Symplectic and Toric Manifolds

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*Reading on symplectic and toric manifolds with Prof. Xiang Tang; based on Ana Cannas da Silva's *Lectures on Symplectic Geometry* [4] and *Symmetric Toric Manifolds* [5].

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Part I

Symplectic Manifolds

Chapter 1

Symplectic Forms

1.1 Skew-Symmetric Bilinear Maps

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map, i.e., linear in one coordinate while fixing the other. The map Ω is **skew-symmetric**, or **alternating**, if $\Omega(u,v) = -\Omega(v,u)$, for all $u, v \in V$.

In general, for a covariant k-tensor f in k-fold tensor product $V^* \otimes \cdots \otimes V^*$,

$$f : V \times \cdots \times V \to \mathbb{R},$$

we say it is alternating if $\sigma f = (\operatorname{sgn}(\sigma)) f$ for every $\sigma \in S_k$, i.e., $f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = (\operatorname{sgn}(\sigma)) f(v_1, \cdots, v_k)$. Since $\operatorname{sgn}(\sigma) = (-1)^{\#\text{inversions}}$, an equivalent characterization is that for every pair (i,j), switching the arguments changes the sign, i.e., $f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k)$.

Theorem 1.1.1 (Standard Form for Skew-symmetric Bilinear Maps)**.** *Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that*

$$\begin{aligned}
&\Omega(u_i, v) = 0, \\
&\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j), \\
&\Omega(e_i, f_j) = \delta_{ij},
\end{aligned}$$

for all i and all $v \in V$, for all i, j, and for all i, j.

Remark 1.1.2.

1. The basis in theorem is not unique, though it is traditionally also called a "canonical" basis.
2. We recall that if V is an n-dimensional vector space with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and B is a bilinear map on V, we have the **matrix of the bilinear form on the basis** A, defined by $A_{ij} = B(\mathbf{e}_i, \mathbf{e}_j)$. With notation $\mathbf{x} = \begin{bmatrix} | \\ x \\ | \end{bmatrix}$ for coordinates of x with respect to this basis, and similarly $\mathbf{y}$ for y, we can write

$$B(x,y) = \mathbf{x}^T A \mathbf{y} = \sum_{i,j=1}^{n} x_i A_{ij} y_j$$

Thus, in matrix notation with respect to the canonical basis $u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n$, we have

$$\Omega(u,v) = [-u-] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}.$$

3. The dimension of the subspace $U=\{u \in V \mid \Omega(u, v)=0$, for all $v \in V\}$ does not depend on the choice of basis. $\Longrightarrow k:=\dim U$ is an invariant of (V, Ω).
4. Since $k+2n=m=\dim V$, n is an invariant of (V, Ω); $2n$ is called the **rank** of Ω.

♠

1.2 Symplectic Vector Spaces

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega: V \times V \rightarrow \mathbb{R}$ be a bilinear map.

Definition 1.2.1. *The map $\widetilde{\Omega}: V \rightarrow V^*$ is the linear map defined by $\widetilde{\Omega}(v)(u)=\Omega(v, u)$.*

The kernel of $\widetilde{\Omega}$ is the subspace U above:

$$\ker(\widetilde{\Omega})=\{v \in V \mid \forall u \in V, \widetilde{\Omega}(v)(u)=\Omega(v, u)=0\}=U$$

Definition 1.2.2. *A skew-symmetric bilinear map Ω is **symplectic** (or **nondegenerate**) if $\widetilde{\Omega}$ is bijective, i.e., $U=\{0\}$ (injectivity plus indentical dimension). The map Ω is then called a **linear symplectic structure** on V, and (V, Ω) is called a **symplectic vector space**.*

The following are immediate properties of a linear symplectic structure Ω:

- **Duality**: the map $\widetilde{\Omega}: V \overset{\simeq}{\rightarrow} V^*$ is a bijection.
- By Theorem 1.1.1, $k=\dim U=0$, so $\dim V=2n$ is **even**.
- By Theorem 1.1.1, a symplectic vector space (V, Ω) has a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ satisfying

$$\Omega\left(e_i, f_j\right)=\delta_{ij} \quad \text{and} \quad \Omega\left(e_i, e_j\right)=0=\Omega\left(f_i, f_j\right).$$

  Such a basis is called a **symplectic basis** of (V, Ω). We have

$$\Omega(u, v)=[-u-]\left[\begin{array}{cc}0 & \mathrm{Id} \\ -\mathrm{Id} & 0\end{array}\right]\left[\begin{array}{c}\mid \\ v \\ \mid\end{array}\right]$$

  Not all subspaces W of a symplectic vector space (V, Ω) look the same:

- A subspace W is called **symplectic** if $\left.\Omega\right|_W$ is nondegenerate. For instance, the span of e_1, f_1 is symplectic. In this case, $\widetilde{(\Omega|_W)}: W \rightarrow W^*$ with $W=\{ae_1+bf_1\}$ should be shown to have $U=\{u \in W \mid \Omega(u, v)=0$, for all $v \in W\}=\{0\}$. First, trvially $0 \in U$. Now, suppose $u=ae_1+bf_1$ is some vector in W such that $\Omega(u, v)=0$ for all $v \in W$, then we let $v=e_1$ to get $-b=0$ and $v=f_1$ to get $a=0$ and see that u has to be 0.
- A subspace W is called **isotropic** if $\left.\Omega\right|_W \equiv 0$. For instance, the span of e_1, e_2 is isotropic. That's because for $x=x_1e_1+x_2e_2$, $y=y_1e_1+y_2e_2$, we have $\Omega(x, y)=\sum_{i=1}^2 \sum_{j=1}^2 x_i \Omega(e_i, e_j) y_j=0$.

Homework 1 describes subspaces W of (V, Ω) in terms of the relation between W and W^{Ω}.

The **prototype of a symplectic vector space** is $\left(\mathbb{R}^{2n}, \Omega_0\right)$ with Ω_0 such that the basis

$$\begin{aligned} e_1=(1,0, \ldots, 0), \quad \ldots, \quad & e_n=(0, \ldots, 0, \overbrace{1}^{n}, 0, \ldots, 0), \\ f_1=(0, \ldots, 0, \underbrace{1}_{n+1}, 0, \ldots, 0), \ldots, \quad & f_n=(0, \ldots, 0,1),\end{aligned}$$

is a symplectic basis. The map Ω_0 on other vectors is determined by its values on a basis and bilinearity.

Definition 1.2.3. *A **symplectomorphism** φ between symplectic vector spaces (V,Ω) and (V',Ω') is a linear isomorphism $\varphi : V \xrightarrow{\cong} V'$ such that $\varphi^*\Omega' = \Omega$. ($\varphi^*$ is the induced map. Recall that $f^* : h \mapsto f \circ h$ and $f_* : h \mapsto h \circ f$. Thus, $\varphi^*\Omega' = \Omega$ reads as $(\varphi^*\Omega')(u,v) = \Omega'(\varphi(u),\varphi(v)) = \Omega(u,v)$.) If a symplectomorphism exists, (V,Ω) and (V',Ω') are said to be **symplectomorphic**.*

The relation of being symplectomorphic is clearly an equivalence relation in the set of all even-dimensional vector spaces. Furthermore, by Theorem 1.1.1, every $2n$-dimensional symplectic vector space (V,Ω) is symplectomorphic to the prototype $\left(\mathbb{R}^{2n},\Omega_0\right)$; a choice of a symplectic basis for (V,Ω) yields a symplectomorphism to $\left(\mathbb{R}^{2n},\Omega_0\right)$. Hence, nonnegative even integers classify equivalence classes for the relation of being symplectomorphic.

1.3 Symplectic Manifolds

Definition 1.3.1. *Let ω be a de Rham 2-form on a manifold M, i.e.,*

$$\omega \in \Omega^2(M) = \underbrace{\Gamma}_{\text{all } C^\infty \text{ sections}} \overbrace{\left(\bigwedge{}^2 T^*M\right)}^{\substack{\text{alternating}\\ \text{cov. 2-tensor}}} .$$

*In words, for each $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric (= alternating) bilinear (= 2-tensor) on the tangent space to M at p, and ω_p varies smoothly in p (i.e., ω written in $a_{ij}(p)dx_p^i \wedge dx_p^j$ has each of its components $a^{ij} : M \to \mathbb{R}$ smooth). We say that ω is **closed** if it satisfies the differential equation $d\omega = 0$, where d is the de Rham differential (i.e., exterior derivative; see [8] p.365 theorem 14.24). Moreover, ω is **symplectic** if ω is closed and ω_p is symplectic for all $p \in M$.*

If ω is symplectic, then $\dim T_pM = \dim M$ must be even.

Definition 1.3.2. *A **symplectic manifold** is a pair (M,ω) where M is a manifold and ω is a symplectic form.*

Example 1.3.3 ([8] Example 22.2)**.** Let V be a vector space of dimension $2n$, with a basis denoted by $(A_1,B_1,\ldots,A_n,B_n)$. Let $\left(\alpha^1,\beta^1,\ldots,\alpha^n,\beta^n\right)$ denote the corresponding dual basis for V^*, and let $\omega \in \Lambda^2\left(V^*\right)$ be the 2-covector defined by

$$\omega = \sum_{i=1}^n \alpha^i \wedge \beta^i .$$

It is obvious an alternating covariant 2-tensor by definition from [8] p.355 (14.3) and p.351 proposition 14.3 (a). Note that the action of ω on basis vectors is given by

$$\omega\left(A_i,A_j\right) = \omega\left(B_i,B_j\right) = 0, \quad \omega\left(A_i,B_j\right) = -\omega\left(B_j,A_i\right) = \delta_{ij}.$$

In fact, we recall from [8] example 14.2 and p.355 (14.3) that for two 1-forms ω, η, one has

$$\omega \wedge \eta(v,w) = 2 \cdot \frac{1}{2}[\omega \otimes \eta(v,w) - \eta \otimes \omega(w,v)] = \omega(w)\eta(v) - \eta(v)\omega(w).$$

Therefore,

$$\alpha^k \wedge \beta^k(A_i,A_j) = \alpha^k \wedge \alpha^k(B_i,B_j) = 0,$$

and

$$\alpha^k \wedge \beta^k(A_i,B_j) = \alpha^k(A_i)\beta^k(B_j) - \underbrace{\alpha^k(B_j)\beta^k(A_i)}_{=0} = \begin{cases} \delta_{ki} = \delta_{kj}, & i = j \\ 0, & i \neq j \end{cases}$$

$$\Longrightarrow \omega(A_i,B_j) = -\omega(B_j,A_i) = \sum_{k=1}^n \alpha^k \wedge \beta^k(A_i,B_j) = \delta_{ij}.$$

Suppose $v = a^i A_i + b^i B_i \in V$ satisfies $\omega(v, w) = 0$ for all $w \in V$. Then $0 = \omega(v, B_i) = a^i$ and $0 = \omega(v, A_i) = -b^i$, which implies that $v = 0$. Thus ω is nondegenerate, and so is a symplectic tensor (a nondegenerate 2-covector). ♣

Example 1.3.4. From above example, we natrually have the following prototype. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic and is called **standard symplectic form on** $\mathbb{R}^{2n}$, and the set

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_n}\right)_p \right\}$$

is a symplectic basis of T_pM. ♣

Example 1.3.5. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, \ldots, z_n$. The form

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is symplectic. In fact, this form equals that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + iy_k$. ♣

Example 1.3.6. Let $M = \mathbb{S}^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to $\mathbb{S}^2$ at p may then be identified with vectors orthogonal to p. The standard symplectic form on $\mathbb{S}^2$ is induced by the inner and exterior products:

$$\omega_p(u, v) := \langle p, u \times v \rangle, \quad \text{for } u, v \in T_p\mathbb{S}^2 = \{p\}^{\perp}.$$

This form is closed because it is of top degree (there are no nonzero differential forms of degree $> m$ for an m-manifold because if $\deg dx^I > m$, then in the expression dx^I at least two of the 1-forms dx^{i_α} must be the same, forcing dx^I and thus $\omega = \sum a_I dx^I$ be zero. Thus $d\omega \in \Omega^3(M)$ is zero.) The form is nondegenerate: if we let $u \neq 0$ and $v = u \times p$, then $\langle p, u \times v \rangle \neq 0$. Thus, any nonzero u will make ω_p not identically vanishing. ♣

1.4 Symplectomorphisms

Definition 1.4.1. *Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a* ***symplectomorphism*** *if $\varphi^*\omega_2 = \omega_1$ (Recall that, by definition of pullback, at tangent vectors $u, v \in T_pM_1$, we have $(\varphi^*\omega_2)_p(u, v) = (\omega_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v))$.)*

We would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem (proved in Lecture 8 and stated below) takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional symplectic manifold looks locally like $(\mathbb{R}^{2n}, \omega_0)$. More precisely, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 1.4.2 (Darboux)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ as in Darboux theorem is called a **Darboux chart**. By Darboux theorem, the **prototype of a local piece of a $2n$-dimensional symplectic manifold** is $M = \mathbb{R}^{2n}$, with linear coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and with symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

1.5 Symplectic Linear Algebra

Given a linear subspace Y of a symplectic vector space (V, Ω), its **symplectic orthogonal**, or **symplectic complement** Y^Ω is the linear subspace defined by

$$Y^\Omega := \{v \in V \mid \Omega(v, u) = 0 \text{ for all } u \in Y\}.$$

Exercise 1.5.1. *Show that* $\dim Y + \dim Y^\Omega = \dim V$.

Solution. Let $Y \subseteq V$ be a subspace, and consider the map

$$\begin{array}{rcl} \Phi : V & \longrightarrow & Y^* = \operatorname{Hom}(Y, \mathbb{R}) \\ v & \longmapsto & \Omega(v, \cdot)|_Y \end{array}$$

(It is the map $\Phi(v) = (v \lrcorner \omega)\mid_Y$ where $\lrcorner$ stands for the **interior multiplication**; see [8] p.358.) Suppose φ is an arbitrary element of Y^*, and let $\tilde{\varphi} \in V^*$ be any extension of φ to a linear functional on all of V. Since the map $\widetilde{\Omega} : V \to V^*$ defined by $v \mapsto v \lrcorner \Omega$ is an isomorphism, there exists $v \in V$ such that $v \lrcorner \Omega = \widetilde{\varphi}$. It follows that $\Phi(v) = \varphi$, and therefore Φ is surjective. By the rank-nullity law, $Y^\Omega = \operatorname{Ker} \Phi$ has dimension equal to $\dim V - \dim Y^* = \dim V - \dim Y$. ♦

Exercise 1.5.2. *Show that* $\left(Y^\Omega\right)^\Omega = Y$.

Solution. We first show that $Y \subseteq (Y^\Omega)^\Omega$: that is, we need to show that for each $v \in Y$, we have $\Omega(u, v) = 0$ for every $u \in Y^\Omega$. Now, $u \in Y^\Omega$ implies that $\Omega(u, v) = 0$, so $\Omega(v, u) = -\Omega(u, v) = 0$.

Note that exercise 1.5.1 gives that $\dim Y + \dim Y^\Omega = \dim V$ and $\dim Y^\Omega + \dim \left(Y^\Omega\right)^\Omega = \dim V$. Thus, $\dim Y = \dim \left(Y^\Omega\right)^\Omega = k$. Combining previous fact $Y \subseteq (Y^\Omega)^\Omega$ with this, we see $\left(Y^\Omega\right)^\Omega = Y$. That's because we can find k linear independent vectors as the basis of Y; and any set of k linearly independent vectors can serve as the basis for $(Y^\Omega)^\Omega$. ♦

Exercise 1.5.3. *Show that, if Y and W are subspaces, then*

$$Y \subseteq W \Longleftrightarrow W^\Omega \subseteq Y^\Omega.$$

Solution. Given $Y \subseteq W$, we see $\forall u \in W$, $\Omega(v, u) = 0$ implies $\forall u \in Y$, $\Omega(v, u) = 0$. Thus, every v in W^Ω is also in Y^Ω. $W^\Omega \subseteq Y^\Omega$.

Now suppose $W^\Omega \subseteq Y^\Omega$. Then by the conclusion we just showed, $(Y^\Omega)^\Omega \subseteq (W^\Omega)^\Omega$. Apply exercise 1.5.2 to conclude. ♦

Symplectic complements differ from orthogonal complements in one important respect: although it is always true that $Y \cap Y^\perp = \{0\}$ in an inner product space, this need not be true in a symplectic vector space. Indeed, if Y is 1-dimensional, i.e., $Y = \{av\}$ the fact that Ω is alternating forces $\Omega(v, av) = a\Omega(v, v) = 0$ for every $av \in Y$, so $Y \subseteq Y^\Omega$. We thus showed the following exercise. Carrying this idea a little further, a linear subspace $Y \subseteq V$ is said to be

- **symplectic** if $Y \cap Y^{\Omega} = \{0\}$;
- **isotropic** if $Y \subseteq Y^{\Omega}$
- **coisotropic** if $Y \supseteq Y^{\Omega}$;
- **Lagrangian** if $Y = Y^{\Omega}$.

Exercise 1.5.4. *Every codimension 1 subspace Y is coisotropic.*

Solution. We just showed that. ♦

Exercise 1.5.5. *Show that, if Y is isotropic, then $\dim Y \leq \frac{1}{2}\dim V$.*

Solution. Since $Y \subseteq Y^{\Omega}$ and thus $\dim Y \leq \dim Y^{\Omega}$, exercise 1.5.1 implies that $\dim V = \dim Y + \dim Y^{\Omega} \geq \dim Y + \dim Y \implies \frac{1}{2}\dim V \geq \dim Y$. ♦

Exercise 1.5.6. *For any symplectic vector space (V, Ω) there exists a Lagrangian subspace $L \in \mathrm{Lag}(V)$, where $\mathrm{Lag}(V)$ is the collection of all Lagrangians, called **Lagrangian Grassmannian**.*

Proof. Let L be an isotropic subspace of V, which is maximal in the sense that it is not contained in any isotropic subspace of strictly larger dimension. Then L is Lagrangian: For if $L^{\Omega} \neq L$, then choosing any $v \in L^{\Omega} \backslash L$ would produce a larger isotropic subspace $L \oplus \operatorname{span}(v)$. ■

Exercise 1.5.7. *Let (V, Ω) be a symplectic vector space, and let $Y \subseteq V$ be a linear subspace. Then*

(a) Above definition of symplectic subspace is equivalent to our first definition, i.e., Y is symplectic if and only if $\Omega|_Y$ is nondegenerate.

(b) Y is symplectic if and only if $V = Y \oplus Y^{\Omega}$.

(c) Y is symplectic if and only if Y^{Ω} is symplectic.

(d) Above definition of isotropic subspace is equivalent to our first definition, i.e., Y is isotropic if and only if $\Omega|_Y = 0$.

(e) Y is coisotropic if and only if Y^{Ω} is isotropic.

(f) Y is Lagrangian if and only if it is isotropic and $\dim Y = \frac{1}{2}\dim V$.

(g) Y is Lagrangian if and only if Y is isotropic and coisotropic.

Solution. (a) is immediately obtained as the subspace U for $(Y, \Omega|_Y)$ is exactly $Y \cap Y^{\Omega}$. For (b), $V = Y \oplus Y^{\Omega}$ directly gives $Y \cap Y^{\Omega}$, and to show the converse it suffices to show every v is written as $y + y'$ for $y \in Y$, $y' \in Y^{\Omega}$. We show that the combined basis of $\{e_1, \cdots, e_k\}$ of Y and $\{f_1, \cdots, f_k\}$ of Y^{Ω} gives a basis of V. Due to exercise 1.5.1, $k + l = n$, and we only need to show each f_i is independent of $\{e_1, \cdots, e_n\}$. But this is trivial, since $Y \cap Y^{\Omega} = \{0\}$. (c) and (e) are due to exercise 1.5.2. (d) and (g) are immediate by definitions. We show (f): one direction is straightforward; for the other direction, $\frac{1}{2}\dim V = \dim Y$ gives $\dim Y = \dim Y^{\Omega}$, which combined with $Y \subseteq Y^{\Omega}$ to give $Y = Y^{\Omega}$ in the same argument as the exercise 1.5.2. ♦

Exercise 1.5.8. *Show that, if Y is a lagrangian subspace of (V, Ω), then any basis $e_1, \ldots, e_n$ of Y can be extended to a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of (V, Ω). Hint: Choose f_1 in W^{Ω}, where W is the linear span of $\{e_2, \ldots, e_n\}$.*

Exercise 1.5.9. *Show that, if Y is a lagrangian subspace, (V, Ω) is symplectomorphic to the space $(Y \oplus Y^*, \Omega_0)$, where Ω_0 is determined by the formula*

$$\Omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v).$$

In fact, for any vector space E, the direct sum $V = E \oplus E^*$ has a canonical symplectic structure determined by the formula above. If $e_1, \ldots, e_n$ is a basis of E, and $f_1, \ldots, f_n$ is the dual basis, then $e_1 \oplus 0, \ldots, e_n \oplus 0, 0 \oplus f_1, \ldots, 0 \oplus f_n$ is a symplectic basis for V.

Chapter 2

Symplectic Form on the Cotangent Bundle

2.1 Cotangent Bundle

Let X be any n-dimensional manifold and $M = T^*X$ its cotangent bundle. If the manifold structure on X is described by coordinate charts $(\mathcal{U}, x_1, \ldots, x_n)$ with $x_i : \mathcal{U} \to \mathbb{R}$, then at any $x \in \mathcal{U}$, the differentials $(dx_1)_x, \ldots (dx_n)_x$ form a basis of T_x^*X. Namely, if $\xi \in T_x^*X$, then $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$ for some real coefficients $\xi_1, \ldots, \xi_n$. This induces a map

$$\begin{aligned} T^*\mathcal{U} &\longrightarrow \mathbb{R}^{2n} \\ (x, \xi) &\longmapsto (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n). \end{aligned}$$

The chart $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ is a coordinate chart for T^*X; the coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ are the cotangent coordinates associated to the coordinates $x_1, \ldots, x_n$ on $\mathcal{U}$. The transition functions on the overlaps are smooth: given two charts $(\mathcal{U}, x_1, \ldots, x_n), (\mathcal{U}', x_1', \ldots, x_n')$, and $x \in \mathcal{U} \cap \mathcal{U}'$, if $\xi \in T_x^*X$, then

$$\xi = \sum_{i=1}^n \xi_i (dx_i)_x = \sum_{i,j} \xi_i \left(\frac{\partial x_i}{\partial x_j'} \right) (dx_j')_x = \sum_{j=1}^n \xi_j' (dx_j')_x$$

where $\xi_j' = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x_j'} \right)$ is smooth. Hence, T^*X is a $2n$-dimensional manifold. We will now construct a major class of examples of symplectic forms. The **canonical forms** on cotangent bundles are relevant for several branches, including analysis of differential operators, dynamical systems and classical mechanics.

2.2 Tautological and Canonical Forms in Coordinates

Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate chart for X, with associated cotangent coordinates $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. Define a 2 -form ω on $T^*\mathcal{U}$ by

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

In order to check that this definition is coordinate-independent, consider the 1-form on $T^*\mathcal{U}$

$$\alpha = \sum_{i=1}^n \xi_i dx_i = \sum_{i=1}^n \xi_i dx_i + \sum_{i=1}^n 0 d\xi_i$$

Since the coefficients for $d\xi_i$'s are all zero, we from [8] p.363 get

$$d\alpha = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\underbrace{\frac{\partial \xi_j}{\partial x_i}}_{=0} dx_i \wedge dx_j + \sum_{i=1}^{n}\underbrace{\frac{\partial \xi_j}{\partial \xi_i}}_{=\delta_{ij}} d\xi_i \wedge dx_j\right) = \sum_{j=1}^{n} d\xi_j \wedge dx_j = -\omega.$$

Claim: The form α is intrinsically defined (and hence the form ω is also intrinsically defined).

Proof. Let $(\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ and $(\mathcal{U}', x_1', \ldots, x_n', \xi_1', \ldots, \xi_n')$ be two cotangent coordinate charts. On $\mathcal{U} \cap \mathcal{U}'$, the two sets of coordinates are related by $\xi_j' = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x_j'}\right)$. Since $dx_j' = \sum_i \left(\frac{\partial x_j'}{\partial x_i}\right) dx_i$, we have

$$\begin{aligned}
\sum_j \xi_j' dx_j' &= \sum_j \sum_i \sum_k \xi_i \left(\frac{\partial x_i}{\partial x_j'}\right)\left(\frac{\partial x_j'}{\partial x_k}\right) dx_k \\
&= \sum_k \sum_i \xi_i \sum_j \left(\frac{\partial x_i}{\partial x_j'}\right)\left(\frac{\partial x_j'}{\partial x_k}\right) dx_k \\
&= \sum_k \sum_i \xi_i \left(\frac{\partial x_i}{\partial x_k}\right) dx_k \\
&= \sum_k \xi_k dx_k
\end{aligned}$$

∎

The 1-form α is the **tautological form** or **Liouville 1-form** and the 2-form ω is the **canonical symplectic form**. The following section provides an alternative proof of the intrinsic character of these forms.

2.3 Coordinate-Free Definitions

Let

$$\begin{array}{lll} M = T^*X & p = (x,\xi) & \xi \in T_x^*X \\ \downarrow \pi & \quad\downarrow & \\ X & \quad x & \end{array}$$

be the natural projection. The tautological 1-form α may be defined pointwise by

$$\alpha_p = \alpha_{(x,\xi)} = (d\pi_p)^* \xi \quad \in T_p^* M,$$

where $(d\pi_p)^*$ is the transpose of $d\pi_p$, that is, $(d\pi_p)^* \xi = \xi \circ d\pi_p$:

$$\begin{array}{lll} p = (x,\xi) & T_pM & T_p^*M \\ \downarrow \pi & \downarrow d\pi_p & \uparrow (d\pi_p)^* \\ x & T_xX & T_x^*X \end{array}$$

Equivalently,

$$\alpha_p(v) = \xi\left((d\pi_p)\, v\right), \quad \text{for } v \in T_pM.$$

Exercise 2.3.1. *Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a chart on X with associated cotangent coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$. Show that on $T^*\mathcal{U}, \alpha = \sum_{i=1}^{n} \xi_i dx_i$.*

Solution. Let $\left(x^i\right)$ be local coordinates on our base manifold M and let $\left(x^i, \xi_j\right)$ be the induced coordinates on the cotangent bundle T^*M. Let $\pi : T^*M \rightarrow M$ be the projection $\left(x^i, \xi_j\right) \mapsto \left(x^i\right)$. It induces a $C^{\infty}(M)$-linear map on 1-forms, which I will write as $\pi^* : \Omega^1(M) \rightarrow \Omega^1\left(T^*M\right)$. In coordinates, this sends a 1-form $\phi = \phi_i \, \mathrm{d}x^i$ (summation convention) to $\left(\phi_i \circ \pi\right) \mathrm{d}x^i$. As usual this induces a $\mathbb{R}$ linear map on the fibres, namely $\pi^*_{(x,\xi)} : T^*_x M \rightarrow T^*_{(x,\xi)}\left(T^*M\right)$, sending the covector p to the covector $(p, 0)$. (We must be careful and distinguish between covectors and 1-forms here, to avoid confusion.)

The tautological 1-form on $T^*\boldsymbol{M}$ is defined to be $\pi^*_{(x,\xi)}\xi$ at each point (x, ξ) in T^*M. ξ by definition is an element of $T^*_x M$, so it typechecks. Thus the point (x, ξ) is mapped to the covector $(\xi, 0)$ in $T^*_{(x,\xi)}\left(T^*M\right)$, and so the tautological 1-form in coordinates is given by $\xi_i \, \mathrm{d}x^i$, as claimed (The coefficient of $\mathrm{d}\xi_j$ is 0.) ♦

The **canonical symplectic 2-form** ω on T^*X is defined as

$$\omega = -d\alpha.$$

Locally, $\omega = \sum_{i=1}^n dx_i \wedge d\xi_i$.

Exercise 2.3.2. *Show that the tautological 1-form α is uniquely characterized by the property that, for every 1 -form $\mu : X \rightarrow T^*X, \mu^*\alpha = \mu$. (See Lecture 3.) Note that μ induces the pullback as a smooth map between manifolds.*

Solution. For a 1-form $\mu : X \rightarrow T^*X$ that sends x to (x, μ_x), we by definition have $\alpha_p = \mu_x \circ d\pi_p$ for any $p = (x, \mu_x) \in T^*X$. The pullback $\mu^*\alpha$ at a point $x \in X$ is defined in [8] p.360. It is another 1-form, on X, given by $\left(\mu^*\alpha\right)_x(v) = \alpha_{\mu(x)}\left(d\mu_x(v)\right) = \mu_x\left(d\pi_{\mu(x)} \circ d\mu_x(v)\right)$ for $v \in T_xX$. Since $\pi \circ \mu = \mathrm{id}_X$, it follows that $d\pi_{\mu(x)} \circ d\mu_x = \mathrm{id}_{T_xX}$, and thus $\left(\mu^*\alpha\right)_x(v) = \mu_x(v)$. Thus, $\mu^*\alpha_x = \mu_x$, i.e., for each point $x \in X$, the two 1-forms send x to the same mapping, so $\mu^*\alpha = \mu$.

Conversely, Notice that if β is another 1-form satisfying $\mu^*\beta = \mu$ for all 1-forms $\mu : X \rightarrow T^*X$, then $\mu^*(\alpha - \beta) = 0$ for all $\mu : X \rightarrow T^*X$. Now it suffices to show $\theta = 0$ if $\forall \mu : X \rightarrow M$, $\mu^*\theta = 0$, i.e., $\forall \mu : X \rightarrow M$, $\forall x \in X$, $\forall v \in T_xX$,

$$0 = (\mu^*\theta)_x(v) = \theta_{\mu(x)}(d\mu_x(v))$$

We let μ_i be such that $\mu_i(x) = \left.\frac{\partial}{\partial x_i}\right|_x$. That is, μ_i, $i = 1, \cdots, n$, form the coordinate frame for T^*X (see [8] Example 8.10.) We also set $v = \left.\frac{\partial}{\partial x_j}\right|_x$. Then by definition of differential in [8] p.62-63,

$$d(\mu_i)_x\left(\left.\frac{\partial}{\partial x_j}\right|_x\right) = \sum_{k=1}^n \frac{\partial \hat{\mu}_i^k}{\partial x_j}(\hat{x}) \left.\frac{\partial}{\partial x_k}\right|_{\mu_i(x)} + \sum_{k=1}^n \frac{\partial \hat{\mu}_i^{n+k}}{\partial x_j}(\hat{x}) \left.\frac{\partial}{\partial \xi_k}\right|_{\mu_i(x)}.$$

Note that $\hat{\mu}_i$ sends $\hat{x} = (x_1, \cdots, x_n)$ to $(x_1, \cdots, x_n, 0, \cdots, \overbrace{1}^{i-\text{th}}, \cdots, 0)$. Thus, $\partial\hat{\mu}_i^k/\partial x_j = \delta_{jk}$ for $k = 1, \cdots, n$; and $\partial\hat{\mu}_i^{n+k}/\partial x_j = 0$ for $k = 1, \cdots, n$. Therefore,

$$d(\mu_i)_x\left(\left.\frac{\partial}{\partial x_j}\right|_x\right) = \left.\frac{\partial}{\partial x_j}\right|_{\mu_i(x)},$$

and

$$0 = \theta_{\mu_i(x)}\left(d(\mu_i)_x\left(\left.\frac{\partial}{\partial x_j}\right|_x\right)\right) = \theta_{\mu_i(x)}\left(\left.\frac{\partial}{\partial x_j}\right|_{\mu_i(x)}\right)$$

for every $j = 1, \cdots, n$. Since $\theta_{\mu_i(x)}$ sends the whole basis to 0, $\theta_{\mu_i(x)}$ sends every v to 0. ♦

2.4 Naturality of the Tautological and Canonical Forms

Let X_1 and X_2 be n-dimensional manifolds with cotangent bundles $M_1 = T^*X_1$ and $M_2 = T^*X_2$, and tautological 1 -forms α_1 and α_2. Suppose that $f : X_1 \to X_2$ is a diffeomorphism. Then there is a natural diffeomorphism (the **cotangent bundle isomorphism**)

$$f_\sharp : M_1 \to M_2$$

which **lifts** f; namely, if $p_1 = (x_1, \xi_1) \in M_1$ for $x_1 \in X_1$ and $\xi_1 \in T^*_{x_1} X_1$, then we define

$$f_\sharp (p_1) = p_2 = (x_2, \xi_2), \quad \text{with } \begin{cases} x_2 = f(x_1) \in X_2 \text{ and} \\ \xi_1 = (df_{x_1})^* \xi_2, \end{cases}$$

where $(df_{x_1})^* : T^*_{x_2} X_2 \overset{\simeq}{\Rightarrow} T^*_{x_1} X_1$, so $f_\sharp|_{T^*_{x_1} X_1}$ is the inverse map of $(df_{x_1})^*$.

Exercise 2.4.1. *Check that $f_\sharp$ is a diffeomorphism. Here are some hints:*

1. *The following diagram commutes:*

$$\begin{array}{ccc} M_1 & \xrightarrow{f_\sharp} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

2. *$f_\sharp : M_1 \to M_2$ is bijective;*
3. *$f_\sharp$ and $f_\sharp^{-1}$ are smooth.*

Proposition 2.4.2. *The lift $f_\sharp$ of a diffeomorphism $f : X_1 \to X_2$ pulls the tautological form on T^*X_2 back to the tautological form on T^*X_1, i.e.,*

$$(f_\sharp)^* \alpha_2 = \alpha_1.$$

Proof. At $p_1 = (x_1, \xi_1) \in M_1$, this identity says

$$(\star): \qquad (df_\sharp)^*_{p_1} (\alpha_2)_{p_2} = (\alpha_1)_{p_1}$$

where $p_2 = f_\sharp (p_1)$. Using the following facts,

(1) Definition of $f_\sharp : p_2 = f_\sharp (p_1) \Longleftrightarrow p_2 = (x_2, \xi_2)$ where $x_2 = f(x_1)$ and $(df_{x_1})^* \xi_2 = \xi_1$.

(2) Definition of tautological 1-form: $(\alpha_1)_{p_1} = (d\pi_1)^*_{p_1} \xi_1$ and $(\alpha_2)_{p_2} = (d\pi_2)^*_{p_2} \xi_2$.

(3) The following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{f_\sharp} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

the proof of $(\star)$ is:

$$\begin{aligned} &(df_\sharp)^*_{p_1} (\alpha_2)_{p_2} \overset{(2)}{=\!=} (df_\sharp)^*_{p_1} (d\pi_2)^*_{p_2} \xi_2 \xlongequal{(g\circ f)^* = f^* \circ g^*} (d(\pi_2 \circ f_\sharp))^*_{p_1} \xi_2 \\ &\overset{(3)}{=\!=} (d(f \circ \pi_1))^*_{p_1} \xi_2 \xlongequal{(g\circ f)^* = f^* \circ g^*} (d\pi_1)^*_{p_1} (df)^*_{x_1} \xi_2 \\ &\overset{(1)}{=\!=} (d\pi_1)^*_{p_1} \xi_1 \overset{(2)}{=\!=} (\alpha_1)_{p_1}. \end{aligned}$$

∎

Corollary 2.4.3. *The lift $f_\sharp$ of a diffeomorphism $f: X_1 \to X_2$ is a symplectomorphism, i.e., f is a diffeomorphism and*

$$(f_\sharp)^* \omega_2 = \omega_1,$$

where ω_1, ω_2 are the canonical symplectic forms. In summary, a diffeomorphism of manifolds induces a canonical symplectomorphism of cotangent bundles:

$$\begin{array}{cccc} f_\sharp: & T^*X_1 & \longrightarrow & T^*X_2 \\ f: & X_1 & \longrightarrow & X_2 \end{array}$$

Example 2.4.4. Let $X_1 = X_2 = S^1$. Then T^*S^1 is an infinite cylinder $S^1 \times \mathbb{R}$. The canonical 2-form ω is the area form $\omega = d\theta \wedge d\xi$. If $f: S^1 \to S^1$ is any diffeomorphism, then $f_\sharp: S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is a symplectomorphism, i.e., is an area-preserving diffeomorphism of the cylinder. ♣

If $f: X_1 \to X_2$ and $g: X_2 \to X_3$ are diffeomorphisms, then $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. In terms of the group $\operatorname{Diff}(X)$ of diffeomorphisms of X and the group $\operatorname{Sympl}(M, \omega)$ of symplectomorphisms of (M, ω), we say that the map

$$\begin{aligned} \operatorname{Diff}(X) &\longrightarrow \operatorname{Sympl}(M, \omega) \\ f &\longmapsto f_\sharp \end{aligned}$$

is a group homomorphism. This map is clearly injective. Is it surjective? Do all symplectomorphisms $T^*X \to T^*X$ come from diffeomorphisms $X \to X$? No: for instance, translation along cotangent fibers is not induced by a diffeomorphism of the base manifold. A criterion for which symplectomorphisms arise as lifts of diffeomorphisms is discussed in Homework 3.

2.5 Symplectic Volume

Exercise 2.5.1. *Given a vector space V, the exterior algebra of its dual space is*

$$\wedge^*(V^*) = \bigoplus_{k=0}^{\dim V} \wedge^k(V^*),$$

*where $\wedge^k(V^*)$ is the set of maps $\alpha: \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ which are linear in each entry, and for any permutation $\pi, \alpha(v_{\pi_1}, \ldots, v_{\pi_k}) = (\operatorname{sign} \pi) \cdot \alpha(v_1, \ldots, v_k)$. The elements of $\wedge^k(V^*)$ are known as **skew-symmetric k-linear maps** or **k-forms** on V.*

(a) Show that any $\Omega \in \wedge^2(V^*)$ is of the form $\Omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*$, where $u_1^*, \ldots, u_k^*, e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*$ is a basis of V^* dual to the standard basis ($k + 2n = \dim V$).

*(b) In this language, a symplectic map $\Omega: V \times V \to \mathbb{R}$ is just a nondegenerate 2-form $\Omega \in \wedge^2(V^*)$, called a **symplectic form** on V. Show that, if Ω is any symplectic form on a vector space V of dimension $2n$, then the n-th exterior power $\Omega^n = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n}$ does not vanish.*

*(c) Deduce that the nth exterior power ω^n of any symplectic form ω on a $2n$-dimensional manifold M is a **volume form** (A volume form is a nonvanishing form of top degree.) Hence, any symplectic manifold (M, ω) is canonically oriented by the symplectic structure. The form $\frac{\omega^n}{n!}$ is called the **symplectic volume** or the **Liouville volume** of (M, ω). Does the Möbius strip support a symplectic structure?*

(d) Conversely, given a 2-form $\Omega \in \wedge^2(V^*)$, show that, if $\Omega^n \neq 0$, then Ω is symplectic. [Hint: Standard form.]

Solution. (a): Our standard basis for V is $u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n$. By [8] p.360, we see that $\Omega \in \wedge^2(V^*)$ can be written as

$$\Omega = \sum_I{}' \omega_I u_{I_u}^* \wedge e_{I_e}^* \wedge f_{I_f}^*$$

where $I_u \cup I_e \cup I_f = I$ iterates through all the strictly increasing indices. Now consider the coefficients for those $u_i^* \wedge u_j^*$: they are $\Omega(u_i, u_j) = 0$. Similarly, the coefficients for $u_i^* \wedge e_j^*$ and $u_i^* \wedge f_j^*$ are also 0. Then consider the coefficients for $e_i^* \wedge e_j^*$: they are $\Omega(e_i, e_j) = 0$. Similarly, coefficients for $f_i^* \wedge f_j^*$ are also 0. The remaining coefficients are those for $e_i^* \wedge f_j^*$ for all $(i,j) \in [\![1,n]\!]^2$ (since all of the f's are ordered behind e's). Since $\Omega(e_i, f_j) = \delta_{ij}$, we see

$$\Omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*.$$

(b): We're given that Ω is nondegenerate and thus a symplectic basis $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ for $2n$-dimensional vector space V. Then we by (a) have $\Omega = \sum_{i=1}^n e_i^* \wedge f_i^*$. Then $\omega^n = \sum_I e_{i_1}^* \wedge f_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge f_{i_n}^*$, where $I = (i_1, \cdots, i_n)$ ranges over all multi-indices of length n, i.e., $\in [\![1,n]\!]^n$. Any term in this sum for which I has a repeated index is zero because $\alpha \wedge \alpha = 0$ (note that α is of rank 1; [8] prop.14.11(c) gives $\alpha \wedge \alpha = (-1)^{1\cdot 1}\alpha \wedge \alpha \Rightarrow \alpha \wedge \alpha = 0$). The surviving terms are those for which I is a permutation of $[\![1,n]\!]$, and these terms are all equal to each other because 2-forms commute with each other under wedge product (see [8] prop.14.11(c)). Thus

$$\Omega^n = n!(e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*) \neq 0.$$

(c): By (b), $(\omega^n)_p = (\omega_p)^n \neq 0$ for every $p \in M$, so $\omega^n \neq 0$. It is of degree $2n$. It is a volume form. Due to [8] prop.15.5, this form determines an orientation for M. Thus, non-orientable two-dimensional manifolds like Möbius strip and Klein bottle cannot have symplectic structures.

(d): Suppose Ω is degenerate, then there is a nonzero vector v such that $\widetilde{\Omega} = i_v\Omega = 0$. By [8] Lemma 14.11, we see that

$$\begin{aligned}
i_v\Omega^n &= i_v(\Omega \wedge \Omega^{n-1}) = (i_v\Omega) \wedge \Omega^{n-1} + \Omega \wedge (i_v\Omega^{n-1}) \\
&= (i_v\Omega) \wedge \Omega^{n-1} + \Omega \wedge ((i_v\Omega) \wedge \Omega^{n-2} + \Omega \wedge (i_v\Omega^{n-2})) \\
&= (i_v\Omega) \wedge \Omega^{n-1} + (i_v\Omega) \wedge \Omega^{n-1} + \Omega^2 \wedge (i_v\Omega^{n-2}) \\
&= \cdots = (i_v\Omega) \wedge \Omega^{n-1} + (i_v\Omega) \wedge \Omega^{n-1} + \cdots + (i_v\Omega) \wedge \Omega^{n-1} + \underbrace{\Omega^{n-1} \wedge (i_v\Omega)}_{=(-1)^{2(n-1)\cdot 1}(i_v\Omega)\wedge\Omega^{n-1}} \\
&= n(i_v\Omega) \wedge \Omega^{n-1} \\
&= 0
\end{aligned}$$

Note that $\Omega^2 \neq 0$. (we cannot apply [8] prop.14.11(c) here since $(-1)^{2\cdot 2} = 1$ instead of -1 now.) We can extend v to a basis $(v_1, \cdots, v_{2n})$ of V where $v_1 = v$. Then $\Omega^n(v_1, \cdots, v_{2n}) = 0$, which implies $\Omega^n = 0$. ◆

Exercise 2.5.2. *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let ω^n be the volume form obtained by wedging ω with itself n times.*

(a) Show that, if M is compact, the de Rham cohomology class $[\omega^n] \in H^{2n}(M; \mathbb{R})$ is non-zero. [Hint: Stokes' theorem.]

(b) Conclude that $[\omega]$ itself is non-zero (in other words, that ω is not exact).

(c) Show that if $n > 1$ there are no symplectic structures on the sphere $\mathbb{S}^{2n}$.

Part II

Symplectomorphisms

Chapter 3

Lagrangian Submanifolds

3.1 Some Notations on Submanifolds

We recall the following terminologies from [8]:

proper map: A map is proper if the preimage of any compact set is compact.

Rank: Given a smooth map $F : M \to N$ for smooth manifolds M and N with or without boundary and a point $p \in M$, we define the **rank of** F **at** p to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$; it is the rank of the Jacobian matrix $J_{\hat{F}(\hat{p})}$ of F in any smooth chart, or the dimension of $\operatorname{Im}(dF_p) \subseteq T_{F(p)}N$. If F has the same rank r at every point, we say that it has **constant rank**, and write $\operatorname{rank} F = r$.

Smooth immersion: if dF_p is surjective at each point, or equivalently, $\operatorname{rank} F = \dim N$.

Smooth submersion: if dF_p is injective at each point, or equivalently, $\operatorname{rank} F = \dim M$.

Local diffeomorphism: $F : M \to N$ is a local diffeomorphism if every point $p \in M$ has a neighborhood U such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Topological embedding: A map $f : X \to Y$ is a topological embedding if it is a homeomorphism onto its image $F(X) \subseteq Y$.

Smooth embedding: both a smooth immersion and a topological embedding.

Embedded submanifold: let M be a smooth manifold with or without boundary. Subset $S \subseteq M$ equipped with subspace topology and a smooth structure with respect to which the inclusion $i : S \hookrightarrow M$ is a smooth embedding. Embedded submanifold is also called regular submanifold.

Properly embedded submanifold: An embedded submanifold is properly embedded if the inclusion i is also a proper map.

Immersed submanifold: An immersed submanifold of M is a subset $S \subseteq M$ endowed with a topology (not neccessarily the subspace topology) with respect to which it is a topological manifold, and a smooth structure with respect to which the inclusion map i is a smooth immersion. Note that every embedded submanifold is immersed directly by definition.

We will use the following conventions for this note.

(1) An **embedding** is assumed to be a smooth embedding. An **immersion** is assumed to be a smooth immersion.

(2) A **closed embedding** is a proper injective smooth immersion. Its name is justified: a closed embedding is an embedding by [8] p.87 proposition 4.22.

(3) A **submanifold** of M is assumed to be an embedded submanifold with $i(X)$ closed in M where i is the inclusion $i : X \hookrightarrow M$. Note that by [8] proposition 5.5, a map $i : X \to M$ is a closed embedding if and only if i is an embedding and its image $i(X)$ is closed in M.

