Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs*

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Abstract

We apply numerical dynamic programming to multi-asset dynamic portfolio optimization problems with proportional transaction costs. Examples include problems with one safe asset plus two to six risky stocks, and seven to 360 trading periods in a finite horizon problem. These examples show that it is now tractable to solve such problems.

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JEL Classification: C61, C63, G11

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1 Introduction

Multi-stage portfolio optimization problems with transaction costs assume that n risky assets ("stocks") and/or a riskless asset ("bank account" paying a fixed interest rate r) are available for trading over the time $[0, T]$. Adjustments of the assets are made through N stages in $[0, T]$, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, to maximize the investor's expected utility of terminal wealth (T is the terminal time) and/or consumptions at each period. The portfolio re-allocation will incur transaction costs. We always assume that the stocks may be bought or sold in arbitrary amounts (not necessarily integral number of shares, and they will be bounded if there are "no-shorting" or "no-borrowing" constraints).

Multi-stage portfolio optimization problems with transaction costs have been studied in many papers. The problem with one risky asset has been well studied; see Zabel (1973), Constantinides (1976, 1986), Gennotte and Jung (1994), and Boyle and Lin (1997). Kamin (1975) considered the case with only two risky assets. Constantinides (1979) and Abrams and Karmarkar (1980) established some properties of the "no-trade" region (NTR) for multiple assets, but present only numerical examples with one safe and one risky asset. Brown and Smith (2011) evaluated some heuristic strategies and their bounds based on simulation, but their method cannot give the optimal portfolios.

In the continuous-time version, there are many papers about the portfolio optimization problem with transaction costs with two or fewer risky assets; see Davis and Norman (1990), Duffie and Sun (1990), Akian et al. (1996), Janecek and Shreve (2004), and Liu (2004). Muthuraman and Kumar (2006, 2008) gave numerical examples at most three risky assets. In Muthuraman and Zha (2008), they provided a computational scheme by combining simulation with the boundary update procedure while the objective is to maximize the long-term expected growth rate, and presented some computational results with $k \geq 3$. But, since they incorporated simulation into the computational scheme, the accuracy of solutions cannot be guaranteed.

To the best of our knowledge, when the number of correlated risky assets is bigger than three and $T \geq 6$, our DP method is the first one to explicitly give a good numerical solution with transaction costs and non-quadratic utility functions. See Cai (2009) and Cai and Judd (2010).

2 Portfolio Models

Assume that there are k risky assets ("stocks") and one riskless asset ("bond") available for investment. The investor's objective is to maximize the expected utility at the terminal time T. In $[0, T]$, there are $N + 1$ stages, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. At each stage t_i (for $i = 0, \ldots, N - 1$), the investor has an opportunity to reallocate the portfolio, which will incur transaction costs. For simplicity, we assume these stages are equally separated with a unit length of time, and then we use $t = i$ as the i-th stage in this paper.

Let $R = (R_1, \ldots, R_k)^\top$ be the random one-period return vector of the stocks, and R_f be the return of the bond. The portfolio fraction for asset i at the beginning of period t right before reallocation is denoted x_{ti}, and let $x_t = (x_{t,1}, \ldots, x_{t,k})^\top$. Let W_t be the total wealth at the beginning of period t right before reallocation. The difference between the total wealth and the wealth invested in stocks is invested in the bond. Let $\delta_{t,i} W_t$ denote the amount of dollars for buying or selling part of the i-th stock at stage t, expressed as a fraction of wealth, while $\delta_{t,i} > 0$ means buying, and $\delta_{t,i} < 0$ means selling. We assume that $f(\delta_{t,i} W_t) = \tau |\delta_{t,i} W_t|$ with a constant $\tau > 0$ is the transaction cost function for buying or selling part of the i-th stock using $\delta_{t,i} W_t$ dollars. Let $\mathbf{e}$ denote the column vector of all elements equal to

1. Then the multi-stage portfolio optimization problem can be expressed as

$$\begin{aligned} V_0(W_0, x_0) = \max_{\delta_t} \quad & \mathbb{E}\left\{u(W_T)\right\} \\ \text{s.t.} \quad & W_{t+1} = \mathbf{e}^\top X_{t+1} + R_f(1 - \mathbf{e}^\top x_t - y_t)W_t), \\ & X_{t+1,i} = R_i(x_{t,i} + \delta_{t,i})W_t, \\ & y_t = \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\ & x_{t+1,i} = X_{t+1,i}/W_{t+1}, \\ & t = 0, \ldots, T-1; \quad i = 1, \ldots, k, \end{aligned}$$

where $\mathbb{E}\{\cdot\}$ is the expectation operator, $X_{t+1} = (X_{t+1,1}, \ldots, X_{t+1,k})^\top$ is the vector of the amount of dollars invested in the risky assets at stage $t+1$, and $\delta_t = (\delta_{t,1}, \ldots, \delta_{t,k})^\top$. Note that all of X_{t+1}, x_{t+1}, W_{t+1}, y_{t+1} and δ_{t+1} are random as R is random. So when R has a continuous distribution, R must be discretized or simulated so that the model is solvable. But the exponential growth of scenarios over the number of periods T allows us to solve this model directly only when T is very limited, typically $T \leq 4$. This limitation makes it necessary to change this model into a DP model discussed in Section 3.

To be more general, we can assume that R_f and the multivariate distribution of R are dependent on a vector discrete time stochastic process θ_t, denoted by $R_f(\theta_t)$ and $R(\theta_t)$ respectively, for $t = 0, \ldots, T-1$. Then the above model becomes

$$\begin{aligned} V_0(W_0, x_0, \theta_0) = \max_{\delta_t} \quad & \mathbb{E}\left\{u(W_T)\right\} \\ \text{s.t.} \quad & W_{t+1} = \mathbf{e}^\top X_{t+1} + R_f(\theta_t)(1 - \mathbf{e}^\top x_t - y_t)W_t, \\ & X_{t+1,i} = R_i(\theta_t)(x_{t,i} + \delta_{t,i})W_t, \\ & y_t = \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\ & x_{t+1,i} = X_{t+1,i}/W_{t+1}, \\ & t = 0, \ldots, T-1; \quad i = 1, \ldots, k. \end{aligned}$$

When there is a consumption decision at each stage, a dynamic portfolio optimization problem is to find an optimal portfolio and a consumption decision C_t at each stage t such that we have a maximal expected total utility, i.e.,

$$\max \ \mathbb{E}\left\{\beta^T u(W_T) + \sum_{t=0}^{T-1}\beta^t u(C_t)\right\},$$

where u is the given utility function, β is the discount factor, $W_{t+1} = \mathbf{e}^\top X_{t+1} + R_f(1 - \mathbf{e}^\top x_t - y_t)W_t - C_t)$, and X_t, x_t and y_t have the same definitions in the equality constraints of the first model.

All these models have their corresponding DP models in the next section to avoid the exponential growth of scenarios over the number of periods T, so that these problems can be solved for a few periods.

