Solving Dynamic Programming Problems on a Computational Grid[☆]

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Abstract

We implement a dynamic programming (DP) algorithm on a computational grid consisting of loosely coupled processors, possibly including clusters and individual workstations. The grid changes dynamically during the computation, as processors enter and leave the pool of workstations. The algorithm is implemented using the runtime support library MW running on the HTCondor grid computing platform. We describe our application of value function iteration to several large dynamic programming problems. Some are optimal growth problems and some are dynamic portfolio problems. We present examples that solve in hours on HTCondor but would take weeks if executed on a single workstation. It is clear that the use of HTCondor can increase a researcher's computational productivity by at least two orders of magnitude.

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1. Introduction: Motivation and Model

This paper introduces one of the first big uses of parallel computation in economics. It is common that many high-dimensional economic and financial problems require weeks or months of computation to solve them, because of the "curse of dimensionality". It is natural to do parallelization to break the "curse of dimensionality".

Dynamic programming (DP) is the essential tool in solving problems of dynamic and stochastic controls in economic analysis. Many DP problems are solved by value function iteration, where the period t value function is computed from the period $t+1$ value function, and the value function is known at the terminal time T. A set of discrete and approximation nodes will be chosen and the period t value function at those nodes will be computed and then we can use some approximation methods to approximate the value function. For every approximation node, there is a time-consuming optimization problem to be solved. Moreover, these optimization problems are independent. Therefore, it will be efficient if we solve them in parallel.

This paper is constructed as follows. Section 2 gives an introduction of HTCondor-MW system. Section 3 describes DP algorithms. Section 4 introduces two types of parallel DP algorithms in the HTCondor-MW system. Section 5 and 6, respectively, give computational results of the parallel DP algorithms in the HTCondor-MW system for solving multidimensional optimal growth problems and dynamic portfolio optimization problems.

2. A Grid Platform

The HTCondor system is a high-throughput computing, open-source software framework for distributed parallelization of computationally in-

tensive tasks on a cluster of computers. The HTCondor software is freely available to all; see http://research.cs.wisc.edu/htcondor/index.html for details. HTCondor acts as a management tool for identifying, allocating and managing available resources to solve large distributed computations. For example, if a workstation on a network is currently unused, HTCondor will detect that fact, and send it a task. HTCondor will continue to use that workstation until a higher-priority user (such as a student sitting at the keyboard) appears, at which time HTCondor ends its use of the workstation. This is called "cycle scavenging" and allows a system to take advantage of essentially free computing time. HTCondor can also be used on a dedicated cluster.

The HTCondor team at the University of Wisconsin-Madison has developed several "flavors" of HTCondor, each fine-tuned for some specific type of parallel programming. In this paper we use the HTCondor Master-Worker (MW) system for parallel algorithms to solve DP problems. The HTCondor MW system consists of two entities: a master process and a cluster of worker processes. The master process decomposes the problem into small tasks and puts those tasks in a queue. Each worker process first examines the queue, takes the "top" problem off the queue and solves it. The worker then sends the results to the master, examines the queue of unfinished tasks, and repeats this proccess until the queue is empty. The workers' execution is a simple cycle: take a task off master's queue, do the task, and then send the results to the master. While the workers are solving the tasks, the master collects the results and puts new tasks on the queue. This is a file-based, remote I/O scheme that serves as the message-passing mechanism between the master and the workers.

The MW paradigm helps the user circumvent the parallel programming

challenges, such as load balancing, termination detection, and the distribution of information across compute nodes. Moreover, computation in the MW paradigm is fault-tolerant: if a worker cannot complete a task, due to machine failure or interruption by another user, the master can detect this and put that task back on the queue for another worker to execute. The user can request any number of workers, independent of the number of tasks. HTCondor can make use of a heterogeneous collection of computers, where the fast computers will solve more tasks but slower computers can still contribute.

HTCondor is an example of "High Throughput Computing" (HTC) and is a valuable alternative to "High Performance Computing" (HPC). HPC is typified by supercomputing where a large, but fixed, number of processors are assigned to a problem, and communication time across the processors is very small. HPC may be fast once the processors are accessible to the user, but a user often must reserve time on the computer with larger requests causing longer lags in getting the desired computing time. HTC is a paradigm that uses a flexible population of processors which is adjusted for processor availability and for the needs of the computation process. Furthermore, it is opportunistic, not requiring advance reservations, and utilizes any resource that becomes available. Interprocessor communication may be much greater in HTC environments but many problems can still efficiently use hundreds of processors in a high latency environment. In this paper, we show that DP is that kind of problem.

In real life, the time that a user cares about is the time between his submission of a job to a system and when he receives the results. On this dimension, HTC may dominate HPC. Moreover, it is very difficult for the typical economist to get access to supercomputers. In those cases, HTC is

the only option, but is also a viable option.

3. Dynamic Programming

In economics and finance, we often encounter a finite horizon optimal decision-making problem that can be expressed in the following general model:

$$V_0(x_0, \theta_0) = \max_{a_t \in \mathcal{D}(x_t, \theta_t, t)} \mathbb{E}\left\{ \sum_{t=0}^{T-1} \beta^t u_t(x_t, a_t) + \beta^T V_T(x_T, \theta_T) \right\},$$

where x_t is a continuous state process with an initial state x_0, θ_t is a discrete state process with an initial state θ_0, and a_t is an action variable (x_t, θ_t and a_t can be vectors), $u_t(x, a)$ is a utility function at time $t < T$ and $V_T(x, \theta)$ is a given terminal value function, β is the discount factor ($0 < \beta \leq 1$), $\mathcal{D}(x_t, \theta_t, t)$ is a feasible set of a_t, and $\mathbb{E}\{\cdot\}$ is the expectation operator.

The DP model for the finite horizon problems is the basic Bellman equation,

$$V_t(x, \theta) = \max_{a \in \mathcal{D}(x, \theta, t)} u_t(x, a) + \beta \mathbb{E}\{V_{t+1}(x^+, \theta^+)\},$$

for $t = 0, 1, \ldots, T-1$, where (x^+, θ^+) is the next-stage state conditional on the current-stage state (x, θ) and action a, and $V_t(x, \theta)$ is called the value function at stage t wihile the terminal value function $V_T(x, \theta)$ is given.

