Joint modeling of SPX and VIX

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Overview of this talk

- What are SPX and VIX?
- The volatility surface
- Stochastic volatility
- Spanning payoffs
- Arbitrage relationships between SPX and VIX
- Joint modeling of SPX and VIX

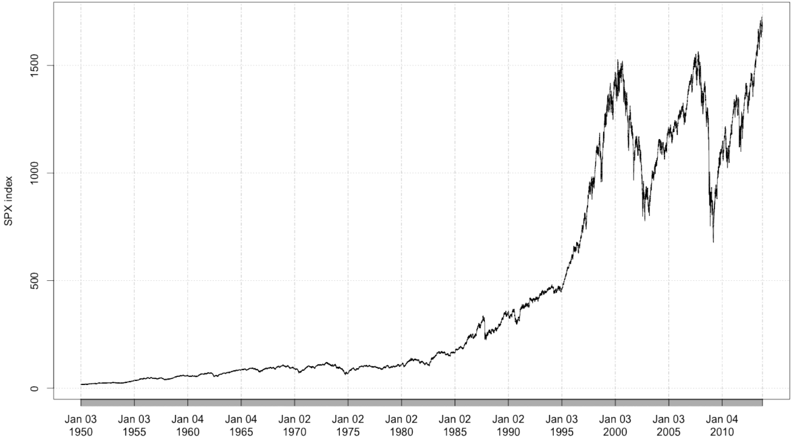
The SPX index

SPX is one of the tickers for the S&P 500 index.

From "S&P 500" Wikipedia: The Free Encyclopedia.

> *The S&P 500, or the Standard & Poor's 500, is a stock market index based on the market capitalizations of 500 large companies having common stock listed on the NYSE or NASDAQ. The S&P 500 index components and their weightings are determined by S&P Dow Jones Indices. It differs from other U.S. stock market indices such as the Dow Jones Industrial Average and the Nasdaq Composite due to its diverse constituency and weighting methodology. It is one of the most commonly followed equity indices and many consider it the best representation of the U.S. stock market as well as a bellwether for the U.S. economy.*

Time series of SPX since 1950

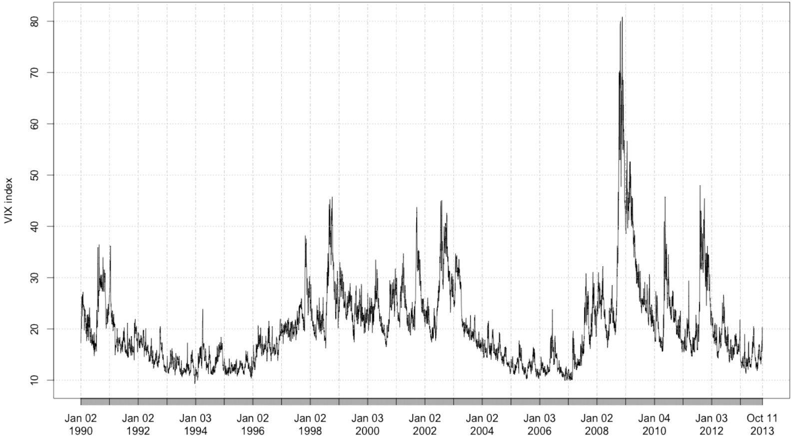


The VIX index

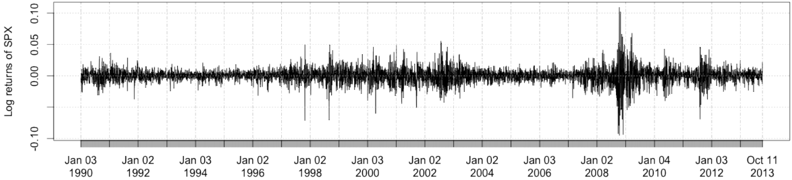
From "VIX" Wikipedia: The Free Encyclopedia.

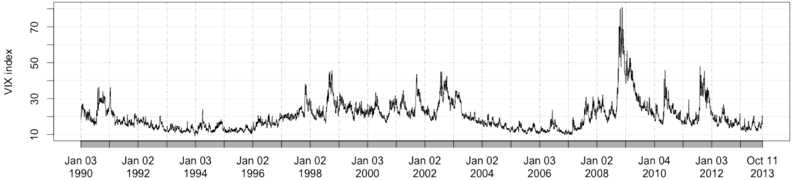
> *VIX is a trademarked ticker symbol for the Chicago Board Options Exchange Market Volatility Index, a popular measure of the implied volatility of S&P 500 index options. Often referred to as the fear index or the fear gauge, it represents one measure of the market's expectation of stock market volatility over the next 30 day period.*

Time series of VIX since 1990



VIX is a measure of volatility





Options

From "Option (finance)" Wikipedia: The Free Encyclopedia.

> *In finance, an option is a contract which gives the buyer (the owner) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date. The seller incurs a corresponding obligation to fulfill the transaction that is to sell or buy if the owner elects to "exercise" the option prior to expiration. The buyer pays a premium to the seller for this right. An option which conveys to the owner the right to buy something at a specific price is referred to as a call; an option which conveys the right of the owner to sell something at a specific price is referred to as a put. Both are commonly traded, but for clarity, the call option is more frequently discussed.*

Options on SPX and VIX

- In particular, there are options on SPX and options on VIX.
- We saw that the VIX index reflects the volatility of SPX.
- The values of options on SPX and options on VIX should be related.
- In the following, we will see some of the ways in which these option values are related.
    - In fact, we will present a model that can fit SPX and VIX options prices simultaneously.

