A Note on Algebraic and Geometric Characteristics of Archetypal Riemann Surfaces

Anthony Hong

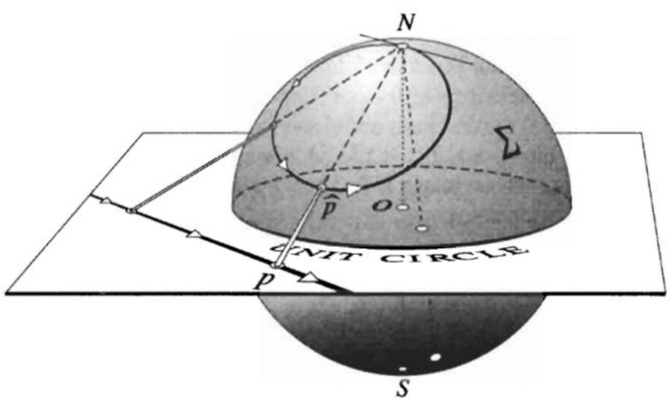
Content

Section 0. Preliminary Notions

A **Riemann surface** is a connected complex 1-manifold, a connected topological 2-manifold with an **analytical atlas**. Note that the difference between a smooth 2-manifold and a complex 1-manifold does not lie in its local homeomorphism to $\mathbb{R}^2$ or $\mathbb{C}$ but in the difference of their atlases, the former possessing smooth compatibility (each transition map is smooth) and the latter possessing holomorphic compatibility (each transition map is holomorphic). A **holomorphic map** $F: R_1 \to R_2$ is a map from one Riemann surface to another (this suffices for our discussions) where each coordinate representations of $\hat{F}: \mathbb{C} \supseteq U \to V \subseteq \mathbb{C}$ is holomorphic in the usual complex analysis sense. If the two Riemann surfaces have a **biholomorphic** map between them, i.e., there is a holomorphic map $F: R_1 \to R_2$ with a holomorphic inverse $F: R_2 \to R_1$ (this automatically requires F to be bijective), we say that R_1 and R_2 are **conformally equivalent**, and F is called a **conformal automorphism**. We denote the **automorphism group** as $Aut(R)$, which is the group of all conformal automorphisms of a Riemann surface R with composition as the binary operation.

A note about $\hat{\mathbb{C}}$. It is convenient to compactify $\mathbb{C} \approx \mathbb{R}^2$ by adding a point denoted as ∞, resulting in a one-point compactification $\hat{\mathbb{C}} = \mathbb{C}_\infty = \mathbb{C} \sqcup \{\infty\}$, known as **Riemann sphere**. $\hat{\mathbb{C}}$ can be considered as a complex 1-manifold due to the **stereographic projection**, a bijection $P: \mathbb{S}^2 \to \mathbb{C} \sqcup \{\infty\}$ defined below:

$$P(X = (X_1, X_2, X_3)) = \begin{cases} \infty, & X = N \coloneqq (0,0,1) \\ z = x + iy, & if\ line\ segment\ \overline{(x, y, 0) - N} \cap \mathbb{S}^2 = X \end{cases}$$



The above is a picture illustrating the stereographic projection of a point $X = \hat{p}$ to a point $p = (x, y, 0) \leftrightarrow z = x + iy$ on the plane ([1] pp. 140). Direct computation shows that for $X =$

$(X_1, X_2, X_3) \in \mathbb{S}^2$,

$$P(X) = \frac{X_1 + iX_2}{1 - X_3}, \qquad P^{-1}(z) = \left(\frac{2Re(z)}{|z|^2+1}, \frac{2IM(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

