The Social Cost of Stochastic and Irreversible Climate Change*

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Abstract

There is great uncertainty about the impact of anthropogenic carbon on future economic wellbeing. We use DSICE, a DSGE extension of the DICE2007 model of William Nordhaus, which incorporates beliefs about the uncertain economic impact of possible climate tipping events and uses empirically plausible parameterizations of Epstein-Zin preferences to represent attitudes towards risk. We find that the uncertainty associated with anthropogenic climate change imply carbon taxes much higher than implied by deterministic models. This analysis indicates that there is much greater urgency to immediately enacting significant GHG policies than implied by DICE2007 and similar models that ignore uncertainty.

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1 Introduction

The global climate system is complex and its response to future increases in anthropogenic GHGs is poorly understood. Any rational policy choice must consider the great uncertainty about the magnitude and timing of global warming's impact on economic productivity. Policy analyses will often begin with estimating the expected social marginal cost of the an extra ton of carbon in the atmosphere. In 2010 the U.S. Government Interagency Working Group on Social Cost of Carbon (IWG, 2010) released an analysis of the social cost of carbon (SCC), and concluded that the 2010 SCC lies in the range \$5 - \$65 for the year 2010, with central price of \$21. Only recently, the Australian government has introduced a carbon tax of \$23. These examples demonstrate an increasing awareness of policy makers about climate policy and carbon pricing.

The IWG report relied on SCC studies in the climate and economics literature, with Nordhaus (2008) and Anthoff et al. (2009) being particularly important for the IWG. However, the IWG report, along with the majority of the literature, assumed that the economy and climate systems evolve deterministically. More generally, no study of the SCC models the stochastic nature of the climate and economic systems in the manner typically used in modern macroeconomics. Such an analysis would necessarily combine an advanced formulation of risk preferences consistent with empirical evidence with policy decision making processes at a time scale compatible with real world decision making. This is necessary for any realistic assessment of the costs of anthropogenic perturbations of current and future climate.

One dimension of realism is the intrinsically uncertain nature of future states of the economy. Analyses of any climate policy should account for its possible impacts on economic decision makers who face economic risks. Many issues, such as how GHG policies should react to economic fluctuations, can be studied only in models with time periods of one year or less, not in models where the time step is measured in decades.

Another dimension includes the uncertainty about the future evolution of the climate system. Several prominent studies (e.g., Nordhaus 2008) assume that damages are a function of contemporaneous temperature. However, possible climate change externalities are more complex. Many scientists are worried about climate change triggering abrupt and irreversible events leading to significant and long-lasting damages (see, e.g., Kriegler et al. 2009). Some elements of the climate system which might exhibit such a triggering effect are called tipping elements. In a prominent study, Lenton et al. (2008) characterizes tipping points for some major elements of the climate system. Examples of some tipping elements are: the weakening or shut down of the North Atlantic thermohaline circulation (THC), the melting of the Greenland ice sheet (GIS) and the West Antarctic ice sheet (WAIS), the die-back of the Amazon rainforest (AMAZ) and the increasing frequency and amplitude of El Niño-Southern Oscillation (ENSO).

While the likelihood of tipping points may be a function of contemporaneous temperature, their effects are long lasting and might be independent of future temperatures. It is assumed that some of these tipping points might occur even in this century, but also that their duration and post-tipping impact are uncertain (see Lenton, 2011). A faithful representation of the possibility of tipping points for the calculation of SCC would require a fully stochastic formulation of irreversibility, and accounting for the deep layer of uncertainties regarding the duration of the tipping process and also its economic impact.

The third component towards a more realistic assessment of the SCC is the modeling of the cost of risk in a manner more compatible with empirical evidence about social risk preferences. Since riskiness is inherent in the nature of tipping points, the socio-economic effects of stochastic and irreversible climate change will be affected by preferences about risks. We know from the equity premium literature that the standard formulations of preferences might misspecify how people feel about risk. Kreps and Porteus (1978) have argued that there could be value in early resolution of uncertainty, and Epstein-Zin (1989) preferences have explored the implications of this for asset pricing.

We account for all three components of a coherent analysis of climate policy under intrinsic uncertainty. This study builds on Cai, Judd and Lontzek (2012b) which combines standard features of DSGE models - productivity shocks, dynamic optimizing agents, and short time periods - with DICE2007, a seminal and basic Integrated Assessment Model (IAM). This study uses the computational framework of Cai, Judd and Lontzek (2012b) to address basic questions about uncertainty and the social cost of carbon. In particular, we present a dynamic stochastic model which incorporates the mutual interplay between climate and economics. We model a stochastic business cycle at the annual time scale, a stochastic tipping point system, and use Epstein-Zin preferences to model social preferences.

In general, IAMs study the interplay between the climate and the economic system. All large-scale IAMs have a complex representation of the climate system but many consider economic activity as exogenously given by some pre-defined range of scenarios. Examples are MAGICC (Wigley and Raper, 1997), ICAM (Dowlatabati and Morgan (1993) and IMAGE (Batjes and Goldewijk, 1994). These models neglect endogenous and forward-looking decisions and are unable to account for economic reactions to climate policies. They cannot study dynamic decision-making in an evolving and uncertain world. On the contrary, only a few models rely on solving an intertemporal optimization problem assuming that climate and the economy are endogenous and mutually dependent. Examples are DICE (Nordhaus, 2008), MERGE (Manne and Richels 2005) and RICE (Nordhaus and Yang, 1996). It is the latter class of models which is suitable for an uncertainty analysis in the climate policy decision making process.

Uncertainty analysis is regarded as an elementary part of these IAMs. Unfortunately, uncertainty analysis is mainly restricted to parametric uncertainty. Examples are Nordhaus (1994), Nordhaus (2008) and Pizer (1999). For general reviews, see Heal and Kriström (2002) and Pindyck (2007). Under parametric uncertainty, the modeler doesn't know the value of key parameters, e.g., climate sensitivity or some damage function parameter. The modeler has an idea about the distribution of these parameters and conducts a Monte Carlo analysis with many simulations, each being itself a deterministic run of the model with a picked set of parameter values. This method can provide valuable information about a possible range of the climate-economy system, but it always assumes that economic actors have perfect knowledge about all parameters. This approach focuses on the uncertainty of the "modeler", rather than uncertainty faced by the "decision maker". It is unclear if the analysis of parametric uncertainty should rely on certainty equivalent formulations, which in many contexts are not reliable analyses of how risk-averse agents respond to uncertainties in a dynamic world. More reliable analyses have recently been studied by a much smaller part of the literature (e.g. Kelly and Kolstad (1999) and Crost and Traeger (2011). These studies follow the stochastic approach, i.e. they assume the realization of the world to be uncertain due to some random events. Crost & Traeger (2011) e.g. formulate a stochastic IAM and find optimal control policies which significantly differ from the pure Monte Carlo analysis. However, on a larger scale IAM community has so far not produced a stochastic IAM flexible enough to represent uncertainty in a quantitatively realistic manner. Most existing stochastic dynamic optimization approaches to model uncertainty in IAM suffer from one or more of the following features: 1) long duration of time periods (e.g., 5 or 10 years), 2) short time horizon or small number of periods (e.g., 2 periods), 3) reduced dimension of a full scale IAM (e.g., a two-dimensional version of the seven-dimensional model), and finally 4) expandability of the model (e.g., the flexibility of the model and computational method dealing with a deepened complexity of the state space).

Several studies build on DICE2007 (Nordhaus, 2008) and attempt to study optimal climate policies under uncertainty. Examples include Kelly and Kolstad (1999) and Bahn et al. (2008). However, just as in the DICE2007 model, these studies assume 10-year time units. We argue that the representation of time should be compatible with the frequencies of both the natural and social processes related to climate change. Any IAM with 10-year time periods represents neither social nor physical processes because nontrivial dynamics and feedbacks may occur in either system during a single decade. Dynamic stochastic general equilibrium (DSGE) models in economics use relatively short time periods, always at most a year. Decadal time periods as in DICE2007 are too long for serious, quantitative analysis of policy questions. For example, if one wants to know how carbon prices should react to business cycle shocks, the time period needs to be at most a year. No one would accept a policy that takes ten years to respond to current shocks to economic conditions. Cai, Judd and Lontzek (2012c) shows that annual time periods produce a significantly different carbon price with the numbers given by DICE2007. Cai, Judd and Lontzek (2012a) develops DICE-CJL, a continuous-time extension of DICE2007, and then demonstrate that many substantive results depend critically on the time step, strongly supporting our contention that short time periods are necessary for a quantitatively reliable analysis.

Other IAM analyses that examine uncertainty assume a very small number of decision periods, often with very large time periods. For example, Webster et al. (2012) models a 7-period horizon version of DICE with 50-year time units, while the model of Fisher and Narain (2003) has only 2 periods. In general, while a small number of decision periods might deliver some important qualitative insights of the model, it is highly unlikely that such a model can capture the full dynamic evolution of the climate and economic systems.

We argue that any reliable IAM should incorporate the ability to study the optimal decision making over a very long time horizon with several hundred decision periods. A stochastic formulation of such a model necessitates a solution method using dynamic programming. As of today there are only a handful of models following that approach. Lontzek and Narita (2011) study a continuous-time, infinite horizon model integrating climate and the economy. A recent study by Lemoine and Traeger (2012) applies discrete-time dynamic programming methods. Lemoine and Traeger (2012) formulate a stochastic IAM with ambiguity and a tipping point allowing for learning about a tipping threshold. Both models build on DICE2007 but examine a lower-dimensional climate system for reasons of computational tractability. The DICE's six-dimensional state transition system is one in which all elements interact and there is no lower dimensional system that is equivalent. Therefore, we argue that, except for a measure zero set of initial conditions and impulses, a lower-dimensional representation of a full dynamic system cannot capture the full spectrum of the DICE results.

It is highly desirable to produce a DSGE IAM, incorporating stochastic evolution of the climate and economic system. However, it is often argued that a complete stochastic analysis of IAMs is not possible due to computational complexity. In fact, in a recent report (EPA, 2010) the Environmental Protection Agency assesses that currently the Integration of DSGE models with long-run inter temporal models, e.g., IGEM (Goettle et al. 2009), is beyond the scientific frontier.

A truly stochastic model would also enable a sound analysis of abrupt and irreversible climate change which so far has only been studied in very simple models. Full-size versions of IAMs have only dealt with this topic by using probabilistic assessment studies. For example, Nordhaus (2008) uses a certainty-equivalent approach to tipping points and argues that the additional carbon tax resulting from tipping points exhibits a ramp structure, i.e., it is initially mild but rises substantially over time with global warming. A recent study by Lontzek, Cai and Judd (2012) uses the same model as Nordhaus (2008) but uses a stochastic formulation of abrupt and irreversible climate change and obtains a completely different result. The possibility of a low-probability and low-impact tipping event results in a flat profile for the additional carbon tax. This example demonstrates how implications for climate policy depend on the appropriate model formulation of the underlying research question. It is reasonable to assume that a fully stochastic version of any IAM will unveil new and interesting results.

