Intermediate Financial Economics
Lecture II: Risk Aversion

Dr. Zhang

This Lecture:

- I. Expected utility
- II. Risk aversion and utility functions.

I. Expected Utility

- A. Utility function
  - 1. Definition: A mapping that represents some preference ordering of consumption bundles in a world without uncertainty.
  - 2. The existence of a utility function implies that the preference is rational.
    - A preference is rational if it is complete and transitive.
  - Is there a function that represents preferences in the case with uncertainty?
  - What are the things that people have preferences over in the world with uncertainty?

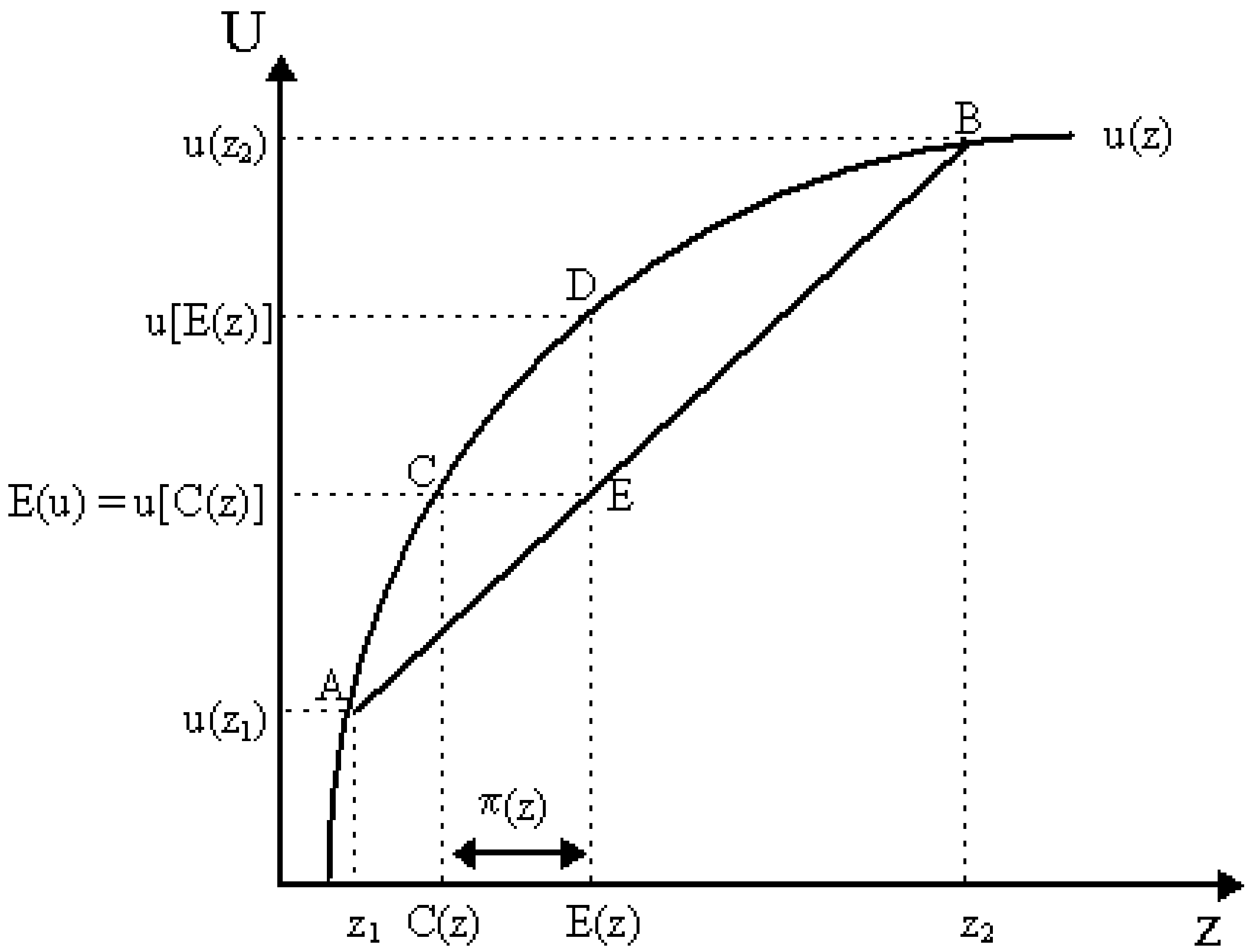
B. The Case with Uncertainty

- What are things that people have preferences over?
  - With certainty: Consumption bundles (certain number for each good)
  - With uncertainty: Lotteries (probability distributions over consumption of each good)
- What represents preferences over lotteries in the world with uncertainty?
  - Expected utility: a linear combination of utilities in each state of nature. $U(c_1,..c_n)=p_1u_1(c_1)+\ldots+p_nu_n(c_n)=E[u(c)]$.
  - $U(c_1,..c_n)$ is also called the Von Neumann-Morgenstern utility.
  - If the utility in each state doesn't depend on the state $u_1(\cdot)=u_2(\cdot)=\ldots=u_n(\cdot)=u(\cdot)$, then $u(\cdot)$ is said to be state independent.
- Under what conditions does expected utility exist?
  - Some assumptions on the preference: continuous, independent,…

II. Risk Aversion

- Suppose your current wealth is y.
- Consider a gamble z that would add to your current wealth with two potential outcomes:
  - Heads z_2, tail z_1, $z_2 > z_1$.
- Definition of risk aversion: one prefers the expected value of the gamble to the gamble itself.
- Mathematical definition: $u(y + E[z]) \geq E[u(y + z)]$
  - This inequality coincides with the mathematical definition of a concave function.
  - Also called the Jensen's inequality for concave functions.
  - We assume that $u'(\cdot)>0$.

Risk Aversion, Certainty Equivalence, and Risk Premium (with y=0)



Risk Aversion and Concavity

- An agent is risk averse if and only if his utility function is concave.