Riemannian metric g is a smooth symmetric covariant 2-tensor field on smooth manifold M that is positive definite at each point, i.e., $g \in \mathcal{T}^2(M) = \Gamma(T^2T^*M)$, meaning that for each $p \in M$, g assigns to it an element in $T^2(T_p^*M) = T_p^*M \otimes T_p^*M \cong L(T_pM, T_pM; \mathbb{R}) = \left\{T_pM \times T_pM \xrightarrow{linear} \mathbb{R}\right\}$. By [1] pp.328 we see that locally $g = g_{ij}dx^i \otimes dx^j$ with Einstein summation convention, so it's just a metric $\langle v, w\rangle_{g,p}$ dependent on point p on the manifold for each tangent space T_pM. Now for Riemann surfaces this can be much simpler. For each point $z \in \Omega$, g assigns a metric for the vector space $T_z\Omega \cong T_z\mathbb{R}^2 \cong \mathbb{R}^2 \cong \mathbb{C}$:

$$\rho(z)^2|dz|^2 = \rho(z)^2(dx^2 + dy^2)$$

Where $\mathrm{f}(z)$ is a smooth positive function on U and now $(g_{ij})_{2\times 2} = \rho(z)I_2$. This is called a **conformal metric**.

$$g_z(v, w) = \rho(z)^2|dz|^2(v, w) = \rho(z)^2(dx^2 + dy^2)(v, w)$$

$$= \rho(z)^2(v_xw_x + v_yw_y) = \rho(z)^2(v \cdot w) \xRightarrow{\rho(z)>0} g$$

induces $\|v\|_z = \rho(z)|v|$ where $|\cdot|$ is just the Euclidean metric and is also denoted as $|dz|$ because that's when $\rho(z) = 1$. We shall also use ρ refer to the metric instead of the smooth function in definition. Examples relevant to this report includes Poincare's model on $\triangle$, where $\|v\|_z = \frac{|dz|}{1-|z|^2}$ and Poincare's model on $\mathbb{H}$, where $\|v\|_z = \frac{|\mathrm{d}z|}{\mathrm{y}}$, $y = Im(z)$.

Distance between $z_1, z_2 \in$ domain Ω is defined as

$$d_\rho(z_1, z_2) = \inf_{\gamma:[a,b]\xrightarrow{\mathcal{C}^1}\Omega \text{ } from \text{ } z_1 to \text{ } z_2} \left\{l_\rho(\gamma) := \int_a^b \langle\dot\gamma(t), \dot\gamma(t)\rangle_{g,z} dt = \int_\gamma \langle\dot\gamma(t), \dot\gamma(t)\rangle_{g,z} |dz|\right\}$$

In fact, the curve γ with the shortest length $l_\rho(\gamma)$ is called **geodesic** or the "straight lines" in the corresponding model. For example, Poincare's model on $\triangle$ has (see [6] pp 47)

$$d_\rho(z_1, z_2) = \frac{1}{2}\ln\left(\frac{1 + \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|}{1 - \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|}\right)$$

where a "straight line" is represented as an arc of a circle whose ends are perpendicular to the disk's boundary (left figure of the below).

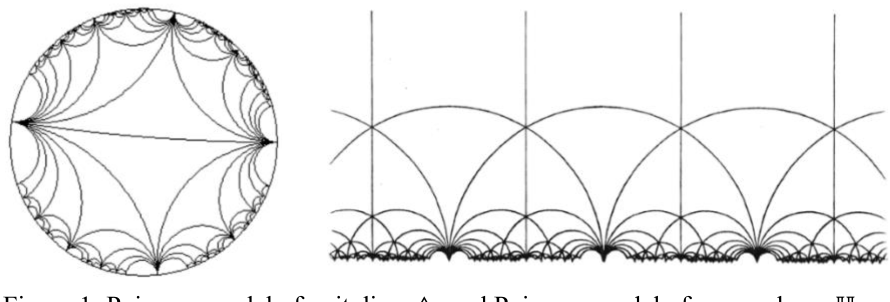


Figure 1: Poincare model of unit disc $\triangle$ and Poincare model of upper plane $\mathbb{H}$

Poincare's model on $\mathbb{H}$ has[1]

$$d_\rho(z_1, z_2) = \frac{1}{2}\ln\left(\frac{|z_2 - z_1| + |z_2 - \bar{z}_2|}{2\sqrt{Im(z_1)Im(z_2)}}\right)$$

The "straight lines" are circular arcs perpendicular to the x-axis (half-circles whose centers are on the x-axis) and straight vertical rays perpendicular to the x-axis (right figure of the above).

