Smolyak Method for Solving Dynamic Economic Models: Lagrange Interpolation, Anisotropic Grid and Adaptive Domain

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Introduction

Curse of dimensionality and Smolyak sparse grids

- Tensor-product rules are intractable even for moderately large problems, *for example, if we have 5 grid points for one variable, we have* 5^d *grid points for* d *variables.*

  $\Rightarrow$ *Exponential growth in complexity - curse of dimensionality!*

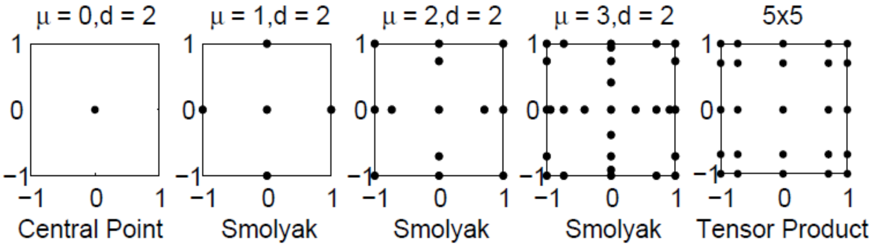
- In a seminal work, Russian mathematician Sergey Smolyak (1963) introduced **a sparse grid technique** for representing, interpolating and integrating multidimensional functions.
- The Smolyak technique builds on non-product rules and does not suffer from the curse of dimensionality (for smooth functions).

Introduction

Idea of the Smolyak method

- Not all tensor product terms are equally important for the quality of approximation.
- Low-order terms are more important than high-order terms (this is like Taylor series).
- The Smolyak technique orders all tensor-product elements by their potential importance and selects a relatively small number of the most important elements.
- A parameter, called a *level of approximation* (like the order of Taylor expansion), controls how many tensor-product elements are included into the Smolyak grid.
- By increasing the level of approximation μ, we add new elements and improve the quality of approximation.

Examples of Smolyak grids under the approximation levels $\mu = 0, 1, 2, 3$ for the two-dimensional case.



Tensor-product grid with 5^d points vs. Smolyak grid

d	Tensor-product grid with 5^d points	Smolyak grid		
		$\mu = 1$	$\mu = 2$	$\mu = 3$
1	5	3	5	9
2	25	5	13	29
10	9,765,625	21	221	1581
20	95,367,431,640,625	41	841	11,561

- The number of points in the Smolyak grids grows polynomials with dimensionality d.
    - for $\mu = 1$, we have $1 + 2d$ elements (grows linearly);
    - for $\mu = 2$, we have $1 + 2d$ $1 + 4d + 4d\,(d-1)$ (grows quadratically).
- A relatively small number of Smolyak grid points contrasts sharply with a huge number of tensor-product grid points.

Introduction

Smolyak's (1963) method in economics

- Krueger and Kubler (2004) introduce the Smolyak method to economics and show how to solve large-scale OLG models.
- Malin, Krueger and Kubler (2011) use the Smolyak method to solve large-scale international RBC models.
- Winschel and Krätzig (2010) use the Smolyak method for econometric estimations.
- Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012) solve a new Keynesian model.

The Smolyak method proved to be accurate and reliable in applications.

But the computational expense of the existing Smolyak method still grows rapidly with the dimensionality of the problem, especially if one targets a high quality of approximation.

- *Krueger and Kubler (2004):* The solution method becomes expensive when the length of agents' life (the number of state variables) ≥ 20.
- *Malin et al. (2011):* Document a high computational expense even for low levels of Smolyak polynomial approximations such as level 2 (their Fortran code runs for 10 hours for a model with 20 state variables).

In this paper, we show a novel variant of the Smolyak method that has lower cost and produces more accurate solutions than the one existing in the literature.

- Our starting point is the original work of Smolyak (1963).
- Smolyak (1963) offers an efficient choice for grid points and basis functions and establishes accuracy bounds for certain classes of functions but do not show how to construct them operationally.
- The Smolyak method for interpolation was implemented in subsequent literature using nested sets; see Delvos (1982), Wasilkowski and Woźniakowski (1999) and Berthelmann et al. (2000).
- We argue that this conventional implementation of the Smolyak method is inefficient and expensive.
- We go back to the original work of Smolyak and implement his construction in an efficient manner does not use nested sets.

Introduction

Our results: toward more efficient Smolyak interpolation

1. *Efficient construction of Smolyak polynomials.*
   - The nested-set construction of Smolyak polynomials is inefficient: it first creates a long list of repeated elements and then eliminates the repeated elements from the list.
   - We construct Smolyak polynomials using disjoint sets $\Longrightarrow$ **we avoid costly repetitions of elements.**
2. *A Lagrange-style technique for computing coefficients.*
   - The conventional Smolyak method computes polynomial coefficients using a formula with a large number of nested loops.
   - We compute the coefficients by precomputing a solution to the inverse problem $\Longrightarrow$ **a simple, general and cheap technique.**
3. *Anisotropic grid: different approximation levels for different variables.*
   - The conventional Smolyak method is symmetric (with the same number of grids and polynomial functions for all variables).
   - We develop an anisotropic version of the Smolyak method $\Longrightarrow$ **we can vary the quality of approximation across variables**.

Our results: adapting Smolyak method to economic applications

4. *Adaptive domain.*
   - The Conventional Smolyak method constructs grid points in a normalized multidimensional hypercube $[-1,1]^d$.
   - **We show how to effectively adapt the Smolyak hypercube domain to the high-probability set of the given model**.

5. *Iterative procedure.*
   - The Conventional Smolyak method of Krueger and Kubler (2004) and Malin et al. (2011) uses *time iteration*: given functional forms for future variables, they solve for current variables using a numerical solver.
   - **We replace time-iteration with a fixed-point iteration which is cheap and simple to implement**. The fixed-point iteration involves just straightforward computations and avoids the need for a numerical solver under time iteration (this modification, although minor in substance, is still important for reducing the cost)**.**

Numerical examples

Malin et al. (2011): Fortran code for a second-level approximation runs for 10 hours for a model with 20 state variables.

Our analysis:

- MATLAB code for a second-level approximation for the same model runs for 45 minutes.
- We also produce very accurate third-level approximations for the model with 20 state variables (it takes us about 45 hours).
- Anisotropic grid and adaptive domain allow us to decrease approximation errors between a half and one order of magnitude (without increasing the cost).
- In a different paper, Valero, Maliar and Maliar (2013), we show how to parallelize the Smolyak method both on a desktop computer using multiple CPUs and GPUs and on Blacklight supercomputer.

Introduction

Two different classes of sparse grid techniques in the literature:

1. Global polynomial approximation (we fall into this category); *Smolyak (1963), Delvos (1982), Wasilkowski and Woźniakowski (1999) and Berthelmann et al. (2000), etc.*
2. Piecewise approximations based on families of local basis functions; *Griebel (1998), Bungartz and Griebel, (2004), Ma and Zabaras (2009), Pflüger (2010), Heinkenschloss and Kouri (2012), etc.*

Our advantages:

- a global polynomial approximation is continuously differentiable everywhere.
- our complexity grows polynomially, while the complexity of piecewise basis functions grows exponentially.

However, piecewise approximations allow to refine approximations locally while our possibilities of local refinement are limited.

Unidimensional nested sets

- Construct sets of points $i = 1, 2, \ldots$ that satisfy two conditions:
  - *Condition 1.* Sets $i = 1, 2, \ldots$ have $m(i) = 2^{i-1} + 1$ points for $i \geq 2$ and $m(1) \equiv 1$.
  - *Condition 2.* Each subsequent set $i + 1$ contains all points of the previous set i. Such sets are called *nested*.
- There are many ways to construct the sets of points, satisfying Conditions 1 and 2.
- As an example, let us consider grid points $\left\{-1, \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right\}$ in the interval $[-1, 1]$ and create 3 nested sets of points:
  $i = 1 : S_1 = \{0\}$;
  $i = 2 : S_2 = \{-1, 0, 1\}$;
  $i = 3 : S_3 = \left\{-1, \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right\}$.