Novel findings for the SCC can be expected from using more flexible alternatives to the standard preference specifications. Jensen and Traeger (2011) e.g. study an IAM with iid shocks to the rate of exogenous technological change. In a similar model, Crost and Traeger (2011) study iid shocks to the damage function. Both studies find that disentangling risk aversion and intertemporal substitution significantly affects the optimal abatement decision.

We argue that, since riskiness is inherent in the nature of tipping points, the socio-economic effects of stochastic abrupt and irreversible climate change will be affected by preferences about risks. Hence, it is natural to combine Epstein-Zin preferences (Kreps and Porteus 1978, Epstein and Zin 1989) with the risk of when a tipping point will occur and how large the damage will be. It seems likely, that including risk-sensitive preferences will surely imply greater willingness to pay to avoid adverse climate change.

For this study we use DSICE (Cai, Judd and Lontzek, 2012b), a DSGE full-dimensional extension of DICE2007. DSICE and DICE2007 are therefore comparable which facilitates the comparison of our carbon tax numbers with the ones obtained, e.g., by the U.S. Government Interagency Working Group on Social Cost of Carbon (IWG, 2010). We include Epstein-Zin preferences as well as a multi-stage tipping point. We are thus endowed with a model structure that has a more realistic description of risk preferences, the stochastic processes of damages and business cycle fluctuations. We model stochastic and irreversible climate change as a low-probability and low-impact event. Thus, we abstain from dramatic catastrophe assumptions such as in the studies of fat-tailed catastrophic events by Weitzman. Whereas e.g. Weitzman (2009) assumes a very high damage from a climate catastrophe, we follow rather a conservative approach. In particular, we study a range of low to moderate impacts of a tipping-point event. At the same time, in accordance to scientific findings, we assume a gradual, stochastic post-tipping damage path which eventually makes the tipping process last up to two centuries. Furthermore, we assume business cycle shocks that are moderate and bounded. We solve DSICE with dynamic programming using advanced computational methods. The solution to DSICE is reliable and quickly obtained.

As in Lontzek, Cai and Judd (2012), we find that the threat of a tipping point induces immediately stringent carbon pricing, even for low-probability and low-impact tipping events. We find that including Epstein-Zin preferences into DSICE significantly increases the carbon tax. For example, larger values of the intertemporal elasticity of substitution and the degree of risk aversion lead to higher carbon taxes. Furthermore, in addition to stochastic and irreversible climate change, we also study cases with significant uncertainty about the post-tipping damage impact. Climate scientists (e.g. Schneider, 1983) argue that all uncertainties within the climate system amplify the range of possible impacts on the economy and that damage to the economy is associated with the highest range of uncertainty. We find that the uncertainty about the damage is also a critical factor leading to a sharp increase in the carbon tax. Furthermore, for low degrees of risk aversion (i.e. smaller than 2), the carbon tax is not affected much by the variance of the uncertain damage. In contrast, high degrees of risk aversion significantly amplify the effect of damage uncertainty on the carbon tax. We also investigate a disaster scenario. We find that if there is a very unlikely tipping point event (about 0.1% probability of tipping until 2100), and it is has major permanent but uncertain impact on productivity (mean is 20% and volatility is 10%), today's carbon tax of a highly risk-averse policy maker is $521.

We proceed as follows: Sections 2 and 3 discuss our modeling of the stochastic climate, stochastic economy and the preferences. Section 4 presents the DP framework of the DSICE model with Epstein-Zin preferences, and then introduces the multi-dimensional numerical dynamic programming method we use to solve it. Section 5 presents our results and Section 6 concludes.

2 Irreversible Climate Change

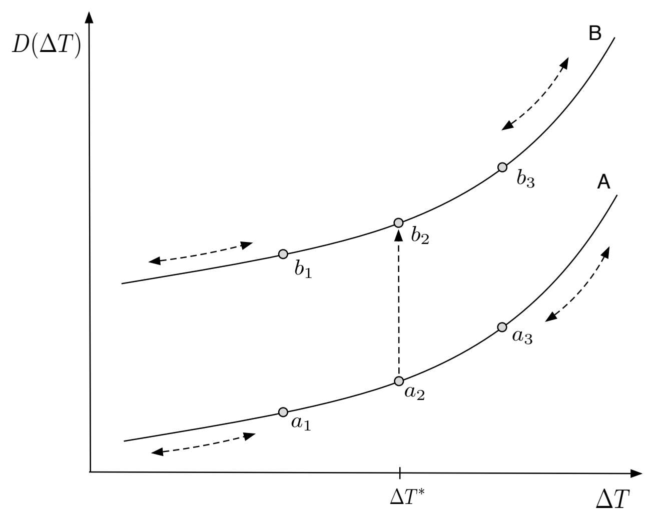
In order to implement the risk of a tipping point event into an IAM, we model a hazard rate which, for any given combination of the state space in any year, gives us the probability of the tipping event occurring in that year. Consequently, the tipping point event becomes stochastic.

So far, most studies model abrupt climate change as a deterministic process. For example, Mastrandrea and Schneider (2001) couple an older version of the DICE model with a simple climate model which simulates the functioning of the THC. Information obtained from the additional climate model is used to enhance the exponential component in the DICE damage function. This specification gives rise to a steeper damage function (in terms of global warming) but is inconsistent with the nature of the externality of an irreversible catastrophe. Nordhaus (2010) also models sea level rise deterministically by increasing the curvature of the damage function.

Another branch of the literature on catastrophic climate change in IAMs (e.g., Yohe et al. 2004) models tipping point events as happening when a threshold temperature is passed (or some other condition is met). In our view this approach (or at least the simplest version of it) is not appealing because it implies that if we have been at that threshold temperature level, then we can immediately infer (learn) that we are safe as long as we stay at or below that level. Modeling abrupt climate change with a known threshold location is a special case of our hazard function approach. In that special case (e.g., Keller et al., 2004) the hazard rate equals zero up to the threshold temperature level and is equal to one beyond that level. In contrast, knowing that there is a critical point but not knowing where it is would not imply a hazard rate. The latter formulation implies that once we have reached a temperature level with no tipping event, we conclude that there cannot be a tipping point at that temperature level. This is not in the statistical nature of a hazard rate. Instead we assume that even if a tipping point has not yet occurred, it may still occur later even if there is no further warming, or even if there is cooling. This surely will affect the optimal climate policy, as is shown by the following example.

For example, the optimal mitigation policy in DICE2007 increases rapidly over time, with small efforts today but much greater effort at the end of the current century. This "ramp" structure of mitigation policy implied by DICE2007-style models arises because of the nature of the externality, i.e., a high atmospheric temperature today reduces output today, but there is no direct impact on future damages, which depend only on contemporaneous temperatures. However, by including the possibility of an stochastic and irreversible climate change, one actually changes the nature of the climate externality. Figure 1 highlights the two aspects of the global warming externality.

Figure 1: The nature of externality from abrupt and irreversible climate change



Curve A represents the pre-tipping damage factor as a function of temperature change (ΔT). It is monotonically increasing and convex. In a pre-tipping regime any movement along A (i.e., $a_1 \leftrightarrow a_2 \leftrightarrow a_3$) can occur, depending on the change in global average temperature. As a consequence, the resulting damage rate $D(\Delta T)$ will exhibit a smooth pattern. The same logic applies in a post-tipping regime, which is denoted by B. Any movement along B (i.e., $b_1 \leftrightarrow b_2 \leftrightarrow b_3$) can occur, depending on the change in global average temperature after a tipping point has occurred. Now, assume that a tipping point has occurred at a_2, which corresponds to a global warming level of ΔT^* [1]. In that case, the damage rate resulting from the change in the climate system will be represented by B (i.e., $a_2 \rightarrow b_2$). However, the irreversible nature of abrupt climate change prevents a movement back to A in case of global cooling and temperature levels below the realized tipping point level. Instead the damage factor will remain to move along B.[2] This stochastic aspect of a

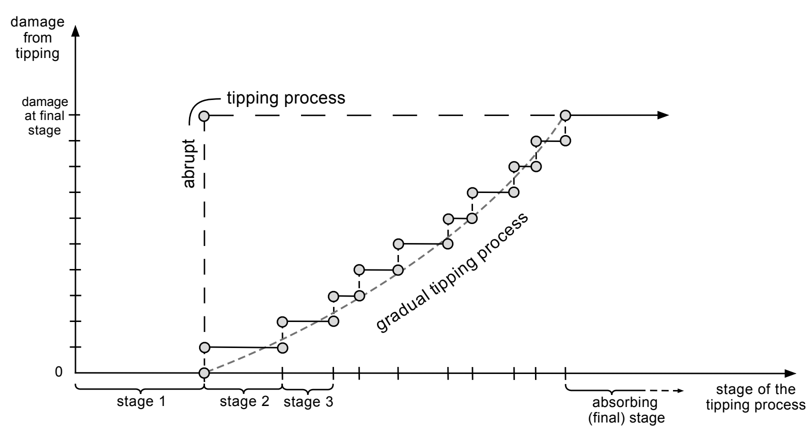
[1] Note that our specification allows for a tipping point to occur even in a phase of global cooling. This feature results from the stochastic nature of a hazard rate formulation of abrupt climate change.

[2] For simplicity, we assume here the most rapid (abrupt) post-tipping impact path which maturates after only one period. This assumption will be relaxed later.

tipping point can only be captured by specifying a hazard rate and modeling abrupt climate change as a jump process where the Markovian hazard rate only depends on contemporaneous conditions. It cannot be captured by changing the shape (e.g., increasing the exponents) of the damage function of global warming.

Irreversibility of post-tipping impacts is a reasonable assumption for the time horizons which are typically being considered in IAMs. Abruptness however, implying a 1-period post-tipping transition scale is a very simplifying assumption that cannot be confirmed by scientific studies. Lenton et al. (2008) e.g. characterizes the transition scales for various tipping elements. While some tipping elements are believed to exhibit a rapid transition of about 10 years (e.g. Arctic summer sea-ice), other elements are believed to exhibit a rather slow transition of more than 300 years (e.g. Greenland ice sheet). As a consequence, the shape of the post-tipping impact path of the underlying tipping element will differ. In this study we to do not aim at modeling one specific tipping element. Nevertheless, we want to allow for the heterogeneity in transition scales. For this purpose, we are studying a wide range of transition scales as part of a sensitivity analysis. In addition, we are assuming that each transition scale is stochastic. Figure 2 illustrates the assumptions about the tipping process in this study. We show two possible damage paths from a tipping point event.