Lastly, we introduce the concept of isometry, following [6]. The **pullback metric** (borrowed from Riemann geometry) arises from a map $f: \Omega_1 \to \Omega_2$ and is given by

$$f^*\rho(z)|dz| = \rho\big(f(z)\big) \cdot \left|\frac{\partial f}{\partial z}\right|$$

where $\partial/\partial z$ (and $\partial/\partial \bar{z}$) is the **Wirtinger derivatives**:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right);\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

And it can be shown that if f is holomorphic then $f'(z) = \frac{\partial f}{\partial z}$.

Suppose $f: (\Omega_1, \rho_1(z)|dz|) \xrightarrow{holo,\, onto} (\Omega_2, \rho_2(z)|dz|)$. If $f^*\rho_2(z) = \rho_1(z)$ for all $z \in \Omega_1$, then f is called an **isometry**. It can be shown that an isometry preserves distance and a distance-preserving holomorphic map is an isometry i.e., $d_{\rho_1}(z_1, z_2) = d_{\rho_2}(f(z_1), f(z_2))$.

[1] https://en.wikipedia.org/wiki/Poincar%C3%A9_half-plane_model

Section 1. Uniformization Theorem

Theorem 1.1. [Uniformization Theorem]

The simply connected Riemann surface R_1 *is conformally equivalent to either* $\hat{\mathbb{C}}$, $\mathbb{C}$, *or* $\triangle$; *and an arbitrary Riemann surface* R_2 *is conformally equivalent to* R_1/Γ *where* $R_1 = \hat{\mathbb{C}}$, $\mathbb{C}$, *or* $\triangle$, *and* Γ *is a discrete subgroup of* $Aut(R_1)$. □

Corollary 1.2.

The universal covering space of any Riemann surface is conformally equivalent to either $\hat{\mathbb{C}}$, $\mathbb{C}$, *or* $\triangle$. □

Every Riemann surface R as a connected topological 2-manifold with additional structure has a universal covering space by the theorem of existence of universal covering E (see [2] pp. 298 Theorem 11.43). By [2] pp.302 pb 11-1 (b) we see E is also a topological 2-manifold and can be attached with a unique analytic atlas to become a complex 1-manifold and thus a simply connected Riemann surface by a naïve application of discussion in [4] pp. 542. The uniformization theorem then leads us to classify Riemann surfaces into three types by their universal covering spaces: elliptic, parabolic, and hyperbolic, respectively. It further follows that every Riemann surface admits a Riemannian metric of constant curvature, where the curvature can be taken to be 1 (elliptic), 0 (parabolic) and -1 (hyperbolic).

Remark 1.3.

There is first a remark about the distinction of the three types to make the theorem indeed a classification: $\mathbb{S}^2 \approx \hat{\mathbb{C}}$ is the only compact one among the three and can then not be conformally equivalent to the other two. $\triangle$ cannot be the holomorphic image of $\mathbb{C}$ by Liouville's Theorem (for it were we would have a bounded non-constant entire function).

There is a higher dimensional generalization in terms of the genus.

Theorem 1.4. [Uniformization Theorem for Compact Riemann Surfaces] ([5] Thm 4.4.1)

Let R_1 *be a compact Riemann surface of genus* p. *Then there exists a conformal diffeomorphism*

$$f: R_1 \to R_2$$

where R_2

i. *a compact Riemann surface of the form* $\mathbb{H}/\Gamma$ *where* Γ *is the discrete subgroup of* $PSL(2,\mathbb{R})$ *if* $p \geq 2$;

ii. *a compact Riemann surface* $\mathbb{C}/M$ *where* M *is the discrete subgroup of* $\mathbb{C}$ *(see section 2.7 of [5]) if* $p = 1$;

iii. *The Riemann sphere* $\mathbb{S}^2$ *if* $p = 0$. □

Corollary 1.5.

