SUBJECT: Physics

TITLE: Investigation into the nature of in-space propulsion systems and the distances they can travel.

RESEARCH QUESTION: How do variations of travel time change which in-space propulsion system can travel the furthest distance in a vacuum experiencing no gravity?

WORD COUNT: 3798

Table of Contents

Section 1 - Introduction

This extended essay will focus on analyzing different types of in-space propulsion systems in order to illustrate under what conditions certain systems perform better. We define this performance as the amount of distance the system travels. We define this goal specifically in the research question:

How do variations of travel time change which in-space bi/monopropellant propulsion system can travel the furthest distance in a vacuum experiencing no gravity?

In-space propulsion systems are systems which propels payloads, masses that are attached to the system, in space. They are used in all sorts of applications from satellites to travelling to the moon. We will in particular focus on in-space travel which experiences minimal gravity, such as when travelling to planets in our solar system or the voyager missions.

A simulation will be created based on predicted mathematical models of the distance travelled by a propulsion system in a vacuum with no gravity affecting it. The results from this simulation will inform us about the research question and hopefully allow us to draw some conclusions.

It is anticipated that propulsion systems that use up alot of fuel to generate a large amount of thrust will almost always be faster than propulsion systems that rely on a smaller amount of fuel to generate lower thrust. However, these slower systems, as they are more fuel-efficient, will outperform the bulkier systems in cases of very long travel times. As well, bipropellant systems are thought to be more complex and use more fuel but generate more thrust than monopropellant systems.

Section 2 - Modeling Distance Travelled of a Propulsion System

As stated in our introduction, in order to create a simulation to answer our research question, we first need to derive and understand the mathematical models for the system we are analysing.

For our purposes we seek the distance travelled by a propulsion system. The simplest way to do this is to model the change in velocity over time of the system and then use the relationship, distance x:

$$x = \int v dt$$

Part A - Understanding Propulsion Systems and Modeling Challenges

Before we derive any models we should first understand how a propulsion system works. At its most basic rocket propulsion operates under the principle of Isacc Newton's third law "for every action there is an equal and opposite reaction." The action is the ejection of the propellant from the propulsion system and the reaction is that the propulsion system, along with an attached payload, accelerates opposite to where the propellant was ejected. This reactionary force, for the purpose of this discussion, is called *thrust*. This force from the propellant is created from a controlled explosion inside a chamber in the propulsion system, called the combustion chamber. This explosion creates hot air that is expelled at V (exhaust velocity in m/s) and at a rate of q (flow rate in kg/s). While the explosion creates a different internal pressure to that of the vacuum of space we assume that the extra velocity caused by this difference in pressure is included in a slightly modified *effective* exhaust velocity $(V e)$.

Knowing these variables we can use newton's second law $F = ma$, where F is force caused by the ejection of propellant, m is mass of propellant and a is the acceleration of the propellant, to solve for force of ejection. Given we can define flow rate q as the amount of mass ejected in one second and that the velocity of the propellant in this time goes from $0\ m/s$ to $V e$, acceleration is $\frac{V e\ m/s}{1\ s}$ and mass is $q \times 1\ m$. Thus we can simplify this into:

$$F_{ejection} = V e \times q$$

Given that this essay deals solely with in-space vacuum flight we can assume that this is the only force applied on the rocket. Thus the net force/thrust of the rocket is $F_{thrust} = F_{ejection}$.

$$F_{thrust} = Ve \times q$$

An important thing to note is that system is constantly losing mass due to expelling propellant, thus given F_{thrust} stays constant but mass m is constantly changing, by newton's second law $F = ma$ acceleration is constantly changing as well. This changinging acceleration presents a challenge when attempting to model the systems. We cannot use traditional methods like the SUVAT equations. One way to tackle a problem like this from my experiences with IB Physics is to use linear momentum. Surprisingly this is how we will tackle deriving the Rocket equation.

Part B - Deriving the Rocket Equation

We can represent the linear momentum (p) as $p = mv$ where m is mass and v is velocity. The law of conservation of momentum also states that given no external forces the total momentum of a system ($p = mv$) will remain constant. We can use this property to equate the momentum before the ejection of fuel and the momentum after as there are no external forces acting on the system.

The momentum before is the mass of the payload and the fuel (M) by the current velocity of the rocket (V). The momentum of the rocket after a time (dt) is the mass of the payload and the fuel (M) minus the change in mass M due to ejecting the fuel ($\frac{dM}{dt}$) by the velocity of the rocket(V) and it's increase in velocity due to the ejection ($\frac{dv}{dt}$). The momentum of the mass ejected after a time (dt) is simply the change in mass M as defined previously ($\frac{dM}{dt}$) by the velocity of the rocket (V) and it's decrease in speed due to being ejected at velocity (Ve). This can be summarized as:

$$p_{before} = p_{After}$$

$$MV = (M - \tfrac{dM}{dt})(V + \tfrac{dv}{dt}) + \tfrac{dM}{dt}(V - Ve)$$

$$MV = (M - \tfrac{dM}{dt})(V + \tfrac{dv}{dt}) + \tfrac{dM}{dt}(V - Ve)$$

$$MV = MV - V\frac{dM}{dt} + M\frac{dv}{dt} - \frac{dv}{dt}\frac{dM}{dt} + V\frac{dM}{dt} - Ve\frac{dM}{dt}$$

Which simplifies to:

$$0 = M\frac{dv}{dt} - \frac{dv}{dt}\frac{dM}{dt} - Ve\frac{dM}{dt}$$

As the both $\frac{dv}{dt}$ and $\frac{dM}{dt}$ are very small numbers their product is insignificant to the equation and we can assume they equal to zero. This leaves us with:

