Measures, Dimensions and Analytic Capacity

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March 19, 2024

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Contents

Chapter 1

Measures and Integrations

We shall work in a metric space X with a metric d, although most of the measure theory presented here goes through in more general settings.

The closed and open balls with centre $x \in X$ and radius $r, 0 < r < \infty$, are denoted by

$$B(x,r) = \{y \in X : d(x,y) \leq r\}$$
$$U(x,r) = \{y \in X : d(x,y) < r\}$$

In $\mathbb{R}^n$ we also set

$$B(r) = B(0,r), U(r) = U(0,r), S(x,r) = \partial B(x,r) \text{ and } S(r) = S(0,r).$$

The diameter of a non-empty subset A of X is

$$d(A) = \sup\{d(x,y) : x, y \in A\}.$$

We agree $d(\varnothing) = 0$. If $x \in X$ and A and B are non-empty subsets of X, the distance from x to A and the distance between A and B are, respectively,

$$d(x,A) = \inf\{d(x,y) : y \in A\},$$
$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$$

For $\varepsilon > 0$ the closed ε-neighbourhood of A is

$$A(\varepsilon) = \{x \in X : d(x,A) \leq \varepsilon\}.$$

1.1 Measures

A measure for us will be a non-negative, monotonic, subadditive set function vanishing for the empty set.

Definition 1.1.1. A set function $\mu : \{A : A \subset X\} \to [0,\infty] = \{t : 0 \leq t \leq \infty\}$ is called a **measure** if
(1) $\mu(\varnothing) = 0$,

(2) $\mu(A) \leq \mu(B) \quad$ whenever $A \subset B \subset X$,

(3) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) \quad$ whenever $A_1, A_2, \cdots \subset X$.

Usually in measure theory a measure means a non-negative countably additive set function defined on some σ-algebra of subsets of X, which need not be the whole power set $\{A : A \subset X\}$. However, considering measures in the sense of Definition 1.1.1 is a convenience rather than a restriction. That is, if ν is a countably additive non-negative set function on a σ-algebra $\mathcal{A}$ of subsets of X, it can be extended to a measure ν^* on X (in the sense of Definition 1.1.1) by

$$\nu^*(A) = \inf\{\nu(B) : A \subset B \in \mathcal{A}\}.$$

Exercise 1.1.2. Show that ν^* defined above is a measure agreeing with ν on $\mathcal{A}$, and, moreover, that ν^* is Borel regular if $\mathcal{A}$ is contained in the family of Borel sets.

On the other hand, a measure μ gives a countably additive set function when restricted to the σ-algebra of μ measurable sets.

Definition 1.1.3. A set $A \subset X$ is μ **measurable** if

$$\mu(E) = \mu(E \cap A) + \mu(E \backslash A) \quad \text{for all } E \subset X.$$

We collect the well-known basic properties of measurable sets in the following theorem.

Theorem 1.1.4. Let μ be a measure on X and let $\mathcal{M}$ be the family of all μ measurable subsets of X.

(1) $\mathcal{M}$ is a σ-algebra, that is,

(i) $\varnothing \in \mathcal{M}$ and $X \in \mathcal{M}$,

(ii) if $A \in \mathcal{M}$, then $X \backslash A \in \mathcal{M}$,

(iii) if $A_1, A_2, \cdots \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

(2) If $\mu(A) = 0$, then $A \in \mathcal{M}$.

(3) If $A_1, A_2, \cdots \in \mathcal{M}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right)$$

(4) If $A_1, A_2, \cdots \in \mathcal{M}$, then

(i) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu\left(A_i\right)$ provided $A_1 \subset A_2 \subset \ldots$,

(ii) $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu\left(A_i\right)$ provided $A_1 \supset A_2 \supset \ldots$ and $\mu\left(A_1\right) < \infty$.

It is also good to remember that the first statement of (4) holds without the measurability assumption if μ is **regular**, that is, for every $A \subset X$ there is a μ measurable set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

Recall that the family of **Borel sets** in X is the smallest σ-algebra containing the open (or equivalently closed) subsets of X. We shall often consider measures with some of the following properties.

Definition 1.1.5. Let μ be a measure on X.

(1) μ is **locally finite** if for every $x \in X$ there is $r > 0$ such that

$$\mu(B(x, r)) < \infty.$$

(2) μ is a **Borel measure** if all Borel sets are μ measurable, i.e., $\mathrm{Bor}(X) \subseteq \mathcal{M}$.

(3) μ is **Borel regular** if it is a Borel measure and if for every $A \subset X$ there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

(4) μ is a **Radon measure** if it is a Borel measure and

(i) $\mu(K) < \infty$ for compact sets $K \subset X$,

(ii) $\mu(V) = \sup\{\mu(K) : K \subset V$ is compact $\}$ for open sets $V \subset X$,

(iii) $\mu(A) = \inf\{\mu(V) : A \subset V, V$ is open $\}$ for $A \subset X$.

We shall give a few simples examples. Many others will be encountered later on.

Example 1.1.6.
(1) The **Lebesgue measure** $\mathcal{L}^n$ on $\mathbb{R}^n$ is a Radon measure.

(2) The **Dirac measure** δ_a at a point $a \in X$ is defined by $\delta_a(A) = 1$, if $a \in A, \delta_a(A) = 0$, if $a \notin A$ (that is, $\delta_a(A) = \chi_A(a)$). It is a Radon measure on any metric space X.

(3) The **counting measure** n on X is defined by letting $n(A)$ be the number of elements in A, possibly ∞. It is Borel regular on any metric space X, but it is a Radon measure only if every compact subset of X is finite, that is, X is discrete.

In general, Radon measures are always Borel regular as a rather immediate consequence of the definition. The converse is not true as the above example (3) shows. Clearly in $\mathbb{R}^n$ the local finiteness means that compact sets have finite measure.

A measure μ on X is called a **metric outer measure** if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $A, B \subseteq X$ are **positively separated**, i.e., $d(A, B) > 0$.

Theorem 1.1.7 (Carathéodory's criterion)**.** Suppose μ is a measure in X, then it is a Borel measures if and only if it is a metric outer measure.

Proof. Theorem 1.5 of [1] (but note that measure here is called outer measure in [1]) shows that every metric outer measure is a Borel measure.

Conversely, suppose μ is a Borel measure. Let $A, B \subseteq X$ with $m := d(A, B) > 0$. Then define

$$U = \bigcup_{x \in A} B\left(x, \frac{m}{2}\right).$$

Clearly, $A \subseteq U$, $B \cap U = \varnothing$, and U is open and thus $U \in \operatorname{Bor}(X) \subseteq \mathcal{M}_\mu$. Therefore, by measurability of U, we have

$$\mu(A \cup B) = \mu((A \cup B) \cap U) + \mu((A \cup B) \cap U^c) = \mu(A) + \mu(B).$$

□

Given a measure μ and a subset A of X we can form a new measure by restricting μ to A.

Definition 1.1.8. The **restriction of a measure** μ to a set $A \subset X, \mu \llcorner A$, is defined by

$$(\mu \llcorner A)(B) = \mu(A \cap B) \quad \text{for } B \subset X.$$

It is clear that $\mu L A$ is a measure. Many of the relations between μ and $\mu \llcorner A$ are easy to derive. For example,

Theorem 1.1.9.

(1) Every μ measurable set is also $\mu \llcorner A$ measurable.

(2) If μ is Borel regular and A is μ measurable with $\mu(A) < \infty$, then $\mu \llcorner A$ is Borel regular.

Proof. The first statement is readily checked from the definitions. Note that A can be quite arbitrary there. We prove the second part.

Let B be a Borel set with $A \subset B$ and $\mu(A) = \mu(B)$. Then $\mu(B \backslash A) = 0$. Given $C \subset X$ let D be a Borel set with $B \cap C \subset D$ and $\mu(B \cap C) = \mu(D)$. Then $C \subset D \cup (X \backslash B) = E$, say, and

$$\begin{aligned}(\mu \llcorner A)(E) &\leq \mu(B \cap E) = \mu(B \cap D) \leq \mu(D) \\ &= \mu(B \cap C) = \mu(A \cap C) = (\mu \llcorner A)(C).\end{aligned}$$

Thus $(\mu \llcorner A)(E) = (\mu \llcorner A)(C)$, and so $\mu \llcorner A$ is Borel regular. □

The following approximation theorem will be extremely useful, see [2] Theorem 2.2.2 for instance.