3 DP Models for Portfolio Problems with Transaction Costs

For dynamic portfolio problems with transaction costs, we can choose the state variables as the total wealth W_t and allocation fractions $x_t = (x_{t,1}, \ldots, x_{t,k})^\top$ invested in the risky assets. Here W_t and x_t are the values right before reallocation at time t. Thus, the DP model becomes

$$V_t(W_t, x_t) = \max_{\delta_t} \ \ \mathbb{E}\left\{V_{t+1}(W_{t+1}, x_{t+1})\right\},$$

where

$$\begin{aligned}
y_t &\equiv \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\
X_{t+1,i} &\equiv R_i(x_{t,i} + \delta_{t,i})W_t, \\
W_{t+1} &\equiv \mathbf{e}^\top X_{t+1} + R_f(1 - \mathbf{e}^\top x_t - y_t)W_t, \\
x_{t+1,i} &\equiv X_{t+1,i}/W_{t+1},
\end{aligned}$$

for $i = 1, \ldots, k$. The terminal value function is $V_T(W, x) = u(W)$ for some given utility function u. Sometimes, the terminal value function is chosen as $V_T(W, x) = u((1 - \tau\mathbf{e}^\top x)W)$, if we assume that all risky assets have to be converted into the riskless asset before consumption. Later, we just assume that $V_T(W,x) = u(W)$ for simplicity.

If we do not allow shorting stocks or borrowing cash, then we just need to add the constraints $x_t + \delta_t \geq 0$ and $1 - \mathbf{e}^\top x_t \geq y_t$. And the range of x_t and x_{t+1} is $[0,1]^k$ while $\mathbf{e}^\top x_{t+1} \leq 1$.

3.1 Portfolio with a CRRA Utility

In economics and finance, we usually assume a CRRA utility function, i.e., $u(W) = W^{1-\gamma}/(1-\gamma)$ for some constant $\gamma > 0$ and $\gamma \neq 1$, or $u(W) = \log(W)$ for $\gamma = 1$. Thus, for $u(W) = W^{1-\gamma}/(1-\gamma)$, if we assume that $V_{t+1}(W_{t+1}, x_{t+1}) = W_{t+1}^{1-\gamma} \cdot g_{t+1}(x_{t+1})$, then

$$V_t(W_t, x_t) = \max_{\delta_t} \quad \mathbb{E}\left\{W_{t+1}^{1-\gamma} \cdot g_{t+1}(x_{t+1})\right\},$$

where

$$\begin{aligned}
s_{t+1,i} &\equiv R_i(x_{t,i} + \delta_{t,i}), \\
y_t &\equiv \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\
\Pi_{t+1} &\equiv \mathbf{e}^\top s_{t+1} + R_f(1 - \mathbf{e}^\top x_t - y_t), \\
W_{t+1} &\equiv \Pi_{t+1} W_t, \\
x_{t+1,i} &\equiv s_{t+1,i}/\Pi_{t+1},
\end{aligned}$$

for $i = 1, \ldots, k$. By changing W_{t+1} to $\Pi_{t+1}W_t$ in the objective function, we get $V_t(W_t, x_t) = W_t^{1-\gamma} \cdot g_t(x_t)$, where

$$g_t(x_t) = \max_{\delta_t} \quad \mathbb{E}\left\{\Pi_{t+1}^{1-\gamma} \cdot g_{t+1}(x_{t+1}) \mid x_t\right\}. \tag{1}$$

Therefore, by induction, from $V_T(W,x) = u(W) = W^{1-\gamma} \cdot 1/(1-\gamma)$, we showed that

$$V_t(W_t, x_t) = W_t^{1-\gamma} \cdot g_t(x_t)$$

for any time $t = 0, 1, \ldots, T$, while $g_t(x)$ has the iterative formula given in the above optimization problem and $g_T(x) = 1/(1-\gamma)$ (or $g_T(x) =$

$(1-\tau\mathbf{e}^\top|x|)^{1-\gamma}/(1-\gamma)$ when we assume that all risky assets have to be converted into the riskless asset before consumption), if we assume a proportional transaction cost and a power utility $u(W) = W^{1-\gamma}/(1-\gamma)$ with a constant relative risk aversion coefficient $\gamma > 0$ and $\gamma \neq 1$.

For $u(W) = \log(W)$, we can also show by induction that

$$V_t(W_t, x_t) = \log(W_t) + g_t(x_t),$$

where

$$g_t(x_t) = \max_{\delta_t} \quad \mathbb{E}\left\{\log(\Pi_{t+1}) + g_{t+1}(x_{t+1}) \mid x_t\right\}, \tag{2}$$

while $g_T(x) = 0$ (or $g_T(x) = \log(1-\tau\mathbf{e}^\top|x|)$, when we assume that all risky assets have to be converted into the riskless asset before consumption).

Since W_t and x_t are separable for CRRA utilities, we could just do a backward recursion on the functions $g_t(x)$ instead of $V_t(W, x)$.

When there is no riskless asset, we just need to cancel the R_f term and replace m_t with 0 in the above models for g_t or ψ_t, while we should have $\mathbf{e}^\top x_t = 1$. (The state variable vector $x_t = (x_{t,1}, \ldots, x_{t,k})$ should be changed to $(x_{t,1}, \ldots, x_{t,k-1})$, and there is the same cutoff in x_{t+1}).

If we do not allow shorting stocks or borrowing cash, then the range of x_t is $[0,1]^k$, and in the models we just need to add the constraints $x_t + \delta_t \geq 0$ and $y_t \leq 1 - \mathbf{e}^\top x_t$, so that $x_{t+1} \in [0,1]^n$ and $\mathbf{e}^\top x_{t+1} \leq 1$. And we still have the property of separation of W and x in the value functions $V(W, x)$.

We know that there is a NTR, Ω_t, for any $t = 0, 1, \ldots, T-1$. When $x_t \in \Omega_t$, the investor will not trade at all, and when $x_t \notin \Omega_t$, the investor will trade to some point on the boundary of Ω_t. That is, Ω_t is defined as

$$\Omega_t = \{x_t : \ \delta_t^* = 0\},$$

where δ_t^* is the optimal control for the given x_t. See Kamin (1975), Constantinides (1976, 1979, 1986), Davis and Norman (1990), Muthuraman and Kumar (2006), and so on.

Abrams and Karmarkar (1980) showed that NTR is a connected set and that it is a cone when the utility function is assumed to be positively homogeneous (a function $u(x)$ is positively homogeneous if there exists a positive value function $\psi(x)$ such that $u(ax) = \psi(a)u(x)$ for any $a > 0$). Moreover, in the case of proportional transaction costs and concave utility functions, NTR can take on many forms ranging from a simple halfline to a nonconvex set. So we should use numerical methods to compute NTR.

From the separability of W and x, we see that the optimal portfolio rules are independent of wealth W_t. Thus the "no-trade" regions Ω_t are also independent of W_t, for the CRRA utility functions.

3.2 Serially Correlated Asset Returns with Stochastic Parameters

In the previous model, we assume that the interest rate is fixed and the risky asset returns have the same multivariate distribution throughout time, and they are not serially correlated. But in real-life models, the riskless return R_f and the multivariate distribution of risky asset returns are stochastic and serial-correlated. Assume that they are dependent on some stochastic parameters. Let all these parameters be denoted as a vector θ_t at time t. They could be discrete Markov chains with a given transition probability matrix from the previous stage to the current stage, or continuously distributed, conditional on their previous-stage values. Let us denote $R_f(\theta_t)$ for the bond return and $R(\theta_t)$ for the stocks return vector respectively, for $t = 0, \ldots, T-1$.