  - i.e. $u(E[z]) \geq E[u(z)]$ iff $u''(\cdot) \leq 0$.

- Special case: risk neutrality.
- Definition of risk neutrality:
  - $u(E[z]) = E[u(z)]$

- An agent is risk neutral if and only if his utility function is linear.
  - i.e. $u(E[z]) = E[u(z)]$ iff $u''(\cdot) = 0$.

Certainty Equivalence and Risk Premium

- Definition of *certainty equivalence*:
  - The certain income that gives the individual the same welfare as he would derive from taking the additional risky consumption plan z at the (deterministic) initial consumption y.
- Mathematically, it is $C(y,z)$ such that $E[u(y+z)]=u(y+C(y,z))$.
- If a risk averse individual is asked to take a gamble,
  - He would rather have $E[z]$ instead of the gamble.
  - He is willing to pay to avoid the gamble.
  - What is the most he is willing to pay out of his expected income $E[z]$ such that he feels exactly indifferent from taking and not taking the gamble?
- Definition of *risk premium* for the additional risky consumption plan z at the (deterministic) initial consumption y:
  - is the maximum amount $\pi(y, z)$ that an individual is indifferent from having z and having $E[z]-\pi(y, z)$.
- Mathematically, risk premium is $\pi(y, z)$ such that

$$E[u(y+z)] = u(y+E[z]-\pi(y, z))$$

- So $E[z]-\pi(y, z)=C(y,z)$.

Properties of Risk Premium

- Property: $\pi(y, z) = \pi(y+E[z], z-E[z])$.
  - So it is enough to consider a risky consumption plan $z' = z-E[z]$ with zero mean and initial wealth $y' = y+E[z]$, because other risk consumption plans with non-zero means are included.
  - The situation gives the same risk premium with the one where the risky consumption plan is z and initial wealth is y.
- Property: An agent is risk averse if and only if his risk premium is greater than or equal to zero. That is, $E[u(y+z)] \leq u(y+E[z])$ is equivalent to $\pi(y,z) \geq 0$.

The equivalence can be seen by the definition of risk premium $E[u(y+z)] = u(y+E[z] - \pi(y, z))$

- <u>Example 1</u>: Suppose y=5. z is -3 or 3 with equal probabilities.

a) u(x)=lnx. What is π? What is C?

b) u(x)=-(lnx)$^{-1}$ What is π? What is C?

a) E[u(z)]=(ln2)/2+(ln8)/2=ln(5 – π). Then π=1. C=-1.

b) π=2.17. C=-2.17.