3.2 Lagrangian Submanifolds of T^*X

Definition 3.2.1. *Let (M, ω) be a $2n$-dimensional symplectic manifold. A submanifold Y of M is a* ***Lagrangian submanifold*** *if, at each $p \in Y, T_pY$ is a lagrangian subspace of T_pM, i.e., $\omega_p|_{T_pY} \equiv 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is lagrangian if and only if $i^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$.*

Let X be an n-dimensional manifold, with $M = T^*X$ its cotangent bundle. If $x_1, \ldots, x_n$ are coordinates on $U \subseteq X$, with associated cotangent coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ on T^*U, then the tautological 1-form on T^*X is

$$\alpha = \sum \xi_i dx_i$$

and the canonical 2 -form on T^*X is

$$\omega = -d\alpha = \sum dx_i \wedge d\xi_i.$$

The **zero section** of T^*X

$$X_0 := \{(x, \xi) \in T^*X \mid \xi = 0 \text{ in } T_x^*X\}$$

is an n-dimensional submanifold of T^*X whose intersection with T^*U is given by the equations $\xi_1 = \ldots = \xi_n = 0$. Clearly $\alpha = \sum \xi_i dx_i$ vanishes on $X_0 \cap T^*U$. In particular, if $i_0 : X_0 \hookrightarrow T^*X$ is the inclusion map, we have $\forall p \in X_0 \cap T^*U$, $\alpha_p(d(i_0)_p(u), d(i_0)_p(v)) = 0$, where $p \in X_0$ on which $d(i_0)_p : T_pX_0 \to T_{i(p)}M$ is defined and $p \in T^*U$ on which α_p is defined. Thus, $i_0^*\alpha = 0$ and

$$i_0^*\omega = i_0^*(-d\alpha) = -i_0^*(d\alpha) \xlongequal{\text{[8] prop.14.26}} -d(i_0^*\alpha) = 0.$$

The submanifold X_0 is then Lagrangian.

What are all the Lagrangian submanifolds of T^*X which are "C^1-close to X_0"*?

Let X_μ be (the image of) another section, that is, an n-dimensional submanifold of T^*X of the form

$$X_\mu = \{(x, \mu_x) \mid x \in X, \mu_x \in T_x^*X\} \qquad (\star)$$

where the covector μ_x depends smoothly on x, and $\mu : X \to T^*X$ is a de Rham 1-form. Relative to the inclusion $i : Y \hookrightarrow T^*X$ of submanifold $Y \subseteq T^*X$ and the cotangent projection $\pi : T^*X \to X, Y$ is of the form $(\star)$ if and only if $\pi \circ i : Y \to X$ is a diffeomorphism.

When is such an X_μ Lagrangian?

Proposition 3.2.2. *Let X_μ be of the form $(\star)$, and let μ be the associated de Rham 1-form. Denote by $s_\mu : X \to T^*X, x \mapsto (x, \mu_x)$, the 1-form μ regarded exclusively as a map. Notice that the image of s_μ is X_μ. Let α be the tautological 1-form on T^*X. Then*

$$s_\mu^*\alpha = \mu.$$

*Recall the C^k norm from Wolfram; we form a similar C^k-norm for section X of cotangent bundle T^*M by $\|X\|_{C^k} = \sum_{n=1}^k \sup_{p\in M} \|\nabla^n X(p)\|$ Two sections X and Y are C^k-close if the C^k-norm of their difference is small. This means that not only are the sections themselves close, but their derivatives up to order k are also close at each point on the manifold.

Proof. By definition of α (previous lecture), $\alpha_p = (d\pi_p)^* \xi$ at $p = (x, \xi) \in M$. For $p = s_\mu(x) = (x, \mu_x)$, we have $\alpha_p = (d\pi_p)^* \mu_x$. Then

$$\left(s_\mu^* \alpha\right)_x = (ds_\mu)_x^* \alpha_p = (ds_\mu)_x^* (d\pi_p)^* \mu_x = (d\underbrace{(\pi \circ s_\mu)}_{\mathrm{id}_X})_x^* \mu_x = \mu_x.$$

∎

Suppose that X_μ is an n-dimensional submanifold of T^*X of the form ($\star$), with associated de Rham 1-form μ. Then $s_\mu : X \to T^*X$ is an embedding with image X_μ, and there is a diffeomorphism $\tau : X \to X_\mu, \tau(x) := (x, \mu_x)$, such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "s_\mu"] \arrow[dr, "\tau"', "\simeq"] & & T^*X \\ & X_\mu \arrow[ur, hook, "i"'] & \end{tikzcd}$$

We want to express the condition of X_μ being lagrangian in terms of the form μ :

$$\begin{aligned} X_\mu \text{ is lagrangian } & \Longleftrightarrow i^* d\alpha = 0 \\ & \Longleftrightarrow \tau^* i^* d\alpha = 0 \\ & \Longleftrightarrow (i \circ \tau)^* d\alpha = 0 \\ & \Longleftrightarrow s_\mu^* d\alpha = 0 \\ & \Longleftrightarrow d s_\mu^* \alpha = 0 \\ & \Longleftrightarrow d\mu = 0 \\ & \Longleftrightarrow \mu \text{ is closed .} \end{aligned}$$

Therefore, we have

Proposition 3.2.3. *there is a one-to-one correspondence between the set of lagrangian submanifolds of T^*X of the form $(\star)$ and the set of closed 1-forms on X.*

When X is simply connected, $H^1_{\text{deRham}}(X) = 0$, so every closed 1-form μ is equal to df for some $f \in C^\infty(X)$. Any such primitive f is then called a **generating function** for the lagrangian submanifold X_μ associated to μ. (Two functions generate the same lagrangian submanifold if and only if they differ by a locally constant function.) On arbitrary manifolds X, functions $f \in C^\infty(X)$ originate lagrangian submanifolds as images of df.

Exercise 3.2.4. *Check that, if X is compact (and not just one point) and $f \in C^\infty(X)$, then $\#\left(X_{df} \cap X_0\right) \geq 2$.*

There are lots of lagrangian submanifolds of T^*X not covered by the description in terms of closed 1 -forms, starting with the cotangent fibers.

3.3 Conormal Bundles

Let S be any k-dimensional submanifold of an n-dimensional manifold X.

Definition 3.3.1. *The **conormal space** at $x \in S$ is*

$$N_x^* S = \left\{\xi \in T_x^* X \mid \xi(v) = 0, \text{ for all } v \in T_x S\right\}.$$

*The **conormal bundle** of S is*

$$N^* S = \left\{(x, \xi) \in T^* X \mid x \in S, \xi \in N_x^* S\right\}.$$

Exercise 3.3.2. *The conormal bundle N^*S is an n-dimensional submanifold of T^*X. Hint: Use coordinates on X [*] adapted to S.*

Proposition 3.3.3. *Let $i : N^*S \hookrightarrow T^*X$ be the inclusion, and let α be the tautological 1-form on T^*X. Then*

$$i^*\alpha = 0.$$

Proof. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on X centered at $x \in S$ and adapted to S, so that $\mathcal{U} \cap S$ is described by $x_{k+1} = \ldots = x_n = 0$ (we can do this by [8] Theorem 5.8 (local slice criterion for embedded submanifolds)). Let $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the associated cotangent coordinate system. The submanifold $N^*S \cap T^*\mathcal{U}$ is then described by

$$x_{k+1} = \ldots = x_n = 0 \quad \text{and} \quad \xi_1 = \ldots = \xi_k = 0.$$

Since $\alpha = \sum \xi_i dx_i$ on $T^*\mathcal{U}$, we conclude that, at $p \in N^*S$,

$$(i^*\alpha)_p = \alpha_p|_{T_p(N^*S)} = \sum_{i>k} \xi_i dx_i \Big|_{\operatorname{span}\left\{\frac{\partial}{\partial x_i}, i \leq k\right\}} = 0.$$

■

Corollary 3.3.4. *For any submanifold $S \subset X$, the conormal bundle N^*S is a lagrangian submanifold of T^*X.*

Taking $S = \{x\}$ to be one point, the conormal bundle $L = N^*S = T_x^*X$ is a cotangent fiber. Taking $S = X$, the conormal bundle $L = X_0$ is the zero section of T^*X

3.4 Applications to Symplectomorphisms

Let (M_1, ω_1) and (M_2, ω_2) be two $2n$-dimensional symplectic manifolds. Given a diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$, when is it a symplectomorphism? (i.e., when is $\varphi^*\omega_2 = \omega_1$?)

Consider the two projection maps

$$\begin{array}{ccccc} (p_1, p_2) & & M_1 \times M_2 & & (p_1, p_2) \\ \downarrow & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \downarrow \\ p_1 & M_1 & & M_2 & p_2 \end{array}$$

Then $\omega = (\mathrm{pr}_1)^* \omega_1 + (\mathrm{pr}_2)^* \omega_2$ is a 2 -form on $M_1 \times M_2$ which is closed,

$$d\omega = (\mathrm{pr}_1)^* \underbrace{d\omega_1}_{0} + (\mathrm{pr}_2)^* \underbrace{d\omega_2}_{0} = 0$$

and symplectic by exercise 2.5.1 (d),

$$\omega^{2n} = \binom{2n}{n} \left((\mathrm{pr}_1)^* \omega_1\right)^n \wedge \left((\mathrm{pr}_2)^* \omega_2\right)^n \neq 0.$$

More generally, if $\lambda_1, \lambda_2 \in \mathbb{R} \backslash \{0\}$, then $\lambda_1 (\mathrm{pr}_1)^* \omega_1 + \lambda_2 (\mathrm{pr}_2)^* \omega_2$ is also a symplectic form on $M_1 \times M_2$. Take $\lambda_1 = 1, \lambda_2 = -1$ to obtain the **twisted product form** on $M_1 \times M_2$:

$$\widetilde{\omega} = (\mathrm{pr}_1)^* \omega_1 - (\mathrm{pr}_2)^* \omega_2.$$

[*] A coordinate chart $(\mathcal{U}, x_1, \ldots, x_n)$ on X is adapted to a k-dimensional submanifold S if $S \cap \mathcal{U}$ is described by $x_{k+1} = \ldots = x_n = 0$.

The graph of a diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$ is the $2n$-dimensional submanifold of $M_1 \times M_2$:

$$\Gamma_\varphi := \operatorname{Graph} \varphi = \{(p, \varphi(p)) \mid p \in M_1\}.$$

The submanifold Γ_φ is an embedded image of M_1 in $M_1 \times M_2$, the embedding being the map

$$\begin{aligned} \gamma : M_1 &\longrightarrow M_1 \times M_2 \\ p &\longmapsto (p, \varphi(p)). \end{aligned}$$

Proposition 3.4.1. *A diffeomorphism φ is a symplectomorphism if and only if Γ_φ is a Lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.*

Proof. The graph Γ_φ is Lagrangian if and only if $\gamma^* \widetilde{\omega} = 0$. But

$$\begin{aligned} \gamma^* \widetilde{\omega} &= \gamma^* \operatorname{pr}_1^* \omega_1 - \gamma^* \operatorname{pr}_2^* \omega_2 \\ &= (\operatorname{pr}_1 \circ \gamma)^* \omega_1 - (\operatorname{pr}_2 \circ \gamma)^* \omega_2 \end{aligned}$$

and $\operatorname{pr}_1 \circ \gamma$ is the identity map on M_1 whereas $\operatorname{pr}_2 \circ \gamma = \varphi$. Therefore,

$$\gamma^* \widetilde{\omega} = 0 \quad \Longleftrightarrow \quad \varphi^* \omega_2 = \omega_1$$

■

3.5 Tautological Form and Symplectomorphisms

Exercise 3.5.1. *Let (M, ω) be a symplectic manifold, and let α be a 1 -form such that*

$$\omega = -d\alpha$$

Show that there exists a unique vector field v such that its interior product with ω is α, i.e., $\imath_v \omega = -\alpha$. Prove that, if g is a symplectomorphism which preserves α (that is, $g^* \alpha = \alpha$), then g commutes with the one-parameter group of diffeomorphisms generated by v, i.e.,

$$(\exp tv) \circ g = g \circ (\exp tv).$$

[Hint: Recall that, for $p \in M$, $(\exp tv)(p)$ is the unique curve in M solving the ordinary differential equation

$$\begin{cases} \frac{d}{dt}(\exp tv(p)) = v(\exp tv(p)) \\ (\exp tv)(p)|_{t=0} = p \end{cases}$$

for t in some neighborhood of 0. Show that $g \circ (\exp tv) \circ g^{-1}$ is the one-parameter group of diffeomorphisms generated by $g_* v$. (The push-forward of v by g is defined by $(g_* v)_{g(p)} = dg_p(v_p)$.) Finally check that g preserves v (that is, $g_* v = v$).

Solution. Since ω is a symplectic form, we by definition see that $\forall p \in M$,

$$\begin{aligned} \widetilde{\omega}_p : T_p M &\to T_p^* M \\ v &\mapsto i_v \omega \end{aligned}$$

is an isomorphism. Thus, $\exists! v$ s.t. $i_v \omega = -\alpha$. For second part, note that $\exp tv$ is the flow generated by v, and [8] p.215 Cor.9.14 then gives that $g \circ (\exp tv) \circ g^{-1}$ is the flow generated by $g_* v$. To show $g_* v = v$, we observe that

$$i_v \omega = \alpha = g^* \alpha = g^*(i_v \omega) \overset{(*)}{=\!=} i_{f_* v}(f^* \omega) \overset{(**)}{=\!=} i_{f_* v} \omega \implies f_* v = v \text{ by uniqueness.}$$

where $(*)$ is due to pullback commutes with interior multiplication, and $(**)$ is due to g being a symplectomorphism.] ♦

Exercise 3.5.2. *Let X be an arbitrary n-dimensional manifold, and let $M = T^*X$. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on X, and let $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ be the corresponding coordinates on $T^*\mathcal{U}$. Show that, when α is the tautological 1-form on M (which, in these coordinates, is $\sum \xi_i dx_i$), the vector field v in the previous exercise is just the vector field $\sum \xi_i \frac{\partial}{\partial \xi_i}$. Let $\exp tv, -\infty < t < \infty$, be the one-parameter group of diffeomorphisms generated by v. Show that, for every point $p = (x, \xi)$ in M,*

$$(\exp tv)(p) = p_t \quad \textit{where} \quad p_t = \left(x, e^t \xi\right).$$

Solution. Simply plug in $W = \frac{\partial}{\partial x_i}$ and $W = \frac{\partial}{\partial \xi_i}$ into $i_v\omega = \left[-\sum_{i=1}^n dx_i \wedge d\xi_i\right](V, W) = \alpha = \left[\sum_{i=1}^n \xi_i dx_i\right](W)$ to get $V = \sum \xi_i \frac{\partial}{\partial \xi_i}$. Now to finish, plug in $V = \sum \xi_i \frac{\partial}{\partial \xi_i} = (0, \xi)$ for the following equation to solve:

$$\begin{cases} \frac{d}{dt}(\boldsymbol{x}(t), \boldsymbol{\xi}(t)) = (x, \xi)(\boldsymbol{x}(t), \boldsymbol{\xi}(t)) \\ (\boldsymbol{x}(t), \boldsymbol{\xi}(t))(p)|_{t=0} = p \end{cases}$$

that is

$$\begin{cases} \frac{d}{dt} 0 = 0 \cdot \boldsymbol{x}(t) \\ \frac{d}{dt} \boldsymbol{\xi}(t) = \xi \cdot \boldsymbol{\xi}(t) \\ \boldsymbol{x}(0) = x \\ \boldsymbol{\xi}(0) = \xi \end{cases}$$

(where we used bold forms to avoid notation abuse.) The solution is $e^{1/\xi}\boldsymbol{\xi}(t) = e^{1/\xi}\xi e^t \implies \boldsymbol{\xi}(t) = \xi e^t \implies p_t = (x, e^t\xi)$. ♦

Exercise 3.5.3. *Let M be as in exercise 2. Show that, if g is a symplectomorphism of M which preserves α, then*

$$g(x, \xi) = (y, \eta) \implies g(x, \lambda\xi) = (y, \lambda\eta)$$

for all $(x, \xi) \in M$ and $\lambda \in \mathbb{R}$. Conclude that g has to preserve the cotangent fibration, i.e., show that there exists a diffeomorphism $f : X \to X$ such that $\pi \circ g = f \circ \pi$, where $\pi : M \to X$ is the projection map $\pi(x, \xi) = x$. Finally prove that $g = f_\#$, the map $f_\#$ being the symplectomorphism of M lifting f. Hint: Suppose that $g(p) = q$ where $p = (x, \xi)$ and $q = (y, \eta)$. Combine the identity

$$(dg_p)^* \alpha_q = \alpha_p$$

with the identity

$$d\pi_q \circ dg_p = df_x \circ d\pi_p.$$

(The first identity expresses the fact that $g^*\alpha = \alpha$, and the second identity is obtained by differentiating both sides of the equation $\pi \circ g = f \circ \pi$ at p.)

Solution. By exercise 1, we have

$$(\exp tv) \circ g = g \circ (\exp tv)$$
$$(y, e^t\eta) = g(x, e^t\xi)$$

Now, for any $\lambda \in \mathbb{R}$, let $t = \ln\lambda$. Then

$$g(x, \lambda\xi) = g(x, e^t\xi) = (y, e^t\eta) = (y, \lambda\eta).$$

For the second claim, just define f as $x \mapsto y$ whenever $g(x, \xi) = (y, \eta)$.

For the third claim, we observe

$$(dg_p)^* \alpha_q = \alpha_p$$

and
$$d\pi_q \circ dg_p = df_x \circ d\pi_p.$$
To show $g = f_\#$, note that
$$g(x,\xi) = (y,\eta)$$
$$f_\#(x,(df_x)^*\eta) = (f(x),\eta)$$
Thus, it suffices to show $(df_x)^*\eta = \xi$. The coordinate-free pointwise definition of the tautological 1-form α is given by $\alpha_p = (d\pi_p)^*\xi$ and $\alpha_q = (d\pi_q)^*\eta$. Thus,
$$(d\pi_p)^*\xi = (dg_p)^*\alpha_q = (\underbrace{d\pi_q \circ dg_p}_{df_x \circ d\pi_p})^*\eta = (d\pi_p)^*(df_x)^*\eta$$
Then notice that df_x and dg_p being isomorphisms plus $d\pi_q \circ dg_p = df_x \circ d\pi_p$ together imply that $d\pi_p$ is injective and thus above equation implies $\xi = (df_x)^*\eta$. ♦

Exercise 3.5.4. *Let M be as in exercise 2, and let h be a smooth function on X. Define $\tau_h : M \to M$ by setting*
$$\tau_h(x,\xi) = (x, \xi + dh_x)\,.$$
Prove that
$$\tau_h^*\alpha = \alpha + \pi^* dh$$
where π is the projection map $\begin{array}{cc} M & (x,\xi) \\ \downarrow \pi & \downarrow \\ X & x \end{array}$

Deduce that
$$\tau_h^*\omega = \omega,$$
i.e., that τ_h is a symplectomorphism.

Solution. Note that $h \in \Omega^0(M)$ and $df \in \Omega^1(M)$ By a local computation, we see that
$$d(\tau_h)_p(v_x, v_\xi) = (v_x, v_\xi + *)$$
where $\tau_h(p) = q$, $p = (x,\xi)$, $q = (x, \xi + dh_x)$, $v = (v_x, v_\xi)$, and $*$ means some unimportant term. Now, the coordinate-free pointwise definition of the tautological 1-form α is given by $\alpha_p = (d\pi_p)^*\xi = \xi_x \circ d\pi_p$ and $\alpha_q = (d\pi_q)^*(\xi + dh_x) = (\xi_x + dh_x) \circ d\pi_q$. Thus, to show $\tau_h^*\alpha = \alpha + \pi^* dh$, we compute that
$$\text{LHS} = \alpha_q(v_x, v_\xi + *) = (\xi_x + dh_x)(d\pi_q(v_x, v_\xi + *)) = (\xi_x + dh_x)(v_x) = \xi_x(v_x) + (dh)_x(v_x)$$
and
$$\text{RHS} = \xi_x(d\pi_p(v_x, v_\xi)) + (dh)_{\pi(p)=x}(d\pi_p(v_x, v_\xi)) = \xi_x(v_x) + (dh)_x(v_x)$$
To show τ_h is a symplectomorphism, we observe that
$$\tau_h^*\omega = \tau_h^*(-d\alpha) = -d\tau_h^*\alpha = -d(\alpha + \pi^* dh) = -d\alpha - \underbrace{\pi^* ddh}_{=0 \text{ as } d\circ d=0} = \omega$$
♦

Chapter 4

Generating Functions

4.1 Constructing Symplectomorphisms

We start with a review on the following identification:

Let X_1 and X_2 be two n-dimensional manifolds. Then

$$\begin{aligned} TX_1 \times TX_2 &\simeq T(X_1 \times X_2) \\ T^*X_1 \times T^*X_2 &\simeq T^*(X_1 \times X_2) \end{aligned}$$

For the first one, we consider the following two maps

$$\begin{aligned} f : T_{(p,q)}(X_1 \times X_2) &\to T_pX_2 \times T_qX_2 \\ v &\mapsto \big(d(\pi_1)_{(p,q)}(v), d(\pi_2)_{(p,q)}(v)\big) \\ g : T_pX_2 \times T_qX_2 &\to T_{(p,q)}(X_1 \times X_2) \\ (v_1, v_2) &\mapsto d(i_1)_p(v_1) + d(i_2)_q(v_2) \end{aligned} \tag{4.1}$$

where $\pi_{1,2} : X_1 \times X_2 \to X_{1,2}$ are canonical projections and $i_1 : X_1 \to X_1 \times X_2$ sends x_1 to (x_1, q) and $i_2 : X_2 \to X_1 \times X_2$ sends x_2 to (p, x_2). Since $\pi_1 \circ i_1 = \mathrm{id}_1, \pi_2 \circ i_2 = \mathrm{id}_2$ and $\pi_1 \circ i_2$, $\pi_2 \circ i_1$ are constant maps, we have

$$\begin{aligned} (f \circ g)(v_1, v_2) &= f\left(d\left(i_1\right)_p\left(v_1\right) + d\left(i_2\right)_q\left(v_2\right)\right) \\ &= \left(d\left(\pi_1\right)_{(p,q)}\left(d\left(i_1\right)_p\left(v_1\right) + d\left(i_2\right)_q\left(v_2\right)\right), d\left(\pi_2\right)_{(p,q)}\left(d\left(i_1\right)_p\left(v_1\right) + d\left(i_2\right)_q\left(v_2\right)\right)\right) \\ &= \left(d\left(\pi_1 \circ i_1\right)_p\left(v_1\right) + d\left(\pi_1 \circ i_2\right)_q\left(v_2\right)\right), d\left(\pi_2 \circ i_1\right)_p\left(v_1\right) + d\left(\pi_2 \circ i_2\right)_q\left(v_2\right)\right)\right) \\ &= (v_1, v_2) \end{aligned}$$

Thus f is surjective. Since $T_{(p,q)}(X_1 \times X_2)$ and $T_pX_1 \times T_qX_2$ have the same dimension, f is a linear isomorphism. Denoting above f, g as $f_{(p,q)}$ and $g_{(p,q)}$, we define $\Phi : T(X_1 \times X_2) \to TX_1 \times TX_2$ as the map bundling up the maps $f_{(p,q)}$'s with Ψ bundling $g_{(p,q)}$'s. We show the smoothness of Φ. That Φ and Ψ are inverses of each other and g is similarly smooth are then clear. With local coordinates $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$, we have $\hat{\pi}_1(x_1, \cdots, x_n, y_1, \cdots, y_n) = (x_1, \cdots, x_n)$, $\hat{\pi}_2(x_1, \cdots, x_n, y_1, \cdots, y_n) = (y_1, \cdots, y_n)$. Then

$$d(\pi_1)_{(p,q)}\left(\underbrace{u^i \left.\frac{\partial}{\partial x_i}\right|_{(p,q)} + w^i \left.\frac{\partial}{\partial y_i}\right|_{(p,q)}}_{=v}\right) = \sum_{i=1}^n \left(u^i \sum_{j=1}^n \underbrace{\frac{\partial \hat{\pi}_1^j}{\partial x_i}\widehat{(p,q)}}_{=\delta_{ij}} \left.\frac{\partial}{\partial x_j}\right|_p + w^i \sum_{j=1}^n \underbrace{\frac{\partial \hat{\pi}_1^j}{\partial y_i}\widehat{(p,q)}}_{=0} \left.\frac{\partial}{\partial x_j}\right|_q\right) = u^i \left.\frac{\partial}{\partial x_i}\right|_p$$

and similarly,

$$d(\pi_2)_{(p,q)}(v) = w^i \left.\frac{\partial}{\partial y_i}\right|_q$$

Thus, in local coordinates,

$$\Phi\left(x_1, \cdots, x_n, y_1, \cdots, y_n, u^1, \cdots, u^n, w^1, \cdots, w^n\right) = \left(x_1, \cdots, x_n, u^1, \cdots, u^n, y_1, \cdots, y_n, w^1, \cdots, w^n\right)$$

which is of course smooth.

For the second diffeomorphism $T^*(X_1 \times X_2) \simeq T^*X_1 \times T^*X_2$, consider the following maps

$$\begin{aligned} f : T^*_{(p,q)}(X_1 \times X_2) &\to T^*_p X_1 \times T^*_q X_2 \\ \omega &\mapsto \left(d(i_1)^*_p(\omega), d(i_2)^*_q(\omega)\right) \\ g : T^*_p X_1 \times T^*_q X_2 &\to T^*_{(p,q)}(X_1 \times X_2) \\ (\alpha, \beta) &\mapsto d(\pi_1)^*_{(p,q)}(\alpha) + d(\pi_2)^*_{(p,q)}(\beta) \end{aligned} \tag{4.2}$$

Bundling them and writing them in coordinates will give us $\Phi : T^*(X_1 \times X_2) \to T^*X_1 \times T^*X_2$ defined by

$$\Phi((p,q),\omega) = ((p,\alpha),(q,\beta))$$

where $\omega = \sum_{i=1}^n a_i dx^i + \sum_{j=1}^n b_j dy^j, \alpha = \sum_{i=1}^n a_i dx^i$, and $\beta = \sum_{j=1}^n b_j dy^j$. And the inverse map $\Psi : T^*M \times T^*N \to T^*(M \times N)$ is defined by:

$$\Psi((p,\alpha),(q,\beta)) = ((p,q),\omega)$$

where $\alpha = \sum_{i=1}^m a_i dx^i$ and $\beta = \sum_{j=1}^n b_j dy^j$, and $\omega = \alpha + \beta$.

It is easy to verify the bijectivity and smoothness.

Remark 4.1.1. Changing the setting "Let X_1 and X_2 be two n-dimensional manifolds" to "X_1 and X_2 be manifolds of dimension n and m" won't change the conclusion. ♠

Now for the two n-dimensional manifolds X_1 and X_2, with cotangent bundles $M_1 = T^*X_1$, $M_2 = T^*X_2$, tautological 1-forms α_1, α_2, and canonical 2-forms ω_1, ω_2. Under the natural identification

$$M_1 \times M_2 = T^*X_1 \times T^*X_2 \simeq T^*\left(X_1 \times X_2\right),$$

the tautological 1-form * on $T^*\left(X_1 \times X_2\right)$ is

$$\alpha = \left(\mathrm{pr}_1\right)^* \alpha_1 + \left(\mathrm{pr}_2\right)^* \alpha_2$$

where $\mathrm{pr}_i : M_1 \times M_2 \to M_i, i = 1, 2$ are the two projections. The canonical 2-form on $T^*\left(X_1 \times X_2\right)$ is

$$\omega = -d\alpha = -d\mathrm{pr}_1^*\alpha_1 - d\,\mathrm{pr}_2^*\,\alpha_2 = \mathrm{pr}_1^*\,\omega_1 + \mathrm{pr}_2^*\,\omega_2.$$

In order to describe the twisted form $\widetilde{\omega} = \mathrm{pr}_1^*\,\omega_1 - \mathrm{pr}_2^*\,\omega_2$, we define an **involution** of $M_2 = T^*X_2$ by

$$\begin{aligned} \sigma_2 : \quad M_2 &\longrightarrow M_2 \\ (x_2, \xi_2) &\longmapsto (x_2, -\xi_2) \end{aligned}$$

which yields $\sigma_2^*\alpha_2 = -\alpha_2$.* Let $\sigma = \mathrm{id}_{M_1} \times \sigma_2 : M_1 \times M_2 \to M_1 \times M_2$. Then*

$$\sigma^*\widetilde{\omega} = \mathrm{pr}_1^*\,\omega_1 + \mathrm{pr}_2^*\,\omega_2 = \omega$$

*See appendix for the computation.
*Same.
*Same.

Proposition 4.1.2. *If Y is a Lagrangian submanifold of $(M_1 \times M_2, \omega)$, then its "twist" $Y^\sigma := \sigma(Y)$ is a Lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$:*

Proof. Dimensionality is obvious; for any $u, v \in T_p Y^\sigma$, there are $q \in Y$ and $u', v' \in T_q Y$ such that $\sigma(q) = p$ and $u = d\sigma_q(u'), v = d\sigma_q(v')$ (σ is a diffeomorphism). Then for $i : Y^\sigma \to M_1 \times M_2$ and $i_Y : Y \to M_1 \times M_2$ we have $i = \sigma \circ i_Y$ and

$$\widetilde{\omega}_p(di_p(u), di_p(v)) = \widetilde{\omega}_{\sigma(q)}\left(d(\sigma \circ i_Y)_q(u'), d(\sigma \circ i_Y)_q(v')\right) \xlongequal{\sigma^*\widetilde{\omega}} \omega_q\left(d(i_Y)_q(u'), d(i_Y)_q(v')\right) \xlongequal{Y \text{ Lagrangian}} 0.$$

■

Recipe for producing symplectomorphisms $M_1 = T^*X_1 \to M_2 = T^*X_2$:

1. Start with a Lagrangian submanifold Y of $(M_1 \times M_2, \omega)$.
2. Twist it to obtain a Lagrangian submanifold Y^σ of $(M_1 \times M_2, \widetilde{\omega})$.
3. Check whether Y^σ is the graph of some diffeomorphism $\varphi : M_1 \to M_2$.
4. If it is, then φ is a symplectomorphism (by Proposition 3.4.1).

Let $i : Y^\sigma \hookrightarrow M_1 \times M_2$ be the inclusion map

$$\begin{tikzcd} & Y^\sigma \arrow[dl, "\mathrm{pr}_1 \circ i"'] \arrow[dr, "\mathrm{pr}_2 \circ i"] & \\ M_1 \arrow[rr, "\varphi?"'] & & M_2 \end{tikzcd}$$

Step 3 amounts to checking whether $\mathrm{pr}_1 \circ i$ and $\mathrm{pr}_2 \circ i$ are diffeomorphisms. If yes, then $\varphi := (\mathrm{pr}_2 \circ i) \circ (\mathrm{pr}_1 \circ i)^{-1}$ is a diffeomorphism.

In order to obtain lagrangian submanifolds of $M_1 \times M_2 \simeq T^*(X_1 \times X_2)$, we can use the method of generating functions.

4.2 Method of Generating Functions

For any $f \in C^\infty(X_1 \times X_2)$, df is a closed 1-form on $X_1 \times X_2$ ($d(df) = 0$). The **Lagrangian submanifold generated by** f is

$$Y_f := \left\{ \left((x,y), (df)_{(x,y)}\right) \mid (x,y) \in X_1 \times X_2 \right\}.$$

We adopt the notation

$$\begin{aligned} d_x f &:= (df)_{(x,y)} \in T^*_{(x,y)}(X_1 \times X_2) \simeq T^*_x X_1 \times T^*_y X_2 \text{ projected to } T^*_x X_1 \times \{0\}, \\ d_y f &:= (df)_{(x,y)} \in T^*_{(x,y)}(X_1 \times X_2) \simeq T^*_x X_1 \times T^*_y X_2 \text{ projected to } \{0\} \times T^*_y X_2 \end{aligned} \tag{4.3}$$

By (4.2), we have

$$(df)_{(x,y)} = d(\pi_1)^*_{(x,y)}\left(\underbrace{d(i_1)^*_x\left((df)_{(x,y)}\right)}_{=:d_x f}\right) + d(\pi_2)^*_{(x,y)}\left(\underbrace{d(i_2)^*_y\left((df)_{(x,y)}\right)}_{=:d_y f}\right) \tag{4.4}$$

Let $(\mathcal{U}_1, x_1, \ldots, x_n), (\mathcal{U}_2, y_1, \ldots, y_n)$ be coordinate charts for X_1, X_2, with $(T^*\mathcal{U}_1, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ and $(T^*\mathcal{U}_2, y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$ the associated charts for M_1, M_2. By [8] p.363,

$$(df)_{(x,y)} = df_{(x,y)} = \sum_i \frac{\partial f}{\partial x_i} dx_i|_{(x,y)} + \sum_i \frac{\partial f}{\partial y_i} dy_i|_{(x,y)}.$$

Then, $d(i_1)_x^*(dx_i|_{(x,y)}) = dx_i|_x$, $d(i_1)_x^*(dy_i|_{(x,y)}) = 0$; similarly for the other component. Thus, in coordinates,

$$d_x f = \sum_i \frac{\partial f}{\partial x_i} dx_i|_x, \quad d_y f = \sum_i \frac{\partial f}{\partial x_i} dy_i|_y$$

This enables us to write

$$Y_f = \{(x, y, d_x f, d_y f) \mid (x, y) \in X_1 \times X_2\}$$

and

$$Y_f^\sigma = \{(x, y, d_x f, -d_y f) \mid (x, y) \in X_1 \times X_2\}.$$

Remark 4.2.1. Here is another way to think about $d_x f$ and $d_y f$. If we for each point (x, y) define $F : X \to \mathbb{R}$ by $F(x) = f(x, y)$, then we observe that $F = f \circ i_1$. Then

$$dF_x = d_{i_1(x)} f \circ d_x i_1 = d_{(x,y)} f \circ d_x i_1 = d_x i_1^* \circ d_{(x,y)} f$$

Similarly, $G(y) = f(x, y)$ gives

$$dG_y = d_y i_2 \circ d_{(x,y)} f$$

♠

When Y_f^σ is in fact the graph of a diffeomorphism $\varphi : M_1 \to M_2$, we call φ the **symplectomorphism generated by** f, and call f the **generating function of** $\varphi : M_1 \to M_2$.

So when is Y_f^σ the graph of a diffeomorphism $\varphi : M_1 \to M_2$?

The set Y_f^σ is the graph of $\varphi : M_1 \to M_2$ if and only if, for any $(x, \xi) \in M_1$ and $(y, \eta) \in M_2$, we have

$$\varphi(x, \xi) = (y, \eta) \iff \xi = d_x f \text{ and } \eta = -d_y f.$$

(Note that this condition implicitly contains the following: $\forall (x, \xi) \in T_x X_1$, $\exists (x, y) \in X_1 \times X_2$ s.t. $(df)_{(x,y)}$ projects to $d_x f = \xi$; similar condition holds for the other component.)

Therefore, by noticing that , given a point $(x, \xi) \in M_1$, to find its image $(y, \eta) = \varphi(x, \xi)$ we must solve the "Hamilton" equations

$$\begin{cases} \xi_i = \dfrac{\partial f}{\partial x_i}(x, y) & (\star) \\ \eta_i = -\dfrac{\partial f}{\partial y_i}(x, y). & (\star\star) \end{cases}$$

If there is a solution $y = \varphi_1(x, \xi)$ of $(\star)$, we may feed it to $(\star\star)$ thus obtaining $\eta = \varphi_2(x, \xi)$, so that $\varphi(x, \xi) = (\varphi_1(x, \xi), \varphi_2(x, \xi))$. Now by the implicit function theorem, in order to solve $(\star)$ locally for y in terms of x and ξ, we need the condition

$$\det \left[\frac{\partial}{\partial y_j} \left(\frac{\partial f}{\partial x_i} \right) \right]_{i,j=1}^n \neq 0.$$

This is a necessary local condition for f to generate a symplectomorphism φ. Locally this is also sufficient, but globally there is the usual bijectivity issue.

Example 4.2.2. Let $X_1 = \mathcal{U}_1 \simeq \mathbb{R}^n, X_2 = \mathcal{U}_2 \simeq \mathbb{R}^n$, and $f(x, y) = -\frac{|x-y|^2}{2} = -\frac{\sum (y_i - x_i)^2}{2}$, the square of euclidean distance up to a constant. The "Hamilton" equations are

$$\begin{cases} \xi_i = \dfrac{\partial f}{\partial x_i} = y_i - x_i \\ \eta_i = -\dfrac{\partial f}{\partial y_i} = y_i - x_i \end{cases} \iff \begin{cases} y_i = x_i + \xi_i \\ \eta_i = \xi_i \end{cases}$$

The symplectomorphism generated by f is

$$\varphi(x,\xi) = (x+\xi,\xi).$$

If we use the euclidean inner product to identify $T^*\mathbb{R}^n$ with $T\mathbb{R}^n$, and hence regard φ as $\widetilde{\varphi} : T\mathbb{R}^n \to T\mathbb{R}^n$ and interpret ξ as the velocity vector, then the symplectomorphism φ corresponds to free translational motion in euclidean space. ♣

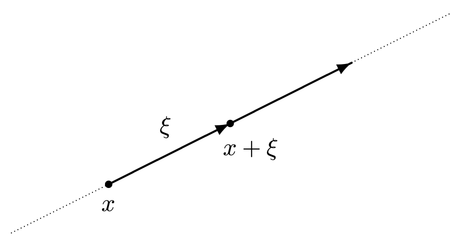


Figure 4.1: Free translational motion

4.3 Application to Geodesic Flow

Let V be an n-dimensional vector space. A positive inner product G on V is a bilinear $\operatorname{map} G : V \times V \to \mathbb{R}$ which is

symmetric :	$G(v,w) = G(w,v),$		and
positive-definite :	$G(v,v) > 0$	when	$v \neq 0$

Definition 4.3.1. *A* ***Riemannian metric*** *on a manifold X is a function g which assigns to each point $x \in X$ a positive inner product g_x on T_xX.*

A Riemannian metric g is ***smooth*** *if for every smooth vector field $v : X \to TX$ the real-valued function $x \mapsto g_x(v_x, v_x)$ is a smooth function on X. (see [8] p.317 proposition 12.19 for equivalent characterizations of smoothness of tensor fields)*

Definition 4.3.2. *A* ***Riemannian manifold*** *(X,g) is a manifold X equipped with a smooth Riemannian metric g.*

The **arc-length** of a piecewise smooth curve $\gamma : [a,b] \to X$ on a riemannian manifold (X,g) is

$$\int_a^b \sqrt{g_{\gamma(t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}dt.$$

Definition 4.3.3. *The* ***Riemannian distance*** *between two points x and y of a connected Riemannian manifold (X,g) is the infimum $d(x,y)$ of the set of all arc-lengths for piecewise smooth curves joining x to y.*

A smooth curve joining x to y is a **minimizing geodesic*** if its arc-length is the riemannian distance $d(x,y)$.

A Riemannian manifold (X,g) is **geodesically convex** if every point x is joined to every other point y by a unique minimizing geodesic.

*This definition actually refers to **minimizing curve** as at [7] p.151; but note that every minimizing curve is a geodesic (see p.103) when put with unit-speed paramatrization (p.156) and every geodesic is locally-minimizing (p.165).

Example 4.3.4. On $X = \mathbb{R}^n$ with $TX \simeq \mathbb{R}^n \times \mathbb{R}^n$, let $g_x(v,w) = \langle v, w\rangle, g_x(v,v) = |v|^2$, where $\langle\cdot,\cdot\rangle$ is the euclidean inner product, and $|\cdot|$ is the euclidean norm. Then $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ is a geodesically convex riemannian manifold, and the riemannian distance is the usual euclidean distance $d(x,y) = |x-y|$. ♣

Suppose that (X,g) is a geodesically convex Riemannian manifold. Consider the function

$$f : X \times X \longrightarrow \mathbb{R}, \quad f(x,y) = -\frac{d(x,y)^2}{2}.$$

What is the symplectomorphism $\varphi : T^*X \to T^*X$ generated by f ? The metric $g_x : T_xX \times T_xX \to \mathbb{R}$ induces an identification

$$\begin{aligned} \widetilde{g}_x : T_xX &\xrightarrow{\simeq} T_x^*X \\ v &\longmapsto g_x(v,\cdot) \end{aligned}$$

Use $\widetilde{g}$ to translate φ into a map $\widetilde{\varphi} : TX \to TX$. We need to solve

$$\begin{cases} \widetilde{g}_x(v) & = \xi_i & = & d_x f(x,y) \\ \widetilde{g}_y(w) & = \eta_i & = & -d_y f(x,y) \end{cases}$$

for (y,η) in terms of (x,ξ) in order to find φ, or, equivalently, for (y,w) in terms (x,v) in order to find $\widetilde{\varphi}$. Let γ be the geodesic with initial conditions $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. Then the symplectomorphism φ corresponds to the map

$$\begin{aligned} \widetilde{\varphi} : TX &\longrightarrow TX \\ (x,v) &\longmapsto \left(\gamma(1), \frac{d\gamma}{dt}(1)\right). \end{aligned}$$

This is called the **geodesic flow** on X.

4.4 Geodesic Flow

Let (X,g) be a Riemannian manifold. The arc-length of a smooth curve $\gamma : [a,b] \to X$ is arc-length of

$$\gamma := \int_a^b \left|\frac{d\gamma}{dt}\right| dt, \quad \text{where} \quad \left|\frac{d\gamma}{dt}\right| := \sqrt{g_{\gamma(t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}.$$

Exercise 4.4.1. *Show that the arc-length of γ is independent of the parametrization of γ, i.e., show that, if we reparametrize γ by $\tau : [a',b'] \to [a,b]$, the new curve $\gamma' = \gamma\circ\tau : [a',b'] \to X$ has the same arc-length. A curve γ is called a curve of constant velocity when $\left|\frac{d\gamma}{dt}\right|$ is independent of t. Show that, given any curve $\gamma : [a,b] \to X$ (with $\frac{d\gamma}{dt}$ never vanishing), there is a reparametrization $\tau : [a,b] \to [a,b]$ such that $\gamma\circ\tau : [a,b] \to X$ is of constant velocity.*

Solution. See [8] Proposition 13.25 and [7] Proposition 2.49. ♦

Exercise 4.4.2. *Given a smooth curve $\gamma : [a,b] \to X$, the action of γ is $\mathcal{A}(\gamma) := \int_a^b \left|\frac{d\gamma}{dt}\right|^2 dt$. Show that, among all curves joining x to y, γ minimizes the action if and only if γ is of constant velocity and γ minimizes arc-length. [Hint: Suppose that γ is of constant velocity, and let $\tau : [a,b] \to [a,b]$ be a reparametrization. Show that $\mathcal{A}(\gamma\circ\tau) \geq \mathcal{A}(\gamma)$, with equality only when τ = identity.]*

Solution. Action-minimizing and arc-length minimizing are the same thing. According to what is hinted, this question probably wants to say that the miniziing curve is among the constant-velocity ones. i.e., constant velocity is a necessary condition for a curve to be minimizing. Then [7] Theorem 6.4 establishes a (stronger) result for this. ⧫

Exercise 4.4.3. *Assume that (X, g) is geodesically convex, that is, any two points $x, y \in X$ are joined by a unique (up to reparametrization) minimizing geodesic; its arc-length $d(x, y)$ is called the riemannian distance between x and y. Assume also that (X, g) is geodesically complete, that is, every geodesic can be extended indefinitely. Given $(x, v) \in TX$, let $\exp(x, v) : \mathbb{R} \to X$ be the unique minimizing geodesic of constant velocity with initial conditions $\exp(x, v)(0) = x$ and $\frac{d\exp(x,v)}{dt}(0) = v$. Consider the function $f : X \times X \to \mathbb{R}$ given by $f(x, y) = -\frac{1}{2} \cdot d(x, y)^2$. Let $d_x f$ and $d_y f$ be the components of $df_{(x,y)}$ with respect to $T^*_{(x,y)}(X \times X) \simeq T^*_x X \times T^*_y X$. Recall that, if*

$$\Gamma_f^\sigma = \{(x, y, d_x f, -d_y f) \mid (x, y) \in X \times X\}$$

is the graph of a diffeomorphism $f : T^*X \to T^*X$, then f is the symplectomorphism generated by f. In this case, $\varphi(x, \xi) = (y, \eta)$ if and only if $\xi = d_x f$ and $\eta = -d_y f$. Show that, under the identification of TX with T^*X by g, the symplectomorphism generated by f coincides with the map $TX \to TX, (x, v) \mapsto \exp(x, v)(1)$ [Hint: The metric g provides the identifications $T_x X v \simeq \xi(\cdot) = g_x(v, \cdot) \in T^*_x X$. We need to show that, given $(x, v) \in TX$, the unique solution of

$$(\star)\begin{cases} g_x(v, \cdot) = d_x f(\cdot) \\ g_y(w, \cdot) = -d_y f(\cdot) \end{cases}$$

is $(y, w) = \left(\exp(x, v)(1), d\frac{\exp(x,v)}{dt}(1)\right)$. Look up the Gauss lemma in a book on riemannian geometry. It asserts that geodesics are orthogonal to the level sets of the distance function. To solve the first line in $(\star)$ for y, evaluate both sides at $v = \frac{d\exp(x,v)}{dt}(0)$. Conclude that $y = \exp(x, v)(1)$. Check that $d_x f(v') = 0$ for vectors $v' \in T_x X$ orthogonal to v (that is, $g_x(v, v') = 0$); this is a consequence of $f(x, y)$ being the arc-length of a minimizing geodesic, and it suffices to check locally. The vector w is obtained from the second line of $(\star)$. Compute $-d_y f\left(\frac{d\exp(x,v)}{dt}(1)\right)$. Then evaluate $-d_y f$ at vectors $w' \in T_y X$ orthogonal to $\frac{d\exp(x,v)}{dt}(1)$; this pairing is again 0 because $f(x, y)$ is the arc-length of a minimizing geodesic. Conclude, using the nondegeneracy of g, that $w = \frac{d\exp(x,v)}{dt}(1)$. For both steps, it might be useful to recall that, given a function $h : X \xrightarrow{dt} \mathbb{R}$ and a tangent vector $v \in T_x X$, we have $dh_x(v) = \frac{d}{du}[h(\exp(x, v)(u))]_{u=0}$.]