Conventional Smolyak grid using nested sets

Tensor products of unidimensional nested sets

		$i_2 = 1$	$i_2 = 2$	$i_2 = 3$
	$S_{i_1} \backslash S_{i_2}$	0	$-1, 0, 1$	$-1, \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1$
$i_1 = 1$	0	$(0,0)$	$(0,-1), (0,0), (0,1)$	$(0,-1), (0, \frac{-1}{\sqrt{2}}), (0,0), (0, \frac{1}{\sqrt{2}}), (0,1)$
$i_1 = 2$	-1 0 1	$(-1,0)$ $(0,0)$ $(1,0)$	$(-1,-1), (-1,0), (-1,1)$ $(0,-1), (0,0), (0,1)$ $(1,-1), (1,0,), (1,1)$	$(-1,-1), (-1, \frac{-1}{\sqrt{2}}), (-1,0), (-1, \frac{1}{\sqrt{2}}), (-1,1)$ $(0,-1), (0, \frac{-1}{\sqrt{2}}), (0,0), (0, \frac{1}{\sqrt{2}}), (1,0)$ $(1,-1), (1, \frac{-1}{\sqrt{2}}), (1,0), \left(1, \frac{1}{\sqrt{2}}\right) (1,1)$
$i_1 = 3$	-1 $\frac{-1}{\sqrt{2}}$ 0 $\frac{1}{\sqrt{2}}$ 1	$(-1,0)$ $\left(\frac{-1}{\sqrt{2}}, 0\right)$ $(0,0)$ $\left(\frac{1}{\sqrt{2}}, 0\right)$ $(1,0)$	$(-1,-1), (-1,0), (-1,1)$ $(\frac{-1}{\sqrt{2}}, -1), (\frac{-1}{\sqrt{2}}, 0), (\frac{-1}{\sqrt{2}}, 1)$ $(0,-1), (0,0), (1,0)$ $(\frac{1}{\sqrt{2}}, -1), (\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, 1)$ $(1,-1), (1,0), (1,1)$	$(-1,-1), (-1, \frac{-1}{\sqrt{2}}), (-1,0), (-1, \frac{1}{\sqrt{2}}), (-1,1)$ $(\frac{-1}{\sqrt{2}}, -1), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, 0), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, 1)$ $(0,-1), (0, \frac{-1}{\sqrt{2}}), (0,0), (0, \frac{1}{\sqrt{2}}), (1,0)$ $(\frac{1}{\sqrt{2}}, -1), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 1)$ $(1,-1), (1, \frac{-1}{\sqrt{2}}), (1,0), \left(1, \frac{1}{\sqrt{2}}\right) (1,1)$

Conventional Smolyak grid using nested sets

Smolyak sparse grid

- Smolyak (1963) rule used to select tensor products:

$$d \leq i_1 + i_2 \leq d + \mu,$$

  where $\mu \in \{0, 1, 2, ...\}$ is the approximation level, and d is the dimensionality (in our case, $d = 2$).
- In terms of the above table, the sum of indices of a column i_1 and a raw i_2, must be between d and $d + \mu$.
- Let $\mathcal{H}^{d,\mu}$ denote the Smolyak grid for a problem with dimensionality d and approximation level μ.

Conventional Smolyak grid using nested sets

Smolyak sparse grid: $d = 2$.

- If $\mu = 0 \Longrightarrow 2 \leq i_1 + i_2 \leq 2$. The only cell that satisfies this restriction is $i_1 = 1$ and $i_2 = 1 \Longrightarrow$ the Smolyak grid has just one grid point
$$\mathcal{H}^{2,0} = \{(0,0)\}.$$
- If $\mu = 1 \Longrightarrow 2 \leq i_1 + i_2 \leq 3$. The 3 cells that satisfy this restriction: (a) $i_1 = 1$, $i_2 = 1$; (b) $i_1 = 1$, $i_2 = 2$; (c) $i_1 = 2$, $i_2 = 1$, and the corresponding 5 Smolyak grid points are
$$\mathcal{H}^{2,1} = \{(0,0), (-1,0), (1,0), (0,-1), (0,1)\}.$$
- If $\mu = 2 \Longrightarrow 2 \leq i_1 + i_2 \leq 4$. There are 6 cells satisfy this restriction $\Longrightarrow$ 13 Smolyak grid points:

$$\mathcal{H}^{2,2} = \Big\{(-1,1), (0,1), (1,1), (-1,0), (0,0), (1,0), (-1,-1), (0,-1), (1,-1), (\frac{-1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, 0), (0, \frac{-1}{\sqrt{2}}), (0, \frac{1}{\sqrt{2}})\Big\}.$$

Conventional Smolyak polynomials using nested sets

Let $\mathcal{P}^{d,\mu}$ denote a Smolyak polynomial function in dimension d, with approximation level μ,

$$\mathcal{P}^{d,\mu}(x_1,...,x_d;b)$$
$$= \sum_{\max(d,\mu+1)\leq|i|\leq d+\mu} (-1)^{d+\mu-|i|}\binom{d-1}{d+\mu-|i|} p^{|i|}(x_1,...,x_d),$$

where $p^{|i|}(x_1,...,x_d)$ is the sum of $p^{i_1,...,i_d}(x_1,...,x_d)$ with $i_1+...+i_d=|i|$ defined as

$$p^{i_1,...,i_d}(x_1,...,x_d) = \sum_{\ell_1=1}^{m(i_1)} ... \sum_{\ell_d=1}^{m(i_d)} b_{\ell_1...\ell_d}\psi_{\ell_1}(x_1)\cdots\psi_{\ell_d}(x_d),$$

where $m(i_1),...,m(i_d)$ = number of basis functions in dimensions $1,...,d$; $m(i)\equiv 2^{i-1}+1$ for $i\geq 2$ and $m(1)\equiv 1$; $\psi_{\ell_1}(x_1),...,\psi_{\ell_d}(x_d)$ = unidimensional basis functions; $\ell_d=1,...,m(i_d)$; and $b_{\ell_1...\ell_d}$ are polynomial coefficients.

Conventional formula for Smolyak polynomial coefficients

- Let us build multidimensional Smolyak grid points and basis functions using unidimensional Chebyshev polynomials and their extrema.
- Then, there is an explicit formula for $b_{\ell_1 \dots \ell_d}$:

$$b_{\ell_1 \dots \ell_d} = \frac{2^d}{(m(i_1)-1)\cdots(m(i_d)-1)} \cdot \frac{1}{c_{\ell_1}\cdots c_{\ell_d}} \times \sum_{j_1=1}^{m(i_1)} \cdots \sum_{j_d=1}^{m(i_d)} \frac{\psi_{\ell_1}(\zeta_{j_1})\cdots\psi_{\ell_d}(\zeta_{j_d})\cdot f(\zeta_{j_1},\dots,\zeta_{j_d})}{c_{j_1}\cdots c_{j_d}}, \quad (1)$$

  where $\zeta_{j_1}, \dots, \zeta_{j_d}$ are grid points in dimensions $j_1, \dots, j_d$; $c_j = 2$ for $j = 1$ and $j = m(i_d)$; $c_j = 1$ for $j = 2, \dots, m(i_d) - 1$.
- If along any dimension d, $m(i_d) = 1 \Longrightarrow$ this dimension is dropped from computation, i.e., $m(i_d) - 1$ and $c_{j_d} = c_1$ are set to 1.
- Formula (1) is well known in the related literature; it is used, e.g., in Malin et al. (2011).