- <u>Example 2</u>: Suppose u(x)= – exp(–ax). Suppose z is any random variable. What is the risk premium π(y,z)?

  – E[exp(–a(y+z)] = – exp(–ay – aE[z]+aπ(y,z)).

  π(y, z)= E[z]+(ln E[exp(–az)])/a

Suppose z is normally distributed with E[z] and Var(z).

$$\pi(y,z) = E[z] + \frac{1}{a}\ln\exp(-aE[z] + a^2Var(z)/2) = aVar(z)/2$$

  - The risk premium in this case does not depend on y.
  - The higher the risk, the more risk premium.
  - The more risk averse a person, the more risk premium she demands.

Risk Premium: An Approximation

- **Theorem:** For small z with $E(z)=0$, we have

$\pi(y, z) \approx A(y)Var(z)/2$.

Proof: A Taylor series expansion of $E[u(y+z)]$ is

$E[u(y+z)] = u(y)+u''(y)Var(z)/2 + o(Var(z))$

Similarly, a Taylor series expansion of $u(y-\pi)$ gives

$u(y-\pi) = u(y)-u'(y)\pi + O(\pi^2)$.

Equating the right hand sides of the two equations and solving for π, we have

$\pi=(-u''(y)/u'(y))\, Var(z)/2+o(Var(z))$

Then $\pi(y, z) \approx A(y)Var(z)/2$ for z with small variance.

Measuring Risk Aversion

- Mathematically, the degree of risk aversion is associated with the second-order derivative of of u(·).
  - Arrow-Pratt measure of **absolute risk aversion**:

$$A(x) = -\frac{u''(x)}{u'(x)}$$

  - where dividing by u'(x) is for the measure to be invariant to affine transformations of u(·).
  - Geometrically, it is the curvature of u(·). The higher – u"(·) is, the more curved u(·) is.
  - Alternative measure: **risk tolerance:** T(x)=1/A(x).
- The concept of risk premium helps make the math more intuitive.
- **The Pratt Theorem**:
  - u_1 and u_2 are strictly increasing, twice differentiable, and have continuous second derivatives. The following are equivalent:
    - i) $A_1(y) \geq A_2(y)$ for all y.
    - ii) $\pi_1(y, z) \geq \pi_2(y, z)$ for all y and random variable z.
    - iii) $u_1(\cdot)$ is a concave transformation of $u_2(\cdot)$

Example 2 Cont.

- Let's verify that Pratt Theorem
- Suppose two persons both have the same type of utility function $u(x) = -\exp(-ay)$, except that $a_1 \geq a_2$.
- $a_1 \geq a_2$ is equivalent to $A_1(y) \geq A_2(y)$, since $A(y)=a$.
- We have shown that $\pi(y,z) = aVar(z)/2$.
- So we have $\pi_1(y, z) \geq \pi_2(y, z)$.
- Mathematically, that function u_1 is more curved than u_2 means that u_1 is a concave transformation of u_2, i.e. $u_1 = f(u_2)$, where f is concave.
- In our example,

$$f(u_2) = -(-u_2)^{a_1/a_2} = -\exp(-a_1 y) = u_1$$

Is f concave? Yes. So u_1 is more concave/curved than u_2.

Pratt Theorem: General Proof

- iii) is equivalent to $u_1 = f \circ u_2$ for f concave and strictly increasing.
- i)=>iii): The f function in iii) is defined by

$f(t)=u_1(u_2^{-1}(t))$. We need to show that f is strictly increasing and concave.

