Linear-Quadratic DP

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Continuous states: Linear-Quadratic Dynamic Programming

- Problem:

$$\max_{u_t} \sum_{t=0}^{T} \beta^t \left(\frac{1}{2} x_t^\top Q_t x_t + u_t^\top R_t x_t + \frac{1}{2} u_t^\top S_t u_t \right) + \frac{1}{2} x_{T+1}^\top W_{T+1} x_{T+1}$$

$$x_{t+1} = A_t x_t + B_t u_t,$$

- Bellman equation:

$$V(x,t) = \max_{u_t} \frac{1}{2} x^\top Q_t x + u_t^\top R_t x + \frac{1}{2} u_t^\top S_t u_t + \beta V(A_t x + B_t u_t, t+1).$$

Finite Horizon

- Key fact: We know solution is quadratic, solve for the unknown coefficients
- The guess $V(x,t) = \frac{1}{2}x^\top W_{t+1}x$ implies f.o.c.

$$0 = S_t u_t + R_t x + \beta B_t^\top W_{t+1}(A_t x + B_t u_t),$$

  - F.o.c. implies the time t control law

$$\begin{aligned} u_t &= -(S_t + \beta B_t^\top W_{t+1} B_t)^{-1}(R_t + \beta B_t^\top W_{t+1} A_t)x \\ &\equiv U_t x. \end{aligned}$$

  - Substitution into Bellman implies *Riccati equation* for W_t:

$$W_t = Q_t + \beta A_t^\top W_{t+1} A_t + (\beta B_t^\top W_{t+1} A_t + R_t^\top)U_t$$

  - Value function method iterates (12.6.4) beginning with known W_{T+1} matrix of coefficients.

Autonomous, Infinite-horizon case

- Assume $R_t = R$, $Q_t = Q$, $S_t = S$, $A_t = A$, and $B_t = B$
- The guess $V(x) \equiv \frac{1}{2}x^\top Wx$ implies the *algebraic Riccati equation*

$$\begin{aligned} W &= Q + \beta A^\top WA - (\beta B^\top WA + R^\top) \\ &\quad \times (S + \beta B^\top WB)^{-1}(\beta B^\top WB + R^\top). \end{aligned}$$

- Two convergent procedures:
  - Value function iteration:

$$\begin{aligned} W_0 &: \text{a negative definite initial guess} \\ W_{k+1} &= Q + \beta A^\top W_k A - (\beta B^\top W_k A + R^\top) \\ &\quad \times (S + \beta B^\top W_k B)^{-1}(\beta B^\top W_k B + R^\top). \end{aligned}$$

  - Policy function iteration:

$$\begin{aligned} W_0 &: \text{initial guess} \\ U_{i+1} &= -(S + \beta B^\top W_i B)^{-1}(R + \beta B^\top W_i A) : \text{optimal policy for } W_i \\ W_{i+1} &= \frac{\frac{1}{2}Q + \frac{1}{2}U_{i+1}^\top S U_{i+1} + U_{i+1}^\top R}{1-\beta} : \text{value of } U_i \end{aligned}$$

Lessons

- We used a functional form to solve the dynamic programming problem
- We solve for unknown coefficients
- We did not restrict either the state or control set
- Can we do this in general?