Figure 2: Possible Abrupt and Gradual Tipping Damage Paths



Stage 1 represents the pre-tipping regime. Note that while the damage from tipping in stage 1 is zero, there will still be some smooth damage from global warming along the lines of the DICE model. First, consider the abrupt post-tipping damage path (dashed black line). This path will constitute our benchmark case and serve as point of comparison. It essentially assumes an instant full transition after the tipping point. It implies that the full impact of tipping is realized immediately. In order to approximate a gradual post-tipping impact path, we assume that it consists of ten sequential stages. Figure 1 depicts an example of a possible gradual post-tipping damage path. Note the difference in the lengths of stages 2-10. We assume that each stage occurs stochastically, given some expectation of the entire duration of the gradual post-tipping damage path. In accordance with the literature, we will study transitions of 25, 50, 100 and 200 years. Our multi-stage tipping process can be thought of representing several consecutive stages of one single tipping element, e.g., sea-level rise due to the melting of GIS. The GIS is a huge ice mass holding an equivalent of about 7m of global sea level (Lenton, 2008). An additional 1 or 2°C of global warming might suffice to trigger an irreversible meltdown of the GIS. Thus, is is quite likely that a GIS tipping point could occur in this century. A tipping of the GIS could lead to a global sea level rise of about 50-100 cm per century (Lenton, 2008). Alternatively, the multi-stage tipping process could be thought of representing a sequence of different sequential tipping point events. In a recent study Lenton (2012) discusses the domino effect applied to tipping elements, arguing that several tipping elements are sequential and one tipping point might trigger a cascade of additional tipping.

3 A Stochastic IAM with Epstein-Zin Preferences

The DSICE model (Cai, Judd and Lontzek, 2012b) is a dynamic stochastic general equilibrium model integrating climate and the economy. DSICE is basically a DGSE-extension of the DICE-CJL model (Cai, Judd and Lontzek, 2012a), which itself is a numerically stable version of DICE2007 with a flexible time-period length. This version of DSICE incorporates both, uncertainty about the future state of climate and the economy. A complete description of the DICE-CJL model with all equations as well as parameters can be found in Cai, Judd and Lontzek (2012a). Furthermore, the DSICE model with separable utility is described in Cai, Judd and Lontzek (2012b). In this section we briefly describe the DSICE model focusing mainly on the stochastic climate, the stochastic economy and Epstein-Zin preferences.

3.1 The Stochastic Climate

Let $\mathbf{M}_t = (M_t^{\mathrm{AT}}, M_t^{\mathrm{UP}}, M_t^{\mathrm{LO}})^\top$ be a three-dimensional vector describing the masses of carbon concentrations in the atmosphere, and upper and lower levels of the ocean. These concentrations evolve over time according to:

$$\mathbf{M}_{t+1} = \mathbf{\Phi}^{\mathrm{M}}\mathbf{M}_t + (\mathcal{E}_t, 0, 0)^\top ,$$

where

$$\mathbf{\Phi}^{\mathrm{M}} = \begin{bmatrix} 1-\phi_{12} & \phi_{12}\varphi_1 & 0 \\ \phi_{12} & 1-\phi_{12}\varphi_1-\phi_{23} & \phi_{23}\varphi_2 \\ 0 & \phi_{23} & 1-\phi_{23}\varphi_2 \end{bmatrix},$$

with $\varphi_1 = M_*^{\mathrm{AT}}/M_*^{\mathrm{UP}}$ and $\varphi_2 = M_*^{\mathrm{UP}}/M_*^{\mathrm{LO}}$, where M_*^{AT}, M_*^{UP} and M_*^{LO} are the preindustrial equilibrium states of the carbon cycle system. The anthropogenic sources of carbon are represented by the $\mathcal{E}_t$, which will be specified in the next subsection.

The DICE climate system also includes temperatures in the atmosphere and ocean, which are represented by the vector $\mathbf{T}_t = (T_t^{\mathrm{AT}}, T_t^{\mathrm{LO}})^\top$. The temperatures dynamically evolve according to:

$$\mathbf{T}_{t+1} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{T}_t + \left(\xi_1 \mathcal{F}_t\left(M_t^{\mathrm{AT}}\right), 0\right)^\top,$$

where the heat diffusion process between ocean and air is represented by the matrix

$$\mathbf{\Phi}^{\mathrm{T}} = \begin{bmatrix} 1 - \xi_1\eta/\xi_2 - \xi_1\xi_3 & \xi_1\xi_3 \\ \xi_4 & 1 - \xi_4 \end{bmatrix},$$

where ξ_2 is the climate sensitivity parameter (we choose $\xi_2 = 3$ in our examples of this paper). Atmospheric temperature is affected by external forcing, F_t^{EX}, and by the interaction between radiation and atmospheric CO2, implying that total radiative forcing at t is

$$\mathcal{F}_t\left(M^{\mathrm{AT}}\right) = \eta \log_2\left(M^{\mathrm{AT}}/M_0^{\mathrm{AT}}\right) + F_t^{\mathrm{EX}}$$

The impact of global warming on the economy is reflected by a convex damage function of temperature in the atmosphere. This is a standard feature of the DICE model family. As discussed in the introduction, we modify the standard damage function by explicitly modeling the possibility of a climate shock (i.e. tipping point) to account for the threat of abrupt and irreversible climate change. Each climate shock occurs at a random time. Thus, the stochastic damage factor in DSICE is given by

$$\Omega\left(T_t^{\mathrm{AT}}, J_t\right) = \frac{1 - J_t}{1 + \pi_1 T_t^{\mathrm{AT}} + \pi_2 (T_t^{\mathrm{AT}})^2},$$

where the denominator represents the standard damage function from the DICE2007 model. In the numerator, J_t is a discrete Markov chain with nondecreasing values over time, with $J_t = 0$ in the pre-tipping regime (stage 1) and $0 < J_t < 1$ in all subsequent stages of the post-tipping regime. Assume that all possible values of J_t are $0 = \mathcal{J}_1 < \mathcal{J}_2 < \cdots < \mathcal{J}_n < 1$, then the Markov chain probability transition matrix from year t to year $t+1$ is an upper-triangular matrix with nonnegative elements:

$$\begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & \cdots & p_{1n} \\ & p_{2,2} & \cdots & \cdots & p_{2n} \\ & & \ddots & \vdots & \vdots \\ & & & p_{n-1,n-1} & p_{n-1,n} \\ & & & & 1 \end{bmatrix},$$

where its (i,j) element is the conditional transition probability from state $\mathcal{J}_i$ to $\mathcal{J}_j$, and

$$\sum_{j=i}^{n} p_{i,j} = 1,$$

for all $i = 1, \ldots, n$. These nonnegative probabilities may be dependent on time t and the contemporaneous surface temperature T_t^{AT}. In principle, they could be a function of the entire state and control space. The zero probabilities of the lower part of the transition matrix represents the irreversibility of the tipping process. So we call J_t the persistent climate damage state representing the irreversibility nature of the tipping point. Since this tipping structure is sequential, the decision maker faces always one tipping point stage at a time but has full information about the tipping system transition matrix. In addition, the final state is an absorbing state. In section 5 we will study an 11-stage tipping process and a 2-stage process as our benchmark case.

3.2 The Stochastic Economy

Capital k_t transits to the next period in a standard fashion:

$$k_{t+1} = (1-\delta)k_t + \mathcal{Y}_t(k_t, T_t^{\mathrm{AT}}, \mu_t, \zeta_t, J_t) - c_t,$$

where $\mathcal{Y}_t$ denotes the stochastic production function. The latter accounts for the costs of mitigation as a fraction of output. Furthermore, it includes the damage resulting from global warming as well as both, an economic shock and a climate shock:

$$\mathcal{Y}_t(k_t, T_t^{\mathrm{AT}}, \mu_t, \zeta_t, J_t) = (1 - \theta_{1,t}\mu_t^{\theta_2})\zeta_t A_t k_t^{\alpha} l_t^{1-\alpha} \Omega\left(T_t^{\mathrm{AT}}, J_t\right),$$

where ζ_t is a discrete-time bounded mean-reverting continuous productivity shock representing economic fluctuations (see Cai, Judd and Lontzek 2012b), and its transition function from stage t to $t+1$ is $\zeta_{t+1} = g^{\zeta}(\zeta_t, \omega_t^{\zeta})$ where ω_t^{ζ} is an i.i.d. random process. We assume that the economic shock and the climate shock are independent.

Given the stochastic production function, annual total carbon emissions are stochastic and given by

$$\mathcal{E}_t\left(k_t, \mu_t, \zeta_t\right) = \sigma_t(1-\mu_t)\zeta_t A_t k_t^{\alpha} l_t^{1-\alpha} + E_t^{\mathrm{Land}}, \tag{1}$$

where σ_t denotes the carbon intensity of output, μ_t denotes the fraction of mitigated emission and E_t^{Land} is an exogenous rate of emissions from biological processes.

3.3 Epstein-Zin Preferences

The standard separable utility function in the finite-horizon DICE2007-class of models is

$$u(c_t, l_t) = \frac{\left(c_t/l_t\right)^{1-\psi}}{1-\psi} l_t,$$

where c_t denotes the consumption level and l_t is total labor supply. It is assumed that a social planner maximizes the present-discounted utility stream up to a terminal time T. We will instead incorporate Epstein-Zin preferences into DSICE. There is no simple utility function for Epstein-Zin preferences, so we will proceed to describe the optimization problem in the standard recursive manner.

The dynamic optimization problem has seven continuous state variables: the capital stock, k, the three-dimensional carbon system $\mathbf{M}$, the two-dimensional temperature vector, $\mathbf{T}$, and the stochastic productivity state, ζ. Furthermore, J is the discrete shock to the climate. The recursive formulation of the social planner's objective is

$$\begin{aligned} U_t\left(k, \mathbf{M}, \mathbf{T}, \zeta, J\right) &= \max_{c,\mu} \left\{ (1-\beta) \frac{\left(c_t/l_t\right)^{1-\psi}}{1-\psi} l_t + \right. \\ & \left. \beta \left[\mathbb{E} \left\{ \left(U_{t+1}\left(k^+, \mathbf{M}^+, \mathbf{T}^+, \zeta^+, J^+\right)\right)^{1-\gamma} \right\} \right]^{\frac{1-\psi}{1-\gamma}} \right\}^{\frac{1}{1-\psi}} \end{aligned}$$

where ψ is the inverse of the intertemporal elasticity of substitution, μ_t is the mitigation rate of emissions, $\mathbb{E}\{\cdot\}$ is the expectation operator, β is the discount factor and γ is called the risk aversion parameter. The actual risk premia will depend on interactions between ψ and γ. The special case of $\psi = \gamma$ is the time separable specification where both parameters represent both risk aversion and the elasticity of substitution. In general, increasing γ will correspond to higher prices of risk.

Epstein-Zin preferences are flexible specifications of decision makers' preferences regarding uncertainty. They are special cases of Kreps-Porteus preferences, which were designed to model preferences over the resolution of risk. Traeger (2012) presents a nice exposition of the features of Epstein-Zin preferences and other intertemporally separable preferences, and their value for analyzing the cost of carbon.

The key fact justifying the use of Epstein-Zin preferences is in the data on aggregate risk preferences. We know from the literature on the equity premium that the willingness to pay to reduce risk is far higher than implied by standard utility functions. Therefore, we expect an increase in the optimal carbon tax in the face of tipping point risk if we use empirically calibrated specifications of Epstein-Zin preferences. As of today, Epstein-Zin preferences have not been analyzed in a full-dimensional DSGE version of a major IAM, such as DICE2007. On top of this, we account for an additional stochastic dimension by modeling abrupt and irreversible climate change with a system of tipping elements.