3.1. Numerical DP Algorithms

In DP problems, if state variables and control variables are continuous, then value functions must be approximated in some computationally tractable manner. It is common to approximate value functions with a finitely parameterized collection of functions; that is, $V(x, \theta) \approx \hat{V}(x, \theta; \mathbf{b})$, where $\mathbf{b}$ is a vector of parameters. The functional form $\hat{V}$ may be a linear combination of polynomials, or it may represent a rational function or neural network representation, or it may be some other parameterization specially

Algorithm 1 Parametric Dynamic Programming with Value Function Iteration for Problems with Multidimensional Continuous and Discrete States

Initialization. Given a finite set of $\theta \in \Theta = \{\theta^j : 1 \leq j \leq D\} \subset \mathbb{R}^d$ and the probability transition matrix $P = \left(p_{j,j'}\right)_{D\times D}$ where $p_{j,j'}$ is the transition probability from $\theta^j \in \Theta$ to $\theta^{j'} \in \Theta$ for $1 \leq j, j' \leq D$. Choose a functional form for $\hat{V}(x, \theta; \mathbf{b})$ for all $\theta \in \Theta$, and choose the approximation grid, $\mathbb{X}_t = \{x_t^i : 1 \leq i \leq N_t\} \subset \mathbb{R}^n$. Let $\hat{V}(x, \theta; \mathbf{b}^T) = V_T(x, \theta)$. Then for $t = T-1, T-2, \ldots, 0$, iterate through steps 1 and 2.

Step 1. Maximization step. Compute

$$v_{i,j} = \max_{a \in \mathcal{D}(x^i, \theta^j, t)} u_t(x^i, \theta^j, a) + \beta \mathbb{E}\{\hat{V}(x^+, \theta^+; \mathbf{b}^{t+1})\},$$

for each $x^i \in \mathbb{X}_t$ and $\theta^j \in \Theta$, $1 \leq i \leq N_t$, $1 \leq j \leq D$, where the next-stage discrete state θ^+ is random with probability mass function $\Pr(\theta^+ = \theta^{j'} \mid \theta^j) = p_{j,j'}$ for each $\theta^{j'} \in \Theta$, and x^+ is the next-stage state transition from x^i and may be also random.

Step 2. Fitting step. Using an appropriate approximation method, for each $1 \leq j \leq D$, compute $\mathbf{b}_j^t$, such that $\hat{V}(x, \theta_j; \mathbf{b}_j^t)$ approximates $\{(x^i, v_{i,j}): 1 \leq i \leq N_t\}$ data, i.e., $v_{i,j} \approx \hat{V}(x^i, \theta^j; \mathbf{b}_j^t)$ for all $x^i \in \mathbb{X}_t$. Let $\mathbf{b}^t = \left\{\mathbf{b}_j^t : 1 \leq j \leq D\right\}$.

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designed for the problem. After the functional form is fixed, we focus on finding the vector of parameters, $\mathbf{b}$, such that $\hat{V}(x, \theta; \mathbf{b})$ approximately satisfies the Bellman equation (Bellman, 1957). Algorithm 1 is the parametric DP method with value function iteration for finite horizon problems with both multidimensional continuous and discrete states. (More detailed discussion of numerical DP can be found in Cai (2009), Judd (1998) and Rust (2008).) In the algorithm, n is the dimension for the continuous states x, and d is the dimension for discrete states $\theta \in \Theta = \{\theta^j : 1 \leq j \leq D\} \subset \mathbb{R}^d$, where D is the number of different discrete state vectors. The transition probabilities from θ^j to $\theta^{j'}$ for $1 \leq j, j' \leq D$ are given.

3.2. Approximation

An approximation scheme has two ingredients: basis functions and approximation nodes. Approximation nodes can be chosen as uniformly spaced nodes, Chebyshev nodes, or some other specified nodes. From the viewpoint of basis functions, approximation methods can be classified as either spectral methods or finite element methods. A spectral method uses globally nonzero basis functions $\phi_j(x)$ such that $\hat{V}(x;\mathbf{b}) = \sum_{j=0}^{m} b_j\phi_j(x)$. Examples of spectral methods include ordinary polynomial approximation, ordinary Chebyshev polynomial approximation, shape-preserving Chebyshev polynomial approximation (Cai and Judd, 2012b), and Chebyshev-Hermite approxiamtion (Cai and Judd, 2012c). In contrast, a finite element method uses local basis functions $\phi_j(x)$ that are nonzero over sub-domains of the approximation domain. Examples of finite element methods include piecewise linear interpolation, shape-preserving rational function spline interpolation (Cai and Judd, 2012a), cubic splines, and B-splines. See Cai (2009), Cai and Judd (2010), and Judd (1998) for more details.

3.2.1. Chebyshev Polynomial Approximation

Chebyshev polynomials on $[-1,1]$ are defined as $\mathcal{T}_j(x) = \cos(j\cos^{-1}(x))$, while general Chebyshev polynomials on $[x_{\min}, x_{\max}]$ are defined as $\mathcal{T}_j((2x - x_{\min} - x_{\max})/(x_{\max} - x_{\min}))$ for $j = 0, 1, 2, \ldots$. These polynomials are orthogonal under the weighted inner product: $\langle f, g\rangle = \int_{x_{\min}}^{x_{\max}} f(x)g(x)w(x)dx$ with the weighting function $w(x) = \left(1 - ((2x - x_{\min} - x_{\max})/(x_{\max} - x_{\min}))^2\right)^{-1/2}$. A degree m Chebyshev polynomial approximation for $V(x)$ on $[x_{\min}, x_{\max}]$ is

$$\hat{V}(x;\mathbf{b}) = \sum_{j=0}^{m} b_j \mathcal{T}_j\left(\frac{2x - x_{\min} - x_{\max}}{x_{\max} - x_{\min}}\right), \tag{1}$$

where $\mathbf{b} = \{b_j\}$ are the Chebyshev coefficients.

If we choose the Chebyshev nodes on $[x_{\min}, x_{\max}]$: $x^i = (z_i + 1)(x_{\max} - x_{\min})/2 + x_{\min}$ with $z_i = -\cos((2i-1)\pi/(2m'))$ for $i = 1, \ldots, m$', and Lagrange data $\{(x^i, v_i) : i = 1, \ldots, m'\}$ are given (where $v_i = V(x^i)$), then the coefficients b_j in (1) can be easily computed by the following formula,

$$\begin{aligned} b_0 &= \frac{1}{m'} \sum_{i=1}^{m'} v_i, \\ b_j &= \frac{2}{m'} \sum_{i=1}^{m'} v_i \mathcal{T}_j(z_i), \qquad j = 1, \ldots, m. \end{aligned} \tag{2}$$

The method is called the Chebyshev regression algorithm in Judd (1998).

When the number of Chebyshev nodes is equal to the number of Chebyshev coefficients, i.e., $m' = m + 1$, then the approximation (1) with the coefficients given by (2) becomes Chebyshev polynomial interpolation (which is a Lagrange interpolation), as $\hat{V}(x^i; \mathbf{b}) = v_i$, for $i = 1, \ldots, m'$.

