Lecture Note on Complex Analysis

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Contents

Chapter 1

Complex Numbers and Complex Functions

1.1 Complex Numbers

We begin by recalling the definition of the complex numbers.

Definition 1.1.1. *The set of* ***complex numbers*** *is*

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R},\ i^2 = -1\}.$$

Addition and multiplication are defined by

$$(x + iy) + (u + iv) = (x + u) + i(y + v),$$

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu).$$

Proposition 1.1.2. *Equipped with the above addition and multiplication, $\mathbb{C}$ is a field: it is a commutative ring in which every nonzero element has a multiplicative inverse.*

Proof. It is straightforward to check that $\mathbb{C}$ is a commutative ring with identity $1 = 1 + 0i$. Let $z = x + iy \in \mathbb{C}$ with $z \neq 0$. Then $(x, y) \neq (0, 0)$, so $x^2 + y^2 > 0$. Define

$$z^{-1} := \frac{x - iy}{x^2 + y^2}.$$

A direct computation shows

$$z \cdot z^{-1} = (x + iy)\frac{x - iy}{x^2 + y^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1,$$

so every nonzero complex number has an inverse. □

Example 1.1.3. The powers of i repeat with period four:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^{n+4} = i^n \quad \text{for all } n \in \mathbb{Z}.$$

◇

Definition 1.1.4. *Let $z = x + iy \in \mathbb{C}$.*

- *The **real part** of z is* $\mathrm{Re}z := x$.
- *The **imaginary part** of z is* $\mathrm{Im}z := y$.
- *The **complex conjugate** of z is* $\bar{z} := x - iy$.
- *The **modulus** (or **norm**) of z is* $|z| := \sqrt{x^2 + y^2}$.

Proposition 1.1.5. *For all $z, w \in \mathbb{C}$, the following identities hold:*

$$\begin{aligned} \mathrm{Re}z &= \frac{z + \bar{z}}{2}, & \mathrm{Im}z &= \frac{z - \bar{z}}{2i}, \\ \bar{\bar{z}} &= z, & \overline{z + w} &= \bar{z} + \bar{w}, \\ \overline{zw} &= \bar{z}\,\bar{w}, & |z|^2 &= z\bar{z}, \\ |zw| &= |z|\,|w|, & |z| &= |\bar{z}|. \end{aligned}$$

Moreover,

$$\max\{|\mathrm{Re}z|, |\mathrm{Im}z|\} \leq |z|.$$

For $z \neq 0$, we also have

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

Proof. All statements follow from the definitions by direct algebraic computation. For instance, if $z = x + iy$, then

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\mathrm{Re}z,$$

and similarly $z - \bar{z} = 2iy = 2i\mathrm{Im}z$. The inequality

$$\max\{|\mathrm{Re}z|, |\mathrm{Im}z|\} \leq |z|$$

follows from $|\mathrm{Re}z| = |x| \leq \sqrt{x^2 + y^2} = |z|$ and likewise for $\mathrm{Im}z$. □

1.2 Arguments and Roots

Geometrically, a nonzero complex number z can be viewed as a vector from the origin to the point $(\mathrm{Re}z, \mathrm{Im}z)$ in the plane. The *argument* of z is the angle this vector makes with the positive real axis.

Definition 1.2.1. *Let $z \in \mathbb{C}$.*

- *If $z \neq 0$, we define the **argument set***

  $$\arg(z) := \{\, \theta \in \mathbb{R} : z/|z| = \cos\theta + i\sin\theta \,\}.$$

  We set $\arg(0) := \{0\}$ *for convenience.*

- *The **principal argument** of $z \neq 0$ is*

  $$\mathrm{Arg}(z) := \arg(z) \cap (-\pi, \pi],$$

  so that $\mathrm{Arg}(z) \in (-\pi, \pi]$ *is a single real number. We define* $\mathrm{Arg}(0) := 0$.

It is easy to check the following basic facts.