Example: Smolyak polynomial

Example: $d = 2$ and $\mu = 1$

- For the case of $\mu = 1$, we have that $\max(d, \mu + 1) = 2 \leq |i| \leq d + \mu = 3$, which means that $i_1 + i_2 \leq d + \mu = 3$.
- This is satisfied in three cases: (a) $i_1 = i_2 = 1$; (b) $i_1 = 1$, $i_2 = 2$; (c) $i_1 = 2$, $i_2 = 1$.

$$\text{(a) } p^{1,1} = \sum_{\ell_1=1}^{m(1)} \sum_{\ell_2=1}^{m(1)} b_{\ell_1 \ell_2} \psi_{\ell_1}(x) \psi_{\ell_2}(y) = b_{11},$$

$$\text{(b) } p^{1,2} = \sum_{\ell_1=1}^{m(1)} \sum_{\ell_2=1}^{m(2)} b_{\ell_1 \ell_2} \psi_{\ell_1}(x) \psi_{\ell_2}(y) = b_{11} + b_{12} \psi_2(y) + b_{13} \psi_3(y),$$

$$\text{(c) } p^{2,1} = \sum_{\ell_1=1}^{m(2)} \sum_{\ell_2=1}^{m(1)} b_{\ell_1 \ell_2} \psi_{\ell_1}(x) \psi_{\ell_2}(y) = b_{11} + b_{21} \psi_2(x) + b_{31} \psi_3(x).$$

where we assume that $\psi_1 \equiv 1$.

Example: Smolyak polynomial

Example: $d = 2$ and $\mu = 1$ (cont.)

- Collecting the elements p^{i_1, i_2} with the same $i_1 + i_2 \equiv |i|$, we obtain

$$\begin{aligned} p^{|2|} &\equiv p^{1,1}, \\ p^{|3|} &\equiv p^{2,1} + p^{1,2}. \end{aligned}$$

- Smolyak polynomial function $\mathcal{P}^{2,1}(x, y; b)$:

$$\begin{aligned} \sum_{2 \leq |i| \leq 3} (-1)^{3-|i|} \binom{1}{3-|i|} p^{|i|} &= \sum_{2 \leq |i| \leq 3} (-1)^{3-|i|} \frac{1}{(3-|i|)!} p^{|i|} \\ = (-1) \cdot p^{|2|} + 1 \cdot p^{|3|} &= (-1) \cdot p^{1,1} + 1 \cdot (p^{2,1} + p^{1,2}) \\ = -b_{11} + b_{11} + b_{21}\psi_2(x) + b_{31}\psi_3(x) &+ b_{11} + b_{12}\psi_2(y) + b_{13}\psi_3(y) \\ = b_{11} + b_{21}\psi_2(x) + b_{31}\psi_3(x) &+ b_{12}\psi_2(y) + b_{13}\psi_3(y). \end{aligned}$$

Example: $d = 2$ and $\mu = 1$

- Smolyak polynomial for $\mu = 1$:

$$\mathcal{P}^{2,1}(x, y; b) = b_{11} + b_{21}\psi_2(x) + b_{31}\psi_3(x) + b_{12}\psi_2(y) + b_{13}\psi_3(y)$$

- Chebyshev family is $\{\psi_1(x), \psi_2(x), \psi_3(x)\} = \{1, x, 2x^2 - 1\}$
- Extrema of Chebyshev polynomials are $\{\zeta_1, \zeta_2, \zeta_3\} = \{0, -1, 1\}$.

Formula (1) implies

$$\begin{aligned} b_{21} &= \frac{2^2}{3-1} \cdot \frac{1}{c_2 \cdot c_1} \sum_{j_1=1}^{3} \frac{\psi_2(\zeta_{j_1}) \cdot \psi_1(\zeta_1) \cdot f(\zeta_{j_1}, \zeta_1)}{c_{j_1} \cdot 1} \\ &= \frac{\psi_2(\zeta_1) \cdot f(\zeta_1, \zeta_1)}{c_1} + \frac{\psi_2(\zeta_2) \cdot f(\zeta_2, \zeta_1)}{c_2} + \frac{\psi_2(\zeta_3) \cdot f(\zeta_3, \zeta_1)}{c_3} \\ &= \frac{-1 \cdot f(-1, 0)}{2} + \frac{1 \cdot f(1, 0)}{2}; \end{aligned}$$

Example: Smolyak coefficients

Example: $d = 2$ and $\mu = 1$ (cont.)
Coefficient b_{12} is obtained similarly,

$$b_{12} = -\frac{f(0,-1)}{2} + \frac{f(0,1)}{2}.$$

Coefficient b_{31} is given by

$$\begin{aligned}
b_{31} &= \frac{2}{3-1} \cdot \frac{1}{c_3 \cdot c_1} \sum_{j_1=1}^{3} \frac{\psi_3(\zeta_{j_1}) \cdot \psi_1(\zeta_1) \cdot f(\zeta_{j_1}, \zeta_1)}{c_{j_1}} \\
&= \frac{1}{2}\left[\frac{1 \cdot f(-1,0)}{2} - f(0,0) + \frac{1 \cdot f(1,0)}{2}\right] \\
&= -\frac{f(0,0)}{2} + \frac{f(-1,0) + f(1,0)}{4},
\end{aligned}$$

and b_{13} is obtained similarly

$$b_{13} = -\frac{f(0,0)}{2} + \frac{f(0,-1) + f(0,1)}{4}.$$

Example: Smolyak coefficients

Example: $d = 2$ and $\mu = 1$ (cont.)
To find b_{11}, observe that

$$\mathcal{P}^{2,1}(0,0;b) = b_{11} + \frac{f(0,0)}{2} - \frac{f(-1,0)+f(1,0)}{4} + \frac{f(0,0)}{2} - \frac{f(0,-1)+f(0,1)}{4}.$$

Since under interpolation, we must have $\mathcal{P}^{2,1}(0,0;b) = f(0,0)$, the last formula implies

$$b_{11} = \frac{f(-1,0)+f(1,0)+f(0,-1)+f(0,1)}{4}.$$

Inefficiency of conventional Smolyak interpolation

- **Inefficiency:** First, we create a list of tensor products with many repeated elements and then, we eliminate the repetitions.
- **Repetitions of grid points.**
  - $\mathcal{H}^{2,1}$: $(0,0)$ is listed 3 times $\Longrightarrow$ must eliminate 2 grid points out of 7.
  - $\mathcal{H}^{2,2}$: must eliminate 12 repeated points out of 25 points.
  - *But grid points must be constructed just once (fixed cost), so repetitions are not so important for the cost*.
- **Repetitions of basis functions**.
  - $\mathcal{P}^{2,1}$ lists 7 basis functions from sets $\{1\}$, $\{1, \psi_2(x), \psi_3(x)\}$, $\{1, \psi_2(y), \psi_3(y)\}$ and eliminates 2 repeated functions $\{1\}$ by assigning a weight (-1) to $p^{|2|}$.
  - $\mathcal{P}^{2,2}$: must eliminate 12 repeated basis functions out of 25.
  - *Smolyak polynomials must be constructed many times (in every grid point, integration node and time period) and each time we suffer from repetitions.*
- **The number of repetitions increases in μ and d** $\Longrightarrow$ important for high-dimensional applications.

Smolyak method with Lagrange interpolation

We now present an alternative variant of the Smolyak method.

- First, instead of nested sets, we use disjoint sets, which allows us to avoid repetitions.
- Second, we find the coefficients using Lagrange-style interpolation. This technique works for any basis function and not necessarily orthogonal ones. Most of the computations can be done up-front (precomputed).
- Our version of the Smolyak method will be more simple and intuitive and easier to program.

Step 1. Smolyak grid using disjoint sets

Unidimensional grid points using disjoint sets

- We construct the Smolyak grid using disjoint sets.
- We consider grid points $\left\{-1, \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right\}$ in the interval $[-1, 1]$ and create 3 unidimensional sets of elements (grid points), A_1, A_2, A_3, which are disjoint, i.e., $A_i \cap A_j = \{\varnothing\}$ for any i and j.
  $i = 1 : A_1 = \{0\}$;
  $i = 2 : A_2 = \{-1, 1\}$;
  $i = 3 : A_3 = \left\{\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$.