Solution. The answer is adapted from Stackexchange.

First we solve the first line for $y \in X$. Since X is geodesically convex there exists a unique minimizing geodesic connecting x and y, which thus coincides with the unique geodesic with initial position x and velocity $u \in T_x M$. Thus, a reparametrization gives $\exp_x(u) = y$. By remark 4.2.1, we see that

$$\begin{aligned} d_x f(v) = dF_x(v) &\overset{\text{[8]Cor.3.25}}{=\!=\!=} (F \circ \gamma_v)'(0) = \left.\frac{d}{dt}\right|_{t=0} F(\gamma_v(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left[-\frac{1}{2} d(\gamma_v(t), y)^2\right] = \left.\frac{d}{dt}\right|_{t=0} \left[-\frac{1}{2} d(\exp_x(tv), \exp_x(u))^2\right] \end{aligned}$$

We need the following claim to proceed as in the hint.

Claim: For $u, v \in T_x M$,

$$\left.\frac{d}{dt}\right|_{t=0} \left[-\frac{1}{2} d\left(\exp_x(tv), \exp_x(u)\right)^2\right] = g_x(u, v)$$

proof: Let $s \in \mathbb{R}$ be sufficiently small so that $\exp_x(su)$ is contained in a geodesic ball centered at x and $\exp_x(u) = y \in \partial B_s(x)$ (see Figure 4.2).

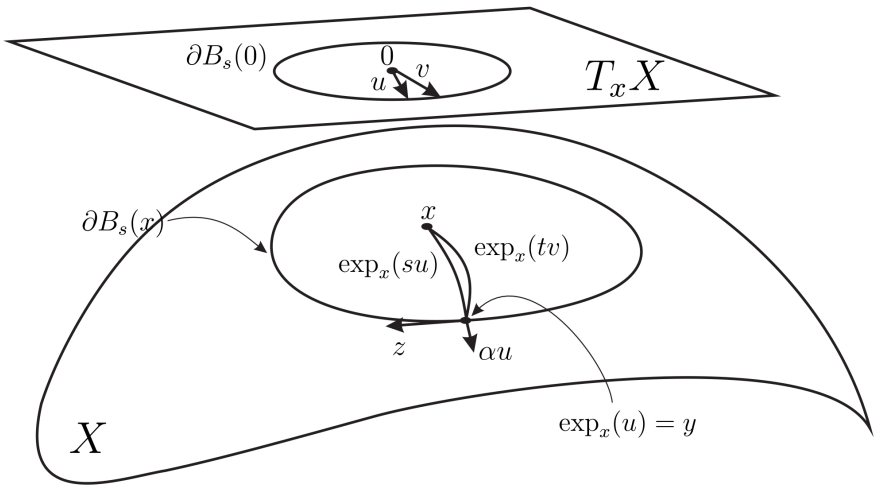


Figure 4.2: Exponential map.

Then we have that $d\left(\exp_x(su), \exp_x(u)\right) = |1-s||u|$. Indeed, let $\gamma : \mathbb{R} \to X$ be the geodesic starting at x with velocity u, then

$$L\left(\gamma|_{[s,1]}\right) = \int_s^1 |\dot{\gamma}(t)| dt = |u| \int_s^1 dt = |1-s||u|$$

Since the radial geodesic $\exp_x(su)$ is the unique minimizing geodesic from $\exp_x(u)$ to $\exp_x(su)$ (see [7]Proposition 6.11), it follows that

$$d\left(\exp_x(su), \exp_x(u)\right) = |1-s||u|$$

Decomposing $v = \alpha u + z$, where z is the tangent vector to the geodesic sphere $\partial B_s(x)$. By **Gauss Lemma** (see proof of [7] Theorem 6.9) this is an orthogonal decomposition, plus there is a curve $\beta : \mathbb{R} \to X$ starting at x with velocity z such that $f(\beta(t), y)$ is constant.

Now define $\hat{f} : X \to \mathbb{R}$, by $x \mapsto f\left(x, \exp_x(u)\right)$ and $l :\to X$, by $t \mapsto \exp_x(tv)$ then

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (f \circ l)(t) &= df_0(l(0)) \cdot l'(0) = df_0(x)(v) \\
&= df_0(x)(\alpha u + z) \\
&= \alpha df_0(x)(u) + \underbrace{df_0}_{0}(x)(z) \\
&= \alpha \left.\frac{d}{ds}\right|_{s=0} \left[-\frac{1}{2} d\left(\exp_x(su), \exp_x(u)\right)\right] \\
&= \alpha \left.\frac{d}{ds}\right|_{s=0} \left[-\frac{1}{2}|1-s|^2|u|\right] \\
&= \alpha|u|^2 \\
&= g_x(u, v)
\end{aligned}$$

as we wanted. //

We now follow the hint. We evaluate both sides of the first line of ($\star$) to get

$$|v| = d_x f(v) = \left.\frac{d}{dt}\right|_{t=0} \left[-\frac{1}{2} d(\exp_x(tv), \exp_x(u))^2\right] = g_x(u, v)$$

For any $v' \perp v$ we evaluate the first line of $(\star)$ to get

$$0 = d_x f\left(v'\right) = g_x\left(u, v'\right)$$

By arbitrariness of v', $u = cv$ and thus $|v| = g_x(u,v) = g_x(cv,v) = c|v| \implies c = 1$, $u = v$. Then $y = \exp_x(v)$.

To solve w from the second line of $(\star)$, let $W = \frac{d}{dt}\big|_{t=1} \exp_x(tv)$ and our goal is to show $w = W$. Note that for any $w' \perp w$ we have $g_y(w,w') = 0 \implies d_y f(w') = 0$. Thus $w = cW$. Since geodesics have constant velocity $|W|^2 = |v|^2$. Hence the left hand side of $(**)$ is

$$g_x(w, W) = k|W|^2 = k|v|^2$$

while the right hand side is

$$\begin{aligned}
-d_y f(W) &= \left.\frac{d}{ds}\right|_{s=0} \left(\frac{1}{2} d\left(x, \exp_x(1+s)(v)\right)^2\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(\frac{1}{2} d\left(\exp_x(0v), \exp_x(1+s)(v)\right)^2\right) \\
&= v \left.\frac{d}{ds}\right|_{s=0} \left(\frac{1}{2}(1+s)^2|v|^2\right) \\
&= |v|^2
\end{aligned}$$

where we used the same first argument of the claim in $(\star)$. Thus we have

$$k|v|^2 = |v|^2 \Rightarrow k = 1$$

and $w = W$ as required. ♦

4.5 Appendix

4.5.1 Tautological 1-form on $T^*(X_1 \times X_2)$

Since $(\mathrm{pr}_1 \circ \mu)(p,q) = (p, d(i_1)_p^*(\mu_{(p,q)})) =: (p, \zeta_1)$ and $(\mathrm{pr}_2 \circ \mu)(p,q) = (q, d(i_2)_q^*(\mu_{(p,q)})) =: (q, \zeta_2)$, we see that

$$\begin{aligned}
&\left((\mathrm{pr}_1 \circ \mu)^* \alpha_1\right)_{(p,q)}(v) + \left((\mathrm{pr}_2 \circ \mu)^* \alpha_2\right)_{(p,q)}(v) \\
=&(\alpha_1)_{(p,\zeta_1)}\left(d(\mathrm{pr}_1 \circ \mu)_{(p,q)}(v)\right) + (\alpha_1)_{(q,\zeta_2)}\left(d(\mathrm{pr}_2 \circ \mu)_{(p,q)}(v)\right) \\
=&\underbrace{d(i_1)_p^*(\mu_{(p,q)})}_{=\zeta_1}\left(d(\pi_1)_p(d(\mathrm{pr}_1 \circ \mu)_{(p,q)}(v))\right) + \underbrace{d(i_2)_q^*(\mu_{(p,q)})}_{=\zeta_2}\left(d(\pi_2)_q(d(\mathrm{pr}_2 \circ \mu)_{(p,q)}(v))\right) \\
=&\mu_{(p,q)}\left(d(i_1)_p(d(\pi_1)_p(d(\mathrm{pr}_1 \circ \mu)_{(p,q)}(v))) + d(i_2)_q(d(\pi_2)_q(d(\mathrm{pr}_2 \circ \mu)_{(p,q)}(v)))\right) \\
=&\mu_{(p,q)}\left(d(i_1 \circ \pi_1 \circ \mathrm{pr}_1 \circ \mu)_{(p,q)}(v) + d(i_2 \circ \pi_2 \circ \mathrm{pr}_2 \circ \mu)_{(p,q)}(v)\right)
\end{aligned}$$

We note that, for example, $i_1 \circ \pi_1 \circ \mathrm{pr}_1 \circ \mu$ sends (x_1, q) to $((x_1,q),(\xi,\eta)) \simeq ((x_1,\xi),(q,\eta))$ to (x_1,ξ) and then to x_1 and lastly to (x_1, q). Thus, in local coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n)$,

$$d(i_1 \circ \pi_1 \circ \mathrm{pr}_1 \circ \mu)_{(p,q)}\left(\overbrace{u^i \left.\frac{\partial}{\partial x_i}\right|_{(p,q)} + w^i \left.\frac{\partial}{\partial y_i}\right|_{(p,q)}}^{=v}\right) = u^i \left.\frac{\partial}{\partial x_i}\right|_{(p,q)}$$

Similarly,

$$d(i_1 \circ \pi_1 \circ \mathrm{pr}_1 \circ \mu)_{(p,q)}(v) = w^i \left.\frac{\partial}{\partial y_i}\right|_{(p,q)}$$

Thus,

$$\begin{aligned}&\left((\operatorname{pr}_1 \circ\mu)^*\alpha_1\right)_{(p,q)}(v) + \left((\operatorname{pr}_2 \circ\mu)^*\alpha_2\right)_{(p,q)}(v) = \mu_{(p,q)}(v)\\ \Longrightarrow\ &\mu = (\operatorname{pr}_1 \circ\mu)^*\alpha_1 + (\operatorname{pr}_2 \circ\mu)^*\alpha_2 = \mu^*\left(\operatorname{pr}_1^*\alpha_1 + \operatorname{pr}_2^*\alpha_2\right)\end{aligned}$$

verifying the characteristic property of $\operatorname{pr}_1^*\alpha_1 + \operatorname{pr}_2^*\alpha_2$. By uniqueness of the tautological one-form satisfying that property, we see $\alpha = \operatorname{pr}_1^*\alpha_1 + \operatorname{pr}_2^*\alpha_2$.

4.5.2 Pullback of the involution $\sigma_2 : M_2 \to M_2$

We show

$$\sigma_2^*\alpha_2 = -\alpha_2$$

for involution $\sigma_2 : M_2 \to M_2$; $(x_2, \xi_2) \mapsto (x_2, -\xi_2)$. That is, we want to show $-\sigma_2^*\alpha_2$ possesses the characteristic property of tautological one-form:

$$\forall \mu : X_2 \to T^*X_2 = M_2, \quad \mu^*(-\sigma_2^*\alpha_2) = \mu.$$

Since $\mu^*(-\sigma_2^*\alpha_2) = -(\sigma_2 \circ \mu)^*\alpha_2 = -\sigma_2 \circ \mu$ where we used the fact that α_2 is a tautological one-form, we see $(-\sigma_2 \circ \mu)_{x_2} = -(\sigma_2(\mu_{x_2})) = -(-\mu_{x_2}) = \mu_{x_2} \implies -\sigma \circ \mu = \mu \implies \mu^*(-\sigma_2^*\alpha_2) = \mu$.

4.5.3 The twist form

Using the fact that the canonical 2-form is the exterior derivative of the tautological 1-form, we have:

$$\sigma_2^*\omega_2 = \sigma_2^*\left(-d\alpha_2\right) = -d\left(\sigma_2^*\alpha_2\right) = -d\left(-\alpha_2\right) = d\alpha_2 = -\omega_2$$

The map σ acts as $\sigma\left(\left(x_1, \xi_1\right), \left(x_2, \xi_2\right)\right) = \left(\left(x_1, \xi_1\right), \left(x_2, -\xi_2\right)\right)$. Let's compute the pullback $\sigma^*\widetilde{\omega}$:

$$\sigma^*\widetilde{\omega} = \sigma^*\left(\operatorname{pr}_1^*\omega_1 - \operatorname{pr}_2^*\omega_2\right)$$

Since $\operatorname{pr}_1 \circ \sigma = \operatorname{pr}_1$ and $\operatorname{pr}_2 \circ \sigma = \sigma_2 \circ \operatorname{pr}_2$, we have:

$$\sigma^*\operatorname{pr}_1^*\omega_1 = \operatorname{pr}_1^*\omega_1$$

and

$$\sigma^*\operatorname{pr}_2^*\omega_2 = \left(\operatorname{pr}_2 \circ\sigma\right)^*\omega_2 = \left(\sigma_2 \circ \operatorname{pr}_2\right)^*\omega_2 = \operatorname{pr}_2^*\left(\sigma_2^*\omega_2\right) = \operatorname{pr}_2^*\left(-\omega_2\right) = -\operatorname{pr}_2^*\omega_2$$

Putting these together, we get:

$$\sigma^*\widetilde{\omega} = \sigma^*\left(\operatorname{pr}_1^*\omega_1 - \operatorname{pr}_2^*\omega_2\right) = \operatorname{pr}_1^*\omega_1 - \left(-\operatorname{pr}_2^*\omega_2\right) = \operatorname{pr}_1^*\omega_1 + \operatorname{pr}_2^*\omega_2$$

Thus, we have shown that:

$$\sigma^*\widetilde{\omega} = \operatorname{pr}_1^*\omega_1 + \operatorname{pr}_2^*\omega_2$$

Chapter 5

Recurrence

5.1 Periodic Points

Let X be an n-dimensional manifold. Let $M = T^*X$ be its cotangent bundle with canonical symplectic form ω.

Suppose that we are given a smooth function $f : X \times X \to \mathbb{R}$ which generates a symplectomorphism $\varphi : M \to M, \varphi(x, d_x f) = (y, -d_y f)$, by the recipe of the previous lecture.

What are the fixed points of φ?

Define $\psi : X \to \mathbb{R}$ by $\psi(x) = f(x, x)$.

Proposition 5.1.1. *There is a one-to-one correspondence between the fixed points of φ and the critical points of ψ.*

Proof. We first compute $d\psi_{x_0} = df_{(x_0,x_0)} \circ d\iota_{x_0}$, where $\iota(x) = (x, x)$. Then by using the identification (4.1), we see

$$\begin{aligned} d\iota_{x_0} : T_{x_0}X &\to T_{(x_0,x_0)}(X \times X) \\ v &\mapsto d(i_1)_{x_0}(v) + d(i_2)_{x_0}(v) \end{aligned}$$

Then by (4.4) and the fact that $\pi_i \circ i_1 = \mathrm{Id}_1$, $\pi_2 \circ i_2 = \mathrm{Id}_2$, and $\pi_1 \circ i_2$, $\pi_2 \circ i_1$ are constant maps,

$$\begin{aligned} d\psi_{x_0}(v) &= df_{(x_0,x_0)}(d(i_1)_{x_0}(v) + d(i_2)_{x_0}(v)) \\ &= d(i_1)^*_{x_0}(df_{(x_0,x_0)}) \Bigg(\underbrace{d(\pi_1)_{(x_0,x_0)} \circ d(i_1)_{x_0}(v)}_{=d(\mathrm{Id}_1)(v)=v} \Bigg) + d(i_1)^*_{x_0}(df_{(x_0,x_0)}) \Bigg(\underbrace{d(\pi_1)_{(x_0,x_0)} \circ d(i_1)_{x_0}(v)}_{=d(\mathrm{Id}_1)(v)=v} \Bigg) \\ &= d(i_1)^*_{x_0}(df_{(x_0,x_0)})(v) + = d(i_2)^*_{x_0}(df_{(x_0,x_0)})(v) \\ &= (d_x f + d_y f)|_{(x,y)=(x_0,x_0)}(v) \end{aligned}$$

Therefore, $d\psi_{x_0} = (d_x f + d_y f)|_{(x,y)=(x_0,x_0)}$. Let $\xi = d_x f|_{(x,y)=(x_0,x_0)}$. Then x_0 is a critical point of $\psi \iff d_{x_0}\psi = 0 \iff d_y f|_{(x,y)=(x_0,x_0)} = -\xi$. Thus, x_0 being a critical point makes $(x, y, d_x f, -d_y f) = (x_0, x_0, \xi, \xi)$ in Γ_f^σ. But Γ_f^σ is the graph of φ, so $\varphi(x_0, \xi) = (x_0, \xi)$ is a fixed point. This argument also works backwards. ■

Consider the iterates of φ,

$$\varphi^{(N)} = \underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{N} : M \longrightarrow M, \quad N = 1, 2, \ldots,$$

each of which is a symplectomorphism of M. According to the previous proposition, if $\varphi^{(N)}: M \rightarrow M$ is generated by $f^{(N)}$, then there is a correspondence

$$\left\{\text{fixed points of } \varphi^{(N)}\right\} \stackrel{1-1}{\longleftrightarrow}\left\{\begin{array}{c}\text{critical points of} \\ \psi^{(N)}: X \rightarrow \mathbb{R}, \psi^{(N)}(x)=f^{(N)}(x, x)\end{array}\right\}$$

Knowing that φ is generated by f, does $\varphi^{(2)}$ have a generating function? The answer is a partial yes: Fix $x, y \in X$. Define a map

$$\begin{array}{rcl}X & \longrightarrow & \mathbb{R} \\ z & \longmapsto & f(x, z)+f(z, y).\end{array}$$

Suppose that this map has a unique critical point z_0, and that z_0 is nondegenerate. Let

$$f^{(2)}(x, y):=f\left(x, z_0\right)+f\left(z_0, y\right)$$

Proposition 5.1.2. *The function $f^{(2)}: X \times X \rightarrow \mathbb{R}$ is smooth and is a generating function for $\varphi^{(2)}$ if we assume that, for each $\xi \in T_x^* X$, there is a unique $y \in X$ for which $d_x f^{(2)}=\xi$.*

Proof. Mimicing the computation in the last proposition, we see taht the critical point z_0 is given implicitly by $0+\left.d_y f\right|_{(x, y)=\left(x, z_0\right)}+\left.d_x f\right|_{(x, y)=\left(z_0, y\right)}+0=d_y f\left(x, z_0\right)+d_x f\left(z_0, y\right)=0$. The nondegeneracy condition is

$$\operatorname{det}\left[\frac{\partial}{\partial z_i}\left(\frac{\partial f}{\partial y_j}(x, z)+\frac{\partial f}{\partial x_j}(z, y)\right)\right] \neq 0.$$

By the implicit function theorem, $z_0=z_0(x, y)$ is smooth. As for the second assertion, $f^{(2)}(x, y)$ is a generating function for $\varphi^{(2)}$ if and only if

$$\varphi^{(2)}\left(x, d_x f^{(2)}\right)=\left(y,-d_y f^{(2)}\right)$$

(assuming that, for each $\xi \in T_x^* X$, there is a unique $y \in X$ for which $d_x f^{(2)}=\xi$). Since φ is generated by f, and z_0 is critical, we obtain

$$\begin{aligned}\varphi^{(2)}\left(x, d_x f^{(2)}(x, y)\right) &=\varphi(\varphi(x, \underbrace{d_x f^{(2)}(x, y)}_{=d_x f\left(x, z_0\right)})=\varphi\left(z_0,-d_y f\left(x, z_0\right)\right) \\ &=\varphi\left(z_0, d_x f\left(z_0, y\right)\right)=(y, \underbrace{-d_y f\left(z_0, y\right)}_{=-d_y f^{(2)}(x, y)}).\end{aligned}$$

■

Exercise 5.1.3. *What is a generating function for $\varphi^{(3)}$? [Hint: Suppose that the function*

$$\begin{array}{rcl}X \times X & \longrightarrow & \mathbb{R} \\ (z, u) & \longmapsto & f(x, z)+f(z, u)+f(u, y)\end{array}$$

has a unique critical point $\left(z_0, u_0\right)$, and that it is a nondegenerate critical point. Let $f^{(3)}(x, y)=f\left(x, z_0\right)+f\left(z_0, u_0\right)+f\left(u_0, y\right)$.]

5.2 Billiards

Let $\chi : \mathbb{R} \to \mathbb{R}^2$ be a smooth plane curve which is 1-periodic, i.e., $\chi(s+1) = \chi(s)$, and parametrized by arc-length, i.e., $\left|\frac{d\chi}{ds}\right| = 1$. Assume that the region Y enclosed by χ is convex, i.e., for any $s \in \mathbb{R}$, the tangent line $\left\{ \chi(s) + t\frac{d\chi}{ds} \,\middle|\, t \in \mathbb{R} \right\}$ intersects $X := \partial Y (=$ the image of $\chi)$ at only the point $\chi(s)$.

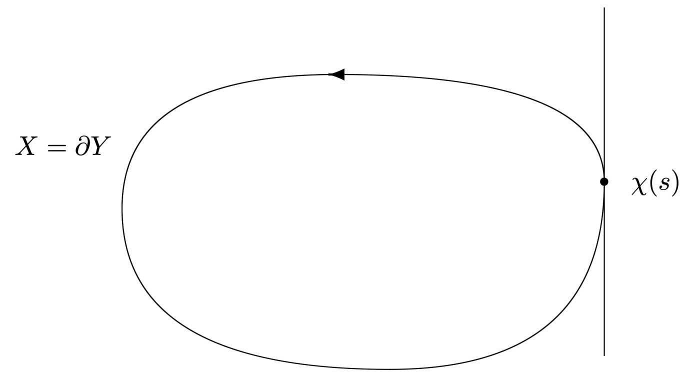


Figure 5.1: billiard table

Suppose that we throw a ball into Y rolling with constant velocity and bouncing off the boundary with the usual law of reflection. This determines a map

$$\begin{aligned} \varphi : \mathbb{R}/\mathbb{Z} \times (-1,1) &\longrightarrow \mathbb{R}/\mathbb{Z} \times (-1,1) \\ (x,v) &\longmapsto (y,w) \end{aligned}$$

by the rule

when the ball bounces off $\chi(x)$ with angle $\theta = \arccos v$, it will next collide with $\chi(y)$ and bounce off with angle $\nu = \arccos w$.

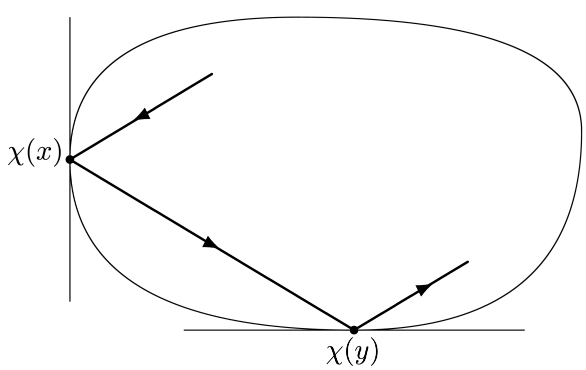


Figure 5.2: ball bouncing off

Let $f : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be defined by $f(x,y) = -|\chi(x) - \chi(y)|$; f is smooth off the diagonal. Use χ to identify $\mathbb{R}/\mathbb{Z}$ with the image curve X to get $f : X \times X \to \mathbb{R}$. Suppose that $\varphi(x,v) = (y,w)$, i.e., (x,v) and

(y, w) are successive points on the orbit described by the ball. Because for $t \neq 0$ we have

$$\begin{aligned}\frac{d(|t|)}{dt} &= \begin{cases}1, & t>0\\ -1, & t<0\end{cases}\\ &= \frac{t}{|t|},\end{aligned}$$

then

$$\begin{cases}\dfrac{\partial f}{\partial x} = -\dfrac{\chi(x)-\chi(y)}{|\chi(x)-\chi(y)|}\cdot\dfrac{d\chi}{ds}(x) = \cos\theta = v\\ \dfrac{\partial f}{\partial y} = -\dfrac{\chi(y)-\chi(x)}{|\chi(x)-\chi(y)|}\cdot\dfrac{d\chi}{ds}(y) = -\cos\nu = -w.\end{cases}$$

We conclude that f is a generating function for φ. Similar approaches work for higher dimensional billiards problems. Periodic points are obtained by finding critical points of

$$\begin{aligned}&\underbrace{X\times\ldots\times X}_{N} \longrightarrow \mathbb{R}, \quad N>1\\ &(x_1,\ldots,x_N)\longmapsto f(x_1,x_2)+f(x_2,x_3)+\ldots+f(x_{N-1},x_N)+f(x_N,x_1)\\ &= -|x_1-x_2|-\ldots-|x_{N-1}-x_N|-|x_N-x_1|,\end{aligned}$$

that is, by finding the N-sided (generalized) polygons inscribed in X of critical perimeter. Notice that

$$\mathbb{R}/\mathbb{Z}\times(-1,1)\simeq\{(x,v)\,|x\in X, v\in T_xX, |\,v\,|<1\}\simeq A$$

is the open unit tangent ball bundle of a circle X, that is, an open annulus A. The map $\varphi : A \to A$ is area-preserving.

5.3 Poincaré Recurrence

Theorem 5.3.1 (Poincaré Recurrence Theorem)**.** *Suppose that $\varphi : A \to A$ is an area-preserving diffeomorphism of a finite-area manifold A. Let $p \in A$, and let $\mathcal{U}$ be a neighborhood of p. Then there is $q \in \mathcal{U}$ and a positive integer N such that $\varphi^{(N)}(q) \in \mathcal{U}$.*

Proof. Let $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_1 = \varphi(\mathcal{U}), \mathcal{U}_2 = \varphi^{(2)}(\mathcal{U}), \ldots$ If all of these sets were disjoint, then, since Area $(\mathcal{U}_i) =$ Area $(\mathcal{U}) > 0$ for all i, we would have Area $A \geq$ Area $(\mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \ldots) = \sum_i$ Area $(\mathcal{U}_i) = \infty$.

To avoid this contradiction we must have $\varphi^{(k)}(\mathcal{U}) \cap \varphi^{(l)}(\mathcal{U}) \neq \emptyset$ for some $k > l$, which implies $\varphi^{(k-l)}(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. ■

Hence, eternal return applies to billiards...

Remark 5.3.2. Above theorem clearly generalizes to volume-preserving diffeomorphisms in higher dimensions. ♠

Theorem 5.3.3 (Poincaré's Last Geometric Theorem)**.** *Suppose $\varphi : A \to A$ is an area-preserving diffeomorphism of the closed annulus $A = \mathbb{R}/\mathbb{Z} \times [-1, 1]$ which preserves the two components of the boundary, and twists them in opposite directions. Then φ has at least two fixed points.*

This theorem was proved in 1913 by Birkhoff, and hence is also called the **Poincaré-Birkhoff theorem**. It has important applications to dynamical systems and celestial mechanics. The Arnold conjecture (1966) on the existence of fixed points for symplectomorphisms of compact manifolds (see Lecture 9) may be regarded as a generalization of the Poincaré-Birkhoff theorem. This conjecture has motivated a significant amount of recent research involving a more general notion of generating function; see, for instance, original references of [4] [34, 45].

Part III

Local Forms

Chapter 6

Preparation for the Local Theory

6.1 Isotopies and Vector Fields

Let M be a manifold, and $\rho : M \times \mathbb{R} \to M$ a map, where we set $\rho_t(p) = \rho^{(p)}(t) := \rho(p,t)$.

Definition 6.1.1. *The map ρ is an* ***isotopy*** *if each $\rho_t : M \to M$ is a diffeomorphism, and $\rho_0 = \mathrm{id}_M$.*

Given an isotopy ρ, we obtain a **time-dependent vector field**, that is, a family of vector fields $v_t, t \in \mathbb{R}$, which at $p \in M$ satisfy

$$v_t(p) = \frac{d}{ds}\rho^{(q)}(s)\bigg|_{s=t} \qquad \text{where} \qquad q = \rho_t^{-1}(p)$$

i.e.,

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \qquad \text{or } v_t(\rho_t(q)) = \frac{d}{ds}\rho^{(q)}(s)\bigg|_{s=t}$$

Conversely, given a time-dependent vector field v_t, if M is compact or if the v_t's are compactly supported, there exists an isotopy ρ satisfying the previous ordinary differential equation.

Proposition 6.1.2. *Suppose that M is compact. Then we have a one-to-one correspondence*

$$\{\textit{isotopies of } M\} \overset{1-1}{\longleftrightarrow} \{\textit{ time-dependent vector fields on } M\}$$
$$\rho_t, t \in \mathbb{R} \longleftrightarrow v_t, t \in \mathbb{R}$$

Proof. The $\Longrightarrow$ direction is [8] problem 9-21 (1). The $\Longleftarrow$ direction is by [8] problem 9-21 (2) plus [8] problem 9-20. The proof of problem 9-20 is by mimicing the proof of the escape lemma (see this proof for example) to obtain a time-dependent analog and apply it the integral curve given by [8] Theorem 9.48 (a). Problem 9-21 is by checking various conditions of the definition of time-dependent vector field and smooth isotopy. ∎

Definition 6.1.3. *When $v_t = v$ is independent of t, the associated isotopy is called the* ***exponential map*** *or the* ***flow*** *of v and is denoted $\exp tv$; i.e., $\{\exp tv : M \to M \mid t \in \mathbb{R}\}$ is the unique smooth family of diffeomorphisms satisfying*

$$\exp tv|_{t=0} = \mathrm{id}_M \qquad \textit{and} \qquad \frac{d}{dt}(\exp tv)(p) = v(\exp tv(p))$$

Definition 6.1.4. *The* ***Lie derivative*** *(see [8] p.321) is the operator*

$$\mathcal{L}_v : \Omega^k(M) \longrightarrow \Omega^k(M) \qquad \textit{defined by} \qquad \mathcal{L}_v\omega := \frac{d}{dt}(\exp tv)^*\omega\bigg|_{t=0}.$$

When a vector field v_t is time-dependent, its flow, that is, the corresponding isotopy ρ, still locally exists by Picard's theorem. More precisely, in the neighborhood of any point p and for sufficiently small time t, there is a one-parameter family of local diffeomorphisms ρ_t satisfying

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \quad \text{and} \quad \rho_0 = \mathrm{id}.$$

*Hence, we say that the **Lie derivative by** v_t is*

$$\mathcal{L}_{v_t} : \Omega^k(M) \longrightarrow \Omega^k(M) \quad \text{defined by} \quad \mathcal{L}_{v_t}\omega := \frac{d}{ds}\psi_{s,t}^*\omega\bigg|_{s=t}$$

where $\psi_{s,t}$ is the flow of v_t, i.e., $s \mapsto \psi_{s,t_0}(p)$ is the unique maximal integral curve of v_t with value p at time $s = t_0$:

$$\frac{d}{ds}\psi_{s,t_0}(p)\bigg|_{s=t} = v_t\left(\psi_{t,t_0}(p)\right) \quad \text{and} \quad \psi_{t_0,t_0}(p) = p.$$

This is related to the previous ρ by $\rho_s = \psi_{s,0}$. If M is compact, the flow is globally defined and we have $\psi_{s,t} = \psi_{s,t_0} \circ \psi_{t_0,t}$, thus $\psi_{s,t}^{-1} = \psi_{t,s}$, and hence $\psi_{s,t} = \rho_s \circ \rho_t^{-1}$. We can use this last expression to write $\mathcal{L}_{v_t}\omega$ alternatively in terms of ρ. See [8] p.236.

Proposition 6.1.5. *The **Cartan magic formula** (see [8] 14.25) is*

$$\mathcal{L}_v\omega = \iota_v d\omega + d\iota_v\omega$$

and another useful formula (see [8] 12.36) is

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega \qquad (\star)$$

where ρ is the (local) isotopy generated by v_t.

We will need the following improved version of formula $(\star)$.

Proposition 6.1.6. *For a smooth family $\omega_t, t \in \mathbb{R}$, of d-forms, we have*

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right).$$

Proof. If $f(x,y)$ is a real function of two variables, by the chain rule we have

$$\frac{d}{dt}f(t,t) = \frac{d}{dx}f(x,t)\bigg|_{x=t} + \frac{d}{dy}f(t,y)\bigg|_{y=t}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\rho_t^*\omega_t &= \underbrace{\frac{d}{dx}\rho_x^*\omega_t\bigg|_{x=t}}_{\rho_x^*\mathcal{L}_{v_x}\omega_t|_{x=t} \text{ by } (\star)} + \underbrace{\frac{d}{dy}\rho_t^*\omega_y\bigg|_{y=t}}_{\rho_t^*\frac{d\omega_y}{dy}\big|_{y=t}} \\ &= \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right).\end{aligned}$$

■

6.2 Tubular Neighborhood Theorem

Let M be an n-dimensional manifold, and let X be a k-dimensional submanifold where $k < n$ and with inclusion map

$$i : X \hookrightarrow M.$$

At each $x \in X$, the tangent space to X is viewed as a subspace of the tangent space to M via the linear inclusion $di_x : T_xX \hookrightarrow T_xM$, where we denote $x = i(x)$. The quotient $N_xX := T_xM/T_xX$ is an $(n-k)$-dimensional vector space, known as the **normal space** to X at x. The **normal bundle** of X is

$$NX = \{(x, v) \mid x \in X, v \in N_xX\}.$$

The set NX has the structure of a vector bundle over X of rank $n-k$ under the natural projection, hence as a manifold NX is n-dimensional. The zero section of NX,

$$i_0 : X \hookrightarrow NX, \quad x \mapsto (x, 0),$$

embeds X as a closed submanifold of NX. A neighborhood $\mathcal{U}_0$ of the zero section X in NX is called **convex** if the intersection $\mathcal{U}_0 \cap N_xX$ with each fiber is convex.

Theorem 6.2.1 (Tubular Neighborhood Theorem). *There exist a convex neighborhood $\mathcal{U}_0$ of X in NX, a neighborhood $\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that*

$$\begin{tikzcd} NX \supseteq \mathcal{U}_0 \arrow[rr, "\varphi", "\simeq"'] & & \mathcal{U} \subseteq M \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes.

Outline of proof.

- Case of $M = \mathbb{R}^n$, and X is a compact submanifold of $\mathbb{R}^n$.

Theorem 6.2.2 (ε-Neighborhood Theorem). *Let $\mathcal{U}^\varepsilon = \{p \in \mathbb{R}^n : |p - q| < \varepsilon$ for some $q \in X\}$ be the set of points at a distance less than ε from X. Then, for ε sufficiently small, each $p \in \mathcal{U}^\varepsilon$ has a unique nearest point $q \in X$ (i.e., a unique $q \in X$ minimizing $|q - p|$). Moreover, setting $q = \pi(p)$, the map $\mathcal{U}^\varepsilon \xrightarrow{\pi} X$ is a (smooth) submersion with the property that, for all $p \in \mathcal{U}^\varepsilon$, the line segment $(1-t)p + tq, 0 \le t \le 1$, is in $\mathcal{U}^\varepsilon$.*

The proof is part of the homework. Here are some hints.

At any $x \in X$, the normal space N_xX may be regarded as an $(n-k)$ dimensional subspace of $\mathbb{R}^n$, namely the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x :

$$N_xX \simeq \{v \in \mathbb{R}^n : v \perp w, \text{ for all } w \in T_xX\}.$$

We define the following open neighborhood of X in NX :

$$NX^\varepsilon = \{(x, v) \in NX : |v| < \varepsilon\}.$$

Let

$$\begin{array}{rccc} \exp : & NX & \longrightarrow & \mathbb{R}^n \\ (x, v) & \longmapsto & x + v. \end{array}$$

Restricted to the zero section, exp is the identity map on X. Prove that, for ε sufficiently small, $\exp$ maps NX^ε diffeomorphically onto $\mathcal{U}^\varepsilon$, and show also that the diagram

$$\begin{tikzcd} NX^\varepsilon \arrow[rr, "\exp"] \arrow[dr, "\pi_0"'] & & \mathcal{U}^\varepsilon \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

commutes.

- Case where X is a compact submanifold of an arbitrary manifold M. Put a riemannian metric g on M, and let $d(p,q)$ be the riemannian distance between $p, q \in M$. The ε-neighborhood of a compact submanifold X is
$$\mathcal{U}^\varepsilon = \{p \in M \mid d(p,q) < \varepsilon \text{ for some } q \in X\}.$$

  Prove the ε-neighborhood theorem in this setting: for ε small enough, the following assertions hold.

  - Any $p \in \mathcal{U}^\varepsilon$ has a unique point $q \in X$ with minimal $d(p,q)$. Set $q = \pi(p)$.

  - The map $\mathcal{U}^\varepsilon \xrightarrow{\pi} X$ is a submersion and, for all $p \in \mathcal{U}^\varepsilon$, there is a unique geodesic curve γ joining p to $q = \pi(p)$.

  - The normal space to X at $x \in X$ is naturally identified with a subspace of T_xM :
$$N_xX \simeq \{v \in T_xM \mid g_x(v,w) = 0, \text{ for any } w \in T_xX\}.$$

  Let $NX^\varepsilon = \left\{(x,v) \in NX \mid \sqrt{g_x(v,v)} < \varepsilon\right\}$.

  - Define $\exp : NX^\varepsilon \to M$ by $\exp(x,v) = \gamma(1)$, where $\gamma : [0,1] \to M$ is the geodesic with $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. Then $\exp$ maps NX^ε diffeomorphically to $\mathcal{U}^\varepsilon$.

- General case.

  When X is not compact, adapt the previous argument by replacing ε by an appropriate continuous function $\varepsilon : X \to \mathbb{R}^+$which tends to zero fast enough as x tends to infinity.

■

Restricting to the subset $\mathcal{U}_0 \subseteq NX$ from the tubular neighborhood theorem, we obtain a submersion $\mathcal{U}_0 \xrightarrow{\pi_0} X$ with all fibers $\pi_0^{-1}(x) = \mathcal{U}_0 \cap N_xX$ convex. We can carry this fibration (i.e., a fibre structure) to $\mathcal{U}$ by setting $\pi = \pi_0 \circ \varphi^{-1}$:

$$\begin{array}{ccccc} \mathcal{U}_0 & \subseteq NX \text{ is a fibration} & \Longrightarrow & \mathcal{U} & \subseteq M \text{ is a fibration} \\ \pi_0 \big\downarrow & & & \pi \big\downarrow & \\ X & & & X & \end{array}$$

This is called the **tubular neighborhood fibration**.

6.3 Homotopy Formula

Let $\mathcal{U}$ be a tubular neighborhood of a submanifold X in M. The restriction $i^* : H^d_{\text{deRham}}(\mathcal{U}) \to H^d_{\text{deRham}}(X)$ by the inclusion map is surjective. As a corollary of the tubular neighborhood fibration, i^* is also injective: this follows from the homotopy-invariance of de Rham cohomology (see the proof of proposition 6.3.2).

Corollary 6.3.1. *For any degree ℓ, $H^\ell_{\text{deRham}}(\mathcal{U}) \simeq H^\ell_{\text{deRham}}(X)$.*

At the level of forms, this means that, if ω is a closed ℓ-form on $\mathcal{U}$ and $i^*\omega$ is exact on X, then ω is exact ($i^*[\omega] = 0$ i.e., i^* is injective). We will need the following related result.

Proposition 6.3.2. *If a closed ℓ-form ω on $\mathcal{U}$ has restriction $i^*\omega = 0$, then ω is exact, i.e., $\omega = d\mu$ for some $\mu \in \Omega^{\ell-1}(\mathcal{U})$. Moreover, we can choose μ such that $\mu_x = 0$ at all $x \in X$.*

Proof. Via $\varphi : \mathcal{U}_0 \xrightarrow{\simeq} \mathcal{U}$, it is equivalent to work over $\mathcal{U}_0$. Define for every $0 \leq t \leq 1$ a map

$$\rho_t : \begin{aligned} \mathcal{U}_0 &\longrightarrow \mathcal{U}_0 \\ (x, v) &\longmapsto (x, tv). \end{aligned}$$

This is well-defined since $\mathcal{U}_0$ is convex. The map ρ_1 is the identity, $\rho_0 = i_0 \circ \pi_0$, and each ρ_t fixes X, that is, $\rho_t \circ i_0 = i_0$. We hence say that the family $\{\rho_t \mid 0 \leq t \leq 1\}$ is a **homotopy** from $i_0 \circ \pi_0$ to the identity fixing X. The map $\pi_0 : \mathcal{U}_0 \to X$ is called a **retraction** because $\pi_0 \circ i_0$ is the identity. The submanifold X is then called a **deformation retract** of $\mathcal{U}$.

A (de Rham) **homotopy operator** between $\rho_0 = i_0 \circ \pi_0$ and $\rho_1 = \mathrm{id}$ is a linear map

$$Q : \Omega^d\left(\mathcal{U}_0\right) \longrightarrow \Omega^{d-1}\left(\mathcal{U}_0\right)$$

satisfying the **homotopy formula**

$$\mathrm{Id} - \left(i_0 \circ \pi_0\right)^* = dQ + Qd.$$

When $d\omega = 0$ and $i_0^*\omega = 0$, the operator Q gives $\omega = dQ\omega$, so that we can take $\mu = Q\omega$. A concrete operator Q is given by the formula:

$$Q\omega = \int_0^1 \rho_t^*\left(l_{v_t}\omega\right) dt,$$

where v_t, at the point $q = \rho_t(p)$, is the vector tangent to the curve $\rho_s(p)$ at $s = t$. The proof that Q satisfies the homotopy formula is below.

In our case, for $x \in X, \rho_t(x) = x$ (all t) is the constant curve, so v_t vanishes at all x for all t, hence $\mu_x = 0$, i.e., $[Q\omega]_x = 0$.

(Also see [8] 17.9-17.11.) ■

To check that Q above satisfies the homotopy formula, we compute

$$\begin{aligned} Qd\omega + dQ\omega &= \int_0^1 \rho_t^*\left(\imath_{v_t} d\omega\right) dt + d\int_0^1 \rho_t^*\left(\imath_{v_t}\omega\right) dt \\ &= \int_0^1 \rho_t^*(\underbrace{\imath_{v_t} d\omega + dv_{v_t}\omega}_{\mathcal{L}_{v_t}\omega}) dt \end{aligned}$$

where $\mathcal{L}_v$ denotes the Lie derivative along v (reviewed in the next section), and we used the Cartan magic formula: $\mathcal{L}_v\omega = \imath_v d\omega + dv_v\omega$. The result now follows from

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega$$

and from the fundamental theorem of calculus:

$$Qd\omega + dQ\omega = \int_0^1 \frac{d}{dt}\rho_t^*\omega dt = \rho_1^*\omega - \rho_0^*\omega$$

6.4 Tubular Neighborhood in $\mathbb{R}^n$

(see [8] p.138-140)

1. Let X be a k-dimensional submanifold of an n-dimensional manifold M. Let x be a point in X. The **normal space** to X at x is the quotient space

$$N_x X = T_x M / T_x X,$$

and the **normal bundle** of X in M is the vector bundle NX over X whose fiber at x is $N_x X$.

(a) Prove that NX is indeed a vector bundle.

(b) If M is $\mathbb{R}^n$, show that $N_x X$ can be identified with the usual "normal space" to X in $\mathbb{R}^n$, that is, the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x.

2. Let X be a k-dimensional compact submanifold of $\mathbb{R}^n$. Prove the **tubular neighborhood theorem** in the following form.

(a) Given $\varepsilon > 0$ let $\mathcal{U}_\varepsilon$ be the set of all points in $\mathbb{R}^n$ which are at a distance less than ε from X. Show that, for ε sufficiently small, every point $p \in \mathcal{U}_\varepsilon$ has a unique nearest point $\pi(p) \in X$.

(b) Let $\pi : \mathcal{U}_\varepsilon \to X$ be the map defined in (a) for ε sufficiently small. Show that, if $p \in \mathcal{U}_\varepsilon$, then the line segment $(1-t) \cdot p + t \cdot \pi(p), 0 \le t \le 1$, joining p to $\pi(p)$ lies in $\mathcal{U}_\varepsilon$.

(c) Let $NX_\varepsilon = \{(x, v) \in NX$ such that $|v| < \varepsilon\}$. Let $\exp : NX \to \mathbb{R}^n$ be the map $(x, v) \mapsto x + v$, and let $\nu : NX_\varepsilon \to X$ be the map $(x, v) \mapsto x$. Show that, for ε sufficiently small, exp maps NX_ε diffeomorphically onto $\mathcal{U}_\varepsilon$, and show also that the following diagram commutes:

$$\begin{tikzcd} NX_\varepsilon \arrow[rr, "\exp"] \arrow[dr, "\nu"'] & & \mathcal{U}_\varepsilon \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

3. Suppose that the manifold X in the previous exercise is not compact. Prove that the assertion about exp is still true provided we replace ε by a continuous function

$$\varepsilon : X \to \mathbb{R}^+$$

which tends to zero fast enough as x tends to infinity.

Chapter 7

Moser Theorems

7.1 Notions of Equivalence for Symplectic Structures

Let M be a $2n$-dimensional manifold with two symplectic forms ω_0 and ω_1, so that (M, ω_0) and (M, ω_1) are two symplectic manifolds.