Proposition 1.2.2. *Let $z \in \mathbb{C}$ with $z \neq 0$.*

(1) If $\theta \in \arg(z)$, then

$$\arg(z) = \theta + 2\pi\mathbb{Z} = \{\theta + 2k\pi : k \in \mathbb{Z}\}.$$

In particular,

$$\arg(z) = \mathrm{Arg}(z) + 2\pi\mathbb{Z}.$$

(2) For any $\theta \in \arg(z)$, we have the polar form

$$z = |z|(\cos\theta + i\sin\theta).$$

(3) For $z, w \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$,

$$\arg(zw) = \arg(z) + \arg(w),$$

in the sense of sum of subsets of $\mathbb{R}$. This is a nontrivial fact; see, e.g., [11], Theorem 2.46. Heuristically, if

$$z = |z|(\cos\alpha + i\sin\alpha), \quad w = |w|(\cos\beta + i\sin\beta),$$

then

$$zw = |z||w|\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big),$$

so $\alpha + \beta \in \arg(zw)$.

(4) As a consequence, for any $\zeta, \omega \in \mathbb{R}$,

$$(\cos\zeta + i\sin\zeta)(\cos\omega + i\sin\omega) = \cos(\zeta+\omega) + i\sin(\zeta+\omega),$$

and for any $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

which is called the De Moivre's formula.

Example 1.2.3. Computing $(1-i)^8$ is tedious using binomial expansion, but it is easy using polar form and De Moivre's formula. We write $1-i$ in polar form. Its modulus is $|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$, and it is easy to see its principal argument is $\operatorname{Arg}(1-i) = -\frac{\pi}{4}$. Thus $1 - i = \sqrt{2}\, e^{-i\pi/4}$. By De Moivre's formula,

$$(1-i)^8 = (\sqrt{2})^8 e^{-i2\pi} = 2^4 \cdot 1 = 16.$$

$\diamond$

We now formalize the notion of choosing an argument *continuously* on some domain.

Definition 1.2.4. *Let $X \subset \mathbb{C} \setminus \{0\}$. A function $f : X \to \mathbb{R}$ is called* ***an argument function*** *on X if*

$$f(z) \in \arg(z) \quad \textit{for all } z \in X.$$

If, in addition, f is continuous on X, then f is called ***a branch of the argument*** *on X.*

Example 1.2.5. The canonical example of an argument function is the principal argument

$$\operatorname{Arg} : \mathbb{C} \setminus \{0\} \to \mathbb{R}.$$

It is not continuous on all of $\mathbb{C} \setminus \{0\}$: it has a jump along the negative real axis $(-\infty, 0)$, where the value changes abruptly from π to $-\pi$. However,

$$\operatorname{Arg} : \mathbb{C} \setminus (-\infty, 0] \to (-\pi, \pi]$$

is continuous (exercise) and is often called the **principal branch of the argument**. $\diamond$

It is natural to ask whether one can "move" the discontinuity to another ray.

Example 1.2.6. Let $v \in \mathbb{C} \setminus \{0\}$, and let

$$\ell_v := \{rv : r \geq 0\}$$

be the ray spanned by v. We claim that there exists a branch of the argument

$$\operatorname{Arg}_v : \mathbb{C} \setminus \ell_v \longrightarrow \mathbb{R}.$$

We may assume that $\operatorname{Arg}(v) \in (-\pi, \pi)$; if not, replace Arg by an equivalent branch differing by $2\pi k$ for some $k \in \mathbb{Z}$. Define, for $z \in \mathbb{C} \setminus \{0\}$,

$$\operatorname{Arg}_v(z) := \begin{cases} \operatorname{Arg}(z), & \text{if } \operatorname{Arg}(z) \in (-\pi, \operatorname{Arg}(v)], \\ \operatorname{Arg}(z) - 2\pi, & \text{if } \operatorname{Arg}(z) \in (\operatorname{Arg}(v), \pi]. \end{cases}$$

One checks that $\operatorname{Arg}_v(z) \in \arg(z)$ for all $z \neq 0$. Moreover, as z crosses the ray ℓ_v, the definition forces a jump of size 2π, so the function cannot be extended continuously across ℓ_v, but it is continuous on $\mathbb{C} \setminus \ell_v$. We leave the details as an exercise. ◇

In particular, if $v = -1$ then $\ell_v = (-\infty, 0]$ and $\operatorname{Arg}_{-1} = \operatorname{Arg}$.