This gives us:

$$0 = M\frac{dv}{dt} - Ve\frac{dM}{dt}$$

$$Ve\frac{dM}{dt} = M\frac{dv}{dt}$$

As $\frac{dv}{dt}$ is simply the acceleration, we have a force in the form $F = ma$ equal to the exhaust velocity by the flow rate. This is equivalent to the formula from part A:

$$Ve \times q = F_{thrust}$$

We are left with a very simple differential equation, dv is positive because it increases with dt, dM is negative because it decrease with dt :

$$M dv = -Ve\, dM$$

We now have an opportunity to use this equation to model the change in velocity over time. Where in we can integrate dv. Isolating for dv:

$$dv = \frac{-Ve\, dM}{M}$$

We can then integrate this:

$$\int dv = \int(-Ve\frac{dm}{M})$$

$$[v + c] = [-Ve\, ln(M) + c]$$

As we will assume all propulsion systems start at rest, the definite integral for when $\int dv = v_{current}$ is when the limit starts at 0 and ends at $v_{current}$. For these limits, M has not expelled any mass as of yet when $v = 0$ and M has expelled the amount of mass required for when $v = v_{current}$. We can define this as $M_{original}$ and M_{final} . Thus we can solve for the define integral

$$\int_{0}^{v_{current}} dv = \int_{M_{original}}^{M_{final}} (-Ve\frac{dm}{M})$$

$$v_{current} - 0 = Ve\,ln(M_{original}) - Ve\,ln(M_{final})$$

$$v_{current} = Ve\,ln(\frac{M_{original}}{M_{final}})$$

$$v = Ve\,ln(\frac{M_{original}}{M_{final}})$$

Thus we have modelled the velocity of a propulsion system. It is important to note that when solving for change in velocity, ie, the system doesn't start at rest, you get the more tradition rocket equation as derived by Tsiolkovsky [1, 2]:

$$\int_{v_{original}}^{v_{final}} dv = \int_{M_{original}}^{M_{final}} (-Ve\frac{dm}{M})$$

$$\Delta v \;\; = Ve\,ln(\frac{M_{original}}{M_{final}})$$

Part C - Deriving the Rocket Equation as a Change in Time

Now that we have derived the velocity of a propulsion system that starts at rest given original and final mass, we must now find a way to define this as a change in time so we can apply the formula:

$$x = \int vdt$$

This shouldn't be too difficult given we can think of $M_{original}$ as the mass of the payload $(M_{payload})$ and the mass of the fuel it brings (M_{fuel}) . Then we can think of M_{final} as the decrease of $M_{original}$ by the mass of the fuel expelled $M_{expelled}$ This gives us:

$$\int_{0}^{v} dv = \int_{M_{payload}+M_{Fuel}}^{M_{payload}+M_{Fuel}-M_{Expelled}} -Ve\frac{dm}{M}$$

$$v - 0 = Ve\,ln(M_{payload} + M_{Fuel}) - Ve\,ln(M_{payload} + M_{Fuel} - M_{Expelled})$$

$$v = V e\ ln(\frac{M_{payload} + M_{Fuel}}{M_{payload} + M_{Fuel} - M_{Expelled}})$$

If we assume that the system will be ejecting fuel for the entire period of time it plans to travel (T), than it will optimally bring qT worth of fuel where q is the flow rate of the ejection which we assume to be consistent. Thus we say $M_{Fuel} = qT$. As well, the amount of mass expelled if we assume the system will consistently eject fuel is qt where t is the amount of time that has currently elapsed. Thus, after a lot of work we can model the velocity of a simple propulsion system in a vacuum that is constantly ejecting fuel as a change in time as:

$$v = V e\ ln(\frac{M+qT}{M+qT-qt})$$

Part D - Modeling the Distance Travelled by a Propulsion System

Given we derived the velocity as a change in time we can use $x = \int v dt$ to solve for distance x across a time T, which importantly is the same T that affects the amount of fuel that the system brings.

$$x = \int_0^T V e\ ln(\frac{M+qT}{M+qT-qt})\ dt$$

Solving for the integral gives us:

$$x = \frac{(V e)(qT - M ln(\frac{M+qT}{M}))}{q}$$

Section 3 - Creating and Designing the Simulation

With the modeling out of the way we can now start designing the simulation.

Part A - Variables in the Simulation

In order to use the formula $\frac{(Ve)(qT - M\ln(\frac{M+qT}{M}))}{q}$ we will need the following variables:

Table 1 - Variables used in the Simulation

Variable	Description	Units	Source
Ve	The *effective* exhaust velocity of the fuel expelled	m/s	External data sheet
q	The flow rate that fuel is expelled, assumed to be consistent	kg/s	External data sheet
T	The length of time that the system will travel for	s	Will be set by the simulation
M	The payload mass + the mass of the propulsions system	kg	External data sheet and set by the simulation
x	The distance the system travels	m	Will be solved for by the simulation

We will source Ve, q and part of M, from a data sheet produced by the company aerojet rocketdyne, which will be discussed in greater detail in section 4.

Part B - Special Considerations

While the mathematical model and the simulation will never perfectly match up with real conditions, there are some important considerations that will greatly improve the accuracy of the simulation. As well, the data sheet we use from aerojet rocketdyne has with it some important considerations:

- All propulsion systems have a recorded limit on the amount of uses or "pulses" they can eject fuel for. Once they reach this limit they are no longer functioning or can no longer consistently function. The data sheet gives us this number of pulses as well as the length of time for each of these pulses. We can use this to find the total length of time the system can functionally expel fuel and hence follow the equation we derived. After it has exceeded this time the system will no longer increase it's velocity but rather continue to travel at the same velocity for the rest of the journey. [1]

- We assume that the time in between these pulses are negligible.
- All systems have a given mass in kg so we will add this to a base of 1000kg to get the payload mass for each specific system. 1000kg was chosen arbitrarily.
- Ve is not listed in the data sheet. Instead something called specific impulse I_{sp} is listed. Luckily we can calculate Ve given I_{sp} . I_{sp} is a measure of how many seconds the expelled propellant can accelerate its own mass to 1g (the acceleration of earth's gravity). The reason it seems like such a strange measure is because it was designed so scientists working in different unit measures could perform calculations in their unit of choice, as gravity can be expressed in different units. For our purposes, to get exhaust velocity in standard metric units we will use the SUVAT equations where initial velocity is 0, a constant acceleration, 9.81, a time, specific impulse (I_{sp}), and we are solving for final velocity, exhaust velocity(Ve) . [1, 6]

  $$v = u + at$$

  $$Ve = (9.81)(I_{sp})$$

  Thus we can calculate for Ve given I_{sp} .