Theorem 1.1.10. Let μ be a Borel regular measure on X, A a μ measurable set, and $\varepsilon > 0$.

(1) If $\mu(A) < \infty$, there is a closed set $C \subset A$ such that $\mu(A \backslash C) < \varepsilon$.

(2) If there are open sets $V_1, V_2, \ldots$ such that $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu\left(V_i\right) < \infty$ for all i, then there is an open set V such that $A \subset V$ and $\mu(V \backslash A) < \varepsilon$.

Note. The result holds for any Borel measure provided A is a Borel set. In $\mathbb{R}^{\boldsymbol{n}}$ it follows immediately that the set C in (1) can be taken to be compact. This holds of course in any σ**-compact space** X, where every closed set is a countable union of compact sets.

Corollary 1.1.11. A measure μ on $\mathbb{R}^n$ is a Radon measure if and only if it is locally finite and Borel regular.

The proof is left as an exercise.

In what follows we shall mainly work with Borel regular measures or Radon measures for convenience. But often they could quite easily be replaced by Borel measures or locally finite Borel measures, for example with the help of Exercise 1.1.2.

We shall often encounter measures μ which are carried by a proper subset F of X, that is, $\mu(X \backslash F) = 0$. It is not hard to see that in the case where μ is a Borel measure and X is separable, there exists a unique smallest closed set with this property.

Definition 1.1.12. If μ is a Borel measure on a separable metric space X, the **support** of μ, spt μ, is the smallest closed set F such that $\mu(X \backslash F) = 0$. In other words,

$$\begin{aligned}\operatorname{spt} \mu &= X \backslash \bigcup \{V : V \text{ open}, \mu(V) = 0\} \\ &= X \backslash \{x : \exists r > 0 \text{ such that } \mu(B(x, r)) = 0\}.\end{aligned}$$

Example 1.1.13.
(1) Let f be a non-negative continuous function on $\mathbb{R}^n$. Define a measure μ_f by

$$\mu_f(A) = \int_A f d\mathcal{L}^n$$

for $\mathcal{L}^n$ measurable sets A. Then the support of μ_f agrees with that of f :

$$\operatorname{spt} \mu_f = \operatorname{spt} f = \mathrm{Cl}\{x : f(x) \neq 0\},$$

where Cl refers to closure.

(2) Let $Q = \{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers, and

$$\mu = \sum_{i=1}^{\infty} 2^{-i} \delta_{q_i},$$

where δ_{q_i} is the Dirac measure at q_i. Then μ is a finite Radon measure on $\mathbb{R}$ with spt $\mu = \mathbb{R}$. Nevertheless, μ is carried by the countable set Q in the sense that $\mu(\mathbb{R} \backslash Q) = 0$.

1.2 Integration

The integral

$$\int_A f d\mu = \int_A f(x) d\mu x$$

with respect to a measure μ over a set A of a function f is defined in the usual way, as well as the μ measurability and integrability of f. When the domain of the integration A is the whole space X, we often omit it using the notation

$$\int f d\mu = \int_X f d\mu$$

In $\mathbb{R}^n$ we abbreviate the Lebesgue integral

$$\int_A f(x) dx = \int_A f(x) d\mathcal{L}^n x.$$

The integral $\int f d\mu$ is defined for any non-negative μ measurable function on X. Even when $f : X \to [0, \infty]$ is not μ measurable we can define the lower and upper integrals by

$$\int_* f d\mu = \sup_{\varphi} \int \varphi d\mu \text{ and } \int^* f d\mu = \inf_{\psi} \int \psi d\mu,$$

where φ and ψ run through the μ measurable functions $X \to [0, \infty]$ such that $\varphi \leq f \leq \psi$.

The μ integrability of $f : X \to \overline{\mathbb{R}}$ (in the last two chapters of $f : X \to \mathbb{C}$) means that f is μ measurable and $\int |f| d\mu < \infty$. As usual, for $1 \leq p < \infty$ the space of μ measurable functions $f : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) with $\int |f|^p d\mu < \infty$ is denoted by $L^p(\mu)$, and $L^\infty(\mu)$ is the space of functions which are essentially bounded with respect to μ.

A function $f : A \to \overline{\mathbb{R}}$ is a **Borel function** if A is a Borel set and the sets $\{x \in A : f(x) < c\}$ are Borel sets for all $c \in \mathbb{R}$. A mapping $f : X \to Y$ between metric spaces X and Y is a Borel mapping if $f^{-1}(U)$ is a Borel set for every open set $U \subset Y$.

We shall mention here only a few of the well-known properties of the integral. The following form of Fubini's theorem will be frequently used.

Theorem 1.2.1. Suppose that X and Y are separable metric spaces, and μ and ν are locally finite Borel measures on X and Y, respectively. If f is a non-negative Borel function on $X \times Y$, then

$$\iint f(x, y) d\mu x d\nu y = \iint f(x, y) d\nu y d\mu x$$

In particular, when f is the characteristic function of a Borel set A,

$$\int \mu(\{x : (x, y) \in A\}) d\nu y = \int \nu(\{y : (x, y) \in A\}) d\mu x$$

There are many more general forms of Fubini's theorem, see [2] §2.6. To formulate an extension, define the product measure $\mu \times \nu$ by

$$(\mu \times \nu)(C) = \inf \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i)$$

where the infimum is taken over all sequences $A_1, A_2, \ldots$ of μ measurable sets and $B_1, B_2, \ldots$ of ν measurable sets such that

$$C \subset \bigcup_{i=1}^{\infty} A_i \times B_i$$

Here $0 \cdot \infty = \infty \cdot 0 = 0$. It is easy to see that $\mu \times \nu$ is a measure over $X \times Y$. Moreover, if both μ and ν are either Borel, Borel regular, or Radon measures, $\mu \times \nu$ has the same property. The statement of Theorem 1.14 is valid for all $\mu \times \nu$ measurable functions f which are non-negative or $\mu \times \nu$ integrable (i.e. $\int |f| d(\mu \times \nu) < \infty$), and the iterated integrals agree with the $\mu \times \nu$ integral:

$$\int f d(\mu \times \nu) = \iint f(x, y) d\mu x d\nu y$$

The assumption that X and Y are separable, which of course implies that $X \times Y$ is separable, guarantees that the Borel sets and functions are $\mu \times \nu$ measurable.

As an application of Fubini's theorem we record the following useful formula.

Theorem 1.2.2. Let μ be a Borel measure and f a non-negative Borel function on a separable metric space X. Then

$$\int f d\mu = \int_0^\infty \mu(\{x \in X : f(x) \geq t\}) dt.$$

Proof. Let $A = \{(x, t) : f(x) \geq t\}$. Then

$$\begin{aligned}
&\int_0^\infty \mu(\{x \in X : f(x) \geq t\}) dt = \int_0^\infty \mu(\{x : (x, t) \in A\}) dt \\
&= \int \mathcal{L}^1(\{t \in [0, \infty) : (x, t) \in A\}) d\mu x = \int \mathcal{L}^1([0, f(x)]) d\mu x \\
&= \int f(x) d\mu x
\end{aligned}$$

□

Another way to look at the Radon measures and integrals with respect to them is to consider them as linear functionals on $C_0(X)$, the space of compactly supported continuous real-valued functions on X. That is, if μ is a Radon measure on X, we can associate to it the linear functional

$$L : C_0(X) \to \mathbb{R}, \quad Lf = \int f d\mu.$$

This is obviously **positive** in the sense that

$$Lf \geq 0 \text{ for } f \geq 0.$$

In the case where X is locally compact the converse also holds, see e.g. [7] 2.14.

Theorem 1.2.3 (Riesz representation theorem). Let X be a locally compact metric space and $L : C_0(X) \to \mathbb{R}$ a positive linear functional. Then there is a unique Radon measure μ such that

$$Lf = \int f d\mu \quad \text{for } f \in C_0(X)$$

1.3 Image measures

We can map measures from one metric space X to another, Y.