Remark 1.2.7. *The "slit" $\mathbb{C} \setminus \ell_v$ in Example **??** is essential. In fact, there is no branch of the argument on the unit circle*

$$S^1 := \{z \in \mathbb{C} : |z| = 1\}.$$

Exercise 1.2.8. *Recall that an argument function $g : X \to \mathbb{R}, X \subset \mathbb{C}\backslash\{0\}$, is any function satisfying $g(z) \in \arg(z)$ for all $z \in X$.*

(a) Let $g : S^1\backslash\{-1\} \to \mathbb{R}$ be a continuous argument function. Show that $g(z) = \operatorname{Arg}(z) + 2\pi n$ for all $z \in S^1\backslash\{-1\}$, where $n \in \mathbb{Z}$ is a constant.

(b) Show that there is no continuous argument function $g : S^1 \to \mathbb{R}$.

Solution. (a): By the first part of Proposition 1.2.2 we have $g(z) = \operatorname{Arg}(z) + 2\pi n(z)$ for some $n(z) \in \mathbb{Z}$. Define $f : (-\pi, \pi) \to \mathbb{Z}$ by setting

$$f\left(e^{i\theta}\right) := n\left(e^{i\theta}\right).$$

As g and Arg are continuous on $S^1\backslash\{-1\}$, so is n - therefore, f is continuous as a composition of continuous functions. It follows from the second part of Exercise 2.2.9 that f is constant - so $n(z) = n$ for some $n \in \mathbb{Z}$.

(b): Suppose $g : S^1 \to \mathbb{R}$ is a continuous argument function. By the first part we must have for some $n \in \mathbb{Z}$ that $g(z) = \operatorname{Arg}(z) + 2\pi n$ whenever $z \neq -1$. By continuity we must have

$$-\pi + 2\pi n = g(-1) = \pi + 2\pi n,$$

which is absurd. So no such g exists. ■

Exercise 1.2.9. *Show that*

(a) $\arg(\bar{z}) = -\arg(z)$ *for all* $z \in \mathbb{C}\backslash\{0\}$,

(b) $\arg(-z) = \arg(z) + \pi$ *for all* $z \in \mathbb{C}\backslash\{0\}$,

(c) $\arg\left(z^{-1}\right) = -\arg(z)$ *for all* $z \in \mathbb{C}\backslash\{0\}$.

Use $z = -1$ to show that none of these statements are true for the principal argument $\operatorname{Arg}(z)$.

We now introduce the exponential notation for points on the unit circle.

Definition 1.2.10. *For $\theta \in \mathbb{R}$ we define*

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

Due to Proposition 1.2.2 (4), we have

$$e^{i(\zeta+\omega)} = e^{i\zeta}e^{i\omega} \quad \text{and} \quad \left(e^{i\theta}\right)^n = e^{in\theta} \quad \text{for all } n \in \mathbb{Z}.$$

Lemma 1.2.11. *Let $r, s > 0$ and $\zeta, \omega \in \mathbb{R}$. Then*

$$re^{i\omega} = se^{i\zeta} \quad \Longleftrightarrow \quad r = s \text{ and } \omega \in \zeta + 2\pi\mathbb{Z}.$$

Proof. Suppose first that $r = s$ and $\omega \in \zeta + 2\pi\mathbb{Z}$. Then

$$re^{i\omega} = re^{i(\zeta+2k\pi)} = re^{i\zeta}e^{i2k\pi} = re^{i\zeta} = se^{i\zeta}.$$

Conversely, assume $re^{i\omega} = se^{i\zeta}$. Taking moduli of both sides gives

$$r = |re^{i\omega}| = |se^{i\zeta}| = s.$$

Thus $r = s > 0$ and we may divide both sides by r to obtain

$$e^{i\omega} = e^{i\zeta}.$$

Comparing real parts yields $\cos\omega = \cos\zeta$, and comparing imaginary parts yields $\sin\omega = \sin\zeta$. By the 2π-periodicity of $\cos$ and $\sin$, this implies $\omega \in \zeta + 2\pi\mathbb{Z}$. $\square$

Using polar coordinates and Lemma 1.2.11, we can solve equations of the form $z^n = w$.

Proposition 1.2.12. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $w \in \mathbb{C}^*$. Write w in polar form*

$$w = |w|e^{i\omega}, \qquad \omega = \operatorname{Arg}(w).$$

Then the complex numbers z satisfying $z^n = w$ are precisely

$$z_k = \sqrt[n]{|w|}\, e^{i(\omega+2\pi k)/n}, \qquad k = 0, 1, \dots, n-1.$$

These n numbers are distinct, and every solution of $z^n = w$ is of this form.