With wealth W_t, allocation fractions x_t and parameters θ_t as the state variables, the DP model becomes

$$V_t(W_t, x_t, \theta_t) = \max_{\delta_t} \quad \mathbb{E}\left\{V_{t+1}(W_{t+1}, x_{t+1}, \theta_{t+1}) \mid W_t, x_t, \theta_t\right\},$$

where the terminal value function is $V_T(W, x, \theta) = u(W)$, and

$$\begin{aligned} y_t &\equiv \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\ X_{t+1,i} &\equiv R_i(\theta_t)(x_{t,i} + \delta_{t,i})W_t, \\ W_{t+1} &\equiv \mathbf{e}^\top X_{t+1} + R_f(\theta_t)(1 - \mathbf{e}^\top x_t - y_t)W_t, \\ x_{t+1,i} &\equiv X_{t+1,i}/W_{t+1}. \end{aligned}$$

Like what we discussed in Section 3.1, when the utility function is $u(W) = W^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$ and $\gamma \neq 1$, we have $V_t(W_t, x_t, \theta_t) = W_t^{1-\gamma} \cdot g_t(x_t, \theta_t)$ where

$$g_t(x_t, \theta_t) = \max_{\delta_t} \ \mathbb{E}\left\{\Pi_{t+1}^{1-\gamma} \cdot g_{t+1}(x_{t+1}, \theta_{t+1}) \mid x_t, \theta_t\right\}, \tag{3}$$

where

$$\begin{aligned} s_{t+1,i} &\equiv R_i(\theta_t)(x_{t,i} + \delta_{t,i}), \\ \Pi_{t+1} &\equiv \mathbf{e}^\top s_{t+1} + R_f(\theta_t)(1 - \mathbf{e}^\top x_t - y_t), \end{aligned}$$

with $g_T(x, \theta) = 1/(1-\gamma)$. Similarly, we can also have the separability of W and (x, θ) when $u(W) = \log(W)$. Moreover, the separability still holds if we add "no-shorting" and/or "no-borrowing" constraints.

From the separability of W and (x, θ), we see that the optimal portfolio rules are independent of wealth W_t. Thus NTR is also independent of W_t, for the CRRA utility functions, but it will be dependent on θ_t.

3.3 Portfolio with Transaction Costs and Consumption

When there is a consumption rate decision C_t at each period t, using W_t and x_t as state variables, the DP model becomes

$$V_t(W_t, x_t) = \max_{C_t, \delta_t} \ u(C_t)h + \beta\mathbb{E}\left\{V_{t+1}(W_{t+1}, x_{t+1})\right\},$$

where h is the length of one period,

$$\begin{aligned} y_t &\equiv \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\ X_{t+1,i} &\equiv R_i(x_{t,i} + \delta_{t,i})W_t, \\ W_{t+1} &\equiv \mathbf{e}^\top X_{t+1} + R_f(W_t(1 - \mathbf{e}^\top x_t - y_t) - C_t h), \\ x_{t+1,i} &\equiv X_{t+1,i}/W_{t+1}, \end{aligned}$$

and the terminal value function $V_T(W, x)$ is given.

Let $C_t = c_t W_t$. Like what we discussed in Section 3.1, when the utility function is $u(C) = C^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$ and $\gamma \neq 1$, and the terminal value function is $V_T(W, x) = W^{1-\gamma} \cdot g_T(x_t)$ for some given $g_T(x)$, we have

$$V_t(W_t, x_t) = W_t^{1-\gamma} \cdot g_t(x_t),$$

where

$$g_t(x_t) = \max_{c_t, \delta_t} \; u(c_t)h + \beta \mathbb{E}\left\{ \Pi_{t+1}^{1-\gamma} \cdot g_{t+1}(x_{t+1}) \mid x_t, c_t \right\}. \tag{4}$$

Here,

$$\begin{aligned} s_{t+1,i} &\equiv R_i(x_{t,i} + \delta_{t,i}), \\ \Pi_{t+1} &\equiv \mathbf{e}^\top s_{t+1} + R_f(1 - \mathbf{e}^\top x_t - y_t - c_t h). \end{aligned}$$

Similarly, we can also have the separability of W and x when $u(C) = \log(C)$ and $V_T(W, x) = \log(W) + g_T(x_t)$. Moreover, the separability still holds if we add "no-shorting" and/or "no-borrowing" constraints. If R and R_f are dependent on a Markov chain parameter vector θ_t, then we still have the separability of W and (x, θ).

From the separability of W and x, we see that the optimal portfolio rules are independent of wealth W_t. Thus NTR, Ω_t, is also independent of W_t, for the CRRA utility functions. Here Ω_t is defined as

$$\Omega_t = \{x_t/(1 - c_t^* h) : \; \delta_t^* = 0\},$$

where c_t^* and δ_t^* are the optimal controls for the given x_t.

4 Numerical DP Algorithms

If state and control variables in a DP problem are continuous, then the value function must be approximated in some computationally tractable manner. It is common to approximate value functions with a finitely parameterized collection of functions; that is, we use some functional form $\hat{V}(x;\mathbf{b})$, where $\mathbf{b}$ is a vector of parameters, and approximate a value function, $V(x)$, with $\hat{V}(x;\mathbf{b})$ for some parameter value $\mathbf{b}$. For example, $\hat{V}$ could be a linear combination of polynomials where $\mathbf{b}$ would be the weights on polynomials. After the functional form is fixed, we focus on finding the vector of parameters, $\mathbf{b}$, such that $\hat{V}(x;\mathbf{b})$ approximately satisfies the Bellman equation (Bellman, 1957).

Numerical solutions to a DP problem are based on the Bellman equation:

$$
\begin{aligned}
V_t(x,\theta) = \max_{a\in\mathcal{D}(x,\theta,t)} \quad & u_t(x,a) + \beta\mathbb{E}\left\{V_{t+1}(x^+,\theta^+) \mid x,\theta,a\right\}, \qquad (5)\\
\text{s.t.} \quad & x^+ = \mathcal{G}_t(x,\theta,a,\omega),\\
& \theta^+ = \mathcal{H}_t(\theta,a,\epsilon),
\end{aligned}
$$

where x is the continuous state, θ is the discrete state, $V_t(x,\theta)$ is called the value function at time $t \le T$ (the terminal value function $V_T(x,\theta)$ is given), a is the action variable vector in its feasible set $\mathcal{D}(x,\theta,t)$, x^+ is the next-stage continuous state with its transition function $\mathcal{G}_t$ at time t, θ^+ is the next-stage discrete state with its transition function $\mathcal{H}_t$ at time t, ω and ϵ are two random variables, and $u_t(x,a)$ is the utility function at time t, β is the discount factor, and $\mathbb{E}\{\cdot\}$ is the expectation operator.

The following is the algorithm of parametric DP with value function iteration for finite horizon problems.