3.2.2. Multidimensional Complete Chebyshev Approximation

In a d-dimensional approximation problem, let the domain of the value function be

$$\left\{x = (x_1, \ldots, x_n) : x_j^{\min} \le x_j \le x_j^{\max},\, j = 1, \ldots, n\right\},$$

for some real numbers $x_j^{\min}$ and $x_j^{\max}$ with $x_j^{\max} > x_j^{\min}$ for $j = 1, \ldots, n$. Let $x^{\min} = (x_1^{\min}, \ldots, x_n^{\min})$ and $x^{\max} = (x_1^{\max}, \ldots, x_n^{\max})$. Then we denote $[x^{\min}, x^{\max}]$ as the domain. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector of nonnegative integers. Let $\mathcal{T}_\alpha(z)$ denote the product $\mathcal{T}_{\alpha_1}(z_1) \cdots \mathcal{T}_{\alpha_n}(z_n)$ for $z = (z_1, \ldots, z_n) \in [-1, 1]^n$. Let

$$Z(x) = \left(\frac{2x_1 - x_1^{\min} - x_1^{\max}}{x_1^{\max} - x_1^{\min}}, \ldots, \frac{2x_n - x_n^{\min} - x_n^{\max}}{x_n^{\max} - x_n^{\min}} \right)$$

for any $x = (x_1, \ldots, x_n) \in [x^{\min}, x^{\max}]$.

Using these notations, the degree-m complete Chebyshev approximation for $V(x)$ is

$$\hat{V}_m(x;\mathbf{b}) = \sum_{0\le|\alpha|\le m} b_\alpha \mathcal{T}_\alpha\left(Z(x)\right), \tag{3}$$

where $|\alpha| = \sum_{j=1}^n \alpha_j$ for the nonnegative integer vector $\alpha = (\alpha_1, \ldots, \alpha_n)$. So the number of terms with $0 \le |\alpha| = \sum_{j=1}^n \alpha_i \le m$ is $\binom{m+n}{n}$ for the degree-m complete Chebyshev approximation in $\mathbb{R}^n$.

3.3. Numerical Integration

In the objective function of the Bellman equation, we often need to compute the conditional expectation of $V(x^+)$. When the random variable is continuous, we have to use numerical integration to compute the expectation. Gaussian quadature rules are often applied in computing the integration.

3.3.1. Gauss-Hermite Quadrature

In the expectation operator of the objective function of the Bellman equation, if the random variable has a normal distribution, then it will be good to apply the Gauss-Hermite quadrature formula to compute the numerical integration. That is, if we want to compute $\mathbb{E}\{f(Y)\}$ where Y has a distribution $\mathcal{N}(\mu, \sigma^2)$, then

$$\begin{aligned}
\mathbb{E}\{f(Y)\} &= (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} f(y) e^{-(y-\mu)^2/(2\sigma^2)} dy \\
&= (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} f(\sqrt{2}\,\sigma\, x + \mu) e^{-x^2} \sqrt{2}\sigma dx \\
&\doteq \pi^{-\frac{1}{2}} \sum_{i=1}^{m} \omega_i f(\sqrt{2}\sigma x_i + \mu),
\end{aligned}$$

where ω_i and x_i are the Gauss-Hermite quadrature with m weights and nodes over $(-\infty, \infty)$. See Cai (2009), Judd (1998), Stroud and Secrest (1966) for more details.

If Y is log normal, i.e., $\log(Y)$ has a distribution $\mathcal{N}(\mu, \sigma^2)$, then we can assume that $Y = e^X$ where $X \sim \mathcal{N}(\mu, \sigma^2)$, thus

$$\mathbb{E}\{f(Y)\} = \mathbb{E}\{f(e^X)\} \doteq \pi^{-\frac{1}{2}} \sum_{i=1}^{m} \omega_i f\left(e^{\sqrt{2}\sigma x_i + \mu}\right).$$

3.3.2. Multidimensional Integration

If we want to compute a multidimensional integration, we could apply the product rule. For example, suppose that we want to compute $\mathbb{E}\{f(X)\}$, where X is a random vector with multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$ over $\mathbb{R}^n$, where μ is the mean column vector and Σ is the covariance matrix, then we could do the Cholesky factorization first, i.e., find a lower triangular matrix L such that $\Sigma = LL^\top$. This is feasible as Σ must be a positive semi-definite matrix from the covariance property. Thus,

$$\begin{aligned}
\mathbb{E}\{f(X)\} &= ((2\pi)^n \det(\Sigma))^{-1/2} \int_{\mathbb{R}^n} f(y) e^{-(y-\mu)^\top \Sigma^{-1}(y-\mu)/2} dy \\
&= \left((2\pi)^n \det(L)^2\right)^{-1/2} \int_{\mathbb{R}^n} f\left(\sqrt{2}Lx + \mu\right) e^{-x^\top x} 2^{n/2} \det(L) dx \\
&\doteq \pi^{-\frac{n}{2}} \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \omega_{i_1} \cdots \omega_{i_d} f\Bigg(\sqrt{2} l_{1,1} x_{i_1} + \mu_1, \\
&\qquad \sqrt{2}(l_{2,1} x_{i_1} + l_{2,2} x_{i_2}) + \mu_2, \cdots, \sqrt{2}\left(\sum_{j=1}^{n} l_{n,j} x_{i_j}\right) + \mu_n\Bigg), \quad (4)
\end{aligned}$$

where ω_i and x_i are the Gauss-Hermite quadrature with m weights and nodes over $(-\infty, \infty)$, $l_{i,j}$ is the (i,j)-element of L, and $\det(\cdot)$ means the matrix determinant operator.

4. Parallel Dynamic Programming

The numerical DP algorithms can be applied easily in the HTCondor MW system for DP problems with multidimensional continuous and discrete states. To solve these problems, numerical DP algorithms with value

function iteration have the maximization step that is mostly time-consuming in numerical DP. That is,

$$v_{i,j} = \max_{a \in \mathcal{D}(x^i, \theta^j, t)} u(x^i, \theta^j, a) + \beta \mathbb{E}\{\hat{V}(x^+, \theta^+; \mathbf{b}^{t+1})\},$$

for each continuous state point x^i in the finite set $\mathbb{X}_t \subset \mathbb{R}^n$ and each discrete state vector $\theta^j \in \Theta$, where N_t is the number of points of $\mathbb{X}_t$ and D is the number of points of Θ. So there are $N_t \times D$ small-size maximization problems. Thus, if the $N_t \times D$ is large (that is very possible in high-dimensional problems), then it will take a huge amount of time to do the DP maximization step. However, these $N_t \times D$ small-size maximization problems can be naturally parallelized in the HTCondor MW system, in which one or several maximization problem(s) could be treated as one task.