Options on SPX from Bloomberg

Options on VIX from Bloomberg

	Calls								Strike		Puts							
	Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt			Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt
	16 Oct 13 (2d); CSize 100; R .16; UXV3 16.80								5		16 Oct 13 (2d); CSize 100; R .16; UXV3 16.80							
1)	VIX 10 C16	.55y	.60y	.56y	167.88	.66	30418	12581	16.0	31)	VIX 10 P16	.30	.35	.85y	135.15	-.28	16342	1529
2)	VIX 10 C17	.75	.85	.35y	176.70	.48	22867	13062	17.0	32)	VIX 10 P17	1.60y	1.75y	1.60y	142.79	-.53	13530	1023
3)	VIX 10 C18	.50	.60	.25y	186.17	.36	32530	21787	18.0	33)	VIX 10 P18	2.45y	2.65y	2.35y	154.83	-.71	5917	8918
4)	VIX 10 C19	.15y	.20y	.15y	203.24	.25	10681	14729	19.0	34)	VIX 10 P19	2.40	2.45	3.48y	173.46	-.80	3978	2801
5)	VIX 10 C20	.25	.30	.15y	228.09	.18	40054	15429	20.0	35)	VIX 10 P20	4.30y	4.60y	4.10y	202.83	-.84	1334	2183
	20 Nov 13 (37d); CSize 100; R .19; UXX3 17.05								5		20 Nov 13 (37d); CSize 100; R .19; UXX3 17.05							
6)	VIX 11 C16	2.30	2.40	1.85y	79.87	.66	7275	78165	16.0	36)	VIX 11 P16	1.10	1.20	1.45y	76.05	-.36	18601	1598
7)	VIX 11 C17	1.50y	1.55y	1.50y	90.42	.56	6507	11375	17.0	37)	VIX 11 P17	1.75	1.80	2.07y	85.33	-.43	4197	1242
8)	VIX 11 C18	1.20y	1.30y	1.25y	96.36	.49	4383	13699	18.0	38)	VIX 11 P18	2.45	2.55	2.81y	87.79	-.53	153	7832
9)	VIX 11 C19	1.40	1.45	1.10y	99.94	.43	1383	13898	19.0	39)	VIX 11 P19	3.60y	3.80y	3.70y	97.51	-.57	857	4080
10)	VIX 11 C20	.95y	1.00y	1.25	104.47	.38	23908	14719	20.0	40)	VIX 11 P20	4.00	4.10	4.50y	101.47	-.62	473	1568
	18 Dec 13 (65d); CSize 100; R .22; UXZ3 17.38								5		18 Dec 13 (65d); CSize 100; R .22; UXZ3 17.38							
11)	VIX 12 C16	2.75	2.90	2.50y	70.16	.67	231	26975	16.0	41)	VIX 12 P16	1.20	1.30	1.50y	68.43	-.32	1452	5763
12)	VIX 12 C17	2.40	2.50	2.05y	77.58	.59	2772	44021	17.0	42)	VIX 12 P17	1.80	1.90	2.10y	74.26	-.39	2548	7543
13)	VIX 12 C18	2.05	2.15	1.79y	80.27	.53	3379	33474	18.0	43)	VIX 12 P18	2.75y	2.90y	2.79y	75.82	-.48	3452	3341
14)	VIX 12 C19	1.45y	1.65y	1.60y	81.90	.48	560	17875	19.0	44)	VIX 12 P19	3.50y	3.70y	3.50y	77.50	-.54	45	1217
15)	VIX 12 C20	1.55	1.65	1.45y	86.68	.42	2407	57501	20.0	45)	VIX 12 P20	4.00	4.10	4.30y	81.81	-.59	1036	1068
	22 Jan 14 (100d); CSize 100; R .26; UXF4 18.05								5		22 Jan 14 (100d); CSize 100; R .26; UXF4 18.05							
16)	VIX 1/14 C16	3.00y	3.20y	3.06y	59.74	.71	21	340	16.0	46)	VIX 1/14 P16	1.15	1.25	1.30y	60.13	-.28	2220	3492

Option valuation

- In mathematical finance, the value of an option is given by the expectation (under the risk neutral measure) of the final payoff conditional on the information available at the current time t.
- Specifically, for a European call option expiring at time T,

$$C(S,K,T) = \mathbb{E}\left[(S_T - K)^+ \middle| \mathcal{F}_t \right].$$

The Black-Scholes model

- Black and Scholes model the evolution of the underlying as

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ_t$$

with the volatility σ constant.

- The price of a European option is then given by the Black-Scholes formula:

$$C(S, K, T) = \mathbb{E}\left[(S_T - K)^+ \middle| \mathcal{F}_t \right] = PV \, \{ F \, \mathcal{N}(d_1) - K \, \mathcal{N}(d_2) \}$$

where F is the forward price, $\mathcal{N}(\cdot)$ is the cumulative normal distribution function and with $\tau = T - t$,

$$d_1 = \frac{\log F/K}{\sigma \sqrt{\tau}} + \frac{\sigma \sqrt{\tau}}{2}; \quad d_2 = \frac{\log F/K}{\sigma \sqrt{\tau}} - \frac{\sigma \sqrt{\tau}}{2}.$$

Implied volatility

From "Implied volatility" Wikipedia: The Free Encyclopedia.