The universal covering space of a compact Riemann surface is conformally equivalent to either $\hat{\mathbb{C}}$, $\mathbb{C}$, *or* $\triangle$. □

Thus corollary 1.5 is also a corollary of corollary 1.2. There is also a non-analytic counterpart of corollary 1.5 (drop Riemann and change conformal equivalence to homeomorphism) below, also observing that $\hat{\mathbb{C}} \approx \mathbb{S}^2$, $\mathbb{C} \approx \mathbb{R}^2$, and $\mathbb{B}^2 = \triangle$.

Theorem 1.6. ([2] Thm 12.29)

Let M *be a compact surface. Then its universal covering space is homeomorphic to either*

i. $\mathbb{S}^2$ *if* $M \approx \mathbb{S}^2$ *or* $\mathbb{RP}^2$

ii. $\mathbb{R}^2$ *if* $M \approx \mathbb{T}^2$ *or* $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx \mathbb{K}$

iii. $\mathbb{B}^2$ *if* M *is any other surface*. □

Section 2. Automorphism Groups

To compute the automorphism groups of $\hat{\mathbb{C}}$, $\mathbb{C}$, and $\triangle$, we consider the **Mobius transformation**, or **linear fractional transformation**:

$$\phi(z)=\frac{az+b}{cz+d}$$

where $a,b,c,d\in\mathbb{C}$ and $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=ad-bc\neq 0$ ($\phi(z)$ is constant if $ad=bc$). We may extend $\phi(z)$ to the whole Riemann sphere by letting

$$c\neq 0:\phi\left(-\frac{d}{c}\right)=\infty;\ \phi(\infty)=\frac{a}{c}$$

$$c=0:\phi(\infty)=\infty$$

In this way, $\phi(z)$ is a holomorphic self-map of complex 1-manifold $\hat{\mathbb{C}}$. The inverse of ϕ is given by

$$\varphi(z)=\frac{dz-b}{-cz+a}$$

which is a Mobius transformation and is thus a holomorphic map too (Besides, if $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=ad-bc=1$ we have $\det\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}=ad-bc=1$). Thus, ϕ is a conformal automorphism of $\hat{\mathbb{C}}$. However, it is not trivially true that every automorphism is of this type. Check https://www.math3ma.com/blog/automorphisms-of-the-riemann-sphere to see a proof of this fact, established as the corollary of the proposition that A function f is meromorphic (holomorphic functions with isolated singularities that are only poles) in $\hat{\mathbb{C}}$ if and only if it is a rational map. In summary,

$$Aut(\hat{\mathbb{C}})=\left\{\phi:\hat{\mathbb{C}}\to\hat{\mathbb{C}}\,\middle|\,a,b,c,d\in\mathbb{C},\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=ad-bc\neq 0\right\}$$

$$\leftrightarrow\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in M(2,\mathbb{C})\,\middle|\,det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=ad-bc\neq 0\right\}=GL(2,\mathbb{C})$$

by an easily verified homomorphism

$$h:\begin{pmatrix} a & b \\ c & d\end{pmatrix}\mapsto\phi(z)=\frac{az+b}{cz+d}$$

Then the kernel and Image of h are

$$Ker(h)=h^{-1}\left(\left\{\phi(z)=\frac{az+b}{cz+d}\ \middle|\ a=d\neq 0,c=b=0\right\}\right)=\{\lambda I_2:\lambda\in\mathbb{C}^*\}$$

$$Im(h)=Aut(\hat{\mathbb{C}})$$

By the first isomorphism theorem, we see

$$Aut(\hat{\mathbb{C}})=Im(h)\cong GL(2,\mathbb{C})/Ker(h)=GL(2,\mathbb{C})/\{\lambda I_2:\lambda\in\mathbb{C}^*\}=PSL(2,\mathbb{C})\qquad(1)$$

where **projective special linear group** $PSL(2,\mathbb{C})=SL(2,\mathbb{C})/\{I_2,-I_2\}$ is the quotient of the **special linear group** $SL(2,\mathbb{C})=\{M\in GL(2,\mathbb{C}):\det M=1\}$. A counterpart is the **general special linear group** $GSL(2,\mathbb{C})=GL(2,\mathbb{C})/\{\lambda I_2:\lambda\in\mathbb{C}^*\}$.