Proof. Let $z = re^{i\zeta}$ with $r > 0$, $\zeta \in \mathbb{R}$. Then

$$z^n = r^n e^{in\zeta}.$$

If $z^n = w = |w|e^{i\omega}$, then by Lemma 1.2.11 we must have

$$r^n = |w| \quad \text{and} \quad n\zeta \in \omega + 2\pi\mathbb{Z}.$$

Hence $r = \sqrt[n]{|w|}$ and

$$\zeta = \frac{\omega + 2\pi k}{n} \quad \text{for some } k \in \mathbb{Z}.$$

This yields

$$z = \sqrt[n]{|w|}\, e^{i(\omega+2\pi k)/n}.$$

If k and k' differ by a multiple of n, say $k' = k + mn$, then

$$e^{i(\omega+2\pi k')/n} = e^{i(\omega+2\pi k)/n}e^{i2\pi m} = e^{i(\omega+2\pi k)/n}.$$

Thus it suffices to take $k = 0, 1, \dots, n-1$ to obtain all distinct solutions. $\square$

Definition 1.2.13. *Let $n \geq 2$. A **root function** (or n-**th root function**) on a set $X \subset \mathbb{C}^*$ is a function*

$$F : X \to \mathbb{C}$$

*such that $F(z)^n = z$ for all $z \in X$. If F is continuous on X, we call F a **branch of the** n-**th root** on X.*

The most commonly used choice is the **principal n-th root**, defined using the principal argument: for $z \neq 0$,

$$\sqrt[n]{z} := \sqrt[n]{|z|}\, e^{i\,\mathrm{Arg}(z)/n}.$$

It is continuous on $X = \mathbb{C} \setminus (-\infty, 0]$.

Remark 1.2.14. *If $r \in \mathbb{R}$ with $r \geq 0$, then $\mathrm{Arg}(r) = 0$ and hence*

$$\sqrt[n]{r} = \sqrt[n]{|r|}\, e^{i\,\mathrm{Arg}(r)/n} = \sqrt[n]{r}\, e^{0} = \sqrt[n]{r}.$$

Thus the principal n-th root on $\mathbb{C}$ extends the usual real n-th root on $[0, \infty)$.

Remark 1.2.15. *You may be familiar with the rule*

$$\sqrt{rs} = \sqrt{r}\sqrt{s} \quad \text{for } r, s \in [0, \infty).$$

The principal square root on $\mathbb{C}$ agrees with the usual square root on $[0, \infty)$, so the above identity holds for nonnegative real numbers. However, it does not *extend to all complex numbers. For instance,*

$$\sqrt{-1}\,\sqrt{-1} = e^{i\,\mathrm{Arg}(-1)/2}\, e^{i\,\mathrm{Arg}(-1)/2} = e^{i\pi/2}e^{i\pi/2} = e^{i\pi} = -1 \neq \sqrt{1}.$$

Remark 1.2.16. *Recall that the principal argument Arg has a jump discontinuity along the negative real axis $(-\infty, 0)$. Essentially for this reason, the principal n-th root*

$$\sqrt[n]{\cdot} : \mathbb{C} \to \mathbb{C}$$

is also discontinuous along $(-\infty, 0)$ for every $n \geq 2$. In particular, it is not *a branch of the n-th root on all of $\mathbb{C}$. However, $\sqrt[n]{\cdot}$ is continuous on the slit plane $\mathbb{C} \setminus (-\infty, 0]$, and hence is a branch of the n-th root on $\mathbb{C} \setminus (-\infty, 0]$.*

Just as for the argument, we can move the "slit" to different rays.

Exercise 1.2.17. *Let $v \in \mathbb{C} \setminus \{0\}$, and let $\ell_v := \{rv : r > 0\}$ be the ray spanned by v. Given $n \geq 2$, show that there exists a branch of the n-th root on $\mathbb{C} \setminus \ell_v$. Hint: Use the function Arg_v from Theorem* **??**.

Exercise 1.2.18. *Show that there exists no branch of the square root on S^1 (in particular, there is no branch of the square root on $\mathbb{C} \setminus \{0\}$). Hint: Given a branch of the square root on S^1, use it to construct a branch of the argument on S^1, contradicting the earlier exercise about S^1.*

1.3 Complex-Valued Functions

Before discussing limits and continuity of complex-valued functions, we recall the basic topological notions on the complex plane. These arise from the Euclidean metric and are identical to the corresponding notions in $\mathbb{R}^2$, since $\mathbb{C} \cong \mathbb{R}^2$ as metric spaces.