Step 1. Smolyak grid using disjoint sets

Tensor products of unidimensional disjoint sets of points

		$i_2 = 1$	$i_2 = 2$	$i_2 = 3$
	$A_{i_1} \backslash A_{i_2}$	0	$-1, 1$	$\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$
$i_1 = 1$	0	$(0, 0)$	$(0, -1), (0, 1)$	$\left(0, \frac{-1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}\right)$
$i_1 = 2$	-1 1	$(-1, 0)$ $(1, 0)$	$(-1, -1), (-1, 1)$ $(1, -1), (1, 1)$	$\left(-1, \frac{-1}{\sqrt{2}}\right), \left(-1, \frac{1}{\sqrt{2}}\right)$ $\left(1, \frac{-1}{\sqrt{2}}\right), \left(1, \frac{1}{\sqrt{2}}\right)$
$i_1 = 3$	$\frac{-1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$	$\left(\frac{-1}{\sqrt{2}}, 0\right)$ $\left(\frac{1}{\sqrt{2}}, 0\right)$	$\left(\frac{-1}{\sqrt{2}}, -1\right), \left(\frac{-1}{\sqrt{2}}, 1\right)$ $\left(\frac{1}{\sqrt{2}}, -1\right), \left(\frac{1}{\sqrt{2}}, 1\right)$	$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

We select elements that belong to the cells with the sum of indices of a column and a row, $i_1 + i_2$, between d and $d + \mu$. This leads to the same Smolyak grids as before. However, in our case, no grid points are repeated.

Smolyak sparse grid

- We use the same Smolyak rule for constructing multidimensional grid points
$$d \leq i_1 + i_2 \leq d + \mu$$
That is, we select elements that belong to the cells in the above table for which the sum of indices of a column and a row, $i_1 + i_2$, is between d and $d + \mu$.
- This leads to the same Smolyak grids $\mathcal{H}^{2,0}$, $\mathcal{H}^{2,1}$ and $\mathcal{H}^{2,2}$ as under the construction built on nested sets. However, in our case, no grid points are repeated.

Step 2. Smolyak polynomials using disjoint sets

Disjoint sets of basis functions

The same construction as the one we used for constructing the grid points.

$i = 1 : A_1 = \{1\};$
$i = 2 : A_2 = \{\psi_2(x), \psi_3(x)\};$
$i = 3 : A_3 = \{\psi_4(x), \psi_5(x)\}.$

Tensor products of unidimensional disjoint sets of basis functions

	$A_{i_1} \backslash A_{i_2}$	$i_2 = 1$ 1	$i_2 = 2$ $\psi_2(y), \psi_3(y)$	$i_2 = 3$ $\psi_4(y), \psi_5(y)$
$i_1 = 1$	1	1	$\psi_2(y), \psi_3(y)$	$\psi_4(y), \psi_5(y)$
$i_1 = 2$	$\psi_2(x)$ $\psi_3(x)$	$\psi_2(x)$ $\psi_3(x)$	$\psi_2(x)\psi_2(y), \psi_2(x)\psi_3(y)$ $\psi_3(x)\psi_2(y), \psi_3(x)\psi_3(y)$	$\psi_2(x)\psi_4(y), \psi_2(x)\psi_5(y)$ $\psi_3(x)\psi_4(y), \psi_3(x)\psi_5(y)$
$i_1 = 3$	$\psi_4(x)$ $\psi_5(x)$	$\psi_4(x)$ $\psi_5(x)$	$\psi_4(x)\psi_2(y), \psi_4(x)\psi_3(y)$ $\psi_5(x)\psi_2(y), \psi_5(x)\psi_3(y)$	$\psi_4(x)\psi_4(y), \psi_4(x)\psi_5(y)$ $\psi_5(x)\psi_4(y), \psi_5(x)\psi_5(y)$

For example, for $\mu = 1$, we get
$\mathcal{P}^{2,1}(x, y; b) = b_{11} + b_{21}\psi_2(x) + b_{31}\psi_3(x) + b_{12}\psi_2(y) + b_{13}\psi_3(y)$.

Step 3. Lagrange-style interpolation for finding coefficients

- Simply find the coefficients so that a polynomial with M basis functions passes through M given grid points.
- Let $f : [-1,1]^d \to \mathbb{R}$ be a smooth function.
- Let $\mathcal{P}(\cdot; b)$ be a polynomial function, $\mathcal{P}(x; b) = \sum_{n=1}^{M} b_n \Psi_n(x)$, where $\Psi_n : [-1,1]^d \to \mathbb{R}$ is a d-dimensional basis function; $b \equiv (b_1, ..., b_M)$ is a coefficient vector.
- We construct a set of M grid points $\{x_1, ..., x_M\}$ within $[-1,1]^d$, and we compute b so that the true function, f, and its approximation, $\mathcal{P}(\cdot; b)$ coincide in all grid points:

$$\begin{bmatrix} f(x_1) \\ \cdots \\ f(x_M) \end{bmatrix} = \begin{bmatrix} \widehat{f}(x_1; b) \\ \cdots \\ \widehat{f}(x_M; b) \end{bmatrix} = \overbrace{\begin{bmatrix} \Psi_1(x_1) & \cdots & \Psi_M(x_1) \\ \cdots & \ddots & \cdots \\ \Psi_1(x_M) & \cdots & \Psi_M(x_M) \end{bmatrix}}^{\equiv \mathcal{B}} \cdot \begin{bmatrix} b_1 \\ \cdots \\ b_M \end{bmatrix}$$

Lagrange-style interpolation

- Provided that the matrix of basis functions $\mathcal{B}$ has full rank, we have a system of M linear equations with M unknowns that admits a unique solution for b

$$\begin{bmatrix} b_1 \\ \cdots \\ b_M \end{bmatrix} = \begin{bmatrix} \Psi_1(x_1) & \cdots & \Psi_M(x_1) \\ \cdots & \ddots & \cdots \\ \Psi_1(x_M) & \cdots & \Psi_M(x_M) \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ \cdots \\ f(x_M) \end{bmatrix}.$$

  By construction, approximation $\mathcal{P}(\cdot; b)$ coincides with true function f in all grid points, i.e., $\widehat{f}(x_n; b) = f(x_n)$ for all $x_n \in \{x_1, ..., x_M\}$.
- For orthogonal basis functions, matrix $\mathcal{B}$ is well-conditioned.

Lagrange-style interpolation

Example: $d = 2$ and $\mu = 1$.

Just compute 5 coefficients in Smolyak polynomial:
$P^{2,1}(x, y; b) = b_{11} + b_{21}x + b_{31}\left(2x^2 - 1\right) + b_{12}y + b_{13}\left(2y^2 - 1\right)$ to match function f in 5 Smolyak grid points $\{(0,0), (-1,0), (1,0), (0,-1), (0,1)\}$

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{12} \\ b_{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} f(0,0) \\ f(-1,0) \\ f(1,0) \\ f(0,-1) \\ f(0,1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{f(-1,0)+f(1,0)+f(0,-1)+f(0,1)}{4} \\ \frac{-f(-1,0)+f(1,0)}{2} \\ -\frac{f(0,0)}{2} + \frac{f(-1,0)+f(1,0)}{4} \\ \frac{-f(0,-1)+f(0,1)}{2} \\ -\frac{f(0,0)}{2} + \frac{f(0,-1)+f(0,1)}{4} \end{bmatrix}.$$

Our Smolyak polynomial using disjoint sets

- We hope you liked our construction without cumbersome formulas.
- However, for those of you who do like more the Smolyak formula, we present our version of the Smolyak formula using disjoint sets.
- Our formula produces the same Smolyak polynomial but avoids repetitions:

$$\mathcal{P}^{d,\mu}\left(x_1,...,x_d;b\right)=\sum_{d\leq|i|\leq d+\mu} q^{|i|}\left(x_1,...,x_d\right), \tag{2}$$

where $q^{|i|}\left(x_1,...,x_d\right)$ is the sum of $q^{i_1,...,i_d}\left(x_1,...,x_d\right)$ whose indices satisfy $i_1+...+i_d=|i|$,

$$q^{i_1,...,i_d}\left(x_1,...,x_d\right)$$

$$=\sum_{\ell_1=m(i_1-1)+1}^{m(i_1)} \cdots \sum_{\ell_d=m(i_d-1)+1}^{m(i_d)} b_{\ell_1...\ell_d}\psi_{\ell_1}\left(x_1\right)\cdots\psi_{\ell_d}\left(x_d\right), \tag{3}$$

where $\psi_{\ell_1}\left(x_1\right),...,\psi_{\ell_d}\left(x_d\right)$ are unidimensional basis functions, in dimensions $1,...,d$; $\psi_{\ell_1}\left(x_1\right)\cdots\psi_{\ell_d}\left(x_d\right)$ is a d-dimensional basis function; by convention, $m\left(0\right)=0$ and $m\left(1\right)=1$.