Part C - Programming the Simulation

We can now write a simple simulation to calculate distance travelled given an array of travel times by using the variables from Table 1 and the considerations from part B. The results from this simulation are graphed into a line graph using the python library matplotlib. It is very important to note that each "Time of Journey" or travel time is measuring the amount of distance the system can travel for that time given it started from a distance of 0 meters with enough fuel for that journey. The variations of this distance is what is plotted.

Section 4 - Running the Simulation

We will now actually run the simulation using data gathered from the data sheet by aerojet rocketdyne [7].

Part A - Data Collection Methodology

While the data sheet includes data from a wide array of different propulsion systems, we will only use the sections on mono and bi propellant propulsion systems as they have the most complete information. 5 pieces of data were collected for each propulsion system, Specific Impulse, Flow Rate, Mass of System, Total Starts/Pulses, Firing Time. Where not all 5 pieces of data were clearly stated, the propulsion system was not included in the raw data tables. Careful attention was paid to units which varied from metric to imperial. All units were converted to metric with the same prefix value.

For values which are given in a range, the highest value of that range will be taken. This is because a range of values in this context is interpreted as the range of possible values the system can be configured to operate at rather than the possible error of the system. This is supported by having ranges of values for some variables that are too impractically large to be interpreted as error, for example) 0-900 seconds firing time or 0.0005 - 0.00009 flow rate. It is also supported by foot notes such as: “Note: thrust levels up to 9lbf have been qualified and flown” which seems to suggest the range of values might extend beyond what is recorded. The error is calculated as ± 1 to the most precise digit and all values were rounded to 3 sig figs or less to ensure consistency.

Part B - Raw Data

Table 2: Monopropellant Propulsion Systems

Name	**Specific Impulse (I_{sp}) (s)**	**Flow Rate (q) (kg/s)**	**Mass (M) (kg)**	**Total Starts**	**Firing Time(s)**
MR-401 0.09 N (0.02 lbf)	184 ± 1	0.0000504 ± 0.000001	0.60 ± 0.01	5960 ± 10	900 ± 1
MR-106L 22N (5.0 lbf)	235 ± 1	0.0140 ± 0.0001	0.59 ± 0.01	121000 ± 1000	4000 ± 10
MR-103G 1N (0.2 lbf)	224 ± 1	0.000500 ± 0.000001	0.33 ± 0.01	835000 ± 1000	1000 ± 10

MR-103J 1N (0.2 lbf)	224 ± 1	0.000500 ± 0.000001	0.37 ± 0.01	1000000 ± 10000	3600 ± 10
MR-111G 4N (1.0 lbf)	229 ± 1	0.00200 ± 0.00001	0.37 ± 0.01	420000 ± 1000	10000 ± 100
MR-107T 110N (25 lbf)	225 ± 1	0.0558 ± 0.0001	1.01 ± 0.01	36500 ± 100	100 ± 1
MR-107S 275N (60 lbf)	236 ± 1	0.155 ± 0.001	1.01 ± 0.01	30300 ± 100	100 ± 1
MR-107U 300N (68 lbf)	229 ± 1	0.098 ± 0.001	1.38 ± 0.01	4410 ± 10	100 ± 1
MR-107V 300N (68 lbf)	229 ± 1	0.098 ± 0.001	1.01 ± 0.01	10200 ± 100	100 ± 1
MR-104H 510N (115 lbf)	237 ± 1	0.250 ± 0.001	2.40 ± 0.01	6520 ± 10	2650 ± 10
MR-104J 440N (100 lbf)	223 ± 1	0.284 ± 0.001	6.44 ± 0.01	6600 ± 10	2010 ± 10
MRM-106F 40N (9.0-lbf)	231 ± 1	0.0177 ± 0.0001	2.23 ± 0.01	1570 ± 10	1000 ± 10
MRM-122 130N (30-lbf)	228 ± 1	0.0635 ± 0.0001	2.98 ± 0.01	7010 ± 10	2140 ± 10