4.1. Type-I Parallelization

When D is large but N_t has a medium-size, we could separate the $N_t \times D$ maximization problems into D tasks, where each task corresponds to a discrete state vector θ^j and all continuous state nodes set $\mathbb{X}_t$. Algorithm 2 is the architecture for the master processor, and Algorithm 3 is the corresponding architecture for the workers.

4.2. Type-II Parallelization

If there are too many nodes for continuous states, i.e., N_t is large, or the maximization step for each node is a bit time-consuming, then it will be better to break the task for one θ^j into subtasks. If the fitting method requires all points $\{(x^i, v_{i,j})\colon\ 1 \le i \le N_t\}$ to contruct the approximation, then each worker cannot do step 3 and 4 along with step 1 and 2 in Algorithm 3, as it has only an incomplete set of approximation nodes x^i for one given θ^j. Thus we have Algorithm 4 for the master process and Algorithm 5 for the workers.

Algorithm 2 Type-I Parallel Dynamic Programming with Value Function Iteration for the Master

Initialization. Given a finite set of $\theta \in \Theta = \{\theta^j : 1 \leq j \leq D\} \subset \mathbb{R}^d$. Set $\mathbf{b}^T$ as the parameters of the terminal value function. For $t = T-1, T-2, \ldots, 0$, iterate through steps 1 and 2.

Step 1. Separate the maximization step into D tasks, one task per $\theta \in \Theta$. Each task contains parameters $\mathbf{b}^{t+1}$, stage number t and the corresponding task identity for some θ^j. Then send these tasks to the workers.

Step 2. Wait until all tasks are done by the workers. Then collect parameters $\mathbf{b}_j^t$ from the workers, for all $1 \leq j \leq D$, and let $\mathbf{b}^t = \left\{\mathbf{b}_j^t : 1 \leq j \leq D\right\}$.

Algorithm 3 Type-I Parallel Dynamic Programming with Value Function Iteration for the Workers

Initialization. Given a finite set of $\theta \in \Theta = \{\theta^j : 1 \leq j \leq D\} \subset \mathbb{R}^d$ and the probability transition matrix $P = \left(p_{j,j'}\right)_{D\times D}$ where $p_{j,j'}$ is the transition probability from $\theta_j \in \Theta$ to $\theta_{j'} \in \Theta$ for $1 \leq j, j' \leq D$. Choose a functional form for $\hat{V}(x, \theta; \mathbf{b})$ for all $\theta \in \Theta$.

Step 1. Get parameters $\mathbf{b}^{t+1}$, stage number t and the corresponding task identity for one $\theta^j \in \Theta$ from the master, and then choose the approximation grid, $\mathbb{X}_t = \{x_t^i : 1 \leq i \leq N_t\} \subset \mathbb{R}^n$.

Step 2. For this given θ^j, compute

$$v_{i,j} = \max_{a \in \mathcal{D}(x^i, \theta^j, t)} u(x^i, \theta^j, a) + \beta \mathbb{E}\{\hat{V}(x^+, \theta^+; \mathbf{b}^{t+1})\},$$

for each $x^i \in \mathbb{X}_t$, $1 \leq i \leq N_t$, where the next-stage discrete state $\theta^+ \in \Theta$ is random with probability mass function $\mathbb{P}(\theta^+ = \theta_{j'} \mid \theta_j) = p_{j,j'}$ for each $\theta_{j'} \in \Theta$, and x^+ is the next-stage state transition from x^i and may be also random.

Step 3. Using an appropriate approximation method, compute $\mathbf{b}_j^t$ such that $\hat{V}(x, \theta^j; \mathbf{b}_j^t)$ approximates $\{(x^i, v_{i,j})\colon\ 1 \leq i \leq N_t\}$, i.e., $v_{i,j} \approx \hat{V}(x^i, \theta^j; \mathbf{b}_j^t)$ for all $x^i \in \mathbb{X}_t$.

Step 4. Send $\mathbf{b}_j^t$ and the corresponding task identity for θ^j to the master.

Algorithm 4 Type-II Parallel Dynamic Programming with Value Function Iteration for the Master

Initialization. Given a finite set of $\theta \in \Theta = \{\theta^j : 1 \leq j \leq D\} \subset \mathbb{R}^d$. Choose a functional form for $\hat{V}(x, \theta; \mathbf{b})$ for all $\theta \in \Theta$, and choose the approximation grid, $\mathbb{X}_t = \{x_t^i : 1 \leq i \leq N_t\} \subset \mathbb{R}^n$. Set $\mathbf{b}^T$ as the parameters of the terminal value function. For $t = T-1, T-2, \ldots, 0$, iterate through steps 1 and 2.

Step 1. Separate $\mathbb{X}_t$ into M disjoint subsets with almost equal sizes: $\mathbb{X}_{t,1}, \ldots, \mathbb{X}_{t,M}$, and separate the maximization step into $M \times D$ tasks, one task per $(\mathbb{X}_{t,m}, \theta^j)$ with $\theta^j \in \Theta$, for $m = 1, \ldots, M$ and $j = 1, \ldots, D$. Each task contains the parameters $\mathbf{b}^{t+1}$, the stage number t and the corresponding task identity for $(\mathbb{X}_{t,m}, \theta^j)$. Then send these tasks to the workers.

Step 2. Wait until all tasks are done by the workers. Then collect all $v_{i,j}$ from the workers, for $1 \leq i \leq N_t$, $1 \leq j \leq D$.

Step 3. Using an appropriate approximation method, for each $\theta^j \in \Theta$, compute $\mathbf{b}_j^t$ such that $\hat{V}(x, \theta_j; \mathbf{b}_j^t)$ approximates $\{(x^i, v_{i,j})\colon\ 1 \leq i \leq N_t\}$, i.e., $v_{i,j} \approx \hat{V}(x^i, \theta^j; \mathbf{b}_j^t)$ for all $x^i \in \mathbb{X}_t$. Let $\mathbf{b}^t = \left\{\mathbf{b}_j^t : 1 \leq j \leq D\right\}$.