Definition 1.3.1. *The **metric** on $\mathbb{C}$ is*

$$d(z, w) := |z - w|,$$

*where $|\cdot|$ denotes the complex modulus. For $z \in \mathbb{C}$ and $r > 0$, the **open ball** (or disk) of radius r centered at z is*

$$B(z, r) := \{w \in \mathbb{C} : |w - z| < r\}.$$

This metric induces the usual topology on $\mathbb{C}$. Using open disks, we define interior, closure, boundaries, and accumulation points.

Definition 1.3.2. *Let $A \subset \mathbb{C}$.*

*(1) A point $z \in A$ is an **interior point** of A if there exists $r > 0$ such that $B(z, r) \subset A$. The set of interior points is denoted $\mathrm{int}(A)$.*

(2) *A point $z \in \mathbb{C}$ is a **closure point** of A if every ball $B(z,r)$ intersects A. The set of closure points is $\overline{A}$.*

(3) *A point $z \in \mathbb{C}$ is an **accumulation point** (or **limit point**) of A if every ball $B(z,r)$ contains a point of A different from z itself.*

(4) *The **boundary** of A is*
$$\partial A := \overline{A} \setminus \operatorname{int}(A).$$

Note that z is an accumulation point of A if and only if there exists a sequence $\{z_n\} \subset A \setminus \{z\}$ such that $z_n \to z$.

We now introduce limits in $\mathbb{C}$ using the metric topology.

Definition 1.3.3 (Limit of a Sequence)**.** *Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. We say that $z \in \mathbb{C}$ is the **limit** of $\{z_n\}$, written $z = \lim_{n\to\infty} z_n$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*
$$z_n \in B(z,\varepsilon) \quad \textit{for all } n \geq n_0,$$
or equivalently,
$$|z_n - z| < \varepsilon \quad \textit{for all } n \geq n_0.$$

We may express limits in terms of real and imaginary parts.

Proposition 1.3.4. *Let $\{z_n\} \subset \mathbb{C}$ and $z \in \mathbb{C}$. Then*
$$\lim_{n\to\infty} z_n = z \quad \iff \quad \lim_{n\to\infty} \operatorname{Re}(z_n) = \operatorname{Re}(z) \quad \textit{and} \quad \lim_{n\to\infty} \operatorname{Im}(z_n) = \operatorname{Im}(z).$$

Proof. Write $z_n = x_n + iy_n$, $z = x + iy$. From the inequality
$$\max\{|x_n - x|,\ |y_n - y|\} \leq |z_n - z| \leq |x_n - x| + |y_n - y|,$$
we see that $|z_n - z| \to 0$ if and only if both $|x_n - x| \to 0$ and $|y_n - y| \to 0$. □

We equip $\mathbb{C}$ with the metric topology and its subsets with the subspace topology.

Definition 1.3.5 (Continuity)**.** *Let $X \subset \mathbb{C}$, and let $f : X \to \mathbb{C}$ be a function. We say that f is **continuous at** $z \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*
$$f(B(z,\delta) \cap X) \subset B(f(z),\varepsilon).$$
Equivalently,
$$|f(w) - f(z)| < \varepsilon \quad \textit{whenever } w \in X,\ |w - z| < \delta.$$
*If f is continuous at every point of $A \subset X$, we say that f is **continuous on** A.*

Sequences give an equivalent characterization of continuity.

Theorem 1.3.6 (Sequential Characterization of Continuity)**.** *Let $X \subset \mathbb{C}$, and let $f : X \to \mathbb{C}$. Then f is continuous at $z \in X$ if and only if for every sequence $\{z_n\} \subset X$ with $z_n \to z$, we have*
$$f(z_n) \to f(z).$$
Symbolically (but somewhat informally),
$$f\bigl(\lim_{n\to\infty} z_n\bigr) = \lim_{n\to\infty} f(z_n).$$

Proof. Left as an exercise (standard ε–δ argument). □

A useful consequence is that continuity of a complex function can be checked by examining its real and imaginary parts.