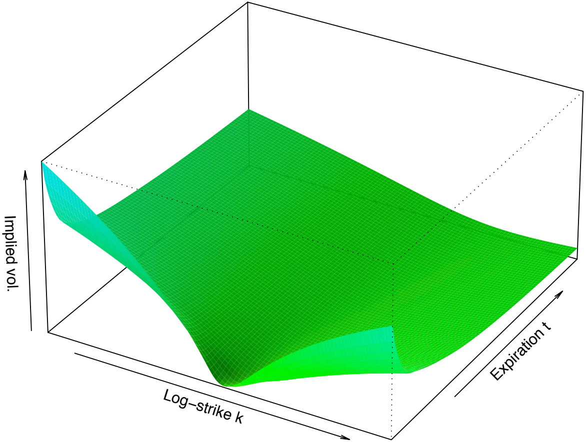
> *In financial mathematics, the implied volatility of an option contract is that value of the volatility of the underlying instrument which, when input in an option pricing model (such as Black-Scholes) will return a theoretical value equal to the current market price of the option. A non-option financial instrument that has embedded optionality, such as an interest rate cap, can also have an implied volatility. Implied volatility, a forward-looking and subjective measure, differs from historical volatility because the latter is calculated from known past returns of a security.*

The volatility surface

- We already saw that empirically, the volatility of SPX is not constant.
- If the Black-Scholes model were correct, options of all strikes and expirations would have the same implied volatility.
  - Empirically, options with different strikes and expirations have different implied volatilities.
- The surface formed by mapping implied volatility as a function of strike and expiration is know as *the volatility surface*.
- The volatility surface encodes the prices of options in a convenient way.
  - In particular, the shape of the volatility surface tends to be quite stable.

Figure 3.2 from TVS: 3D plot of volatility surface

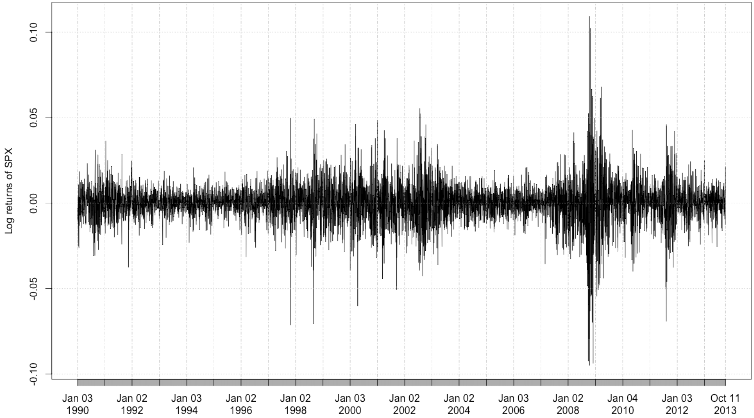
Here's a 3D plot of the volatility surface as of September 15, 2005:



$k := \log K/F$ is the log-strike and t is time to expiry.

SPX log-returns again

Figure 1: Note the intermittency and volatility clustering!



Stochastic volatility

- So volatility is stochastic and mean-reverting.
  - Volatility moves around
  - Big moves follow big moves, small moves follow small moves
- This motivates a large class of models known as *stochastic volatility* models.

The stochastic volatility (SV) process

We suppose that the stock price S and its variance $v = \sigma^2$ satisfy the following SDEs:

$$dS_t = \mu_t \, S_t \, dt + \sqrt{v_t} \, S_t \, dZ_1 \tag{1}$$

$$dv_t = \alpha(S_t, v_t, t) \, dt + \eta \, \beta(S_t, v_t, t) \, \sqrt{v_t} dZ_2 \tag{2}$$

with

$$\mathbb{E}\left[dZ_1 \, dZ_2\right] = \rho \, dt$$

where μ_t is the (deterministic) instantaneous drift of stock price returns, η is the volatility of volatility and ρ is the correlation between random stock price returns and changes in v_t. dZ_1 and dZ_2 are Wiener processes.

The stock price process

- The stochastic process (1) followed by the stock price is equivalent to the Black-Scholes (BS) process.
  - This ensures that the standard time-dependent volatility version of the Black-Scholes formula may be retrieved in the limit $\eta \to 0$.
- In practical applications, this is desirable for a stochastic volatility option pricing model as practitioners' intuition for the behavior of option prices is invariably expressed within the framework of the Black-Scholes formula.

The variance process

- The stochastic process (2) followed by the variance is very general.
- We don't assume anything about the functional forms of $\alpha(\cdot)$ and $\beta(\cdot)$.
- In particular, we don't assume a square-root process for variance.

The Heston model

In the Heston model,

$$\alpha = -\lambda\,(v - \bar{v});\ \beta = 1$$

So that (again with $r = 0$)

$$\begin{aligned} dS_t &= \sqrt{v_t}\,S_t\,dZ_1 \\ dv_t &= -\lambda\,(v - \bar{v})\,dt + \eta\,\sqrt{v_t}dZ_2 \end{aligned}$$

with

$$\mathbb{E}\left[dZ_1\,dZ_2\right] = \rho\,dt$$

The corresponding valuation equation with European boundary conditions may be solved using Fourier techniques leading to a quasi-closed form solution – the famous Heston formula.

The SABR model

The SABR model is usually written in the form

$$\begin{aligned} dS_t &= \sigma\, S_t^\beta\, dZ_1 \\ d\sigma_t &= \alpha\,\sigma\, dZ_2 \end{aligned}$$

with $\mathbb{E}\left[dZ_1\, dZ_2\right] = \rho\, dt$.

Hence the name "stochastic alpha beta rho model".

