risk is not priced.

Risk Corrections in Terms of Returns

- Dividing both sides of (4) by p_i , we have $1=E[mR_i]$, which can be written as

(11) $$1 = E[m]E[R_i] + cov(m, R_i)$$

- Dividing both sides by E[m] and using (6), we have

$$E[R_i] - R_f = -R_f cov(m, R_i)$$

or

$$E[R_i] - R_f = -\frac{\beta R_f}{u'(c_t)} cov(u'(c_{t+1}), R_i)$$

- The asset that is positively correlated with the marginal utility is a valuable asset. Its risk premium is negative.
- The asset that is negatively correlated with the marginal utility is a risky asset. Its risk premium is positive.

A4. The Case of Log-Normal Consumption

- Suppose consumption is log-normally distributed
  - meaning that the log of consumption is normally distributed.
- Suppose the utility is CRRA: $u(c)=c^{1-a}/(1-a)$
- From (6), we have a multi-period version

(12) $$R_t^f = E_t^{-1}\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-a}\right]$$

- Since the ratio $c_{t+1}/c_t = \exp(\ln c_{t+1} - \ln c_t) = \exp(\Delta \ln c_{t+1})$, where $\Delta \ln c_{t+1}$, the exponential growth rate of consumption, is normally distributed, then

$$R_t^f = E_t^{-1}\left[\beta \exp(-a\Delta \ln\ c)\right]$$

Determinants of The Riskfree Interest Rate

- Fact: If $\Delta \ln c_{t+1} \sim N(\mu, \sigma^2)$, then

  $$E[\exp(a\, \Delta \ln c_{t+1})] = \exp(a\mu + a^2 \sigma^2/2).$$

- Applying this formula, we have

  $$R_t^f = \beta^{-1} \exp(a\mu - a^2\sigma^2/2)$$

- Taking the log to both sides, we obtain the geometric (or continuously compounded) riskless rate of return

(13) $$r_t^f \equiv \ln R_t^f = -\ln \beta + a\mu - a^2\sigma^2/2$$

- Determinants:
  - Time impatience: more patient, lower interest rate.
  - Average consumption growth: higher growth, higher interest rate.
  - Risk: higher risk, lower interest rate.
  - Sensitivity to consumption growth: more risk averse, more sensitive.

A5. General Equilibrium

- The SDF depends on equilibrium consumption of the representative agent.
- Yet we haven't said anything about what consumption should be.
- Now we impose the market clearing conditions:
  - Asset markets: total net asset holdings = total net supply of assets. $\Sigma_k h_k^i = 0$ for all asset i.
  - For the representative agent model, h^i =0 for all i.
  - So there is no trade whatsoever in equilibrium!
  - Goods markets: total consumption = total endowment.
  - For the rep. agent model, $c_t = e_t$. That is, Robinson Crusoe finds no one to trade, and eats everything he can find on the island.

Understanding The Models

- How could assets be priced even when there is no trade?
  - The Euler equation must hold such that the agent does not have any incentive to trade.
  - If it didn't hold, the agent would want to trade. But the market could not clear. So this would not be an equilibrium.
- Caution is needed to interpret the prices.
- The consumption-based asset pricing models focus on the demand side.
  - The supply side of the goods markets is completely abbreviated to the exogenous endowment process.
  - The supply side of the asset markets is taken to be exogenous.
- All models are simplifications and approximations of the real world.
  - Good models identify a few crucial elements, demonstrate their relations, and conform to empirical evidence.

EXAMPLE 5

- Consider a two-period model with a representative agent, who has log utility. Suppose ß=1. Suppose today's endowment is 1. Future's endowment is 1 if the economy is bad, and is 3 if the economy is good. The bad state and the good state occur with equal probabilities.
- In the asset market, Stock U pays 0 in the bad state, and 2 in the good state, whereas Stock V pays 2 in the bad state and 0 in the good state. There is also a riskless bond that pays 1 regardless of which states.
- What are the prices and returns of the bond and the stock?
- $m=c_0/c_1 = 1$ in the bad state, and $=1/3$ in the good state.
- $P^f=2/3$. $R^f=1.5$.
- $P^u=1/3$. $R^u=3$.
- $P^v=1$. $R^v=1$.

Example 5 Cont.

- The expected payoffs of the stocks are the same with the bond. Their prices are lower since they are risky.
- Stock U is riskier than V because it pays off well in the state when the person needs it the least.
- Risk corrections in terms of prices:
  - risk corrections for Stock U = $p^u - E[x]/R^f = 1/3 - 2/3 = -1/3$.
  - risk corrections for Stock V = $p^v - E[x]/R^f = 1 - 2/3 = 1/3$.
- Risk premia (corrections in terms of returns) are
  - $R^u - R^f = 1.5$
  - $R^v - R^f = -0.5$