Dynamic Programming with Shape Preservation[1]

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[1] Joint work with Yongyang Cai.

Shape-preserving Chebyshev Interpolation

LP model for shape-preserving Chebyshev Interpolation:

$$
\begin{aligned}
\min_{c_j} \quad & \sum_{j=0}^{m-1} (c_j^+ + c_j^-) + \sum_{j=m}^{n} (j+1-m)^2 (c_j^+ + c_j^-) \\
\text{s.t.} \quad & \sum_{j=0}^{n} c_j T_j'(y_i) > 0 > \sum_{j=0}^{n} c_j T_j''(y_i), \qquad i = 1, \ldots, m', \\
& \sum_{j=0}^{n} c_j T_j(z_i) = v_i, \quad i = 1, \ldots, m, \\
& c_j - \hat{c}_j = c_j^+ - c_j^-, \quad j = 0, \ldots, m-1, \\
& c_j = c_j^+ - c_j^-, \quad j = m, \ldots, n, \\
& c_j^+ \geq 0, \quad c_j^- \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

Optimal Growth Models

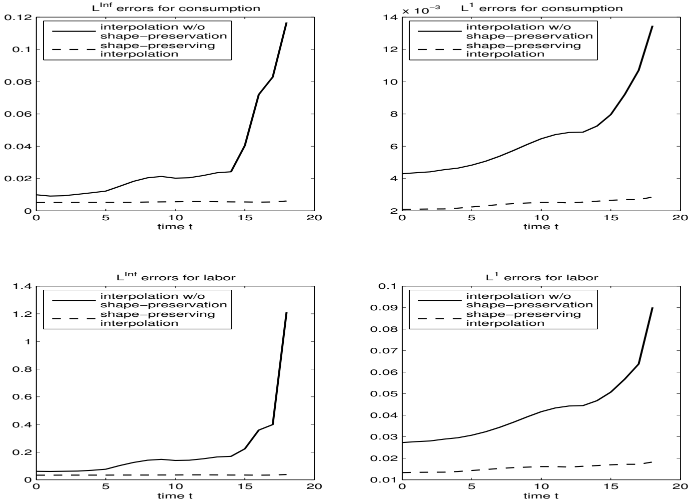
- Optimal Growth Problem:

$$V_0(k_0) = \max_{c,l} \quad \sum_{t=0}^{T-1} \beta^t u(c_t, l_t) + \beta^T V_T(k_T),$$
$$\text{s.t.} \quad k_{t+1} = F(k_t, l_t) - c_t, \quad 0 \leq t < T$$

- DP model of optimal growth problem:

$$V_t(k) = \max_{c,l} \quad u(c, l) + \beta V_{t+1}(F(k, l) - c)$$

Errors of NDP with Chebyshev interpolation (shape-preserving or not)



Multi-Stage Portfolio Optimization

- W_t: wealth at stage t; stocks' random return: $R = (R_1, \ldots, R_n)$; bond's riskfree return: R_f;
- $S_t = (S_{t1}, \ldots, S_{tn})^\top$: money in the stocks; $B_t = W_t - e^\top S_t$: money in the bond,
- $W_{t+1} = R_f(W_t - e^\top S_t) + R^\top S_t$
- Multi-Stage Portfolio Optimization Problem:

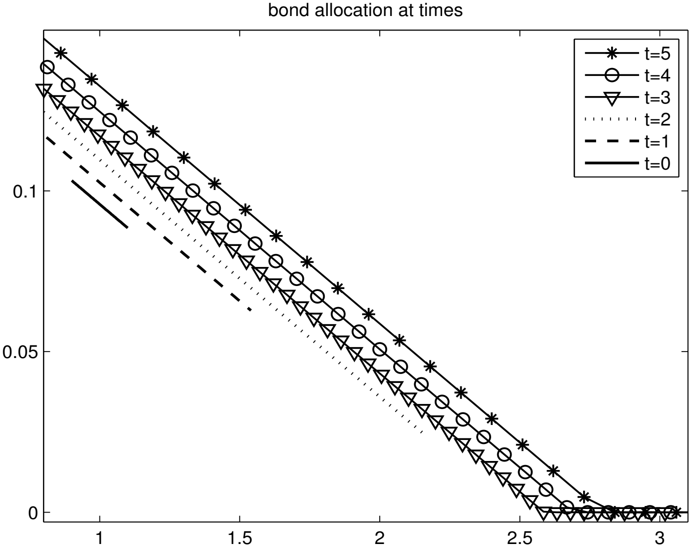
$$V_0(W_0) = \max_{X_{\boldsymbol{t}}, 0 \leq t < T} E\{u(W_T)\}$$

- Bellman Equation:

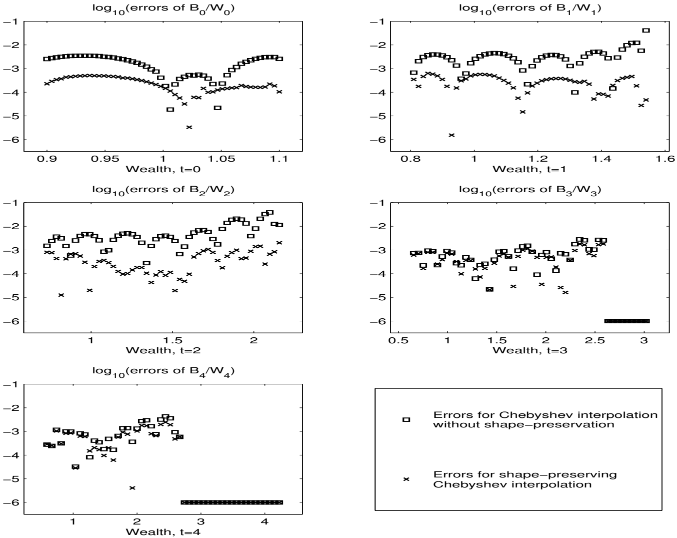
$$V_t(W) = \max_{S} \quad E\{V_{t+1}(R_f(W - e^\top S) + R^\top S)\}$$

W: state variable; S: control variables.

Exact optimal bond allocation



Errors of Optimal Stock Allocations (shape-preserving or not)



Shape-preserving Hermite Spline Interpolation

- Using Hermite data $\{(x_i, v_i, s_i) : i = 1, \ldots, m\}$,

$$\hat{V}(x; \mathbf{c}) = c_{i1} + c_{i2}(x - x_i) + \frac{c_{i3} c_{i4}(x - x_i)(x - x_{i+1})}{c_{i3}(x - x_i) + c_{i4}(x - x_{i+1})},$$

when $x \in [x_i, x_{i+1}]$, where

$$\begin{aligned} c_{i1} &= v_i, \\ c_{i2} &= \frac{v_{i+1} - v_i}{x_{i+1} - x_i}, \\ c_{i3} &= s_i - c_{i2}, \\ c_{i4} &= s_{i+1} - c_{i2}, \end{aligned}$$

for $i = 1, \ldots, m - 1$.

- $\hat{V}(x;\mathbf{c})$ is shape-preserving: When $x \in (x_i, x_{i+1})$, if the value function is increasing and concave, then the rational function spline interpolation is also increasing and concave in each (x_i, x_{i+1}).
- $\hat{V}(x;\mathbf{c})$ is a rational function on each interval (x_i, x_{i+1}), and $\mathcal{C}^1$ globally.

Errors of Optimal Bond Allocations (Lagrange vs Hermite vs Shape-preserving+Hermite)