- Note that this formulation is in general inconsistent with our original formulation (1) because the stock price is conditionally lognormal only if $\beta = 1$. We get the CEV model in the limit $\alpha \to 0$.
- There is an accurate asymptotic formula for BS implied volatility (the SABR formula) in terms of the parameters of the model permitting easy calibration to the volatility smile.

Model-independent arbitrage relationships

- We will see that VIX represents the volatility of SPX in a precise way.
  - There should therefore be relationships between the prices of options on SPX and options on VIX.
- If we assume diffusion (that is, no jumps), we can derive many model-independent relationships between financial assets.
  - In particular, the fair values of variance swaps may expressed in terms of the market prices of European options - independent of any model!
- It is also possible to generate upper and lower bounds for the prices of options on VIX given the prices of all SPX options.

Spanning generalized European payoffs

- In what follows we will assume that European options with all possible strikes and expirations are traded.
- We will show that any twice-differentiable payoff at time T may be statically hedged using a portfolio of European options expiring at time T.

Proof from [Carr and Madan]

The value of a claim with a generalized payoff $g(S_T)$ at time T is given by

$$\begin{aligned} g(S_T) &= \int_0^\infty g(K)\,\delta(S_T - K)\,dK \\ &= \int_0^F g(K)\,\delta(S_T - K)\,dK + \int_F^\infty g(K)\,\delta(S_T - K)\,dK \end{aligned}$$

Integrating by parts gives

$$\begin{aligned} g(S_T) &= g(F) - \int_0^F g'(K)\,\theta(K - S_T)\,dK \\ &\quad + \int_F^\infty g'(K)\,\theta(S_T - K)\,dK. \end{aligned}$$

... and integrating by parts again gives

$$
\begin{aligned}
g(S_T) &= \int_0^F g''(K)\,(K - S_T)^+\,dK + \int_F^\infty g''(K)\,(S_T - K)^+\,dK \\
&\quad + g(F) + g'(F)\,\left[(F - S_T)^+ - (S_T - F)^+\right] \\
&= \int_0^F g''(K)\,(K - S_T)^+\,dK + \int_F^\infty g''(K)\,(S_T - K)^+\,dK \\
&\quad + g(F) + g'(F)\,(F - S_T) \qquad (3)
\end{aligned}
$$

Then, with $F = \mathbb{E}[S_T]$,

$$
\mathbb{E}\left[g(S_T)\right] = g(F) + \int_0^F dK\,\tilde{P}(K)\,g''(K) + \int_F^\infty dK\,\tilde{C}(K)\,g''(K) \qquad (4)
$$

- Equation (3) shows how to build any curve using hockey-stick payoffs (if $g(\cdot)$ is twice-differentiable).

Remarks on spanning of European-style payoffs

- From equation (3) we see that any European-style twice-differentiable payoff may be replicated using a portfolio of European options with strikes from 0 to ∞.
    - The weight of each option equal to the second derivative of the payoff at the strike price of the option.
- This portfolio of European options is a static hedge because the weight of an option with a particular strike depends only on the strike price and the form of the payoff function and not on time or the level of the stock price.
- Note further that equation (3) is *completely model-independent*.

Example: European options

- In fact, using Dirac delta-functions, we can extend the above result to payoffs which are not twice-differentiable.
- For example with $g(S_T) = (S_T - L)^+$, $g''(K) = \delta(K - L)$ and equation (4) gives:

$$
\begin{aligned}
\mathbb{E}\left[(S_T - L)^+\right] &= (F - L)^+ + \int_0^F dK\, \tilde{P}(K)\, \delta(K - L) \\
&\quad + \int_F^\infty dK\, \tilde{C}(K)\, \delta(K - L) \\
&= \begin{cases} (F - L) + \tilde{P}(L) & \text{if} \quad L < F \\ \tilde{C}(L) & \text{if} \quad L \geq F \end{cases} \\
&= \tilde{C}(L)
\end{aligned}
$$

with the last step following from put-call parity as before.

- The replicating portfolio for a European option is just the option itself.

The log contract

Now consider a contract whose payoff at time T is $\log(S_T/F)$. Then $g''(K) = -\,1/{S_T}^2\big|_{S_T=K}$ and it follows from equation (4) that

$$\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\int_0^F \frac{dK}{K^2}\,\tilde{P}(K) - \int_F^\infty \frac{dK}{K^2}\,\tilde{C}(K)$$

Rewriting this equation in terms of the log-strike variable $k := \log\left(K/F\right)$, we get the promising-looking expression

$$\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\int_{-\infty}^0 dk\,p(k) - \int_0^\infty dk\,c(k) \qquad (5)$$

with

$$c(y) := \frac{\tilde{C}(F e^y)}{F e^y};\; p(y) := \frac{\tilde{P}(F e^y)}{F e^y}$$

representing option prices expressed in terms of percentage of the strike price.

Variance swaps

Assume zero interest rates and dividends. Then $F = S_0$ and applying Itô's Lemma, path-by-path

$$\begin{aligned} \log\left(\frac{S_T}{F}\right) &= \log\left(\frac{S_T}{S_0}\right) \\ &= \int_0^T d\log\left(S_t\right) \\ &= \int_0^T \frac{dS_t}{S_t} - \int_0^T \frac{{\sigma_t}^2}{2} dt \qquad (6) \end{aligned}$$

- The second term on the RHS of equation (6) is immediately recognizable as half the total variance (or quadratic variation) $W_T := \langle x \rangle_T$ over the interval $[0, T]$.