Corollary 1.3.7. *Let $f : X \to \mathbb{C}$. Then f is continuous at $z \in X$ if and only if the real–valued functions*

$$\mathrm{Re} f : X \to \mathbb{R}, \qquad \mathrm{Im} f : X \to \mathbb{R}$$

are continuous at z.

The next proposition shows that the usual algebraic and compositional operations preserve continuity.

Proposition 1.3.8. *Let $f, g : X \to \mathbb{C}$ be continuous at $z \in X$, and let $h : g(X) \to \mathbb{C}$ be continuous at $g(z)$. Then the following functions are continuous at z:*

$$f + g, \qquad fg, \qquad h \circ g.$$

If in addition $g(z) \neq 0$, then the quotient

$$\frac{f}{g}(w) := f(w)\,(g(w))^{-1}$$

is continuous at z.

An immediate consequence is that all polynomials are continuous on $\mathbb{C}$. Rational functions are continuous wherever their denominators do not vanish.

Proposition 1.3.9. *The following functions are continuous:*

(1) $z \mapsto \bar{z}$ and $z \mapsto |z|$ are continuous on all of $\mathbb{C}$.

(2) $z \mapsto z^{-1}$ is continuous on $\mathbb{C} \setminus \{0\}$.

(3) $z \mapsto \mathrm{Arg}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$.

(4) For $n \geq 2$, the principal n-th root $z \mapsto \sqrt[n]{z}$ is continuous on $\mathbb{C} \setminus (-\infty, 0)$.

These continuity statements follow from earlier results on the argument function and the algebraic structure of polar form.

We now refine the notion of limit of a function so that the limit point does not need to belong to the domain.

Definition 1.3.10 (Limit of a Map Along a Set)**.** *Let $X \subset \mathbb{C}$ and $f : X \to \mathbb{C}$. Let $z_0 \in \overline{X}$. We say that f **has limit** $w \in \mathbb{C}$ **at** z_0 **along** X if the following holds:*

For every sequence $\{z_n\} \subset X$ with $z_n \to z_0$, we have

$$f(z_n) \to w.$$

In this case we write

$$w = \lim_{\substack{z \to z_0 \\ z \in X}} f(z), \qquad \text{or simply } w = \lim_{z \to z_0} f(z) \text{ along } X.$$

Remark 1.3.11. *If $z_0 \in X$, then the only possible limit along X is $f(z_0)$. Thus f is continuous at z_0 if and only if f has limit $f(z_0)$ along X.*

Remark 1.3.12. *Definition 1.3.10 is especially useful when $z_0 \in \overline{X} \setminus X$, so that $f(z_0)$ is not even defined. Since $z_0 \in \overline{X}$, there exist sequences in X converging to z_0, and we can study the behavior of f near z_0 via these sequences. A central example: given a function $f : \mathbb{C} \to \mathbb{C}$ and fixed $z \in \mathbb{C}$, define*

$$g_z(w) := \frac{f(w) - f(z)}{w - z}, \qquad w \in \mathbb{C} \setminus \{z\}.$$

Although g_z is not defined at $w = z$, the limit

$$\lim_{\substack{w \to z \\ w \in \mathbb{C} \setminus \{z\}}} g_z(w) = \lim_{\substack{w \to z \\ w \neq z}} \frac{f(w) - f(z)}{w - z}$$

may exist; this is precisely the ***complex derivative*** *$f'(z)$, to be studied in the next chapter.*

Limits along sets allow us to extend functions continuously to boundary points.

Exercise 1.3.13. *Let $f : X \to \mathbb{C}$ be continuous on X, and suppose that f has a limit $w \in \mathbb{C}$ along X at a point $z_0 \in \overline{X} \setminus X$. Define*

$$\tilde{f}(z) := \begin{cases} f(z), & z \in X, \\ w, & z = z_0. \end{cases}$$

Show that $\tilde{f} : X \cup \{z_0\} \to \mathbb{C}$ is continuous at z_0.

This construction is fundamental: many analytic functions are defined at boundary points precisely via such limiting procedures.

Exercise 1.3.14. *Let $X \subset \mathbb{C}$ be a set, and let $f : \bar{X} \to \mathbb{C}$ be continuous in $\bar{X}$. Then,*

$$f(\bar{X}) \subset \overline{f(X)}$$

Chapter 2

Complex Differentiation

2.1 Complex Differentiation

In this chapter we start the study of *analytic functions*, that is, complex–valued functions which admit a complex derivative at every point of their domain.