$f'(t)=u_1'(u_2^{-1}(t))/u_2'(u_2^{-1}(t))>0.$

$f''(t)=[u_1''(y) - u_2''(y)\, u_1'(y)/ u_2'(y)]/[u_2'(y)]^2$, where $y=u_2^{-1}(t)$. It can be rewritten as

$f''(t)=[A_2(y) - A_1(y)]\, u_1'(y) /(u_2'(y))^2 \leq 0.$

- iii) => ii): By definition of risk premium, we have $E[u_1(y+z)]=u_1(y+E[z] - \pi_1)$.

$$E[u_1(y+z)]=E\{f[u_2(y+z)]\} \leq f\{E[u_2(y+z)]\}=f\{u_2(y+E[z]- \pi_2)\}$$
$$=u_1(y+E[z]- \pi_2).$$

We have $u_1(y+E[z] - \pi_1) \leq u_1(y+E[z]- \pi_2)$, which implies $\pi_1 \geq \pi_2$

- ii)=>i): we prove this by proving that not i) implies not ii). Suppose ii) holds: $\pi_1(y, z) \geq \pi_2(y, z)$.
- Not i) is equivalent to A_1 (y) < A_2 (y) for some y in the neighborhood $[y_0 - d, y_0 + d]$.
- But from the proof of i)=> iii), we know that this condition implies $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$ in the same neighborhood.
- Then from the proof of iii)=>ii), we know that $\pi_2(y,z) > \pi_1(y,z)$, for y in this neighborhood.
- But this contradicts ii).

CARA, IARA, AND DARA

- Constant absolute risk aversion (CARA) means $A'(x)=0$.
- Increasing absolute risk aversion (IARA) means $A'(x)>0$.
- Decreasing absolute risk aversion (DARA) means $A'(x)<0$.
  - Experimental and empirical evidence is consistent with DARA.

- An alternative measure:
  - Arrow-Pratt measure of relative risk aversion (RRA):

$$R(x) = -x\frac{u''(x)}{u'(x)}$$

CRRA

- The class of utilities of constant relative risk aversion (CRRA), meaning that $R'(x)=0$, is popular,
  - because CRRA implies DARA,
    - To see this, note that $R'(x)=A(x)+xA'(x)=0$ for CRRA. But since both x and $A(x)$ are positive, we have $A'(x) < 0$.
  - because DARA seems to describe people's risk attitude,
  - and because CRRA has nice math properties.

Increasing (Decreasing) Risk Aversion and Increasing (Decreasing) Risk Premium

- **Corollary**: For a strictly increasing and twice-differentiable utility function u with continuous second derivative, $\pi(y, z)$ is increasing (decreasing) in y for every z if and only if $A(y)$ is increasing (decreasing) in y.

Proof: Let us define $u_1(y)= u(y+d)$ for some $d \geq 0$. Let $u_2(y) = u(y)$. We have $A_1(y)=A(y+d)$ and $A_2(y)=A(y)$. Similarly $\pi_1(y, z) = \pi(y+d, z)$ and $\pi_2(y,z) = \pi(y, z)$. Then we know from the Pratt Theorem that $A_1(y) \geq A_2(y)$ if and only if $\pi_1(y, z) \geq \pi_2(y,z)$. But this means that $A(y)$ is increasing in y if and only if $\pi(y, z)$ is increasing in y. The other case can be proved similarly.

B. Utility Functions

- Power utility (CRRA),

$$u(x) = \frac{x^{1-a}-1}{1-a}, \; a > 0.$$

  - where a is the coefficient of relative risk aversion.
  - Logarithmic utility (a special case when a=1)

$$u(x) = \ln x$$

  - Quadratic utility (a special case when a=3)

$$u(x) = -x^{-2}$$

which represents the same risk preference as $u(x) = -\frac{x^{-2}-1}{2}$

  - The class of power utility has hyperbolic absolute risk aversion, or linear risk tolerance. Also known as the HARA utility or utility with LRT.

- Negative exponential utility (CARA)

$$u(x) = -e^{-ax}, \; a > 0.$$

  - where a is the coefficient of absolute risk aversion.

Implication of DARA, IARA, and CARA

- Consider a one-period model. There are two assets: a risky asset with gross random (risky) return R, and a riskless asset with gross return R_f. His endowed wealth is W_0. He has to decide how to divide up the wealth to invest in different assets, in order to maximizes his expected utility of final wealth.
- His optimization problem is

$$\max_a E[u(W)]$$

$$s.t.\ W = aR + (W_0 - a)R_f$$

where "a" denotes the invested money in the risky asset.