4 The Dynamic Programing Problem

We solve the stochastic Integrated Assessment Model with Epstein-Zin preferences numerically. This section shows the dynamic programing formulation of the model.

Furthermore, we present the numerical algorithm and specifically focus on some essential aspects of our computational method.

4.1 DP formulation

We simplify the mathematical expressions by letting

$$V_t\left(k, \mathbf{M}, \mathbf{T}, \zeta, J\right) = \frac{\left[U_t\left(k, \mathbf{M}, \mathbf{T}, \zeta, J\right)\right]^{1-\psi}}{\left(1-\psi\right)\left(1-\beta\right)}$$

denote the value function. We assume that the simple one-period utility function is

$$u(c,l) = \frac{(c/l)^{1-\psi}}{1-\psi} l,$$

and then rewrite the dynamic programming problem in the following form:

$$\begin{aligned}
V_t(k, \mathbf{M}, \mathbf{T}, \zeta, J) = \max_{c,\mu} \quad & u(c_t, l_t) + \frac{\beta}{1-\psi} \times \\
& \left[\mathbb{E}\left\{\left(\left(1-\psi\right) V_{t+1}\left(k^+, \mathbf{M}^+, \mathbf{T}^+, \zeta^+, J^+\right)\right)^{\frac{1-\gamma}{1-\psi}}\right\}\right]^{\frac{1-\psi}{1-\gamma}}, \\
\text{s.t.} \quad & k^+ = (1-\delta)k_t + \mathcal{Y}_t(k, T^{\mathrm{AT}}, \mu, \zeta, J) - c_t, \\
& \mathbf{M}^+ = \mathbf{\Phi}^{\mathrm{M}}\mathbf{M} + \left(\mathcal{E}_t\left(k, \mu, \zeta\right), 0, 0\right)^\top, \\
& \mathbf{T}^+ = \mathbf{\Phi}^{\mathrm{T}}\mathbf{T} + \left(\xi_1 \mathcal{F}_t\left(M^{\mathrm{AT}}\right), 0\right)^\top, \\
& \zeta^+ = g^\zeta(\zeta, \omega^\zeta), \\
& J^+ = g^J(J, \mathbf{T}, \omega^J), \qquad (2)
\end{aligned}$$

for $t = 0, 1, \ldots, 599$. The terminal value function V_{600} is the terminal value function given in Cai, Judd and Lontzek (2012b). In the model, consumption c and emission control rate μ are two control variables, $(k, \mathbf{M}, \mathbf{T}, \zeta, J)$ is 8-dimensional state vector at year t (where $\mathbf{M} = (M^{\mathrm{AT}}, M^{\mathrm{UP}}, M^{\mathrm{LO}})^\top$ is the three-layer CO_2 concentration and $\mathbf{T} = (T^{\mathrm{AT}}, T^{\mathrm{LO}})^\top$ is the two-layer global mean temperature), and $(k^+, \mathbf{M}^+, \mathbf{T}^+, \zeta^+, J^+)$ is its next-year state vector.

4.2 The Dynamic Programming Algorithm

In dynamic programming problems, when the value function is continuous, it has to be approximated. In this study, we use a finitely parameterized collection of functions to approximate a value function, $V(x, J) \approx \hat{V}(x, J; \mathbf{b})$, where x is the continuous state vector (in DSICE, it is the 7-dimensional vector $(k, \mathbf{M}, \mathbf{T}, \zeta)$, J is the discrete state vector (in DSICE, it is the persistent climate damage level), and $\mathbf{b}$ is a vector of parameters. The functional form $\hat{V}$ may be a linear combination of polynomials, or it may represent a rational function or neural network representation, or it may be some other parameterization especially designed for the problem. After the functional form is fixed, we focus on finding the vector of parameters, $\mathbf{b}$, such that $\hat{V}(x, J; \mathbf{b})$ approximately satisfies the Bellman equation. Numerical DP with value function iteration can solve the Bellman equation approximately (Judd, 1998). Thus, the Bellman equation (2) can be rewritten in a general form:

$$
\begin{aligned}
V_t(x, J) = \max_{a \in \mathcal{D}(x,J,t)} \quad & u_t(x, a) + \beta \mathbb{E}\left\{V_{t+1}(x^+, J^+)\right\}, \\
\text{s.t.} \quad & x^+ = f(x, J, a, \omega^\zeta), \\
& J^+ = g(x, J, \omega^J),
\end{aligned}
$$

where $V_t(x, J)$ is the value function at time $t \leq T$ (the terminal value function $V_T(x, J)$ is given), (x^+, J^+) is the next-stage state, $\mathcal{D}(x, J, t)$ is a feasible set of a, ω^J and ω^ζ are random variables, β is a discount factor and $u_t(x, a)$ is the utility function at time t. The following is the algorithm of parametric DP with value function iteration for finite horizon problems. Detailed discussion of numerical DP can be found in Cai (2009), Judd (1998) and Rust (2008).

Algorithm 1. *Numerical Dynamic Programming with Value Function Iteration for Finite Horizon Problems*

Initialization. *Choose the approximation nodes,* $\mathbb{X}_t = \{x_{i,t} : 1 \leq i \leq m_t\}$ *for every* $t < T$, *and choose a functional form for* $\hat{V}(x, J; \mathbf{b})$, *where* $J \in \Theta$. *Let* $\hat{V}(x, J; \mathbf{b}_T) \equiv V_T(x, J)$. *Then for* $t = T-1, T-2, \ldots, 0$, *iterate through steps 1 and 2.*

Step *1. Maximization step. Compute*

$$
\begin{aligned}
v_{i,j} = \max_{a \in \mathcal{D}(x_i,J_j,t)} \quad & u_t(x_i, a) + \beta \mathbb{E}\left\{\hat{V}(x^+, J^+; \mathbf{b}_{t+1})\right\} \\
\text{s.t.} \quad & x^+ = f(x_i, J_j, a, \omega^\zeta), \\
& J^+ = g(x_i, J_j, \omega^J),
\end{aligned}
$$

for each $J_j \in \Theta$, $x_i \in \mathbb{X}_t$, $1 \leq i \leq m_t$.

Step *2. Fitting step. Using an appropriate approximation method, compute the* $\mathbf{b}_t$ *such that* $\hat{V}(x, J_j; \mathbf{b}_t)$ *approximates* $(x_i, v_{i,j})$ *data for each* $J_j \in \Theta$.

We implement our numerical dynamic programming algorithm to solve the DSICE model with Epstein-Zin preferences. The code is written in Fortran and uses the methods presented in Judd (1998), Cai (2009), and Cai and Judd (2010, 2012a, 2012b, 2012c), and we use NPSOL (Gill, P., et al., 1994) as the optimization solver in the maximization step. In particular, for each discrete state value, we choose the degree four complete Chebyshev polynomials to approximate the value function. The multidimensional tensor grid with five Chebyshev nodes on each continuous dimension in the continuous state ranges gives us our set of approximation nodes. The continuous state ranges are constructed from the solution of the DICE-CJL model (Cai, Judd and Lontzek, 2012a), a continuous-time reformation of DICE2007 (Nordhaus, 2008). The results of the maximization step at each of the approximation nodes are used to compute the Chebyshev coefficients via a regression procedure. We assume a one-year time period and iterate backwards for 600 years, beginning with a terminal value function. See Cai, Judd and Lontzek (2012b) for a description of how we construct the terminal value function and the continuous state ranges and how we verify the accuracy of solutions given by our algorithm.

It is important to use one year instead of ten years as the time unit. Cai, Judd and Lontzek (2012a, 2012c) shows that annual time periods produce a significantly different carbon price with the numbers given by DICE2007 using ten-year time periods. For the models with uncertain state variables, Cai, Judd and Lontzek (2012b) also shows that annual time periods have a significantly higher carbon price (about 24% higher in the first period) than ten-year time periods.

After computing the approximate value function at each time t for $t = 0, 1, ..., 599$, we simulate the optimal path corresponding to a sequence of shocks, both economic and climate. That is, given the current state along a path, we compute the optimal decisions and then use the realized shock to compute next period's state. We start this process with the given initial continuous state and $(\zeta_0, J_0) = (1, 1)$, and run it until the terminal time. The next section presents the statistical results of 1000 simulated paths of the optimal solution.

5 Numerical Examples

We now analyze how the optimal carbon tax is affected by different preference parameters combined with various tipping point events. We first solve a benchmark case with standard parameter assumptions. Using the benchmark case, we explain the major drivers of the carbon tax results in this paper. Later examples will deviate from the benchmark case by assuming different parameter specifications as well as different characteristics of the post-tipping impact and the tipping process itself, allowing us to determine what are the critical factors behind our carbon tax results.

5.1 Parameter Choices

The parameters of the deterministic climate system in DSICE are the same as in the original DICE2007 model and described in Cai, Judd and Lontzek (2012) in more detail. We stay as close as possible to the climate aspects of DICE2007 model, and focus on how productivity risks and preferences interact to compute the willingness to pay for reducing GHG's in order to avoid future climate damages.

5.1.1 Parameters Describing Preferences

We cover a broad range of values for the degree of risk aversion γ and the reciprocal of the intertemporal elasticity of substitution ψ, representing empirical work that aims to estimate these parameters. Basal and Yaron (2004) finds $\gamma = 10$ and $\psi = 2/3$, Vissing-Jørgensen and Attanasio (2003) finds γ between 5 and 10 and $\psi < 1$, and Vissing-Jørgensen (2002) and Campbell and Cochrane (1999) also find cases of $\psi > 1$. Nordhaus (2008) essentially chooses $\psi = \gamma = 2$. We therefore choose a range for γ between 0.5 and 20 and for ψ between 0.5 and 2.

5.1.2 Parameters Describing The Tipping System

Kriegler et al. (2009) points out that the lack of data and limited understanding of the underlying processes make it difficult to assess the likelihood of changes in the earth system due to global warming. As of today, most probability assessments of abrupt climate change come from expert elicitation surveys (e.g., Kriegler et al. 2009 and Zickfeld et al. 2007). In these elicitations, the experts are asked about their beliefs about probabilities of a tipping point occurring at, or before some time (typically the year 2100 or 2200) based on an emission scenario. These experts' opinions reveal their probabilistic beliefs for a tipping point occurring within a time frame under alternative assumptions about temperature changes within that specific time frame.

The absence of precise knowledge about the physical system is not particularly important for this problem. Decisions today must be based on current beliefs about the climate system, and will reflect the imprecise nature of those beliefs. The only assumption we are tacitly making is that these expert opinions are the ones held by the social planner.