Example of efficient formula for Smolyak polynomial

Theorem

Polynomial functions produced by our and conventional Smolyak formulas are equivalent.

Example: $d = 2$ and $\mu = 1$ (revisited).

- We have $d = 2 \leq i_1 + i_2 \leq d + \mu = 3$ which is satisfied in three cases: (a) $i_1 = i_2 = 1$; (b) $i_1 = 1$, $i_2 = 2$; (c) $i_1 = 2$, $i_2 = 1$.

$$q^{1,1} = \sum_{\ell_1 = m(0)+1}^{m(1)} \sum_{\ell_2 = m(0)+1}^{m(1)} b_{\ell_1 \ell_2} \psi_{\ell_1}(x) \psi_{\ell_2}(y) = b_{11},$$

$$q^{1,2} = \sum_{\ell_1 = m(0)+1}^{m(1)} \sum_{\ell_2 = m(1)+1}^{m(2)} b_{\ell_1 \ell_2} \psi_{\ell_1}(x) \psi_{\ell_2}(y) = b_{12} \psi_2(y) + b_{13} \psi_3(y)$$

$$q^{2,1} = \sum_{\ell_1 = m(1)+1}^{m(2)} \sum_{\ell_2 = m(0)+1}^{m(1)} b_{\ell_1 \ell_2} \psi_{\ell_1}(x) \psi_{\ell_2}(y) = b_{21} \psi_2(x) + b_{31} \psi_3(x)$$

Example of efficient formula for Smolyak polynomial

Example: $d = 2$ and $\mu = 1$ (revisited).

The restriction $i_1 + i_2 \leq 3$ is satisfied in 3 cases: (a) $i_1 = i_2 = 1$; (b) $i_1 = 1$, $i_2 = 2$; (c) $i_1 = 2$, $i_2 = 1$.

$$\begin{aligned}
\text{(a) } q^{1,1} &= b_{11}, \\
\text{(b) } q^{1,2} &= b_{12}\psi_2(y) + b_{13}\psi_3(y), \\
\text{(c) } q^{2,1} &= b_{21}\psi_2(x) + b_{31}\psi_3(x).
\end{aligned}$$

Smolyak polynomial function for the case of $\mu = 1$ is given by

$$\begin{aligned}
\mathcal{P}^{2,1}(x, y; b) &= \sum_{|i| \leq d+\mu} q^{|i|} = q^{1,1} + q^{2,1} + q^{1,2} \\
&= b_{11} + b_{21}\psi_2(x) + b_{31}\psi_3(x) + b_{12}\psi_2(y) + b_{13}\psi_3(y).
\end{aligned}$$

Our formula gives us the same polynomial as the one produced by the conventional Smolyak formula but avoids costly repetitions of elements.

Nested-set versus disjoint-set constructions: comparison of costs

- Recall that $m(i_j) \equiv 2^{i_j-1}+1$ for $i_j \geq 2$ & $m(1) \equiv 1$ & $m(0) = 0$.
- **Number of terms.**

Nested-set construction:

$$\sum_{\max(d,\mu+1)\leq|i|\leq d+\mu} \prod_{j=1}^{d} m(i_j)$$

Disjoint-set construction:

$$\sum_{d\leq|i|\leq d+\mu} \prod_{j=1}^{d} [m(i_j) - m(i_j - 1)]$$

- **Example:** $d = 2$ and $\mu = 1$.

$$-\sum_{2\leq|i|\leq 3} \prod_{j=1}^{2} m(i_j) = m(1)\,m(1) + m(1)\,m(2) + m(2)\,m(1)$$

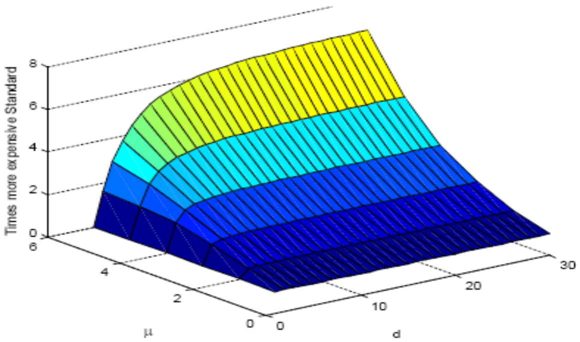
$$= 1\cdot 1 + 1\cdot 3 + 3\cdot 1 = 7.$$

$$-\sum_{2\leq|i|\leq 3} \prod_{j=1}^{2} [m(i_j) - m(i_j - 1)] = [m(1) - m(0)]\cdot[m(1) - m(0)]$$

$$+2\cdot[m(1) - m(0)]\cdot[m(2) - m(1)] = 1 + 2\cdot 2 = 5.$$

Nested sets versus disjoint sets: comparison of costs

- A ratio of the number of terms under the two constructions, $\mathcal{R}^{d,\mu}$.



- For any $i_j \geq 2$, we reduce the number of terms by at least a factor of $\frac{2}{3}$ when we use disjoint sets instead of nested sets.

Adapting Smolyak methods to economic applications

- Our previous analysis was intended to make the Smolyak interpolation method more efficient (i.e., less costly).
- Now, we show how to better adapt the Smolyak method to economic applications.
- We will consider 3 ways to enhance the performance of the Smolyak method.
  - Anisotropic grid (different treatment of variables).
  - Adaptive domain (fitting the hypercube to a high probability set of a given economic model).
  - Efficient iterative procedure for finding fixed point coefficients.

Anisotropic grid

- The conventional Smolyak method treats all dimensions symmetrically: it uses the same number of grid points and basis functions for all variables.
- In economic applications, it may be of value to give different treatments to different variables.
- *Why?*
  - Decisions functions may have more curvature in some variables than in others.
  - Some variables may have a larger range of values than the others.
  - Some variables may be more important than the others.
- **Literature**.
  - *Gerstner and Griebel (1998, 2003)*: dimension-adaptive tensor-product quadrature to integrate high-dimensional functions.
  - *Kouri (2012):* anisotropic grids to solve PDE with uncertain coefficients.

Anisotropic grid

- Let μ_i be an approximation level in dimension i.
- Let $\mu = \left(\mu_1, ..., \mu_d\right)$.
- Let $\mu^{\max} = \max\left\{\mu_1, ..., \mu_d\right\}$
- Note that $\mu_j = i_j^{\max} - 1$ where $i_j^{\max}$ is the maximum index of the sets considered for dimension j.
- Smolyak grid is called *asymmetric (anisotropic)* if there is at least one dimension j such that $i_j \neq i_k$ for $\forall k \neq j$.
- $\mathcal{H}^{d,\left(\mu_1,...,\mu_d\right)} \equiv$ a d-dimensional anisotropic Smolyak grid of approximation levels $\mu = \left(\mu_1, ..., \mu_d\right)$.
- $\mathcal{P}^{d,\left(\mu_1,...,\mu_d\right)} \equiv$ the corresponding Smolyak polynomial.