Algorithm 1. *Numerical Dynamic Programming with Value Function Iteration for Finite Horizon Problems*

Initialization. *Choose the approximation nodes,* $\mathbb{X}_t = \{x_t^i : 1 \le i \le m_t\} \subset \mathbb{R}^d$, *for every* $t < T$, *and choose a functional form for* $\hat{V}(x,\theta;\mathbf{b})$. *Let* $\hat{V}(x,\theta;\mathbf{b}^T) \equiv V_T(x,\theta)$. *Then for* $t = T-1, T-2, \ldots, 0$, *iterate through* *steps 1 and 2.*

Step *1. Maximization Step. Compute*

$$\begin{aligned} v_{i,j} = \max_{a\in\mathcal{D}(x_i,\theta_j,t)} & \quad u_t(x_i,a) + \beta\mathbb{E}\left\{\hat{V}(x^+,\theta^+;\mathbf{b}^{t+1})\right\} \\ \text{s.t.} & \quad x^+ = \mathcal{G}_t(x_i,\theta_j,a,\omega), \\ & \quad \theta^+ = \mathcal{H}_t(\theta_j,a,\epsilon), \end{aligned}$$

for each $\theta_j \in \Theta$, $x_i \in \mathbb{X}_t$, $1 \le i \le m_t$.

Step *2. Fitting Step. Using an appropriate approximation method, compute the* $\mathbf{b}^t$ *such that* $\hat{V}(x,\theta_j;\mathbf{b}^t)$ *approximates* $(x_i, v_{i,j})$ *data for each* $\theta_j \in \Theta$.

Algorithm 1 shows that there are three main components in value function iteration for deterministic DP problems: optimization, approximation, and integration. More detailed discussion of numerical DP can be found in Cai (2009), Judd (1998) and Rust (2008).

4.1 Approximation

An approximation scheme consists of two parts: basis functions and approximation nodes. Approximation nodes can be chosen as uniformly spaced nodes, Chebyshev nodes, or some other specified nodes. From the viewpoint of basis functions, approximation methods can be classified as either spectral methods or finite element methods. A spectral method uses globally nonzero basis functions $\phi_j(x)$ such that $\hat{V}(x;\mathbf{b}) = \sum_{j=0}^{n} b_j\phi_j(x)$ is a degree-n approximation. Examples of spectral methods include ordinary polynomial approximation, Chebyshev polynomial approximation, and shape-preserving Chebyshev polynomial approximation (Cai and Judd, 2012a), and Hermite approximation (Cai and Judd, 2012c). In contrast, a finite element method uses locally basis functions $\phi_j(x)$ that are nonzero over sub-domains of the approximation domain. Examples of finite element methods include piecewise linear interpolation, cubic splines, B-splines, and shape-preserving rational splines (Cai and Judd, 2012b). See Cai (2009), Cai and Judd (2010), and Judd (1998) for more details.

We prefer Chebyshev polynomials when the value function is smooth. Chebyshev polynomials on $[-1,1]$ are defined as $\mathcal{T}_j(x) = \cos(j\cos^{-1}(x))$, while general Chebyshev polynomials on $[x_{\min}, x_{\max}]$ are defined as $\mathcal{T}_j((2x - x_{\min} - x_{\max})/(x_{\max} - x_{\min}))$ for $j = 0, 1, 2, \ldots$. In a d-dimensional approximation problem, let the domain of the value function be

$$\left\{x = (x_1, \ldots, x_d) : x_j^{\min} \leq x_j \leq x_j^{\max},\ j = 1, \ldots d\right\},$$

for some real numbers $x_j^{\min}$ and $x_j^{\max}$ with $x_j^{\max} > x_j^{\min}$ for $j = 1, \ldots, d$. Let $x^{\min} = (x_1^{\min}, \ldots, x_d^{\min})$ and $x^{\max} = (x_1^{\max}, \ldots, x_d^{\max})$. Then we denote $[x^{\min}, x^{\max}]$ as the domain. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a vector of nonnegative integers. Let $\mathcal{T}_\alpha(z)$ denote the product $\mathcal{T}_{\alpha_1}(z_1)\cdots\mathcal{T}_{\alpha_d}(z_d)$ for $z = (z_1, \ldots, z_d) \in [-1,1]^d$. Let

$$Z(x) = \left(\frac{2x_1 - x_1^{\min} - x_1^{\max}}{x_1^{\max} - x_1^{\min}}, \ldots, \frac{2x_d - x_d^{\min} - x_d^{\max}}{x_d^{\max} - x_d^{\min}}\right)$$

for any $x = (x_1, \ldots, x_d) \in [x^{\min}, x^{\max}]$.

Using these notations, the degree-n complete Chebyshev approximation for $V(x)$ is

$$\hat{V}_n(x; \mathbf{b}) = \sum_{0 \leq |\alpha| \leq n} b_\alpha \mathcal{T}_\alpha\left(Z(x)\right), \tag{6}$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$ for the nonnegative integer vector $\alpha = (\alpha_1, \ldots, \alpha_d)$. So the number of terms with $0 \leq |\alpha| = \sum_{j=1}^d \alpha_i \leq n$ is $\binom{n+d}{d}$ for the degree-n complete Chebyshev approximation in $\mathbb{R}^d$.

4.2 Numerical Integration

In the objective function of the Bellman equation, we often need to compute the conditional expectation of $V(x^+ \mid x, a)$. When the random variable is continuous, we have to use numerical integration to compute the expectation.

One naive way is to apply Monte Carlo or pseudo Monte Carlo methods to compute the expectation. By the central limit theorem in statistics, the numerical error of the integration computed by (pseudo) Monte Carlo methods has a distribution that is close to normal. So there is no bound for the numerical error occurred by (pseudo) Monte Carlo methods. Moreover, the optimization problem often needs hundreds or thousands of evaluations of the objective function. This implies that once one evaluation of the objective function has a big numerical error, the previous iterations to solve the optimization problem may make no sense. Therefore, the iterations may never converge to the optimal solution. Thus it is not practical to apply (pseudo) Monte Carlo methods to the optimization problem generally, unless the stopping criterion of the optimization problem is set very loosely.

Therefore, it will be good to have a numerical integration method with a bounded numerical error. Here we present a common numerical integration method when the random variable is normal.

In the expectation operator of the objective function of the Bellman equation, if the random variable has a normal distribution, then it will be good to apply the Gauss-Hermite quadrature formula to compute the numerical integration. That is, if we want to compute $\mathbb{E}\{f(Y)\}$ where Y has a distribution $\mathcal{N}(\mu, \sigma^2)$, then

$$\begin{aligned}
\mathbb{E}\{f(Y)\} &= (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} f(y) e^{-(y-\mu)^2/(2\sigma^2)} dy \\
&= (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} f(\sqrt{2}\,\sigma\,x + \mu) e^{-x^2} \sqrt{2}\sigma dx \\
&\doteq \pi^{-\frac{1}{2}} \sum_{i=1}^{m} \omega_i f(\sqrt{2}\sigma x_i + \mu),
\end{aligned}$$

where ω_i and x_i are the Gauss-Hermite quadrature weights and nodes over $(-\infty, \infty)$. See Stroud and Secrest (1966).