Table 3: Bipropellant Propulsion Systems

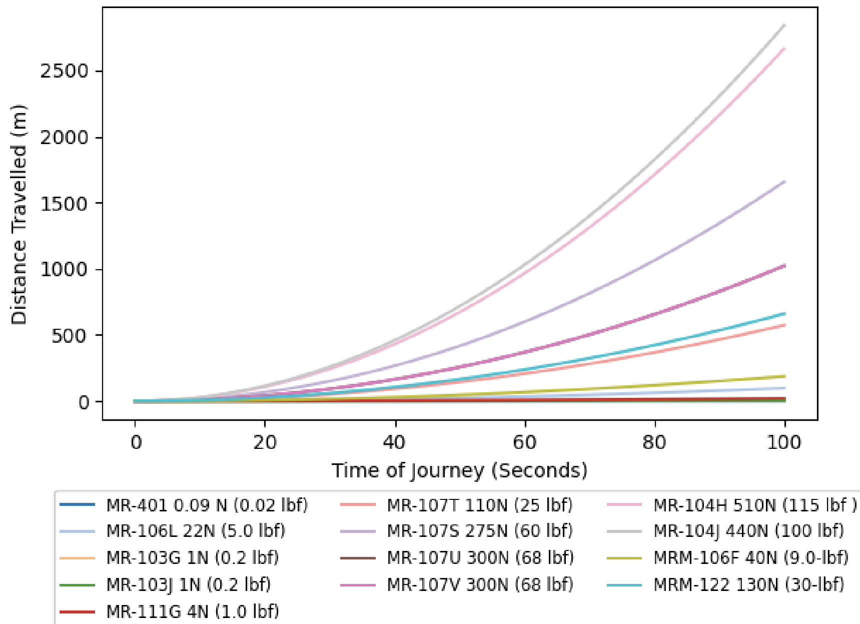
Name	**Specific Impulse** (I_{sp}) **(s)**	**Flow Rate** (q) **(kg/s)**	**Mass** (M) **(kg)**	**Total Starts**	**Firing Time(s)**
AJ10-220 62.3 N (14.0 lbf)	285 ± 1	0.0223 ± 0.0001	1.95 ± 0.01	65000	300 ± 1
R-1E 110N (25 lbf)	280 ± 1	0.0404 ± 0.0001	2.00 ± 0.01	330000	*
R-42 890N (200 lbf	305 ± 1	0.300 ± 0.001	4.53 ± 0.01	150	3940 ± 10
R-6F 22N (5lbf)	305 ± 1	0.00744 ± 0.00001	0.965 ± 0.001	19881	*
R-42DM 890N (200 lbf)	327 ± 1	0.277 ± 0.001	7.30 ± 0.01	60	1000 ± 10
R-40 3,870N (870 lbf)	281 ± 1	1.4 ± 0.1	10.5 ± 0.01	50000	23000 ± 100
R-40B 4,000N (900 lbf)	293 ± 1	1.4 ± 0.1	10.5 ± 0.01	50000	23000 ± 100
AJ10-190 Space Shuttle	316 ± 1	8.61 ± 0.01	118 ± 1	1000	54000 ± 100

*Data Sheet listed these values as "unlimited" to stimulate that a value of a billion will be used

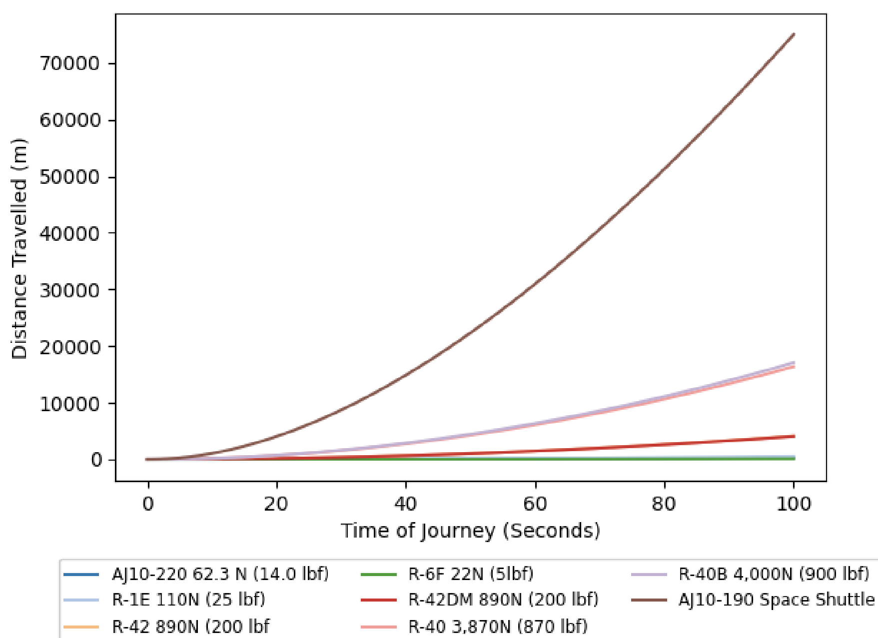
Section 5 - Results and Data Analysis

Part A - Across 100 seconds

Distance Travelled by Monopropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 100 seconds (figure 1)



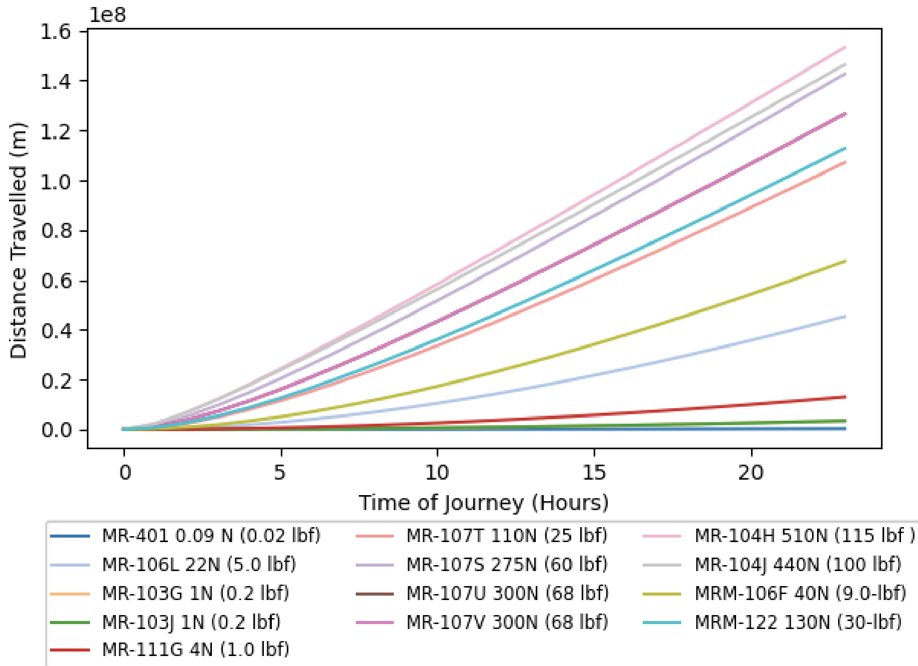
Distance Travelled by Bipropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 100 seconds (figure 2)