Zickfeld et al. (2007) presents experts' subjective cumulative probabilities that a collapse of THC will occur or be irreversibly triggered at or before 2100 under alternative assumptions about temperature at 2100. The survey data obtained in Zickfeld et al. (2007) on THC collapse reveal a huge range of assigned probabilities. For example, some of these experts believe that even with an additional warming of $6°C$ it is impossible for the THC to reach a tipping point by 2100. On the contrary, one third of the experts believe that the probability is $> 60\%$ and two experts even suggest 90%.[3] For less global warming by 2100 these elicited probabilities are accordingly lower. The average expert assigns about 18% chance of tipping at $4°C$ by 2100 and 4% at $2°C$. A similar large range of experts' opinions is obtained by Kriegler et al. (2009) which presents the tipping probabilities for five tipping elements. For the medium temperature corridor (i.e. a warming of $2 - 4°C$ by 2200), this study finds a range of roughly 50% for the THC, 80% for GIS, WAIS, and AMAZ, and about 40% for ENSO. Overall, the numbers from both studies mirror a remarkably huge range of expert opinions about the tipping point probabilities. This incongruity

[3] The probability numbers have been eyeballed from Zickfeld et al. (2007).

in probability assessment among the experts makes the analysis of a tipping point even more interesting from a uncertainty point of view.

In order to derive the hazard rates for our abrupt climate change examples, we use the intrinsic information provided in the expert elicitation studies by Zickfeld et al. (2007) and Kriegler et al. (2009). We follow the approach in Lontzek, Cai and Judd (2012) which is the first study to incorporate the results of expert elicitation study into a stochastic IAM. The approach in Lontzek, Cai and Judd (2012) is to use the experts' opinions on the cumulative probability of a tipping point occurring by 2100 or 2200, and infer hazard rates for tipping elements from these opinions and their assumptions about the temperature path. The resulting hazard rate becomes a function of the contemporaneous temperature level (measured as the deviation from preindustrial temperature).

5.2 Benchmark Case

We first examine the case of a tipping event with a most rapid post-tipping impact. In fact, we assume that the entire post-tipping impact will be realized within one year. This strong assumption should serve as a point of comparison when we extended the duration of the entire post-tipping impact path in later examples. In addition, for illustrative purposes, we first do not account for business cycle shocks, which we include in a further step. Recall, that the two-stage probability transition matrix of J_t from year t to year $t+1$ is

$$\begin{bmatrix} 1-p_t & p_t \\ 0 & 1 \end{bmatrix},$$

where its (i, j) element is the transition probability from stage i to j. The probability of entering stage two, p_t, depends positively on the surface temperature at time t.

$$p_t = 1 - \exp\left\{-\nu \max\left\{0, \, (T_t^{\mathrm{AT}} - 1)\right\}\right\}, \tag{3}$$

where ν is called the hazard rate parameter. Note, that p_t is endogenous, depending positively on the surface temperature at time t.

As discussed in Section 2, we specify the hazard rates for the tipping events in this study based on expert elicitation studies by Zickfeld et al. (2007) and Kriegler et al. (2009). The hazard rates in our numerical examples are motivated by the range of hazard rates that experts put on the various tipping events. Our objective is not to model any specifically tipping element explicitly but rather to show how that range of numbers is related to the SCC.

Figure 3: Carbon tax in the benchmark scenario

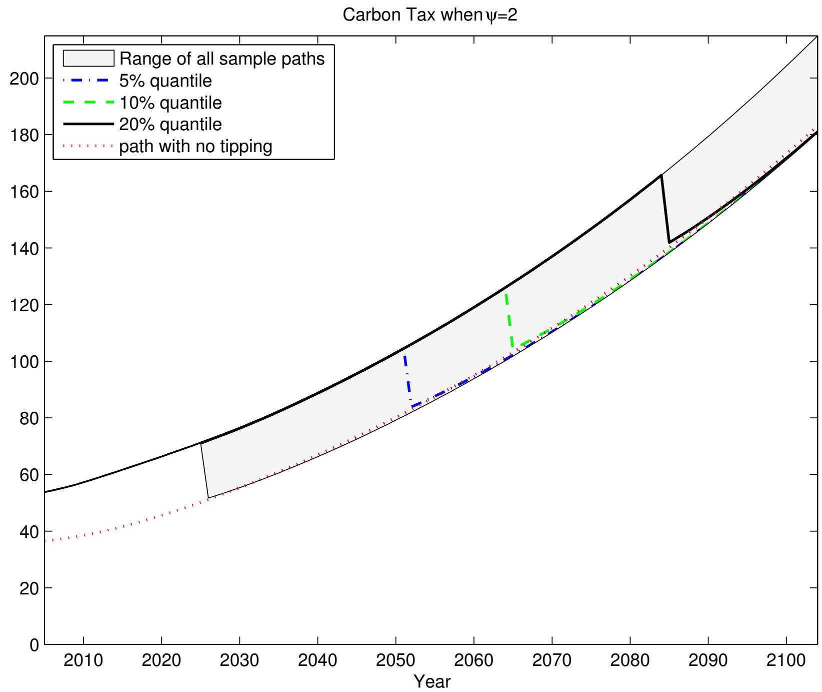


Figure 3 shows the statistical results of the simulation runs for the optimal carbon tax policy for our benchmark scenario: $\psi = 2$, $\gamma = 10$, the hazard rate parameter $\nu = 0.00574$, and the damage level is 2.5% (i.e., $J_t = 0.025$ when the tipping event happens). We depict the pre-tipping path, the earliest tipping sample path as well as some relevant quantiles. Furthermore, we depict the basic deterministic carbon tax scenario in which a tipping point does not exist. Thus, we can compare the cases in which the economy is threatened by a tipping point with the case in which no tipping point (and the associated risk and impact) exists. The upper envelope in Figure 3 represents at each time t, the carbon tax if there has not yet been a tipping event. We call this the pre-tipping carbon tax. In contrast, the lower envelope in general represents the carbon tax in the post-tipping regime. Figure 3 also displays the timing of some sample tipping events. For example, the first vertical drop (which is at about 2027 in Figure 3) is the first tipping out of our 1000 simulations. By the middle of this century about 5% of the simulated paths have generated a tipping point and by the end of the 21st century more than 20% of the paths have exhibited a tipping point. Furthermore, note that in the initial period (2005), the optimal carbon tax is \$54, while it is \$37 in the case when the tipping point does not exist. Thus, in face of a low probability - low impact event the immediate additional preventive carbon tax is \$17.

Figure 3 also resembles the two distinct aspects of the abrupt climate change externality which have been discussed by Lontzek, Cai and Judd (2012). On the one hand, the DICE-like “ramp” structure of the carbon tax implies a rather low carbon tax for low global warming but then it becomes more stringent over time when global warming becomes more severe. This is shown by the no-tipping path in Figure 3. On the other hand, the threat of an irreversible and abrupt climate change results in a nearly constant additional carbon tax (i.e., the “ramp” is not the dominant feature.) to delay the tipping point from occurring. This result is derived from the drop in the carbon tax immediately after the tipping point event. Figure 3 implies a carbon tax markup of about 25%. As in Lontzek, Cai and Judd (2012), we conclude that the optimal carbon tax response to the threat of abrupt and irreversible climate change depends on the dynamic pattern of the adverse impacts. If these impacts are permanent, as it is the case here, the optimal policy is one with substantial carbon taxation immediately.

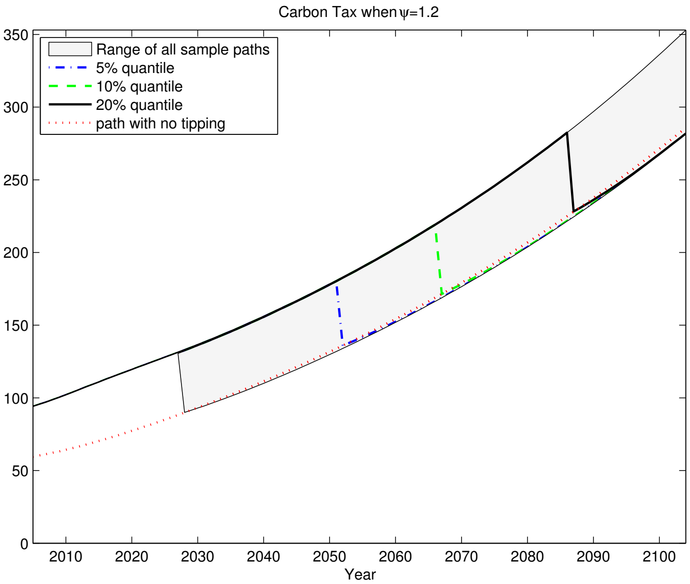
The results in Figure 3 are based on the benchmark Epstein-Zin parameter, $\psi = 2$ and $\gamma = 10$. We have argued previously that riskiness is inherent in the nature of tipping points. In the following, we therefore analyze how the optimal carbon tax is affected by stochastic abrupt and irreversible climate change under different preferences about risk and inter temporal substitution. Furthermore, we conduct some other sensitivity analyses.

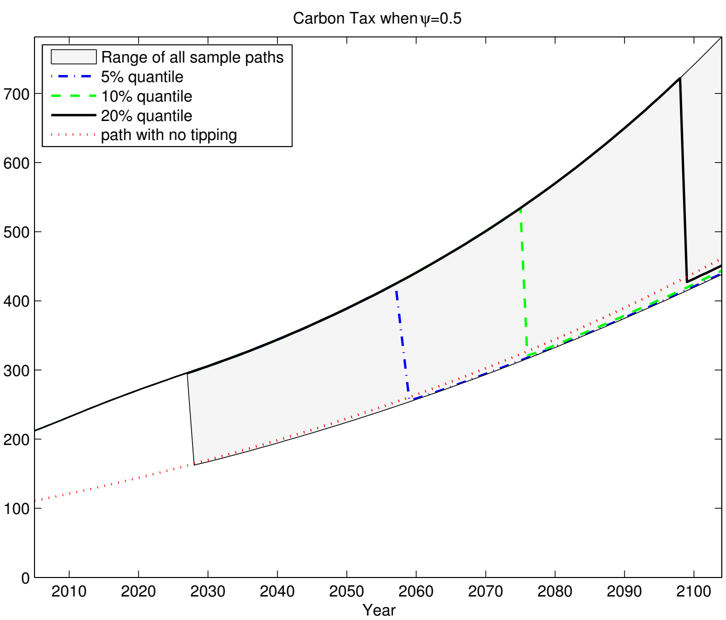
5.3 Perturbations of the Benchmark Case

5.3.1 Effect of Intertemporal Elasticity of Substitution

First, we consider the effect of a higher intertemporal elasticity of substitution on the carbon tax and analyze the cases with $\psi = 1.2$ and $\psi = 0.5$. The general finding from Figure 4 is that the carbon tax will increase as the intertemporal elasticity increases (i.e., ψ decreases). This can be best seen in the initial period, where the carbon tax almost doubles when moving from $\psi = 2$ to $\psi = 1.2$.