Tensor products of sets of unidimensional elements

	$A_{i_1} \backslash A_{i_2}$	$i_2 = 1$ 0	$i_2 = 2$ $-1, 1$
$i_1 = 1$	0	$(0, 0)$	$(0, -1), (0, 1)$
$i_1 = 2$	-1 1	$(-1, 0)$ $(1, 0)$	$(-1, -1), (-1, 1)$ $(1, -1), (1, 1)$
$i_1 = 3$	$\frac{-1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$	$\left(\frac{-1}{\sqrt{2}}, 0\right)$ $\left(\frac{1}{\sqrt{2}}, 0\right)$	$\left(\frac{-1}{\sqrt{2}}, -1\right), \left(\frac{-1}{\sqrt{2}}, 1\right)$ $\left(\frac{1}{\sqrt{2}}, -1\right), \left(\frac{1}{\sqrt{2}}, 1\right)$
$i_1 = 4$	$\frac{-\sqrt{2+\sqrt{2}}}{2}$ $\frac{-\sqrt{2-\sqrt{2}}}{2}$ $\frac{\sqrt{2-\sqrt{2}}}{2}$ $\frac{\sqrt{2+\sqrt{2}}}{2}$	$\left(\frac{-\sqrt{2+\sqrt{2}}}{2}, 0\right)$ $\left(\frac{-\sqrt{2-\sqrt{2}}}{2}, 0\right)$ $\left(\frac{\sqrt{2-\sqrt{2}}}{2}, 0\right)$ $\left(\frac{\sqrt{2+\sqrt{2}}}{2}, 0\right)$	$\left(\frac{-\sqrt{2+\sqrt{2}}}{2}, -1\right), \left(\frac{-\sqrt{2+\sqrt{2}}}{2}, 1\right)$ $\left(\frac{-\sqrt{2-\sqrt{2}}}{2}, -1\right), \left(\frac{-\sqrt{2-\sqrt{2}}}{2}, 1\right)$ $\left(\frac{\sqrt{2-\sqrt{2}}}{2}, -1\right), \left(\frac{\sqrt{2-\sqrt{2}}}{2}, 1\right)$ $\left(\frac{\sqrt{2+\sqrt{2}}}{2}, -1\right), \left(\frac{\sqrt{2+\sqrt{2}}}{2}, 1\right)$

Anysotropic Smolyak sets

- The Smolyak rule: select elements that satisfy

$$d \leq i_1 + i_2 \leq d + \mu^{\max}$$

- If $\mu = (1, 0)$, then $\mu^{\max} = 1$ and $2 \leq i_1 + i_2 \leq 3$. The 3 cells that satisfy this restriction are (a) $i_1 = 1$, $i_2 = 1$; (b) $i_1 = 1$, $i_2 = 2$; (c) $i_1 = 2$, $i_2 = 1$,

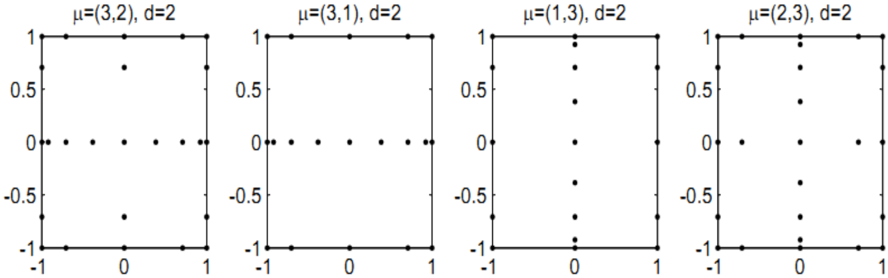
$$\mathcal{H}^{2,\{1,0\}} = \{(0,0), (-1,0), (1,0)\}.$$

- If $\mu = (2, 1)$, then $\mu^{\max} = 2$ and $2 \leq i_1 + i_2 \leq 4$, there are 5 cells that satisfy this restriction (a) $i_1 = 1$, $i_2 = 1$; (b) $i_1 = 1$, $i_2 = 2$; (c) $i_1 = 2$, $i_2 = 1$; (d) $i_1 = 1$, $i_2 = 3$; (e) $i_1 = 2$, $i_2 = 2$; and 11 points:

$$\mathcal{H}^{2,\{2,1\}} = \Big\{(-1,1), (0,1), (1,1), (-1,0), (0,0), (1,0), (-1,-1), (0,-1), (1,-1), (\frac{-1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, 0)\Big\}.$$

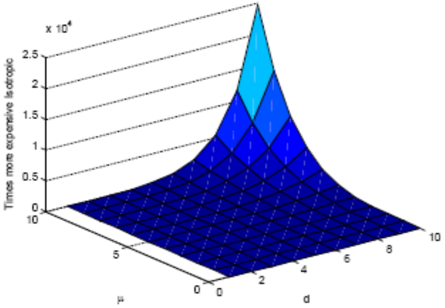
- If $\mu = (3, 1)$, then $\mu^{\max} = 3$ and $2 \leq i_1 + i_2 \leq 5$, there are 19 points.

Anisotropic grids: an illustration



Isotropic versus anisotropic grids: comparison of costs

- We compare isotropic and anisotropic grids in which no elements are repeated. The number of elements depends on the specific restrictions $\{i_1^{\max}, ..., i_d^{\max}\}$ imposed.
- We show a limiting case when we have $i_1^{\max} = \mu + 1$ in dimension 1 and we have $i_j^{\max} = 1$ for all other dimensions, $j = 2, ..., d$.



The savings can be orders of magnitude when d and μ are large.

Anisotropic grids: accuracy bounds

We generalize Barthelmann et al. (2000) to anisotropic grids.

Consider the spaces $F_1^k = C^k([-1,1])$, with the norm $\|f\| = \max\left\{\|D^\alpha f\|_\infty \,|\alpha = 0, ..., k\right\}$ and for $d > 1$. $F_d^k = C^k([-1,1]^d)$, with the norm $\|f\| = \max\left\{\|D^\alpha f\|_\infty \,|\alpha \in \mathbb{N}_0^d,\ \alpha_i \leq k\right\}$. Finite linear combinations of functions $f_1 \otimes f_2 \otimes \ldots \otimes\ f_d$ with $f_i \in F_d^k$ are dense in F_d^k and $\|f_1 \otimes f_2 \otimes \ldots \otimes\ f_d\| = \|f_1\|\,\|f_2\| \ldots\ \|f_d\|$. Let I_d denote the embedding $F_d^k \to C^k([-1,1]^d)$. Moreover, let $\|S\| = \sup\left\{\|S(f)\|_\infty \,|f \in F_d^k, \|f\| \leq 1\right\}$ for $S : F_d^k \to C^k([-1,1]^d)$. We use $c_{d,k}$ to denote constants that only depend on d and k.

Theorem

For the space F_d^k we obtain

$$\left\|f_{1,\ldots,d} - \widehat{f}_{1,\ldots,d}\right\| \leq c\,\|f_{1,\ldots,d}\|\,\mu 2^{-2k} \sum_{j=1}^{d-1} \sum_{s=j}^{q-d+j} \prod_{v=1,\forall v:\ i_v \leq i_v^{\max}}^{j} \left\|\widehat{f}_v^{i_v} - \widehat{f}_v^{i_v-1}\right\| \tag{4}$$

Anisotropic grids: intuition for the theorem

The theorem provides a useful intuition.

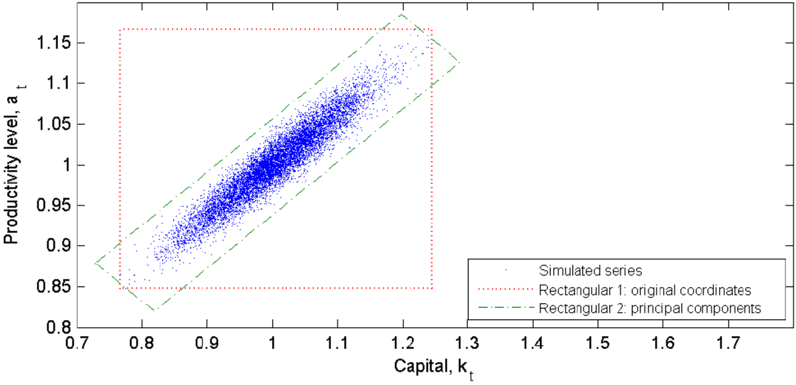
- The only term affected by anisotropy is $\prod_{v=1,\forall v \ i_v \leq i_v^{\max}}^{j} \left\| \widehat{f}_v^{i_v} - \widehat{f}_v^{i_v - 1} \right\|$.
- It shows how the quality of approximation improves when we increase the level of approximation.