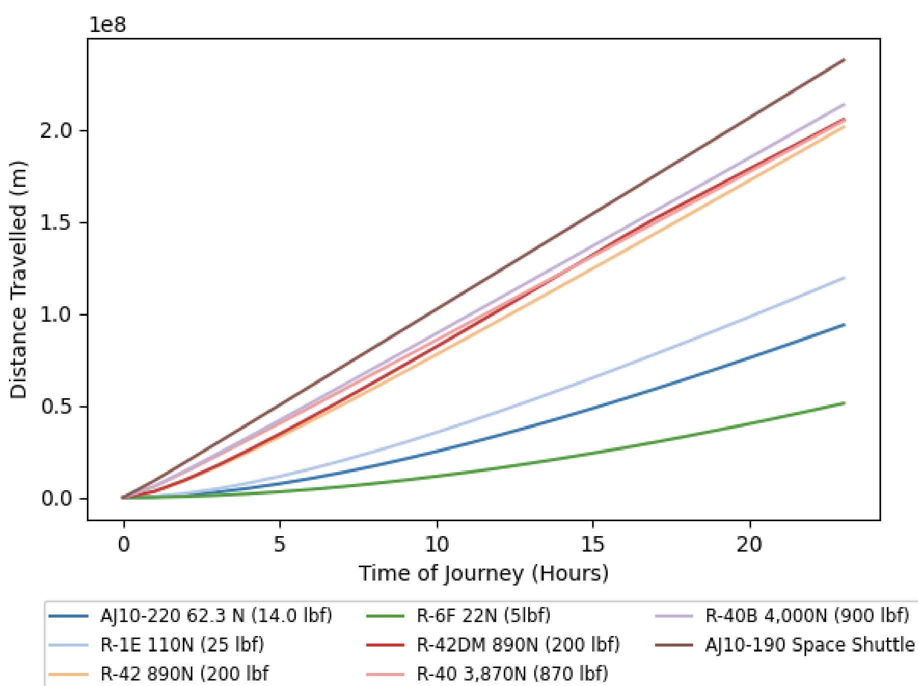
The first major observation is that bipropellant systems seem to travel much further generally, as shown when comparing systems in figure 1 and figure 2, the furthest travelled bipropellant system was AJ10-190 reaching 70000 meters in 100 seconds compared to the furthest travelled monopropellant system, MR-104J, which reached 2500 meters in 100 seconds. I suspect this pattern will continue. As expected, systems which expel a lot of propulsion performed exceptionally well for small periods of time, with both MR-104J and AJ10-190 have the highest flow rate in their categories. Interestingly MR-104J actually has one of the smallest specific impulse (Table 2 and 3).

Part B - Across 1 Day

Distance Travelled by Monopropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 1 Day (figure 3)



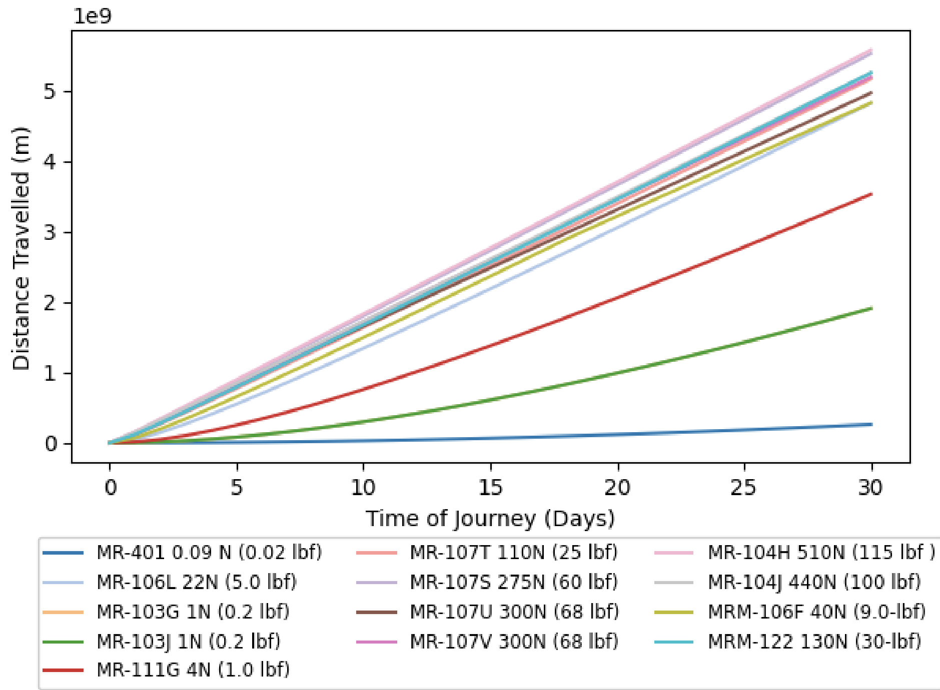
Distance Travelled by Bipropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 1 Day (figure 4)