Algorithm 5 Type-II Parallel Dynamic Programming with Value Function Iteration for the Workers

Initialization. Given a finite set of $\theta \in \Theta = \{\theta^j : 1 \le j \le D\} \subset \mathbb{R}^d$ and the probability transition matrix $P = \left(p_{j,j'}\right)_{D\times D}$ where $p_{j,j'}$ is the transition probability from $\theta^j \in \Theta$ to $\theta^{j'} \in \Theta$ for $1 \le j, j' \le D$. Choose the approximation grid, $\mathbb{X}_t = \{x_t^i : 1 \le i \le N_t\} \subset \mathbb{R}^n$, which is the same with the set $\mathbb{X}_t$ in the master.

Step 1. Get the parameters $\mathbf{b}^{t+1}$, stage number t and the corresponding task identity for one $(\mathbb{X}_{t,m}, \theta^j)$ with $\theta^j \in \Theta$ from the master.

Step 2. For this given θ^j, compute

$$v_{i,j} = \max_{a\in\mathcal{D}(x^i,\theta^j,t)} u(x^i, \theta^j, a) + \beta\mathbb{E}\{\hat{V}(x^+, \theta^+; \mathbf{b}^{t+1})\},$$

for all $x^i \in \mathbb{X}_{t,m}$, where the next-stage discrete state $\theta^+ \in \Theta$ is random with probability mass function $\mathbb{P}(\theta^+ = \theta_{j'} \mid \theta_j) = p_{j,j'}$ for each $\theta^{j'} \in \Theta$, and x^+ is the next-stage state transition from x^i and may be also random.

Step 3. Send $v_{i,j}$ for these given $x^i \in \mathbb{X}_{t,m}$ and θ^j, to the master process.

If it is quick to compute $\mathbf{b}_j^t$ in the fitting step (e.g., Chebyshev polynomial approximation using Chebyshev regression algorithm), then we can just let the master do the fitting step like the type-II parallel DP algorithm. However, if the fitting step is time-consuming, then the master could send these fitting jobs for each discrete state θ^j to the workers, and then collect the the new approximation parameters.

4.3. Sparsity

In many cases, the probability transition matrix is sparse. Thus, when we compute the expectation in the objective function of the maximzation problems, we only need to calculate the value functions with nonzero transition probabilities for the given θ^j. That is,

$$\mathbb{E}\{\hat{V}(x^+, \theta^+; \mathbf{b}^{t+1})\} = \sum_{1\le j'\le D,\, p_{j,j'}\neq 0} p_{j,j'}\mathbb{E}\{\hat{V}(x^+, \theta_{j'}; \mathbf{b}_{j'}^{t+1})\}.$$

This follows that the master just need to include those $\mathbf{b}_{j'}^{t+1}$ with nonzero transition probability $p_{j,j'}$ (instead of the whole set of parameters, $\mathbf{b}^{t+1}$) in the tasks corresponding to θ^j, i.e., $\left\{\mathbf{b}_{j'}^{t+1} : p_{j,j'} > 0,\, 1 \leq j' \leq D\right\}$ where $p_{j,j'} = \mathbb{P}(\theta^+ = \theta_{j'} \mid \theta_j)$, and then send the subset of $\mathbf{b}^{t+1}$ to the workers in Step 1 of Algorithm 2 or 4, in order to save data transportation amount between the master and the workers.

5. Application on Stochastic Optimal Growth Models

We consider a multi-dimensional stochastic optimal growth problem. We assume that there are d sectors, and let $k_t = (k_{t,1}, \ldots, k_{t,d})$ denote the capital stocks of these sectors which is a d-dimensional continuous state vector at time t. Let $\theta_t = (\theta_{t,1}, \ldots, \theta_{t,d}) \in \Theta = \{\theta_t^j : 1 \leq j \leq D\} \subset \mathbb{R}^d$ denote current productivity levels of the sectors which is a d-dimensional discrete state vector at time t, and assume that θ_t follows a Markov process with a stable probability transition matrix, denoted as $\theta_{t+1} = g(\theta_t, \xi_t)$ where ξ_t are i.i.d. disturbances. Let $l_t = (l_{t,1}, \ldots, l_{t,d})$ denote elastic labor supply levels of the sectors which is a d-dimensional continuous control vector variable at time t. Assume that the net production function of sector i at time t is $f(k_{t,i}, l_{t,i}, \theta_{t,i})$, for $i = 1, \ldots, d$. Let $c_t = (c_{t,1}, \ldots, c_{t,d})$ and $I_t = (I_{t,1}, \ldots, I_{t,d})$ denote, respectively, consumption and investment of the sectors at time t. We want to find an optimal consumption and labor supply decisions such that expected total utility over a finite-horizon time is

maximized, i.e.,

$$\begin{aligned} V_0(k_0,\theta_0) \;=\; & \max_{k_t,I_t,c_t,l_t} \; \mathbb{E}\left\{\sum_{t=0}^{T-1}\beta^t u(c_t,l_t)+\beta^T V_T(k_T,\theta_T)\right\}, \\ & \text{s.t.} \quad k_{t+1,j}=(1-\delta)k_{t,j}+I_{t,j}+\epsilon_{t,j}, \quad j=1,\ldots,d, \\ & \qquad \Gamma_{t,j}=\frac{\zeta}{2}k_{t,j}\left(\frac{I_{t,j}}{k_{t,j}}-\delta\right)^2, \quad j=1,\ldots,d, \\ & \qquad \sum_{j=1}^{d}\left(c_{t,j}+I_{t,j}-\delta k_{t,j}\right)=\sum_{j=1}^{d}\left(f(k_{t,j},l_{t,j},\theta_{t,j})-\Gamma_{t,j}\right), \\ & \qquad \theta_{t+1}=g(\theta_t,\xi_t), \end{aligned}$$

where k_0 and θ_0 are given, δ is the depreciation rate of capital, $\Gamma_{t,j}$ is the investment adjustment cost of sector j, and ζ governs the intensity of the friction, $\epsilon_t=(\epsilon_{t,1},\ldots,\epsilon_{t,d})$ are serially uncorrellated i.i.d. disturbances with $\mathbb{E}\{\epsilon_{t,i}\}=0$, and $V_T(k,\theta)$ is a given terminal value function. For this finite-horzion model, Cai and Judd (2012c) solve some of its simplified problem. An infinite-horizon version of this model is introduced in Den Haan et al (2011), Juillard and Villemot (2011), and a nonlinear programming method for dynamic programming is introduced in Cai et al. (2012d) to solve the multi-country growth model with infinite horizon.