Definition 2.1.1 (Complex differentiability)**.** *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$. We say that f is* ***complex differentiable*** *at $z \in U$ if the limit*

$$f'(z) := \lim_{\substack{w \to z \\ w \in U,\ w \neq z}} \frac{f(w) - f(z)}{w - z}$$

exists in $\mathbb{C}$. If f is complex differentiable at every point of U, we say that f is ***analytic****, or* ***holomorphic****, on U. If $U = \mathbb{C}$, we say that f is* ***entire****.*

Proposition 2.1.2 (Analytic functions are continuous)**.** *Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be differentiable at $z \in U$. Then f is continuous at z.*

Proof. The proof is identical to the one-variable real case. If $w \neq z$, we can write

$$|f(w) - f(z)| = |w - z| \, \frac{|f(w) - f(z)|}{|w - z|}.$$

As $w \to z$, the factor

$$\frac{|f(w) - f(z)|}{|w - z|} \longrightarrow |f'(z)| \in \mathbb{R},$$

while $|w - z| \to 0$. Hence $|f(w) - f(z)| \to 0$, which is precisely continuity at z. □

A convenient characterization of differentiability is the following "linear approximation" lemma.

Proposition 2.1.3. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$, and $z \in U$.*

1. *If f is differentiable at z, then there exists a function $\varepsilon : U \to \mathbb{C}$ with $\varepsilon(w) \to 0$ as $w \to z$, such that*

   $$f(w) - f(z) = \big(f'(z) + \varepsilon(w)\big)(w - z), \qquad w \in U.$$

2. *Conversely, assume there exists $\alpha \in \mathbb{C}$ and a function $\varepsilon : U \to \mathbb{C}$ with $\varepsilon(w) \to 0$ as $w \to z$, and*

   $$f(w) - f(z) = (\alpha + \varepsilon(w))(w - z), \qquad w \in U \setminus \{z\}.$$

   Then f is differentiable at z, and $f'(z) = \alpha$.

Proof. If f is differentiable at z, set $\alpha := f'(z)$ and

$$\varepsilon(w) := \begin{cases} \dfrac{f(w)-f(z)}{w-z} - \alpha, & w \neq z, \\ 0, & w = z. \end{cases}$$

Then $\varepsilon(w) \to 0$ as $w \to z$ and the first formula holds.

Conversely, if

$$f(w) - f(z) = (\alpha + \varepsilon(w))(w - z),$$

for $w \neq z$, then

$$\left| \frac{f(w)-f(z)}{w-z} - \alpha \right| = |\varepsilon(w)| \to 0$$

as $w \to z$. Hence the limit defining $f'(z)$ exists and equals α. □

Proposition 2.1.4 (Differentiation rules). *Let $U \subset \mathbb{C}$ be open, $z \in U$, and $f, g : U \to \mathbb{C}$ differentiable at z. Let $\lambda \in \mathbb{C}$. Then the functions*

$$\lambda f, \quad f + g, \quad fg$$

are differentiable at z. If in addition $g(z) \neq 0$, then f/g is differentiable at z. Their derivatives satisfy

$$(\lambda f)'(z) = \lambda f'(z), \qquad (f+g)'(z) = f'(z) + g'(z),$$

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z),$$

and, when $g(z) \neq 0$,

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

Corollary 2.1.5. *Polynomials are analytic on $\mathbb{C}$; the derivative of*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

is

$$p'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1.$$

Example 2.1.6. The function $f(z) = \bar{z}$ is not complex differentiable at any $z \in \mathbb{C}$: take a limit along the real axis (i.e., $a_n + bi \to a + bi$) to get 1, and along the imaginary axis (i.e., $a + bi_n \to a + bi$) to get -1. This is striking, because in real coordinates the map is just

$$(x, y) \mapsto (x, -y),$$

an invertible linear map $\mathbb{R}^2 \to \mathbb{R}^2$, hence smooth in the real-variable sense. ◇

Example 2.1.7. The function $f(z) = |z|^2 = x^2 + y^2$ is not complex differentiable at any $z \neq 0$. Indeed, if f were differentiable at some $z \neq 0$, then so would

$$z \mapsto \bar{z} = \frac{|z|^2}{z},$$

contradicting Example 2.1.6. (You may check directly that f is differentiable at 0 with $f'(0) = 0$.) ◇