By https://www.math3ma.com/blog/automorphisms-of-the-complex-plane, we see

$$Aut(\mathbb{C})=\{f(z)=az+b:a,b\in\mathbb{C},a\neq 0\}$$

$$\cong\left\{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}:a,b\in\mathbb{C},a\neq 0\right\}\coloneqq P(2,\mathbb{C})\qquad(2)$$

where $P(2,\mathbb{C})$ is called a **parabolic group** of 2-by-2 complex matrices. Two subgroups of the parabolic subgroup are its **Levi component**

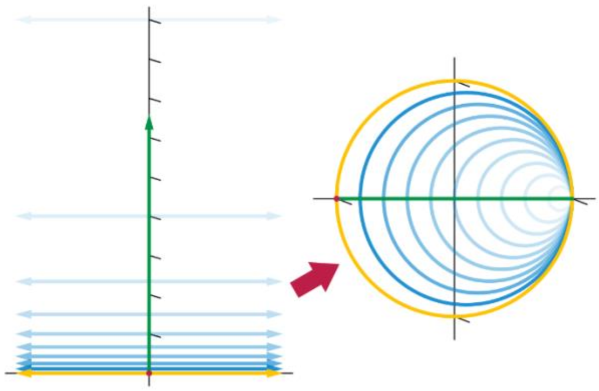
$$L = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^* \right\}$$

describing the dilations $f(z) = az$, or in fact a composition of rotation and dilation, and its **unipotent radical**

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{C} \right\}$$

describing the translation $f(z) = z + b$. The geometric terminologies corresponding to these linear transformations will come again when we talk about the isometries.

Lastly, note that the **Cayley transform** $C(z) = \frac{z-i}{z+i}$ is a biholomorphic map between $\mathbb{H}$ and $\triangle$, so $Aut(\mathbb{H}) \cong Aut(\triangle)$. The following picture by KSmrq from Wiki webpage well illustrates this transform:



Now we compute $Aut(\triangle)$. By [6] pp.14 Theorem 3 we see that

$$Aut(\triangle) = \left\{ \rho_\phi(z) \frac{z-a}{1-\bar{a}z} \middle| a \in \triangle, \phi \in (0,2\pi] \right\}$$

where $\rho_\phi(z) := e^{i\phi} \cdot z$ is the rotation by ϕ. To compute its matrix group representation $PSU(2,\mathbb{C})$, we first denote **special unitary subgroup** of $SL(2,\mathbb{C})$ as

$$SU(2,\mathbb{C}) = \{M \in SL(2,\mathbb{C}) : M^*M := (\bar{M})^T M = I, \det M = 1\}$$
$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

which can thus be viewed as a group structure on the compact $\mathbb{S}^3 \subseteq \mathbb{R}^4 \cong \mathbb{C} \times \mathbb{C}$. We also define the **projective special unitary subgroup** as $PSU(2,\mathbb{C}) = SU(2,\mathbb{C})/\{I_2, -I_2\}$.

Proposition 2.7.

$$\begin{aligned} Aut(\triangle) &= \left\{ \phi(z) = \frac{az+b}{-\bar{b}z+\bar{a}} \middle| a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} \\ &= h(SU(2,\mathbb{C})) \cong SU(2,\mathbb{C})/\ker h = PSU(2,\mathbb{C}) \end{aligned} \tag{3}$$

Proof:

Notice that the matrix of

$$\psi(z) = \frac{z-a}{1-\bar{a}z}$$

is

$$\begin{pmatrix} 1 & -a \\ -\bar{a} & 1 \end{pmatrix}$$

and noticing that $a \in \triangle \Rightarrow |a| < 1$, we normalize it as

$$A = \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} 1 & -a \\ -\bar{a} & 1 \end{pmatrix}$$

so that $A \in SU(2, \mathbb{C})$ as dilation does not change the Mobius transformation (the Mobius transformation of A is still $\psi(z)$). Then we observe that a rotation $\rho_\phi(z) \coloneqq e^{i\phi} \cdot z$ is just the multiplication by matrix

$$B = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}$$

and $B \in SU(2, \mathbb{C})$. $BA \in SU(2, \mathbb{C}) \Rightarrow (BA)_{11}, (BA)_{12} \in \mathbb{C}, |(BA)_{11}|^2 + |(BA)_{11}|^2 = 1$, so $\rho_\phi \circ \psi(z) = [h(B)] \circ [h(A)](z) = h(BA)(z)$ shows that

$$Aut(\triangle) = \left\{ \phi(z) = \frac{az+b}{-\bar{b}z+\bar{a}} \middle| a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} \quad \square$$

Computation of $Aut(\mathbb{H})$ can be seen in [7] or [8] or Lemma 2.3.5 of [5]:

$$Aut(\mathbb{H}) \cong PGL_2^+(\mathbb{R}) \cong PSL_2(\mathbb{R})$$

Section 3. A Summary of Algebraic and Geometric Properties

We summarize the above computations of automorphism groups with two geometric characteristic classes of the four objects, the isometry groups and curvature, in the following table:

Proposition 3.1.

Models	Automorphisms	Isometries	Curvature
$\left(\hat{\mathbb{C}}, \frac{2\lvert dz\rvert}{1+\lvert z\rvert^2}\right)$	$Aut(\hat{\mathbb{C}}) \cong PSL(2,\mathbb{C})$	$Isom(\hat{\mathbb{C}}) \cong O(3)$[2]	1[3]
$(\mathbb{C}, \lvert dz\rvert)$	$Aut(\mathbb{C}) \cong P(2,\mathbb{C})$	$Isom(\mathbb{C}) \cong E(2)$[4]	0
$\left(\triangle, \frac{\lvert dz\rvert}{1-\lvert z\rvert^2}\right)$	$Aut(\triangle) \cong PSU(2,\mathbb{C})$	$Isom^+(\triangle) \cong PSU(1,1)$[5]	-1[6]
$\left(\mathbb{H}, \frac{\lvert dz\rvert}{\mathrm{Im}(z)}\right)$	$Aut(\mathbb{H}) \cong PSL(2,\mathbb{R})$	$Isom^+(\mathbb{H}) \cong PSL(2,\mathbb{R})$[7]	-1[8]

Before discussing the geometry in the next section, we note an issue about the notation. The unitary group of signature (n,m) is the group of matrices preserving a Hermitian form of signature (n,m), and is isomorphic to this group preserving a particularly simple such form (a diagonal one):

$$\mathrm{U}(n,m) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}_{n+m}(\mathbb{C}) : AA^* - BB^* = I_n,\ DD^* - CC^* = I_m,\ AC^* = BD^* \right\}$$

The special unitary group $\mathrm{SU}(n,m) = \{M \in \mathrm{U}(n,m) : \det(M) = 1\}$ is restricted to those matrices of determinant 1.

The projective special unitary group $\mathrm{PSU}(n,m) = SU(n,m)/Z$ is the quotient of this group be its subgroup of scalar matrices $Z = \left\{ \begin{bmatrix} zI_n & 0 \\ 0 & zI_m \end{bmatrix} : \lvert z\rvert = 1, z^{n+m} = 1 \right\}$.

Therefore, for $n = m = 1$ we have

$$Z = \left\{ \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} : \left(e^{i\theta}\right)^2 = 1 \right\} = \{\pm I_2\}$$

and

$$PSU(1,1) = SU(1,1)/Z = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \lvert a\rvert^2 - \lvert b\rvert^2 = 1 \right\} / \{\pm I_2\}$$

[2] see for example 18.5.1, 18.5.2, and 18.5.3 of Marcel Berger's *Geometry II*, Corrected Fourth Printing 2009.