5.1. Dynamic Programming Model

The DP formulation of the multi-dimensional stochastic optimal growth problem is

$$\begin{aligned} V_t(k,\theta) \;=\; & \max_{c,l,I} \; u(c,l)+\beta\mathbb{E}\left\{V_{t+1}(k^+,\theta^+)\mid\theta\right\}, \\ & \text{s.t.} \quad k_j^+=(1-\delta)k_j+I_j+\epsilon_j, \quad j=1,\ldots,d, \\ & \qquad \Gamma_j=\frac{\zeta}{2}k_j\left(\frac{I_j}{k_j}-\delta\right)^2, \quad j=1,\ldots,d, \\ & \qquad \sum_{j=1}^{d}\left(c_j+I_j-\delta k_j\right)=\sum_{j=1}^{d}\left(f(k_j,l_j,\theta_j)-\Gamma_j\right), \\ & \qquad \theta^+=g(\theta,\xi_t), \end{aligned}$$

for $t = 0, \ldots, T-1$, where $k = (k_1, \ldots, k_d)$ is the continuous state vector and $\theta = (\theta_1, \ldots, \theta_d) \in \Theta = \{(\vartheta_{j,1}, \ldots, \vartheta_{j,d}) : 1 \leq j \leq D\}$ is the discrete state vector, $c = (c_1, \ldots, c_d)$, $l = (l_1, \ldots, l_d)$, and $I = (I_1, \ldots, I_d)$ are control variables, $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$ are i.i.d. disturbance with mean 0, and $k^+ = (k_1^+, \ldots, k_d^+)$ and $\theta^+ = \left(\theta_1^+, \ldots, \theta_d^+\right) \in \Theta$ are the next-stage state vectors. Numerically, $V(k, \theta)$ is approximated with given values at finite nodes, so the approximation is only good at a finite range. That is, the state variable must be in a finite range $[\underline{k}, \bar{k}]$, then we should have the restriction $k^+ \in [\underline{k}, \bar{k}]$. Here $\underline{k} = (\underline{k}_1, ..., \underline{k}_d), \bar{k} = (\bar{k}_1, ..., \bar{k}_d)$, and $k^+ \in [\underline{k}, \bar{k}]$ denotes that $k_i^+ \in [\underline{k}_i, \bar{k}_i]$ for all $1 \leq i \leq d$. Moreover, we should add $c > 0$ and $l > 0$ in the constraints.

5.2. Numerical Example

In the following numerical example, we see the application of parallelization of numerical DP algorithms for the DP model of the multi-dimensional stochastic optimal growth problem. We let $T = 5$, $\beta = 0.8$, $\delta = 0.025$, $\zeta = 0.5$, $[\underline{k}, \bar{k}] = [0.2, 3.0]^d$, $f(k_i, l_i, \theta_i) = \theta_i A k_i^{\psi} l_i^{1-\psi}$ with $\psi = 0.36$ and $A = (1-\beta)/(\psi\beta) = 1$, for $i = 1, \ldots, d$, and

$$u(c, l) = \sum_{i=1}^{d} \left[\frac{(c_i/A)^{1-\gamma} - 1}{1-\gamma} - (1-\psi) \frac{l_i^{1+\eta} - 1}{1+\eta} \right],$$

with $\gamma = 2$ and $\eta = 1$.

In this example, we let $d = 4$. So this is a DP example with 4-dimensional continuous states and 4-dimensional discrete states. Here we assume that the possible values of θ_i and θ_i^+ are

$$\vartheta_1 = 0.85,\ \vartheta_2 = 0.9,\ \vartheta_3 = 0.95,\ \vartheta_4 = 1.0,\ \vartheta_5 = 1.05,\ \vartheta_6 = 1.1,\ \vartheta_7 = 1.15,$$

and the probability transition matrix from θ_i to θ_i^+ is a 7×7 tridiagonal

matrix:

$$P = \begin{bmatrix} 0.75 & 0.25 & & & & & \\ 0.25 & 0.50 & 0.25 & & & & \\ & 0.25 & 0.50 & 0.25 & & & \\ & & & 0.25 & \ddots & \ddots & \\ & & & & \ddots & 0.50 & 0.25 \\ & & & & & 0.25 & 0.75 \end{bmatrix},$$

for each $i = 1, \ldots, 4$, and we assume thar $\theta_1^+, \ldots, \theta_d^+$ are independent of each other. That is,

$$\Pr[\theta^+ = (\vartheta_{i_1}, \ldots, \vartheta_{i_4}) \mid \theta = (\vartheta_{j_1}, \ldots, \vartheta_{j_4})] = P_{i_1,j_1} P_{i_2,j_2} P_{i_3,j_3} P_{i_4,j_4},$$

where P_{i_α,j_α} is the (i_α, j_α) element of P, for any $i_\alpha, j_\alpha = 1, \ldots, 7$, $\alpha = 1, \ldots, 4$.

In addition, we assume that $\epsilon_1, \ldots, \epsilon_4$ are i.i.d., and each ϵ_i has 3 discrete values:

$$\delta_1 = -0.01,\ \delta_2 = 0.0,\ \delta_3 = 0.01,$$

while their probabilities are $q_1 = 0.25$, $q_2 = 0.5$ and $q_3 = 0.25$, respectively. That is,

$$\Pr[\epsilon = (\delta_{n_1}, \ldots, \delta_{n_4})] = q_{n_1} q_{n_2} q_{n_3} q_{n_4},$$

for any $n_\alpha = 1, 2, 3$, $\alpha = 1, \ldots, 4$. Moreover, $\epsilon_1, \ldots, \epsilon_4$ are assumed to be independent of $\theta_1^+, \ldots, \theta_4^+$.

Therefore,

$$\begin{aligned} & \mathbb{E}\{V(k^+, \theta^+) \mid \theta = (\vartheta_{j_1}, \ldots, \vartheta_{j_4})\} \\ = & \sum_{n_1,n_2,n_3,n_4=1}^{3} q_{n_1} q_{n_2} q_{n_3} q_{n_4} \sum_{i_1,i_2,i_3,i_4=1}^{6} P_{i_1,j_1} P_{i_2,j_2} P_{i_3,j_3} P_{i_4,j_4} \times \\ & V(\hat{k}_1^+ + \delta_{n_1}, \ldots, \hat{k}_4^+ + \delta_{n_4}, \vartheta_{i_1}, \ldots, \vartheta_{i_4}), \end{aligned} \tag{5}$$

where $\hat{k}_\alpha^+ = (1-\delta)k_\alpha + I_\alpha$, for any $i_\alpha = 1, \ldots, 7$, $\alpha = 1, \ldots, 4$.

From the formula (5), it seems that we should compute the value function V at a large number of points up to $3^4 * 7^4 = 194,481$ in order to evaluate the expectation. But in fact, we can take advantage of the sparsity of the probability transition matrix P. After canceling the zero probability terms, the evaluation of the expectation will need to compute the value function at a number of points ranging from $3^4 * 2^4 = 1,296$ to $3^4 * 3^4 = 6,561$, which is far less that the case without using the sparsity. Moreover, the data transportation amount between the master and workers is also far less than the case without using the sparsity.