Example: *Suppose we have a quadratic function. Then, the approximation is perfect alone the dimension i_v, the terms in the product are zero starting from the third order and hence, the product is zero. A further increase in the order of approximation will not change the maximum norm. This is precisely the kind of case we want to show.*

Adaptive domain

- The Smolyak construction (built on extrema of Chebyshev polynomials) tells us how to represent and interpolate functions defined on a normalized d-dimensional hypercube ($[-1,1]^d$).
- The solution domain of a typical dynamic economic model does not have the shape of a hypercube but can be of any shape in a d-dimensional space.
- We describe how to effectively adapt a multidimensional hypercube to an unstructured solution domain of a given problem.
- By choosing an appropriate system of coordinates, we are able to reduce the Euler equation residuals significantly (without increasing the running time).

Conventional hypercube vs. a hypercube obtained after the change of variables



How to construct the conventional hypercube

- Obtain simulated data: $\{k_t, \theta_t\}_{t=1}^{T}$.
- Define $[\underline{k}, \overline{k}]$ and $[\underline{\theta}, \overline{\theta}]$ as intervals for the state variables that we get in the simulation.
- Where $\overline{k}$,and $\overline{\theta}$ represent the maxima in each coordinate and $\underline{k}$,and $\underline{\theta}$ represent the minima in each coordinate. Consider a linear transformation $(k, \theta) \in [\underline{k}, \overline{k}] \times [\underline{\theta}, \overline{\theta}]$

$$x = 2\frac{k - \underline{k}}{\overline{k} - \underline{k}} - 1 \text{ and } y = 2\frac{\theta - \underline{\theta}}{\overline{\theta} - \underline{\theta}} - 1. \tag{5}$$

- For instance, $[\underline{k}, \overline{k}] = [0.8k_s, 1.2k_s]$, with k_s being capital in the steady state, and $[\underline{\theta}, \overline{\theta}] = [\exp(\frac{-0.8\sigma}{1-\rho}), \exp(\frac{0.8\sigma}{1-\rho})]$.

The procedure is similar to the one used in Judd, Maliar and Maliar (2011, 2013) for constructing cluster and EDS grids.

1. Obtain simulated data: $\{k_t, \theta_t\}_{t=1}^{T}$.
2. Normalize the data

$$\widetilde{k}_t = \frac{k_t - \mu_k}{\sigma_k} \text{ and } \widetilde{\theta}_t = \frac{\theta_t - \mu_\theta}{\sigma_\theta}, \tag{6}$$

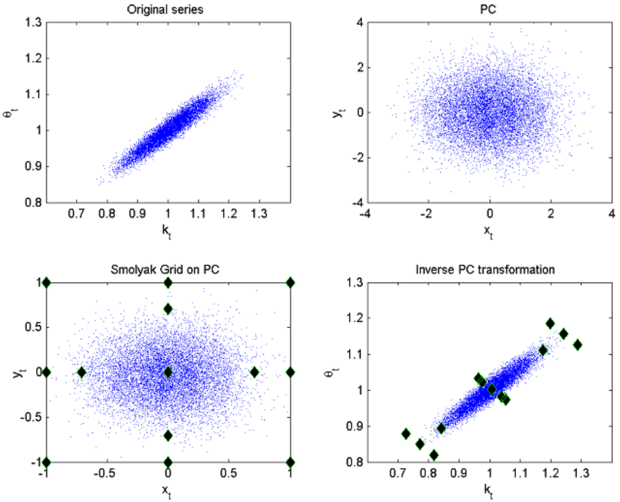
where μ_k and μ_θ are means, and σ_k and σ_θ are standard deviations of of capital and productivity.

3. Consider the **singular value decomposition** of the matrix of the normalized data $D \equiv \begin{bmatrix} \widetilde{k}_1 & \widetilde{\theta}_1 \\ \vdots & \vdots \\ \widetilde{k}_T & \widetilde{\theta}_T \end{bmatrix}$. $D = USV^\top$.

How to construct an adaptive hypercube

4. $Z \equiv DV$, the variables $\left\{z_t^1, z_t^2\right\}_{t=1}^{T}$ are called *principal components* of D and are orthogonal (uncorrelated).
5. In order to use Chebyshev polynomials, it is necessary to stay in the intervals between -1 and 1, therefore the same linear transformation as above is applied. Once we are in $[-1,1]^2$ we can define interpolation nodes $\mathcal{H}^{2,\mu}$. In this step we collect $\underline{z}^1$, $\overline{z}^1$, $\underline{z}^2$ and $\overline{z}^2$ with the idea of inverting the procedure.
6. We can invert the process to obtain the interpolation nodes $\mathcal{H}^{2,\mu}$ in the same space that the real data $\left\{k_t, \theta_t\right\}_{t=1}^{T}$ are.

Smolyak grid on principal components



Adaptive domain: accuracy bounds

Theorem

Assume that we interpolate function on hypercube domains H^1, H^2 such that $H^1 \subset H^2$. Then, the approximation errors satisfy $\| f_{1,\ldots,d} - A(q,d)(f) \|_{H^1} \geq \| f_{1,\ldots,d} - A(q,d)(f) \|_{H^2}$.

The intuition is that when we reduce the domain, we always reduce the approximation errors according to the studied maximum norm because if the maximum error is in the domain H^2 the error does not change and if it is in H^1 but not in H^2, the errors decrease.

Standard neoclassical stochastic growth model

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = (1-\delta) k_t + \theta_t f(k_t)$$

$$\ln \theta_t = \rho \ln \theta_{t-1} + \sigma \varepsilon_t \text{ with } \varepsilon_t \sim N(0,1)$$

where:

- $c_t, k_{t+1} \geq 0$, k_0 and θ_0 are given;
- $\beta \in (0,1]$ is the discount factor;
- $U(c_t)$ is the utility function, increasing and concave;
- $\delta \in (0,1]$ is the depreciation rate of capital;
- $f(k_t, \theta_t)$ is the production function; E_0 is the operator of conditional expectation;
- $\rho \in (-1,1)$ and $\sigma > 0$.

The algorithm iterating on Euler equation

Smolyak algorithm
Initialization.
a. Choose the approximation level, μ.
b. Construct the Smolyak grid $\mathcal{H}^{2,\mu} = \{(x_n, y_n)\}_{n=1,\dots,M}$ on $[-1,1]^2$.
c. Compute the Smolyak basis functions in each grid point n. The resulting $M \times M$ matrix is $\mathcal{B}$.
d. Fix $\Phi : (k,\theta) \to (x,y)$, where $(k,\theta) \in \mathbb{R}^2_+$ and $(x,y) \in [-1,1]^2$. Use Φ^{-1} to compute (k_n, θ_n) that corresponds to (x_n, y_n) in $\mathcal{H}^{2,\mu}$.
e. Choose integration nodes, ϵ_j, and weights, ω_j, $j = 1, \dots, J$.
f. Construct future productivities, $\theta'_{n,j} = \theta_n^{\rho} \exp\left(\epsilon_j\right)$ for all j;
g. Choose an initial guess $b^{(1)}$.

The algorithm iterating on Euler equation

Step 1. Computation of a solution for K.

a. At iteration i, for $n = 1, ..., M$, compute

– $k_n' = \mathcal{B}_n b^{(i)}$, where $\mathcal{B}_n$ is the nth row of $\mathcal{B}$.

– $\left(x_n', y_{n,j}'\right)$ that correspond to $\left(k_n', \theta_{n,j}'\right)$ using Φ.

– Compute the Smolyak basis functions in each point $\left(x_n', y_{n,j}'\right)$.

– The resulting $M \times M \times J$ matrix is $\mathcal{B}'$.

– $k_{n,j}'' = \mathcal{B}_{n,j}' b^{(i)}$, where $\mathcal{B}_{n,j}'$ is the nth row of $\mathcal{B}'$ in state j.