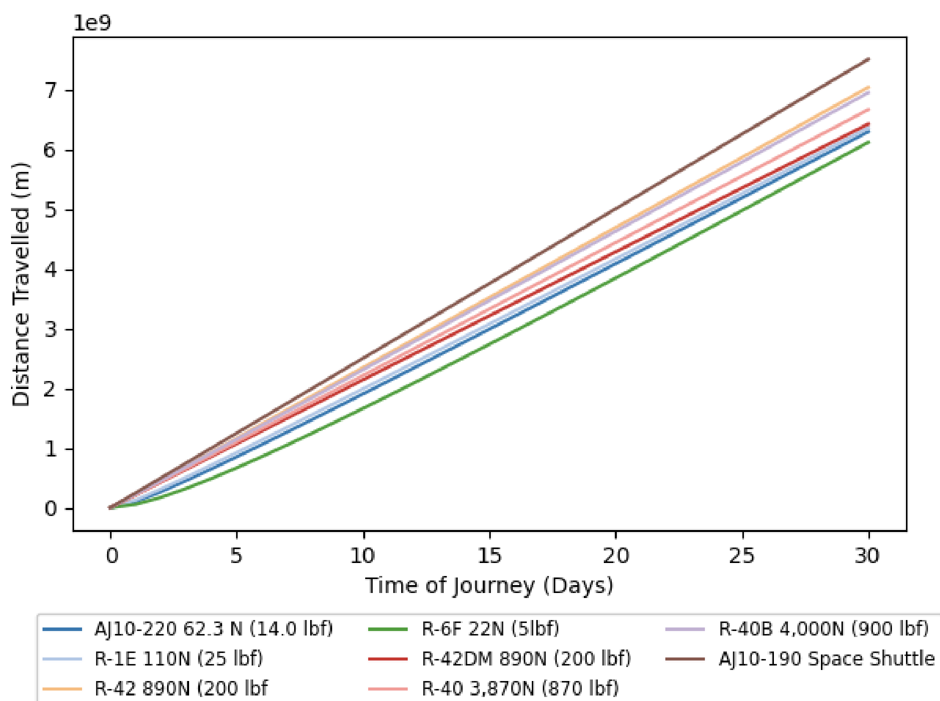
For the monopropellant systems (figure 3), MR-104H surpassed MR-104J for almost all times across 1 day. Looking at the difference between the two, while MR-104J has a slightly higher flow rate, MR-104H has a much higher specific impulse. MR-104H can be thought of as having a better balance between the two (specific impulse and flow rate) which seems to be better suited towards journey times across 1 days. For the bipropellant systems, it can still be seen that they continue to travel further than monopropellant systems. AJ10-190 is still the furthest travelling system for journey times varying across a day, however, other systems are closing the gap (figure 4). It's also interesting to see the current poor performers such as MR-401 and R-6F, which both have the smallest flow rate and some of the lowest specific impulses (Table 2 and 3).

Part C - Across 1 Month

Distance Travelled by Monopropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 1 Month (figure 5)



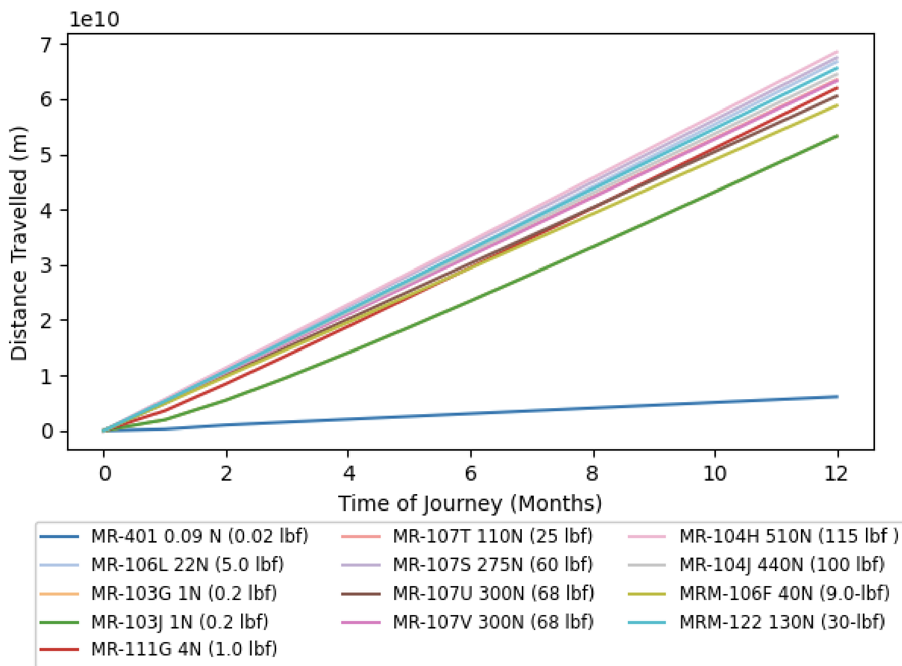
Distance Travelled by Bipropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 1 Month (figure 6)



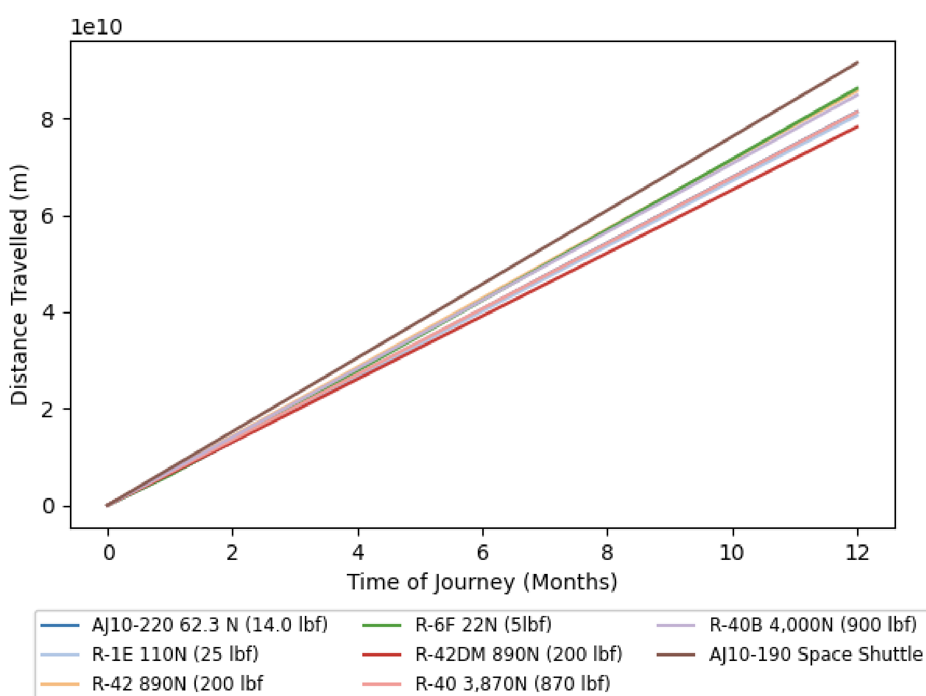
For the monopropellant systems (figure 5), MR-104H is still the top performing, however, it's quite interesting that MRM-122 surpassed MR-107V. It would at first appear that MRM-122 has no advantage over MR-107V, it has a slightly smaller specific impulse, a heavier mass and two thirds the flow rate of MR-107V. This is the first indication that a smaller flow rate could perhaps make a system travel further for longer journey times. Despite this fact, the top performer is still MR-104H which has the highest flow rate. The reason it still travels the furthest is most likely because it also has the highest specific impulse. This hints to the fact that a higher specific impulse is almost always a benefit to a system, while a flow rate can vary from an advantage to a disadvantage depending on the length of time the system must travel. This is also evident for Bipropellant systems (figure 6), where R-42 is now the second best performer, surpassing R-40B and R-40. This is probably because both R-40B and R-40 have a more than 4 times higher flow rate while R-42 has a higher specific impulse. This hints that having a higher flow rate can outweigh having a low specific impulse for the short time period, such as when R-40 and R-40B performed better than R-42 (Figure 2,4).