[3] by $\kappa(z) = -\frac{\Delta \ln \rho(z)}{\rho(z)^2}$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$

[4] see for example 3.37 and 3.38 of Peter J. Oliver's Classical Invariant Theory

[5] see Theorem 16.5 of https://mathweb.ucsd.edu/~jmckerna/Teaching/19-20/Spring/120B/l_16.pdf. It can be shown that $Isom(\triangle) =$

[6] see https://thatsmaths.files.wordpress.com/2013/10/134-half-plane-curvature.pdf

[7] see Theorem 1.16 of http://math.uchicago.edu/~may/REU2012/REUPapers/Khan.pdf

[8] see Wikipedia Poincare unit disc-metric and curvature

Section 4. A Toy Model in Hyperbolic Geometry

We start with some geometric properties of the Mobius transformation:

Proposition 4.1.

The Mobius transformation can be decomposed to maps of the following two types:

(1) $w = kz + h,\ k \neq 0$

(2) $w = \frac{1}{z}$

Proof: In fact, when $c = 0$, $\phi(z)$ is already of the form (1):

$$w = \frac{a}{d}z + \frac{b}{d}$$

when $c \neq 0$, $\phi(z)$ can be written as

$$w = \frac{a}{c} + \frac{bc - ad}{c(cz + d)} = \frac{bc - ad}{c} \cdot \frac{1}{cz + d} + \frac{a}{c}$$

let $\xi = cz + d$ and $\eta = \frac{1}{\xi}$ then w is the composition of types (1) and (2):

$$w = \frac{bc - ad}{c} \cdot \eta + \frac{a}{c}$$

□

Therefore, knowing about the geometric properties of transformations in types (1) and (2) help us understand the geometric properties of the Mobius transformation. In fact, (1) is a composition of **dilation** (by the norm of k), **rotation** (by the angle of k), and **translation** (by the constant h); (2) is called the **inversion transformation**. We may say a bit more about the inversion transformation: it maps a circle not passing through the origin to a circle not passing through the origin; a circle passing through the origin to a line not passing through the origin; a line not passing through the origin to a circle passing through the origin; a line passing through the origin to a line passing through the origin.

These familiar terms in Euclidean space, for example $\mathbb{R}^2$, or understood as $\mathbb{C}$, change their geometric visualizations when we enter into the hyperbolic world. However, diving deeply into hyperbolic geometry requires a whole lecture, so we shall only talk about, in a non-rigorous way, one of the basic facts as an illustrative example, which also uses up our previous definitions in section 0. Many of them are simple special cases of the study of Riemannian geometry. We first return to an observation we claimed in section 0.

Proposition 4.2.

The geodesics of $\mathbb{H}$ are lines perpendicular to the real line, and half-circles orthogonal to the real line.

Proof: We first show that the vertical lines are geodesics. Let z be a point on the y-axis. We will attempt to determine the geodesic through z with a vertical tangent at z.

Assume for the sake of contradiction that there exists a geodesic that is not simply the line $l = \{z \in \mathbb{H} | Re(z) = 0\}$. The map given by the reflection of the hyperbolic plane across the y-axis is an isomorphism, so the reflection of this geodesic across the axis would have the same length. Since geodesics in $\mathbb{H}$ are distance-reducing paths, this reflection would also be a geodesic, with the same initial point and tangent vector. This contradicts the uniqueness of geodesics, so the only geodesic through z with a vertical tangent vector is a vertical line.

We have shown in previous sections that Mobius transformations are isometries of $\mathbb{H}$.

Combining this with proposition 4.1 diversifies examples of transformations of geometric objects. All geodesics are isometric, so the geodesics of $\mathbb{H}$ are all constructions isometric to the vertical line through the origin. That is to say, the geodesics of the upper half plane are all of the possible results when a Mobius transformation is applied to this vertical line. These results are lines perpendicular to the real line and half-circles orthogonal to the real line by proposition 4.1. □

Corollary 4.3.

Mobius transformations map geodesics to geodesics in the upper plan $\mathbb{H}$. □

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[6] Steven G. Krantz. *Complex Analysis: The Geometric Viewpoint*. 2nd edition, AMS, 2004

[7] https://www.math3ma.com/blog/automorphisms-of-the-upper-half-plane

[8] https://people.reed.edu/~jerry/311/lec10.pdf