The continuous value function approximation is the complete degree-6 Chebyshev polynomial approximation method (3) with $7^4 = 2401$ Chebyshev nodes for continuous state variables, the optimizer is NPSOL (Gill, P., et al., 1994), and the terminal value function is chosen as

$$V_T(k, \theta) = u(f(k, \mathbf{e}, \mathbf{e}), \mathbf{e})/(1-\beta),$$

where $\mathbf{e}$ is the vector with 1's everywhere. Here $\mathbf{e}$ is chosen because it is the steady state labor supply for the corresponding infinite-horizon problem and is also the average value of θ.

5.3. Parallelization Results

We use the master algorithm 2 and the worker algorithm 3 to solve the optimal growth problem. Since the number of possible values of θ_i is 7 for $i = 1, \ldots, 4$, the total number of HTCondor-MW tasks for one value function iteration is $7^4 = 2401$, and each task computes 2401 small-size maximzation problems as there are 2401 Chebyshev nodes.

Under HTCondor, we assign 50 workers to do this parallel work. Table 1 lists some statistics of our parallel DP algorithm under HTCondor-MW

Table 1: Statistics of Parallel DP under HTCondor-MW for the growth problem

Wall clock time for all 3 VFIs	8.28 hours
Wall clock time for 1st VFI	0.34 hours
Wall clock time for 2nd VFI	3.92 hours
Wall clock time for 3rd VFI	4.01 hours
Total time workers were up (alive)	16.9 days
Total cpu time used by all workers	16.5 days
Minimum task cpu time	6.4 seconds
Maximum task cpu time	661 seconds
Average task wall clock time	199 seconds
Standard deviation task wall clock time	165 seconds
Mean uptime for the workers	8.13 hours
Standard deviation uptime for the workers	0.14 hours
Mean cpu time for the workers	7.92 hours
Standard deviation cpu time for the workers	0.23 hours
Number of (different) workers	50
Average Number Present Workers	49
Overall Parallel Performance	98.6%

system for the growth problem after running 3 value function iterations (VFI). The last line of Table 1 shows that the parallel efficiency of our parallel numerical DP method is very high (up to 98.6%) for this example. We see that the total cpu time used by all workers to solve the optimal growth problem is nearly 17 days, i.e., it will take nearly 17 wall clock days to solve the problem without using parallellization. However, it takes only 8.28 wall clock hours to solve the problem if we use the parallel algorithm and 50 worker processors.

Table 2 gives the parallel efficiency with various number of worker processors for this optimal growth model. We see that it has an almost linear speed-up when we add the number of worker processors from 50 to 200. We see that the wall clock time to solve the problem is only 2.26 hours now if the number of worker processors increases to 200.

Table 2: Parallel efficiency for various number of worker processors

# Worker processors	Parallel efficiency	Average task wall clock time (second)	Total wall clock time (hour)
50	98.6%	199	8.28
100	97%	185	3.89
200	91.8%	186	2.26

6. Applications on Dynamic Portfolio Problem with Transaction Costs

We consider a dynamic portfolio problem with transaction costs. We assume that an investor begins with some initial wealth W_0, invests it in several assets, and manages it at every time t so as to maximize the expected utility of wealth at a terminal time T. We assume a power utility function for terminal wealth, $u(W) = W^{1-\gamma}/(1-\gamma)$ where $\gamma > 0$ and $\gamma \neq 1$. Let $R = (R_1, \ldots, R_n)^\top$ be the random one-period return of n risky assets, and R_f be the return of the riskless asset. The portfolio share for asset i at the beginning of period t is denoted $x_{t,i}$, and let $x_t = (x_{t,1}, \ldots, x_{t,n})^\top$. The difference between wealth and the wealth invested in stocks is invested in bonds. At the beginning of every period, the investor has a chance to re-balance the portfolio with a proportional transaction cost rate τ for buying or selling stocks. Let $\delta^+_{t,i}W$ denote the amount of asset i purchased, expressed as a fraction of wealth, and let $\delta^-_{t,i}W$ denote the amount sold, where $\delta^+_{t,i}, \delta^-_{t,i} \geq 0$, for periods $t = 0, \ldots, T-1$.

We assume that the riskless return R_f and the risky assets' return R may be dependent on a discrete time stochastic process θ_t (could be a vector), denoted by $R_f(\theta_t)$ and $R(\theta_t)$ respectively, for $t = 0, \ldots, T-1$. Then the

dynamic portfolio problem becomes

$$
\begin{aligned}
V_0(W_0, x_0, \theta_0) \;=\; & \max_{\delta^+, \delta^- \geq 0} \; \mathbb{E}\left\{u(W_T)\right\}, \qquad (6)\\
& \text{s.t.} \quad W_{t+1} = \mathbf{e}^\top X_{t+1} + R_f(\theta_t)(1 - \mathbf{e}^\top x_t - y_t)W_t,\\
& \qquad\; X_{t+1,i} = R_i(\theta_t)(x_{t,i} + \delta^+_{t,i} - \delta^-_{t,i})W_t,\\
& \qquad\; y_t = \mathbf{e}^\top(\delta^+_t - \delta^-_t + \tau(\delta^+_t + \delta^-_t)),\\
& \qquad\; x_{t+1,i} = X_{t+1,i}/W_{t+1},\\
& \qquad\; \theta_{t+1} = g(\theta_t, \xi_t),\\
& \qquad\; t = 0, \ldots, T-1; \quad i = 1, \ldots, n,
\end{aligned}
$$

where $\mathbf{e}$ is the column vector with 1's everywhere, $X_{t+1} = (X_{t+1,1}, \ldots, X_{t+1,n})^\top$, $\delta^+_t = (\delta^+_{t,1}, \ldots, \delta^+_{t,n})^\top$, and $\delta^-_t = (\delta^-_{t,1}, \ldots, \delta^-_{t,n})^\top$. Here, W_{t+1} is time $t+1$ wealth, $X_{t+1,i}$ is time $t+1$ wealth in asset i, $y_t W_t$ is the change in bond holding, and $x_{t+1,i}$ is the allocation of risky asset i.