Figure 4: Carbon Tax for different intertemporal Elasticity



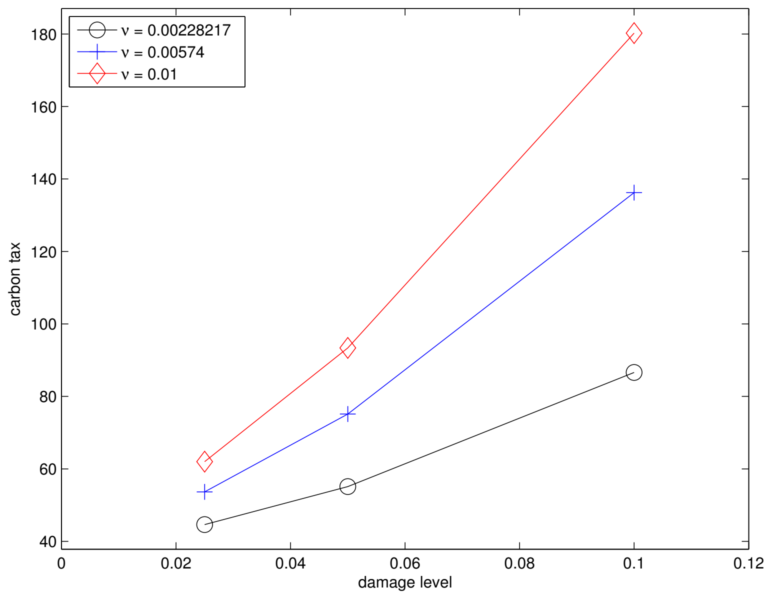


A further doubling of the tax in the initial period occurs when going from $\psi = 1.2$ to $\psi = 0.5$. Moreover, the difference between the pre-tipping path (100% quantile line) and the post-tipping path (0% quantile line) is increasing as the intertemporal elasticity increases. This additional preventive carbon tax around 2050 rises from about \$20 ($\psi = 2$) , to \$40 ($\psi = 1.2$) and to about \$160 ($\psi = 0.5$). The strong effect of the IES on the carbon tax occurs because a low intertemporal elasticity of substitution, by definition, enhances the consumption-smoothing choices of the decision maker. In DSICE, these choices are savings and mitigation. On the one hand, more capital is built up to compensate for a potential loss in disposable output in case of a tipping point. On the other hand, mitigation efforts are very high and as a result, very few carbon emissions are released into the atmosphere.

5.3.2 Effect of Hazard Rates and Damage Level

While the previous example showed how the intertemporal elasticity of substitution affects the carbon tax, this section studies how the carbon tax is affected by different hazard rate parameters and damage levels. So far, we have considered a 2.5% impact of a tipping point on the economy. This is within the lower bound of what is assumed in the studies of catastrophic events. Only a few studies provide an estimate of the loss in output from a tipping point catastrophe, such as the THC collapse. Keller et al. (2004) estimate the loss in GDP from a tipping event in the range of 1% - 3%. Other studies, such as Mastrandrea and Schneider (2001) and Nordhaus (1994) use much higher estimates. In the DICE model catastrophic damages can amount up to 30% of GDP (Nordhaus, 2008) and in PAGE (Hope, 2006) up to 5%. To study the sensitivity of the carbon tax, we present the optimal carbon tax numbers for much higher impact levels on the economy. In particular, 5% and 10%. Lontzek, Cai and Judd (2012) have shown that the near constancy of the optimal anti-tipping effort is retained for different levels, and argued that this additional carbon tax does not depend on the magnitude of the damage but rather is inherent in the stochastic structure of the jump process shock. In this example, we retain the inverse of intertemporal elasticity of substitution at $\psi = 2$, and the risk-averse coefficient at $\gamma = 10$. Figure 5 shows the carbon tax at the initial year for different hazard rate parameters and damage levels. For example, the optimal carbon tax in face of a 10% damage and a hazard rate parameter of $\nu = 0.01$ is \$180, while it is only \$45 in face of a 2.5% damage and $\nu = 0.00228217$. Two general insights can be drawn from all four plots. First, the carbon tax increases more than proportionally with higher damage levels and second, the carbon tax also increases if the hazard rate parameter increases.

Figure 5: Effect of Hazard Rate Parameter and Damage Level

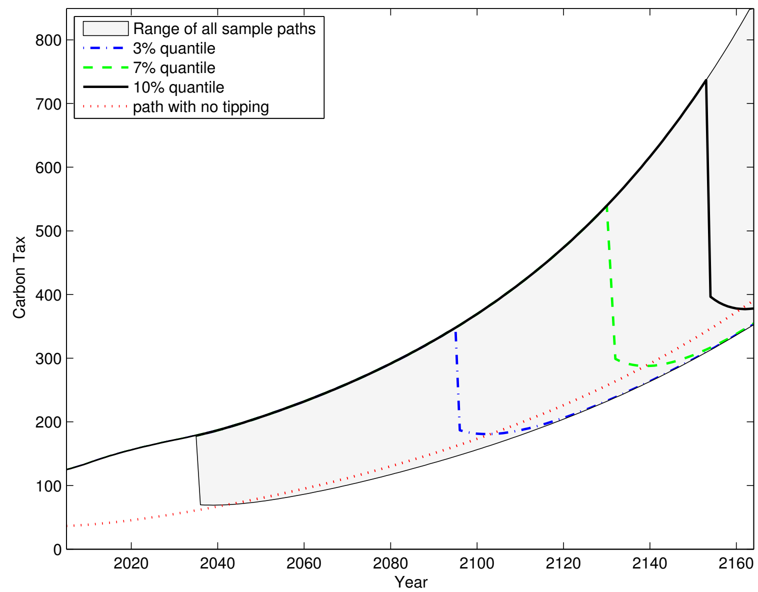


5.4 Putting a Price on a Disaster

So far we have considered tipping scenarios with a relatively low post-tipping impact. Recall that, in our benchmark scenario the post-tipping damage is 2.5% while our hazard rate parameter is $\nu = 0.00574$. We have observed the flattening of the additional carbon tax when the society accounts for a possible tipping point level. Our benchmark parameter specification resembles a "low probability - low impact" scenario for tipping points. Above, we also discussed cases with an impact level of up to 10% and saw that higher impact levels drastically raise the carbon tax. While an impact level of 10% is certainly not very low, it is still far below the levels of some extreme-catastrophic scenarios analyzed in the literature (see e.g. Weitzman, 2009).

As an example of a very high impact from a tipping point, we run the case for $J = 0.2$, i.e., a permanent 20% impact level. At the same time, we substantially reduce the hazard rate parameter to account for the most conservative lower assumptions about tipping point probabilities. In particular, we assume $\nu = 0.0008$, which results in a hazard rate of about 0.1% at 2100. This specification attributes a rather thin-tailed nature to our disaster case. Furthermore, we retain our benchmark preference parameters, $\psi = 2$, and $\gamma = 10$. Figure 6 presents the optimal carbon tax in the face of a single tipping event resulting in a permanent damage of 20%.

Figure 6: Disaster Case



The carbon tax in the first period is \$124, compared to our benchmark tax of \$54, which resembles the combined effect of much higher damages and a much lower hazard rate. This remarkably high increase in the carbon tax shows that the optimal climate policy towards catastrophic events implies a very high price for delaying (preventing) a low probability - low impact event. The pre-tipping carbon tax rises to \$369 by the end of this century and further exceeds \$800 by 2160. Furthermore, note that the earliest sample tipping point occurs at around 2035, while by 2100 only slightly more than 3% of the tipping paths exhibit a tipping point. Thus, given a 97% chance that no tipping will occur within this century, it is still optimal to increase today's carbon tax from \$54 to 124%. In a series of seminal studies, (e.g. Weitzman, 2009), Weitzman has argued that under certain circumstances, the probability of a catastrophic event calls for immediate stringent climate policies. The qualitative nature of our results, is similar to Weitzman's findings, despite our much more moderate assumptions about the post-catastrophic impact levels. One other crucial difference between these two approaches is that while Weitzman (2009) conducts a prevention-focused analysis of the catastrophe, the hazard-rate structure in our model implies a delay-focused climate policy.

5.5 A Tipping Point with Uncertain Damage Level

Our examples have so far assumed we knew the post-tipping damage, but this is unreasonable. We cannot precisely aggregate the multitude of sectoral and regional impacts. Because we don't know when a tipping point will occur, perhaps 2050, or

2100, or later, the impact will depend on many characteristics of the climate system and economy at that time. In fact, Steven Schneider's popular "cascading pyramid of uncertainties" implies that all possible uncertainties amplify the range of possible impacts on the economy (Schneider, 1983). We next examine a specification where we are uncertain about the impact of a tipping point conditional on when it occurs, creating an additional layer of uncertainty

In our first example, we assume that if the tipping event occurs, then the damage is random with

$$J_t = \begin{cases} 0.05 + \varsigma, & \text{with probability } 50\%, \\ 0.05 - \varsigma, & \text{with probability } 50\%, \end{cases} \tag{4}$$

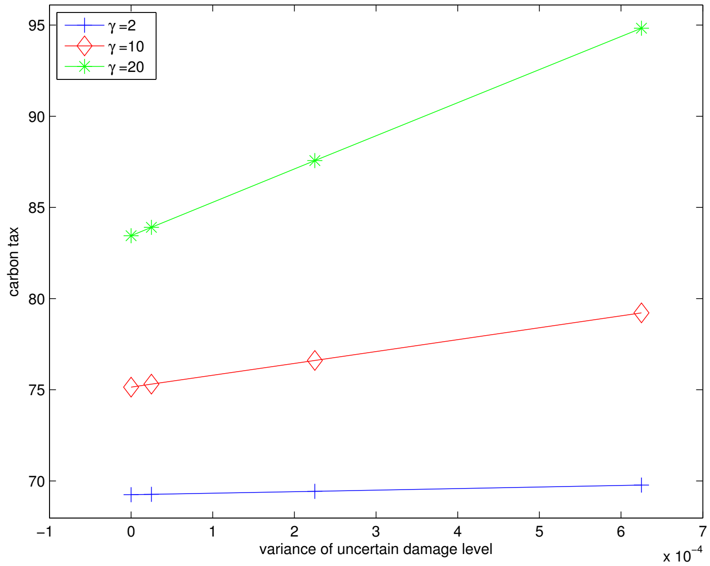
where $0 \leq \varsigma < 0.05$ is called as the volatility of the uncertain damage level. Thus, we have three values for J_t, one value before tipping and two after, where both of the post-tipping values are absorbing states. Therefore, J_t is a three-state Markov chain with the probability transition matrix

$$\begin{bmatrix} 1 - p_t & 0.5p_t & 0.5p_t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

at year t, where its (i, j) element is the transition probability from state i to j, and p_t is given in the formula (3).

In the previous sections we observed a strong effect of the intertemporal elasticity of substitution on the carbon tax. Here, we investigate the joint effect between risk aversion and the uncertain damage level on the carbon tax. We choose the inverse of intertemporal elasticity of substitution as $\psi = 2$, the hazard rate parameter $\nu = 0.00574$. We assume that the tipping point has an uncertain damage level according to equation (4). Figure 7 shows the numbers of carbon tax in the first period for various risk-aversion coefficients and variances of the uncertain damage level.

Figure 7: Effect of Risk-Aversion Coefficient and Variance of Uncertain Damage Level



We see that when γ is small, the carbon tax has only a very minor increase as the variance of uncertain damage level increases. However, for larger values of γ, the carbon tax increases in a clearly visible amount as the variance increases. Therefore, the effect of a rising uncertainty about future damage from abrupt climate change is amplified with higher degrees of risk aversion. In models assuming DICE-like separable preferences, it is rather the mean of the uncertain damage and not the variance which affects the carbon tax. We observe this finding as well for low degrees of risk aversion (i.e. smaller than 2).