If Y is log normal, i.e., $\log(Y)$ has a distribution $\mathcal{N}(\mu, \sigma^2)$, then we can assume that $Y = e^X$ where $X \sim \mathcal{N}(\mu, \sigma^2)$, thus

$$\mathbb{E}\{f(Y)\} = \mathbb{E}\{f(e^X)\} \doteq \pi^{-\frac{1}{2}} \sum_{i=1}^{m} \omega_i f\left(e^{\sqrt{2}\sigma x_i + \mu}\right).$$

If we want to compute a multidimensional integration, we could apply the product rule. For example, suppose that we want to compute $\mathbb{E}\{f(X)\}$, where X is a random vector with multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$ over $(-\infty, +\infty)^d$, where μ is the mean column vector and Σ is the covariance matrix, then we could do the Cholesky factorization first, i.e., find a lower triangular matrix L such that $\Sigma = LL^\top$. This is feasible as Σ must be a positive semi-definite matrix from the covariance property. Thus,

$$
\begin{aligned}
\mathbb{E}\{f(X)\} &= \left((2\pi)^d \det(\Sigma)\right)^{-1/2} \int_{R^d} f(y) e^{-(y-\mu)^\top \Sigma^{-1}(y-\mu)/2} dy \\
&= \left((2\pi)^d \det(L)^2\right)^{-1/2} \int_{R^d} f\left(\sqrt{2}Lx + \mu\right) e^{-x^\top x} 2^{d/2} \det(L) dx \\
&\doteq \pi^{-\frac{d}{2}} \sum_{i_1=1}^{m} \cdots \sum_{i_d=1}^{m} \omega_{i_1} \cdots \omega_{i_d} f\Bigg(\sqrt{2}L_{1,1}x_{i_1} + \mu_1, \\
&\quad \sqrt{2}(L_{2,1}x_{i_1} + L_{2,2}x_{i_2}) + \mu_2, \cdots, \sqrt{2}(\sum_{j=1}^{d} L_{d,j}x_{i_j}) + \mu_d\Bigg),
\end{aligned}
$$

where ω_i and x_i are the Gauss-Hermite quadrature weights and nodes over $(-\infty, \infty)$, $L_{i,j}$ is the (i, j)-element of L, and $\det(\cdot)$ means the matrix determinant operator.

4.3 Parallelization

Parallelization allows researchers to solve huge problems and is the foundation of modern scientific computation. Our work shows that parallelization can also be used effectively in solving the dynamic portfolio optimization problems using value function iteration. The key fact is that at each maximization step, there are many independent optimization problems, one for each (x_i, θ_j). In our portfolio problems there are often thousands of such independent problems, and future problems will easily have millions of independent problems. See Cai et al. (2012d) for more detailed discussion.

5 Numerical DP in Portfolio Problems

In the models of Section 3, the constraints are nonlinear because of the existence of the absolute operator. We can change the nonlinear constraints into linear constraints. Let $\delta_t = \delta_t^+ - \delta_t^-$ with $\delta_t^+, \delta_t^- \geq 0$, we have $|\delta_t| = \delta_t^+ + \delta_t^-$. Here $\delta_t^+ = (\delta_{t,1}^+, \ldots, \delta_{t,k}^+)^\top$ is the vector of fractions of wealth W_t for buying stocks, and $\delta_t^- = (\delta_{t,1}^-, \ldots, \delta_{t,k}^-)^\top$ is the vector of fractions of wealth W_t for selling stocks. Thus, by using δ_t^+ and δ_t^- instead of δ_t in the models, all the constraints are linear, while

$$y_t \equiv \mathbf{e}^\top((\delta_t^+ - \delta_t^-) + \tau(\delta_t^+ + \delta_t^-)).$$

In fact, in the models of Section 3, variables y_t, s_{t+1}, Π_{t+1}, and x_{t+1} can be substituted. Thus, the maximization problems (1) and 3 become models with $2k$ control variables (δ_t^+ and δ_t^-), and $2k$ bound constraints ($\delta_t^+, \delta_t^- \geq 0$), where k is the number of risky assets. And the maximization problem (4) becomes a model with $2k+1$ control variables (c_t, δ_t^+ and δ_t^-), and $2k+1$ bound constraints ($c_t, \delta_t^+, \delta_t^- \geq 0$). If we do not allow shorting stocks or borrowing cash, then there are $(k+1)$ linear constraints: $x_t + \delta_t^+ - \delta_t^- \geq 0$ and $y_t \leq 1 - \mathbf{e}^\top x_t$ (or $y_t \leq 1 - \mathbf{e}^\top x_t - c_t$ when there is a consumption decision), where y_t should be replaced by $\mathbf{e}^\top((\delta_t^+ - \delta_t^-) + \tau(\delta_t^+ + \delta_t^-))$.

If the return for a risky asset i, R_i, is unbounded (e.g., log-normal), then we have $\mathbb{P}(\Pi_t \leq 0) > 0$ when $x_{t,i} + \delta_{t,i} < 0$. It follows that for the CRRA utility functions, the optimal solution must have $x_{t,i} + \delta_{t,i}^* \geq 0$ unless the asset i can be replicated by other assets, which is a degenerate case. Since the expectation of next-time value function is computed numerically by numerical DP, we must add the constraint $x_{t,i} + \delta_{t,i}^+ - \delta_{t,i}^- \geq 0$ to avoid an unreasonable approximation solution. If the no-boundedness of returns applies for all risky assets, then for the CRRA utility functions, we must have both "no-shorting" and "no-borrowing" constraints: $x_t + \delta_t^+ - \delta_t^- \geq 0$ and $y_t \leq 1 - \mathbf{e}^\top x_t$ (or $y_t \leq 1 - \mathbf{e}^\top x_t - c_t$ when there is a consumption decision). Thus, we can set the domain of x_t as $[0,1]^k$ for each stage t.

In the portfolio optimization problems, R is often assumed to be log-normal and correlated. Assume that the random log-returns of the stocks, $\log(R) = (\log(R_1), \cdots, \log(R_k))^\top \in \mathbb{R}^k$, have a multivariate normal distribution $\mathcal{N}((\mu - \frac{\sigma^2}{2})h, (\Lambda\Sigma\Lambda)h)$ in $\mathbb{R}^k$, where h is the length of a period, $\mu = (\mu_1, \cdots, \mu_k)^\top$ is the drift, $\sigma = (\sigma_1, \cdots, \sigma_k)^\top$ is the volatility, and Σ is the correlation matrix of the log-returns, and $\Lambda = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$. Let the interest rate be r, then the continuously compounded risk-free return over a period is $R_f = \exp(rh)$.

Since the correlation matrix is positive definite, we can apply the Cholesky factorization $\Sigma = LL^\top$, where $L = (L_{i,j})_{k\times k}$ is a lower triangular matrix. Then

$$\log(R_i) = (\mu_i - \frac{\sigma_i^2}{2})h + \sigma_i\sqrt{h}\sum_{j=1}^{i} L_{i,j}z_j,$$

where z_i are independent standard normal random variables, for $i = 1, \ldots, k$. Therefore, for the optimization problems (1), (3) and (4), we can apply a product Gauss-Hermite quadrature to estimate the conditional expectation of $\Pi_{t+1}^{1-\gamma} \cdot g_{t+1}(x_{t+1})$ while x_t is given.