Definition 1.3.1. The image of a measure μ under a mapping $f : X \to Y$ is defined by

$$f_\sharp \mu(A) = \mu\left(f^{-1} A\right) \quad \text{for } A \subset Y.$$

It is apparent that $f_\sharp\mu$ is a measure on Y. It is also immediate that A is $f_\sharp\mu$ measurable whenever $f^{-1}(A)$ is μ measurable. Hence if μ is a Borel measure and f a Borel function, $f_\sharp\mu$ is a Borel measure. The following simple criterion on the Radonness of $f_\sharp\mu$ will suffice for us. For more general results, see e.g. [2] 2.2.17.

Theorem 1.3.2. Let X and Y be separable metric spaces. If $f : X \to Y$ is continuous and μ is a Radon measure on X with compact support, then $f_\sharp\mu$ is a Radon measure. Moreover, spt $f_\sharp\mu = f(\operatorname{spt}\mu)$.

Proof. Replacing X by the subspace spt μ we may assume X is compact. Statement (i) of Definition 1.1.5(4) is trivial, as μ, and hence also $f_\sharp\mu$, are finite measures. We leave (ii) as an exercise and prove only (iii). $\square$

Let $A \subset Y$ and $\varepsilon > 0$. Since μ is a Radon measure there is an open set $U \subset X$ such that $f^{-1}A \subset U$ and $\mu(U) \le \mu\left(f^{-1}A\right) + \varepsilon$. Set $V = Y \backslash f(X \backslash U)$. Then V is open, as X is compact, $A \subset V$ and

$$\begin{aligned} f_\sharp\mu(V) &= \mu\left(f^{-1}(Y \backslash f(X \backslash U))\right) \\ &= \mu\left(X \backslash f^{-1}(f(X \backslash U))\right) \le \mu(U) \\ &\le \mu\left(f^{-1}A\right) + \varepsilon = f_\sharp\mu(A) + \varepsilon. \end{aligned}$$

This yields (iii). We leave the last statement on supports also as an exercise.

The following theorem can be proven via a rather straightforward approximation by simple functions. It can also be easily deduced from Theorem 1.2.2.

Theorem 1.3.3. Suppose $f : X \to Y$ is a Borel mapping, μ is a Borel measure on X, and g is a non-negative Borel function on Y. Then

$$\int g d f_\sharp\mu = \int (g \circ f) d\mu$$

When Y is locally compact, all this could also be done in the reverse order: letting

$$Lg = \int (g \circ f) d\mu \quad \text{for } g \in C_0(Y)$$

we obtain a linear functional on $C_0(Y)$ which by the Riesz representation theorem 1.16 corresponds to a Radon measure $f_\sharp\mu$.

It is clear that pulling back measures is not nearly as natural as pushing them forward: the formula $\mu(A) = \nu(fA)$ does not usually define a Borel measure even for very nice measures ν if f fails to be injective. Still it is often possible to find such pull-backs abstractly. The following proof can be found in Schwartz's *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures* $\S I.5$.

Theorem 1.3.4. Let X and Y be compact metric spaces and $f : X \to Y$ a continuous surjection. For any Radon measure ν on Y there exists a Radon measure μ on X such that $f_\sharp\mu = \nu$.

1.4 Weak convergence

Next we consider a convergence of measures.

Definition 1.4.1. Let $\mu, \mu_1, \mu_2, \ldots$ be Radon measures on a metric space X. We say that the sequence (μ_i) converges weakly to μ,

$$\mu_i \xrightarrow{\mathbf{w}} \mu,$$

if

$$\lim_{i \to \infty} \int \varphi d\mu_i = \int \varphi d\mu \quad \text{for all } \varphi \in C_0(X)$$

Example 1.4.2.
(1) In $\mathbb{R}, \delta_i \xrightarrow{\mathbf{w}} 0$ as $i \to \infty$.

(2) Let

$$\mu_k = \frac{1}{k}\sum_{i=1}^{k} \delta_{i/k}.$$

Then $\mu_k \xrightarrow{\mathbf{w}} \mathcal{L}^1 \llcorner [0,1]$.

The weak convergence is useful because a very general compactness theorem holds. We prove it only for $\mathbb{R}^n$.

Theorem 1.4.3. If $\mu_1, \mu_2, \ldots$ are Radon measures on $\mathbb{R}^n$ with

$$\sup\left\{\mu_i(K) : i = 1, 2, \ldots\right\} < \infty$$

for all compact sets $K \subset \mathbb{R}^n$, then there is a weakly convergent subsequence of (μ_i).

Proof. The space $C_0\left(\mathbb{R}^n\right)$ is separable under the norm

$$\|\varphi\| = \max\left\{|\varphi(x)| : x \in \mathbb{R}^n\right\},$$

whence it has a countable dense subset D. For example, choosing functions $\varphi_i \in C_0\left(\mathbb{R}^n\right), i = 1, 2, \ldots$, with $\varphi_i = 1$ on $B(i)$, one can by the Weierstrass approximation theorem take for D the set of all products $\varphi_i P$ where $i = 1, 2, \ldots$ and P runs through polynomials with rational coefficients. For each $\varphi \in D$ the bounded sequence $\left(\int \varphi d\mu_i\right)$ of real numbers has a convergent sub-sequence. Using the diagonal method we can thus extract a sub-sequence (μ_{i_k}) such that the limit

$$L\varphi = \lim_{k\to\infty} \int \varphi d\mu_{i_k}$$

exists and is finite for all $\varphi \in D$. The denseness of D then implies that this actually holds for all $\varphi \in C_0\left(\mathbb{R}^n\right)$, and the Riesz representation theorem 1.16 gives the limit measure. □

As Example 1.4.2 shows $\mu_i \xrightarrow{w} \mu$ need not imply that $\mu_i(A) \to \mu(A)$ even when $A = \mathbb{R}^n$. However, the following semicontinuity properties hold.

Theorem 1.4.4. Let $\mu_1, \mu_2, \ldots$ be Radon measures on a locally compact metric space. If $\mu_i \xrightarrow{w} \mu, K \subset X$ is compact and $G \subset X$ is open, then

$$\begin{aligned}\mu(K) &\geq \limsup_{i\to\infty} \mu_i(K),\\ \mu(G) &\leq \liminf_{i\to\infty} \mu_i(G).\end{aligned}$$

Proof.
(1) Let $\varepsilon > 0$. By property (4) (iii) of Definition 1.1.5 there is an open set V such that $K \subset V$ and $\mu(V) \leq \mu(K) + \varepsilon$. By Urysohn's lemma, see e.g. Rudin $[1, 2.12]$, there is $\varphi \in C_0(X)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on K and $\operatorname{spt}\varphi \subset V$. Thus

$$\begin{aligned}\mu(K) \geq \mu(V) - \varepsilon &\geq \int \varphi d\mu - \varepsilon\\ &= \lim_{i\to\infty} \int \varphi d\mu_i - \varepsilon \geq \limsup_{i\to\infty} \mu_i(K) - \varepsilon,\end{aligned}$$

and (1) follows.

(2) is proven similarly through approximation of G with compact sets from inside. □

1.5 Approximate identities

We shall now show that arbitrary Radon measures in $\mathbb{R}^n$ can be approximated weakly by smooth functions, that is, by measures of the form $A \mapsto \int_A g d\mathcal{L}^n$ where $g \in C^\infty(\mathbb{R}^n)$, the space of infinitely differentiable real-valued functions on $\mathbb{R}^n$. First we define convolutions.

Definition 1.5.1. Let f and g be real-valued functions on $\mathbb{R}^n$ and μ a Radon measure on $\mathbb{R}^n$. The **convolutions** $f * g$ of f and g, and $f * \mu$ of f and μ, are defined by

$$f * g(x) = \int f(x-y)g(y)dy,$$
$$f * \mu(x) = \int f(x-y)d\mu y,$$

provided the integral exists.