- The first term on the RHS represents the payoff of a hedging strategy which involves maintaining a constant dollar amount in stock (if the stock price increases, sell stock; if the stock price decreases, buy stock so as to maintain a constant dollar value of stock).
- Since the log payoff on the LHS can be hedged using a portfolio of European options as noted earlier, it follows that the total variance W_T may be replicated in a completely model-independent way so long as the stock price process is a diffusion.
  - In particular, volatility may be stochastic or deterministic and equation (6) still applies.

The log-strip hedge for a variance swap

Now taking the risk-neutral expectation of (6) and comparing with equation (5), we obtain

$$\mathbb{E}\left[\int_0^T \sigma_{S_t}^2 \, dt\right] = -2\,\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] = 2\left\{\int_{-\infty}^0 dk\, p(k) + \int_0^\infty dk\, c(k)\right\} \tag{7}$$

- We see that the fair value of total variance is given by the value of an infinite strip of European options in a completely *model-independent* way so long as the underlying process is a diffusion.

The VIX computation

- In 2004, the CBOE listed futures on the VIX.
- Originally, the VIX computation was designed to mimic the implied volatility of an at-the-money 1 month option on the OEX index. It did this by averaging volatilities from 8 options (puts and calls from the closest to ATM strikes in the nearest and next to nearest months).
- The CBOE changed the VIX computation: "CBOE is changing VIX to provide a more precise and robust measure of expected market volatility and to create a viable underlying index for tradable volatility products."

  - Note that VIX is a measure of implied volatility.
  - Historical volatility is a very noisy estimator of volatility and arriving at a definition on which everyone could agree would be difficult.

The new VIX formula

Here is the new VIX definition (converted to our notation) as specified in the CBOE white paper:

$$VIX^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} Q_i(K_i) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2 \tag{8}$$

where Q_i is the price of the out-of-the-money option with strike K_i and K_0 is the highest strike below the forward price F .

We recognize (8) as a straightforward discretization of the log-strip and makes clear the reason why the CBOE implies that the new index permits replication of volatility.

Specifically, (with obvious notation)

$$\begin{aligned}
\frac{VIX^2\,T}{2} &= \int_0^F \frac{dK}{K^2}\,P(K) + \int_F^\infty \frac{dK}{K^2}\,C(K) \\
&= \int_0^{K_0} \frac{dK}{K^2}\,P(K) + \int_{K_0}^\infty \frac{dK}{K^2}\,C(K) + \int_{K_0}^F \frac{dK}{K^2}\,(P(K) - C(K)) \\
&=: \int_0^\infty \frac{dK}{K^2}\,Q(K) + \int_{K_0}^F \frac{dK}{K^2}\,(K - F) \\
&\approx \int_0^\infty \frac{dK}{K^2}\,Q(K) + \frac{1}{K_0^2}\int_{K_0}^F dK\,(K - F) \\
&= \int_0^\infty \frac{dK}{K^2}\,Q(K) - \frac{1}{K_0^2}\,\frac{(K_0 - F)^2}{2}.
\end{aligned}$$

One possible discretization of this last expression is

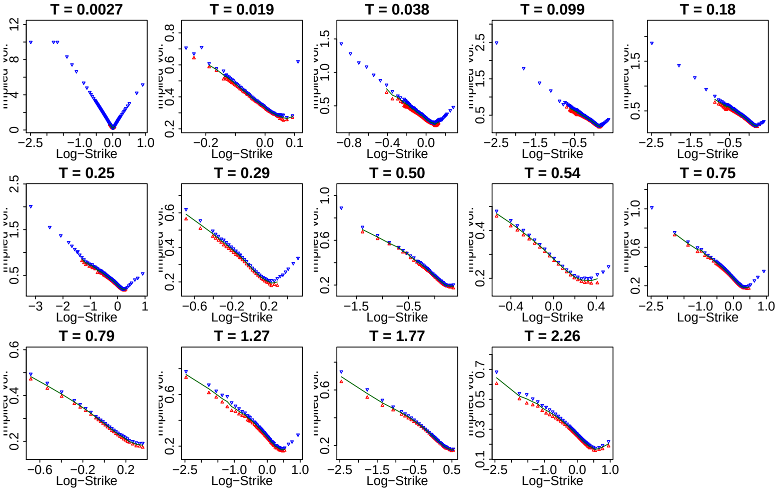
$$VIX^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2}\,Q_i(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2$$

as in the VIX specification (8).

Summary so far

- So now we understand precisely how SPX and VIX are related.
  - VIX^2 is (the fair strike of) a variance swap.
  - The fair value of VIX^2 may be estimated in a model-independent manner (assuming diffusion) by computing the value of the so-called log-strip of options.
- We now look at one day in history so see how all of this works out in practice.