6.1. Dynamic Programming Model

The DP model of the multi-stage portfolio optimization problem (6) is

$$
V_t(W, x, \theta) \;=\; \max_{\delta^+, \delta^- \geq 0} \; \mathbb{E}\left\{V_{t+1}(W^+, x^+, \theta^+)\right\},
$$

for $t = 0, 1, \ldots, T-1$, while the terminal value function is $V_T(W, x, \theta) = W^{1-\gamma}/(1-\gamma)$. Given the isoelasticity of V_T, we know that the value function can be rewritten as

$$
V_t(W_t, x_t, \theta_t) = W_t^{1-\gamma} \cdot H_t(x_t, \theta_t),
$$

for some functions $H_t(x_t, \theta_t)$, where W_t and x_t are respectively wealth and allocation fractions of stocks right before re-balancing at stage $t = 0, 1, \ldots, T$,

and

$$
\begin{aligned}
H_t(x,\theta) \;=\; & \max_{\delta^+,\delta^-} \;\; \mathbb{E}\left\{\Pi^{1-\gamma}\cdot H_{t+1}(x^+,\theta^+)\right\}, \qquad (7)\\
& \text{s.t.} \quad \delta^+ \geq 0,\ \delta^- \geq 0,\\
& \qquad\; x+\delta^+-\delta^- \geq 0,\\
& \qquad\; y \leq 1-\mathbf{e}^\top x,\\
& \qquad\; \theta^+ = g(\theta,\xi_t),
\end{aligned}
$$

where $H_T(x,\theta) = 1/(1-\gamma)$, and

$$
\begin{aligned}
y &\equiv \mathbf{e}^\top(\delta^+-\delta^-+\tau(\delta^++\delta^-)),\\
s_i &\equiv R_i(\theta)(x_i+\delta_i^+-\delta_i^-),\\
\Pi &\equiv \mathbf{e}^\top s + R_f(\theta)(1-\mathbf{e}^\top x-y),\\
x_i^+ &\equiv s_i/\Pi,
\end{aligned}
$$

for $i = 1,\ldots,n$ and $t = 0,1,\ldots,T-1$. A detailed discussion of this model can be seen in Cai (2009).

Since W_t and x_t are separable, we can just assume that $W_t = 1$ dollar for simplicity. Thus, at time t, δ^+ and δ^- are the amounts for buying and selling stocks respectively, y is the change in bond holding, s is the next-stage amount vector of dollars on the stocks, Π is the total wealth at the next stage, and x^+ is the new fraction vector of the stocks at the next stage. In this model, the state variables, x and x^+, are continuous in $[0,1]^n$.

6.2. Numerical Examples

We choose a portfolio with $n = 6$ stocks and one riskless bond. The investor wants to maximize the expected terminal utility after $T = 6$ years with the terminal utility, $u(W) = W^{1-\gamma}/(1-\gamma)$, with $\gamma = 4$. At the beginning of each year $t = 0,1,\ldots T-1$, the investor has a chance to rebalance the

portfolio with a proportional transaction cost rate $\tau = 0.002$ for buying or selling stocks. We assume that the stock returns are independent each other, and stock i has a log-normal annual return, i.e., $\log(R_i) \sim \mathcal{N}(\mu_i - \sigma_i^2/2,\, \sigma_i^2)$ with $\mu_i = 0.07$ and $\sigma_i = 0.25$, for $i = 1, \ldots, n$. We assume that the bond has a riskless annual return $\exp(r_t)$, while the interest rate r_t is a discrete Markov chain, with $r_t = 0.01, 0.02, 0.03, 0.04$ or 0.05, and its transition probability matrix is

$$P = \begin{bmatrix} 0.7 & 0.3 & & & \\ 0.3 & 0.4 & 0.3 & & \\ & 0.3 & 0.4 & 0.3 & \\ & & 0.3 & 0.4 & 0.3 \\ & & & 0.3 & 0.7 \end{bmatrix}.$$

We use the degree-4 complete Chebyshev polynomials (3) as the approximation method, and choose 5 Chebyshev nodes on each dimension, so that we can apply the Chebyshev regression algorithm to compute the approximation coefficients in the fitting step of numerical DP algorithms. Thus, the number of approximation nodes is $5^6 = 15,625$ for each discrete state, so the total number of small-size maximzation problems for one value function iteration is $5 \times 5^6 = 78,125$. We use the product Gaussian-Hermite quadrature formula (4) with 5 nodes for each dimension, so the number of quadrature nodes is $5^6 = 15,625$ for each discrete state. Therefore, after using the sparsity of the probability transition matrix, the computation of the expectation in the objective function of the maximization problem (7) includes $2 \times 5^6 = 31,250$ or $3 \times 5^6 = 46,875$ evaluations of the approximated value function at stage $t + 1$ for each approximation node. We use NPSOL as our optimization solver for solving the maximization problem (7) .

Table 3: Statistics of Parallel DP under HTCondor-MW for the 7-asset portfolio problem with stochastic interest rate

Wall clock time for all 6 VFIs	3.6 hours
Wall clock time for 1st VFI	4.8 minutes
Wall clock time for 2nd VFI	43.4 minutes
Wall clock time for 3rd VFI	40.6 minutes
Wall clock time for 4th VFI	41.5 minutes
Wall clock time for 5th VFI	42.9 minutes
Wall clock time for 6th VFI	43.7 minutes
Total time workers were up (alive)	29.3 days
Total cpu time used by all workers	27.4 days
Minimum task cpu time	3 secponds
Maximum task cpu time	286 seconds
Mean uptime for the workers	3.5 hours
Standard deviation uptime for the workers	23.5 minutes
Mean cpu time for the workers	3.3 hours
Standard deviation cpu time for the workers	22.6 minutes
Number of (different) workers	200
Average Number Present Workers	194
Overall Parallel Performance	94.2%

6.3. Parallelization Results

We apply Algorithm 4 and 5 to solve the high-dimensional dynamic portfolio problem. Here we set the number of HTCondor-MW tasks for each discrete state as 625 so that each HTCondor-MW task contains 25 small-size maximzation problems, and the total number of HTCondor-MW tasks is $625 \times 5 = 3,125$. Under HTCondor, we assign 200 workers to do this parallel work.

Table 3 lists some statistics of our parallel DP algorithm under HTCondor-MW system for the portfolio problem with six stocks and one bond with stochastic interest rates. We see that the parallel efficiency of our parallel numerical DP method is up to 94.2% for this example, even when we use 200 workers. Moreover, we see that the total cpu time used by all workers to solve the dynamic portfolio optimization problem is more than 27 days, i.e., it will take more than 27 wall clock days to solve the problem without

using parallellization. However, it takes only about 3.6 wall clock hours to solve the problem if we use the type-II parallel DP algorithm and 200 worker processors.

7. Conclusion

This paper presents the parallel dynamic programming methods in HT-Condor Master-Worker system, and has shown that they have high parallel efficiency and can be applied to solve very time-consuming high-dimensional dynamic programmming problems efficiently.

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