We now consider an **approximate identity** $\{\psi_\varepsilon\}_{\varepsilon>0}$. By this we mean that each ψ_ε is a non-negative continuous function on $\mathbb{R}^n$ such that

$$\operatorname{spt}\psi_\varepsilon \subset B(\varepsilon) \text{ and } \int \psi_\varepsilon d\mathcal{L}^n = 1.$$

Any continuous function $\psi : \mathbb{R}^n \to [0,\infty)$ with spt $\psi \subset B(1)$ and $\int \psi d\mathcal{L}^n = 1$ obviously gives such an approximate identity by

$$\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon).$$

In particular we may take

$$\begin{array}{ll} \psi_\varepsilon(x) = c(\varepsilon)e^{-1/\left(\varepsilon^2-|x|^2\right)} & \text{for } |x| < \varepsilon, \\ \psi_\varepsilon(x) = 0 & \text{for } |x| \geq \varepsilon, \end{array}$$

where $c(\varepsilon)$ is determined by $\int \psi_\varepsilon d\mathcal{L}^n = 1$, to get an approximate identity consisting of C^∞ functions. It is shown in many text-books that for any such approximate identity consisting of C^∞ functions the functions $\psi_\varepsilon * f$, where $f \in L^p(\mathbb{R}^n)$, are also C^∞ and they converge to f in L^p. We now study $\psi_\varepsilon * \mu$ in the same spirit.

Theorem 1.5.2. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be an approximate identity and μ a Radon measure on $\mathbb{R}^n$. Then the functions $\psi_\varepsilon * \mu$ are infinitely differentiable and they converge weakly to μ as $\varepsilon \downarrow 0$, that is,

$$\lim_{\varepsilon\downarrow 0}\int \varphi\left(\psi_\varepsilon * \mu\right) d\mathcal{L}^n = \int \varphi d\mu$$

for all $\varphi \in C_0(\mathbb{R}^n)$. If $\mu(\mathbb{R}^n) < \infty$, this holds for all uniformly continuous bounded functions $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Proof. By studying the difference quotients and using induction one can verify in a straightforward manner that for all $i_j \in \{1,\ldots,n\}, j = 1,\ldots,k$,

$$\partial_{i_1}\ldots\partial_{i_k}\left(\psi_\varepsilon * \mu\right) = \left(\partial_{i_1}\ldots\partial_{i_k}\psi_\varepsilon\right) * \mu$$

where ∂_i means the partial derivative with respect to the i-th coordinate. It follows that $\psi_\varepsilon * \mu$ has partial derivatives of all orders.

To prove the second statement we use Fubini's theorem, change of variable and the facts that spt $\psi_\varepsilon \subset B(\varepsilon)$

and $\int \psi_\varepsilon d\mathcal{L}^n = 1$ to compute

$$\begin{aligned}
&\int \varphi\left(\psi_\varepsilon * \mu\right) d\mathcal{L}^n - \int \varphi d\mu \\
&= \int \varphi(x) \int \psi_\varepsilon(x-y) d\mu y dx - \int \varphi(y) \int \psi_\varepsilon(x) dx d\mu y \\
&= \int \left[\int \varphi(x)\psi_\varepsilon(x-y) dx - \int \varphi(y)\psi_\varepsilon(x) dx\right] d\mu y \\
&= \iint_{B(\varepsilon)} [\varphi(x+y) - \varphi(y)]\psi_\varepsilon(x) dx d\mu y.
\end{aligned}$$

Since φ is uniformly continuous with compact support and $\int \psi_\varepsilon d\mathcal{L}^n = 1$, this goes to zero as $\varepsilon \downarrow 0$. The last statement follows also by the above proof. $\square$

We finish this chapter with some remarks on lower semicontinuous functions. We shall need this concept only for non-negative functions. One way to define them is to say that a non-negative function g on $\mathbb{R}^{\boldsymbol{n}}$ is lower semicontinuous if there are non-negative functions $\varphi_i \in C_0\left(\mathbb{R}^n\right)$, $i = 1, 2, \ldots$, such that $\varphi_1 \leq \varphi_2 \leq \ldots$ and $g = \lim_{i\to\infty} \varphi_i$. An equivalent definition is that the sets $\{x : g(x) > c\}$ are open for all $c \in \mathbb{R}$. Examples are characteristic functions of open sets and $x \mapsto |x|^p, p \in \mathbb{R}$ (with value ∞ at 0 if $p < 0$).

Chapter 2

Hausdorff Measure and Dimension

To get some motivations, see [6]. Mendelbrot also has many writings on the connection between fractals and the nature.

We will introduce Hausdorff measures and dimension for measuring the metric size of quite general sets. They will be one of the basic means for studying geometric properties of sets and expressing results that these studies lead to. Hausdorff measures also provide a fruitful source for getting examples to which several later results on general measures apply. The basic definitions and first results on Hausdorff measures and dimension are due to Carathéodory and Hausdorff. We shall start with a more general construction, called Carathéodory's construction.

2.1 Carathéodory's construction

Let X be a metric space, $\mathcal{F}$ a family of subsets of X and ζ a non-negative function on $\mathcal{F}$. We make the following two assumptions.

(1) For every $\delta>0$ there are $E_1, E_2, \cdots \in \mathcal{F}$ such that $X=\bigcup_{i=1}^{\infty} E_i$ and $d\left(E_i\right) \leq \delta$.

(2) For every $\delta>0$ there is $E \in \mathcal{F}$ such that $\zeta(E) \leq \delta$ and $d(E) \leq \delta$. For $0<\delta \leq \infty$ and $A \subset X$ we define

$$\psi_\delta(A)=\inf \left\{\sum_{i=1}^{\infty} \zeta\left(E_i\right): A \subset \bigcup_{i=1}^{\infty} E_i, d\left(E_i\right) \leq \delta, E_i \in \mathcal{F}\right\}$$

Assumption (1) was only introduced to guarantee that such coverings always exist. The role of (2) is to have $\psi_\delta(\varnothing)=0$ (we fix δ for ψ_δ but δ in (2) can be arbitrarily small). It also allows us to use coverings $\{E_i\}_{i \in I}$ with I finite or countable without changing the value of $\psi_\delta(A)$.

It is easy to see that ψ_δ is monotonic and subadditive so that it is a measure. Usually it is highly non-additive and not a Borel measure. See Exercise below.

Exercise 2.1.1. Let U be an open ball in $\mathbb{R}^n, n \geq 2$, with $d(U)=\delta$. Show that for $0 \leq s \leq 1, \mathcal{H}_\delta^s(U)=\mathcal{H}_\delta^s(\bar{U})=\overline{\mathcal{H}}_\delta^s(\partial U)$.

Evidently,

$$\psi_\delta(A) \leq \psi_{\varepsilon}(A) \quad \text{whenever } 0<\varepsilon<\delta \leq \infty.$$

Hence we can define $\psi=\psi(\mathcal{F}, \zeta)$ by

$$\psi(A)=\lim_{\delta \downarrow 0} \psi_\delta(A)=\sup_{\delta>0} \psi_\delta(A) \quad \text{for } A \subset X.$$

The measure-theoretic behaviour of ψ is much better than that of ψ_δ.

Theorem 2.1.2. (1) ψ is a Borel measure. (2) If the members of $\mathcal{F}$ are Borel sets, ψ is Borel regular.