6 Numerical Examples

In this section, we give several numerical examples for solving the multi-stage portfolio optimization problems with proportional transaction costs and a power utility function $u(W) = W^{1-\gamma}/(1-\gamma)$. In these examples, the length of one period is h, $R_f = \exp(rh)$ for an interest rate r, and R is always assumed to be log-normal with

$$\log(R) \sim \mathcal{N}((\mu - \frac{\sigma^2}{2})h, (\Lambda\Sigma\Lambda)h)$$

in $\mathbb{R}^k$, where $\mu = (\mu_1, \cdots, \mu_k)^\top$ is the drift, $\sigma = (\sigma_1, \cdots, \sigma_k)^\top$ is the volatility, and Σ is the correlation matrix of the log-returns, and $\Lambda = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$.

In these examples, the number of stages is from 7 to 361, and the number of assets is from 3 to 7. By using the numerical DP method, we computed the "no-trade" regions for each stage in these examples. We applied the

Table 1: Parameters and Running Times for Examples of Portfolio Problems without Consumption

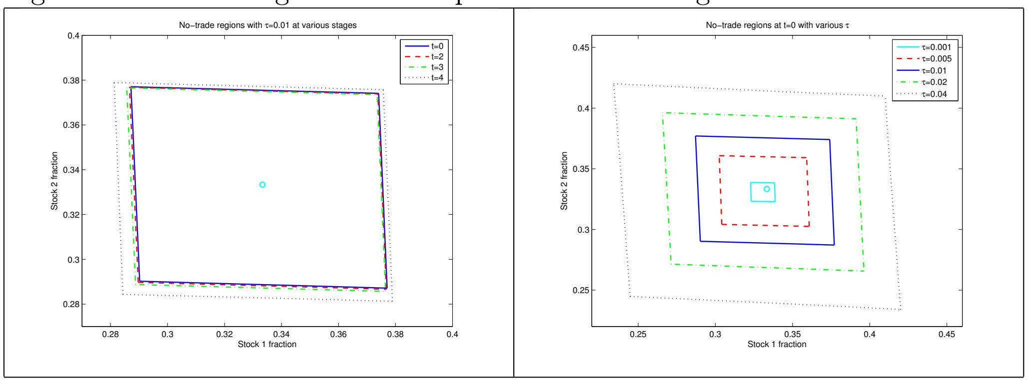
	Example 1	Example 2
k	2	2
h	1	1
T	6	6
γ	3	3
r	0.03	0.04
$\mu^\top$	$(0.07, 0.07)$	$\mu_i = 0.06$ or 0.08 for $i = 1, 2$
$\sigma^\top$	$(0.2, 0.2)$	$(0.2, 0.2)$
Σ	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Running time	0.7 seconds	2.7 seconds

NPSOL optimization package (Gill et al., 1994), and the complete Chebyshev approximation method, and the multi-dimensional product Gauss-Hermite quadrature rule in the numerical DP method for these examples.

6.1 Three-Asset Portfolio Problems without Consumption

We gives two examples to illustrate our numerical DP method for solving the model (1) and (3) respectively, the multi-stage portfolio optimization problems without consumption. In these examples, the assets available for trading include one bond with a constant interest rate r and two stocks with independent log-normal annual returns. We assume that the transaction cost proportion is τ, the utility function at the terminal time $T = 6$ years is $u(W) = W^{1-\gamma}/(1-\gamma)$. In the numerical DP method for these examples, we use the degree-10 complete Chebyshev approximation method with 11^k tensor Chebyshev nodes for k stocks, and the multi-dimensional product Gauss-Hermite quadrature rule with 9 nodes in each dimension. Table 1 lists the parameters and running times on a single core of a Mac laptop with a 2.5 GHz processor.

Figure 1: No trade regions of Example 1 at various stages or with various τ



6.1.1 Example 1

The first example is a simple three asset portfolio problem of the model (1). Figure 1 shows the NTR with $\tau = 0.01$ at stages $t = 0, 2, 3, 4$, and the NTR at stage $t = 0$ with various $\tau = 0.001, 0.005, 0.02, 0.04$. The circle point located inside NTR is the Merton's point. We see that these NTR are close to be square and also symmetric along the 45 degree line. Moreover, the NTR with larger τ contains the NTR with smaller τ.

6.1.2 Example 2

In this example, we solve the model (3), where the drift terms of stocks, $\mu_t = (\mu_{t,1}, \mu_{t,2})^\top$, are discrete Markov chains and independent of each other. The transition probability matrix

$$\mathbb{P}(\mu_{t+1,i} \mid \mu_{t,i}) = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

for $i = 1, 2$, while $\mu_{t,1}$ is independent of $\mu_{t,2}$. Let the proportional transaction cost ratio is $\tau = 0.01$ for buying or selling stocks.

Figure 2 displays NTR for four possible discrete states of $(\mu_{t,1}, \mu_{t,2})$ at stages $t = 0, 1, 4, 5$. The top-right squares are NTR for the state $(\mu_{t,1}, \mu_{t,2}) = (0.08, 0.08)$, and the bottom-left squares are NTR for the state $(\mu_{t,1}, \mu_{t,2}) = (0.06, 0.06)$, and the top-left and the bottom-right squares are respectively

Figure 2: NTR for 2 stocks with stochastic μ and 1 bond

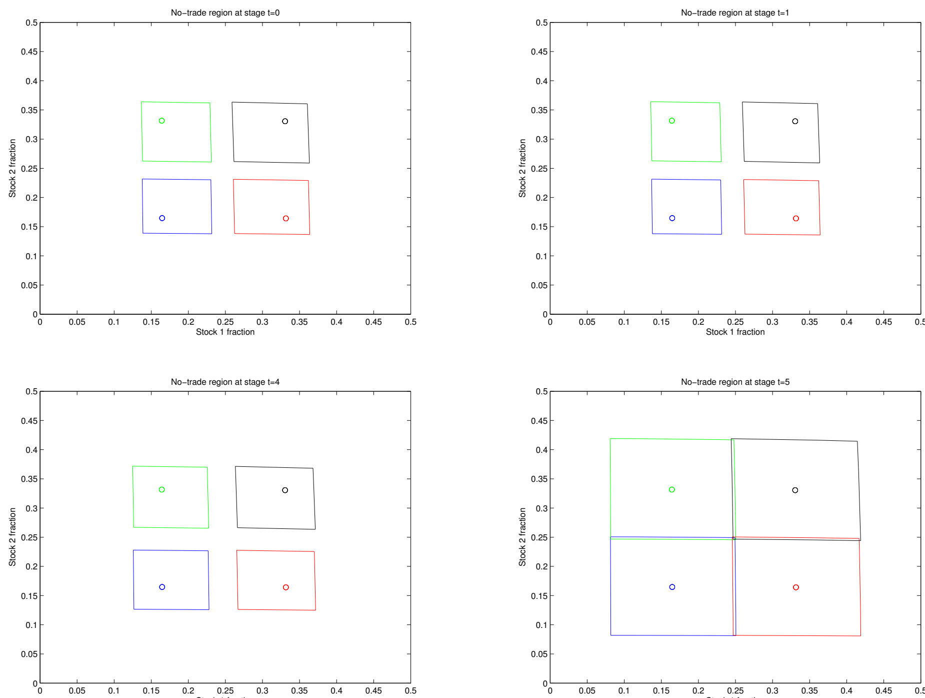


Table 2: Parameters and Running Times for Examples of Portfolio Problems with Consumption