Definition 7.1.1. *We say that*

- (M, ω_0) *and* (M, ω_1) *are* ***symplectomorphic*** *if there is a diffeomorphism* $\varphi : M \to M$ *with* $\varphi^* \omega_1 = \omega_0$*;*
- (M, ω_0) *and* (M, ω_1) *are* ***strongly isotopic*** *if there is an isotopy* $\rho_t : M \to M$ *such that* $\rho_1^* \omega_1 = \omega_0$*;*
- (M, ω_0) *and* (M, ω_1) *are* ***deformation-equivalent*** *if there is a smooth family* ω_t *of symplectic forms joining* ω_0 *to* ω_1*;*
- (M, ω_0) *and* (M, ω_1) *are* ***isotopic*** *if they are deformation-equivalent with* $[\omega_t]$ *independent of* t*.*

Clearly, we have

$$\begin{aligned} \text{strongly isotopic} &\implies \text{symplectomorphic, and} \\ \text{isotopic} &\implies \text{deformation-equivalent .} \end{aligned}$$

We also have

$$\text{strongly isotopic} \implies \text{isotopic}$$

because, if $\rho_t : M \to M$ is an isotopy such that $\rho_1^* \omega_1 = \omega_0$, then the set $\omega_t := \rho_t^* \omega_1$ is a smooth family of symplectic forms joining ω_1 to ω_0 and $[\omega_t] = [\omega_1], \forall t$, by the homotopy invariance of de Rham cohomology. As we will see below, the Moser theorem states that, on a compact manifold, isotopic $\implies$ strongly isotopic.

7.2 Moser Trick

Problem: Given a $2n$-dimensional manifold M, a k-dimensional submanifold X, neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X, and symplectic forms ω_0, ω_1 on $\mathcal{U}_0, \mathcal{U}_1$, does there exist a symplectomorphism preserving X? More precisely, does there exist a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ with $\varphi^* \omega_1 = \omega_0$ and $\varphi(X) = X$? At the two extremes, we have:

Case X	$=$ point:	Darboux theorem $-$ see Lecture 8.
Case X	$= M$:	Moser theorem $-$ discussed here:

Let M be a compact manifold with symplectic forms ω_0 and ω_1.

- Are (M, ω_0) and (M, ω_1) symplectomorphic?

I.e., does there exist a diffeomorphism $\varphi : M \to M$ such that $\varphi_1^* \omega_0 = \omega_1$?

Moser asked whether we can find such an φ which is homotopic to id_M. A necessary condition is $[\omega_0] = [\omega_1] \in H^2(M;\mathbb{R})$ because: if $\varphi \sim \mathrm{id}_M$, then, by the homotopy formula, there exists a homotopy operator Q such that

$$\begin{aligned} \mathrm{id}_{M^*}^* \omega_1 - \varphi^* \omega_1 = dQ\omega_1 + Q \underbrace{d\omega_1}_{=0 \text{ as } \omega = d(-\alpha)} \\ \Longrightarrow \omega_1 = \varphi^* \omega_1 + d\left(Q\omega_1\right) \\ \Longrightarrow [\omega_1] = [\varphi^* \omega_1] = [\omega_0]\,. \end{aligned}$$

- If $[\omega_0] = [\omega_1]$, does there exist a diffeomorphism φ homotopic to id_M such that $\varphi^* \omega_1 = \omega_0$?

Moser proved that the answer is yes, with a further hypothesis as in the following theorem. But McDuff showed that, in general, the answer is no; for a counterexample, see Example 7.23 in *Introduction to Symplectic Topology*.

Theorem 7.2.1 (Moser Theorem - Version I)**.** *Suppose that M is compact, $[\omega_0] = [\omega_1]$ and that the 2-form $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for each $t \in [0,1]$. Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^* \omega_t = \omega_0$ for all $t \in [0,1]$. In particular, $\varphi = \rho_1 : M \xrightarrow{\simeq} M$, satisfies $\varphi^* \omega_1 = \omega_0$. The following argument, due to Moser, is extremely useful; it is known as the **Moser trick**.*

Proof. Suppose that there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^* \omega_t = \omega_0$, $0 \le t \le 1$. Let

$$v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}, \quad t \in \mathbb{R}$$

Then

$$\begin{aligned} 0 = \frac{d}{dt} \overbrace{\left(\rho_t^* \omega_t\right)}^{\omega_0 \text{ constant}} \xlongequal{\text{prop. 6.1.6}} \rho_t^* \left(\mathcal{L}_{v_t} \omega_t + \tfrac{d\omega_t}{dt}\right) \\ \overset{\rho_t \text{diffeo, so} \rho_t^* \text{ iso}}{\Longleftrightarrow} \mathcal{L}_{v_t} \omega_t + \tfrac{d\omega_t}{dt} = 0. \qquad (\star) \end{aligned}$$

Suppose conversely that we can find a smooth time-dependent vector field v_t, $t \in \mathbb{R}$, such that $(\star)$ holds for $0 \le t \le 1$. Then compactness of M with proposition 6.1.2 enables us to integrate v_t to an isotopy $\rho : M \times \mathbb{R} \to M$ (thus $\rho_0 = id_M$) with

$$\frac{d}{dt}\left(\rho_t^* \omega_t\right) = 0 \quad \Longrightarrow \quad \rho_t^* \omega_t = \rho_0^* \omega_0 \xlongequal{\rho_0 = id_M} \omega_0.$$

So everything boils down to solving $(\star)$ for v_t. First, from $\omega_t = (1-t)\omega_0 + t\omega_1$, we conclude that

$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0 \qquad (1)$$

Second, since $[\omega_0] = [\omega_1]$, there exists a 1-form μ such that

$$\omega_1 - \omega_0 = d\mu \qquad (2)$$

Third, by the Cartan magic formula, we have

$$\mathcal{L}_{v_t} \omega_t = d v_{v_t} \omega_t + v_{v_t} \underbrace{d\omega_t}_{0} \qquad (3)$$

Putting equations (1), (2), (3), and $(\star)$ together, we must find v_t such that

$$d v_{v_t} \omega_t + d\mu = 0$$

It is sufficient to solve $\iota_{v_t} \omega_t + \mu = 0$. By the nondegeneracy of ω_t, we can solve this pointwise, to obtain a unique (smooth) v_t. ■

Theorem 7.2.2 (Moser Theorem - Version II). *Let M be a compact manifold with symplectic forms ω_0 and ω_1. Suppose that $\omega_t, 0 \leq t \leq 1$, is a smooth family of closed 2-forms joining ω_0 to ω_1 and satisfying:*

(1) cohomology assumption: $[\omega_t]$ is independent of t, i.e., $\frac{d}{dt}[\omega_t] = \left[\frac{d}{dt}\omega_t\right] = 0$,

(2) nondegeneracy assumption: ω_t is nondegenerate for $0 \leq t \leq 1$.

Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^* \omega_t = \omega_0, 0 \leq t \leq 1$.

Moser trick. We have the following implications from the hypotheses:

(1) $\Longrightarrow \exists$ family of 1-forms μ_t such that

$$\frac{d\omega_t}{dt} = d\mu_t, \quad 0 \leq t \leq 1$$

We can indeed find a smooth family of 1-forms μ_t such that $\frac{d\omega_t}{dt} = d\mu_t$. The argument involves the Poincaré lemma for compactly-supported forms, together with the Mayer-Vietoris sequence in order to use induction on the number of charts in a good cover of M. For a sketch of the argument, see page 95 in [83]. (2) $\Longrightarrow \exists$ unique family of vector fields v_t such that

$$\iota_{v_t}\omega_t + \mu_t = 0 \quad \textbf{(Moser equation)}.$$

Extend v_t to all $t \in \mathbb{R}$. Let ρ be the isotopy generated by v_t (ρ exists by compactness of M). Then we indeed have

$$\frac{d}{dt}\left(\rho_t^* \omega_t\right) = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) = \rho_t^*\left(dv_{v_t}\omega_t + d\mu_t\right) = 0.$$

■

The compactness of M was used to be able to integrate v_t for all $t \in \mathbb{R}$. If M is not compact, we need to check the existence of a solution ρ_t for the differential equation $\frac{d\rho_t}{dt} = v_t \circ \rho_t$ for $0 \leq t \leq 1$

Picture: Fix $c \in H^2(M)$. Define $S_c = \{$ symplectic forms ω in M with $[\omega] = c\}$. The Moser theorem implies that, on a compact manifold, all symplectic forms on the same path-connected component of S_c are symplectomorphic.

7.3 Moser Relative Theorem

Theorem 7.3.1 (Moser Theorem - Relative Version). *Let M be a manifold, X a compact submanifold of $M, i : X \hookrightarrow M$ the inclusion map, ω_0 and ω_1 symplectic forms in M.*

Hypothesis: $\left.\omega_0\right|_p = \left.\omega_1\right|_p, \quad \forall p \in X.$

Conclusion: There exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes, and

$$\varphi^* \omega_1 = \omega_0.$$

Proof. 1. Pick a tubular neighborhood $\mathcal{U}_0$ of X. The 2-form $\omega_1 - \omega_0$ is closed on $\mathcal{U}_0$, and $(\omega_1 - \omega_0)_p = 0$ at all $p \in X$. By the homotopy formula on the tubular neighborhood, there exists a 1-form μ on $\mathcal{U}_0$ such that $\omega_1 - \omega_0 = d\mu$ and $\mu_p = 0$ at all $p \in X$.

2. Consider the family $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\mu$ of closed 2-forms on $\mathcal{U}_0$. Shrinking $\mathcal{U}_0$ if necessary, we can assume that ω_t is symplectic for $0 \leq t \leq 1$.

3. Solve the Moser equation: $v_{v_t}\omega_t = -\mu$. Notice that $v_t = 0$ on X.

4. Integrate v_t. Shrinking $\mathcal{U}_0$ again if necessary, there exists an isotopy $\rho : \mathcal{U}_0 \times [0,1] \to M$ with $\rho_t^* \omega_t = \omega_0$, for all $t \in [0,1]$. Since $\left.v_t\right|_X = 0$, we have $\left.\rho_t\right|_X = \mathrm{id}_X$.

Set $\varphi = \rho_1, \mathcal{U}_1 = \rho_1(\mathcal{U}_0)$. ■

Chapter 8

Darboux-Moser-Weinstein Theory

8.1 Darboux Theorem

Theorem 8.1.1 (Darboux). *Let (M,ω) be a symplectic manifold, and let p be any point in M. Then we can find a coordinate system $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots y_n)$ centered at p such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Remark 8.1.2. As a consequence of Darboux theorem, if we prove for $\left(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i\right)$ a local assertion which is invariant under symplectomorphisms, then that assertion holds for any symplectic manifold. ♠

Proof. Apply the Moser relative theorem to $X = \{p\}$: Use any symplectic basis for T_pM to construct coordinates $(x_1', \ldots, x_n', y_1', \ldots y_n')$ centered at p and valid on some neighborhood $\mathcal{U}'$, so that

$$\omega_p = \sum dx_i' \wedge dy_i'\Big|_p .$$

There are two symplectic forms on $\mathcal{U}'$: the given $\omega_0 = \omega$ and $\omega_1 = \sum dx_i' \wedge dy_i'$. By the Moser theorem, there are neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\varphi(p) = p \quad \text{and} \quad \varphi^*\left(\sum dx_i' \wedge dy_i'\right) = \omega.$$

Since

$$\varphi^*\left(\sum dx_i' \wedge dy_i'\right) \xlongequal{[8]14.16(c)} \sum d\left(x_i' \circ \varphi\right) \wedge d\left(y_i' \circ \varphi\right),$$

we only need to set new coordinates $x_i = x_i' \circ \varphi$ and $y_i = y_i' \circ \varphi$ with $\mathcal{U} = \varphi(\mathcal{U}_1 \cap \mathcal{U}') \subseteq \mathcal{U}_2$. ■

If in the Moser relative theorem we assume instead

Hypothesis: X is an n-dimensional submanifold with $i^*\omega_0 = i^*\omega_1 = 0$ where $i : X \hookrightarrow M$ is inclusion, i.e., X is a submanifold lagrangian for ω_0 and ω_1,

then Weinstein proved that the conclusion still holds. We need some algebra for the Weinstein theorem.

8.2 Lagrangian Subspaces

Subspace complement: Let $U \subseteq V$ be a subspace of vector space V. Then a subspace W is said to be a **complement** of U if $U \cap W = \{0\}$ and $V = U + W$. Note that complements are far from unique. For example, in $V = \mathbb{R}^2$, we have a subspace $U =$a line passing through 0. Then the complement of U can be any line passing through 0 but not parallel with U.

Suppose that U, W are n-dimensional vector spaces, and $\Omega : U \times W \to \mathbb{R}$ is a bilinear pairing; the map Ω gives rise to a linear map $\widetilde{\Omega} : U \to W^*, \widetilde{\Omega}(u) = \Omega(u, \cdot)$. Then Ω is **nondegenerate** if and only if $\widetilde{\Omega}$ is bijective.

Lemma 8.2.1. *Suppose (V, Ω) is a symplectic vector space. Let U be a Lagrangian subspace of V and W be a complement of U. Then $B := \Omega' : U \times W \to \mathbb{R}$ is nondegenerate in the above sense, where $B(u, w) = \Omega(u, w)$, i.e., $B = \Omega' = \Omega|_{U \times W}$.*

Proof. $\ker \widetilde{B} = \{u \in U : B(u, w) = 0\ \forall v \in W\}$. Thus, suppose $u \in U$ such that $B(u, w) = 0\ \forall v \in W$, so $\Omega(u, w) = 0\ \forall v \in W$. Since U is Lagrangian, $\Omega(u, v) = 0\ \forall v \in U$, we see that for any $e = w + v \in U \oplus W = V$, we have $\Omega(u, e) = \Omega(u, w + v) = \Omega(u, w) + \Omega(u, v) = 0$.

We thus showed that $u \in \ker \widetilde{B} \implies u \in \ker \widetilde{\Omega}$, or $\ker \widetilde{B} \subseteq \ker \widetilde{\Omega}$. Now, Ω nondegenerate gives $\ker \widetilde{\Omega} = \{0\} \implies \ker \widetilde{B} = \{0\}$. ■

Proposition 8.2.2. *Suppose that (V, Ω) is a $2n$-dimensional symplectic vector space. Let U be a lagrangian subspace of (V, Ω) (i.e., $\Omega|_{U \times U} = 0$ and U is n-dimensional). Let W be any vector space complement to U, not necessarily lagrangian. Then from W we can canonically build a lagrangian complement to U.*

Proof. By Lemma 8.2.1, the pairing Ω gives a nondegenerate pairing $U \times W \xrightarrow{\Omega'} \mathbb{R}$. Therefore, $\widetilde{\Omega}' : U \to W^*$ is bijective. We look for a lagrangian complement to U of the form

$$W' = \{w + Aw \mid w \in W\},$$

the map $A : W \to U$ being linear. For W' to be lagrangian we need

$$\begin{aligned}
&\forall w_1, w_2 \in W, \quad \Omega\left(w_1 + Aw_1, w_2 + Aw_2\right) = 0 \\
&\implies \Omega\left(w_1, w_2\right) + \Omega\left(w_1, Aw_2\right) + \Omega\left(Aw_1, w_2\right) + \underbrace{\Omega(\underbrace{Aw_1, Aw_2}_{\in U})}_{0} = 0 \\
&\implies \Omega\left(w_1, w_2\right) = \underset{\widetilde{\Omega}}{\Omega}\left(Aw_2, w_1\right) - \Omega\left(Aw_1, w_2\right) \\
&= \widetilde{\Omega}'\left(Aw_2\right)\left(w_1\right) - \widetilde{\Omega}'\left(Aw_1\right)\left(w_2\right).
\end{aligned}$$

Let $A' = \widetilde{\Omega}' \circ A : W \to W^*$, and look for A' such that

$$\forall w_1, w_2 \in W, \quad \Omega\left(w_1, w_2\right) = A'\left(w_2\right)\left(w_1\right) - A'\left(w_1\right)\left(w_2\right).$$

The canonical choice is $A'(w) = -\frac{1}{2}\Omega(w, \cdot)$. Then set $A = \left(\widetilde{\Omega}'\right)^{-1} \circ A'$. ■

Proposition 8.2.3. *Let V be a $2n$-dimensional vector space, let Ω_0 and Ω_1 be symplectic forms in V, let U be a subspace of V lagrangian for Ω_0 and Ω_1, and let W be any complement to U in V. Then from W we can canonically construct a linear isomorphism $L : V \overset{\cong}{\rightrightarrows} V$ such that $L|_U = \mathrm{Id}_U$ and $L^*\Omega_1 = \Omega_0$.*

Proof. By the last proposition, from W we canonically obtain complements W_0 and W_1 to U in V such that W_0 is lagrangian for Ω_0 and W_1 is lagrangian for Ω_1. By Lemma 8.2.1, we get nondegenerate bilinear pairings

$$W_0 \times U \xrightarrow{\Omega_0} \mathbb{R}$$
$$W_1 \times U \xrightarrow{\Omega_1} \mathbb{R}$$

which give isomorphisms

$$\widetilde{\Omega}_0 : W_0 \xrightarrow{\simeq} U^*$$
$$\widetilde{\Omega}_1 : W_1 \xrightarrow{\simeq} U^*.$$

Consider the diagram

$$\begin{tikzcd} W_0 \arrow[r, "\widetilde{\Omega}_0"] \arrow[d, "B"'] & U^* \arrow[d, "\mathrm{id}"] \\ W_1 \arrow[r, "\widetilde{\Omega}_1"'] & U^* \end{tikzcd}$$

where the linear map B satisfies $\widetilde{\Omega}_1 \circ B = \widetilde{\Omega}_0$, or $B := (\widetilde{\Omega}_1)^{-1} \circ \widetilde{\Omega}_0 : W_0 \to W_1$. Then $\Omega_0(w_0, u) = \Omega_1(Bw_0, u)$, $\forall w_0 \in W_0, \forall u \in U$. Extend B to the rest of V by setting it to be the identity on U :

$$L := \mathrm{Id}_U \oplus B : U \oplus W_0 \longrightarrow U \oplus W_1.$$

Finally, we check that $L^*\Omega_1 = \Omega_0$.

$$\begin{aligned} (L^*\Omega_1)(u \oplus w_0, u' \oplus w_0') &= \Omega_1(u \oplus Bw_0, u' \oplus B\omega_0') \\ &= \Omega_1(u, Bw_0') + \Omega_1(Bw_0, u') \\ &= \Omega_0(u, w_0') + \Omega_0(w_0, u') \\ &= \Omega_0(u \oplus w_0, u' \oplus w_0'). \end{aligned}$$

where the second and last equalities use Lagrangianity:

$$\Omega_1(\overbrace{u, u'}^{\in U}) = 0,\ \Omega_1(\overbrace{Bw_0, Bw_0'}^{\in W_1}) = 0,\ \Omega_0(\overbrace{u, u'}^{\in U}) = 0,\ \Omega_0(\overbrace{w_0, w_0'}^{\in W_0}) = 0.$$

■

8.3 Weinstein Lagrangian Neighborhood Theorem

Theorem 8.3.1 (Weinstein Lagrangian Neighborhood Theorem). *Let M be a $2n$-dimensional manifold, X a compact n-dimensional submanifold, $i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e., X is a lagrangian submanifold of both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes and

$$\varphi^* \omega_1 = \omega_0.$$

The proof of the Weinstein theorem uses the Whitney extension theorem.

Theorem 8.3.2 (Whitney Extension Theorem)**.** *Let M be an n-dimensional manifold and X a k-dimensional submanifold with $k < n$. Suppose that at each $p \in X$ we are given a linear isomorphism $L_p : T_pM \overset{\simeq}{\rightrightarrows} T_pM$ such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and L_p depends smoothly on p. Then there exists an embedding $h : \mathcal{N} \to M$ of some neighborhood $\mathcal{N}$ of X in M such that $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for all $p \in X$. The linear maps L serve as "germs" for the embedding.*

Proof of the Weinstein theorem. Put a riemannian metric g on M; at each $p \in M, g_p(\cdot,\cdot)$ is a positive-definite inner product. Fix $p \in X$, and let $V = T_pM$, $U = T_pX$ and $W = U^\perp$ the orthocomplement of U in V relative to $g_p(\cdot,\cdot)$.

Since $i^*\omega_0 = i^*\omega_1 = 0$, the space U is a lagrangian subspace of both $\left(V, \omega_0|_p\right)$ and $\left(V, \omega_1|_p\right)$. By symplectic linear algebra, we canonically get from $U^\perp$ a linear isomorphism $L_p : T_pM \to T_pM$, such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and $L_p^*\omega_1\big|_p = \omega_0|_p$. L_p varies smoothly with respect to p since our recipe is canonical!

By the Whitney theorem, there are a neighborhood $\mathcal{N}$ of X and an embedding $h : \mathcal{N} \hookrightarrow M$ with $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. Hence, at any $p \in X$,

$$(h^*\omega_1)_p = (dh_p)^* \omega_1\big|_p = L_p^*\omega_1\big|_p = \omega_0|_p$$

Applying the Moser relative theorem to ω_0 and $h^*\omega_1$, we find a neighborhood $\mathcal{U}_0$ of X and an embedding $f : \mathcal{U}_0 \to \mathcal{N}$ such that $f|_X = \mathrm{id}_X$ and $f^*(h^*\omega_1) = \omega_0$ on $\mathcal{U}_o$. Set $\varphi = h \circ f$. ■

Sketch of proof for the Whitney theorem. Case $M = \mathbb{R}^n$: For a compact k-dimensional submanifold X, take a neighborhood of the form

$$\mathcal{U}^\varepsilon = \{p \in M \mid \text{ distance } (p, X) \le \varepsilon\}$$

For ε sufficiently small so that any $p \in \mathcal{U}^\varepsilon$ has a unique nearest point in X, define a projection $\pi : \mathcal{U}^\varepsilon \to X, p \mapsto$ point on X closest to p. If $\pi(p) = q$, then $p = q + v$ for some $v \in N_qX$ where $N_qX = (T_qX)^\perp$ is the normal space at q; see Homework 5. Let

$$\begin{aligned} h : \mathcal{U}^\varepsilon &\longrightarrow \mathbb{R}^n \\ p &\longmapsto q + L_q v, \end{aligned}$$

where $q = \pi(p)$ and $v = p - \pi(p) \in N_qX$. Then $h_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. If X is not compact, replace ε by a continuous function $\varepsilon : X \to \mathbb{R}^+$which tends to zero fast enough as x tends to infinity. General case: Choose a riemannian metric on M. Replace distance by riemannian distance, replace straight lines $q + tv$ by geodesics $\exp(q, v)(t)$ and replace $q + L_qv$ by the value at $t = 1$ of the geodesic with initial value q and initial velocity L_qv. ■

Later we will need the following generalization of above Weinstein Lagrangian Neighborhood Theorem. For a proof see, for instance, either of $[47, 58, 107]$ in the references of [4].

Theorem 8.3.3 (Coisotropic Embedding Theorem)**.** *Let M be a manifold of dimension $2n, X$ a compact submanifold of dimension $k \ge n, i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M, such that $i^*\omega_0 = i^*\omega_1$ and X is coisotropic for both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

8.4 Oriented Surfaces

Exercise 8.4.1. *The standard symplectic form on the 2-sphere is the standard area form: If we think of S^2 as the unit sphere in 3 -space*

$$S^2 = \left\{u \in \mathbb{R}^3 \text{ such that } |u| = 1\right\},$$

then the induced area form is given by

$$\omega_u(v, w) = \langle u, v \times w\rangle$$

where $u \in S^2, v, w \in T_u S^2$ are vectors in $\mathbb{R}^3$, $\times$ is the exterior product, and $\langle\cdot,\cdot\rangle$ is the standard inner product. With this form, the total area of S^2 is 4π. Consider cylindrical polar coordinates (θ, z) on S^2 away from its poles, where $0 \leq \theta < 2\pi$ and $-1 \leq z \leq 1$. Show that, in these coordinates,

$$\omega = d\theta \wedge dz$$

Exercise 8.4.2. *Prove the Darboux theorem in the 2-dimensional case, using the fact that every nonvanishing 1 -form on a surface can be written locally as $f dg$ for suitable functions f, g.*

[Hint: $\omega = df \wedge dg$ is nondegenerate $\Longleftrightarrow$ (f, g) is a local diffeomorphism.]

Exercise 8.4.3. *Any oriented 2-dimensional manifold with an area form is a symplectic manifold.*

(a) Show that convex combinations of two area forms ω_0, ω_1 that induce the same orientation are symplectic. This is wrong in dimension 4: find two symplectic forms on the vector space $\mathbb{R}^4$ that induce the same orientation, yet some convex combination of which is degenerate. Find a path of symplectic forms that connect them.

(b) Suppose that we have two area forms ω_0, ω_1 on a compact 2-dimensional manifold M representing the same de Rham cohomology class, i.e., $[\omega_0] = [\omega_1] \in H^2_{\text{deRham}}(M)$. Prove that there is a 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ such that $\varphi_1^*\omega_0 = \omega_1, \varphi_0 = \mathrm{id}$, and $\varphi_t^*\omega_0$ is symplectic for all $t \in [0,1]$.

[Hint: Exercise (a) and the Moser trick.]

Remark 8.4.4. Such a 1-parameter family φ_t is called a strong isotopy between ω_0 and ω_1. In this language, this exercise shows that, up to strong isotopy, there is a unique symplectic representative in each non-zero 2-cohomology class of M. ♠

Chapter 9

Weinstein Tubular Neighborhood Theorem

9.1 Observation from Linear Algebra

Let (V, Ω) be a symplectic linear space, and let U be a lagrangian subspace.

Proposition 9.1.1. *There is a canonical nondegenerate bilinear pairing $\Omega' : V/U \times U \to \mathbb{R}$.*

Proof. Define $\Omega'([v], u) = \Omega(v, u)$ where $[v]$ is the equivalence class of v in V/U. We need to check that Ω' is well-defined and nondegenerate.

Well-defined: suppose $v' \in [v]$, i.e., $\exists u$ such that $v' - v = u$. We want to show $\Omega(v', u) = \Omega(v, u)$. This is true as $\Omega(u + v, u) = \Omega(u, u) + \Omega(v, u) = 0 + \Omega(v, u)$.

Nondegeneracy: suppose $u \in \ker \widetilde{\Omega'}$, i.e., $u \in U$ and $\Omega'(u, [v]) = 0$ for all $[v] \in V/U$. Thus, for all $v \in V$, we have $\Omega'(u, v) = \Omega'(u, [v]) = 0$. This implies $\ker \widetilde{\Omega'} \subseteq \ker \widetilde{\Omega} \xlongequal{\text{nondegeneracy}} \{0\} \implies \ker \widetilde{\Omega'} = \{0\}$. ∎

Consequently, we get

$\implies \widetilde{\Omega}' : V/U \to U^*$ defined by $\widetilde{\Omega}'([v]) = \Omega'([v], \cdot)$ is an isomorphism.

$\implies V/U \simeq U^*$ are canonically identified.

In particular, if (M, ω) is a symplectic manifold, and X is a lagrangian submanifold, then T_xX is a lagrangian subspace of (T_xM, ω_x) for each $x \in X$. The space $N_xX := T_xM/T_xX$ is called the **normal space of X at x**.

$\implies$ There is a canonical identification $N_xX \simeq T_x^*X$.

Therefore, we obtain

Theorem 9.1.2. *The vector bundles NX and T^*X are canonically identified.*

9.2 Tubular Neighborhoods

We have shown the following theorem before.

Theorem 9.2.1 (Standard Tubular Neighborhood Theorem)**.** *Let M be an n dimensional manifold, X a k-dimensional submanifold, NX the normal bundle of X in M, $i_0 : X \hookrightarrow NX$ the zero section, and $i : X \hookrightarrow M$*

inclusion. Then there are neighborhoods $\mathcal{U}_0$ of X in NX,$\mathcal{U}$ of X in M and a diffeomorphism $\psi : \mathcal{U}_0 \to \mathcal{U}$ such that

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\psi"] & & \mathcal{U} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes.

We combine this with the Weinstein lagrangian neighborhood theorem to show:

Theorem 9.2.2 (Weinstein Tubular Neighborhood Theorem)**.** *Let (M, ω) be a symplectic manifold, X a compact lagrangian submanifold, ω_0 the canonical symplectic form on $T^*X, i_0 : X \hookrightarrow T^*X$ the lagrangian embedding (i.e., $(i_0)^*\omega_0 = 0$) as the zero section, and $i : X \hookrightarrow M$ the lagrangian embedding ($i^*\omega = 0$) given by inclusion. Then there are neighborhoods $\mathcal{U}_0$ of X in T^*X,$\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes and

$$\varphi^*\omega = \omega_0.$$

Proof. (1) By the Standard Tubular Neighborhood Theorem:

Since $NX \simeq T^*X$, we can find a neighborhood $\mathcal{N}_0$ of X in T^*X, a neighborhood $\mathcal{N}$ of X in M, and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}$ such that

$$\begin{tikzcd} \mathcal{N}_0 \arrow[rr, "\psi"] & & \mathcal{N} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes.

(2) By the Weinstein Tubular Neighborhood Theorem:

Let $\omega_0 =$canonical form on T^*X and $\omega_1 = \psi^*\omega$ be symplectic forms on $\mathcal{N}_0$. The submanifold X is lagrangian for both ω_0 and ω_1. There exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in $\mathcal{N}_0$ and a diffeomorphism $\theta : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\theta"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i_0"] \arrow[ur, "i_0"'] & \end{tikzcd}$$

commutes and

$$\theta^*\omega_1 = \omega_0.$$

Take $\varphi = \psi \circ \theta$ and $\mathcal{U} = \varphi(\mathcal{U}_0)$. Check that $\varphi^*\omega = \theta^* \underbrace{\psi^*\omega}_{\omega_1} = \omega_0$. ∎

Remark 9.2.3. This theorem classifies lagrangian embeddings: up to local symplectomorphism, the set of lagrangian embeddings is the set of embeddings of manifolds into their cotangent bundles as zero sections.

The classification of **isotropic embeddings** was also carried out by Weinstein in [105,107]. An isotropic embedding of a manifold X into a symplectic manifold (M,ω) is a closed embedding $i : X \hookrightarrow M$ such that $i^*\omega = 0$. Weinstein showed that neighbourhood equivalence of isotropic embeddings is in one-to-one correspondence with isomorphism classes of symplectic vector bundles.

The classification of **coisotropic embeddings** is due to Gotay [47]. A coisotropic embedding of a manifold X carrying a closed 2-form α of constant rank into a symplectic manifold (M,ω) is an embedding $i : X \hookrightarrow M$ such that $i^*\omega = \alpha$ and $i(X)$ is coisotropic as a submanifold of M. Let E be the **characteristic distribution** of a closed form α of constant rank on X, i.e., E_p is the kernel of α_p at $p \in X$. Gotay showed that then E^* carries a symplectic structure in a neighbourhood of the zero section, such that X embeds coisotropically onto this zero section, and, moreover every coisotropic embedding is equivalent to this in some neighbourhood of the zero section. ♠

9.3 Application 1: Tangent Space to the Group of Symplectomorphisms

The symplectomorphisms of a symplectic manifold (M,ω) form the group

$$\operatorname{Sympl}(M,\omega) = \left\{ f : M \xrightarrow{\simeq} M \mid f^*\omega = \omega \right\}.$$

- What is $T_{\text{id}}\left(\operatorname{Sympl}(M,\omega)\right)$?
  (What is the "Lie algebra" of the group of symplectomorphisms?)
- What does a neighborhood of id in $\operatorname{Sympl}(M,\omega)$ look like?

We use notions from the C^1-topology: Let X and Y be manifolds.

Definition 9.3.1. *A sequence of maps $f_i : X \to Y$* ***converges in the C^0-topology*** *to $f : X \to Y$ if and only if f_i converges uniformly on compact sets.*

Definition 9.3.2. *A sequence of C^1 maps $f_i : X \to Y$* ***converges in the C^1 topology*** *to $f : X \to Y$ if and only if it and the sequence of derivatives $df_i : TX \to TY$ converge uniformly on compact sets.*

For these two topologies, think about them locally: compactness helps to let a set to be covered by finitely many charts; then for each chart think about the $\mathbb{R}^n$ case of these topologies.

Let (M,ω) be a compact symplectic manifold and $f \in \operatorname{Sympl}(M,\omega)$. Then by proposition 3.4.1, we see that $\left.\begin{array}{l}\text{Graph } f\\ \text{Graph id } = \Delta\end{array}\right\}$ are lagrangian submanifolds of $(M \times M, \operatorname{pr}_1^*\omega - \operatorname{pr}_2^*\omega)$. ($\operatorname{pr}_i : M \times M \to M, i = 1,2,$ are the projections to each factor.)

By the Weinstein tubular neighborhood theorem, there exists a neighborhood $\mathcal{U}$ of the diagonal $\Delta(\simeq M)$ in $(M \times M, \operatorname{pr}_1^*\omega - \operatorname{pr}_2^*\omega)$ which is symplectomorphic to a neighborhood $\mathcal{U}_0$ of M in (T^*M, ω_0). Let $\varphi : \mathcal{U} \to \mathcal{U}_0$ be the symplectomorphism satisfying $\varphi(p,p) = (p,0), \forall p \in M$.

Suppose that f is sufficiently C^1**-close to id**, i.e., f is in some sufficiently small neighborhood of id in the C^1-topology. Then:

1. We can assume that Graph $f \subseteq \mathcal{U}$. Let $j : M \hookrightarrow \mathcal{U}$ be the embedding as Graph f, $i : M \hookrightarrow \mathcal{U}$ be the embedding as Graph id $= \Delta$.
2. The map j is sufficiently C^1-close to i.

3. By the Weinstein theorem, $\mathcal{U} \simeq \mathcal{U}_0 \subseteq T^*M$, so the above j and i induce $j_0 : M \hookrightarrow \mathcal{U}_0$ embedding, where $j_0 = \varphi \circ j$, $i_0 : M \hookrightarrow \mathcal{U}_0$ embedding as 0 -section . Hence, we have

$$\begin{tikzcd} M \times M \supseteq \mathcal{U} \arrow[rr, "\varphi"] & & \mathcal{U}_0 \subseteq T^*M \\ & X \arrow[ul, "i"] \arrow[ur, "i_0"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} M \times M \supseteq \mathcal{U} \arrow[rr, "\theta"] & & \mathcal{U}_0 \subseteq T^*M \\ & X \arrow[ul, "j"] \arrow[ur, "j_0"'] & \end{tikzcd}$$

where $i(p) = (p, p), i_0(p) = (p, 0), j(p) = (p, f(p))$ and $j_0(p) = \varphi(p, f(p))$ for $p \in M$.

4. The map j_0 is sufficiently C^1-close to i_0.
$\Downarrow$
The image set $j_0(M)$ intersects each T_p^*M at one point μ_p depending smoothly on p.

5. The image of j_0 is the image of a smooth section $\mu : M \to T^*M$, that is, a 1-form $\mu = j_0 \circ (\pi \circ j_0)^{-1}$.

Therefore, Graph $f \simeq \left\{(p, \mu_p) \mid p \in M, \mu_p \in T_p^*M\right\}$.

Exercise 9.3.3. *Vice-versa: if μ is a 1-form sufficiently C^1-close to the zero 1-form, then*

$$\left\{(p, \mu_p) \mid p \in M, \mu_p \in T_p^*M\right\} \simeq \text{Graph } f,$$

for some diffeomorphism $f : M \to M$. By proposition 3.2.3, we have Graph f is lagrangian $\Longleftrightarrow$ μ is closed.

Conclusion. A small C^1-neighborhood of id in $\mathrm{Sympl}(M, \omega)$ is homeomorphic to a C^1-neighborhood of zero in the vector space of closed 1-forms on M. So:

$$T_{\text{id}}\left(\mathrm{Sympl}(M, \omega)\right) \simeq \left\{\mu \in \Omega^1(M) \mid d\mu = 0\right\} = \mathcal{Z}^1(M).$$

In particular, $T_{\text{id}}(\mathrm{Sympl}(M, \omega))$ contains the space of exact 1-forms

$$\mathcal{B}^1(M) = \{\mu = dh \mid h \in C^\infty(M)\} \simeq C^\infty(M)/ \text{ locally constant functions }.$$

9.4 Application 2: Fixed Points of Symplectomorphisms

Theorem 9.4.1. *Let (M, ω) be a compact symplectic manifold with $H^1_{\text{deRham}}(M) = 0$. Then any symplectomorphism of M which is sufficiently C^1-close to the identity has at least two fixed points.*

Proof. Suppose that $f \in \mathrm{Sympl}(M, \omega)$ is sufficiently C^1-close to id. Then Graph $f \simeq$ closed 1-form μ on M.

$$\left.\begin{array}{l} d\mu = 0 \\ H^1_{\text{deRham}}(M) = 0 \end{array}\right\} \Longrightarrow \mu = dh \text{ for some } h \in C^\infty(M).$$

Since M is compact, h has at least 2 critical points (see for example Bishop and Goldberg's *Tensor Analysis on Manifolds* p.145 Theorem 3.10.4.).

$$\begin{array}{ccc} \text{Fixed points of f} & = & \text{critical points of } h \\ \| & & \| \\ \text{Graph } f \cap \Delta & = & \{p : \mu_p = dh_p = 0\}. \end{array}$$

where the bottom equality is due to third step in the last section. ∎

Lagrangian intersection problem:
A submanifold Y of M is C^1-close to X when there is a diffeomorphism $X \to Y$ which is, as a map into M, C^1-close to the inclusion $X \hookrightarrow M$.

Theorem 9.4.2. *Let (M, ω) be a symplectic manifold. Suppose that X is a compact lagrangian submanifold of M with $H^1_{deRham}(X) = 0$. Then every lagrangian submanifold of M which is C^1-close to X intersects X in at least two points.*

Exercise 9.4.3. *Prove above theorem.*

Arnold conjecture:
Let (M, ω) be a compact symplectic manifold, and $f : M \to M$ a symplectomorphism which is "exactly homotopic to the identity" (see below). Then

$$\begin{aligned} \#\{\text{ fixed points of } f\} \geq & \text{ minimal \# of critical points} \\ & \text{ a smooth function on } M \text{ can have} . \end{aligned}$$

Together with Morse theory,* we obtain†

$$\begin{aligned} \#\{\text{ nondegenerate fixed points of } f\} \geq & \text{ minimal \# of critical points} \\ & \text{ a Morse function on } M \text{ can have} \\ & \geq \sum_{i=0}^{2n} \dim H^i(M; \mathbb{R}). \end{aligned}$$

The Arnold conjecture was proved by Conley-Zehnder, Floer, Hofer-Salamon, Ono, Fukaya-Ono, Liu-Tian using Floer homology (which is an ∞-dimensional analogue of Morse theory). There are open conjectures for sharper bounds on the number of fixed points.

Meaning of "f is exactly homotopic to the identity:" Suppose that $h_t : M \to \mathbb{R}$ is a smooth family of functions which is 1-periodic, i.e., $h_t = h_{t+1}$. Let $\rho : M \times \mathbb{R} \to M$ be the isotopy generated by the time-dependent vector field v_t defined by $\omega(v_t, \cdot) = dh_t$. Then "$f$ being exactly homotopic to the identity" means $f = \rho_1$ for some such h_t.

In other words, f is **exactly homotopic to the identity** when f is the time- 1 map of an isotopy generated by some smooth time-dependent 1-periodic hamiltonian function. There is a one-to-one correspondence

$$\text{fixed points of } f \quad \stackrel{1-1}{\longleftrightarrow} \quad \text{period-1 orbits of } \rho : M \times \mathbb{R} \to M$$

because $f(p) = p$ if and only if $\{\rho(t, p), t \in [0, 1]\}$ is a closed orbit.

Proof of the Arnold conjecture in the case when $h : M \to \mathbb{R}$ is independent of t : p is a critical point of $h \Longleftrightarrow dh_p = 0 \Longleftrightarrow v_p = 0 \Longrightarrow \rho(t, p) = p, \forall t \in \mathbb{R} \Longrightarrow p$ is a fixed point of ρ_1.

Exercise 9.4.4. *Compute these estimates for the number of fixed points on some compact symplectic manifolds (for instance, $S^2, S^2 \times S^2$ and $T^2 = S^1 \times S^1$).*

* A Morse function on M is a function $h : M \to \mathbb{R}$ whose critical points (i.e., points p where $dh_p = 0$) are all nondegenerate (i.e., the hessian at those points is nonsingular: $\det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)_p \neq 0$).

† A fixed point p of $f : M \to M$ is nondegenerate if $df_p : T_pM \to T_pM$ is nonsingular.

Part IV

Moment Maps

Chapter 10

Hamiltonian Vector Fields

10.1 Hamiltonian and Symplectic Vector Fields

- What does a symplectic geometer do with a real function?...

Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function. Its differential dH is a 1-form. By nondegeneracy, there is a unique vector field X_H on M such that $X_H = \widetilde{\omega}^{-1}(dH)$, i.e., $\omega(X_H, \cdot) = dH$, or $\iota_{X_H}\omega = dH$. Then integrate X_H. Supposing that M is compact, or at least that X_H is complete, let $\rho_t : M \to M, t \in \mathbb{R}$, be the one-parameter family of diffeomorphisms generated by X_H:

$$\begin{cases} \rho_0 = \mathrm{id}_M \\ \frac{d\rho_t}{dt} \circ \rho_t^{-1} = X_H \end{cases}$$

Claim. Each diffeomorphism ρ_t preserves ω, i.e., $\rho_t^*\omega = \omega, \forall t$.

Proof. We have $\frac{d}{dt}\rho_t^*\omega \xlongequal{\text{[8]prop.12.36}} \rho_t^*\mathcal{L}_{X_H}\omega \xlongequal{\text{Cartan magic formula}} \rho_t^*(\underbrace{d\iota_{X_H}\omega}_{dH} + \iota_{X_H}\underbrace{d\omega}_{\substack{0 \text{ as} \\ \omega \text{ closed}}}) = 0.$ ∎

Therefore, every function on (M, ω) gives a family of symplectomorphisms. Notice how the proof involved both the nondegeneracy and the closedness of ω.

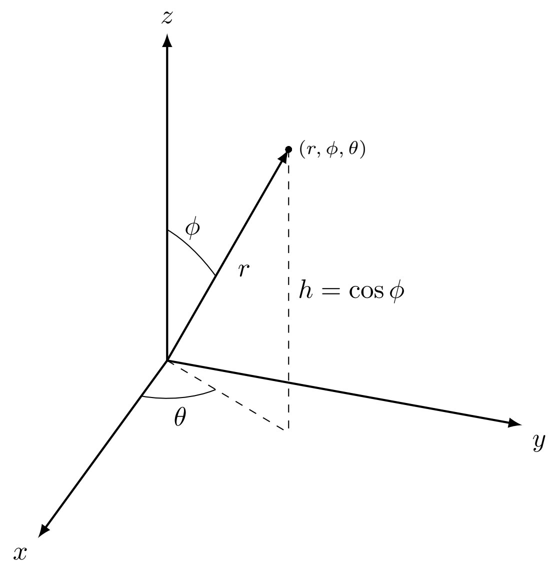
Definition 10.1.1. *A vector field X_H as above is called the **Hamiltonian vector field** with **Hamiltonian function** H.*

Example 10.1.2 (Height function on sphere)**.** We consider the spherical coordinates:

$$\begin{aligned} x &= f_1(\rho, \theta, \phi) = \rho \sin\phi \cos\theta \\ y &= f_2(\rho, \theta, \phi) = \rho \sin\phi \sin\theta \\ z &= f_3(\rho, \theta, \phi) = \rho \cos\phi \end{aligned}$$

Then take the unit sphere S^2 so that $\rho = 1$. A point P on S^2 will be $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$. It has "height" $h = \cos\phi$. We thus define the **height function** as $H(\theta, h) = h$ on the sphere

$$(M, \omega) = \left(S^2, \underbrace{d\theta \wedge dh}_{\substack{\text{i.e. standard form} \\ \text{for chart } (U,(\theta,h))}}\right).$$



We have

$$dH = \frac{\partial H}{\partial \theta} d\theta + \frac{\partial H}{\partial h} dh = dh$$

and

$$\begin{aligned} \iota_{X_H}(d\theta \wedge dh)(v) &= (d\theta \wedge dh)(X_H, v) \\ &= d\theta(X_H)dh(v) - d\theta(v)dh(X_H) \end{aligned}$$

This equals to $dH(v) = dh(v)$ iff

$$\begin{aligned} & d\theta(X_H) = 1,\ 0 = \overbrace{d\theta(v)}^{\neq 0} dh(X_H) \\ \iff & X_H = \frac{\partial}{\partial \theta}. \end{aligned}$$

Thus, $\rho_t(\theta, h) = (\theta + t, h)$, which is rotation about the vertical axis; the height function H is preserved by this motion. ♣

Exercise 10.1.3. *Let X be a vector field on an abstract manifold W. There is a unique vector field $X_\sharp$ on the cotangent bundle T^*W, whose flow is the lift of the flow of X (see chapter 2). Let α be the tautological 1-form on T^*W and let $\omega = -d\alpha$ be the canonical symplectic form on T^*W. Show that $X_\sharp$ is a Hamiltonian vector field with Hamiltonian function $H := \iota_{X_\sharp}\alpha$.*

Solution.

$$\begin{aligned} & \text{[8] 14.35 Cartan magic formula} \\ \implies & \mathcal{L}_{X_\sharp}\alpha = d\iota_{X_\sharp}\alpha + \iota_{X_\sharp} d\alpha = d\iota_{X_\sharp}\alpha - \iota_{X_\sharp}\omega \\ \implies & d\iota_{X_\sharp}\alpha = \mathcal{L}_{X_\sharp}\alpha + \iota_{X_\sharp}\omega \end{aligned}$$

Thus, showing $X_\sharp$ is a Hamiltonian vector field with Hamiltonian function $H := \iota_{X_\sharp}\alpha \iff d(\iota_{X_\sharp}\alpha) = \iota_{X_\sharp}\omega$ is equivalent of showing that $\mathcal{L}_{X_\sharp}\alpha = 0$.