However, with higher degrees of risk aversion, we see that the impact on the carbon tax is proportional to the variance of the uncertainty. This is not surprising since it fits into CAPM-style intuition. The simple CAPM logic tells us that price of risk is related to its covariance with the aggregate endowment. Since the magnitude of the damages is proportional to output, the damages are strongly related to output; in fact, in this case the climate damage is the only stochastic element of output conditional on the tipping event. Therefore, the covariance is unity and, as we expect from CAPM logic, the carbon tax has a price of risk component linear in the variance. Moreover, the sole effect of the risk-aversion coefficient γ is significant: it increases much as γ increases from 2 to 10 or from 10 to 20, independent of which variance of the uncertain damage level is chosen, even if the damage level is certain at 5% (i.e., $\delta = 0$).

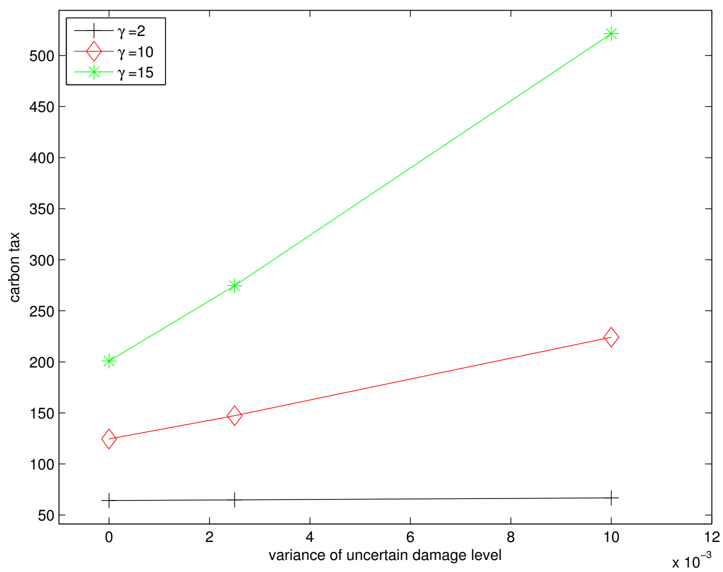
In the previous example the average damage level was 5% and its volatility at most 2.5%. We now return to the disaster case of the previous section and assume both a higher mean and more uncertain impact. In particular, we assume

$$J_t = \begin{cases} 0.2 + \varsigma, & \text{with probability } 50\%, \\ 0.2 - \varsigma, & \text{with probability } 50\%, \end{cases}$$

where $0 \leq \varsigma < 0.2$, while the probability transition matrix is the same with (5). We choose the inverse of intertemporal elasticity of substitution as $\psi = 2$, and the hazard rate parameter $\nu = 0.0008$, which is the same as in the disaster case of the previous subsection 5.4. Again, we study the effect of risk-aversion and volatility of the uncertain damage level on the carbon tax.

Figure 8 shows that for any volatility of the uncertain damage level, the carbon tax in the initial period significantly increases with higher risk-aversion.

Figure 8: Effect of Risk-Aversion Coefficient and Variance of Uncertain Damage Level under Disaster Case



As in the previous example, higher variance amplifies the effect of risk aversion on the carbon tax. For example, when the variance is 1%, the carbon tax in the initial period is \$521 for $\gamma = 15$. This is much higher than the case with $\gamma = 2$ (in which the carbon tax is only \$67). Here again, if the risk-aversion coefficient is small (e.g., $\gamma = 2$), then higher variance brings about only almost no change in the carbon tax in the initial period. The price of risk for this parameter setting as well as for standard separable preferences is zero. However, if $\gamma = 15$, the carbon tax rises sharply from \$201 to \$521 when the volatility increases from 0 to 10%. To sum up this example: in face of a very unlikely tipping point event (about 0.1% probability of tipping at 2100), and uncertainty about its impact (mean of 20% and uncertainty is 10%), a risk-averse policymaker sets the carbon tax of the initial period at \$521. Here, the Epstein-Zin formulation of preferences is a key determinant of our finding in the case of uncertain post-tipping damage. Standard DICE-like separable preferences necessitate unrealistically extreme catastrophic scenarios (e.g. Weitzman, 2009) in order to obtain results similar to ours.

5.6 Gradual Tipping Point Impact

So far, we have studied a simple two-stage tipping process to help us understand the basic drivers of the carbon tax within our model. As we have argues in section 2, tipping elements in the climate system exhibit heterogeneity with respect to their assumed duration of the post-tipping impact paths. According to Lenton et al. (2008) the durations can be as low as less than 10 years on the lower scale, but also last high as more than 300 years. Therefore, a simple two-stage process assuming a most rapid abrupt climate change is not a sensible representation of scientific accord.

In this section, we use DSICE to examine an eleven stage tipping process to study how a sequence of tipping events affects the SCC. Here again, we do not focus on any specific tipping elements, but rather study a range of different assumptions about post-tipping damage and duration of the entire tipping process. We are therefore able to match a broader range of beliefs about the nature of a tipping point scenarios and analyze how they will affect the optimal climate policy in our model.

Since this tipping structure is sequential, the decision maker faces always one tipping point stage at a time but has full information about the tipping system transition matrix. We assume that only stage 1 is associated with an endogenous tipping point probability (i.e. depending on contemporaneous temperature). Each subsequent stage (i.e. stage 2 - final stage) are associated with an exogenous and constant tipping point probability. To model this, we assume that J_t is a discrete Markov chain with 11 possible values of 0, 0.01, 0.02, ..., 0.1, and its probability transition matrix from year t to year $t+1$ is

$$\begin{bmatrix} 1-p_t & p_t & & & & \\ & 1-q_t & q_t & & & \\ & & 1-q_t & \ddots & & \\ & & & \ddots & q_t & \\ & & & & 1-q_t & q_t \\ & & & & & 1 \end{bmatrix},$$

where p_t is the transition probability entering stage 2 from the initial stage, given in the formula (3), and q_t is the transition probability entering stage $i+1$ from the stage i for $i > 1$. We let $q_t = 1 - e^{\chi}$, and then calibrate χ to be in line with the expected duration of the entire tipping process.

In this example, we choose $\chi = 0.2$ so that the expected duration of the whole tipping process is 50 years. Furthermore, we set the inverse of intertemporal elasticity of substitution is $\psi = 2$, the risk-averse coefficient is $\gamma = 10$. Also, similar to our benchmark case, we assume that the hazard rate parameter at stage 1 is $\nu = 0.00574$. Figure 9 displays the optimal carbon tax for this application. As in the previous examples, we do report the statistics of 1000 simulated paths of the carbon tax.

Figure 9: Carbon Tax with an Eleven-Stage Tipping Process

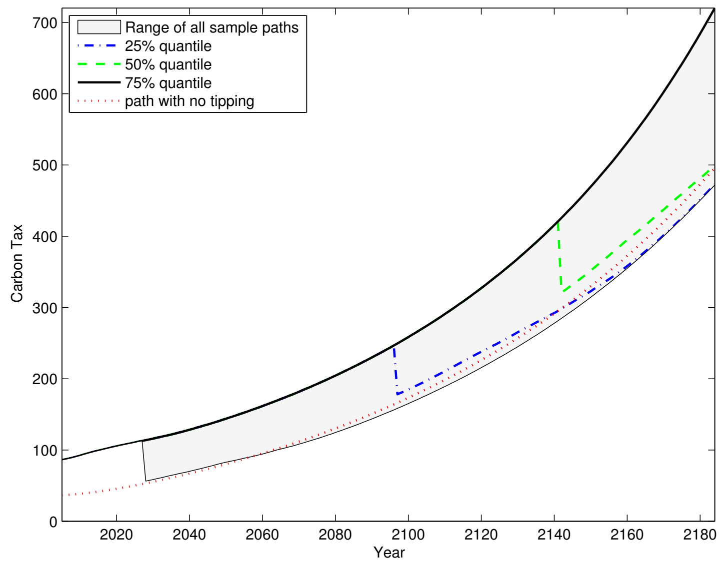
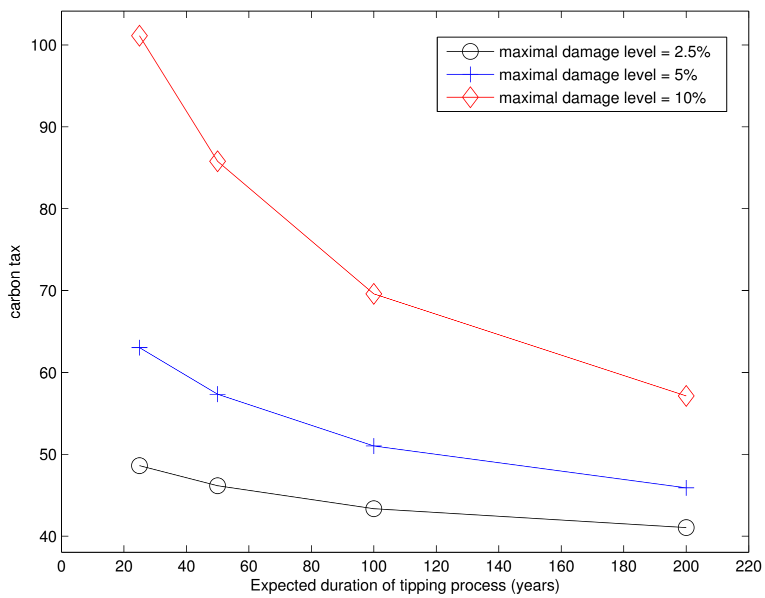


Figure 9 shows that the qualitative nature of the carbon tax path for the 11-stage tipping process is very similar to the one of the 2-stage process. As in the previous examples, the no-tipping path shows a carbon tax of \$37 in 2005. The optimal pre-tipping carbon tax in 2005 is about \$86. This is more than twice as much, but not surprising since we now assume a stage 11 - damage of 10%. Furthermore, the pre-tipping carbon tax rises to about \$250 by the end of the century and by an additional \$250 at 2150. The expected duration of the entire stochastic tipping process is 50 years. Our 1000 simulations show the fastest transition of 18 years and the slowest transition of 132 years (the sample mean duration is 49.4 years). Here again, we are taking account of scientific uncertainty regarding the nature of tipping point events. Furthermore, we observe from Figure 10 how our assumption of an endogenous tipping probability in stage 1 and exogenous probability thereafter translates in the optimal carbon tax numbers. Once, stage two is reached, the decision maker cannot longer prevent the subsequent stages from occurring. Therefore, a sequence of adverse cascading events is triggered and additional mitigation efforts aiming at preventing such an cascade are eliminated. Stages 3 - 11 are unavoidable. It is stage 2 which we want to delay. Note, the asymptotic path of the 25% quantile (blue dotted line) toward the 0% quantile. It reflects the fact that after stage 2 (the vertical drop) is reached, it takes about 50 years for the final stage (stage 11) to occur.

