Intermediate Financial Economics
Lecture III: Risk

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In This Lecture:

- I. Risk.
  - Greater risk: Definition and implication.
  - Equivalence between greater risk and risk aversion.
  - Greater risk v.s. greater variance.
  - Greater risk and risk premium.

A. Greater Risk: Definition

- Motivation: What does it mean when we say a gamble/consumption plan is riskier than another?
- Example 1:
  - a) Gamble A has payoffs (-1,1) with equal probabilities. Gamble B has payoffs (-2, 2) with equal probabilities. B is obviously "riskier" than A.
  - b) What about (-1, 1) v.s. (0, 4) with equal probabilities?
  - Not obvious.
- Definition: A consumption plan Y is riskier than consumption plan X if there exists a random noise Z such that $Y - E[Y]$ has the same distribution with $X - E[X] + Z$, where $E[Y|X]=0$ for every X
  - i.e. $Y - E[Y] =^d X - E[X] + Z$, with $E[Z|X]=0$ for every X.

EXAMPLE 1: B) $X = X - E[X] = \begin{cases} 1 \text{ w.p. } 1/2 \\ -1 \text{ w.p. } 1/2 \end{cases}$ $Y = \begin{cases} 4 \text{ w.p. } 1/2 \\ 0 \text{ w.p. } 1/2 \end{cases}$

- Is there such a Z for this example?

$$Y - E[Y] = \begin{cases} 2 \text{ w.p. } 1/2 \\ -2 \text{ w.p. } 1/2 \end{cases}$$

- Consider $(Z|X = 1) = \begin{cases} 1 \text{ w.p. } 3/4 \\ -3 \text{ w.p. } 1/4 \end{cases}$

$$(Z|X = -1) = \begin{cases} 3 \text{ w.p. } 1/4 \\ -1 \text{ w.p. } 3/4 \end{cases}$$

- We have E[Z|X]=0 for X=1 and X= -1.
- And $X + Z = \begin{cases} 2 \text{ w.p. } 1/2 \\ -2 \text{ w.p. } 1/2 \end{cases}$
- So Y is riskier than X.
- It can be shown in general that Y is riskier than X if and only if Y – E[Y] is riskier than X – E[X].

B. IMPLICATION

- Proposition I: For Y and X that have the same expectation, Y is riskier than X if and only if $E[u(Y)] \leq E[u(X)]$ for all $u(\cdot)$ concave.
  - That is, for two consumption plans with the same expectation, Y is riskier than X if and only if every risk averse individual prefers X to Y.
- Proof: Suppose Y is riskier than X, and $E[Y]=E[X]$. Then $Y =^d X+Z$, where $E[Z|X]=0$ for all X. We have $E[u(Y)]=E[u(X+Z)]=E[E[u(X+Z)|X]]$ by the law of iterated expectation. For concave utility, we have $E[u(X+Z)|X] \leq u(E[Z|X]+X)=u(X)$. Taking expectation on both sides, we have $E[E[u(X+Z)|X]] \leq E[u(X)]$, which says that $E[u(Y)] \leq E[u(X)]$. The other part is more difficult. See Rothchild and Stiglitz (1970).

Proposition II:

- For any consumption plan X, if there is some Z with $E[Z|X]=0$, then $X+bZ$ is riskier than $X+aZ$ for every $b>a\geq 0$.
- Proof: Let $k=a/b$. Then $X+aZ=k(X+bZ)+(1-k)X$. Because k is between 0 and 1, we have for every concave utility that $u(X+aZ) \geq ku(X+bZ)+(1-k)u(X)$. Then

$$E[u(X+aZ)] \geq kE[u(X+bZ)]+(1-k)E[u(X)] \quad (*)$$

Because $X+aZ$ is riskier than X, we have $E[u(X)] \geq E[u(X+aZ)]$. Inequality (*) becomes

$$E[u(X+aZ)] \geq E[u(X+bZ)]$$

for every concave u. So $X+bZ$ is riskier than $X+aZ$ by Prop. I.

C. Greater Variance $\neq$ Greater Risk

- Variance is often used as a measure of risk.
- However, it is not equivalent to the definition of greater risk.
- <u>Example 2</u>: Let X take on the values 1, 3, 4, 6 with equal probabilities. Let Y take value 2 with probability ½, and values 3 and 7, each with probability ¼. Then

E[X]=E[Y]=3.5, and Var[Y]=4.25 > Var[X]=3.25.

However, for log utility which is concave, we have

E[u(Y)]=ln(84)/4>E[u(X)]=ln(72)/4.

- The ordering of riskiness based on variance is generated by quadratic utility $u(c)= -(c - a)^2$, $c \leq a$.
- $E[u(z)]= -E[(z - a)^2]= -\{Var[z] + (E[z] - a)^2 \}$
- We have $E[u(Y)] \leq E[u(X)]$ where $u(c)= -(c - a)^2$, $c \leq a$ if and only if $Var[X] \leq Var[Y]$.
- But the ordering of riskiness based on greater risk is generated by all concave utility functions.
- Therefore greater risk is more general (and complete) than greater variance.
- However, there is an exception.
- Proposition III: X and Y are both normally distributed. Then Y is riskier than X if and only if $Var[X] \leq Var[Y]$.
- Proof: Use Proposition II.

D. Greater Risk and Risk Premium

- Question: If the gamble becomes riskier, would a risk averse person pays more to avoid the gamble?
- Example 3: Suppose y=5. z is -3 and 3 with equal probabilities. We've shown that the risk premium satisfies $E[u(z)]=(\ln 2)/2+(\ln 8)/2=\ln(5-\pi)$, and $\pi=1$. Now suppose z' is -4 or 4 with equal probabilities. z' is riskier than z. We have $\pi'=2$.
  - A risk averse person would be willing to pay more to avoid a riskier gamble.
- Proposition IV: For two risky consumption plan X and Y that has the same expectation, Y is riskier than X if and only if the risk premium of Y is greater than or equal to the risk premium of X.