	Example 3	Example 4
k	2	3
h	1/12 (1 month)	1
T	360	6
γ	2	3
β	$\exp(-0.1h)$	0.95
r	0.07	0.04
$\mu^\top$	$(0.15, 0.15)$	$(0.07, 0.07, 0.07)$
$\sigma^\top$	$(\sqrt{0.17}, \sqrt{0.17})$	$(0.2, 0.2, 0.2)$
Σ	$\begin{bmatrix} 1 & 0.4706 \\ 0.4706 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.4 & 0.4 \\ 0.4 & 1 & 0.16 \\ 0.4 & 0.16 & 1 \end{bmatrix}$
Running time	3.5 minutes	8 minutes

NTR for the state $(\mu_{t,1}, \mu_{t,2}) = (0.06, 0.08)$ and $(\mu_{t,1}, \mu_{t,2}) = (0.08, 0.06)$. The circle points inside the squares are the optimal allocation fractions given the discrete states $(\mu_{t,1}, \mu_{t,2})$ when we assume that there is no transaction cost in trading stocks.

6.2 Portfolio Problems with Consumption

We gives two examples to illustrate our numerical DP method for solving the model (4), the multi-stage portfolio optimization problems with consumption. In these examples, the assets available for trading include one bond with a constant interest rate r and two or three stocks with correlated log-normal annual returns. Table 2 lists the other parameters and running times on a single core of a Mac laptop with a 2.5 GHz processor.

6.2.1 Example 3

This example assume that the total investment horizon is 30 years and each re-balance allocation period is one month, which is the finite-horizon discretized approximation with different notations for the infinite-horizon continuous-time dynamic portfolio problem given by Discussion 1 in Muthuraman and Kumar (2006).

To approximate the infinite-horizon problem, we assume that after the terminal stage $T = 360$, the transaction cost becomes 0, so the investor will always choose the Merton ratio,

$$x^* = \frac{(\Lambda\Sigma\Lambda)^{-1}(\mu - r)}{\gamma},$$

as the optimal portfolio and the optimal consumption rate is

$$c^* \equiv \frac{1}{\gamma}\left[\rho - (1-\gamma)\left(\frac{(\mu - r)^\top x^*}{2} + r\right)\right] = 9.14\%,$$

where $\rho = -\log(\beta)/h = 0.1$ is the continuously compounded discount rate corresponding to the discretely compounded discount factor β. See Merton (1969) for more details. In order to rebalance the portfolio x to the Merton ratio x^* at the terminal time, there are transaction costs: $\tau \mathbf{e}^\top |x - x^*|$. Therefore, the terminal value function is $V_T(W, x) = W^{1-\gamma} \cdot g_T(x)$, with

$$g_T(x) = \left(\tau \mathbf{e}^\top |x - x^*|\right) \int_0^\infty e^{-\rho t} u\left(c^*\right) dt = \left(\tau \mathbf{e}^\top |x - x^*|\right) \frac{\left(c^*\right)^{1-\gamma}}{\rho(1-\gamma)}.$$

In the numerical DP method for this example, we use the degree-14 complete Chebyshev approximation method with 15^2 tensor Chebyshev nodes for two stocks, and the multi-dimensional product Gauss-Hermite quadrature rule with 9 nodes in each dimension. We also ran the same example with $h = 0.25$ year and various τ using numerical DP algorithm.

Figure 3 displays NTR at the stage $t = 0, 345, 355, 358$, when $h = 1/12$ year and the transaction cost proportion is $\tau = 0.01$, and also shows NTR at $t = 0$ with $h = 0.25$ year and various transaction cost proportions $\tau = 0.001, 0.002, 0.005, 0.01, 0.02$. The circle point located inside NTR is the Merton's point $(0.16, 0.16)$. We found that NTR at stage $t = 0$ is close to the solution given in Muthuraman and Kumar (2006). We can also see that the NTR with larger τ contains the NTR with smaller τ.

Figure 3: NTR for 2 correlated stocks and 1 bond with consumption (Example 2)

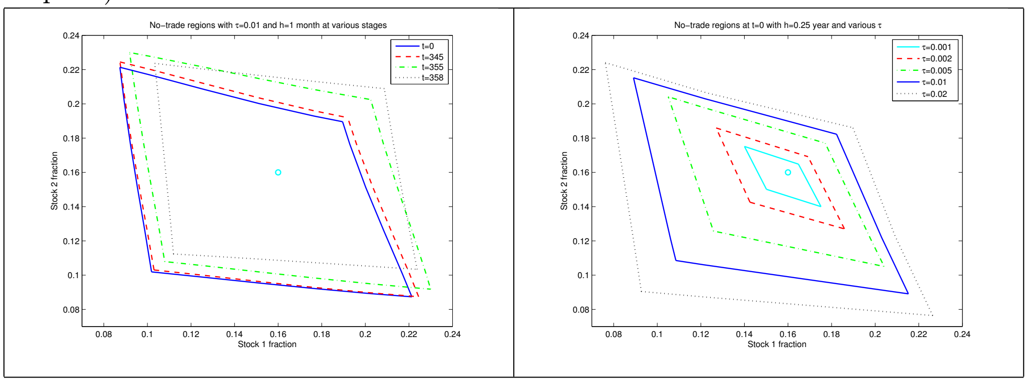


Table 3: Optimal Consumptions at $t = 0$ for Example 2

τ	0.04	0.02	0.01	0.005	0.002	0.0
c_0^*	8.95%	8.99%	9.02%	9.03%	9.04%	9.06%

For the monthly rebalanced portfolio problem with $\tau = 0.01$, the optimal consumption rate decisions when $\delta^* = 0$ at the stages are given as

$$c_{359}^* \approx 9.473\%,\ c_{358}^* \approx 9.469\%,\ c_{357}^* \approx 9.465\%, \ldots,$$
$$c_2^* \approx 9.0135\%,\ c_1^* \approx 9.0132\%,\ c_0^* \approx 9.013\%.$$

Therefore, the optimal consumption rate at the first period in the 360-period investment-consumption problem with transaction costs is a bit smaller than the optimal consumption rate of the infinite-horizon continuous-time investment-consumption problem without transaction costs. For the portfolio problems with $h = 0.25$, the computed optimal consumptions over the first period are listed in Table 3 for various τ. It is shown that NTR at time $t = 0$ converges to a point $(0.159, 0.159)$ (while the Merton's point is $(0.16, 0.16)$), and the optimal consumption rate at $t = 0$ converges to 9.06%, as $\tau \longrightarrow 0$.