SPX volatility smiles as of 15-Sep-2011

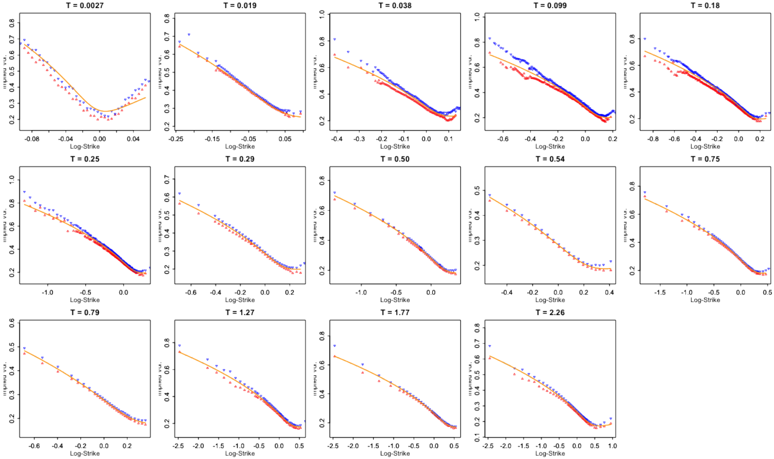


Interpolation and extrapolation

- In order to compute the log-strip, we need to interpolate and extrapolate option prices for each expiration.
  - In general, this is hard to do without introducing arbitrage such as negative calendar spreads or negative butterflies.
- We use the arbitrage-free SVI ("stochastic volatility inspired") parametrization presented in [Gatheral and Jacquier].
  - For each timeslice, with $\sigma_{BS}(k, T)^2\, T =: w(k)$,

$$w(k) = a + b\left\{\rho\,(k - m) + \sqrt{(k - m)^2 + \sigma^2}\right\}$$

SPX volatility smiles as of 15-Sep-2011 with SVI fits

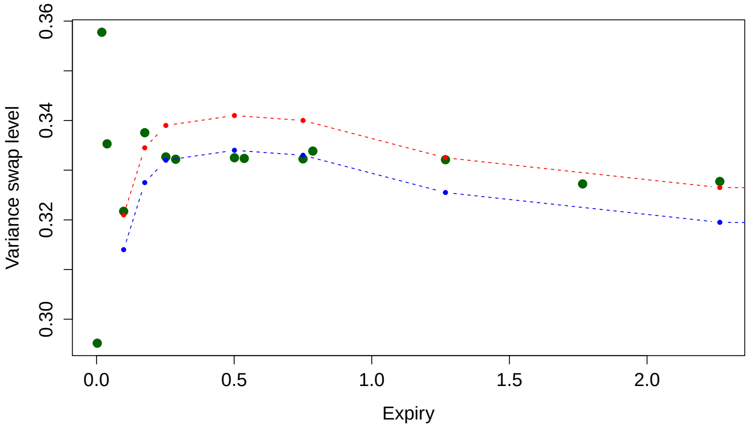


- SVI fits are in orange.

Results

- Computing the log-strip and comparing with market variance swap quotes gives impressive results...

Variance swaps as of 15-Sep-2011

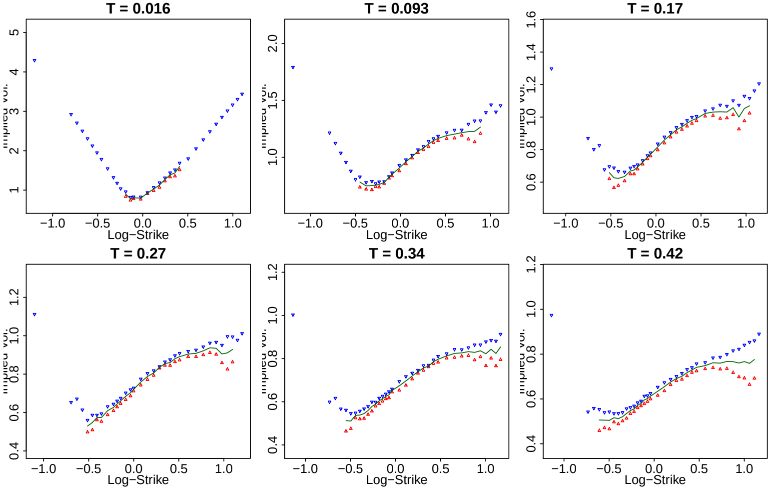


- Green dots are computed using the log-strip of SPX options.
- Blue and red points are bid and ask variance swap quotes from a friendly investment bank.

SPX and VIX

- What about relationships between VIX options and SPX options?
  - Are VIX and SPX options priced consistently with each other in practice?

VIX volatility smiles as of 15-Sep-2011



SPX and VIX smiles

- Note that SPX smiles are downward sloping:
  - Out-of-the-money puts are expensive because investors worry about big downside moves in SPX.
- VIX smiles are upward sloping:
  - Out-of-the-money calls are expensive because investors worry about big upside moves in volatility.
- In fact, when the SPX index falls, volatility (and VIX) typically increases.

VIX futures and options

A time-T VIX future is valued at time t as

$$\mathbb{E}_t\left[\sqrt{\mathbb{E}_T\left[\int_T^{T+\Delta} v_s\, ds\right]}\right]$$

where Δ is around one month (or $\Delta \approx 1/12$).

A VIX option expiring at time T with strike K_{VIX} is valued at time t as

$$\mathbb{E}_t\left[\left(\sqrt{\mathbb{E}_T\left[\int_T^{T+\Delta} v_s\, ds\right]} - K_{VIX}\right)^+\right].$$

VIX futures and options

- Note that we can span the payoff of a forward starting variance swap $\mathbb{E}_t\left[\int_T^{T+\Delta} v_s\, ds\right]$ using VIX options.
- Recall the spanning formula:

$$\mathbb{E}\left[g(S_T)\right] = g(F) + \int_0^F dK\, \tilde{P}(K)\, g''(K) + \int_F^\infty dK\, \tilde{C}(K)\, g''(K).$$