Proof.
(1) The proof that ψ is a measure is straightforward and left to the reader. To show that ψ is a Borel measure, we verify the condition of Theorem 1.7. Let $A, B \subset X$ with $d(A,B) > 0$. Choose δ with $0 < \delta < d(A,B)/2$. If the sets $E_1, E_2, \ldots \in \mathcal{F}$ cover $A \cup B$ and satisfy $d(E_i) \leq \delta$, then none of them can meet both A and B. Hence

$$\begin{aligned}\sum_i \zeta(E_i) &\geq \sum_{A\cap E_i \neq \varnothing} \zeta(E_i) + \sum_{B\cap E_i \neq \varnothing} \zeta(E_i) \\ &\geq \psi_\delta(A) + \psi_\delta(B).\end{aligned}$$

Taking the infimum over all such coverings we have $\psi_\delta(A \cup B) \geq \psi_\delta(A) + \psi_\delta(B)$. But the opposite inequality holds also as ψ_δ is a measure, and so $\psi_\delta(A\cup B) = \psi_\delta(A) + \psi_\delta(B)$. Letting $\delta \downarrow 0$, we obtain $\psi(A\cup B) = \psi(A) + \psi(B)$ as required. (2) If $A \subset X$, choose for every $i = 1, 2, \ldots$ sets $E_{i,1}, E_{i,2}, \cdots \in \mathcal{F}$ such that

$$\begin{aligned}&A \subset \bigcup_j E_{i,j}, d(E_{i,j}) \leq 1/i \text{ and} \\ &\sum_j \zeta(E_{i,j}) \leq \psi_{1/i}(A) + 1/i.\end{aligned}$$

Then $B = \bigcap_i \bigcup_j E_{i,j}$ is a Borel set such that $A \subset B$ and $\psi(A) = \psi(B)$. Thus ψ is Borel regular. ☐

2.2 Hausdorff measures

Let X be separable, $0 \leq s < \infty$, and choose

$$\begin{aligned}&\mathcal{F} = \{E : E \subset X\}, \\ &\zeta(E) = \zeta_s(E) = d(E)^s\end{aligned}$$

with the interpretations $0^0 = 1$ and $d(\varnothing)^s = 0$. The resulting measure ψ is called the **s-dimensional Hausdorff measure** and denoted by $\mathcal{H}^s$. So

$$\mathcal{H}^s(A) = \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(A)$$

where

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_i d(E_i)^s : A \subset \bigcup_i E_i, d(E_i) \leq \delta\right\}$$

The integral dimensional Hausdorff measures play a special role. Let us start from $s = 0$. It is easy to see that $\mathcal{H}^0$ is the counting measure:

$$\mathcal{H}^0(A) = \operatorname{card} A = \text{ the number of points in } A.$$

Next, for $s = 1, \mathcal{H}^1$ also has a concrete interpretation as a generalized length measure. In particular, for a rectifiable curve Γ in $\mathbb{R}^n, \mathcal{H}^1(\Gamma)$ can be shown to equal the length of Γ. (If the length is defined in some other reasonable way; of course, $\mathcal{H}^1(\Gamma)$ can also be taken as the definition of the length of Γ.) For unrectifiable curves $\Gamma, \mathcal{H}^1(\Gamma) = \infty$. More generally, if m is an integer, $1 \leq m < n$, and M is a sufficiently regular m-dimensional surface in $\mathbb{R}^n$ (for example, C^1 submanifold), then the restriction $\mathcal{H}^m LM$ gives a constant multiple of the surface measure on M. This follows for example from the area formula, see [2] 3.2.3. For $s = n$ in $\mathbb{R}^n$,

$$\mathcal{H}^n = 2^n \alpha(n)^{-1} \mathcal{L}^n, \tag{1}$$

whence

$$\mathcal{H}^n(B(x,r)) = (2r)^n \quad \text{for } x \in \mathbb{R}^n, 0 < r < \infty. \tag{2}$$

Often one normalizes Hausdorff measures (as in [2]) so that $\mathcal{H}^n$ will equal $\mathcal{L}^n$, but since we shall not usually be interested in the exact values of Hausdorff measures, we use the simpler definition. The proof of the equality (1) is rather complicated and based on the so-called isodiametric inequality

$$\mathcal{L}^n(A) \leq 2^{-n}\alpha(n)d(A)^n \quad \text{for } A \subset \mathbb{R}^n$$

see [2] 2.10.33. But to see that $\mathcal{H}^n = c\mathcal{L}^n$ with some positive and finite constant is much easier. All we have to do is to verify that both $\mathcal{H}^n$ and $\mathcal{L}^n$ are **uniformly distributed measures** (i.e., Borel regular measures μ on a metric space X such that $0 < \mu(B(x,r)) = \mu(B(y,r)) < \infty$ for $x, y \in X, 0 < r < \infty$) and use the following theorem (That $\mathcal{H}^n$ is Borel regular will be noted in Corollary 2.2.3.) We shall use the formulas (1) and (2) many times, but almost always the weaker information that they hold with some unspecified constants would suffice.

Theorem 2.2.1. Let μ and ν be uniformly distributed Borel regular measures on a separable metric space X. Then there is a constant c such that $\mu = c\nu$.

Proof. Let g and h be the functions giving the μ and ν measures of the balls of radius r :

$$g(r) = \mu(B(x,r)), h(r) = \nu(B(x,r)) \text{ for } x \in X, 0 < r < \infty.$$

Let U be a non-empty bounded open subset of X. Clearly the limit $\lim_{r\downarrow 0}(\nu(U \cap B(x,r))/h(r))$ exists and equals 1 for $x \in U$. Hence by Fatou's lemma and Fubini's theorem

$$\begin{aligned}
\mu(U) &= \int_U \lim_{r\downarrow 0} h(r)^{-1}\nu(U \cap B(x,r))d\mu x \\
&\leq \liminf_{r\downarrow 0} h(r)^{-1} \int \nu(U \cap B(x,r))d\mu x \\
&= \liminf_{r\downarrow 0} h(r)^{-1} \int_U \mu(B(y,r))d\nu y \\
&= \left(\liminf_{r\downarrow 0} g(r)/h(r)\right)\nu(U)
\end{aligned}$$

Interchanging μ and ν we obtain similarly

$$\nu(U) \leq \left(\liminf_{r\downarrow 0} \frac{h(r)}{g(r)}\right)\mu(U).$$

It follows that the limit $c = \lim_{r\downarrow 0}(g(r)/h(r))$ exists and $\mu(U) = c\nu(U)$ for every open set U. That $\mu = c\nu$ then follows by Theorem 1.1.10(2) and the Borel regularity of μ and ν. ☐

For any $s > n, \mathcal{H}^s$ in $\mathbb{R}^n$ is uninteresting since $\mathcal{H}^s(\mathbb{R}^n) = 0$ (see Theorem 2.2.5).

Hausdorff measures behave nicely under translations and dilations in $\mathbb{R}^n$: for $A \subset \mathbb{R}^n, a \in \mathbb{R}^n, 0 < t < \infty$,

$$\begin{aligned}
&\mathcal{H}^s(A + a) = \mathcal{H}^s(A) && \text{where } A + a = \{x + a : x \in A\}, \\
&\mathcal{H}^s(tA) = t^s\mathcal{H}^s(A) && \text{where } tA = \{tx : x \in A\}.
\end{aligned}$$

These are readily verified from the definition. In particular,

$$\mathcal{H}^s(B(x,r)) = c(s,n)r^s \quad \text{for } x \in \mathbb{R}^n, 0 < r < \infty.$$

But, as follows from Theorem 2.2.5, $c(s,n)$ is positive and finite only when $s=n$; for $s>n, c(s,n)=0$, for $s<n, c(s,n)=\infty$. Thus only $\mathcal{H}^n$ is uniformly distributed in $\mathbb{R}^n$. To prove that $0<c(n,n)<\infty$, one can use any of the standard proofs for the fact that the unit ball (or cube) has positive and finite Lebesgue measure.

We shall now derive some simple properties of Hausdorff measures in a general separable metric space X.

Theorem 2.2.2. Let $0 \leq s<n$ and $\zeta(E)=d(E)^s$ for $E \subset X$. If

(1) $\mathcal{F}=\{F \subset X: F$ is closed $\}$ or

(2) $\mathcal{F}=\{U \subset X: U$ is open $\}$ or

(3) $X=\mathbb{R}^n$ and $\mathcal{F}=\{K \subset \mathbb{R}^n: K$ is convex $\}$, then $\psi(\mathcal{F}, \zeta)=\mathcal{H}^s$.

The first and last statement follow from the fact that the closure and convex hull of a set E have the same diameter as E. The second statement holds since for any $\varepsilon>0,\{x: d(x, E)<\varepsilon\}$ is open and has diameter at most $d(E)+2 \varepsilon$. We leave the details as an exercise. Recalling Theorem 2.1.2(2) we have

Corollary 2.2.3. $\mathcal{H}^s$ is Borel regular.