Part D - Across 1 Year

Distance Travelled by Monopropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 1 Year (figure 7)



Distance Travelled by Monopropellant Propulsion Systems In a Vacuum Carrying 1000kg Across 1 Year (figure 8)



It seems pretty clear that there is a steady creep of the systems with smaller flow rate becoming able to travel further distances compared to those with larger flow rate. This can be seen in figure 7 with MR-106L, MRM-122 and MRM-111G. The most stark example of this comes from figure 8 with R-6F, which went from the worst performing system in Figure 6, Figure 4, Figure 2 to the second best performing system in Figure 8. Figure 8 in general is very interesting with the two systems that travelled the furthest for a year being R-6F which as a flow rate of 0.00744 kg/s and a specific impulse of 305s and AJ10-190 which as a flow rate of 8.61 kg/s and a specific impulse of 316s. This puts the interesting notion that travelling an extreme distance doesn't have to be exclusive to low flow rates nor high flow rates. However, in almost all cases where cost efficiency is required it would make sense to go with a less fuel consuming system. In this scenario the fuel carried by the AJ10-190 system is $8.61 kg/s * (31545000)s = 271602450\ kg$! This is compared to R-6F system which only carries $0.00744 * 31545000 = 234694.8\ kg$ over 1000 times less fuel.

Section 6 - Conclusion

In general, the results were as predicted. Systems which were more fuel efficient, which have a low flow rate but a lower exhaust velocity (specific impulse), travelled relatively small distances for short travel times, but travelled relatively large distances for longer travel times. Surprisingly, however, systems that were not fuel efficient at all, where they had a large flow rate to generate a large exhaust velocity, travelled relatively large distances for both short and long travel times. This is a really interesting result, especially because both major examples of this, AJ10-190 and MR-104H, travelled the furthest in **almost all** travel times. This means that even for very large time periods and distances the most time efficient way to travel in-space is using systems that burn a lot of fuel fast, not what you would expect. A reason for this might be that the specific impulse generated by the fuel outweighs the negative of having to carry so much fuel. Interestingly in figure 1 the opposite happens, with MR-104J travelling the furthest distance despite having one of the lowest specific impulses. It seems that having a large enough flow rate outweighs. We can see how flow rate acts almost like a balance with both low and high flow rates harming or benefiting the system depending on the time travelled while specific impulse acts only as a benefit the higher it is. Thus we can answer the research question as the **higher** the system's specific impulse the further the system can travel in any given time period while the **lower** the system's flow rate the further it can travel in a long time period but the less it can travel in a short time period and the **higher** the system's flow rate the further it can travel in a short time period but the less it can travel in a long time period. If the specific impulse is high enough the comparative loss from having a high flow rate can be negated for long time periods such as in figure 7 and 8. While if the flow rate is high enough it can negate a low specific impulse for short time periods such as in figure 1.

While this is an interesting result, this analysis could perhaps be furthered by looking at how much this extra fuel costs for systems with high flow rate. As shown in calculations done based on figure 8, the difference between the amount of fuel the two furthest travelled systems burn to achieve this distance is over 100000%. It would be interesting to see the cost per meter of using one over the other to see under which conditions it might be worth the cost. It would also be insightful to expand this cost analysis to the actual systems themselves. One conclusion of the simulation is that bipropellant systems can travel further distances than monopropellant systems. This can perhaps be attributed to cost as bipropellant systems are often more expensive requiring two different chemicals to be stored and more complex mechanics to combine the two chemicals.

Section 7- Sources Used

[1] http://www.braeunig.us/space/propuls.htm

[2] http://web.mit.edu/16.00/www/aec/rocket.html

[3] https://www.math24.net/rocket-motion/#:~:text=If%20we%20integrate%20the%20differential,dv%3Dudm.

[4] https://www.integral-calculator.com/#

[5] https://en.wikipedia.org/wiki/Hydrazine

[6] https://en.wikipedia.org/wiki/Specific_impulse

[7] https://www.rocket.com/sites/default/files/documents/In-Space%20Data%20Sheets%204.8.20.pdf