6.2.2 Example 4

In this example, we have one bond and three stocks available for trading, with a transaction cost proportion $\tau = 0.01$. The terminal value function is $V_T(W, x) = W^{1-\gamma}/(1-\gamma)$. In the numerical DP method for this example, we use the degree-10 complete Chebyshev approximation method with 11^3 tensor Chebyshev nodes for three stocks, and the multi-dimensional product Gauss-Hermite quadrature rule with 9 nodes in each dimension. Figure 4 displays NTR for $t = 0, 1, 4, 5$.

Notice that the faces of NTR seem to be flat, but in fact there are small perturbation on the faces, which might be due to numerical errors or the possibility that the exact NTR might have curvy faces.

6.3 Seven-Asset Portfolio Problem

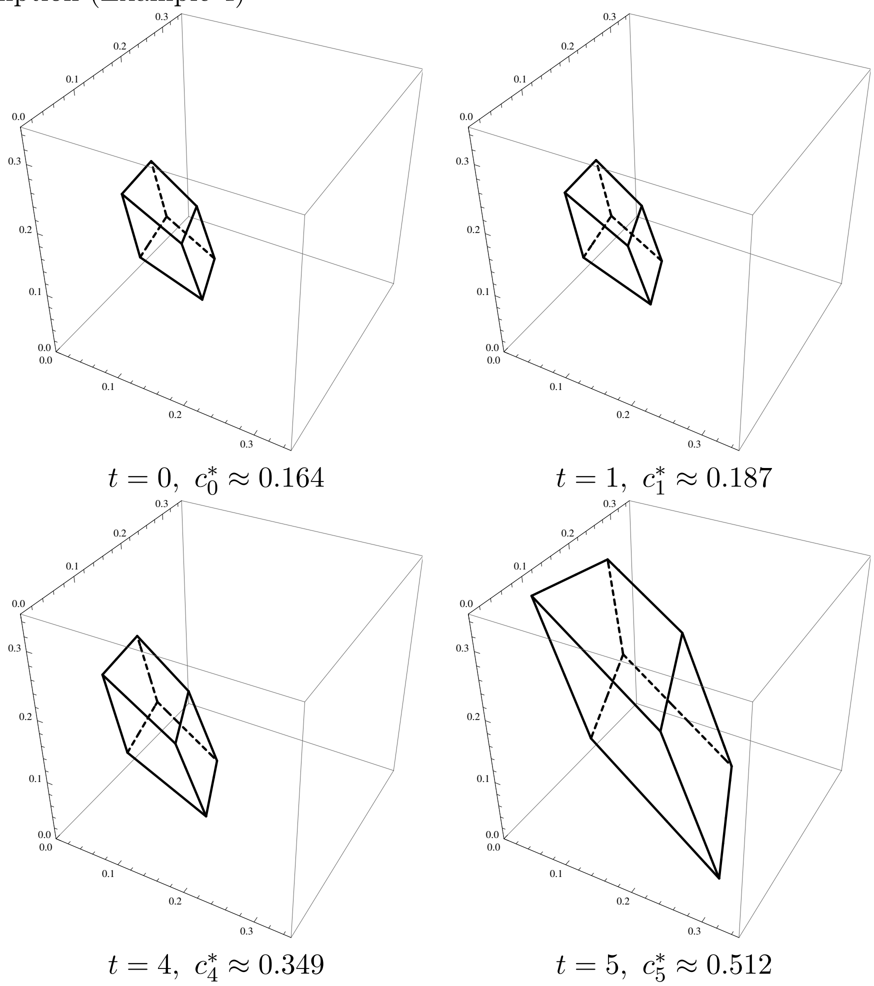
Our last example solves the model (1), where the assets available for trading include one bond with a constant interest rate $r = 0.03$ and six stocks with independent log-normal annual returns. We assume that

$$\begin{aligned}
\mu &= (0.06, 0.064, 0.068, 0.072, 0.076, 0.08)^\top, \\
\sigma &= (0.2, 0.22, 0.24, 0.26, 0.28, 0.3)^\top,
\end{aligned}$$

$\gamma = 5$, $\tau = 0.01$, and $h = 1$ year. We applied the degree-4 complete Chebyshev approximation method with 5 Chebyshev nodes in each dimension, and the multi-dimensional product Gauss-Hermite quadrature rule with 5 nodes in each dimension. We used the parallel DP algorithm in the Condor-MW system (Cai et al., 2012d) and 100 processors to solve this problem, and it took 1.3 hours to get the solutions.

Figure 5 displays NTR for $t = 0, 1, 4, 5$, and we see that the regions are close to hypercubes. In the figure, the circle, the mark and the plus are respectively the Merton ratios for stock 1 and 2, stock 3 and 4, and stock 5 and 6.

Figure 4: NTR for 3 stocks with higher correlation and 1 bond with consumption (Example 4)



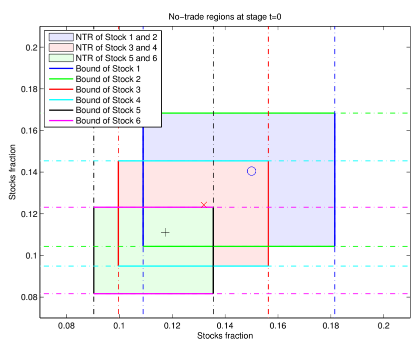
$t = 0,\ c_0^* \approx 0.164$

$t = 1,\ c_1^* \approx 0.187$

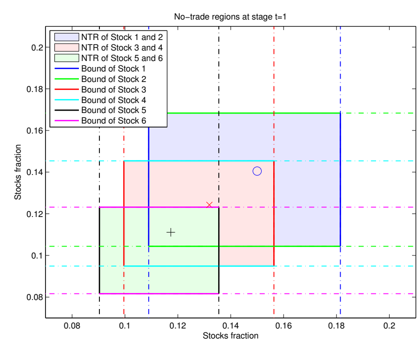
$t = 4,\ c_4^* \approx 0.349$

$t = 5,\ c_5^* \approx 0.512$

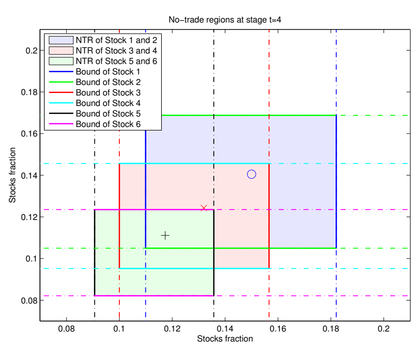
Figure 5: NTR for 6 independent stocks and 1 bond



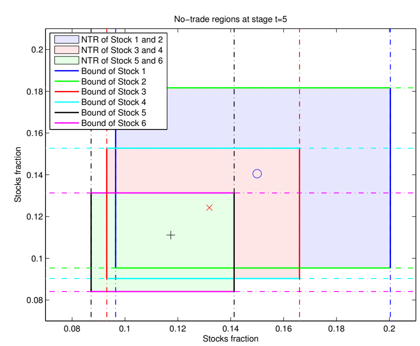
$t = 0$



$t = 1$



$t = 4$



$t = 5$

7 Conclusion

This paper has shown that numerical value function iteration can solve the multi-stage portfolio optimization problems with multiple assets and transaction costs in an efficient and accurate manner, and has given the no-trade regions for various cases, such as a portfolio with a risk-free bond and six risky stocks, or a portfolio with serially-correlated stock returns.

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