Notice that usually $\mathcal{H}^s$ is not a Radon measure since it need not be locally finite. For example, if $s<n$ every non-empty open set in $\mathbb{R}^n$ has non- σ-finite $\mathcal{H}^s$ measure. But taking any $\mathcal{H}^s$ measurable set A in $\mathbb{R}^n$ with $\mathcal{H}^s(A)<\infty$, the restriction $\mathcal{H}^s \llcorner A$ is a Radon measure by Theorem 1.1.9(2) and Corollary 1.1.11.

Often one is only interested in knowing which sets have $\mathcal{H}^s$ measure zero. For this it is enough to use any of the approximating measures $\mathcal{H}_\delta^s$, for example $\mathcal{H}_{\infty}^s$; in fact we don't really need any measure at all.

Lemma 2.2.4. Let $A \subset X, 0 \leq s<\infty$ and $0<\delta \leq \infty$. Then the following conditions are equivalent:

(1) $\mathcal{H}^s(A)=0$.

(2) $\mathcal{H}_\delta^s(A)=0$.

(3) $\forall \varepsilon>0 \exists E_1, E_2, \ldots \subset X$ such that

$$A \subset \bigcup_i E_i \text{ and } \sum_i d\left(E_i\right)^s<\varepsilon .$$

The proof is left as an exercise.

We shall now compare measures $\mathcal{H}^s$ with each other.

Theorem 2.2.5. For $0 \leq s<t<\infty$ and $A \subset X$,

(1) $\mathcal{H}^s(A)<\infty$ implies $\mathcal{H}^t(A)=0$,

(2) $\mathcal{H}^t(A)>0$ implies $\mathcal{H}^s(A)=\infty$.

Proof. To prove (1), let $A \subset \bigcup_i E_i$ with $d\left(E_i\right) \leq \delta$ and $\sum_i d\left(E_i\right)^s \leq \mathcal{H}_\delta^s(A)+1$. Then

$$\mathcal{H}_\delta^t(A) \leq \sum_i d\left(E_i\right)^t \leq \delta^{t-s} \sum_i d\left(E_i\right)^s \leq \delta^{t-s}\left(\mathcal{H}_\delta^s(A)+1\right),$$

which gives (1) as $\delta \downarrow 0$.

(2) is really only a restatement of (1). But we have emphasized this simple theorem by doublestating it, because it leads to one of the most fundamental concepts of this book, the Hausdorff dimension. □

2.3 Hausdorff dimension

According to Theorem 2.2.5, we may define

Definition 2.3.1. The **Hausdorff dimension** of a set $A \subset X$ is

$$\begin{aligned} \dim A &= \sup \{s : \mathcal{H}^s(A) > 0\} = \sup \{s : \mathcal{H}^s(A) = \infty\} \\ &= \inf \left\{t : \mathcal{H}^t(A) < \infty\right\} = \inf \left\{t : \mathcal{H}^t(A) = 0\right\} \end{aligned}$$

(Sometimes some of these sets may be empty, but we leave the obvious interpretations to the reader.)

Clearly the Hausdorff dimension has the natural properties of monotonicity and stability with respect to countable unions:

$$\begin{array}{ll} \dim A \leq \dim B & \text{for } A \subset B \subset X \\ \dim \bigcup_{i=1}^{\infty} A_i = \sup_i \dim A_i & \text{for } A_i \subset X, i = 1, 2, \ldots \end{array}$$

To state the definition in other words, $\operatorname{din} A$ is the unique number (it may be ∞ in some metric spaces) for which

$$\begin{array}{ll} s < \dim A & \text{implies } \mathcal{H}^s(A) = \infty, \\ t > \dim A & \text{implies } \mathcal{H}^t(A) = 0. \end{array}$$

At the borderline case $s = \dim A$ we cannot have any general nontrivial information about the value $\mathcal{H}^s(A)$; all three cases $\mathcal{H}^s(A) = 0$, $0 < \mathcal{H}^s(A) < \infty, \mathcal{H}^s(A) = \infty$ are possible. But if for some given A we can find s such that $0 < \mathcal{H}^s(A) < \infty$, then s must equal $\dim A$. Since $\mathbb{R}^n$ has infinite but σ-finite $\mathcal{H}^n$ measure, it follows that

$$\dim \mathbb{R}^n = n.$$

Hence $0 \leq \dim A \leq n$ for all $A \subset \mathbb{R}^n$. We shall soon see that for all $s \in [0, n], \dim A = s$ for some subset A of $\mathbb{R}^n$.

To find the Hausdorff dimension or to estimate the Hausdorff measures of a given set, it is always possible and often advantageous to use coverings with some simpler sets like balls or, in $\mathbb{R}^n$, dyadic cubes. This is easy to see and we shall return to it in the next chapter.

Recalling Lemma 2.2.4(3) we observe that we do not really need Hausdorff measures to define Hausdorff dimension.

Remark 2.3.2. Although the Hausdorff dimension measures the metric size of any subset of our metric space, the values of the Hausdorff measures often do not give much extra information. This is so since there may be no value s for which the set has positive and finite $\mathcal{H}^s$ measure. But often replacing $\zeta_s(E) = d(E)^s$ by some other function of the diameter, one can find measures measuring the given set in a more delicate manner.

Let $h : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $h(0) = 0$. We take again

$$\mathcal{F} = \{E : E \subset X\} \quad \text{and} \quad \zeta(E) = h(d(E))$$

(with $d(\varnothing) = 0$). Then the corresponding measure $\psi(\mathcal{F}, \zeta) = \Lambda_h$ is called the Hausdorff h measure. Of course, $\Lambda_h = \mathcal{H}^s$ when $h(t) = t^s$.

There are many cases where some other h than t^s is more useful and natural. Among the most important are sets related to Brownian motion in $\mathbb{R}^n$. For example, the trajectories of the Brownian motion in $\mathbb{R}^n$ have positive and σ-finite Λ_h measure almost surely with (for small t)

$$\begin{array}{ll} h(t) = t^2 \log \log t^{-1} & \text{in the case } n \geq 3, \text{ and} \\ h(t) = t^2 \log t^{-1} \log \log \log t^{-1} & \text{in the case } n = 2. \end{array}$$

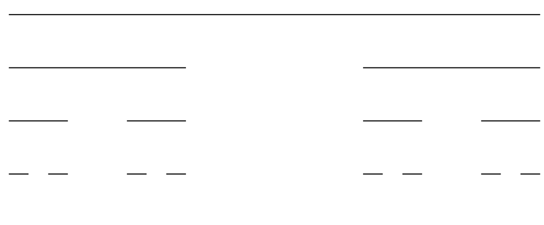
We have now introduced measures for measuring the size of very general sets. It is time to look at some examples with which Hausdorff measures are convenient and useful. We begin with the most classical.

2.4 Cantor sets

2.4.1 Cantor sets in $\mathbb{R}^1$

Let $0 < \lambda < 1/2$. Denote $I_{0,1} = [0,1]$, and let $I_{1,1}$ and $I_{1,2}$ be the intervals $[0, \lambda]$ and $[1-\lambda, 1]$, respectively. We continue this process of selecting two subintervals of each already given interval. If we have defined intervals $I_{k-1,1}, \ldots, I_{k-1,2^{k-1}}$, we define $I_{k,1}, \ldots, I_{k,2^k}$ by deleting from the middle of each $I_{k-1,j}$ an interval of length $(1-2\lambda) d\left(I_{k-1,j}\right) = (1-2\lambda)\lambda^{k-1}$. All the intervals $I_{k,j}$ thus obtained have length λ^k. We define a kind of limit set of this construction by

$$C(\lambda) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j}$$



Then $C(\lambda)$ is an uncountable compact set without interior points and with zero Lebesgue measure. The most commonly used case is the Cantor middle-third set $C(1/3)$, see the figure.