Let τ_t be the flow on $M = T^*W$ and ρ_t be the flow on W. [8] 12.36 gives

$$(\mathcal{L}_{X_\sharp}\alpha)_p = \left.\frac{d}{dt}\right|_{t=0} (\tau_t^*\alpha)_p$$

Since $\tau_t : M = T^*W \to M = T^*W$ as a diffeomorphism is the lift of $\rho_t : W \to W$, Proposition 2.4.2 implies that

$$(\tau_t^* \alpha)_p = \alpha_p$$

which gets rid of the variable t. Then

$$(\mathcal{L}_{X_\sharp} \alpha)_p = 0$$

♦

Remark 10.1.4. If X_H is Hamiltonian, i.e., $dH = \iota_{X_H} H$, then

$$\mathcal{L}_{X_H} H \xlongequal{\text{magic formula}} \iota_{X_H} dH = \iota_{X_H} \iota_{X_H} \omega = 0$$

where the last step is because

$$\iota_{X_H} \iota_{X_H} \omega = \iota_{X_\sharp} \omega(X_H) = \omega(X_H, X_H) \xlongequal{\text{skew-symmetric}} 0.$$

Then by definition of Lie derivative (see [8] p.321), Hamiltonian vector fields preserve their Hamiltonian functions. That is, $H(x) = (\rho_t^* H)(x)$. And each integral curve $\{\rho_t(x) \mid t \in \mathbb{R}\}$ of X_H must be contained in a level set of H:

$$H(x) = (\rho_t^* H)(x) = H(\rho_t(x)), \quad \forall t$$

♠

Definition 10.1.5. *A vector field X on M preserving ω (i.e., such that $\mathcal{L}_X \omega = 0$) is called a* ***symplectic vector field****.*

$$\begin{cases} X \text{ is symplectic} & \Longleftrightarrow \iota_X \omega \text{ is closed} , \\ X \text{ is Hamiltonian} & \Longleftrightarrow \iota_X \omega \text{ is exact} . \end{cases}$$

Locally, by [8] Cor.17.15, we see that every symplectic vector field is Hamiltonian. If $H^1_{\text{deRham}}(M) = 0$, then globally every symplectic vector field is Hamiltonian. In general, $H^1_{\text{deRham}}(M)$ measures the obstruction for symplectic vector fields to be Hamiltonian.

Example 10.1.6. On the 2-torus $(M, \omega) = (\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields $X_1 = \frac{\partial}{\partial \theta_1}$ and $X_2 = \frac{\partial}{\partial \theta_2}$ are symplectic but not Hamiltonian. In fact, $d\iota_{X_1}\omega = d(\omega(X, \cdot)) = d(d\theta_2(\cdot) - 0) = 0$, and $\omega(X_1, \cdot) = d\theta_2(\cdot) \implies \exists H = \theta_2$ such that $\iota_X \omega$ is exact. These are similar for $X_2 = \partial/\partial\theta_2$. ♣

Remark 10.1.7. To summarize, vector fields on a symplectic manifold (M, ω) which preserve ω are called **symplectic**. The following are equivalent:

- X is a symplectic vector field;
- the flow ρ_t of X preserves ω, i.e., $\rho_t^* \omega = \omega$, for all t;
- $\mathcal{L}_X \omega = 0$;
- $\iota_X \omega$ is closed.

A **Hamiltonian** vector field is a vector field X for which

- $\iota_X \omega$ is exact,
  i.e., $\iota_X \omega = dH$ for some $H \in C^\infty(M)$. A primitive H of $\iota_X \omega$ is then called a **Hamiltonian function** of X.

♠

10.2 Classical Mechanics

Consider euclidean space $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $\omega_0 = \sum dq_j \wedge dp_j$. Let $X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$. Then,

$$\begin{aligned} \iota_{X_H} \omega &= \sum_{j=1}^n \iota_{X_H} (dq_j \wedge dp_j) = \sum_{j=1}^n \left[(\iota_{X_H} dq_j) \wedge dp_j - dq_j \wedge (\iota_{X_H} dp_j) \right] \\ &= \sum_{j=1}^n \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) = dH \end{aligned}$$

This shows that

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

is a Hamiltonian vector field with Hamiltonian function H. The curve $\rho_t = (q(t), p(t))$ is an integral curve for X_H, i.e., $X_H(\rho(t)) = \frac{d}{dt} \rho_t = \sum_{i=1}^n \left(\frac{dq_i(t)}{dt} \frac{\partial}{\partial q_i} + \frac{dp_i(t)}{dt} \frac{\partial}{\partial p_i} \right)$, exactly if

$$\begin{cases} \dfrac{dq_i}{dt}(t) = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt}(t) = -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{10.1}$$

This is called the **Hamiltonian equations**.

Remark 10.2.1. The gradient vector field of H relative to the euclidean metric is

$$\nabla H := \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial q_i} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial p_i} \right)$$

If J is the standard (almost) complex structure so that $J\left(\frac{\partial}{\partial q_i}\right) = \frac{\partial}{\partial p_i}$ and $J\left(\frac{\partial}{\partial p_i}\right) = -\frac{\partial}{\partial q_i}$, we have $JX_H = \nabla H$. ♠

The case where $n = 3$ has a simple physical illustration. Newton's second law states that a particle of mass m moving in **configuration space** $\mathbb{R}^3$ with coordinates $q = (q_1, q_2, q_3)$ under a potential $V(q)$ moves along a curve $q(t)$ such that

$$m \frac{d^2 q}{dt^2} = -\nabla V(q)$$

Introduce the **momenta** $p_i = m \frac{dq_i}{dt}$ for $i = 1, 2, 3$, and **energy** function $H(p, q) = \frac{1}{2m} |p|^2 + V(q)$. Let $\mathbb{R}^6 = T^* \mathbb{R}^3$ be the corresponding **phase space**, with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$. Newton's second law in $\mathbb{R}^3$ is equivalent to the Hamilton equations in $\mathbb{R}^6$:

$$\begin{cases} \frac{dq_i}{dt} = \frac{1}{m} p_i = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = m \frac{d^2 q_i}{dt^2} = -\frac{\partial V}{\partial q_i} = -\frac{\partial H}{\partial q_i} \end{cases}$$

The energy H is conserved by the physical motion.

10.3 Brackets

Vector fields are differential operators on functions: if X is a vector field and $f \in C^\infty(M)$, df being the corresponding 1-form, then

$$Xf \xlongequal{\text{[8]Thm14.24(iv)}} \overbrace{df(X)}^{=\iota_X df} \xlongequal{\text{[8]prop.12.32(a)}} \mathcal{L}_X f \tag{10.2}$$

Given two vector fields X, Y, there is a unique vector field W such that

$$\mathcal{L}_W f=\mathcal{L}_X\left(\mathcal{L}_Y f\right)-\mathcal{L}_Y\left(\mathcal{L}_X f\right)$$

The vector field W is called the **Lie bracket** of the vector fields X and Y and denoted $W=[X, Y]$, since $\mathcal{L}_W=\left[\mathcal{L}_X, \mathcal{L}_Y\right]$ is the commutator.

Exercise 10.3.1. *Check that, for any form α,*

$$\iota_{[X, Y]} \alpha=\mathcal{L}_X \iota_Y \alpha-\iota_Y \mathcal{L}_X \alpha=\left[\mathcal{L}_X, \iota_Y\right] \alpha$$

Since each side is an anti-derivation with respect to the wedge product, it suffices to check this formula on local generators of the exterior algebra of forms, namely functions and exact 1-forms.

Solution. See [8] 9.39. ♦

Proposition 10.3.2. *If X and Y are symplectic vector fields on a symplectic manifold (M, ω), then $[X, Y]$ is Hamiltonian with Hamiltonian function $\omega(Y, X)$.*

Proof.

$$\begin{aligned}
\iota_{[X, Y]} \omega & \overset{\text{exercise above}}{=\!=\!=} \mathcal{L}_X \iota_Y \omega-\iota_Y \mathcal{L}_X \omega \\
& \overset{\text{magic formula}}{=\!=\!=} d \iota_X \iota_Y \omega+\iota_X \underbrace{d \iota_Y \omega}_{0 \text{ as symp v.f.}}-\iota_Y \underbrace{d \iota_X \omega}_{0 \text{ as symp v.f.}}-\iota_Y \iota_X \underbrace{d \omega}_{0 \text{ as } \omega \text{ closed}} \\
& =d(\omega(Y, X))
\end{aligned}$$

■

A (real) **Lie algebra** is a (real) vector space $\mathfrak{g}$ together with a Lie bracket $[\cdot, \cdot]$, i.e., a bilinear map $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

(a) $[x, y]=-[y, x], \quad \forall x, y \in \mathfrak{g}$. (**antisymmetry**)

(b) $[x,[y, z]]+[y,[z, x]]+[z,[x, y]]=0, \quad \forall x, y, z \in \mathfrak{g}$. (**Jacobi identity**)

Let

$$\begin{aligned}
\chi(M) & =\{\text{ vector fields on } M\} \\
\chi^{\text{sympl}}(M) & =\{\text{ symplectic vector fields on } M\} \\
\chi^{\text{ham}}(M) & =\{\text{ Hamiltonian vector fields on } M\}
\end{aligned}$$

Corollary 10.3.3. *The inclusions $\left(\chi^{\text{ham}}(M),[\cdot, \cdot]\right) \subseteq\left(\chi^{\text{sympl}}(M),[\cdot, \cdot]\right) \subseteq(\chi(M),[\cdot, \cdot])$ are inclusions of Lie algebras.*

Definition 10.3.4. *The Poisson bracket of two functions $f, g \in C^{\infty}(M ; \mathbb{R})$ is*

$$\{f, g\}:=\omega\left(X_f, X_g\right) \in C^{\infty}(M)$$

Remark 10.3.5. We have $X_{\{f, g\}}=-\left[X_f, X_g\right]$ because Proposition 10.3.2 gives

$$\iota_{\left[x_g, X_f\right]}=d\left(\omega\left(X_f, X_g\right)\right) \Longrightarrow X_{\omega\left(X_f, X_g\right)}=\left[X_g, X_f\right]=-\left[X_f, X_g\right].$$

♠

Theorem 10.3.6. *The bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity, i.e.,*

$$\{f,\{g, h\}\}+\{g,\{h, f\}\}+\{h,\{f, g\}\}=0$$

Proof. eq.(10.2) gives $\mathcal{L}_{X_f} g = \iota_{X_f} dg$ (one could also see this by Cartan's magic formula, noticing the convention at [LeeSM] p.358 that $\iota_V \eta = 0$ for any zero-covector field η, i.e., a function). Then by definition of Hamiltonian function, $\mathrm{d}g = \iota_{X_g}\omega$,

$$\begin{aligned} \mathcal{L}_{X_f} g &= \iota_{X_f} dg \\ &= \iota_{X_f}\iota_{X_g}\omega \\ &= \omega(X_g, X_f) \\ &= -\{f, g\} \end{aligned} \tag{10.3}$$

Jacobi's Identity is now a simple computation.

$$\begin{aligned} \{\{f,g\},h\} + \{\{h,f\},g\} &= \mathcal{L}_{X_h}\{f,g\} + \mathcal{L}_{X_g}\{h,f\} \\ &= \mathcal{L}_{X_h}\mathcal{L}_{X_g} f - \mathcal{L}_{X_g}\mathcal{L}_{X_h} f \\ &= \mathcal{L}_{-[X_g,X_h]} f \quad \text{by [8] prop.9.39} \\ &= \mathcal{L}_{X_{\{g,h\}}} f \quad \text{by Remark 10.3.5} \\ &= -\{\{g,h\},f\} \end{aligned}$$

■

Definition 10.3.7. *A* ***Poisson algebra*** $(\mathcal{P}, \{\cdot,\cdot\})$ *is a commutative associative algebra* $\mathcal{P}$ *with a Lie bracket* $\{\cdot,\cdot\}$ *satisfying the Leibniz rule:*

$$\{f, gh\} = \{f,g\}h + g\{f,h\}$$

Exercise 10.3.8. *Check that the Poisson bracket* $\{\cdot,\cdot\}$ *defined above satisfies the Leibniz rule.*

Solution. Use eq.(10.3) twice:

$$\{f, gh\} = -\mathcal{L}_{X_f} gh \xlongequal{\text{[8]prop.12.32(b)}} -\left(\mathcal{L}_{X_f} g\right) h - g\left(\mathcal{L}_{X_f} h\right) = \{f,g\}h + g\{f,h\}$$

which gives the Leibniz rule. ♦

We conclude that, if (M,ω) is a symplectic manifold, then $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra. Furthermore, we have a Lie algebra anti-homomorphism

$$\begin{array}{rcl} C^\infty(M) & \longrightarrow & \chi(M) \\ H & \longmapsto & X_H \\ \{\cdot,\cdot\} & \rightsquigarrow & -[\cdot,\cdot]. \end{array}$$

10.4 Integrable Systems

Definition 10.4.1. *A* ***Hamiltonian system*** *is a triple* (M,ω,H)*, where* (M,ω) *is a symplectic manifold and* $H \in C^\infty(M;\mathbb{R})$ *is a function, called the* ***Hamiltonian function***.

Theorem 10.4.2. *Let* f *and* H *be two smooth functions. We have* $\{f,H\} = 0$ *if and only if* f *is constant along integral curves of* X_H.

Proof. Let ρ_t be the flow of X_H. Then

$$\begin{aligned} \frac{d}{dt}(f\circ\rho_t) = \frac{d}{dt}(\rho_t^* f) &\xlongequal{\text{[LeeSM]12.36}} \rho_t^*\mathcal{L}_{X_H} f \xlongequal{\text{(eq.10.2)}} \rho_t^*\iota_{X_H} df \\ &= \rho_t^*\iota_{X_H}\iota_{X_f}\omega = \rho_t^*\omega(X_f, X_H) = \rho_t^*\{f,H\} \end{aligned}$$

■

A function f as in above theorem is called an **integral of motion** (or a **first integral** or a **constant of motion**). In general, Hamiltonian systems do not admit integrals of motion which are independent of the Hamiltonian function. Functions $f_1, \ldots, f_n$ on M are said to be **independent** if their differentials $(df_1)_p, \ldots, (df_n)_p$ are linearly independent at all points p in some open dense subset of M. Loosely speaking, a Hamiltonian system is (completely) integrable if it has as many commuting integrals of motion as possible. Commutativity is with respect to the Poisson bracket. Notice that, if $f_1, \ldots, f_n$ are commuting integrals of motion for a Hamiltonian system (M, ω, H), then, at each $p \in M$, their Hamiltonian vector fields generate an isotropic subspace of T_pM:

$$\omega\left(X_{f_i}, X_{f_j}\right) = \{f_i, f_j\} = 0$$

If $f_1, \ldots, f_n$ are independent, then, by symplectic linear algebra, n can be at most half the dimension of M.

Definition 10.4.3. *A Hamiltonian system (M, ω, H) is **(completely) integrable** if it possesses $n = \frac{1}{2}\dim M$ independent integrals of motion, $f_1 = H, f_2, \ldots, f_n$, which are pairwise in involution with respect to the Poisson bracket, i.e., $\{f_i, f_j\} = 0$, for all i, j.*

Example 10.4.4. The simple pendulum (see Homework in the next section) and the harmonic oscillator are trivially integrable systems - any 2-dimensional Hamiltonian system (where the set of non-fixed points is dense) is integrable. ♣

Example 10.4.5. A Hamiltonian system (M, ω, H) where M is 4-dimensional is integrable if there is an integral of motion independent of H (the commutativity condition is automatically satisfied). Homework after chapter on Reduction shows that the spherical pendulum is integrable. ♣

For sophisticated examples of integrable systems, see [2]. Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level set, $f^{-1}(c)$, is a lagrangian submanifold, because it is n-dimensional and its tangent bundle is isotropic.

Lemma 10.4.6. *If the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are complete on the level $f^{-1}(c)$, then the connected components of $f^{-1}(c)$ are homogeneous spaces for $\mathbb{R}^n$, i.e., are of the form $\mathbb{R}^{n-k} \times \mathbb{T}^k$ for some $k, 0 \leq k \leq n$, where $\mathbb{T}^k$ is a k-dimensional torus.*

Proof. Exercise (just follow the flows to obtain coordinates). ■

Any compact component of $f^{-1}(c)$ must hence be a torus. These components, when they exist, are called **Liouville tori**. (The easiest way to ensure that compact components exist is to have one of the f_i's proper.)

Theorem 10.4.7. *(Arnold-Liouville [1]) Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level $f^{-1}(c)$ is a lagrangian submanifold of M.*

*(a) If the flows of $X_{f_1}, \ldots, X_{f_n}$ starting at a point $p \in f^{-1}(c)$ are complete, then the connected component of $f^{-1}(c)$ containing p is a homogeneous space for $\mathbb{R}^n$. With respect to this affine structure, that component has coordinates $\varphi_1, \ldots, \varphi_n$, known as **angle coordinates**, in which the flows of the vector fields $X_{f_1}, \ldots, X_{f_n}$ are linear.*

(b) There are coordinates $\psi_1, \ldots, \psi_n$, known as action coordinates, complementary to the angle coordinates such that the ψ_i 's are integrals of motion and $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ form a Darboux chart.

Therefore, the dynamics of an integrable system is extremely simple and the system has an explicit solution in action-angle coordinates. The proof of part (a) the easy part - of the Arnold-Liouville theorem is sketched above. For the proof of part (b), see [3], [28].

Geometrically, regular levels being lagrangian submanifolds implies that, in a neighborhood of a regular value, the map $f : M \to \mathbb{R}^n$ collecting the given integrals of motion is a **lagrangian fibration**, i.e., it is locally trivial and its fibers are lagrangian submanifolds. Part (a) of the Arnold-Liouville theorem states that there are coordinates along the fibers, the angle coordinates φ_i, in which the flows of $X_{f_1}, \ldots, X_{f_n}$ are linear. Part (b) of the theorem guarantees the existence of coordinates on $\mathbb{R}^n$, the action coordinates ψ_i, which (Poisson) commute among themselves and satisfy $\{\varphi_i, \psi_j\} = \delta_{ij}$ with respect to the angle coordinates. Notice that, in general, the action coordinates are not the given integrals of motion because $\varphi_1, \ldots, \varphi_n, f_1, \ldots, f_n$ do not form a Darboux chart.

10.5 Simple Pendulum

The **simple pendulum** is a mechanical system consisting of a massless rigid rod of length l, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate in the vertical plane. Assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.

(a) Let θ be the oriented angle between the rod (regarded as a line segment) and the vertical direction. Let ξ be the coordinate along the fibers of T^*S^1 induced by the standard angle coordinate on S^1. Show that the function $H : T^*S^1 \to \mathbb{R}$ given by

$$H(\theta, \xi) = \underbrace{\frac{\xi^2}{2ml^2}}_{K} + \underbrace{ml(1 - \cos\theta)}_{V}$$

is an appropriate Hamiltonian function to describe the simple pendulum. More precisely, check that gravity corresponds to the potential energy $V(\theta) = ml(1 - \cos\theta)$ (we omit universal constants), and that the kinetic energy is given by $K(\theta, \xi) = \frac{1}{2ml^2}\xi^2$.

(b) For simplicity assume that $m = l = 1$.

Plot the level curves of H in the (θ, ξ) plane. Show that there exists a number c such that for $0 < h < c$ the level curve $H = h$ is a disjoint union of closed curves. Show that the projection of each of these curves onto the θ-axis is an interval of length less than π. Show that neither of these assertions is true if $h > c$. What types of motion are described by these two types of curves? What about the case $H = c$?

(c) Compute the critical points of the function H. Show that, modulo 2π in θ, there are exactly two critical points: a critical point s where H vanishes, and a critical point u where H equals c. These points are called the stable and unstable points of H, respectively. Justify this terminology, i.e., show that a trajectory of the Hamiltonian vector field of H whose initial point is close to s stays close to s forever, and show that this is not the case for u. What is happening physically?

Chapter 11

Actions

11.1 One-Parameter Groups of Diffeomorphisms

Let M be a manifold and X a complete vector field on M. For each point $p \in M$, there is a unique maximal integral curve $\gamma(t)$ passing through p at $t = 0$, i.e., (1) $\gamma(0) = p$ and (2) $\frac{d}{dt}\gamma(t) = X_{\gamma(t)}$. Then since X is complete, we can define $\rho(t,p) = \gamma(t)$ on the whole domain $\mathcal{D} = \mathbb{R} \times M$. This is the flow ρ generated by X. Observe that $M_t = \{p \in M : (t,p) \in \mathcal{D}\} = M_{-t} = M$, and [8] Theorem 9.12 claims that for $t \in \mathbb{R}$, $\rho_t : M \to M$; $p \mapsto \rho(t,p)$ is a diffeomorphism with inverse $\rho_{-t} : M \to M$. We will also say these diffeomorphisms $\rho_t : M \to M$ form a family of diffeomorphisms generated by the complete vector field X. Due to the way ρ is defined, we have

$$\begin{cases} \rho_0(p) = p \\ \dfrac{d\rho_t(p)}{dt} = X\left(\rho_t(p)\right) \end{cases}$$

Claim. We have the group laws $\rho_0 = \mathrm{Id}$ and $\rho_t \circ \rho_s = \rho_{t+s}$.

Proof. $\rho_0 = \mathrm{Id}$ is clear. Let $\rho_s(q) = p$. We need to show that $(\rho_t \circ \rho_s)(q) = \rho_{t+s}(q)$, for all $t \in \mathbb{R}$. Reparametrize as $\tilde{\rho}_t(q) := \rho_{t+s}(q)$. Then

$$\begin{cases} \tilde{\rho}_0(q) = \rho_s(q) = p \\ \dfrac{d\tilde{\rho}_t(q)}{dt} = \dfrac{d\rho_{t+s}(q)}{dt} = X\left(\rho_{t+s}(q)\right) = X\left(\tilde{\rho}_t(q)\right) \end{cases}$$

i.e., $\tilde{\rho}_t(q)$ is an integral curve of X through p at time 0. By uniqueness we must have $\tilde{\rho}_t(q) = \rho_t(p)$, that is, $\rho_{t+s}(q) = \rho_t\left(\rho_s(q)\right)$. ■

Consequence. We have that $\rho_t^{-1} = \rho_{-t}$. In terms of the group $(\mathbb{R}, +)$ and the group $(\mathrm{Diff}(M), \circ)$ of all diffeomorphisms of M, these results can be summarized as:

Corollary 11.1.1. *The map $\mathbb{R} \to \mathrm{Diff}(M), t \mapsto \rho_t$, is a group homomorphism. The family $\{\rho_t \mid t \in \mathbb{R}\}$ is then called a **one-parameter group of diffeomorphisms of** M and denoted*

$$\rho_t = \exp tX$$

That is, $\exp tX(p)$ is the unique maximal integral curve passing through p at $t = 0$ and $\exp tX|_{t=0}(p) = p$, $\frac{d}{dt}\exp tX(p) = X(\exp tX(p))$.

11.2 Lie Groups: One-Parameter Subgroup and Exponential Map

There are also exponential map for Lie groups, which are manifolds, so we address the ambiguity now (cf. Cor. 11.1.1 and prop. 11.2.3).

A **one-parameter subgroup of** G is defined to be a Lie group homomorphism $\gamma : \mathbb{R} \to G$, with $\mathbb{R}$ considered as a Lie group under addition. By this definition, a one-parameter subgroup is not a Lie subgroup of G, but rather a homomorphism into G. (However, the image of a one-parameter subgroup is a Lie subgroup when endowed with a suitable smooth manifold structure; see [8] Problem 20-1.)

Theorem 11.2.1 (Characterization of One-Parameter Subgroups). *Let G be a Lie group. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity, i.e., passing through $e \in G$ at time 0.*

Proof. See [8] Theorem 20.1. ∎

Given $X \in \mathrm{Lie}(G) = \mathfrak{g}$, the one-parameter subgroup determined by X in this way is called the **one-parameter subgroup generated by** X. Because left-invariant vector fields are uniquely determined by their values at the identity (see [8]; **evaluation map** $\mathfrak{g} \to T_eG$; $X \mapsto X_1$ is an isomorphism), it follows that each one-parameter subgroup is uniquely determined by its initial velocity in T_eG, and thus there are one-to-one correspondences

$$\{\text{one-parameter subgroups of } G\} \overset{1-1}{\longleftrightarrow} \mathrm{Lie}(G) \overset{1-1}{\longleftrightarrow} T_eG. \tag{11.1}$$

Proposition 11.2.2. *Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in T_eH.*

Proof. See [8] Proposition 20.3. ∎

Given a Lie group G with Lie algebra g, we define a map $\exp : g \to G$, called the **exponential map of** G, as follows: for any $X \in \mathfrak{g}$, we set

$$\exp X = \gamma(1)$$

where γ is the one-parameter subgroup generated by X, or equivalently the integral curve of X starting at the identity. The following proposition shows that, like the matrix exponential, this map sends the line through X to the one-parameter subgroup generated by X.

Proposition 11.2.3. *Let G be a Lie group. For any $X \in \mathrm{Lie}(G), \gamma(s) = \exp sX$ is the one-parameter subgroup of G generated by X.*

Proof. Let $\gamma : \mathbb{R} \to G$ be the one-parameter subgroup generated by X, which is the integral curve of X starting at e. For any fixed $s \in \mathbb{R}$, it follows from the rescaling lemma ([8] Lemma 9.3) that $\tilde{\gamma}(t) = \gamma(st)$ is the integral curve of sX starting at e, so

$$\exp sX = \tilde{\gamma}(1) = \gamma(s)$$

∎

Two results will be useful later.

Proposition 11.2.4.

(1) *If H is another Lie group, $\mathfrak{h}$ is its Lie algebra, and $\Phi : G \to H$ is a Lie group homomorphism, the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow[\Phi]{} & H \end{array}$$

(2) *The flow θ of a left-invariant vector field X is given by $\theta_t = R_{\exp tX}$ (right multiplication by $\exp tX$). In particular, the maximal curve passing through g at 0 with velocity X_g is given by $g \cdot \exp(tX)$.*

Proof. See [8] Proposition 20.8. ∎

11.3 Smooth Actions

We have an identification for general manifolds similar to the one (11.1) of Lie algebra with $\exp sX$.

Let M be a manifold.

Definition 11.3.1. *An **action of a Lie group** G **on** M is a group homomorphism*

$$\begin{aligned} \Psi : \quad G &\longrightarrow \operatorname{Diff}(M) \\ g &\longmapsto \Psi_g. \end{aligned}$$

*(We will only consider left actions where Ψ is a homomorphism. A right action is defined with Ψ being an anti-homomorphism.) The **evaluation map** associated with an action $\Psi : G \to \operatorname{Diff}(M)$ is*

$$\begin{aligned} \mathrm{ev}_\Psi : M \times G &\longrightarrow M \\ (p, g) &\longmapsto \Psi_g(p). \end{aligned}$$

The action Ψ is smooth if ev_Ψ is a smooth map.

Example 11.3.2. If X is a complete vector field on M, then

$$\begin{array}{rccc} \rho : & \mathbb{R} & \longrightarrow & \operatorname{Diff}(M) \\ & t & \longmapsto & \rho_t = \exp tX \end{array}$$

is a smooth action of $\mathbb{R}$ on M. ♣

Remark 11.3.3. Every complete vector field gives rise to a smooth action of $\mathbb{R}$ on M. Conversely, every smooth action of $\mathbb{R}$ on M is defined by a complete vector field.

$$\begin{aligned} \{\text{complete vector fields on } M\} &\overset{1-1}{\longleftrightarrow} \{\text{ smooth actions of } \mathbb{R} \text{ on } M\} \\ X &\longmapsto \exp tX \\ X_p = \left.\frac{d\Psi_t(p)}{dt}\right|_{t=0} &\longleftarrow \Psi \end{aligned}$$

That is, if we have a one-parameter family of diffeomorphisms $\Psi_t : M \to M$, we for each point $p \in M$ have a curve $\Psi_t(p)$ in M and thus a tangent vector $X_p = \left.\frac{d\Psi_t(p)}{dt}\right|_{t=0}$. We thus obtain a vector field X, which can be denoted as $\left.\frac{d\Psi_t}{dt}\right|_{t=0}$. The vector field is complete since Ψ_t is defined for each $t \in \mathbb{R}$. The "Lie algebra" of the group $\operatorname{Diff}(M)$ will be the space $\{\text{complete vector fields on } M\} \subseteq \mathfrak{X}(M)$. ♠

11.4 Symplectic and Hamiltonian Actions

Let (M, ω) be a symplectic manifold, and G a Lie group. Let $\Psi : G \longrightarrow \operatorname{Diff}(M)$ be a (smooth) action.

Definition 11.4.1. *The action Ψ is a* ***symplectic action*** *if*

$$\Psi : G \longrightarrow \operatorname{Sympl}(M, \omega) \subset \operatorname{Diff}(M)$$

i.e., G "acts by symplectomorphisms." Then Remark 11.3.3 plus 10.1.7 (specifically, symplectic v.f. $\iff$ the flow preserves the symplectic form) result in the following one-to-one correspondence:

$$\{\textit{complete symplectic vector fields on } M\} \stackrel{1-1}{\longleftrightarrow} \{\textit{ symplectic actions of } \mathbb{R} \textit{ on } M\}$$

Example 11.4.2. On $\mathbb{R}^{2n}$ with $\omega = \sum dx_i \wedge dy_i$, let $X = -\frac{\partial}{\partial y_1}$. The orbits of the action generated by X are lines parallel to the y_1-axis,

$$\{(x_1, y_1 - t, x_2, y_2, \ldots, x_n, y_n) \mid t \in \mathbb{R}\}$$

We show that $X = X_{x_1}$ is Hamiltonian (with Hamiltonian function $H = x_1$): by uniqueness of the Hamiltonian vector field it suffices to show $\imath_{-\partial/\partial y_1}\omega = dx_1$. But

$$\begin{aligned} \omega\left(-\frac{\partial}{\partial y_1}, v\right) &= \sum\left(dx_i\left(-\frac{\partial}{\partial y_1}\right)dy_i(v) - dx_i(v)dy_i\left(-\frac{\partial}{\partial y_1}\right)\right) \\ &= -dx_1(v)dy_1\left(-\frac{\partial}{\partial y_1}\right) = dx_1(v) \end{aligned}$$

This is actually an example of a Hamiltonian action of $\mathbb{R}$. ♣

Example 11.4.3. On S^2 with $\omega = d\theta \wedge dh$ (cylindrical coordinates), let $X = \frac{\partial}{\partial \theta}$. Each orbit is a horizontal circle (called a "parallel") $\{(\theta + t, h) \mid t \in \mathbb{R}\}$. Notice that all orbits of this $\mathbb{R}$-action close up after time 2π, so that this is an action of S^1:

$$\begin{aligned} \Psi : \quad S^1 &\longrightarrow \operatorname{Sympl}\left(S^2, \omega\right) \\ t &\longmapsto \text{rotation by angle } t \text{ around } h\text{-axis} \end{aligned}$$

Since $X = X_h$ is Hamiltonian (with Hamiltonian function $H = h$), this is an example of a Hamiltonian action of S^1. ♣

Definition 11.4.4. *A symplectic action Ψ of S^1 or $\mathbb{R}$ on (M, ω) is Hamiltonian if the vector field generated by Ψ is Hamiltonian. Equivalently, an action Ψ of S^1 or $\mathbb{R}$ on (M, ω) is Hamiltonian if there is $H : M \to \mathbb{R}$ with $dH = \imath_X \omega$, where X is the vector field generated by Ψ.*

What is a "Hamiltonian action" of an arbitrary Lie group? For the case where $G = \mathbb{T}^n = S^1 \times \ldots \times S^1$ is an n-torus, an action $\Psi : G \to \operatorname{Sympl}(M, \omega)$ should be called Hamiltonian when each restriction

$$\Psi^i := \left.\Psi\right|_{i \text{ th } S^1 \text{ factor}} : S^1 \longrightarrow \operatorname{Sympl}(M, \omega)$$

is Hamiltonian in the previous sense with Hamiltonian function preserved by the action of the rest of G.

When G is not a product of S^1's or $\mathbb{R}$'s, the solution is to use an upgraded Hamiltonian function, known as a moment map. Before its definition though, we need a little Lie theory.

11.5 Adjoint and Coadjoint Representations

Let G be a Lie group. Given $g \in G$ let

$$\begin{aligned} L_g : G &\longrightarrow G \\ a &\longmapsto g \cdot a \end{aligned}$$

be left multiplication by g. A vector field X on G is called **left-invariant** if,

$$d(L_g)_a X_a = X_{ga} \ \forall g, a \in G.$$

Since L_g is a diffeomorphism, we can write above condition in push-forward: $(L_g)_* X = X$ for every $g \in G$. There are similar right notions.

Let $\mathfrak{g}$ be the vector space of all left-invariant vector fields on G. Together with the Lie bracket $[\cdot,\cdot]$ of vector fields, $\mathfrak{g}$ forms a Lie algebra, called the Lie algebra $\mathrm{Lie}(G)$ of the Lie group G.

Exercise 11.5.1. *Show that the map*

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & T_eG \\ X & \longmapsto & X_e \end{array}$$

where e is the identity element in G, is an isomorphism of vector spaces.

Solution. This is the so-called evaluation map shown in [8] Theorem 8.37. ♦

Any Lie group G acts on itself by **conjugation**:

$$\begin{aligned} &G \longrightarrow \mathrm{Diff}(G) \\ &g \longmapsto \Psi_g, \qquad \Psi_g(a) = g \cdot a \cdot g^{-1} \end{aligned}$$

The derivative at the identity of

$$\begin{aligned} \Psi_g : G &\longrightarrow G \\ a &\longmapsto g \cdot a \cdot g^{-1} \end{aligned}$$

is an invertible linear map $\mathrm{Ad}_g : \mathfrak{g} \longrightarrow \mathfrak{g}$, i.e., $\in \mathrm{GL}(\mathfrak{g})$ Here we identify the Lie algebra $\mathfrak{g}$ with the tangent space T_eG. Letting g vary, we obtain the **adjoint representation** (or **adjoint action**) of G on $\mathfrak{g}$:

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ g &\longmapsto \mathrm{Ad}_g . \end{aligned}$$

Exercise 11.5.2. *Check for matrix groups that*

$$\frac{d}{dt} \mathrm{Ad}_{\exp tX} Y \bigg|_{t=0} = [X,Y], \quad \forall X, Y \in \mathfrak{g}$$

Hint: For a matrix group G (i.e., a subgroup of $\mathrm{GL}(n;\mathbb{R})$ for some n), we have

$$\mathrm{Ad}_g(Y) = gYg^{-1}, \quad \forall g \in G, \forall Y \in \mathfrak{g}$$

Solution. See Brian Hall p.57: using product rule of matrix-valued function and the fact that $\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X$, we can show

$$\frac{d}{dt} \Psi_{\exp tX} Y \bigg|_{t=0} = [X,Y], \quad \forall X, Y \in \mathfrak{g} \tag{11.2}$$

where $\Psi_g(a) = g \cdot a \cdot g^{-1}$. Then use the hint. ♦

Let $\langle\cdot,\cdot\rangle$ be the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$:

$$\begin{array}{rccc} \langle\cdot,\cdot\rangle : & \mathfrak{g}^* \times \mathfrak{g} & \longrightarrow & \mathbb{R} \\ & (\xi, X) & \longmapsto & \langle \xi, X\rangle = \xi(X). \end{array}$$

Given $\xi \in \mathfrak{g}^*$, we define $\mathrm{Ad}_g^* \xi$ by

$$\left\langle \mathrm{Ad}_g^* \xi, X \right\rangle = \left\langle \xi, \mathrm{Ad}_{g^{-1}} X \right\rangle, \quad \text{for any } X \in \mathfrak{g} \tag{11.3}$$

The collection of maps Ad_g^* forms the **coadjoint representation** (or **coadjoint action**) of G on $\mathfrak{g}^*$:

$$\begin{aligned}\mathrm{Ad}^* : G &\longrightarrow \mathrm{GL}(\mathfrak{g}^*)\\ g &\longmapsto \mathrm{Ad}_g^*\end{aligned}$$

We take g^{-1} in the definition of $\mathrm{Ad}_g^* \xi$ in order to obtain a (left) representation, i.e., a group homomorphism, instead of a "right" representation, i.e., a group antihomomorphism.

Exercise 11.5.3. *Show that* $\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh}$ *and* $\mathrm{Ad}_g^* \circ \mathrm{Ad}_h^* = \mathrm{Ad}_{gh}^*$.

11.6 Hermitian Matrices

Let $\mathcal{H}$ be the vector space of $n \times n$ complex hermitian matrices. The unitary group $\mathrm{U}(n)$ acts on $\mathcal{H}$ by conjugation: $A \cdot \xi = A\xi A^{-1}$, for $A \in \mathrm{U}(n), \xi \in \mathcal{H}$. For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ complex hermitian matrices whose spectrum is λ.

Proposition 11.6.1. *Let $A^* = \overline{A^T} = \overline{A}^T$ be the hermitian, or adjoint of A. Several properties are noted first:*

(1) Let $A, B \in M_n(\mathbb{C})$, then $(AB)^* = B^* A^*$.

(2) Let $A \in GL_n(\mathbb{C})$, then $AA^{-1} = I \implies \left(A^{-1}\right)^* A^* = I^* = I \implies (A^*)^{-1} = (A^{-1})^*$.

(3) Let $\xi \in \mathcal{H}$ and $A \in U(n)$, so $(A^*)^{-1} = A$ and $A^{-1} = A^*$. Then $(A\xi A^{-1})^* = ((A\xi)A^{-1})^* = (A^*)^{-1}\xi^* A^* = A\xi A^{-1}$. This shows $A \cdot \xi \in \mathcal{H}$.

(4) See [11] Theorem 10.18 or Theorem 10.19 to see that a complex square matrix A is Hermitian if and only if it is unitarily diagonalizable with real eigenvalues.

(5) See [11] Theorem 10.18 or Theorem 10.19 to see that a unitary matrix A has all of its complex eigenvalues norm one.

Exercise 11.6.2. *Show that the orbits of the $\mathrm{U}(n)$-action are the manifolds $\mathcal{H}_\lambda$.*

For a fixed $\lambda \in \mathbb{R}^n$, what is the stabilizer of a point in $\mathcal{H}_\lambda$?

[Hint: If $\lambda_1, \ldots, \lambda_n$ are all distinct, the stabilizer of the diagonal matrix is the torus $\mathbb{T}^n$ of all diagonal unitary matrices.]

Solution. Let A be any matrix. The characteristic polynomial is $\chi_A(\lambda) = \det(A - \lambda I)$ Consider a similar matrix $A' = C^{-1}AC$. Then

$$\chi_{A'}(\lambda) = \det\left(A' - \lambda I\right) = \det\left(C^{-1}(A - \lambda I)C\right) = \det\left(C^{-1}\right)\det(A - \lambda I)\det(C) = \chi_A(\lambda).$$

so conjugation preserves spectrum. The orbit $\mathcal{O}_\xi = G \cdot \xi = \{A\xi A^{-1} | A \in G = U(n)\}$ is $\mathcal{H}_\lambda$ with $\lambda = \lambda(\xi)$.

By proposition 14.2.1 (1), $\mathcal{H}_\lambda$ is equipped with a manifold structure.

Let $\lambda \in \mathbb{R}^n$ s.t. $\lambda_i \neq \lambda_j$ for $i \neq j$. Then

$$\mathrm{Stab}(\mathrm{diag}(\lambda)) = \{A \in U(n) | A\,\mathrm{diag}(\lambda)A^{-1} = \mathrm{diag}(\lambda)\}$$

Observe that

$$\begin{pmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} = \begin{pmatrix}\lambda_1 a_{11} & \cdots & \lambda_n a_{1n}\\ \vdots & \ddots & \vdots\\ \lambda_1 a_{n1} & \cdots & \lambda_n a_{nn}\end{pmatrix}$$

and

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 a_{11} & \cdots & \lambda_1 a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_n a_{n1} & \cdots & \lambda_n a_{nn} \end{pmatrix}$$

Thus, $A\operatorname{diag}(\lambda)A^{-1} = \operatorname{diag}(\lambda) \implies A\operatorname{diag}(\lambda) = \operatorname{diag}(\lambda)A \implies a_{ij}(\lambda_i - \lambda_j) = a_{ji}(\lambda_j - \lambda_i) = 0,\ \forall i \neq j \implies a_{ij} = 0,\ \forall i \neq j$ because $\lambda_i - \lambda_j \neq 0$ for $i \neq j$. That is,

$$\operatorname{Stab}(\operatorname{diag}(\lambda)) = \text{all diagonal unitary matrices in } U(n).$$

By prop.11.6.1(5), we see the diagonals of these matrices all have norm one, i.e., $\in S^1 \simeq \mathbb{R}/\mathbb{Z}$. So we have also say the stabilizer is just a torus $\mathbb{T}^n$ when the spectrum λ has all λ_i's distinct.

If not all of λ_i's in λ are distinct, say $\lambda_{i_1} = \cdots = \lambda_{i_k}$ then

$$\operatorname{Stab}(\operatorname{diag}(\lambda)) = \{A \in U(n) | \text{all entries except submatrix by } i_1, \cdots, i_k \text{ and diagonals are zero}\}.$$

♦

Exercise 11.6.3. *Show that the symmetric bilinear form on $\mathcal{H}$, $\beta(X,Y) = \operatorname{tr}(XY)$, is nondegenerate.*

For $\xi \in \mathcal{H}$, define a skew-symmetric bilinear form ω_ξ on $\mathfrak{u}(n) = T_1\mathrm{U}(n) = i\mathcal{H}$ (space of skew-hermitian matrices) by

$$\omega_\xi(X,Y) = i\operatorname{tr}([X,Y]\xi), \quad X, Y \in i\mathcal{H}$$

Check that $\omega_\xi(X,Y) = i$ trace $(X(Y\xi - \xi Y))$ and $Y\xi - \xi Y \in \mathcal{H}$.

Show that the kernel of $\widetilde{\omega}_\xi$ is $K_\xi := \{Y \in \mathfrak{u}(n) \mid [Y,\xi] = 0\}$.

Solution. The form β is clearly bilinear and it is symmetric bc. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. By choosing $Y = E_{ij} + E_{ji} \in \mathcal{H}$, we have $\operatorname{tr}(XY) =$ j-th element of Col-i of X+i-th element of Row-j of $X = X_{ij} + X_{ji} = X_{ij} + \overline{X_{ij}} = 2\operatorname{Re}(X_{ij}) = 0$. Similarly, by choose $Y = \mathbf{i}E_{ij} - \mathbf{i}E_{ji} \in \mathcal{H}$, we have $\operatorname{tr}(XY) = \mathbf{i}(X_{ij} - \overline{X_{ij}}) = -2\operatorname{Im}(X_{ij}) = 0$. Thus, all real and imaginary parts of X's entries are zero. $X = 0$. β is nondegenerate.

Recall $\mathfrak{u}(n) = T_1U(n) = \{X \in GL(n,\mathbb{C}) | X + X^* = 0\} = i\mathcal{H}$. Cyclic property of trace gives $\omega_\xi(X,Y) = i\operatorname{tr}([X,Y]\xi) = i\operatorname{tr}(X(Y\xi - \xi Y))$. To check $Y\xi - \xi Y \in \mathcal{H}$, we let $Y = iA$ for some $A \in \mathcal{H}$ and get $Y^* = (iA)^* = \overline{iA^T} = -iA^*$. Then

$$(Y\xi - \xi Y)^* = \xi^*(-iA^*) - (-iA^*)\xi^* \overset{A,\xi\in\mathcal{H}}{=\!=\!=} -i\xi A + iA\xi = -\xi Y + Y\xi \implies Y\xi - \xi Y \in \mathcal{H}.$$

Let $Y \in \ker \widetilde{\omega}_\xi$. Then

$$\forall X = iB \in i\mathcal{H},\ \omega_\xi(X,Y) = i\operatorname{tr}(iB(Y\xi - \xi Y)) = ii\operatorname{tr}(B(Y\xi - \xi Y)) = -\beta(\overbrace{B}^{\in\mathcal{H}}\overbrace{(Y\xi - \xi Y)}^{\in\mathcal{H}}) = 0$$

Nondegeneracy of β implies $Y\xi - \xi Y = [Y,\xi] = 0$. Thus,

$$K_\xi = \ker \widetilde{\omega}_\xi = \{Y \in \mathfrak{u}(n) \mid [Y,\xi] = 0\}.$$

♦

Exercise 11.6.4. *Show that K_ξ is the Lie algebra of the stabilizer of ξ.*

[Hint: Differentiate the relation $A\xi A^{-1} = \xi$.]

Show that the ω_ξ 's induce nondegenerate 2 -forms on the orbits $\mathcal{H}_\lambda$.
Show that these 2 -forms are closed.
Conclude that all the orbits $\mathcal{H}_\lambda$ are compact symplectic manifolds.