Since it is the expected stochastic damage that matters for the carbon tax in each period we next solve the model for twelve different parameter settings. We study three different maximal damage levels: 2.5%, 5% and 10%. Furthermore, we study four different expected durations of the multi-stage tipping process: 25, 50, 100 and 200 years, since the tipping process has been triggered. Figure 10 depicts the pre-tipping carbon tax in 2005 for these twelve runs.

Figure 10: Effect of Damage Level and Expected Duration of Tipping Process an Eleven-Stage Tipping Process



We see that the 2005 optimal carbon tax is increasing with the final-stage maximal damage level. It is also increasing with a shorter expected duration of the entire tipping-process.

Note further, that the optimal carbon tax (200 periods) at 10% damage is about the same as the carbon tax (50 periods) at 5% damage. The same holds for the carbon tax (200 periods) at 5% damage and the carbon tax (50 periods) at 2.5% damage. These findings confirm our intuition that it is the combination of the post-tipping damage and the duration of the tipping process which drives the optimal additional carbon tax in a pre-tipping regime.

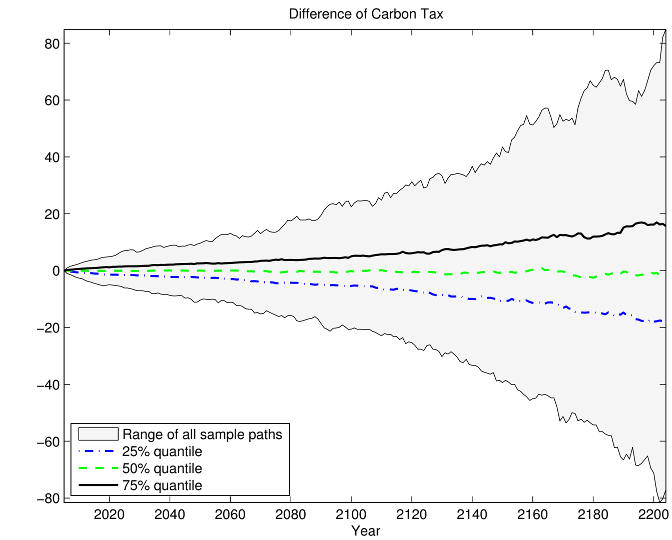
Comparing the 10% damage line (red) in Figure 10 with the blue line at 10% damage in Figure 5, we can provide numbers for the reduction in the additional carbon tax when moving to a multi-step tipping process of different expected duration of the entire process. First, recall that the 2005 pre-tipping additional carbon tax is \$98 (\$37 for no tipping path vs. \$135 for the two-step tipping path). This number reduces to \$64 when moving to a 11-stage process with 25 years of expected transition to stage 11. Furthermore, the additional pre-tipping carbon tax reduces to \$64 (50 years of expected transition) and \$49 (100 years). Finally, even if the expected duration of the post-tipping carbon tax is 200 years, the additional pre-tipping carbon tax in 2005 is still \$20. Of course, if we reduce the stage 11- damage from 10% to 5% and 2.5%, the additional pre-tipping carbon tax decreases. Nevertheless, for an expected duration of 200 years and a 2.5% damage from the final stage of the tipping process, we find that the optimal carbon tax today is about \$41, which is more than a 10% increase to the deterministic case without a tipping process.

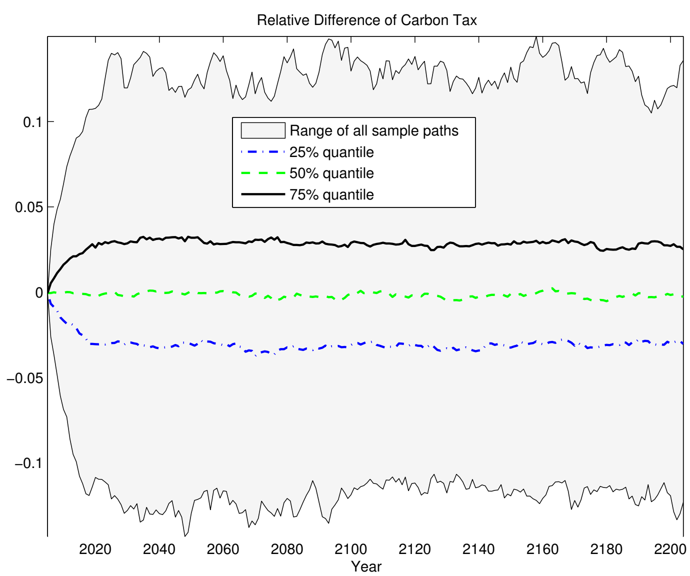
5.7 Stochastic Economic Productivity

Decision makers face at least two kinds of uncertainty: uncertainty about future economic productivity as well as uncertainty about future climate impacts. We next show how economic productivity shocks interact with climate change dynamics.

First, we investigate solely the effect of economic shocks on the carbon tax. We disregard the tipping point for a moment. In the example, we assume that the economic shock ζ_t is a bounded mean-reverting process given in Cai, Judd and Lontzek (2012b): its reverting rate is 0.1, its long-run mean is 1, the standard deviation of the conditional productivity shock is 2%, and ζ_t is contained in the interval $[0.92, 1.08]$. We use our benchmark taste parameters: $\psi = 2$ and $\gamma = 10$, in this example. Figure 11 shows the (relative) difference in the carbon tax to the deterministic model under a long persistence and the benchmark preference parameters. The upper and lower envelopes of our simulations result in roughly a 10% change in the carbon tax. In all simulations, the carbon tax fluctuates between $+15\%$ and -15% relative to the mean, with a standard deviation of about 1.7%, a modest amount of volatility.

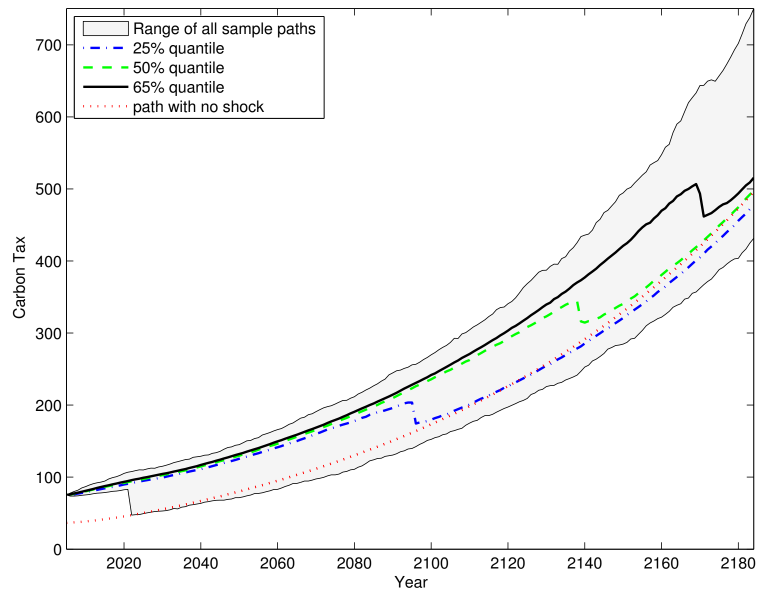
Figure 11: Effect of Economic Shock





Second, we present the results of the economic shock uncertainty and the tipping point uncertainty combined. Figure 12 shows the carbon tax for the case with the benchmark tipping point with 5% damage level and the business cycle uncertainty having the same parameters with the above example.

Figure 12: Effect of Economic Shock



The carbon tax in this example exhibits both features from the "tipping only" and "business cycle only" examples. The constancy in the additional carbon tax from tipping is retained as well as the relatively mild fluctuations of the carbon tax. The latter point can be seen when looking at the positions of the quantiles. As opposed to Figure 3, the optimal carbon taxes for the different quantiles in the post-tipping regime no longer overlap in this example, i.e., they still fluctuate after a tipping point has occurred.

These two examples indicate that the total uncertainty is essentially the sum of the two separate processes in DSICE when we assume a very simple structure to the economic system. Future extensions will incorporate features of endogenous growth and more persistent economic shocks, both of which could produce more interesting interactions between the economic and climate systems.

6 Conclusion

This study has analyzed the optimal level and dynamic properties of the carbon tax in face of stochastic and irreversible climate change and its interaction with economic factors, including business cycle fluctuations and preferences about risk. The underlying model entails several ingredients which we consider essential for a sound analysis of climate policy under catastrophic risk. First, we model a low-probability tipping point process as jump processes with a temperature-dependent hazard rate and a stochastic duration of the entire process. Second, we account for the intrinsically uncertain nature of future states of the economy and incorporate the cyclicality of business conditions into the optimal decision making process of the policy maker. Third, we model the cost of risk in a manner more compatible with empirical evidence about social risk preferences. Riskiness is inherent in the nature of tipping points, and the socio-economic effects of stochastic, sequential and irreversible perturbations to the climate system will be affected by preferences about risks. While these features are familiar model components in economics, it is the joint modeling of them within a DSGE version of a widely accepted IAM, that attributes to the novelty of this study.

In particular, we use DSICE (Cai, Judd and Lontzek, 2012b), a DSGE full-dimensional extension of DICE2007. DSICE and DICE2007 are therefore comparable, which facilitates the comparison of our carbon tax numbers with the ones obtained, e.g., by the U.S. Government Interagency Working Group on Social Cost of Carbon (IWG, 2010). In contrast to other approaches in the literature to study a stochastic version of DICE2007, we are endowed with an annual-frequency, full-dimensional, stochastic IAM with intrinsic uncertainty about annual economic productivity and stochastic climate components. We solve DSICE with dynamic programming using advanced computational methods. The solution to DSICE is reliable and quickly obtained. Furthermore, the advanced computational architecture behind DSICE will enable us to study much more complex and higher-dimensional extensions of the model presented in this study.

We find that the threat of a tipping point induces significant and immediate increases in the social cost of carbon, even for low-probability and low-impact tipping events. We find that incorporating Epstein-Zin preferences into DSICE in a manner that is compatible with the evidence on risk aversion and the intertemporal elasticity of substitution significantly increases in the carbon tax. For example, larger values of the intertemporal elasticity of substitution and the degree of risk aversion lead to higher carbon taxes. Furthermore, in addition to stochastic and irreversible climate change, we also study cases with significant uncertainty about post-tipping damages. We find that uncertainty about the damage is also a critical factor leading to a sharp increase in the carbon tax. Furthermore, for low degrees of risk aversion (i.e. smaller than 2), the carbon tax is not affected much by the volatility of the uncertain damage. In contrast, high degrees of risk aversion significantly amplify the effect of damage uncertainty on the carbon tax. We also investigate a disaster scenario that is unlikely (about 0.1% probability of tipping at 2100), but with large and uncertain impacts (mean is 20% and volatility is 10%), today's carbon tax of a highly risk-averse policy maker is $521.

This application of DSICE also shows that it is quite feasible to solve dynamic stochastic IAMs with current numerical algorithms and computational hardware. Past analyses have been hampered by both software and hardware limitations, but advances over the past twenty years now make it possible to examine important economic questions without making unappealing assumptions for the sake of tractability.

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