We shall now study the Hausdorff measures and dimension of $C(\lambda)$. As usual, it is much simpler to find upper bounds than lower bounds for the Hausdorff measures. This is due to the definition: a judiciously chosen covering will give an upper estimate, but a lower estimate requires finding an infimum over arbitrary coverings. For every $k = 1, 2, \ldots, C(\lambda) \subset \bigcup_j I_{k,j}$, and so

$$\mathcal{H}^s_{\lambda^k}(C(\lambda)) \leq \sum_{j=1}^{2^k} d\left(I_{k,j}\right)^s = 2^k \lambda^{ks} = \left(2\lambda^s\right)^k.$$

In order for this upper bound to be useful, it should stay bounded as $k \to \infty$. The smallest value of s for which this happens is given by $2\lambda^s = 1$, that is,

$$s = \log 2 / \log(1/\lambda).$$

For this choice we have

$$\mathcal{H}^s(C(\lambda)) = \lim_{k \to \infty} \mathcal{H}^s_{\lambda^k}(C(\lambda)) \leq 1.$$

Thus $\dim C(\lambda) \leq s$. Next we shall show

$$\mathcal{H}^s(C(\lambda)) \geq 1/4 \tag{2.1}$$

which will give

$$\dim C(\lambda) = \log 2 / \log(1/\lambda).$$

To prove (2.1), it suffices to show that

$$\sum_j d\left(I_j\right)^s \geq 1/4 \tag{2.2}$$

whenever open intervals $I_1, I_2, \ldots$ cover $C(\lambda)$. Since $C(\lambda)$ is compact, finitely many I_j 's cover $C(\lambda)$ so that we may assume that there were only $I_1, \ldots, I_n$ to begin with. Since $C(\lambda)$ has no interior points, we can, making I_j slightly larger if necessary, assume that the end-points of each I_j are outside $C(\lambda)$. Then there is $\delta > 0$ such that the distance from all these end-points to $C(\lambda)$ is at least δ. Choosing k so large that

$\delta > \lambda^k = d\left(I_{k,i}\right)$, it follows that every interval $I_{k,i}$ is contained in some I_j. We shall now show that for any open interval I and any fixed ℓ,

$$\sum_{I_{\ell,i} \subset I} d\left(I_{\ell,i}\right)^s \leq 4d(I)^s \tag{2.3}$$

This gives (2.2), since

$$4\sum_j d\left(I_j\right)^s \geq \sum_j \sum_{I_{k,i} \subset I_j} d\left(I_{k,i}\right)^s \geq \sum_{i=1}^{2^k} d\left(I_{k,i}\right)^s = 1$$

To verify (2.3), suppose there are some intervals $I_{\ell,i}$ inside I and let n be the smallest integer for which I contains some $I_{n,i}$. Then $n \leq \ell$. Let $I_{n,j_1}, \ldots, I_{n,j_p}$ be all the n-th generation intervals which meet I. Then $p \leq 4$, since otherwise I would contain some $I_{n-1,i}$. Thus

$$4d(I)^s \geq \sum_{m=1}^{p} d\left(I_{n,j_m}\right)^s = \sum_{m=1}^{p} \sum_{I_{\ell,i} \subset I_{n,j_m}} d\left(I_{\ell,i}\right)^s \geq \sum_{I_{\ell,i} \subset I} d\left(I_{\ell,i}\right)^s .$$

Actually it is not hard to show that (2.2) can be improved to $\sum d\left(I_j\right)^s \geq 1$, which gives the precise value $\mathcal{H}^s(C(\lambda)) = 1$, see [1] Theorem 1.14. However, the above argument can be generalized to many situations where the exact value of the measure is practically impossible to compute.

Note that $\dim C(\lambda)$ measures the sizes of the Cantor sets $C(\lambda)$ in a natural way: when λ increases, the sizes of the deleted holes decrease and the sets $C(\lambda)$ become larger, and also $\dim C(\lambda)$ increases. Notice also that when λ runs from 0 to $1/2, \dim C(\lambda)$ takes all the values between 0 and 1.

2.4.2 Generalized Cantor sets in $\mathbb{R}^1$

Instead of keeping constant the ratios of the lengths of the intervals in every two successive stages of the construction, we can vary it in the following way. Let $T = (\lambda_i)$ be a sequence of numbers in the open interval $(0, 1/2)$. We construct a set $C(T)$ otherwise as above, but take the intervals $I_{k,j}$ to have length $\lambda_k d\left(I_{k-1,i}\right)$. Then for every k we get 2^k intervals $I_{k,j}$ of length

$$s_k = \lambda_1 \cdots \lambda_k .$$

Let $h : [0, \infty) \to [0, \infty)$ be a continuous increasing function such that

$$h\left(s_k\right) = 2^{-k}. \tag{2.4}$$

Then by the above argument

$$1/4 \leq \Lambda_h(C(T)) \leq 1.$$

Conversely, we can start from any continuous increasing function $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$ and $h(2r) < 2h(r)$ for $0 < r < \infty$, and inductively select $\lambda_1, \lambda_2, \ldots$ such that (2.4) is valid. Thus for any such h there is a compact set $C_h \subset \mathbb{R}^1$ such that $0 < \Lambda_h\left(C_h\right) < \infty$. Choosing $h(r) = r^s \log(1/r)$ for small values of r, where $0 < s \leq 1$, we have $\dim C_h = s$ and $\mathcal{H}^s\left(C_h\right) = 0$. On the other hand, choosing $h(r) = r^s / \log(1/r)$ for small r, where $0 \leq s < 1, C_h$ has non- σ-finite $\mathcal{H}^s$ measure and dimension s. In particular, the extreme cases $s = 1$ and $s = 0$ give a set of the dimension 1 with zero Lebesgue measure and an uncountable set of dimension zero.

2.4.3 Cantor sets in $\mathbb{R}^n$

We can use the same ideas as above to construct Cantor-type sets in $\mathbb{R}^n$ having a given Hausdorff dimension s. We can start from a ball, cube, rectangle etc. and at each stage of the construction select similar geometric figures inside the previous ones. One can then often use the following proposition.

Suppose for $k = 1, 2, \ldots$ we have compact sets $E_{i_1,\ldots,i_k}, i_j = 1, \ldots, m_j$, such that

$$\begin{aligned}
&E_{i_1,\ldots,i_k,i_{k+1}} \subset E_{i_1,\ldots,i_k}, \\
&d_k = \max_{i_1 \ldots i_k} d\left(E_{i_1,\ldots,i_k}\right) \rightarrow 0 \text{ as } k \rightarrow \infty \\
&\sum_{j=1}^{m_{k+1}} d\left(E_{i_1,\ldots,i_k,j}\right)^s = d\left(E_{i_1,\ldots,i_k}\right)^s \\
&\sum_{B \cap E_{i_1,\ldots,i_k} \neq \emptyset} d\left(E_{i_1,\ldots,i_k}\right)^s \leq c d(B)^s
\end{aligned}$$

for any ball B with $d(B) \geq d_k$, where c is a positive constant. Then

$$0 < \mathcal{H}^s\left(\bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k} E_{i_1,\ldots,i_k}\right) < \infty.$$

We leave the proof as an exercise. Notice that the above conditions are satisfied for example in the following situation: select all the sets $E_{i_1,\ldots,i_k}$ to be balls of radius r_k. Choose the balls $E_{i_1,\ldots,i_k,j}$ fairly uniformly distributed inside $E_{i_1,\ldots,\ell_k}$ and so that $m_{k+1} r_{k+1}^s = r_k^s$. If r_k tends to zero very rapidly (or equivalently, m_k grows very rapidly), the diameter $2r_{k+1}$ of $d\left(E_{i_1,\ldots,i_k,i}\right)$ is much smaller for large k than the distance from $E_{i_1,\ldots,i_k,i}$ to the nearest neighbour $E_{i_1,\ldots,i_k,j}$; this distance is of magnitude $r_k^{1-s/n} r_{k+1}^{s/n}$. Hence sets with large Hausdorff dimension (even equal to n) can look extremely porous at arbitrarily small scales, cf. Figure 2.1.