Solution. Let the action be $\Psi : G \times \mathcal{H} \to \mathcal{H}$ where $G = U(n)$. By equation (14.3) we have

$$\mathrm{Lie}(G_\xi) = \mathfrak{u}(n)_\xi = \{Y \in \mathfrak{u}(n) | Y^\#(\xi) = 0\}$$

Since

$$Y^\#(\xi) = \left.\frac{d}{dt}\right|_{t=0} \Psi_{e^{tY}}(\xi) = \left.\frac{d}{dt}\right|_{t=0} e^{tY}\xi e^{-tY} \xlongequal{\text{(eq.11.2)}} [Y,\xi] \tag{11.4}$$

We see

$$\mathrm{Lie}(G_\xi) = K_\xi.$$

By equation (14.2) we for $\xi \in \mathcal{H}_\lambda$ with $\lambda = \lambda(\xi)$ we have

$$T_\xi \mathcal{H}_\lambda = T_\xi \mathcal{O}_\xi = \{X^\#(\xi) | X \in \mathfrak{u}(n)\} = [\mathfrak{u}(n), \xi]$$

The induced 2-form Ω_ξ on the orbit $\mathcal{H}_\lambda$ is defined by

$$\Omega_\xi\left(X_\xi^\#, Y_\xi^\#\right) = \omega_\xi(X,Y) = i\,\mathrm{tr}([X,Y]\xi).$$

It is nondegenerate because

$$\ker \widetilde{\Omega}_\xi = \{Y_\xi^\# | Y \in K_\xi\} = \{[Y,\xi] | Y \in K_\xi\} = 0$$

It is also closed: by [LeeSM] Prop.14.32, we have

$$\begin{aligned}(d\Omega)(X^\#, Y^\#, Z^\#) =& X^\#(\Omega(Y^\#,Z^\#)) - Y^\#(\Omega(X^\#,Z^\#)) + Z^\#(\Omega(X^\#,Y^\#)) \\ &-\Omega([X^\#,Y^\#],Z^\#) - \Omega([Z^\#,X^\#],Y^\#) - \Omega([Y^\#,Z^\#],X^\#)\end{aligned} \tag{11.5}$$

We evaluate $X^\#(\Omega(Y^\#,Z^\#))$ pointwise:

$$\begin{aligned}X_\xi^\#(\overbrace{\Omega_\xi(Y_\xi^\#,Z_\xi^\#)}^{f:\mathcal{H}_\lambda\to\mathbb{R};\xi\mapsto\Omega_\xi(Y_\xi^\#,Z_\xi^\#)}) &= \left(\left.\frac{d}{dt}\right|_{t=0}\Psi_{e^{tX}}(\xi)\right)(f) = \left(\left.\frac{d}{dt}\right|_{t=0}\overbrace{e^{tX}\xi e^{-tX}}^{\gamma(t)}\right)(f) \\ &= d\gamma_0\left(\left.\frac{d}{dt}\right|_{t=0}\right)(f) = \left(\left.\frac{d}{dt}\right|_{t=0}\right)(f\circ\gamma) \\ &= \left.\frac{d}{dt}\right|_{t=0} i\,\mathrm{tr}\left([Y,Z]e^{tX}\xi e^{-tX}\right) \\ &= i\,\mathrm{tr}\left([Y,Z]\left.\frac{d}{dt}\right|_{t=0}e^{tX}\xi e^{-tX}\right) \quad \text{multiplication and tr are linear} \\ &= i\,\mathrm{tr}\left([Y,Z][X,\xi]\right) = i\,\mathrm{tr}\left(YZX\xi - ZYX\xi - YZ\xi X + ZY\xi X\right) \\ &= i\,\mathrm{tr}\left(\xi(YZX - ZYX - XYZ + XZY)\right) \quad \text{cyclic property of trace} \\ &= i\,\mathrm{tr}\left(\xi[[Y,Z],X]\right)\end{aligned} \tag{11.6}$$

Thus, the Jacobi identity implies that the first line of (11.5) is zero for every ξ on the orbit, and it also kills the next line for every ξ simply because

$$\begin{aligned}&-\Omega_\xi([X_\xi^\#,Y_\xi^\#],Z_\xi^\#) - \Omega_\xi([Z_\xi^\#,X_\xi^\#],Y_\xi^\#) - \Omega([Y_\xi^\#,Z_\xi^\#],X_\xi^\#) \\ =&-i\,\mathrm{tr}\left([[X,Y],Z]\xi + [[Z,X],Y]\xi + [[Y,Z],X]\xi\right)\end{aligned}$$

The orbits $\mathcal{H}_\lambda$ are equipped with a nondegenerate, closed 2 -form ω_ξ, which defines a symplectic structure on each orbit. $\mathcal{H}_\lambda$ is compact because it is the image of a compact Lie group $U(n)$ under the orbit map (which is smooth and thus continuous). ◆

Exercise 11.6.5. *Describe the manifolds $\mathcal{H}_\lambda$.*

When all eigenvalues are equal, there is only one point in the orbit.
Suppose that $\lambda_1 \neq \lambda_2 = \ldots = \lambda_n$. Then the eigenspace associated with λ_1 is a line, and the one associated with λ_2 is the orthogonal hyperplane. Show that there is a diffeomorphism $\mathcal{H}_\lambda \simeq \mathbb{CP}^{n-1}$. We have thus exhibited a lot of symplectic forms on $\mathbb{CP}^{n-1}$, on for each pair of distinct real numbers. What about the other cases?

[Hint: When the eigenvalues $\lambda_1 < \ldots < \lambda_n$ are all distinct, any element in $\mathcal{H}_\lambda$ defines a family of pairwise orthogonal lines in $\mathbb{C}^n$: its eigenspaces.]

Solution. When all eigenvalues are equal, there is only one point in the orbit. That's because $A(\lambda I)A^{-1} = \lambda AA^{-1} = \lambda I$. To show $\mathcal{H}_\lambda \simeq \mathbb{CP}^{n-1}$, we build the bijection

$$\begin{aligned} \mathcal{H}_\lambda &\longleftrightarrow \mathbb{CP}^{n-1} \\ \xi &\xmapsto{f} E_1 \\ \lambda_1 vv^* + \lambda_2(I - vv^*) &\xleftarrow{g} [v] \end{aligned}$$

where E_1 is the eigenspace spanned by the eigenvector of λ_1 for matrix ξ; v as a representative of $[v]$ is chosen to have unit length; vv^* is the projection matrix onto the line spanned by v, with $I - vv^*$ the projection onto the orthogonal complement of v. That is, $\operatorname{Proj}_v w = P_v w = (vv^*)w$ gives the projection of w onto unit vector v where $P_v = vv^*$ is the projection matrix.

Note that v has unit length means $v^*v = 1$, so $g([v])v = \lambda_1 v$. And for $w \in (\mathbb{C}\{v\})^\perp$, $vv^*w = P_v w = 0 \implies g([v])w = \lambda_2 w$. Thus, $g([v]) \in \mathcal{H}_\lambda$. Besides, $g([v])v = \lambda_1 v$ also shows that $f(g([v])) = [v] \implies f$ surjective. But injectivity of f is clear. Thus, f is surjective.

To show the smoothness, note that for an element ξ with $\xi v = \lambda_1 v$, if we apply a perturbation by A to get $A\xi A^{-1}$, then Av will be the eigenvector of $A\xi A^{-1}$ and thus $f(A\xi A^{-1}) = [Av]$, which is also perturbed by A. The other direction $g([v]) = \lambda_1 vv^* + \lambda_2(I - vv^*)$ has a rather explicit functional form that can be shown to be smooth.

For case $\lambda_i \neq \lambda_j$, each $A \in U(n)/G_\xi \cong U(n)/\mathbb{T}^n$ gives a different element in $\mathcal{H}_\lambda$ where G_ξ is the stabilizers of ξ. Thus, $\mathcal{H}_\lambda \simeq U(n)/\mathbb{T}^n$.

For the general case, see [3] Proposition II.1.15 and Example II.1.16:

Assume that λ consists of k distinct values with multiplicities $n_1, \ldots, n_k$. The orbit $\mathcal{H}_\lambda$ is diffeomorphic with the homogeneous space $\mathrm{U}(n)/\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k)$. It also shows that the orbit under consideration is diffeomorphic with the manifolds of flags

$$0 \subset P_1 \subset \cdots \subset P_k = \mathbb{C}^n$$

(with $P_i = Q_1 \oplus \cdots \oplus Q_i$; Q_i is the eigenspace of λ_i with dimension each to its multiplicity n_i). ♦

Chapter 12

Hamiltonian Actions

12.1 Moment and Comoment Maps

Let

- (M, ω) be a symplectic manifold;
- G a Lie group;
- $\Psi : G \to \mathrm{Sympl}(M, \omega)$ a (smooth) symplectic action, i.e., a group homomorphism such that the evaluation map $\mathrm{ev}_\Psi(g, p) := \Psi_g(p)$ is smooth.

Case $G = \mathbb{R}$

We have the following bijective correspondence:

$$\{\text{complete symplectic vector fields on } M\} \overset{1-1}{\longleftrightarrow} \{\text{ smooth symplectic actions of } \mathbb{R} \text{ on } M\}$$
$$X \longmapsto \exp tX$$
$$X_p = \left.\frac{d\Psi_t(p)}{dt}\right|_{t=0} \longleftarrow \Psi$$

The action Ψ is **Hamiltonian** if there exists a function $H : M \to \mathbb{R}$ such that $dH = \iota_X \omega$ where X is the vector field on M generated by Ψ.

Case $G = S^1$

An action of S^1 is an action of $\mathbb{R}$ which is 2π-periodic: $\Psi_{2\pi} = \Psi_0$. The S^1-action is called **Hamiltonian** if the underlying $\mathbb{R}$-action is Hamiltonian.

General case

Let

- (M, ω) be a symplectic manifold,
- G a Lie group,
- $\mathfrak{g}$ the Lie algebra of G,

- $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$, and
- $\Psi : G \longrightarrow \operatorname{Sympl}(M, \omega)$ a smooth symplectic action.

Definition 12.1.1. *The action Ψ is a **Hamiltonian action** if there exists a map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying

1. *For each $X \in \mathfrak{g}$, let*
   - *$\overline{X} : \mathfrak{g}^* \to \mathbb{R};\ \xi \mapsto \xi(X)$ be the evaluation map of X, and we define $\mu^X \in C^\infty(M)$ as $\mu^X = \mu \circ \overline{X}$, i.e.,*

   $$\begin{aligned} \mu^X : M &\longrightarrow \mathbb{R} \\ p &\longmapsto \langle \mu(p), X\rangle = \mu(p)(X) \end{aligned}$$

   - *$X^\#$ be the vector field on M generated by the one-parameter subgroup $\exp tX$:*

   $$X^\#(p) = \left.\frac{d}{dt}\right|_{t=0} \exp tX \cdot p = \left.\frac{d}{dt}\right|_{t=0} \Psi_{\exp tX}(p)$$

   Then
   $$d\mu^X = \iota_{X^\#}\omega,$$
   i.e., μ^X is a Hamiltonian function for the vector field $X^\#$.
2. *μ is equivariant with respect to the given action Ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:*

   $$\mu \circ \Psi_g = \operatorname{Ad}_g^* \circ \mu, \qquad \forall g \in G.$$

*The tuple (M, ω, G, μ) is then called a **Hamiltonian G-space** and μ is a **moment map**.*

For connected Lie groups, Hamiltonian actions can be equivalently defined in terms of a **comoment map**

$$\begin{aligned} \mu^* : \mathfrak{g} &\longrightarrow C^\infty(M) \\ X &\longmapsto \mu^X \end{aligned}$$

with the two conditions rephrased as:

1. $\mu^*(X) = \mu^X$ is a Hamiltonian function for the vector field $X^\#$,
2. μ^* is a Lie algebra homomorphism:

   $$\mu^*[X, Y] = \{\mu^*(X), \mu^*(Y)\}$$

   where $\{\cdot,\cdot\}$ is the Poisson bracket on $C^\infty(p)$.

Remark 12.1.2. We note that the first claim, i.e., $d\mu^X = \iota_{X^\#}\omega$, implies that $X_{\mu^X} = X^\#$. That is, the Hamiltonian vector field for function μ^X is $X^\#$. Similarly, $Y^\# = X_{\mu^Y}$. Then

$$\{\mu^*(X), \mu^*(Y)\} = \{\mu^X, \mu^Y\} = \omega(X_{\mu^X}, X_{\mu^Y}) = \omega(X^\#, Y^\#)$$

Now,

$$\iota_{[X,Y]^\#}\omega \xlongequal{[8]20.18(a)} \iota_{-[X^\#,Y^\#]}\omega \xlongequal{\text{prop.}10.3.2} d(\omega(X^\#, Y^\#)) = d\{\mu^*(X), \mu^*(Y)\}.$$

Thus, $[X, Y]^\#$ is *the* Hamiltonian v.f. of $\{\mu^*(X), \mu^*(Y)\}$; and $\{\mu^*(X), \mu^*(Y)\}$ is *a* Hamiltonian function for $[X, Y]^\#$. ♠

Remark 12.1.3. The proof that the second claim (Lie algebra homomorphism) is equivalent to the equivariance condition:

First assume the G-action is Hamiltonian. Then for any $X, Y \in \mathfrak{g}$,

$$\begin{aligned}
\left\{\mu^X, \mu^Y\right\}(p) &\xlongequal{\text{eq.(10.2)}} X_{\mu^Y}(\mu^X)(p) \xlongequal{d\mu^Y = \iota_{Y^\#}\omega} [Y^\#(\mu^X)](p) \\
&= \left[\left.\frac{d}{dt}\right|_{t=0} \Psi_{\exp tY}(p)\right](\mu^X) \\
&\xlongequal{\text{let } f: t \mapsto \Psi_{\exp tY}(p)} df_0\left(\left.\frac{d}{dt}\right|_{t=0}\right)(\mu^X) \\
&\xlongequal{\text{defn. of differential; see [8] p.55}} \left.\frac{d}{dt}\right|_{t=0} \mu^X \circ f \\
&\xlongequal{\text{defn. of } \mu^X: M \to \mathbb{R}} \left.\frac{d}{dt}\right|_{t=0} \langle \mu(\Psi_{\exp tY}(p)), X\rangle \\
&\xlongequal{\text{equivariance}} \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathrm{Ad}^*_{\exp tY}(\mu(p)), X\right\rangle \qquad (12.1)\\
&\xlongequal{\text{defn. of coadj (11.3)}} \left.\frac{d}{dt}\right|_{t=0} \langle \mu(p), \mathrm{Ad}_{\exp -tY} X\rangle \\
&\xlongequal{\text{by [8]prop.20.8(g) and Thm.20.27}} \left.\frac{d}{dt}\right|_{t=0} \left\langle \mu(p), \left(\overbrace{\exp(\underbrace{\mathrm{ad}(-tY)}_{\in \mathfrak{gl}(\mathfrak{g})})}^{\in GL(\mathfrak{g})}\right) X \right\rangle \\
&= \left\langle \mu(p), \left.\frac{d}{dt}\right|_{t=0} \left(X - t\,\mathrm{ad}(Y)(X) + \frac{t^2}{2!}\mathrm{ad}(Y)^2(X) - \cdots\right)\right\rangle \\
&= \langle \mu(p), -[Y, X]\rangle = \langle \mu(p), [X, Y]\rangle \\
&= \mu^{[X,Y]}(p)
\end{aligned}$$

Conversely, suppose μ^* is a Lie algebra homomorphism. Since G is connected and the exponential map $\exp$ is a local diffeomorphism (see [8] prop.20.8(f)), any element g of G can be written as a product of elements of the form $\exp(X)$. As a result, to prove G-equivariance it is enough to prove

$$\mu(\Psi_{\exp tX}(p)) = \mathrm{Ad}^*_{\exp(tX)}\, \mu(p)$$

We will need a result analogous to [8] proposition 9.13:

Lemma 12.1.4. *Let M and N be two smooth manifolds and $F : M \to N$ be a smooth map. Let G be a Lie group. Consider the smooth actions $\theta : G \times M \to M$ and $\eta : G \times N \to N$. Let $X, Y \in \mathfrak{g}$. Define*

$$X^\#(p) = \left.\frac{d}{dt}\right|_{t=0} \theta_{\exp tX}(p), \quad Y^\#(q) = \left.\frac{d}{dt}\right|_{t=0} \eta_{\exp tY}(q).$$

If $X^\#$ and $Y^\#$ are F-related, i.e., $dF_p(X^\#_p) = Y^\#_{F(p)}$, $\forall p \in M$, then $\eta_g \circ F = F \circ \theta_g$.

sketch of proof: We note that

$$X^\#((\theta_{\exp t_0 X})(p)) = \left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t+t_0)X}(p) = \left.\frac{d}{dt}\right|_{t=t_0} \theta_{\exp(t+t_0)X}(p)$$

which shows that the curve $\gamma(t) : t \mapsto \theta_{\exp tX}(p)$ has the property that $\gamma(0) = p$ and $(d/dt)\gamma(t) = X^\#(\gamma(t))$. By F-relatedness, it is easy to see $\sigma = F \circ \gamma$ is the curve with the property that $\sigma(0) = F(p)$ and $(d/dt)\sigma(t) = Y^\#(\sigma(t))$. Then $t \mapsto \eta_{\exp tY}(p)$ coincide with $\sigma(t)$ by uniquness from ODE theory.

Now consider the action $\Psi : G \times M \to M$ and $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$. Let X^* be the vector field generated by $\mathrm{Ad}^*_{\exp tX}$, i.e.,

$$X^*(\xi) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp tX}(\xi)$$

Notice that in eq.(12.1), we have shown that

$$\underbrace{\left.\frac{d}{dt}\right|_{t=0} \left\langle \mathrm{Ad}^*_{\exp tX}(\xi), Y \right\rangle}_{\langle X^*(\xi), Y\rangle} = \langle \xi, [Y,X] \rangle$$

Using this plus the first line of eq.(12.1) and the preservation of bracket of μ^*, we see that for $Y \in \mathfrak{g}$,

$$\langle X^*(\mu(p)), Y \rangle = \langle \mu(p), [Y,X] \rangle = \mu^{[Y,X]}(p) = \left\{\mu^Y, \mu^X\right\}(p) = [X^\#(\mu^Y)](p) = [X^\#\langle \mu, Y\rangle](p)$$

With the evaluation map $\overline{Y} : \xi \mapsto \xi(Y)$, we see

$$\langle d\mu_p(X^\#(p)), Y \rangle = \left(d\mu_p(X^\#(p))\right)(Y) = \overline{Y}\left(d\mu_p(X^\#(p))\right) = d\overline{Y}_{\mu(p)} d\mu_p X^\#_p = d(\overline{Y} \circ \mu)_p X^\#_p$$

where the second-to-last equality is because $\overline{Y}$ is a linear map. Since $\overline{Y} \circ \mu : M \to \mathbb{R}$ is a smooth function, eq.(10.2) equates the RHS of the last two equations:

$$d(\overline{Y} \circ \mu)_p X^\#_p = [X^\#(\overline{Y} \circ \mu)](p) = [X^\#\langle \mu, Y \rangle](p)$$

Thus,

$$d\mu_p(X^\#(p)) = X^*(\mu(p))$$

Due to the lemma above, the $\Longleftarrow$ direction is proved. ♠

Remark 12.1.5. For abelian group actions, the conjugations become identity maps. So do their differential Ad_g at identity and thus the coadjoint Ad^*_g. So the equivariance condition becomes invariance $\mu \circ \Psi_g = \mu$. We shall see this in the next remark. ♠

Remark 12.1.6. The general definition matches the previous ones for the cases $G = \mathbb{R}, S^1, \mathbb{T}^n$:

Case $G = S^1$ (or $\mathbb{R}$):

We note that now $X^\#$ does not arise from any $X \in \mathfrak{g}$ like what the general case does. $\mathbb{R}$ as an additive Lie group has exponential map $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$. S^1 has exponential map $\mathbb{R} \ni \theta \mapsto e^{i\theta} \in S^1$. The only difference between S^1 and $\mathbb{R}$ actions is that the former is 2π-periodic.

Here $\mathfrak{g} \simeq \mathbb{R}, \mathfrak{g}^* \simeq \mathbb{R}$. A moment map $\mu : M \longrightarrow \mathbb{R}$ satisfies:

1. For the generator $X = 1$ of $\mathfrak{g}$, we have $\mu^X(p) = \mu(p) \cdot 1$, i.e., $\mu^X = \mu$. The vector field $X^\#(p) = \left.\frac{d}{dt}\right|_{t=0} \Psi_{\exp t \cdot 1}(p) = \left.\frac{d}{dt}\right|_{t=0} \Psi_t(p)$ is the standard vector field on M generated by S^1 (or $\mathbb{R}$). Then $d\mu = \iota_{X^\#}\omega$.
2. μ is invariant: $\mathcal{L}_{X^\#}\mu = \iota_{X^\#} d\mu = 0$.

Case $G = \mathbb{T}^n = n$-torus:

Here $\mathfrak{g} \simeq \mathbb{R}^n, \mathfrak{g}^* \simeq \mathbb{R}^n$. A moment map $\mu : M \longrightarrow \mathbb{R}^n$ satisfies:

1. For each basis vector X_i of $\mathbb{R}^n$, μ^{X_i} is a Hamiltonian function for $X_i^\#$.
2. μ is invariant.

♠

12.2 Orbit Spaces

Let $\Psi : G \to \mathrm{Diff}(M)$ be any action.

Definition 12.2.1. *The* ***orbit*** *of* G *through* $p \in M$ *is* $\{\Psi_g(p) \mid g \in G\}$. *The* ***stabilizer*** *(or* ***isotropy****) of* $p \in M$ *is the subgroup* $G_p := \{g \in G \mid \Psi_g(p) = p\}$.

When two points x and y are on the same group orbit, say $y = gx$, then the isotropy groups are conjugate subgroups. More precisely, $G_y = gG_xg^{-1}$. In fact, any subgroup conjugate to G_x occurs as an isotropy group G_y to some point y on the same orbit as x.

Definition 12.2.2. *We say that the action of* G *on* M *is ...*

- ***transitive*** *if there is just one orbit,*
- ***free*** *if all stabilizers are trivial* $\{e\}$*,*
- ***locally free*** *if all stabilizers are discrete.*

Let $\sim$ be the orbit equivalence relation; for $p, q \in M$,

$$p \sim q \quad \Longleftrightarrow \quad p \text{ and } q \text{ are on the same orbit.}$$

The space of orbits $M/\sim = M/G$ is called the **orbit space**. Let

$$\begin{array}{rccc} \pi : & M & \longrightarrow & M/G \\ & p & \longmapsto & \text{orbit through } p \end{array}$$

be the **point-orbit projection**.

Topology of the orbit space

We equip M/G with the weakest topology for which π is continuous, i.e., $\mathcal{U} \subseteq M/G$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in M. This is called the **quotient topology**. This topology can be "bad." For instance:

Example 12.2.3. Let $G = \mathbb{R}$ act on $M = \mathbb{R}$ by

$$t \longmapsto \Psi_t = \text{ multiplication by } e^t.$$

There are three orbits $\mathbb{R}^+, \mathbb{R}^-$and $\{0\}$. Since $\pi^{-1}(\{0\}) = \{0\} \subseteq \mathbb{R}$ is not of the form of (a, b) and is open open, we see the only neighborhood containing $\{0\}$ in orbit space is the whole space. Thus, for any pair of points with $\{0\}$, the separating condition of Hausdorffness fails. The orbit topology is not Hausdorff. ♣

Example 12.2.4. Let $G = \mathbb{C}\backslash\{0\}$ act on $M = \mathbb{C}^n$ by

$$\lambda \longmapsto \Psi_\lambda = \text{ multiplication by } \lambda.$$

The orbits are the punctured complex lines (through non-zero vectors $z \in \mathbb{C}^n$), plus one "unstable" orbit through 0, which has a single point. The orbit space is

$$M/G = \mathbb{CP}^{n-1} \sqcup \{\text{ point }\}$$

The quotient topology restricts to the usual topology on $\mathbb{CP}^{n-1}$. The only open set containing { point } in the quotient topology is the full space. Thus, again the quotient topology in M/G is not Hausdorff.

However, it suffices to remove 0 from $\mathbb{C}^n$ to obtain a Hausdorff orbit space: $\mathbb{CP}^{n-1}$. Then there is also a compact (yet not complex) description of the orbit space by taking only unit vectors:

$$\mathbb{CP}^{n-1} = (\mathbb{C}^n\backslash\{0\})/(\mathbb{C}\backslash\{0\}) = S^{2n-1}/S^1$$

♣

12.3 Preview of Reduction

The transformations

$$\begin{cases} z^j = x^j + iy^j \\ \bar{z}^j = x^j - iy^j \end{cases}$$

and

$$\begin{cases} x^j = r^j \cos\theta^j \\ y^j = r^j \sin\theta^j \end{cases}$$

writes the standard symplectic form on $\mathbb{C}^n$ in Example 1.3.5 as

$$\omega = \frac{i}{2}\sum dz^j \wedge d\bar{z}^j = \sum dx^j \wedge dy^j = \sum r^j dr^j \wedge d\theta^j.$$

Also recall that

$$\begin{cases} \frac{\partial}{\partial x^j} = \cos\theta^j \frac{\partial}{\partial r^j} - \frac{\sin\theta^j}{r^j}\frac{\partial}{\partial\theta^j} \\ \frac{\partial}{\partial y^j} = \sin\theta^j \frac{\partial}{\partial r^j} + \frac{\cos\theta^j}{r^j}\frac{\partial}{\partial\theta^j} \end{cases}$$

Consider the following S^1-action on $(\mathbb{C}^n, \omega)$:

$$t \in S^1 \longmapsto \Psi_t = \text{ multiplication by } t$$

The action Ψ is Hamiltonian with moment map

$$\begin{aligned} \mu : \mathbb{C}^n &\longrightarrow \mathbb{R} \\ z &\longmapsto -\frac{|z|^2}{2} + \text{ constant} \end{aligned}$$

Since under remark 12.1.6,

$$\begin{aligned}
d\mu^X &= d\mu = -\frac{1}{2}d\left(\sum r_j^2\right) \\
X^{\#}(z) &= \left.\frac{d}{d\theta}\right|_{\theta=0}\Psi_{\exp\theta\cdot 1}(z) = \left.\frac{d}{d\theta}\right|_{\theta=0}\overbrace{\Psi_{e^{i\theta}}(z)}^{e^{i\theta}z} = iz^{\dagger} \text{ see remark below.} \\
&= i(x^j+iy^j)_j^n = (-y^j+ix^j)_j^n \xlongequal{x+yi\leftrightarrow(x,y)} \sum_{j=1}^n\left(-y^j\frac{\partial}{\partial x^j} + x^j\frac{\partial}{\partial x^j}\right) \\
&= \sum_{j=1}^n\left((-r^j\sin\theta^j)\left(\cos\theta^j\frac{\partial}{\partial r^j} - \frac{\sin\theta^j}{r^j}\frac{\partial}{\partial\theta^j}\right) + (r^j\cos\theta^j)\left(\sin\theta^j\frac{\partial}{\partial r^j} + \frac{\cos\theta^j}{r^j}\frac{\partial}{\partial\theta^j}\right)\right) \\
&= \frac{\partial}{\partial\theta^1} + \frac{\partial}{\partial\theta^2} + \ldots + \frac{\partial}{\partial\theta^n} \\
(\iota_{X^{\#}}\omega)(v) &= \left(\sum r^j dr^j \wedge d\theta^j\right)(X^{\#}, v) = \sum\left(r^j\underbrace{dr^j(X^{\#})}_{=0}d\theta^j(v) - r^j dr^j(v)\underbrace{d\theta^j(X^{\#})}_{=1}\right) \\
\Longrightarrow \iota_{X^{\#}}\omega &= -\sum r^j dr^j = -\frac{1}{2}\sum d((r^j)^2)
\end{aligned}$$

If we choose the constant in definition of μ to be $\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$$

$\mathbb{CP}^{n-1}$ is thus called a **reduced space**. Notice also that the image of the moment map is half-space.

These particular observations are related to major theorems: Under assumptions,

- [Marsden-Weinstein-Meyer] reduced spaces are symplectic manifolds;
- [Atiyah-Guillemin-Sternberg] the image of the moment map is a convex polytope;
- [Delzant] Hamiltonian $\mathbb{T}^n$-spaces are classified by the image of the moment map.

Remark 12.3.1. The vector field $X^{\#}(z) = iz$ we computed above should not be interpreted as

$$i\left(z^1\frac{\partial}{\partial z^1} + z^2\frac{\partial}{\partial z^2} + \ldots + z^n\frac{\partial}{\partial z^n}\right).$$

Recall

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right)$$

Thus we instead have

$$\begin{aligned}
\left(iz^1, \ldots, iz^n\right) &= \left(ix^1 - y^1, \ldots, ix^n - y^n\right) \\
&\equiv \left(-y^1, \ldots, -y^n, x^1, \ldots, x^n\right) \\
&= \sum_{j=1}^n \left(-y^j\frac{\partial}{\partial x^j} + x^j\frac{\partial}{\partial y^j}\right) \\
&= \sum_{j=1}^n \left(-y^j\left(\frac{\partial}{\partial z^j} + \frac{\partial}{\partial \bar{z}^j}\right) + x^j i\left(\frac{\partial}{\partial z^j} - \frac{\partial}{\partial \bar{z}^j}\right)\right) \\
&= \sum_{j=1}^n \left(\left(-y^j + x^j i\right)\frac{\partial}{\partial z^j} + \left(-y^j - x^j i\right)\frac{\partial}{\partial \bar{z}^j}\right) \\
&= \sum_{j=1}^n \left(i\left(iy^j + x^j\right)\frac{\partial}{\partial z^j} - i\left(-iy^j + x^j\right)\frac{\partial}{\partial \bar{z}^j}\right) \\
&= \sum_{j=1}^n \left(iz^j\frac{\partial}{\partial z^j} - iz^j\frac{\partial}{\partial \bar{z}^j}\right)
\end{aligned}$$

♠

12.4 Classical Examples

Rotations

Example 12.4.1.
Let $G = \mathrm{SO}(3) = \{A \in \mathrm{GL}(3;\mathbb{R}) \mid A^tA = \mathrm{Id} \text{ and } \det A = 1\}$. Then $\mathfrak{g} = \{A \in \mathfrak{gl}(3;\mathbb{R}) \mid A + A^t = 0\}$ is the space of 3×3 skew-symmetric matrices and can be identified with $\mathbb{R}^3$. The Lie bracket on $\mathfrak{g}$ can be identified with the exterior product via

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \longmapsto \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$
$$[A, B] = AB - BA \longmapsto \vec{a} \times \vec{b}$$

Exercise 12.4.2. *Under the identifications $\mathfrak{g}, \mathfrak{g}^* \simeq \mathbb{R}^3$, the adjoint and coadjoint actions are the usual* $\mathrm{SO}(3)$-*action on $\mathbb{R}^3$ by rotations.*

Solution. Two properties will be needed (see wikipedia for a collection of properties of cross product): (1) If we let $v_\times$ be the associated 3×3 skew-symmetric matrix for vector $v\in\mathbb{R}^3$, then for any vector $w\in\mathbb{R}^3$, we have $v_\times w = v\times w$. (2) If R is a rotation, i.e., $\in SO(3)$, then $(Rv)\times(Rw)=R(v\times w)$. This is called the rotational invariance of cross product. See here for the proof. In fact, just think about this as the rotation of the frame.

We claim that $R\cdot v_\times\cdot R^{-1}=(Rv)_\times$ for any $v\in\mathbb{R}^3$ and $R\in SO(3)$. Let $w\in\mathbb{R}^3$. Then, the two properties above imply

$$R\cdot v_\times\cdot R^{-1}w = R\left(v_\times\left(R^{-1}w\right)\right)=R\left(v\times\left(R^{-1}w\right)\right)=(Rv)\times w.$$

Notice that the skew-symmetric matrix corresponding to the rotated vector Rv is $(Rv)_\times$, which satisfies

$$(Rv)_\times w=(Rv)\times w.$$

The claim then follows.

We now compute the adjoint action: let $R\in SO(3)$ and $v\in\mathbb{R}^3$ with its corresponding matrix $v_\times$ in $\mathfrak{g}$, we have

$$\begin{aligned}
\operatorname{Ad}_R v_\times &= \left.\frac{d}{dt}\right|_{t=0}\gamma_{\operatorname{Ad}_R v_\times}(t)\\
&= \left.\frac{d}{dt}\right|_{t=0}\exp(\overbrace{\operatorname{Ad}_R}^{:=d(\tau_R)_e}(tv_\times))\quad \text{where } \tau_g: a\mapsto gag^{-1}\\
&= \left.\frac{d}{dt}\right|_{t=0}\tau_R\left(\exp(tv_\times)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} R\cdot(\exp(tv_\times))\cdot R^{-1}\quad \text{by [8] prop.20.8(g)}\\
&= \left.\frac{d}{dt}\right|_{t=0} R\cdot e^{tv_\times}\cdot R^{-1}\\
&= \left.\frac{d}{dt}\right|_{t=0} R\cdot\left(I+tv_\times+O(t^2)\right)\cdot R^{-1}\\
&= R\cdot v_\times\cdot R^{-1}=(Rv)_\times\quad \text{by the property we showed above.}
\end{aligned}$$

Thus, $\operatorname{Ad}_R: v_\times\mapsto(Rv)_\times$ as a map from $\mathfrak{g}$ to $\mathfrak{g}$. Under the identification $\mathfrak{g}\cong\mathbb{R}^3$, $\operatorname{Ad}_R: v\mapsto Rv$ as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$. Let $\xi\in\mathfrak{g}^*\cong\mathbb{R}^3$ =all row vectors. Then,

$$\operatorname{Ad}^*_R(\xi)(v)=\xi\left(\operatorname{Ad}_{R^{-1}}(v)\right)=\xi R^{-1}v\implies \operatorname{Ad}^*_R(\xi)=\xi R^t.$$

♦

Therefore, the coadjoint orbits are the spheres in $\mathbb{R}^3$ centered at the origin. Homework shows that coadjoint orbits are symplectic. ♣

The name “moment map” comes from being the generalization of linear and angular momenta in classical mechanics.

Linear action*

Consider $\left(\mathbb{R}^{2n},\Omega_0\right)$. The linear symplectic group

$$\operatorname{Sp}(2n)=\{A\in\operatorname{GL}(2n)\mid A^*\Omega_0=\Omega_0\}=\left\{A\mid J_0=A^TJ_0A\right\}$$

*Taken from this note.

acts on $\left(\mathbb{R}^{2n}, \Omega_0\right)$ in the natural way. The Lie algebra of $\mathrm{Sp}(2n)$ is

$$\mathfrak{sp}(2n) = \left\{A \in \mathfrak{gl}(2n) \mid A^t J_0 = -J_0 A\right\}$$

For any $x \in \mathbb{R}^{2n}$ and any $X \in \mathfrak{sp}(2n) \subset \mathfrak{gl}(2n)$,

$$X_M(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX)x = Xx$$

One can check that the map $\mu : \mathbb{R}^{2n} \to \mathfrak{sp}(2n)^*$ defined by

$$\langle \mu(x), X\rangle = \frac{1}{2}\Omega_0(X(x), x)$$

is the moment map of this action.

Compact Lie group*

[Lifting of smooth action to cotangent bundle.] Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action of a compact Lie group G on a smooth manifold M. The action induces a natural action $\gamma : G \to \mathrm{Diff}\left(T^*M\right)$ of G on T^*M by

$$g \cdot (m, \eta) = \left(g \cdot m, \left(dg^{-1}\right)_m^* \eta\right)$$

where $\eta \in T_m^*M$. We shall denote $p = (m, \eta)$ for simplicity.

Observation 1: The projection map $\pi : T^*M \to M$ is G-equivariant:

$$\pi(g \cdot p) = g \cdot m = g \cdot \pi(p)$$

As a consequence, we have

$$d\pi_{g\cdot p} \circ dg_p = dg_m \circ d\pi_p$$

and thus

$$dg_p^* \circ d\pi_{g\cdot p}^* = d\pi_p^* \circ dg_m^*$$

Observation 2: For any $X \in \mathfrak{g}$,

$$dg_p^{-1}\left(X_M(g \cdot p)\right) = \left(Ad_{g^{-1}}X\right)_M(p)$$

This follows from direct computation:

$$\begin{aligned}
\left(Ad_{g^{-1}}X\right)_M(p) &= \left.\frac{d}{dt}\right|_{t=0} \exp\left(tAd_{g^{-1}}X\right) \cdot p \\
&= \left.\frac{d}{dt}\right|_{t=0} g^{-1}\exp(tX)g \cdot p \\
&= dg_p^{-1}\left(X_M(g \cdot p)\right)
\end{aligned}$$

Proposition 12.4.3. *The induced action γ is a Hamiltonian action with moment map $\mu : T^*M \to \mathfrak{g}^*$ given by*

$$\langle \mu(p), X\rangle = \langle \eta, d\pi_p\left(X_M\right)\rangle$$

Proof. Let α be the tautological 1-form on T^*M. Recall that for any $p = (m, \eta) \in M$, $\alpha_p = \left(d\pi_p\right)^* \eta$. So

$$\left(g^*\alpha\right)_p = \left(dg_p\right)^* \alpha_{g\cdot p} = \left(dg_p\right)^* \left(d\pi_{g\cdot p}\right)^* \left(dg^{-1}\right)_m^* \eta = d\pi_p^* \eta = \alpha_p$$

i.e. α is invariant under the G-action. It follows that $\mathcal{L}_{X_M}\alpha = 0$ for all $X \in \mathfrak{g}$. So by Cartan's magic formula,

$$d\iota_{X_M}\alpha = \iota_{X_M}\omega.$$

■

*Same.

Translation

Consider $\mathbb{R}^6$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$ and symplectic form $\omega = \sum dx_i \wedge dy_i$. Let $\mathbb{R}^3$ act on $\mathbb{R}^6$ by translations:

$$\begin{aligned} \vec{a} \in \mathbb{R}^3 \longmapsto & \psi_{\vec{a}} \in \operatorname{Sympl}\left(\mathbb{R}^6, \omega\right) \\ & \psi_{\vec{a}}(\vec{x}, \vec{y}) = (\vec{x} + \vec{a}, \vec{y}) \end{aligned}$$

Then $X^{\#} = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}$ for $X = \vec{a}$, and

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3, \quad \mu(\vec{x}, \vec{y}) = \vec{y}$$

is a moment map, with

$$\mu^{\vec{a}}(\vec{x}, \vec{y}) = \langle \mu(\vec{x}, \vec{y}), \vec{a} \rangle = \vec{y} \cdot \vec{a}.$$

Classically, $\vec{y}$ is called the **momentum vector** corresponding to the **position vector** $\vec{x}$, and the map μ is called the **linear momentum**.

Rotation

The $\mathrm{SO}(3)$-action on $\mathbb{R}^3$ by rotations lifts to a symplectic action ψ on the cotangent bundle $\mathbb{R}^6$. The infinitesimal version of this action is

$$\begin{aligned} \vec{a} \in \mathbb{R}^3 \longmapsto & d\psi(\vec{a}) \in \chi^{\text{sympl}}\left(\mathbb{R}^6\right) \\ & d\psi(\vec{a})(\vec{x}, \vec{y}) = (\vec{a} \times \vec{x}, \vec{a} \times \vec{y}) \end{aligned}$$

Then

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3, \quad \mu(\vec{x}, \vec{y}) = \vec{x} \times \vec{y}$$

is a moment map, with

$$\mu^{\vec{a}}(\vec{x}, \vec{y}) = \langle \mu(\vec{x}, \vec{y}), \vec{a} \rangle = (\vec{x} \times \vec{y}) \cdot \vec{a}$$

The map μ is called the **angular momentum**.

12.5 Coadjoint Actions

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$.

Exercise 12.5.1. *Let ${}^{\mathfrak{g}}X^{\#}$ be the vector field generated by $X \in \mathfrak{g}$ for the adjoint representation of G on $\mathfrak{g}$. Show that*

$${}^{\mathfrak{g}}X^{\#}_Y = [X, Y] \quad \forall Y \in \mathfrak{g}$$

Solution. We have action $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, and Exercise 11.5.2 shows exactly that ${}^{\mathfrak{g}}X^{\#}(Y) = [X, Y]$. ♦

Exercise 12.5.2. *Let $X^{\#}$ be the vector field generated by $X \in \mathfrak{g}$ for the coadjoint representation of G on $\mathfrak{g}^*$. Show that*

$$\left\langle X^{\#}_{\xi}, Y \right\rangle = \langle \xi, [Y, X] \rangle \quad \forall Y \in \mathfrak{g}$$

Solution. We have action

$$\begin{aligned} \mathrm{Ad}^* : G & \longrightarrow GL(\mathfrak{g}^*) \\ g & \mapsto \mathrm{Ad}^*_g := \left(\begin{array}{l} \mathfrak{g}^* \to \mathfrak{g}^* \\ \xi \mapsto \langle \xi, \mathrm{Ad}_{g^{-1}}(\,\cdot\,) \rangle \end{array} \right) \end{aligned}$$

or $\mathrm{Ad}^*: G \times \mathfrak{g}^* \to \mathfrak{g}^*$. Then with exactly the same proof as in (eq.12.1),

$$\begin{aligned}\langle X_\xi^\#, Y\rangle &= \frac{d}{dt}\bigg|_{t=0} \langle \xi, \mathrm{Ad}_{\exp -tX}(Y)\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle \xi, Y - t[X,Y] + O\left(t^2\right)\rangle \\ &= \langle \xi, [Y,X]\rangle\end{aligned}$$

♦

Exercise 12.5.3. *For any $\xi \in \mathfrak{g}^*$, define a skew-symmetric bilinear form on $\mathfrak{g}$ by*

$$\omega_\xi(X,Y) := \langle \xi, [X,Y]\rangle$$

Show that the kernel of ω_ξ is the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer of ξ for the coadjoint representation.

Solution. By (eq.14.3), we have

$$\mathrm{Lie}(\mathfrak{g}_\xi) = \{X \in \mathfrak{g} | X_\xi^\# = 0\} \xlongequal{\text{prev. ex.}} \{X \in \mathfrak{g} | \langle \xi, [Y,X]\rangle = 0, \forall Y \in \mathfrak{g}\} = \ker \widetilde{\omega}_\xi.$$

♦

Exercise 12.5.4. *Show that ω_ξ defines a nondegenerate 2-form on the tangent space at ξ to the coadjoint orbit through ξ.*

Solution. By (eq.14.2), we have

$$T_\xi \mathcal{O}_\xi = \{X_\xi^\# | X \in \mathfrak{g}\}$$

Then, $\forall X_\xi^\#, Y_\xi^\# \in T_\xi\mathcal{O}_\xi$, define

$$\Omega_\xi(X_\xi^\#, Y_\xi^\#) = \omega_\xi(X,Y) = \langle \xi, [X,Y]\rangle.$$

It is nondegenerate because

$$\ker \widetilde{\Omega}_\xi = \{X_\xi^\# | X \in \ker \widetilde{\omega}_\xi\} \xlongequal{\text{prev. ex.}} 0.$$

♦

Exercise 12.5.5. *Show that ω_ξ defines a closed 2-form on the orbit of ξ in $\mathfrak{g}^*$.*

[Hint: The tangent space to the orbit being generated by the vector fields $X^\#$, this is a consequence of the Jacobi identity in $\mathfrak{g}$.]

This ***canonical symplectic form*** *on the coadjoint orbits in $\mathfrak{g}^*$ is also known as the* ***Lie-Poisson*** *or* ***Kostant-Kirillov symplectic structure****.*

Solution. We still use $X^\#, Y^\#, Z^\#$ to denote the their restrictions on the orbit. Then, by [LeeSM] Prop.14.32, we have

$$\begin{aligned}(d\Omega)(X^\#, Y^\#, Z^\#) =& X^\#(\Omega(Y^\#, Z^\#)) - Y^\#(\Omega(X^\#, Z^\#)) + Z^\#(\Omega(X^\#, Y^\#)) \\ &- \Omega([X^\#, Y^\#], Z^\#) - \Omega([Z^\#, X^\#], Y^\#) - \Omega([Y^\#, Z^\#], X^\#) \\ =& \langle X_\xi^\#, [Y,Z]\rangle - \langle Y_\xi^\#, [X,Z]\rangle + \langle Z^\#\xi, [X,Y]\rangle \\ &- \langle \xi, [[X,Y],Z]\rangle - \langle \xi, [[Z,X],Y]\rangle - \langle \xi, [[Y,Z],X]\rangle = 0\end{aligned}$$

We evaluate $X^{\#}(\Omega(Y^{\#}, Z^{\#}))$ pointwise: for every ξ on the coadjoint orbit,

$$\begin{aligned} X^{\#}_{\xi}(\overbrace{\Omega_{\xi}(Y^{\#}_{\xi}, Z^{\#}_{\xi})}^{f:\mathcal{O}_{\xi}\to\mathbb{R};\xi\mapsto\Omega_{\xi}(Y^{\#}_{\xi},Z^{\#}_{\xi})}) &= \left.\frac{d}{dt}\right|_{t=0} f(\mathrm{Ad}^*_{\exp tX}(\xi)) \quad \text{similar to (11.6)} \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\xi(\mathrm{Ad}_{\exp -tX}(\,\cdot\,))) = \left.\frac{d}{dt}\right|_{t=0} \xi(\mathrm{Ad}_{\exp -tX}([Y,Z])) \\ &= \xi\left(-[X,[Y,Z]]\right) \quad \text{pass by linearity and use (eq.12.1)} \end{aligned}$$

The first line vanishes by writing all three components in this way and apply Jacobi identity. The second line also vanishes by Jacobi identity as

$$\begin{aligned} &-\Omega_{\xi}([X^{\#}_{\xi}, Y^{\#}_{\xi}], Z^{\#}_{\xi}) - \Omega([Z^{\#}_{\xi}, X^{\#}_{\xi}], Y^{\#}_{\xi}) - \Omega([Y^{\#}_{\xi}, Z^{\#}_{\xi}], X^{\#}_{\xi}) \\ =&-\langle \xi, [[X,Y],Z]\rangle - \langle \xi, [[Z,X],Y]\rangle - \langle \xi, [[Y,Z],X]\rangle \end{aligned}$$

For other proofs other than this algebraic one, see [6] p.6. ♦

Exercise 12.5.6. *The Lie algebra structure of $\mathfrak{g}$ defines a canonical Poisson structure on $\mathfrak{g}^*$:*

$$\{f,g\}(\xi) := \langle \xi, [df_{\xi}, dg_{\xi}]\rangle$$

for $f, g \in C^{\infty}(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. Notice that $df_{\xi} : T_{\xi}\mathfrak{g}^* \simeq \mathfrak{g}^* \to \mathbb{R}$ is identified with an element of $\mathfrak{g} \simeq \mathfrak{g}^{**}$. Check that $\{\cdot,\cdot\}$ satisfies the Leibniz rule:

$$\{f, gh\} = g\{f,h\} + h\{f,g\}$$

Solution. By definition of the bracket, we have $\{f, gh\}(\xi) = \langle \xi, [df_{\xi}, d(gh)_{\xi}]\rangle$. The differential $d(gh)$ can be expressed as $gdh + hdg$. Substituting this into the expression for $\{f, gh\}$, we obtain $\{f, gh\}(\xi) = \langle \xi, [df_{\xi}, gdh_{\xi} + hdg_{\xi}]\rangle$. Since the Lie bracket is linear, we can expand this as $\langle \xi, g\,[df_{\xi}, dh_{\xi}] + h\,[df_{\xi}, dg_{\xi}]\rangle$. Using the bilinearity of the pairing $\langle \xi, \cdot\rangle$, this yields $g\,\langle \xi, [df_{\xi}, dh_{\xi}]\rangle + h\,\langle \xi, [df_{\xi}, dg_{\xi}]\rangle$, which corresponds to $g\{f,h\}(\xi) + h\{f,g\}(\xi)$. ♦

Exercise 12.5.7. *Show that the* ***jacobiator***

$$J(f,g,h) := \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$$

is a trivector field, i.e., J is a skew-symmetric trilinear map $C^{\infty}(\mathfrak{g}^*) \times C^{\infty}(\mathfrak{g}^*) \times C^{\infty}(\mathfrak{g}^*) \to C^{\infty}(\mathfrak{g}^*)$, which is a derivation in each argument.