- This problem depends on how much initial wealth he has, and how he expects the return and the risk of the risky asset, and the return of the riskless asset.
- The first order condition is $E[u'(W)(R - R_f)] = 0$

LEMMA

- The investment in the risky asset is
  - i) positive iff $E[R]>R_f$
  - ii) negative iff $E[R]<R_f$
  - iii) zero iff $E[R]=R_f$.
- Proof: Taking the derivative of the FOC with respect to a, we have $E[u''(W)(R-R_f)^2]<0$. So the expected utility is a concave function of a.

If $a^*>0$, then the first order derivative of the expected utility must have positive slope at a=0, i.e. $E[u'(W)(R-R_f)] |_{a=0}$

$= E[u'(W_0 R_f)(R-R_f)]>0 \Rightarrow E[R]>R_f$.

The other two items can be proved similarly.

- From now on, we assume that $E[R]>R_f$.

PROPOSITIONS

- Suppose $E[R]>R_f$. DARA implies that the risky asset is a normal good; IARA implies that the risky asset is an inferior good; CARA implies that the demand for the risky asset is invariant to the initial wealth.

- (Definition: A (an) normal (inferior) good is a good whose demand increases (decreases) with wealth.)
- It can be shown that

$$\frac{da}{dW_0} = -R_f \frac{E[u''(W)(R-R_f)]}{E[u''(W)(R-R_f)^2]}$$

- The sign of da/dW_0 depends on the sign of the numerator: $E[u''(W)(R - R_f)]$.

Example 3

- DARA: u(z)=ln z;
- CARA: u(z)=-exp(-x);
- IARA: $u(z)=-(z-c)^2$, $z \leq c$.
- R_f=1.1. R=2 or 0.8 with equal probabilities.
- For DARA, the first order condition is

$$\frac{1}{2}\frac{2-1.1}{2a+1.1(W_0-a)}+\frac{1}{2}\frac{0.8-1.1}{0.8a+1.1(W_0-a)}=0$$

- The solution is $a=11W_0/9$, increasing in W_0.
- For CARA , a=ln3/1.2 ≈ 0.9155, independent of W_0.
- For IARA, $a=4 - 2.2W_0/3$, decreasing in W_0.

IRRA, DRRA, CRRA

- Recall $R(x) = -x\frac{u''(x)}{u'(x)}$
- We have increasing, decreasing, and constant relative risk aversion, which means that R'(x)>0, R'(x)<0, or R'(x)=0, respectively.
- Definition: The wealth elasticity of the demand for the risky asset is defined by the percentage change of the amount invested in the risky asset over the percentage change of initial wealth:

$$\eta = \frac{da/a}{dW_0/W_0} = 1 + \frac{W_0 \cdot da/dW_0 - a}{a}$$

- Proposition: Suppose $E[R]>R_f$. The wealth elasticity of the demand for the risky asset is
  - i) less than 1 for IRRA;
  - ii) more than 1 for DRRA;
  - iii) equal to 1 for CRRA.

How Risk Averse Are You? – A Thought Experiment

- Suppose your risk behavior is consistent with the expected utility theory and your preference can be described by a CRRA utility: $u(x)= x^{1-a}/(1-a)$.
- What is your coefficient of RRA, $R(x)=a$?
- Consider the following gamble:
  - Your wealth could be doubled with probability p, or reduced by one half with probability 1 – p.
  - Suppose you can choose p.
- Question: what p would you pick such that you would feel exactly indifferent between having and not having the gamble?
- The indifference condition gives an equation:

$$p\frac{(2W)^{1-a}}{1-a} + (1-p)\frac{(W/2)^{1-a}}{1-a} = \frac{W^{1-a}}{1-a}$$

BACKING OUT YOUR COEFFICIENT OF RELATIVE RISK AVERSION

- Dividing both sides by $W^{1-a}/(1-a)$, we have

$$p2^{1-a} + (1-p)(1/2)^{1-a} = 1$$

- where a can be solved for inputs of p:

$$a = \frac{\ln \frac{2p}{1-p}}{\ln 2}$$

- The results are

p	0.5	0.666667	0.75	0.8	0.857143	0.9
a	1.0	2.0	2.6	3.0	3.6	4.2

p	0.941176	0.97	0.98	0.99	0.9999	0.999999999
a	5.0	6.0	6.6	7.6	14.3	30.9