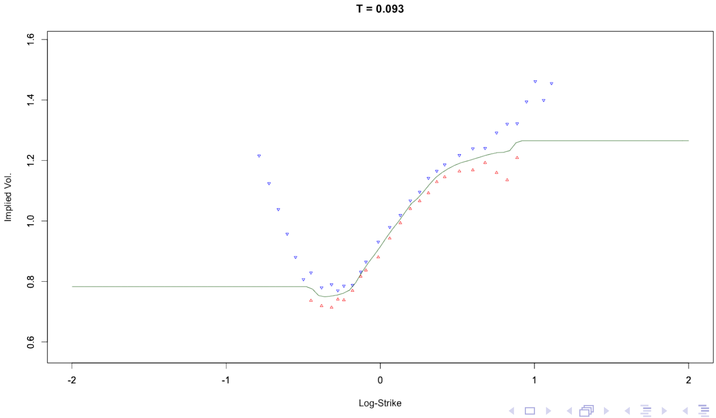
- In this case, $g(x) = x^2$ so

$$\mathbb{E}_t\left[\int_T^{T+\Delta} v_s\, ds\right] = F_{VIX}^2 + 2\int_0^{F_{VIX}} \tilde{P}(K)\, dK + 2\int_{F_{VIX}}^\infty \tilde{C}(K)\, dK.$$

  - F_{VIX} can be computed using put-call parity.
  - We need to interpolate and extrapolate out-of-the-money option prices to get the *convexity adjustment*.

Interpolation and extrapolation

- We can't use SVI because it is convex.
- We choose the simplest possible interpolation/ extrapolation.
  - Monotonic spline interpolation of mid-vols.
  - Extrapolation at constant level.



Forward variance swaps from SPX and VIX options

- We can compute the fair value of forward starting variance swaps in two ways:
  - Using variance swaps from the SPX log-strip.
  - From the linear strip of VIX options.
- We now compare the two valuations as of September 15, 2011.

Forward variance swaps as of 15-Sep-2011

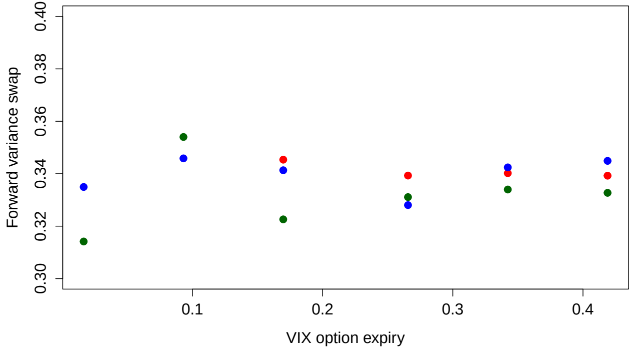


Figure 2: Red dots are forward variance swap estimates from SPX variance swaps; Green dots are interpolation of the SPX log-strip; Blue dots are forward variance swap estimates from the linear VIX option strip.

Consistency of forward variance swap estimates

- Forward variance swap estimates from SPX and VIX are very consistent on this date.
- Sometimes, VIX futures trade at a premium to the forward-starting variance swap.
  - This arbitrage has come and gone over time.
- Taking advantage as a proprietary trader is difficult because you need to cross the bid-ask so often.
  - Buy the long dated variance swap, sell the shorter-dated variance swap.
  - Sell the linear strip of VIX options.
- However, the practical consequence is that buyers of volatility should buy variance swaps, sellers should sell VIX.

Why model SPX and VIX jointly

- We had so much success with model-free computations, why should we model SPX and VIX jointly?
    - We may want to value exotic options that are sensitive to the precise dynamics of the underlying such as barrier options, lookbacks or cliquets.
- But is it possible to come up with a parsimonious, realistic model that fits SPX and VIX jointly?
    - And even if we could, would it be possible to calibrate such a model efficiently?
- We will now see that it is indeed possible to do this!

The DMR model

- In [my Bachelier 2008 presentation], a specific three factor variance curve model was introduced with dynamics motivated by economic intuition for the empirical dynamics of the variance.
- In this *double-mean-reverting* or *DMR* model, the dynamics are given by

$$\mathrm{d}S_t = \sqrt{v_t}\, S_t \,\mathrm{d}W_t^1, \tag{9a}$$

$$\mathrm{d}v_t = \kappa_1 \left(v_t' - v_t\right) \mathrm{d}t + \xi_1\, v_t^{\alpha_1}\, \mathrm{d}W_t^2, \tag{9b}$$

$$\mathrm{d}v_t' = \kappa_2 \left(\theta - v_t'\right) \mathrm{d}t + \xi_2\, v_t'^{\alpha_2}\, \mathrm{d}W_t^3, \tag{9c}$$

where the Brownian motions W_i are all in general correlated with $\mathbb{E}[\mathrm{d}W_t^i\, \mathrm{d}W_t^j] = \rho_{ij}\, \mathrm{d}t$.

Qualitative features of the DMR model

- Instantaneous variance v mean-reverts to a level v' that itself moves slowly over time with the state of the economy, mean-reverting to the long-term mean level θ.
- Also, it is a stylized fact that the distribution of volatility (whether realized or implied) should be roughly lognormal
  - When the model is calibrated to market option prices, we find that indeed $\alpha_1 \approx 1$ consistent with this stylized fact.
- As we will see later, the DMR model calibrated jointly to SPX and VIX options markets fits pretty well.

Computations in the DMR model

- One drawback of the DMR model is that calibration is not easy
    - No closed-form solution for European options exists so finite difference or Monte Carlo methods need to be used to price options.
    - Calibration using conventional techniques is therefore slow.
- In [Bayer, Gatheral and Karlsmark], the DMR model is calibrated using the Monte Carlo scheme of [Ninomiya and Victoir].
    - Joint calibration of the model to SPX and VIX options is possible in less than 5 seconds.