Solution. For $f, g, h, k \in C^{\infty}(\mathfrak{g}^*)$, we want to show:

$$J(fk, g, h) = kJ(f,g,h) + fJ(k,g,h).$$

By definition of J, we have:

$$J(fk,g,h) = \{\{fk,g\},h\} + \{\{g,h\},fk\} + \{\{h,fk\},g\}.$$

We now apply the Leibniz rule for each of the Poisson brackets involved:

$$\begin{aligned} \{\{fk,g\},h\} &= \{f\{k,g\} + k\{f,g\},h\} \\ &= \{f\{k,g\},h\} + \{k\{f,g\},h\} \\ &= \{f,h\}\{k,g\} + \{\{k,g\},h\}f + \{k,h\}\{f,g\} + \{\{f,g\},h\}k \end{aligned}$$

Similarly,

$$\{\{h,fk\},g\} = \{f,g\}\{h,k\} + \{\{h,k\},g\}f + \{k,g\}\{h,f\} + \{\{h,f\},g\}k$$

$$\{\{g,h\},fk\} = \{\{g,h\},k\}f + \{\{g,h\},f\}k$$

Combining these results, we obtain:

$$\begin{aligned} J(fk,g,h) &= \{\{k,g\},h\}f + \{\{h,k\},g\}f + \{\{g,h\},k\}f \\ &\quad + \{\{f,g\},h\}k + \{\{h,f\},g\}k + \{\{g,h\},f\}k \\ &= fJ(k,g,h) + kJ(f,g,h) \end{aligned}$$

This is similar for the other two arguments. ♦

Exercise 12.5.8. *Show that $J \equiv 0$, i.e., $\{\cdot,\cdot\}$ satisfies the Jacobi identity.*

[Hint: Follows from the Jacobi identity for $[\cdot,\cdot]$ in $\mathfrak{g}$. It is enough to check on coordinate functions.]

Solution. Let $\{e_i\}$ denote a basis for the Lie algebra $\mathfrak{g}$. We define the coordinate functions $x_i \in C^\infty(\mathfrak{g}^*)$ by:

$$x_i(\xi) = \langle \xi, e_i \rangle$$

The differential of the linear map x_i at any point ξ are then interpreted as the basis elements e_i of $\mathfrak{g}$, i.e., $dx_i = e_i$. Thus, the Jacobi identity in $\mathfrak{g}$ implies

$$J(x_i, x_j, x_k)(\xi) = \langle \xi, [[e_i,e_j],e_k] + [[e_j,e_k],e_i] + [[e_k,e_i],e_j] \rangle = 0$$

♦

Chapter 13

Existence and Uniquness of Moment Maps

13.1 Lie Algebras of Vector Fields

Let (M,ω) be a symplectic manifold and $v \in \chi(M)$ a vector field on M.

$$\begin{aligned} v \text{ is symplectic } &\Longleftrightarrow \imath_v\omega \text{ is closed} \\ v \text{ is hamiltonian } &\Longleftrightarrow \imath_v\omega \text{ is exact} \end{aligned}$$

The spaces

$$\begin{aligned} \chi^{\text{sympl}}(M) &= \text{ symplectic vector fields on } M \\ \chi^{\text{ham}}(M) &= \text{ hamiltonian vector fields on } M \end{aligned}$$

are Lie algebras for the Lie bracket of vector fields. $C^\infty(M)$ is a Lie algebra for the Poisson bracket, $\{f,g\} = \omega(v_f, v_g)$.$H^1(M;\mathbb{R})$ and $\mathbb{R}$ are regarded as Lie algebras for the trivial bracket. We have two exact sequences of Lie algebras:

$$\begin{aligned} 0 \longrightarrow \chi^{\text{ham}}(M) \hookrightarrow \chi^{\text{sympl}}(M) &\longrightarrow H^1(M;\mathbb{R}) \longrightarrow 0 \\ v &\longmapsto [\imath_v\omega] \end{aligned}$$

$$\begin{aligned} 0 \quad \longrightarrow \quad \mathbb{R} \quad \hookrightarrow \quad C^\infty(M) &\longrightarrow \chi^{\text{ham}}(M) \longrightarrow 0 \\ f &\longmapsto v_f \end{aligned}$$

In particular, if $H^1(M;\mathbb{R}) = 0$, then $\chi^{\text{ham}}(M) = \chi^{\text{sympl}}(M)$. Let G be a connected Lie group. A symplectic action $\Psi : G \to \operatorname{Sympl}(M,\omega)$ induces an infinitesimal action

$$\begin{aligned} d\Psi : \mathfrak{g} &\longrightarrow \chi^{\text{sympl}}(M) \\ X &\longmapsto X^\# = \text{vector field generated by the} \\ &\qquad\qquad \text{one-parameter group } \{\exp tX(e) \mid t \in \mathbb{R}\}. \end{aligned}$$

Exercise 13.1.1. *Check that the map $d\Psi$ is a Lie algebra anti-homomorphism.*

The action Ψ is Hamiltonian if and only if there is a Lie algebra homomorphism $\mu^* : \mathfrak{g} \to C^\infty(M)$ lifting $d\Psi$, i.e., making the following diagram commute.

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \chi^{\text{sympl}}(M) \\ & \nwarrow^{\mu^*} \quad \mathfrak{g} \quad \nearrow^{d\Psi} & \end{array}$$

The map μ^* is then called a comoment map.

Existence of μ^* comoment map	$\Longleftrightarrow$	Existence of μ moment map
Lie algebra homomorphism	$\longleftrightarrow$	equivariance

13.2 Lie Algebra Cohomology

Let $\mathfrak{g}$ be a Lie algebra, and

$$\begin{aligned} C^k &:= \Lambda^k \mathfrak{g}^* = k\text{-cochains on } \mathfrak{g} \\ &= \text{ alternating } k\text{-linear maps } \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{k} \longrightarrow \mathbb{R}. \end{aligned}$$

Define a linear operator $\delta : C^k \to C^{k+1}$ by

$$\delta c\left(X_0, \ldots, X_k\right) = \sum_{i<j} (-1)^{i+j} c\left(\left[X_i, X_j\right], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k\right)$$

Exercise 13.2.1. *Check that $\delta^2 = 0$. The **Lie algebra cohomology groups** (or **Chevalley cohomology groups)** of $\mathfrak{g}$ are the cohomology groups of the complex $0 \xrightarrow{\delta} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \ldots$:*

$$H^k(\mathfrak{g}; \mathbb{R}) := \frac{\ker \delta : C^k \longrightarrow C^{k+1}}{\operatorname{im} \delta : C^{k-1} \longrightarrow C^k}$$

Theorem 13.2.2. *If $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G, then*

$$H^k(\mathfrak{g}; \mathbb{R}) = H^k_{\textit{deRham}}(G)$$

Proof. Exercise. [Hint: by averaging show that the de Rham cohomology can be computed from the subcomplex of G-invariant forms.] ∎

Meaning of $H^1(\mathfrak{g}; \mathbb{R})$ and $H^2(\mathfrak{g}; \mathbb{R})$:

- An element of $C^1 = \mathfrak{g}^*$ is a linear functional on $\mathfrak{g}$. If $c \in \mathfrak{g}^*$, then $\delta c\left(X_0, X_1\right) = -c\left(\left[X_0, X_1\right]\right)$. The **commutator ideal** of $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}] := \{$ linear combinations of $[X, Y]$ for any $X, Y \in \mathfrak{g}\}$. Since $\delta c = 0$ if and only if c vanishes on $[g, \mathfrak{g}]$, we conclude that

$$H^1(\mathfrak{g}; \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^0$$

  where $[\mathfrak{g}, \mathfrak{g}]^0 \subseteq \mathfrak{g}^*$ is the annihilator of $[\mathfrak{g}, \mathfrak{g}]$.

- An element of C^2 is an alternating bilinear map $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

$$\delta c\left(X_0, X_1, X_2\right) = -c\left(\left[X_0, X_1\right], X_2\right) + c\left(\left[X_0, X_2\right], X_1\right) - c\left(\left[X_1, X_2\right], X_0\right)$$

  If $c = \delta b$ for some $b \in C^1$, then

$$c\left(X_0, X_1\right) = (\delta b)\left(X_0, X_1\right) = -b\left(\left[X_0, X_1\right]\right)$$

13.3 Existence of Moment Maps

Theorem 13.3.1. *If $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g},\mathbb{R}) = 0$, then any symplectic G-action is hamiltonian.*

Proof. Let $\Psi : G \to \mathrm{Sympl}(M,\omega)$ be a symplectic action of G on a symplectic manifold (M,ω). Since

$$H^1(\mathfrak{g};\mathbb{R}) = 0 \Longleftrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$$

and since commutators of symplectic vector fields are hamiltonian, we have

$$d\Psi : \mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \longrightarrow \chi^{\text{ham}}(M)$$

The action Ψ is hamiltonian if and only if there is a Lie algebra homomorphism $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(M) \arrow[rr] & & \chi^{\text{ham}}(M) \\
& & \mathfrak{g} \arrow[ul, "?"] \arrow[ur, "d\Psi"'] &
\end{tikzcd}$$

We first take an arbitrary vector space lift $\tau : \mathfrak{g} \to C^\infty(M)$ making the diagram commute, i.e., for each basis vector $X \in \mathfrak{g}$, we choose

$$\tau(X) = \tau^X \in C^\infty(M) \quad \text{such that} \quad v_{(\tau^X)} = d\Psi(X)$$

The map $X \mapsto \tau^X$ may not be a Lie algebra homomorphism. By construction, $\tau^{[X,Y]}$ is a hamiltonian function for $[X,Y]^\#$, and (as computed in Lecture 16) $\{\tau^X,\tau^Y\}$ is a hamiltonian function for $-\left[X^\#,Y^\#\right]$. Since $[X,Y]^\# = -\left[X^\#,Y^\#\right]$, the corresponding hamiltonian functions must differ by a constant:

$$\tau^{[X,Y]} - \{\tau^X,\tau^Y\} = c(X,Y) \in \mathbb{R}$$

By the Jacobi identity, $\delta c = 0$. Since $H^2(\mathfrak{g};\mathbb{R}) = 0$, there is $b \in \mathfrak{g}^*$ satisfying $c = \delta b, c(X,Y) = -b([X,Y])$. We define

$$\begin{aligned} \mu^* : \mathfrak{g} &\longrightarrow C^\infty(M) \\ X &\longmapsto \mu^*(X) = \tau^X + b(X) = \mu^X. \end{aligned}$$

Now μ^* is a Lie algebra homomorphism:

$$\mu^*([X,Y]) = \tau^{[X,Y]} + b([X,Y]) = \{\tau^X,\tau^Y\} = \{\mu^X,\mu^Y\}$$

■

So when is $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$?

A compact Lie group G is semisimple if $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$

Example 13.3.2. The unitary group $\mathrm{U}(n)$ is not semisimple because the multiples of the identity, $S^1 \cdot \mathrm{Id}$, form a nontrivial center; at the level of the Lie algebra, this corresponds to the 1-dimensional subspace $\mathbb{R} \cdot \mathrm{Id}$ of scalar matrices which are not commutators since they are not traceless. Any direct product of the other compact classical groups $\mathrm{SU}(n), \mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ is semisimple $(n > 1)$. Any commutative Lie group is not semisimple. ♣

Theorem 13.3.3 (Whitehead Lemmas)**.** *Let G be a compact Lie group.*

$$G \text{ is semisimple } \Longleftrightarrow H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$$

A proof can be found in [67, pages 93-95].

Corollary 13.3.4. *If G is semisimple, then any symplectic G-action is hamiltonian.*

13.4 Uniquness of Moment Maps

Let G be a compact connected Lie group.

Theorem 13.4.1. *If $H^1(\mathfrak{g};\mathbb{R}) = 0$, then moment maps for hamiltonian G-actions are unique.*

Proof. Suppose that μ_1^* and μ_2^* are two comoment maps for an action Ψ:

$$\begin{tikzcd} C^\infty(M) \arrow[rr] & & \chi^{\mathrm{ham}}(M) \\ & \mathfrak{g} \arrow[ul, "\mu_1^*", "\mu_2^*"'] \arrow[ur, "d\Psi"'] & \end{tikzcd}$$

For each $X \in \mathfrak{g}, \mu_1^X$ and μ_2^X are both hamiltonian functions for $X^\#$, thus $\mu_1^X - \mu_2^X = c(X)$ is locally constant. This defines $c \in \mathfrak{g}^*, X \mapsto c(X)$.

Since μ_1^*, μ_2^* are Lie algebra homomorphisms, we have $c([X,Y]) = 0, \forall X, Y \in \mathfrak{g}$, i.e., $c \in [\mathfrak{g},\mathfrak{g}]^0 = \{0\}$. Hence, $\mu_1^* = \mu_2^*$. ■

Corollary of this proof. In general, if $\mu : M \to \mathfrak{g}^*$ is a moment map, then given any $c \in [\mathfrak{g},\mathfrak{g}]^0, \mu_1 = \mu + c$ is another moment map.

In other words, moment maps are unique up to elements of the dual of the Lie algebra which annihilate the commutator ideal.

The two extreme cases are:

- G semisimple: any symplectic action is hamiltonian , moment maps are unique.
- G commutative: symplectic actions may not be hamiltonian , moment maps are unique up to any constant $c \in \mathfrak{g}^*$.

Example 13.4.2. The circle action on $(\mathbb{T}^2, \omega = d\theta_1 \wedge d\theta_2\ )$ by rotations in the θ_1 direction has vector field $X^\# = \frac{\partial}{\partial \theta_1}$; this is a symplectic action but is not hamiltonian. ♣

Part V

Symplectic Reduction

Chapter 14

The Marsden-Weinstein-Meyer Theorem

14.1 Statement

Theorem 14.1.1 (Marsden-Weinstein-Meyer)**.** *Let (M,ω,G,μ) be a Hamiltonian G-space for a compact Lie group G. Let $i:\mu^{-1}(0)\hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{\text{red}}=\mu^{-1}(0)/G$ is a manifold,*
- *$\pi:\mu^{-1}(0)\to M_{\text{red}}$ is a principal G-bundle, and*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega=\pi^*\omega_{\text{red}}$.*

Definition 14.1.2. *The pair $(M_{\text{red}},\omega_{\text{red}})$ is called the **reduction** of (M,ω) with respect to G,μ, or the **reduced space**, or the **symplectic quotient**, or the **Marsden-Weinstein-Meyer quotient**, etc.*

Combining the equivariance of μ,

$$\mu\circ\Psi_g=\operatorname{Ad}_g^*\circ\mu$$

with linearity of co-adjoint action (so zero is sent to zero), we have

Lemma 14.1.3. *If $\mu(p)=0$. then for any $g\in G,\mu\circ\Psi_g(p)=0$.*

In other words, the G-actions on M induces a G-action on $\mu^{-1}(0)$.

Also, it is easy to check the following.

Lemma 14.1.4. *If an action $\theta:G\times M\to M$ is free then $g_1\cdot p=g_2\cdot p\implies g_1=g_2$. In this case, the orbit map $\theta^{(p)}:G\to G\cdot p$; $g\mapsto g\cdot p$ is bijective.*

Special cases

The simplest case for a compact Lie group is $G=S^1$ and the simplest case for M is when $\dim M=2$. For example, the action $S^1\times\mathbb{C}^n\to\mathbb{C}^n$ with $n=1$ gives $\mu^{-1}(0)/S^1=S^1/S^1$ a single point with zero dimension.

The next simplest example is when $G=S^1$ and $\dim M=4$. In this case the moment map is $\mu:M\to\mathbb{R}$ and the embedded submanifold $\mu^{-1}(0)$ has dimension 3. Let $p\in\mu^{-1}(0)$. Choose local coordinates on M:

- θ along the orbit through p;
- μ given by the moment map; and
- η_1,η_2 pullback of coordinates on two-dimensional $\mu^{-1}(0)/S^1$.

To see how these four coordinates determine a point p in $\mu^{-1}(0) \subseteq M$, we first observe that $\mu : M \to \mathbb{R}$ cuts M into slices of level sets $\mu^{-1}(c)$, and each of them is a 3-manifold. For $\mu^{-1}(0)$, lemma 14.1.3 shows that G acts on $\mu^{-1}(0)$ by the same Ψ. Thus, each point p in $\mu^{-1}(0)$ is classified in different orbits $\mathcal{O}_p = G \cdot p \subseteq \mu^{-1}(0)$. Now, note that lemma 14.1.4 says that the orbit map $\Psi^{(p)}$ corresponds each value of Lie group G with one point in $\mathcal{O}_p$. In all, μ determines whether or not p is in $\mu^{-1}(0)$; η_1, η_2 determines which orbit $\mathcal{O}_p$ in the orbit space $\mu^{-1}(0)/S^1$ does p lie; θ determines the location of that p is its orbit $\mathcal{O}_p$.

Using these coordinates, we can write the symplectic form as

$$\omega = A d\theta \wedge d\mu + B_j d\theta \wedge d\eta_j + C_j d\mu \wedge d\eta_j + D d\eta_1 \wedge d\eta_2$$

Since $d\mu = \iota_{\left(\frac{\partial}{\partial \theta}\right)} \omega$, we must have $A = 1, B_j = 0$. Hence,

$$\omega = d\theta \wedge d\mu + C_j d\mu \wedge d\eta_j + D d\eta_1 \wedge d\eta_2$$

Since ω is symplectic, we must have $D \neq 0$. Therefore, $i^*\omega = D d\eta_1 \wedge d\eta_2$ is the pullback of a symplectic form on M_{red}. The actual proof of the Marsden-Weinstein-Meyer theorem requires the following ingredients.

14.2 Ingredients

Orbits

We recall some more properties of orbit map $\theta^{(p)} : G \to M;\ g \mapsto g \cdot p$ for an action G on M:

Proposition 14.2.1.

(1) [LeeSM] Proposition 7.26 + [LeeSM] Problem 21-17: for a smooth left action θ of a Lie group G on smooth manifold M and a point $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is smooth and has constant rank, so the isotropy group $G_p = \left(\theta^{(p)}\right)^{-1}(p)$ is a properly embedded Lie subgroup of G. Each orbit $G \cdot p = \operatorname{Im}\left(\theta^{(p)}\right)$ is an immersed submanifold of M, which is embedded if the action is proper. Note that if the action is transitive, then $\theta^{(p)}$ is a surjective and thus a smooth submersion by [LeeSM] Theorem 4.14 (Global Rank Theorem), which by [LeeSM] Corollary 5.13 (Submersion Level Set Theorem) implies that $G_p = \left(\theta^{(p)}\right)^{-1}(p)$ also has dimension$= \dim M - \dim G$. If $G_p = \{e\}$, then $\theta^{(p)}$ is an injective smooth immersion.

(2) [LeeSM] Corollary 21.6 says that continuous action by compact Lie group on manifold is proper, and [LeeSM] Proposition 21.7 says that the orbit map is then a proper map and $G \cdot p = \operatorname{Im}\left(\theta^{(p)}\right)$ is closed in M. Since (1) claims that if we have $G_p = \{e\}$ then $\theta^{(p)}$ is an injective smooth immersion, [LeeSM] Proposition 4.22 shows that $\theta^{(p)}$ is an embedding and thus its image $G \cdot p = \operatorname{Im}\left(\theta^{(p)}\right)$ is a properly embedded submanifold (where we also used [LeeSM] Theorem 5.5.)

Remark 14.2.2. Suppose θ is a smooth left action of Lie group G on smooth manifold M and $p \in M$. Then by (1) above (specifically [8] Problem 21-17), $\mathcal{O}_p = G \cdot p$ is a smooth manifold. Consider a new orbit map $\varphi : G \times \mathcal{O}_p \to \mathcal{O}_p$ defined by restricting the action θ. Then by the smoothness and contant rank of $\varphi^{(p)}$ given by (1) above (specifically [8] Proposition 7.26), we see this surjective map φ is a smooth submersion due to [8] Theorem 4.14. Thus, $T_p\mathcal{O}_p = T_{\varphi^{(p)}(e)}\mathcal{O}_p = \operatorname{Im}\left(d\varphi_e^{(p)}\right)$.

Another way to show this claim is by using [8] Theorem 21.18 applied to G-homogeneous space $\mathcal{O}_p$ and considering the commutative diagram below:

$$\begin{tikzcd} & G \ni g \arrow[dl, "\pi"'] \arrow[d, "\varphi^{(p)}"] \\ G/G_p \arrow[r, "F"] & G \cdot p \end{tikzcd} \qquad g \cdot G_p \longmapsto g \cdot p$$

Then $d\varphi_e^{(p)} = d(F\circ\pi)_e = \underbrace{dF_{\pi(e)}}_{\text{diffeo}} \circ \underbrace{d\pi_e}_{C^\infty\text{ subm-}}$ is a smooth submersion. Thus, $T_p\mathcal{O}_p = \operatorname{Im}\left(d\varphi_e^{(p)}\right)$. ♠

The following results will also be useful.

Theorem 14.2.3. *[LeeSM Theorem 21.10 (Quotient Manifold Theorem)] Suppose G is a Lie group acting smoothly, freely, and properly on smooth manifold M. Then the orbit space M/G is a topological manifold of dimension $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

Proposition 14.2.4. *[LeeSM Proposition 5.38] Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi : U \to N$ is any local defining map for S, i.e., $S \cap U$ is a regular level set of Φ, then $T_pS = \ker d\Phi_p : T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap N$.*

We note a convenient alternative characterization of the vector field $X^\#(p)$, which is called the **infinitesimal generator of group action** $\Psi : G\times M \to M$ as at [8] p.529. Consider the orbit map $\Psi^{(p)} : G \to M$; $g \mapsto g\cdot p$. Then the orbit $\mathcal{O}_p$ through p is the image of $\Psi^{(p)}$. Since $\gamma(t) = \exp tX$ is a smooth curve in G whose velocity is $\gamma'(0) = X_e$, it follows from [8] corollary 3.25 that for each $p \in M$ we have

$$d\Psi_e^{(p)}(X_e) = (\Psi^{(p)}\cdot\gamma)'(0) = \left.\frac{d}{dt}\right|_{t=0} \Psi_{\exp tX}(p) = X^\#(p) \tag{14.1}$$

Let $\Psi : G \times M \to M$ be a smooth action of a Lie group over smooth manifold M.

- The orbit

  $$\mathcal{O}_p = G\cdot p = \{g\cdot p \mid g \in G\}$$

  is an embedded submanifold with tangent space at p equals

  $$T_p\mathcal{O}_p = \left\{X^\#(p) \mid X \in \mathfrak{g}\right\} \tag{14.2}$$

  due to Remark 14.2.2 and eq.(14.1).

- The stabilizer subgroup of each $p \in M$,

  $$G_p = \{g \in G \mid g\cdot p = p\}$$

  is a Lie subgroup $H = \left(\Psi^{(p)}\right)^{-1}(p)$ of G which by [8] Theorem 8..46 has Lie algebra $\mathfrak{h}$ equal to

  $$\begin{aligned} \mathfrak{g}_p :=\mathfrak{h} &= \{X \in \mathfrak{g} : X_e \in T_eH\} \\ &= \left\{X \in \mathfrak{g} : X_e \in \ker d\Psi_e^{(p)}\right\} \\ &= \left\{X \in \mathfrak{g} \mid X^\#(p) = 0\right\} \subseteq \mathfrak{g} \end{aligned} \tag{14.3}$$

  where the next-to-last step is due to property 14.2.1 (1) and proposition 14.2.4; and the last step is due to eq.(14.1).

Now suppose (M, ω, G, μ) is a Hamiltonian G-manifold. Let

$$(T_p\mathcal{O}_p)^{\omega_p} = \{v \in T_pM \mid \omega_p(v,w) = 0 \text{ for any } w \in T_p\mathcal{O}_p\}$$

be the **symplectic orthocomplement** of $T_p\mathcal{O}_p$ in (T_pM, ω_p), and let

$$\mathfrak{g}_p^0 = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0 \text{ for any } X \in \mathfrak{g}_p\}$$

be the **annihilator** of $\mathfrak{g}_p$ in $\mathfrak{g}^*$.

Lemma 14.2.5. *For any $p \in M$,*

(1) $\ker\left(d\mu_p\right)=\left(T_p\mathcal{O}_p\right)^{\omega_p}$.

(2) $\operatorname{Im}\left(d\mu_p\right)=\mathfrak{g}_p^0$.

Proof. For any $v\in T_pM$ and any $X\in\mathfrak{g}$ one has

$$\omega_p\left(X^{\#}(p),v\right)=\left(\iota_{X^{\#}}\omega\right)_p(v)=\left(d\mu^X\right)_p(v)=\langle d\mu_p(v),X\rangle$$

The last step comes from the following observation:

$$\begin{aligned}\left(d\mu^X\right)_p(v)&\xlongequal[\gamma\text{ w}/\gamma(0)=p,\gamma'(0)=v]{[8]\ 3.25}\left.\frac{d}{dt}\right|_{t=0}\mu^X(\gamma(t))\\&\xlongequal{\text{eval. map }\overline{X}}\left.\frac{d}{dt}\right|_{t=0}\overline{X}(\mu(\gamma(t)))\\&\xlongequal{\overline{X}\text{ linear}}\overline{X}\left(\left.\frac{d}{dt}\right|_{t=0}\mu(\gamma(t))\right)\\&\xlongequal{[8]\ 3.25}\overline{X}(d\mu_p(v))\\&=\langle d\mu_p(v),X\rangle\end{aligned}$$

Therefore, we have

$$\ker\left(d\mu_p\right)=\left(T_p\mathcal{O}_p\right)^{\omega_p}$$

which is (1). Besides, it is also easy to see

$$\operatorname{Im}\left(d\mu_p\right)\subseteq\mathfrak{g}_p^0$$

Then (2) follows from a dimensionality argument:

$$\dim\operatorname{Im}\left(d\mu_p\right)=\dim T_pM-\dim\ker\left(d\mu_p\right)\xlongequal{(1)}\dim T_p\mathcal{O}_p=\dim\operatorname{Im}\left(A_p\right),$$

where A_p is the linear map

$$A_p:\mathfrak{g}\to T_pM,X\mapsto X^{\#}(p)$$

and thus

$$\dim\operatorname{Im}\left(A_p\right)=\dim\mathfrak{g}-\dim\ker\left(A_p\right)=\dim\mathfrak{g}-\dim\mathfrak{g}_p=\dim\mathfrak{g}_p^0$$

∎

Several Consequences

We have some consequences of above lemma 14.2.5.

Proposition 14.2.6.
The action is locally free at p, i.e., stabilizer subgroup G_p is discrete
$\Longleftrightarrow\mathfrak{g}_p=\{0\}$
$\Longleftrightarrow$ *$d\mu_p$ is surjective*
$\Longleftrightarrow$ *p is a regular point of μ.*

Explaination:

One lemma is useful for understanding discrete subgroup of any topological group. For reasons why it is true, see the note on [9] p.313.

Lemma 14.2.7. *A subgroup of a topological group $\Gamma\leq G$ is* ***discrete*** *if its subspace topology is discrete, i.e., $=$ power set of the underlying space Γ. It can be shown that it is discrete if and only if the identity e is an isolated element of the group, i.e., $\exists$ nbd V of e in G such that $V\cap\Gamma=\{e\}$.*

For the first $\iff$, just note that (1) [8] Proposition 21.28; (2) $\mathfrak{g}_p = \mathrm{Lie}(G_p) \cong T_e(G_p)$; and (3) zero dimensional vector space is exactly 0.

For the second $\iff$: When $\mathfrak{g}_p = \{X \in \mathfrak{g} | X^{\#}(p) = 0\} = 0$, we see that any annihilator has nothing to annihilate, i.e., $\mathfrak{g}_p^0$ includes the whole $\mathfrak{g}^*$, which as a vector space is identified with the tangent space of itself, i.e., $\mathrm{Im}(d\mu_p)$. Conversely, suppose $\forall \xi \in \mathfrak{g}^*$, $\xi(X) = 0$, then it has to be the case $X = 0$ (for if not then there is some basis element b_j with nonzero coefficient a_i of which X consists. Then we can construct a linear map ξ that only evaluates b_i nontrivially to see $\xi(X) \neq 0$.)

The last $\iff$ is just the definition.

Proposition 14.2.8.
G acts freely on $\mu^{-1}(0)$
$\Longrightarrow 0$ is a regular value of μ
$\Longrightarrow \mu^{-1}(0)$ is a closed submanifold of M of codimension equal to $\dim G$.
$\Longrightarrow T_p\mu^{-1}(0) = \ker d\mu_p$ for $p \in \mu^{-1}(0)$
$\Longrightarrow T_p\mu^{-1}(0)$ and $T_p\mathcal{O}_p$ are symplectic orthocomplements in T_pM.
In particular, the tangent space to the orbit through $p \in \mu^{-1}(0)$ is an isotropic subspace of T_pM. Hence, orbits in $\mu^{-1}(0)$ are isotropic.

Explaination:

G acts freely on $\mu^{-1}(0)$, so in particular for any $p \in \mu^{-1}(0)$, the action is locally free at p. Thus, proposition 14.2.6 says that any $p \in \mu^{-1}(0)$ serves as a regular point of μ and 0 is a regular value.

The second and third $\Longrightarrow$'s are just regular level set theorem (see [8] Cor.5.14) and Proposition 14.2.4.

The last $\Longrightarrow$ is due to Lemma 14.2.5 (a).

$T_p\mathcal{O}_p$ is an isotropic subspace of T_pM because the inclusion $\mathcal{O}_p \subseteq \mu^{-1}(0)$ (see Lemma 14.1.3) implies the inclusion $T_p\mathcal{O}_p \subseteq T_p\mu^{-1}(0) = (T_p\mathcal{O}_p)^{\omega_p}$.

Remark 14.2.9. Since any tangent vector to the orbit is the value of a vector field generated by the group, we can confirm that orbits are isotropic directly by computing, with remark 12.1.2, for any $X, Y \in \mathfrak{g}$ and any $p \in \mu^{-1}(0)$,

$$\begin{aligned}
\omega_p\left(X_p^{\#}, Y_p^{\#}\right) &= \text{ Hamiltonian function for } \left[Y^{\#}, X^{\#}\right] \text{ at } p \\
&= \text{ Hamiltonian function for } [Y, X]^{\#} \text{ at } p \\
&= \mu^{[Y,X]}(p) = 0
\end{aligned}$$

♠

Lemma 14.2.10. *Let (V, ω) be a symplectic vector space. Suppose that I is an isotropic subspace, that is, $\omega|_I \equiv 0$. Then ω induces a canonical symplectic form Ω on I^ω/I.*

Proof. Let $u, v \in I^\omega$, and $[u], [v] \in I^\omega/I$. Define $\Omega([u],[v]) = \omega(u, v)$.

- Ω is well-defined:

$$\omega(u+i, v+j) = \omega(u, v) + \underbrace{\omega(u, j)}_{0} + \underbrace{\omega(i, v)}_{0} + \underbrace{\omega(i, j)}_{0}, \quad \forall i, j \in I$$

- Ω is nondegenerate:
Suppose that $[u] \in I^\omega/I$ has $\omega([u],[v]) = 0$, for all $[v] \in I^\omega/I$. Then $\omega(u, v) = 0$, for all $v \in I^\omega$. Then $u \in (I^\omega)^\omega = I$, i.e., $[u] = 0$. ∎

Principal G-Bundle

We recall the concept of principal G-bundle.

Definition 14.2.11. *Let M and F be topological spaces. A **fiber bundle** (E, M, π, F) over **base space** M with **model fiber** F is a topological space E called **total space** together with a surjective continuous **projection** $\pi : E \to M$ with the property that for each $x \in M$, there exist a neighborhood U of x in M and a homeomorphism $\Phi_U : \pi^{-1}(U) \to U \times F$, called a **local trivialization of** E **over** U, such that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

*where π_1 is the projection into first component. We say maps with such commutativity property are **fiber-preserving**. If E, M, and F are smooth manifolds with or without boundary, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then it is called a **smooth fiber bundle**, where in this case π is a smooth submersion and the fibers E_x's are thus closed embedded submanifolds by the regular level set theorem.*

Definition 14.2.12. *A smooth fiber bundle $\pi : P \to M$ with fiber Lie group G is a smooth **principal** G**-bundle** if G acts smoothly and freely on P and the fiber-preserving local trivializations*

$$\Phi_U : \pi^{-1}(U) \to U \times G$$

*are G**-equivariant**:*

$$\Phi_U(g \cdot x) = g \cdot \Phi_U(x), \quad \forall x \in \pi^{-1}(U)$$

where on the RHS, G acts on $U \times G$ by

$$g \cdot (x, h) = (x, gh)$$

Example 14.2.13.

1. A **trivial fiber bundle** is one that admits a local trivialization over the entire base space (a global trivialization). It is said to be **smoothly trivial** if it is a smooth bundle and the global trivialization is a diffeomorphism.
2. Every product space $M \times F$ is a fiber bundle with projection $\pi_1 : M \times F \to M$, called a **product fiber bundle**. It has a global trivialization given by the identity map $M \times F \to M \times F$, so every product bundle is trivial. If $F = G$ a Lie group, then it is a **product** G**-bundle**, a principal G-bundle.
3. Every rank-k vector bundle is a fiber bundle with model fiber $\mathbb{R}^k$.
4. If G is a Lie group and H is a closed subgroup, then the quotient G/H can be given the structure of a manifold such that the projection map $\pi : G \to G/H$ is a **principal** H**-bundle**.
5. The group S^1 of unit complex numbers acts on the complex vector space $\mathbb{C}^{n+1}$ by multiplication. This action induces an action of S^1 on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. The complex projective space $\mathbb{C}P^n$ can be defined as the orbit space of S^{2n+1} by S^1. The natural projection $S^{2n+1} \to \mathbb{C}P^n$ with fiber S^1 turn out to be a principal S^1-bundle. When $n = 1, S^3 \to \mathbb{C}P^1$ with fiber S^1 is called the **Hopf bundle**.

♣

Theorem 14.2.14. *If a Lie group G acts smoothly, properly, and freely on a smooth manifold M, then M/G is a manifold and the map $\pi : M \to M/G$ is a principal G-bundle.*

Proof. [4] gives a proof without assuming many of the results from [8] we listed before. At [8] Problem 21-5, this theorem is listed as an exercise following the quotient manifold theorem. ■

14.3 Proof of the Mardsen-Weinstein-Meyer Theorem

Marsden-Weinstein-Meyer Theorem:
Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then

- the orbit space $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold,
- $\pi : \mu^{-1}(0) \to M_{\text{red}}$ is a principal G-bundle, and
- there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{\text{red}}$.

Proof. Due to Proposition 14.2.8, we see $\mu^{-1}(0)$ is a smooth manifold of dimension $\dim M - \dim G$. For the first two parts of the theorem it is enough to apply Theorem 14.2.14 to the free action of compact Lie group G on $\mu^{-1}(0)$. M_{red} has dimension $\dim M - 2\dim G$. Again Proposition 14.2.8 tells us that at $p \in \mu^{-1}(0)$ the tangent space to the orbit $T_p\mathcal{O}_p$ is an isotropic subspace of the symplectic vector space (T_pM, ω_p), i.e., $T_p\mathcal{O}_p \subseteq (T_p\mathcal{O}_p)^{\omega_p}$, and $(T_p\mathcal{O}_p)^{\omega_p} = \ker d\mu_p = T_p\mu^{-1}(0)$. The lemma 14.2.10 gives a canonical symplectic structure on the quotient $T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. The point $[p] \in M_{\text{red}} = \mu^{-1}(0)/G$ has tangent space $T_{[p]}M_{\text{red}} \simeq T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. And π has differential

$$d\pi_p : T_p\mu^{-1}(0) \ni v \mapsto [v] \in T_{[p]}M_{\text{red}} \simeq T_p\mu^{-1}(0)/T_p\mathcal{O}_p. \tag{14.4}$$

Thus the lemma defines a nondegenerate 2-form ω_{red} on M_{red}. This is well-defined because ω is G-invariant. Therefore, $i^*\omega$, which is simply $\omega|_{T_p\mu^{-1}(0)}$, is equal to $\pi^*\omega_{\text{red}}$, where

$$\begin{array}{ccc} \mu^{-1}(0) & \overset{i}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_{\text{red}} & & \end{array}$$

Hence, using [8] Proposition 14.26 (naturality of exterior derivative) twice and noticing closedness of ω, we see

$$\pi^* d\omega_{\text{red}} = d\pi^*\omega_{\text{red}} = di^*\omega = i^*d\omega = 0.$$

Then the closedness of ω_{red} follows from the injectivity of π^*. ■

Proposition 14.3.1. *Let (M, ω, G, μ) be a Hamiltonian G-space and $(M_{\text{red}}, \omega_{\text{red}})$ be the symplectic reduction. Suppose that another Lie group H acts on (M, ω) in a Hamiltonian way with moment map $\phi : M \to \mathfrak{h}^*$. If H-action commutes with the G-action and ϕ is G-invariant, then the action of H on M_{red} admits a Hamiltonian action of H with moment map ϕ_{red}.*

Proof. The H-action is defined in a natural way:

$$\begin{aligned} H \times M_{\text{red}} &\to M_{\text{red}} \\ h \cdot \mathcal{O}_p &\mapsto \mathcal{O}_{h\cdot p} \end{aligned}$$

This is well-defined, i.e, whatever representative in the orbit $\mathcal{O}_p$ is used to get $\mathcal{O}_{h\cdot p}$, the result is the same. In symbols, for any $q = g \cdot p \in \mathcal{O}_p = G \cdot p$ for some $g \in G$, we have $h \cdot q = g \cdot (h \cdot p) \in \mathcal{O}_{h\cdot p}$.

Since ϕ is G-invariant and thus ϕ is constant on each orbit, we can define $\phi_{\text{red}} : M_{\text{red}} \to \mathfrak{h}^*$ by $\phi_{\text{red}}(\mathcal{O}_p) = \phi(p)$. That is, $\phi_{\text{red}} \circ \pi = \phi \circ i$. We now show that $(M_{\text{red}}, \omega_{\text{red}}, H, \phi_{\text{red}})$ is a Hamiltonian H-space.

<u>Hamiltonian condition</u>: Starting from the Hamiltonian condition on M, i.e., $d\phi^X = \iota_{X^\#}\omega$, and restricting this equation to $\mu^{-1}(0)$, we have

$$d\left(i^*\phi^X\right) = \iota_{X^\#}\left(i^*\omega\right).$$

Observe that $\phi^X_{\text{red}}(\mathcal{O}_p) = \phi^X(p) \implies \phi^X_{\text{red}} \circ \pi = \phi^X \circ i$. Then

$$d(i^*\phi^X)_p = d(\phi^X \circ i)_p = d(\phi^X_{\text{red}} \circ \pi)_p = d\left(\phi^X_{\text{red}}\right)_{\pi(p)} \circ d\pi_p(\,\cdot\,)$$

Since $i^*\omega = \pi^*\omega_{\text{red}}$, we have

$$\left[\iota_{X^\#}\left(i^*\omega\right)\right]_p = \left[\iota_{X^\#}\left(\pi^*\omega_{\text{red}}\right)\right]_p = \left(\omega_{\text{red}}\right)_{\pi(p)}\left(d\pi_p(X^\#), d\pi_p(\,\cdot\,)\right)$$

Since π is a submersion, i.e, the map in (14.4) is surjective, we see every element in T_pM_{red} is of the form $d\pi_p(\,\cdot\,)$. Thus,

$$d\phi^X_{\text{red}} = \iota_{X^\#}\omega_{\text{red}}.$$

Equivariance condition: We want to show $\forall h \in H$, $\mathcal{O}_p \in M_{\text{red}}$,

$$\phi_{\text{red}}\left(h \cdot \mathcal{O}_p\right) = \mathrm{Ad}^*_h\left(\phi_{\text{red}}\left(\mathcal{O}_p\right)\right)$$

LHS is

$$\phi_{\text{red}}(\mathcal{O}_{h\cdot p}) = \phi(h \cdot p),$$

and RHS is

$$\mathrm{Ad}^*_h\left(\phi(p)\right)$$

but they are equal because of the equivariance of ϕ. ■

Chapter 15

Reduction

15.1 Noether Principle

Definition 15.1.1. *Let (M, ω, G, μ) be a Hamiltonian G-space. A smooth function $f : M \to \mathbb{R}$ is called an **integral of motion** of (M, ω, G, μ) if it is G-invariant, i.e., $\forall g \in G,\ f \circ \Psi_g(p) = f(p)$.*

Remark 15.1.2. Recall that if we let θ be a flow on M, we say a smooth tensor field A on M is **invariant under** θ if for each t, the map θ_t pulls A back to itself wherever it is defined, i.e., $(\theta_t^* A)_p = d(\theta_t)_p^* \left(A_{\theta_t(p)}\right) = A_p,\ \forall (t,p) \in \mathcal{D}(\theta)$. For functions, we have $\theta_t^* f(p) = f(\theta_t(p))$. [8] Theorem 12.37 claims that A is invariant under the flow θ of $V \in \mathfrak{X}(M)$ if and only if $\mathcal{L}_V A = 0$. ♠

Remark 15.1.3. Now note that $\theta(t,p) = \Psi_{\exp tX}(p)$ is a flow on M. The integral curves of θ are trajectories of $X^\#$, which is the Hamiltonian vector field of μ^X. Also,

- f is invariant under $\theta \overset{\text{defn.}}{\iff} \forall (t,p),\ f(p) = (\theta_t^* f)(p) := f(\theta_t(p)) = f(\Psi_{\exp tX}(p)) \overset{12.37}{\iff} \mathcal{L}_{X^\#} f = 0$;
- f is G-invariant $\iff f(p) = f(\Psi_g(p))$.

The second invariance is stronger than the first invariance.

However, for connected Lie group G, we can use the fact that the exponential map is a local diffeomorphism to write any element g of G as a product of elements of the form $\exp(X_1) \cdots \exp(X_k)$. Then

$$
\begin{aligned}
f(g \cdot p) &= f(\exp(X_1) \cdots \exp(X_k) \cdot p) = f(\exp(X_1)(\exp(X_2) \cdots \exp(X_k) \cdot p)) \\
&\overset{\text{1st condition}}{=\!=\!=} f(\exp(X_2) \cdots \exp(X_k) \cdot p) = \cdots = f(p).
\end{aligned}
$$

♠

Definition 15.1.4. *Let (M, ω, G, μ) be a Hamiltonian G-space. Let X_f be the unique Hamiltonian vector field associated with a smooth function f. Then if μ is constant on the trajectories of X_f, we call the one-parameter group of diffeomorphisms $\{\exp tX_f | t \in \mathbb{R}\}$ a **symmetry** of (M, ω, G, μ).*

The Noether principle asserts that symmetries give rise to integral of motions and they have a are one-to-one correspondence when G is connected.

Theorem 15.1.5 (Noether)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space. If a function $f : M \to \mathbb{R}$ is G-invariant, then μ is constant on the trajectories of the Hamiltonian vector field of f. The converse is true when G is a connected Lie group.*

Proof. Let X_f be the Hamiltonian vector field of f. Let ρ_t be the flow of X_f. Fix $p \in M$. μ constant along trajectory of X_f means $\forall t$, $\mu(\rho_t(p)) = \mu(p) \iff \forall X, \forall t, \mu^X(\rho_t(p)) = \mu^X(p) \iff \forall X, \rho_t^* \mu^X = \mu^X \iff \forall X, \mathcal{L}_{X_f}\mu^X = 0$. Now, we compute,

$$\begin{aligned} \mathcal{L}_{X_f}\mu^X &\xlongequal{(10.2)} \iota_{X_f} d\mu^X \xlongequal{\text{Hamiltonian}} \iota_{X_f}\iota_{X^\#}\omega \\ &\xlongequal{\text{skew-sym}} -\iota_{X^\#}\iota_{X_f}\omega \xlongequal{\text{Hamiltonian}} -\iota_{X^\#} df \\ &\xlongequal{(10.2)} -\mathcal{L}_{X^\#} f \xlongequal{f\ G\text{-invariant}} 0. \end{aligned}$$

Suppose G is connected and $\mathcal{L}_{X_f}\mu^X = 0$. Then $\mathcal{L}_{X^\#} f = 0 \iff \forall t, X,\ f(p) = f(\Psi_{\exp tX})(p)$. Remark 15.1.3 shows that f is G-invariant. ■

15.2 Elementary Theory of Reduction

Finding a symmetry for a $2n$-dimensional mechanical problem may reduce it to a $(2n-2)$-dimensional problem as follows.

Let (M, ω, H) be a $2n$ dimensional Hamiltonian system, i.e., $G = \mathbb{R}$ and thus $H = \mu^X = \mu$.