Section 8 - Appendix

Part A - Solving for Distance Travelled by a Propulsion System

$$\int V\,e\;ln(\frac{M+qT}{M+qT-qt})\;dt$$

$$= V\,e\int ln(\frac{M+qT}{M+qT-qt})\;dt$$

$$\text{Let }\; u = M + qT - qt$$

$$\frac{du}{dt} = -q$$

$$\therefore V\,e\int ln(\frac{M+qT}{u})(\frac{-1}{q})\frac{du}{dt}\;dt$$

$$=-\frac{V\,e}{q}\int ln(\frac{M+qT}{u})\;du$$

$$=-\frac{V\,e}{q}[\int[ln(M+qT) - ln(u)]\;du$$

$$=-\frac{V\,e}{q}[\int[ln(M+qT)du - \int ln(u)\;du]$$

$$=-\frac{V\,e}{q}[ln(M+qT)u - \int ln(u)du]$$

If we use integration in parts to solve for $\int ln(u)du$

$$w = ln(u) \qquad v' = du$$

$$w' = \frac{1}{u} \qquad v = u$$

$$\therefore \;\; =-\frac{V\,e}{q}[ln(M+qT)u - u\,ln(u) + u])$$

$$=-\frac{V\,e(u)}{q}[ln(M+qT) - ln(u) + 1])$$

Sub back in $u = M + qT - qt$

$$=-\frac{V\,e(M+qT-qt)}{q}[ln(M+qT) - ln(M+qT-qt) + 1])$$

$$=- \frac{Ve(M+qT-qt)}{q}[ln(\frac{M+qT)}{M+qT-qt}) + 1])$$

$$= \frac{-(Ve)(M+qT-qt)[ln(\frac{M+qT)}{M+qT-qt})+1]}{q}$$

If we perform unit analysis we get a mass times a constant divided by mass per second. This gives us an answer in seconds. If we multiply that by speed (Ve) we get in meters total distance traveled.

Now that we solved the integral, we can now calculate the definite integral.

$$x = \int_0^T Ve\, ln(\frac{M+qT}{M+qT-qt})\, dt$$

$$x = \frac{-(Ve)(M+qT-qT)[ln(\frac{M+qT)}{M+qT-qT})+1]}{q} - \frac{-(Ve)(M+qT-q0)[ln(\frac{M+qT)}{M+qT-q0})+1]}{q}$$

$$x = -\frac{Ve}{q}[(M(ln(\frac{M+qT}{M}) + 1) - (M + qT)(ln(\frac{M+qT}{M+qT}) + 1)\ ]$$

$$x = -\frac{Ve}{q}[(M\, ln(\frac{M+qT}{M}) + M) - (M + qT)]$$

$$x = -\frac{Ve}{q}[(M\, ln(\frac{M+qT}{M}) - qT)]$$

$$x = \frac{(Ve)(qT - M\, ln(\frac{M+qT)}{M}))}{q}$$

This is the equation for the distance travelled by a propulsion system given a time and is what is used to make the simulation.

Part B - The Simulation Code in Python

This is the core of the code that is reiterated for each journey time to calculate distance traveled

```
Fuel = flowRate * arrayTime
# For if the propulsion system exceeds it's max operation time
if (BreakDownTime < arrayTime):
    Time = BreakDownTime
    Fuel = flowRate * Time
    ExtraTime = arrayTime - BreakDownTime
```

```
# Uses formula derived to calculate distance travelled
Distance = (ExhaustVelocity * (Fuel - originalMass * np.log((originalMass + Fuel) / originalMass))) / flowRate

# Calculate extra distance travelled if the system keeps its final velocity
if (BreakDownTime < arrayTime):
    FinalVelocity = (ExhaustVelocity * (np.log((originalMass + Fuel) / originalMass))
    ExtraDistance = FinalVelocity*ExtraTime

# Calculates distance including the extra distance
TotalDistance = Distance + ExtraDistance
```

Part C - Sample Calculation With Error Propagation

Calculation for MR-107U 300N (68 lbf) as it has a high level of error:

$T = 100s$ $M_{payload} = 10kg$ $I_{sp} = 229s$ $q = 0.098kg/s$ $M_{system} = 1.38kg$	Absolute Uncertainty: I_{sp} : 1 q : 0.001 M_{system} : 0.01 Relative Uncertainty: I_{sp} : 0.43668% q : 1.0204% M_{system} : 0.72464%
$M = 1.38 + 10$ $M = 11.4$	M Relative Uncertainty: $= 0.72464\%$ M Absolute Uncertainty: $= 0.082609$
$Ve = (229)(9.81)$ $Ve = 2246.5$	Ve Relative Uncertainty: $= 0.43668\%$
Given: $\frac{(Ve)(qT - M ln(\frac{M+qT}{M}))}{q}$ $qT = 0.098 * 100$ $= 9.8\ kg$ $M + qT = 11.4 + 9.8$ $= 21.2\ kg$ $ln(\frac{M+qT}{M}) = ln(\frac{21.2}{10})$	qT Absolute Uncertainty: $= 1.0204\% * 9.8$ $= 0.01$ $M + qT$ Absolute Uncertainty: $= 0.082609 + 0.01$ $= 0.092609$ $M + qT$ Relative Uncertainty: $= 0.092609/21.2$

$= ln(2.12)$ $= 0.75142$ $Mln(\frac{M+qT}{M}) = 11.4 * 0.75142$ $= 8.5662\ kg$ $qT - Mln(\frac{M+qT}{M}) = 9.8 - 8.5662$ $= 1.2338\ kg$ $(Ve)(qT - Mln(\frac{M+qT}{M})) = 2246.5 * 1.2338$ $= 2771.7\ (kgm)/s$ $\frac{(Ve)(qT-Mln(\frac{M+qT}{M}))}{q} = \frac{2771.7}{0.098}$ $= 28283\ m$	$= 0.43683\%$ $Mln(\frac{M+qT}{M})$ Relative Uncertainty: $= 0.43683 + 0.72464 + 0.72464$ $= 1.8861\%$ $Mln(\frac{M+qT}{M})$ Absolute Uncertainty: $= 1.8861\% * 8.5662$ $= 0.16157$ $qT - Mln(\frac{M+qT}{M})$ Absolute Uncertainty: $= 0.16157 * 0.01$ $= 0.17157$ $qT - Mln(\frac{M+qT}{M})$ Relative Uncertainty: $= 0.17157/1.2338$ $= 13.905\%$ $\frac{(Ve)(qT-Mln(\frac{M+qT}{M}))}{q}$ Relative Uncertainty: $= 13.905 + 0.43668 + 1.0204$ $= 15.362\%$

A 15.362% uncertainty isn't great by any means, however, the error is small enough so that any general trend revealed in the data should be accurate.