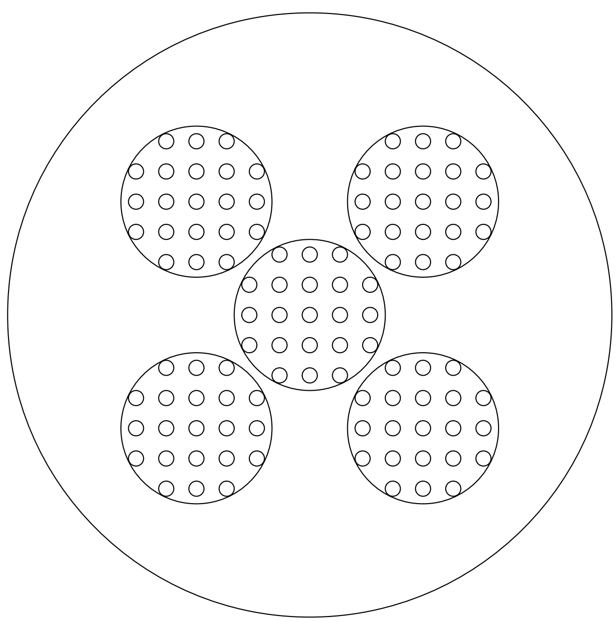


Figure 2.1: A very porous Cantor set.

2.4.4 Self-similar sets

Roughly speaking, a subset of $\mathbb{R}^n$ is selfsimilar if it can be split into parts which are geometrically similar to the whole set. The Cantor sets $C(\lambda)$ in subsection 2.4.1 are simple examples. If the parts $C(\lambda)\cap[0,\lambda]$ and $C(\lambda)\cap[1-\lambda,1]$ are magnified in ratio $1/\lambda$ we get (a translate of) the original Cantor set. We shall briefly describe parts of the more general elegant theory of Hutchinson. For more details see [1]. The self-similarity of $C(\lambda)$ above can be expressed by the formula

$$C(\lambda) = S_1(C(\lambda)) \cup S_2(C(\lambda))$$

where the similarity maps $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ are defined by $S_1(x) = \lambda x$, $S_2(x) = \lambda x + 1 - \lambda$. Another standard example is von Koch's "snowflake" curve, see Figure 4.3. In the construction one replaces at each stage a segment of length d by four segments of length $d/3$ as in the figure. The von Koch curve K is a limit of the polygonal curves thus obtained. It is a non-rectifiable curve having tangents at none of its points. It can also be presented in terms of similarity maps S_i in the form

$$K = S_1K \cup S_2K \cup S_3K \cup S_4K.$$

Here $S_1, \ldots, S_4$ are the orientation-preserving similarities of ratios $1/3$ of the plane which map the first initial segment onto the four next ones.

We now state the basic ideas of Hutchinson's general theory. A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a **similitude** if there is $r, 0 < r < 1$, such that

$$|S(x) - S(y)| = r|x - y| \quad \text{for } x, y \in \mathbb{R}^n.$$

Similitudes are exactly those maps S which can be written as

$$S(x) = rg(x) + z, \quad x \in \mathbb{R}^n,$$

for some $g \in O(n), z \in \mathbb{R}^n$ and $0 < r < 1$. Suppose $\mathcal{S} = \{S_1, \ldots, S_N\}$, $N \geq 2$, is a finite sequence of similitudes with contraction ratios $r_1, \ldots, r_N$. We say that a non-empty compact set K is **invariant** under $\mathcal{S}$ if

$$K = \bigcup_{i=1}^{N} S_iK.$$

Then for any such $\mathcal{S}$ there exists a unique invariant compact set. A quick way to prove this is to use the fact that the family of all non-empty compact subsets of $\mathbb{R}^n$ is a complete metric space with the Hausdorff metric ρ,

$$\rho(E, F) = \max\{d(x, F), d(y, E) : x \in E, y \in F\},$$

see e.g. [2] 2.10.21. The map $\widetilde{S} : E \mapsto \bigcup_{i=1}^N S_iE$ is readily seen to be a contraction in the Hausdorff metric, whence it has a unique fixed point. By definition, this is the invariant set we wanted.

In addition, it follows by the simple general properties of contractions in complete metric spaces that however we choose an initial compact set $F \subset \mathbb{R}^n$, the iterations

$$\tilde{S}^m(F) = \tilde{S} \circ \cdots \circ \widetilde{S}(F) = \bigcup_{i_1=1}^{N} \cdots \bigcup_{i_m=1}^{N} S_{i_1} \circ \cdots \circ S_{i_m}(F)$$

will converge to K. Moreover, for any m the set K satisfies

$$K = \bigcup_{i_1=1}^{N} \cdots \bigcup_{i_m=1}^{N} S_{i_1} \circ \cdots \circ S_{i_m}(K).$$

Since

$$d\left(S_{i_1}\circ\cdots\circ S_{i_m}(K)\right)\leq\left(\max_{1\leq i\leq N} r_i\right)^m d(K)\to 0,\quad \text{as } m\to\infty,$$

an invariant set can be expressed as a union of arbitrarily small sets geometrically similar to itself. We define an invariant set under $\mathcal{S}$ to be **self-similar** if with $s=\dim K$,

$$\mathcal{H}^s\left(S_i(K)\cap S_j(K)\right)=0\quad \text{for } i\neq j.$$

This definition is rather awkward to use, but the following somewhat stronger separation condition, called the **open set condition**, is very convenient: There is a non-empty open set O such that

$$\bigcup_{i=1}^N S_i(O)\subset O \text{ and } S_i(O)\cap S_j(O)=\emptyset\quad \text{for } i\neq j.$$

This is satisfied if the different parts $S_i(K)$ are disjoint as for the classical Cantor sets. Then we can use as O the ε-neighbourhood $\{x: d(x,K)<\varepsilon\}$ for sufficiently small ε. The open set condition also holds in many other interesting cases. For example, in the case of the von Koch curve we can take for O the open triangle which is the interior of the convex hull of the polygonal line consisting of the first four line segments, see Figure 2.2.

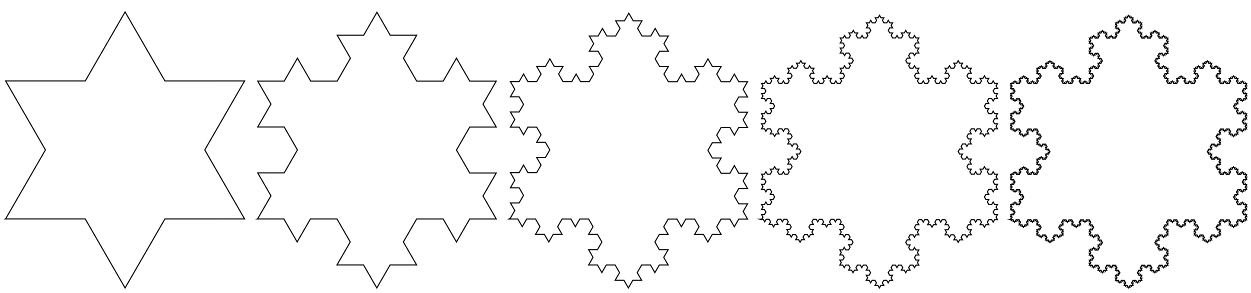


Figure 2.2: von Koch curve (as part of von Koch snowflake)

Under the open set condition the dimension of K is explicitly determined by the contraction ratios $r_1,\ldots,r_N$ of the similitudes S_i in $\mathcal{S}$:

Theorem 2.4.1. If $\mathcal{S}$ satisfies the open set condition, then the invariant set K is self-similar and $0<\mathcal{H}^s(K)<\infty$, whence $s=\dim K$, where s is the unique number for which

$$\sum_{i=1}^N r_i^s=1.$$

Moreover, there are positive and finite numbers a and b such that

$$ar^s\leq\mathcal{H}^s(K\cap B(x,r))\leq br^s\quad \text{for } x\in K, 0<r\leq 1.$$

For a proof see [1]. If in the above $r_1=\cdots=r_N=r$ we have $\dim K=\log N/\log(1/r)$ in accordance with what we previously proved about the Cantor sets $C(\lambda)$. For the von Koch curve K this gives $\dim K=\log 4/\log 3$.

Chapter 3

Other Dimensions and Measures

TO DO.

[4] chapter 5.

Note: this is related to random matrices theory (see e.g. High-dimensional probability)

Chapter 4

Rectifiability

TO DO.

Need to single out some theorems in [4] on rectifiability and possibly extend some discussions.

Chapter 5

Analytic Capacity

TO DO.

Classical and friendly:

[3] Chapter 8 and [8] Chapter 1.

More:

[4]

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