We note that the distinction in Remark 15.1.3 is immaterial as exponential map in $(\mathbb{R}, +)$ is the identity map. Also due to Theorem 10.4.2, we see the following are equivalent:

$$\begin{aligned} & f \text{ is integral of motion} \\ \iff & f \text{ is } G\text{-invariant} \\ \iff & \mu = H \text{ is constant on trajectories of } X_f \\ \iff & \{H, f\} = 0 \iff \{f, H\} = 0 \\ \iff & f \text{ is constant on trajectories of } X_H \end{aligned}$$

An integral of motion f may enable us to understand the trajectories of this system in terms of the trajectories of a $(2n-2)$-dimensional Hamiltonian system $(M_{\text{red}}, \omega_{\text{red}}, H_{\text{red}})$. To make this precise, we will describe this process locally. Suppose that $\mathcal{U}$ is an open set in M with Darboux coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ such that $f = \xi_n$ for this chart, and write H in these coordinates: $H = H(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. Since $f(x_1, \cdots, x_n, \xi_1, \cdots, x_n) = \xi_n$, by the computation in section 10.2, we see that

$$\iota_{X_f}\left(\sum_{i=1}^n dx_i \wedge d\xi_i\right) = df = d\xi_n \iff X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial \xi_i}\right) = \frac{\partial}{\partial x_n}$$

Thus,

$$\{f, H\} = \omega(X_f, X_H) = \omega\left(\frac{\partial}{\partial x_n}, \sum_{i=1}^n \left(\frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i}\right)\right)$$

Note

$$\omega(X, Y) = \sum_{i=1}^n \left(dx_i(X) d\xi_i(Y) - d\xi_i(X) dx_i(Y)\right),$$

and we have $dx_n(X_f) = 1$ with all other differentials evaluate to zero. Thus,

$$\omega(X_f, X_H) = d\xi_n(X_H) = -\frac{\partial H}{\partial x_n}$$

Then by the equivalences we remarked above,

$$f=\xi_n \text{ is an integral of motion } \Longrightarrow \begin{cases} \text{the trajectories of } X_H \text{ lie on the} \\ \text{hyperplane } \xi_n = \text{ constant;} \\ \{\xi_n, H\} = 0 = -\frac{\partial H}{\partial x_n} \\ \Longrightarrow H = H\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_n\right) \end{cases}$$

If we set $\xi_n = c$, the motion of the system on this hyperplane is described by the Hamilton equations same as (10.1):

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{\partial H}{\partial \xi_1}\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c\right) \\ \vdots \\ \dfrac{dx_{n-1}}{dt} = \dfrac{\partial H}{\partial \xi_{n-1}}\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c\right) \\ \dfrac{d\xi_1}{dt} = -\dfrac{\partial H}{\partial x_1}\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c\right) \\ \vdots \\ \dfrac{d\xi_{n-1}}{dt} = -\dfrac{\partial H}{\partial x_{n-1}}\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c\right) \\ \dfrac{dx_n}{dt} = \dfrac{\partial H}{\partial \xi_n} \\ \dfrac{d\xi_n}{dt} = -\dfrac{\partial H}{\partial x_n} = 0. \end{cases}$$

The **reduced phase space** is

$$\mathcal{U}_{\text{red}} = \left\{\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}\right) \in \mathbb{R}^{2n-2} \mid \left(x_1, \ldots, x_{n-1}, a, \xi_1, \ldots, \xi_{n-1}, c\right) \in \mathcal{U} \text{ for some } a\right\}$$

The **reduced Hamiltonian** is

$$\begin{aligned} &H_{\text{red}} : \mathcal{U}_{\text{red}} \longrightarrow \mathbb{R}, \\ &H_{\text{red}}\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}\right) = H\left(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c\right) \end{aligned}$$

In order to find the trajectories of the original system on the hypersurface $\xi_n = c$, we look for the trajectories

$$x_1(t), \ldots, x_{n-1}(t), \xi_1(t), \ldots, \xi_{n-1}(t)$$

of the reduced system on $\mathcal{U}_{\text{red}}$. We integrate the equation

$$\frac{dx_n}{dt}(t) = \frac{\partial H}{\partial \xi_n}\left(x_1(t), \ldots, x_{n-1}(t), \xi_1(t), \ldots, \xi_{n-1}(t), c\right)$$

to obtain the original trajectories

$$\begin{cases} x_n(t) = x_n(0) + \displaystyle\int_0^t \frac{\partial H}{\partial \xi_n}(\ldots) dt \\ \xi_n(t) = c \end{cases}$$

15.3 Reduction for Product Groups

Let G_1 and G_2 be two Lie groups and let $G = G_1 \times G_2$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$$

Suppose each G_i symplectically acts on (M, ω) with moment map $\psi_i : M \to \mathfrak{g}_i$.

We shall see how the following assumptions can help us reduce the action.

- **A1**: The actions of G_1 and G_2 on M commute.
- **A2**: ψ_1 is G_2-invariant and ψ_2 is G_1-invariant.

Lemma 15.3.1. *We assume* ***A1****, then there is a well-defined action of $G_1 \times G_2$ on M given by $(g_1, g_2) \cdot z = g_1 \cdot (g_2 \cdot p) = g_2 \cdot (g_1 \cdot p)$. We claim:*

$$\psi := \psi_1 \times \psi_2 : M \to \mathfrak{g} = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$$

is a moment map for the action of $G_1 \times G_2$ on M.

Proof. For $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_2$, we have $\exp(t(X,Y)) = (\exp(tX), \exp(tY))$ and thus $(X,Y)^\#(p) = X^\#(p) + Y^\#(p)$. Note that $\psi^{(X,Y)}(p) = \langle \psi(p), (X,Y) \rangle = \langle (\psi_1(p), \psi_2(p)), (X,Y) \rangle = \langle \psi_1(p), X \rangle + \langle \psi_2(p), Y \rangle = \psi_1^X(p) + \psi_2^Y(p)$. Therefore, $d\psi^{(X,Y)} = d\psi_1^X + d\psi_2^Y = \iota_{X^\#}\omega + \iota_{Y^\#}\omega = \iota_{(X,Y)^\#}\omega$. ■

There are some remarks we want to make for assumption **A2**.

Remark 15.3.2. If the symplectic manifold arises as the cotangent bundle of a manifold, i.e, $M = T^*X$ and the actions are lifted from commuting actions on X, then we assert that the condition **A2** automatically holds:

In the cotangent case, we can use the explicit formula for the equivariant moment maps ψ_1 and ψ_2. Let $g_2 \in G, \alpha_p \in T_p^*X$ where $p = (x, v)$ and $\xi \in \mathfrak{g}_1$. Then

$$\begin{aligned} \langle \psi_1(g_2 \cdot \alpha_p), \xi \rangle &= \left\langle g_2 \cdot \alpha_p, \xi^\#(g_2 \cdot p) \right\rangle \\ &= \left\langle g_2 \cdot \alpha_p, g_2 \cdot \xi^\#(p) \right\rangle \\ &= \left\langle \alpha_p, \xi^\#(p) \right\rangle \\ &= \langle \psi_1(\alpha_p), \xi \rangle . \end{aligned}$$

There is a similar argument for ψ_2. This proves our assertion. ♠

Remark 15.3.3. In a sense, one needs to only assume that "half" of **A2** holds. Namely, we claim that if ψ_2 is G_1-invariant and G_2 is connected, then ψ_1 is G_2-invariant. Indeed, $d\langle \psi_2, \eta \rangle \cdot \xi^\# = (d\psi_2^\eta)(\xi^\#) = (\iota_{\eta^\#}\omega)(\xi^\#) = 0$ for all $\xi \in \mathfrak{g}_1$ and $\eta \in \mathfrak{g}_2$ and hence

$$\begin{aligned} 0 &\xlongequal{G_1-\text{inv.}} -\mathcal{L}_{\xi^\#}\psi_2^\eta = -\mathcal{L}_{X_{\psi_1^\xi}}\psi_2^\eta \\ &\xlongequal{(10.3)} \{\psi_1^\xi, \psi_2^\eta\} = \mathcal{L}_{X_{\psi_2^\eta}}\psi_1^\xi = \mathcal{L}_{\eta^\#}\psi_1^\xi \end{aligned}$$

Remark 15.1.3 shows that when G_2 is connected, $\mathcal{L}_{\eta^\#}\psi_1^\xi = 0$ gives G_2-invariance of ψ_1^ξ and thus ψ_1, as ξ is arbitrary. ♠

Now we have the ingredients needed to get an equivariant moment map.

Proposition 15.3.4. *Under hypotheses* ***A1*** *and* ***A2****, ψ is an equivariant moment map for the action of $G = G_1 \times G_2$ on M.*

Proof. For all $p \in P$ and $(g_1, g_2) \in G_1 \times G_2$ we have

$$\begin{aligned}(\psi_1 \times \psi_2)((g_1, g_2) \cdot p) &= (\psi_1(g_1 \cdot g_2 \cdot p), \psi_2(g_1 \cdot g_2 \cdot p)) \\ &= (g_1 \cdot \psi_1(p), g_2 \cdot \psi_2(p)) \\ &= (g_1, g_2) \cdot (\psi_1 \times \psi_2)(p)\end{aligned}$$

where we have used equivariance of each of ψ_1 and ψ_2, the fact that the actions commute (**A1**), and condition **A2**, the invariance of ψ_1 and ψ_2. ■

Corollary 15.3.5. *Under hypotheses **A1** and **A2**, (M, ω, G, ψ) is a Hamiltonian G-space.*

Let $Z_1 = \psi_1^{-1}(0)$ and and assume G_1 acts freely on Z_1. Let $M_1 = Z_1/G_1$ be the reduced space and let ω_1 be corresponding reduced symplectic form. We let $\mu_2 : M_1 \to \mathfrak{g}_2^*$ be the moment map $(\psi_2)_{\text{red}} : M_1 \to \mathfrak{g}_2^*$ induced from $\psi_2 : M \to \mathfrak{g}_2^*$ as in Proposition 14.3.1. The main result is the following. For its proof, see [10] Theorem 4.1.3 for example.

Theorem 15.3.6. *Suppose G acts freely on $\psi^{-1}(0,0)$, then G_2 acts freely on $\mu_2^{-1}(0)$, and there is a natural symplectomorphism such that*

$$\mu_2^{-1}(0)/G_2 \simeq \psi^{-1}(0,0)/G.$$

This technique of performing reduction with respect to one factor of a product group at a time is called **reduction in stages**. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H and can also be extended to semidirect products; see [10] chapter 3 and 4.

15.4 Reduction at Other Levels

Suppose that a compact Lie group G acts on a symplectic manifold (M, ω) in a Hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$.

To reduce at the level ξ of μ, we need $\mu^{-1}(\xi)$ to be preserved by G, or else take the G-orbit of $\mu^{-1}(\xi)$, or else take the quotient by the maximal subgroup of G which preserves $\mu^{-1}(\xi)$.

Since μ is equivariant,

$$\begin{aligned}G \text{ preserves } \mu^{-1}(\xi) &\iff G \text{ preserves } \xi \\ &\iff \mathrm{Ad}_g^* \xi = \xi, \forall g \in G\end{aligned}$$

Of course the level 0 is always preserved. Also, when G is a torus, any level is preserved and reduction at ξ for the moment map μ, is equivalent to reduction at 0 for a shifted moment map $\phi : M \to \mathfrak{g}^*, \phi(p) := \mu(p) - \xi$.

Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$ equipped with the **canonical symplectic form** (also know as the **Kostant-Kirillov symplectic form** or the **Lie-Poisson symplectic form**) $\omega_{\mathcal{O}}$ defined in exercise 12.5.5. Let $\mathcal{O}^-$ be the orbit $\mathcal{O}$ equipped with $-\omega_{\mathcal{O}}$. The natural product action of G on $M \times \mathcal{O}^-$ is Hamiltonian with moment map $\mu_{\mathcal{O}}(p, \xi) = \mu(p) - \xi$. If the Marsden-Weinstein-Meyer hypothesis is satisfied for $M \times \mathcal{O}^-$, then one obtains a **reduced space with respect to the coadjoint orbit** $\mathcal{O}$.

15.5 Orbifolds

Example 15.5.1. Let $G = \mathbb{T}^n$ be an n-torus. Consider a Hamiltonian G-space (M, ω) with moment map μ. For any $\xi \in (\mathfrak{t}^n)^*$, suppose $\mu^{-1}(\xi)$ is preserved by the $\mathbb{T}^n$-action. Suppose that ξ is a regular value of μ.

(By Sard's theorem, the singular values of μ form a set of measure zero.) Then $\mu^{-1}(\xi)$ is a submanifold of codimension n. Note that by proposition 14.2.6, we have

$$\begin{aligned}
\xi \text{ regular } &\Longrightarrow d\mu_p \text{ is surjective at all } p \in \mu^{-1}(\xi) \\
&\Longrightarrow \mathfrak{g}_p = 0 \text{ for all } p \in \mu^{-1}(\xi) \\
&\Longrightarrow \text{ the stabilizers on } \mu^{-1}(\xi) \text{ are finite} \\
&\Longrightarrow \mu^{-1}(\xi)/G \text{ is an } \textbf{orbifold}\text{; see [12].}
\end{aligned}$$

Let G_p be the stabilizer of p. By the slice theorem (see [4] Theorem 23.5), $\mu^{-1}(\xi)/G$ is modeled by S/G_p, where S is a G_p-invariant disk in $\mu^{-1}(\xi)$ through p and transverse to $\mathcal{O}_p$. Hence, locally $\mu^{-1}(\xi)/G$ looks indeed like $\mathbb{R}^n$ divided by a finite group action. ♣

Example 15.5.2. Consider the S^1-action on $\mathbb{C}^2$ given by $e^{i\theta} \cdot (z_1, z_2) = \left(e^{ik\theta} z_1, e^{i\theta} z_2\right)$ for some fixed integer $k \geq 2$. This is Hamiltonian with moment map

$$\begin{aligned}
\mu : \mathbb{C}^2 &\longrightarrow \mathbb{R} \\
(z_1, z_2) &\longmapsto -\frac{1}{2}\left(k|z_1|^2 + |z_2|^2\right).
\end{aligned}$$

Any $\xi < 0$ is a regular value and $\mu^{-1}(\xi)$ is a 3-dimensional ellipsoid. The stabilizer of $(z_1, z_2) \in \mu^{-1}(\xi)$ is $\{1\}$ if $z_2 \neq 0$, and is $\mathbb{Z}_k = \left\{ e^{i\frac{2\pi\ell}{k}} \,\middle|\, \ell = 0, 1, \ldots, k-1 \right\}$ if $z_2 = 0$. The reduced space $\mu^{-1}(\xi)/S^1$ is called a **teardrop** orbifold or **conehead**; it has one **cone** (also known as a **dunce cap**) singularity of type k (with cone angle $\frac{2\pi}{k}$). ♣

Example 15.5.3. Let S^1 act on $\mathbb{C}^2$ by $e^{i\theta} \cdot (z_1, z_2) = \left(e^{ik\theta} z_1, e^{i\ell\theta} z_2\right)$ for some integers $k, \ell \geq 2$. Suppose that k and ℓ are relatively prime. Then

$$\begin{aligned}
(z_1, 0) &\text{ has stabilizer } \mathbb{Z}_k \quad (\text{ for } z_1 \neq 0), \\
(0, z_2) &\text{ has stabilizer } \mathbb{Z}_\ell \quad (\text{ for } z_2 \neq 0), \\
(z_1, z_2) &\text{ has stabilizer } \{1\} \quad (\text{ for } z_1, z_2 \neq 0).
\end{aligned}$$

The quotient $\mu^{-1}(\xi)/S^1$ is called a **football** orbifold. It has two cone singularities, one of type k and another of type ℓ. ♣

Example 15.5.4. More generally, the reduced spaces of S^1 acting on $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = \left(e^{ik_1\theta} z_1, \ldots, e^{ik_n\theta} z_n\right)$$

are called **weighted** (or **twisted**) **projective spaces**. ♣

15.6 Spherical Pendulum

This is the Homework 18 in [4].

The spherical pendulum is a mechanical system consisting of a massless rigid rod of length l, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate freely in all directions. Assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.

Let $\varphi, \theta (0 < \varphi < \pi, 0 < \theta < 2\pi)$ be spherical coordinates for the bob. For simplicity assume that $m = l = 1$.

Exercise 15.6.1. *Let η, ξ be the coordinates along the fibers of T^*S^2 induced by the spherical coordinates φ, θ on S^2. Show that the function $H : T^*S^2 \to \mathbb{R}$ given by*

$$H(\varphi, \theta, \eta, \xi) = \frac{1}{2}\left(\eta^2 + \frac{\xi^2}{(\sin\varphi)^2}\right) + \cos\varphi$$

is an appropriate Hamiltonian function to describe the spherical pendulum.

Exercise 15.6.2. *Compute the critical points of the function H. Show that, on S^2, there are exactly two critical points: s (where H has a minimum) and u. These points are called the stable and unstable points of H, respectively. Justify this terminology, i.e., show that a trajectory whose initial point is close to s stays close to s forever, and show that this is not the case for u. What is happening physically?*

Exercise 15.6.3. *Show that the group of rotations about the vertical axis is a group of symmetries of the spherical pendulum. Show that, in the coordinates above, the integral of motion associated with these symmetries is the function*

$$J(\varphi, \theta, \eta, \xi) = \xi$$

Give a more coordinate-independent description of J, one that makes sense also on the cotangent fibers above the North and South poles.

Exercise 15.6.4. *Locate all points $p \in T^*S^2$ where dH_p and dJ_p are linearly dependent:*

(a) Clearly, the two critical points s and u belong to this set. Show that these are the only two points where $dH_p = dJ_p = 0$.

(b) Show that, if $x \in S^2$ is in the southern hemisphere ($x_3 < 0$), then there exist exactly two points, $p_+ = (x, \eta, \xi)$ and $p_- = (x, -\eta, -\xi)$, in the cotangent fiber above x where dH_p and dJ_p are linearly dependent.

(c) Show that dH_p and dJ_p are linearly dependent along the trajectory of the Hamiltonian vector field of H through p_+. Conclude that this trajectory is also a trajectory of the Hamiltonian vector field of J, and, hence, that its projection onto S^2 is a latitudinal circle (of the form $x_3 =$ constant). Show that the projection of the trajectory through $p_-$is the same latitudinal circle traced in the opposite direction.

Exercise 15.6.5. *Show that any nonzero value j is a regular value of J, and that S^1 acts freely on the level set $J = j$. What happens on the cotangent fibers above the North and South poles?*

Exercise 15.6.6. *For $j \neq 0$ describe the reduced system and sketch the level curves of the reduced Hamiltonian.*

Exercise 15.6.7. *Show that the integral curves of the original system on the level set $J = j$ can be obtained from those of the reduced system by "quadrature", in other words, by a simple integration.*

Exercise 15.6.8. *Show that the reduced system for $j \neq 0$ has exactly one equilibrium point. Show that the corresponding relative equilibrium for the original system is one of the horizontal curves in exercise 4.*

Exercise 15.6.9. *The **energy-momentum map** is the map $(H, J) : T^*S^2 \to \mathbb{R}^2$. Show that, if $j \neq 0$, the level set $(H, J) = (h, j)$ of the energy-momentum map is either a circle (in which case it is one of the horizontal curves in exercise 4), or a two-torus. Show that the projection onto the configuration space of the two-torus is an annular region on S^2.*

15.7 Examples of Moment Maps

This is the Homework 19 in [4].

Exercise 15.7.1. *Suppose that a Lie group G acts in a Hamiltonian way on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$. Prove that the diagonal action of G on $M_1 \times M_2$ is Hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by*

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2), \text{ for } p_j \in M_j$$

Solution. Let $\omega = (\mathrm{pr}_1)^* \omega_1 + (\mathrm{pr}_2)^* \omega_2$ be the symplectic form on $M_1 \times M_2$ (see section 3.4.)

Hamiltonian condition:

Let $X \in \mathfrak{g}^*$. $\langle \mu(p_1,p_2), X\rangle = \langle \mu_1(p_1), X\rangle + \langle \mu_2(p_2), X\rangle \implies \mu^X(p_1,p_2) = \mu_1^X(\mathrm{pr}_1(p_1,p_2)) + \mu_2^X(\mathrm{pr}_2(p_1,p_2))$. Then,

$$\begin{aligned}
&\omega\left(X^{\#}, v\right) \\
=&\omega_1\left(d(\mathrm{pr}_1)X^{\#}, d(\mathrm{pr}_1)v\right) + \omega_2\left(d(\mathrm{pr}_2)X^{\#}, d(\mathrm{pr}_2)v\right) \\
=&\omega_1\left(X_{M_1}(p_1), d(\mathrm{pr}_1)v\right) + \omega_2\left(X_{M_2}(p_2), d(\mathrm{pr}_2)v\right) \\
=&\left(\iota_{X_{M_1}}\omega_1\right)\left(d(\mathrm{pr}_1)v\right) + \left(\iota_{X_{M_2}}\omega_2\right)\left(d(\mathrm{pr}_2)v\right) \\
=&(d\mu_1^X)(d(\mathrm{pr}_1)v) + (d\mu_2^X)(d(\mathrm{pr}_2)v) \\
=&d\mu^X(v)
\end{aligned}$$

Equivariant condition:

$$\begin{aligned}
\mathrm{Ad}^*_{g^{-1}}\left(\mu\left(p_1,p_2\right)\right) &= \mathrm{Ad}^*_{g^{-1}}\left(\mu_1\left(p_1\right) + \mu_2\left(p_2\right)\right) \\
&= \mathrm{Ad}^*_{g^{-1}}\left(\mu_1\left(p_1\right)\right) + \mathrm{Ad}^*_{g^{-1}}\left(\mu_2\left(p_2\right)\right) \\
&= \mu_1\left(g\cdot p_1\right) + \mu_2\left(g\cdot p_2\right) \\
&= \mu(g\cdot(p_1,p_2))
\end{aligned}$$

♦

Exercise 15.7.2. *Let $\mathbb{T}^n = \{(t_1,\ldots,t_n)\in\mathbb{C}^n : |t_j| = 1$, for all $j\}$ be a torus acting on $\mathbb{C}^n$ by*

$$(t_1,\ldots,t_n)\cdot(z_1,\ldots,z_n) = \left(t_1^{k_1}z_1,\ldots,t_n^{k_n}z_n\right)$$

where $k_1,\ldots,k_n\in\mathbb{Z}$ are fixed. Check that this action is Hamiltonian with moment map $\mu:\mathbb{C}^n\to(\mathfrak{t}^n)^*\cong\mathbb{R}^n$ given by

$$\mu\left(z_1,\ldots,z_n\right) = -\frac{1}{2}\left(k_1\left|z_1\right|^2,\ldots,k_n\left|z_n\right|^2\right)(+\text{constant})$$

Solution. Let $(E_1,\cdots,E_n)$ be the basis of $\mathfrak{t}^n\cong\mathbb{R}^n$ and $(\varepsilon^1,\cdots,\varepsilon^n)$ be the basis of $(\mathfrak{t}^n)^*\cong\mathbb{R}^n$. For each component z_j in $\mathbb{C}^n$, write:

$$z_j = r_je^{i\theta_j} = r_j\cos\theta_j + ir_j\sin\theta_j,$$

where $r_j = |z_j|$ represents the modulus and θ_j the argument of z_j. In terms of these coordinates, we have

$$\omega = \frac{i}{2}\sum_{j=1}^n dz_j\wedge d\bar{z}_j = \sum_{j=1}^n r_jdr_j\wedge d\theta_j.$$

The proposed moment map becomes

$$\mu(z_j) = \mu(r_j,\theta_j) = -\frac{1}{2}\left(k_1r_1^2,\ldots,k_nr_n^2\right) + \text{ constant } = -\frac{1}{2}\sum_{i=1}^n k_ir_i^2\varepsilon^i + \text{ constant.}$$

Hamiltonian condition:

Let $X = \sum_j X_jE_j\in\mathfrak{t}^n\cong\mathbb{R}^n$ and $z = (z_1,\cdots,z_n)\in\mathbb{C}^n$.

$$\langle\mu(z), X\rangle = -\frac{1}{2}\sum_{j=1}^n k_jr_j^2\varepsilon^j(X) + \text{constant} = -\frac{1}{2}\sum_{j=1}^n k_jr_j^2X_j + \text{constant}.$$

Then

$$d\mu^X = -\frac{1}{2}\sum_{j=1}^{n} k_j X_j d\left(r_j^2\right) = -\sum_{j=1}^{n} k_j X_j r_j dr_j.$$

Since $\frac{d}{dt}\big|_{t=0} \exp tX_j \cdot z_j = \frac{d}{d\theta}\big|_{\theta=0} e^{itX_j k_j} z_j = iX_j k_j z_j$ and by paying attention to remark 12.3.1, we see

$$X^{\#}(z_j) = X^{\#}(r_j, \theta_j) = \sum_{j=1}^{n} ik_j X_j z_j \frac{\partial}{\partial z_j} - ik_j X_j \bar{z}_j \frac{\partial}{\partial \bar{z}_j} = \sum_{j=1}^{n} k_j X_j \frac{\partial}{\partial \theta_j}.$$

Then

$$\iota_{X^{\#}}\omega = \sum_{j=1}^{n} r_j \left(0 - d\theta_j(X^{\#}) dr_j\right) = -\sum_{j=1}^{n} r_j dr_j \left(k_j X_j\right) = -\sum_{j=1}^{n} k_j X_j r_j dr_j = d\mu^X.$$

Equivariant condition:

Note that $\mathbb{T}^n$ is abelian so by Remark 12.1.5, we need to check $\mu((t_1, \cdots, t_n) \cdot (z_1, \cdots, z_n)) = \mu(z_1, \cdots, z_n)$. This is true because $|t_j| = 1 \implies |t_j^{k_j} z_j|^2 = |z_j|^2$.

♦

Exercise 15.7.3. *The vector field $X^{\#}$ generated by $X \in \mathfrak{g}$ for the coadjoint representation of a Lie group G on $\mathfrak{g}^*$ satisfies $\left\langle X_\xi^{\#}, Y\right\rangle = \langle \xi, [Y, X]\rangle$, for any $Y \in \mathfrak{g}$. Equip the coadjoint orbits with the canonical symplectic forms. Show that, for each $\xi \in \mathfrak{g}^*$, the coadjoint action on the orbit $G \cdot \xi$ is Hamiltonian with moment map the inclusion map:*

$$\mu : G \cdot \xi \hookrightarrow \mathfrak{g}^*$$

Solution. Let Ω be the symplectic form on $\mathcal{O}_\xi = G \cdot \xi$. This is given by exercise 12.5.4. Let $X \in \mathfrak{g}^*$ and $Y_\xi^{\#} \in T_\xi \mathcal{O}_\xi$. One can check that μ^X is just the evaluatation map $\overline{X} : \mathfrak{g}^* \to \mathbb{R}$, so it is linear and its differential is itself. Then Hamiltonian condition is satisfied because,

$$d(\mu^X)_\xi(Y_\xi^{\#}) = \mu^X(Y_\xi^{\#}) = \langle \mu(Y_\xi^{\#}), X\rangle = \langle Y_\xi^{\#}, X\rangle = \langle \xi, [X, Y]\rangle = \Omega_\xi(X_\xi^{\#}, Y_\xi^{\#}) = (\iota_{X^{\#}}\Omega)_\xi(Y_\xi^{\#})$$

The equivariance condition is trivially true as $g \cdot \xi = \mathrm{Ad}_g^* \xi$ and μ is identity on the orbit. ♦

Exercise 15.7.4. *Consider the natural action of $\mathrm{U}(n)$ on $(\mathbb{C}^n, \omega_0)$. Show that this action is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ given by*

$$\mu(z) = \frac{i}{2} z z^*, \tag{15.1}$$

where we identify the Lie algebra $\mathfrak{u}(n)$ with its dual via the inner product $(A, B) = \operatorname{trace}(A^* B)$. That is, by (15.7.4) we mean

$$\mu(z) = \frac{1}{2} i \operatorname{trace}\left((zz^*)^*(\,\cdot\,)\right) = \frac{1}{2} i \operatorname{trace}\left(zz^*(\,\cdot\,)\right), \text{ or } \mu^X(z) = \frac{1}{2} i \operatorname{trace}\left(zz^* X\right).$$

[Hint: Denote the elements of $\mathrm{U}(n)$ in terms of real and imaginary parts $g = h + ik$. Then g acts on $\mathbb{R}^{2n}$ by the linear symplectomorphism $\begin{pmatrix} h & -k \\ k & h \end{pmatrix}$. The Lie algebra $\mathfrak{u}(n)$ is the set of skew-hermitian matrices $X = V + iW$ where $V = -V^t \in \mathbb{R}^{n\times n}$ and $W = W^t \in \mathbb{R}^{n\times n}$. Show that the infinitesimal action is generated by the Hamiltonian functions

$$\mu^X(z) = -\frac{1}{2}(x, Wx) + (y, Vx) - \frac{1}{2}(y, Wy)$$

where $z = x + iy, x, y \in \mathbb{R}^n$ and $(\cdot, \cdot)$ is the standard inner product. Show that

$$\mu^X(z) = \frac{1}{2}iz^*Xz = \frac{1}{2}i\,\text{trace}\,(zz^*X)$$

Check that μ is equivariant.]

Solution. Our multiplication in matrix forms is from here.

Let $X \in \mathfrak{u}(n) = \{A \in GL(n, \mathbb{C}) | A + A^* = 0\}$ =set of all skew-Hermitian matrices. We define

$$\mu^X(z) = \langle \mu(z), X \rangle = \frac{i}{2}\,\text{tr}\,(zz^*X)\,.$$

and we want to show that μ is the moment map satisfying equivariance condition. By cyclic property of trace and $X + X^* = 0$ we have

$$\overline{\text{tr}\,(zz^*X)} = \text{tr}\left((zz^*X)^*\right) = \text{tr}\,(X^*zz^*) = \text{tr}\,(zz^*X^*) = -\,\text{tr}\,(zz^*X)$$

This shows that $\text{tr}\,(zz^*X)$ is purely imaginary. Thus, (15.1) is well-defined since multiplying the purely imaginary number $\text{tr}\,(zz^*X)$ with i gives a real number for every $X \in \mathfrak{u}(n)$. We continue:

$$\begin{aligned}
\text{tr}\,(zz^*X) = \text{tr}\,\overbrace{(z^*Xz)}^{\in\mathbb{C}} = z^*Xz = (x - iy)^T(V + iW)(x + iy)\\
=& x^TVx + y^TWx - iy^TVx + ix^TWx + ix^TVy + iy^TWy + y^TVy - x^TWy\\
=& (x^TVx + y^TWx + y^TVy - x^TWy) + i(x^TWx - y^TVx + x^TVy + y^TWy)
\end{aligned}$$

Since this is purely imaginary, the real part vanishes, and we have

$$\mu^X(z) = \frac{i}{2}i(x^TWx - y^TVx + x^TVy + y^TWy) = -\frac{1}{2}(x^TWx - y^TVx + x^TVy + y^TWy)$$

Note that $V = -V^T$ and x^TVy is a number, so $x^TVy = (x^TVy)^T = y^TV^Tx = -y^TVx$. The above becomes

$$\mu^X(z) = \frac{i}{2}i(x^TWx - y^TVx + x^TVy + y^TWy) = -\frac{1}{2}x^TWx + y^TVx + \frac{1}{2}y^TWy.$$

Now, note that $x = (I_n \quad 0)\,z$ and $y = (0 \quad I_n)\,z$. So we can rewrite the above formula further to get

$$p^TVq = z^T\binom{I_n}{0}V\begin{pmatrix} 0 & I_n \end{pmatrix}z = z^T\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}z.$$

Similarly, we get for the other three terms

$$p^TWp = z^T\binom{I_n}{0}W\begin{pmatrix} I_n & 0 \end{pmatrix}z = z^T\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}z,$$

Together, we have

$$\begin{aligned}
\mu^X(z) &= -\frac{1}{2}p^TVq - \frac{1}{2}p^TWp + q^TVp - q^tWq\\
&= \frac{1}{2}z^T\left(-\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix}\right)z\\
&= \frac{1}{2}z^T\begin{pmatrix} -W & -V \\ V & -W \end{pmatrix}z
\end{aligned}$$

Since the action of $U(n)$ on $\mathbb{C}^n$ is just by multiplication, we have

$$X^{\#}(z) = \left.\frac{d}{dt}\right|_{t=0} e^{tX} z = (V+iW)z = \begin{pmatrix} V & -W \\ W & V \end{pmatrix} z$$

We show $d\mu^X = \iota_{X^{\#}}\omega$ now. let $Y \in \mathfrak{X}(\mathbb{C}^n)$. Let $\tilde{z} = Y_z$. Let $\gamma(t) = z + t\tilde{z}$. By [LeeSM] Corollary 3.25, we have

$$\begin{aligned}
(d\mu^X)_z(Y_z) &= \left.\frac{d}{dt}\right|_{t=0} \mu^X \circ \gamma(t) = \left.\frac{d}{dt}\right|_{t=0} \mu^X(z+tY_z) \\
&= \left.\frac{d}{dt}\right|_{t=0} (z+t\tilde{z})^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} (z+t\tilde{z}) \\
&= z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} + \tilde{z}^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} z \\
&= z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} + z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix}^T \tilde{z} \\
&= z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} + z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} \\
&= z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} \tilde{z}
\end{aligned}$$

where we again applied transpose to a number. On the other hand,

$$\begin{aligned}
&\left(\iota_{X^{\#}}\omega(Y)\right)(z) = \omega\left(X^{\#}, Y\right)(z) \\
&= \sum_{j=1}^n dx^j \wedge dy^j \left(X^{\#}, Y\right)(z) \\
&= \sum_{j=1}^n dx^j \left(X^{\#}\right)(z) dy^j(Y)(z) - dx^j(Y)(z) dy^j\left(X^{\#}\right)(z) \\
&= \sum_{j=1}^n \left(\begin{pmatrix} V & -W \\ W & V \end{pmatrix} z\right)_j \tilde{z}_{j+n} - \left(\begin{pmatrix} V & -W \\ W & V \end{pmatrix} z\right)_{j+n} \tilde{z}_j \\
&= \sum_{j=1}^n \left(\begin{pmatrix} V & -W \end{pmatrix} z\right)_j \tilde{z}_{j+n} - \left(\begin{pmatrix} W & V \end{pmatrix} z\right)_j \tilde{z}_j \\
&= \sum_{j=1}^n \left(\begin{pmatrix} V & -W \end{pmatrix} z\right)_j \tilde{z}_{j+n} + (\begin{pmatrix} -W & -V \end{pmatrix} z)_j \tilde{z}_j \\
&= \sum_{j=1}^{2n} \left(\begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} z\right)_j \tilde{z}_j \\
&= z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix}^T \tilde{z} \\
&= z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} \tilde{z}.
\end{aligned}$$

Thus, $\iota_{X^{\#}}\omega = d\mu^X$.

It remains now to show that μ is equivariant. Let $g \in U(n), z \in \mathbb{C}^n$ and $X \in \mathfrak{u}(n)$. Then we can compute

$$\mu\left(\Psi_g(z)\right)(X) = \mu(gz)(X) = \frac{i}{2}\operatorname{tr}\left(gz(gz)^* X\right) = \frac{i}{2}\operatorname{tr}\left(gzz^* g^* X\right).$$

On the other hand, using the fact that the adjoint action to matrix Lie group is conjugation and the definition of the coadjoint action, we get that

$$\begin{aligned}
\operatorname{Ad}_g^*(\mu(z))(X) &= \left\langle \operatorname{Ad}_g^*(\mu(z)), X\right\rangle \\
&= \left\langle \mu(z), \operatorname{Ad}_{g^{-1}} X\right\rangle \\
&= \left\langle \mu(z), g^{-1} X g\right\rangle \\
&= \frac{i}{2} \operatorname{tr}\left(z z^* g^{-1} X g\right).
\end{aligned}$$

Now, using that the trace is invariant under cyclic permutations and that $g^* = g^{-1}$ as $g \in U(n)$, we can conclude that

$$\mu\left(\Psi_g(z)\right)(X) = \operatorname{Ad}_g^*(\mu(z))(X)$$

As X and z were arbitrary, this shows that

$$\mu \circ \Psi_g = \operatorname{Ad}_g^* \circ \mu$$

Hence, μ is equivariant and thus the action is hamiltonian with the claimed moment map. ♦

Exercise 15.7.5. *Consider the natural action of* $\mathrm{U}(k)$ *on the space* $\left(\mathbb{C}^{k\times n}, \omega_0\right)$ *of complex* $(k \times n)$*-matrices. Identify the Lie algebra* $\mathfrak{u}(k)$ *with its dual via the inner product* $(A, B) = \operatorname{trace}(A^* B)$*. Prove that a moment map for this action is given by*

$$\mu(A) = \frac{i}{2} A A^* + \frac{\mathrm{Id}}{2i}, \text{ for } A \in \mathbb{C}^{k\times n}$$

(The choice of the constant $\frac{\mathrm{Id}}{2i}$ *is for convenience in Homework 15.8.)*

[Hint: Exercises 1 and 4.]

Solution. We assume the constant is zero. We have that $\mathbb{C}^{n\times k} = \left(\mathbb{C}^n\right)^k$ and the action of $U(n)$ on $\left(\mathbb{C}^n\right)^k$ is as the action in exercise 15.7.1 where the action of $U(n)$ on $\mathbb{C}^n$ is as in above exercise. So we can apply exercise 15.7.4 inductively k times and get that this action is indeed hamiltonian and that the moment map is given by

$$A \in \mathbb{C}^{n\times k} \mapsto \mu(A) = \sum_{j=1}^k \mu'\left(A_j\right)$$

where we denote by μ' the moment map of above exercise and A_j denotes the j-th column of A. Thus, we have

$$\mu(A) = \sum_{j=1}^k \frac{i}{2} \operatorname{tr}\left(A_j A_j^* \cdot\right) = \frac{i}{2} \operatorname{tr}\left(\sum_{j=1}^k A_j A_j^* \cdot\right)$$

Thus, it is enough to show that

$$\sum_{j=1}^k A_j A_j^* = A A^*$$

For this, we compute the (a, b)-th entry of these matrices. We have

$$\left(\sum_{j=1}^k A_j A_j^*\right)_{ab} = \sum_{j=1}^k A_{aj} \bar{A}_{bj} = \sum_{j=1}^k A_{aj} A_{jb}^* = \left(A A^*\right)_{ab},$$

which finishes the proof. ♦

Exercise 15.7.6. *Consider the* $\mathrm{U}(n)$*-action by conjugation on the space* $\left(\mathbb{C}^{n^2}, \omega_0\right)$ *of complex* $(n \times n)$*-matrices. Show that a moment map for this action is given by*

$$\mu(A) = \frac{i}{2}\left[A, A^*\right]$$

[Hint: Previous exercise and its "transpose" version.]

Solution. For this, we first want to look at the following action that is similar to the previous two exercises.

$$\Psi : U(n) \times \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}, g \cdot A = A g^*$$

We want to argue why this is hamiltonian with moment map $\mu(A) = -\frac{i}{2} \operatorname{tr}\left(A^* A \cdot\right)$. For this, consider first the action of $U(n)$ on $\mathbb{C}^n$ defined by $g \cdot z = \left(z^T g^*\right)^T = \bar{g} z$. Similar to the Section 6.1, this action is hamiltonian with moment map $\mu(z) = \overline{\frac{i}{2} \operatorname{tr}\left(z z^*\right)} = -\frac{i}{2} \operatorname{tr}\left(\bar{z} z^T .\right)$. Now, the action Ψ as above can be written as

$$g \cdot A = \left(\begin{array}{c} \left(\bar{g} \cdot A_1^T\right)^T \\ \vdots \\ \left(\bar{g} \cdot A_n^T\right)^T \end{array}\right)$$

where the A_j are here the rows of A. Thus, we can as in previous exercise use exercise 15.7.1 to get that the action is hamiltonian with moment map

$$\mu(A) = -\frac{i}{2} \sum_{j=1}^n \operatorname{tr}\left(\overline{A_j^T} A_j \cdot\right)$$

To prove our claim about the action Ψ, it remains to show that $\sum_{j=1}^n A_j^* A_j = A^* A$, but this works exactly the same as in the proof of Theorem 6.3 . So, we have indeed that the action of $U(n)$ on $\mathbb{C}^{n \times n}$ given by $g \cdot A = A g^*$ is hamiltonian with moment map

$$\mu(A) = -\frac{i}{2} \operatorname{tr}\left(A^* A \cdot\right)$$

We then move to this exercise.

We can write our action as the composition of the two commuting actions $g \cdot A = g A$ and $g \cdot A = A g^*$. We have seen before that these are hamiltonian. We want to apply product Lie group action to get that the action of $U(n) \times U(n)$ on $\mathbb{C}^{n \times n}$ given by $(g, h) \cdot A = g A h^*$ is also hamiltonian with moment map

$$\mu : \mathbb{C}^{n \times n} \rightarrow \mathfrak{u}(n)^* \times \mathfrak{u}(n)^*, A \mapsto \mu_1(A) \oplus \mu_2(A),$$

where μ_1 and μ_2 are the moment maps of the actions $g \cdot A = g A$ and $g \cdot A = A g^*$. For this to work, we need to check that we can really apply product Lie group action, i.e. we need to show that the moment maps are invariant with respect to the other action. But this is true, as for $g \in U(n)$ we have

$$\begin{aligned} \mu_1\left(A g^*\right) & = \frac{i}{2} \operatorname{tr}\left(\left(A g^*\right)\left(A g^*\right)^* \cdot\right) \\ & = \frac{i}{2} \operatorname{tr}\left(A g^*\left(g^*\right)^* A^* \cdot\right) \\ & = \frac{i}{2} \operatorname{tr}\left(A A^* \cdot\right) \\ & = \mu_1(A) \end{aligned}$$

since $g \in U(n)$ implies that $g^*(g^*)^* = g^*g$ is the identity matrix. Similarly,

$$\mu_2(gA) = -\frac{i}{2}\operatorname{tr}\left((gA)^*(gA)\cdot\right) = -\frac{i}{2}\operatorname{tr}\left(A^*g^*gA\cdot\right) = \mu_2(g),$$

as g^*g is again the identity matrix since $g \in U(n)$. So we have indeed that the action of $U(n)\times U(n)$ on $\mathbb{C}^{n\times n}$ given by $(g,h)\cdot A = gAh^*$ is hamiltonian with moment map μ as defined above. When we now restrict our action to the diagonal $D = \{(g,g) \mid g \in U(n)\}$, the action is still hamiltonian with moment map $\mu : \mathbb{C}^{n\times n} \to \mathfrak{g}^* \times \mathfrak{g}^* \hookrightarrow \mathfrak{d}^*$, where the second map is the inclusion of $\mathfrak{g}^*\times\mathfrak{g}^*$ into $\mathfrak{d}^*$, which makes sense as $\mathfrak{d}$ (the Lie algebra of D) is a Lie subalgebra of $\mathfrak{g}\times\mathfrak{g}$. Now we have that $D \cong G$ and $\mathfrak{d}^* \cong \mathfrak{g}^*$ via

$$f \in \mathfrak{d}^* \mapsto f \circ i \in \mathfrak{g}^*,$$

where $i : \mathfrak{g} \to \mathfrak{d}$ is the map $x \mapsto (x,x)$. Thus, the action as in the statement of the theorem of G on $\mathbb{C}^{n\times n}$ is also hamiltonian with moment map

$$\mu : \mathbb{C}^{n\times n} \to \mathfrak{g}^*, A \mapsto \mu_1(A) + \mu_2(A).$$

Now, we have

$$\mu_1(A) + \mu_2(A) = \frac{i}{2}\operatorname{tr}\left(AA^*\cdot\right) - \frac{i}{2}\operatorname{tr}\left(A^*A\cdot\right)$$

by previous exercise and the arguments at the beginning of this soln. Furthermore, we have

$$\frac{i}{2}\operatorname{tr}\left(AA^*\cdot\right) - \frac{i}{2}\operatorname{tr}\left(A^*A\cdot\right) = \frac{i}{2}\operatorname{tr}\left(\left[A,A^*\right]\cdot\right),$$

♦

15.8 Examples of Reduction

This is the Homework 20 in [4].

Exercise 15.8.1. *For the natural action of* $\mathrm{U}(k)$ *on* $\mathbb{C}^{k\times n}$ *with moment map computed in exercise 5 of Homework 15.7, we have* $\mu^{-1}(0) = \left\{A \in \mathbb{C}^{k\times n} \mid AA^* = \mathrm{Id}\right\}$. *Show that the quotient*

$$\mu^{-1}(0)/\mathrm{U}(k) = \mathbb{G}(k,n)$$

is the grassmannian of k*-planes in* $\mathbb{C}^n$.

Exercise 15.8.2. *Consider the* S^1*-action on* $\left(\mathbb{R}^{2n+2},\omega_0\right)$ *which, under the usual identification of* $\mathbb{R}^{2n+2}$ *with* $\mathbb{C}^{n+1}$*, corresponds to multiplication by* e^{it}. *This action is Hamiltonian with a moment map* $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ *given by*

$$\mu(z) = -\frac{1}{2}|z|^2 + \frac{1}{2}$$

Prove that the reduction $\mu^{-1}(0)/S^1$ *is* $\mathbb{CP}^n$ *with the Fubini-Study symplectic form* $\omega_{red} = \omega_{\mathrm{FS}}$.

[Hint: Let pr: $\mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CPP}^n$ *denote the standard projection. Check that*

$$\mathrm{pr}^*\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar{\partial}\log\left(|z|^2\right)$$

Prove that this form has the same restriction to S^{2n+1} *as* ω_0*.]*

Exercise 15.8.3. *Show that the natural actions of* $\mathbb{T}^{n+1}$ *and* $\mathrm{U}(n+1)$ *on* $(\mathbb{CP},\omega_{\mathrm{FS}})$ *are Hamiltonian, and find formulas for their moment maps.*

[Hint: Previous exercise and exercises 15.7.2 and 15.7.4.]

Part VI

Convexity

Chapter 16

Convexity

16.1 Convexity Theorem

16.2 Effective Actions

16.3 Examples

16.4 Connectedness

Part VII

Symplectic Toric Manifolds

Chapter 17

Classification of Symplectic Toric Manifolds

17.1 Delzant Polytopes

17.2 Delzant Theorem

17.3 Sketch of Delzant Construction

Chapter 18

Delzant Construction

Chapter 19

Duistermaat-Heckman Theorems

19.1 Duistermaat-Heckman Polynomial

19.2 Local Form for Reduced Spaces

19.3 Variation of the Symplectic Volume

19.4 S^1-Equivariant Cohomology

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