Estimation of κ_1, κ_2, θ and ρ_{23}

- In the DMR model, the fair strike of a variance swap is given by the expression

$$\mathbb{E}\left[\int_t^T v_s \, \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \theta\,\tau + (v_t - \theta)\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1} + (v'_t - \theta)\frac{\kappa_1}{\kappa_1 - \kappa_2}\left\{\frac{1 - e^{-\kappa_2 \tau}}{\kappa_2} - \frac{1 - e^{-\kappa_1 \tau}}{\kappa_1}\right\} \tag{10}$$

  which is affine in the state variables v_t and v'_t.
- Fixing θ, κ_1 and κ_2, and given daily variance swap estimates, time series of v_t and v'_t may be imputed by linear regression.
  - Optimal values of θ, κ_1 and κ_2 are obtained by minimizing mean squared differences between the fitted and actual variance swap curves.

Daily model fitting

- The model parameters κ_1, κ_2, θ and ρ_{23} are considered fixed. They are obtained from historical variance swap data.
- The state variables v_t and v_t' are obtained by linear regression against the fair values of variance swaps proxied by the log-strip.
  - Arbitrage-free interpolation and extrapolation of the volatility surface is achieved using the SVI parameterization in [Gatheral and Jacquier].
- The volatility-of-volatility parameters ξ_1 and ξ_2 are obtained by calibrating the DMR model to the market prices of VIX options (using NVs).
- The correlation parameters ρ_{12} and ρ_{13} are then calibrated to SPX options.

VIX smiles

The VIX option smile encodes information about the dynamics of volatility in a stochastic volatility model.

- For example, under stochastic volatility, a very long-dated VIX smile would give us the stable distribution of VIX.
- In the context of the DMR model:
  - The slope of the VIX smile allows us to fix the exponents α_1 and α_2. Increasing α causes the slope of the DMR VIX smile to increase.
    - An exponent of $1/2$ as in the Heston model would induce a VIX smile with a negative slope!
  - The levels of the VIX smile fix ξ_1 and ξ_2 ("volatility of volatility").
  - Only two parameters, ρ_{12} and ρ_{13} are then left to match all of the SPX smiles!

VIX fit as of September 15, 2011

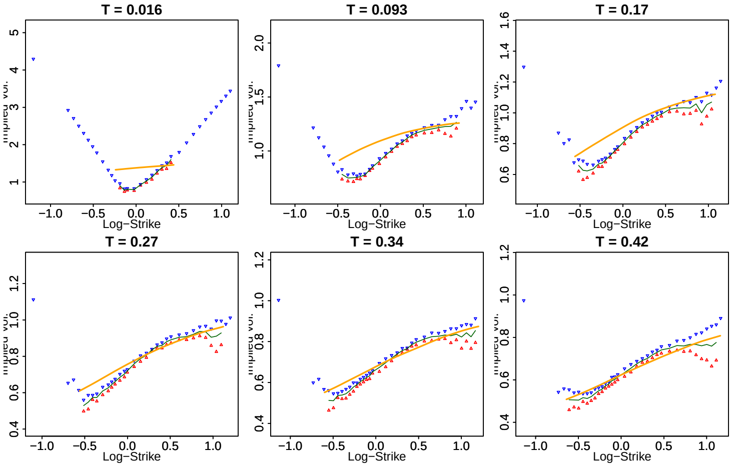
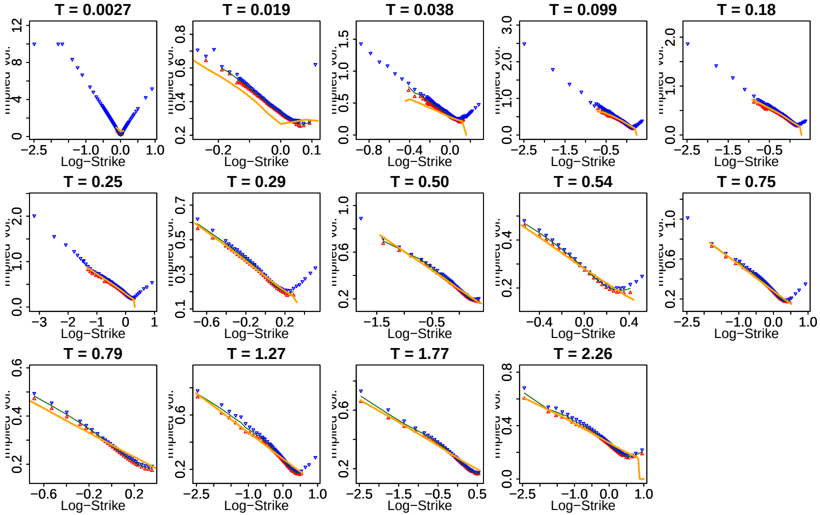


Figure 3: VIX smiles as of September 15, 2011: Bid vols in red, ask vols in blue, and model fits in orange.

SPX fit as of September 15, 2011



Summary

- We explained how VIX can be understood as representing the volatility of SPX in a very precise way.
- We exhibited one particular arbitrage relationship between the SPX and VIX options markets.
  - Two ways to arrive at the fair value of a forward-starting variance swap.
- We presented a model, the DMR model, that can be calibrated to SPX and VIX options markets simultaneously.
  - Fits are pretty good.
  - Exotic options with SPX as underlying can be valued with greater confidence.

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