– $c_n = (1-\delta) k_n + \theta_n f(k_n) - k_n'$;

– $c_{n,j}' = (1-\delta) k_n' + \theta_n^\rho \exp(\epsilon_j) A f(k_n') - k_{n,j}''$ for all j;

– $\widehat{k}_n' \equiv \beta \sum_{j=1}^{J} \omega_j \cdot \left[\frac{u_1\left(c_{n,j}'\right)}{u_1(c_n)} \left[1 - \delta + \theta_n^\rho \exp\left(\epsilon_j\right) f_1\left(k_n'\right)\right] k_n' \right]$

b. Find b that solves the system in Step 1a.

– Compute $\widehat{b}$ that solves $\widehat{k}' = \mathcal{B}\widehat{b}$, i.e., $\widehat{b} = \mathcal{B}^{-1}\widehat{k}_n'$.

– Use damping to compute $b^{(i+1)} = (1-\xi) b^{(i)} + \xi \widehat{b}$, where $\xi \in (0,1]$.

– Check for convergence: end Step 1 if $\frac{1}{M\xi} \sum_{n=1}^{M} \left| \frac{(k_n')^{(i+1)} - (k_n')^{(i)}}{(k_n')^{(i)}} \right| < 10^{-\vartheta}$, $\vartheta > 0$.

Iterate on Step 1 until convergence.

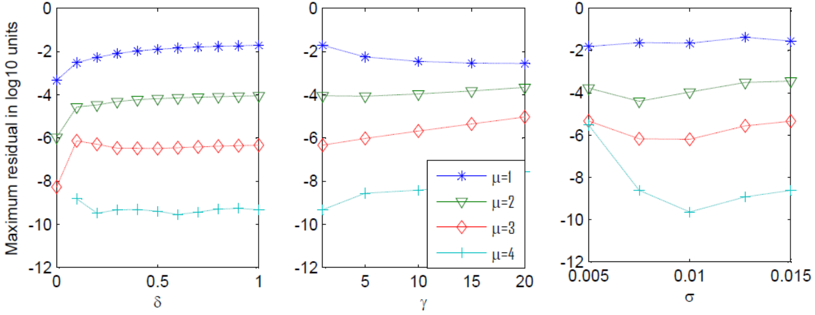
Results for the representative agent model

- CRRA utility function: $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$;
- Cobb-Douglas production function: $f(k) = k^{\alpha}$, with $\alpha = 1/3$;
- AR(1) process: $\ln \theta' = \rho \ln \theta + \sigma \varepsilon$, with $\rho = 0.95$
- Discount factor: $\beta = 0.99$.
- Benchmark values: $\delta = 0.025$, $\gamma = 1$ and $\sigma = 0.01$.
- Then, we consider variations in δ, γ and σ one-by-one holding the remaining parameters at the benchmark values.

$$
\begin{aligned}
\delta &= \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}, \\
\gamma &= \{1, 5, 10, 15, 20\}, \\
\sigma &= \{0.001, 0.005, 0.01, 0.02, 0.03, 0.04, 0.05\}.
\end{aligned}
$$

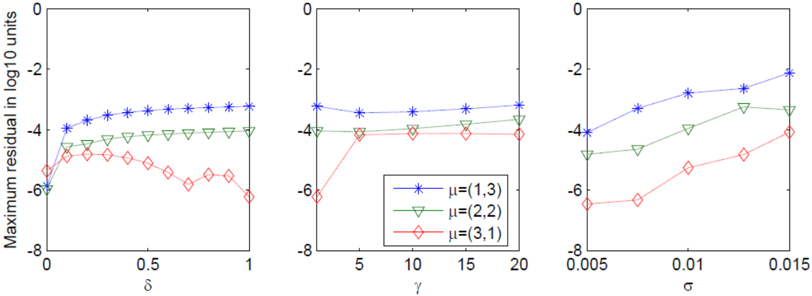
Conventional (isotropic) sparse grids under different approximation levels

- Consider approximation levels $\mu = 1, 2, 3, 4$.

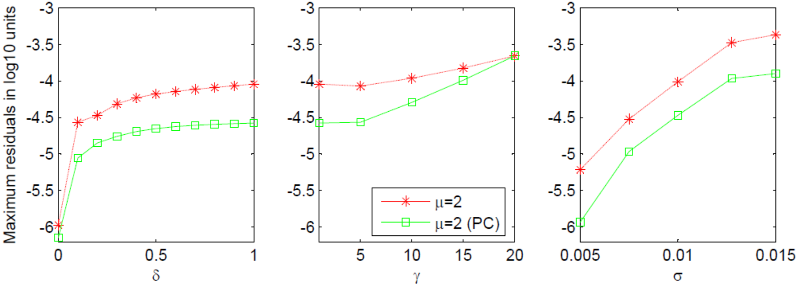


Anisotropic sparse grids

- Consider 2 anisotropic cases: $\mu = (3,1)$ and $\mu = (1,3)$.
- There are 9 elements in the first dimension and 3 elements in the second dimension $\Longrightarrow$ 15 grid points and 15 basis functions.



Adaptive domain



Multicountry model

$$\max_{\left\{c_t^h, k_{t+1}^h\right\}_{t=0,\ldots,\infty}^{h=1,\ldots,N}} E_0 \sum_{h=1}^{N} \tau^h \left(\sum_{t=0}^{\infty} \beta^t u^h \left(c_t^h\right) \right)$$

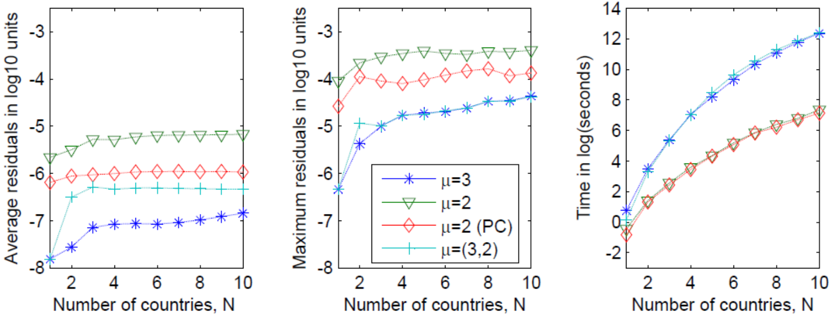
$$\text{s.t.} \quad \sum_{h=1}^{N} c_t^h = \sum_{h=1}^{N} \left[\theta_t^h f^h \left(k_t^h\right) + k_t^h (1-\delta) - k_{t+1}^h \right],$$

$$\ln \theta_t^h = \rho \ln \theta_{t-1}^h + \epsilon_t^h,$$

- c_t^h, k_t^h, a_t^h, u^h, f^h and τ^h = consumption, capital, productivity level, utility function, production function, welfare weight of a country h;
- $\epsilon_t^h \equiv \epsilon_t + \varpi_t^h$, $\epsilon_t \sim N(0, \sigma)$ is a common-for-all-countries shocks, $\varpi_t^h \sim N(0, \sigma)$ is a country-specific productivity shocks;
- Thus, $\left(\epsilon_t^1, \ldots, \epsilon_t^N\right)^\top \sim \mathcal{N}(0_N, \Sigma)$, with $0_N \in \mathbb{R}^N$,

$$\Sigma = \begin{pmatrix} 2\sigma^2 & \ldots & \sigma^2 \\ \ldots & \ldots & \ldots \\ \sigma^2 & \ldots & 2\sigma^2 \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

Results for the multicountry model



Conclusion

- The Smolyak method is designed to deal with high-dimensional problems, but its cost still grows rapidly with dimensionality, especially if we target a high quality of approximation.
- We propose a variant of the Smolyak method that has a better performance (lower cost and higher accuracy).
  - We introduce formula for Smolyak polynomials that avoids repetitions and eliminates unnecessary function evaluations.
  - We propose a simple Lagrange-style technique for finding the polynomial coefficients.
  - We develop an anisotropic version of the Smolyak grid that takes into account an asymmetric structure of variables in economic model.
  - As a solution domain, we use a minimum hypercube that encloses the high-probability set of a given economic model.
- The above four improvements are related to Smolyak interpolation. Our last improvement is concerned with an iterative procedure for solving dynamic economic models. We propose to use fixed-point iteration instead of time iteration.