Theorem 2.1.8 (Chain rule). *Let $U, V \subset \mathbb{C}$ be open, $z \in U$, and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$. Assume that f is differentiable at z, that $f(U) \subset V$, and that g is differentiable at $f(z)$. Then $g \circ f$ is differentiable at z, with*

$$(g \circ f)'(z) = g'(f(z))\, f'(z).$$

Proof. A suggestive but not rigorous argument is

$$\lim_{w\to z}\frac{(g\circ f)(w)-(g\circ f)(z)}{w-z}=\lim_{w\to z}\frac{g(f(w))-g(f(z))}{f(w)-f(z)}\,\frac{f(w)-f(z)}{w-z}=g'(f(z))f'(z).$$

The problem is that $f(w)-f(z)$ might vanish for some $w\neq z$. For a rigorous proof, apply Proposition 2.1.3 to both f and g.

There exist functions $\varepsilon_f,\varepsilon_g$ with $\varepsilon_f(w)\to 0$ as $w\to z$ and $\varepsilon_g(v)\to 0$ as $v\to f(z)$, such that

$$f(w)-f(z)=\big(f'(z)+\varepsilon_f(w)\big)(w-z),$$

$$g(v)-g(f(z))=\big(g'(f(z))+\varepsilon_g(v)\big)(v-f(z)).$$

Substitute $v=f(w)$ in the second formula and combine:

$$g(f(w))-g(f(z))=\big(g'(f(z))+\varepsilon_g(f(w))\big)\big(f(w)-f(z)\big).$$

For $w\neq z$,

$$\frac{g(f(w))-g(f(z))}{w-z}=\big(g'(f(z))+\varepsilon_g(f(w))\big)\big(f'(z)+\varepsilon_f(w)\big).$$

As $w\to z$, continuity of f at z (Proposition 2.1.2) implies $f(w)\to f(z)$, so both $\varepsilon_f(w)$ and $\varepsilon_g(f(w))$ tend to 0. Therefore the last expression tends to $g'(f(z))f'(z)$. $\square$

Proposition 2.1.9 (Derivative of an inverse function)**.** *Let $U\subset\mathbb{C}$ be open, $f:U\to\mathbb{C}$ a map, and let $V\subset\mathbb{C}$ be open. Suppose $g:V\to U$ is a map, continuous at some $w\in V$, such that*

$$f(g(v))=v,\qquad v\in V.$$

If f is differentiable at $g(w)$ and $f'(g(w))\neq 0$, then g is differentiable at w, and

$$g'(w)=\frac{1}{f'(g(w))}.$$

Remark 2.1.10. *It is not assumed that f is a bijection on all of U, and in applications it often is not (for example $f(z)=z^2$ on $\mathbb{C}$). However, from $f\circ g=\mathrm{Id}_V$ one gets three facts:*

(i) $f:U\to V$ is surjective;

(ii) $g:V\to U$ is injective; and

(iii) $f|_{g(V)}:g(V)\to V$ is injective and in fact a bijection with inverse $g:V\to g(V)$.

Reasons: (i): For any $v\in V$, there exists $u=g(v)\in U$ with $f(u)=f(g(v))=v$; (ii): If $g(v_1)=g(v_2)$, then applying f gives $v_1=f(g(v_1))=f(g(v_2))=v_2$; (iii): $f|_{g(V)}\circ g=\mathrm{Id}_V$ is automatic. We show $g\circ f|_{g(V)}=\mathrm{Id}_{g(V)}$: Suppose $g(u)\neq g(v)\in g(V)$. Then $g\circ f|_{g(V)}(g(u))=g\circ f(g(u))=g(\mathrm{id}_V(u))=g(u)$ and is not equal to $g(v)=g\circ f|_{g(V)}(g(v))$.

Proof of Proposition 2.1.9. We need to show

$$\lim_{\substack{v\to w\\ v\in V\setminus\{w\}}}\frac{g(v)-g(w)}{v-w}=\frac{1}{f'(g(w))}.$$

Since g is injective, for $v\neq w$, we have $g(v)\neq g(w)$. Thus

$$\frac{g(v)-g(w)}{v-w}=\frac{g(v)-g(w)}{f(g(v))-f(g(w))}=\left(\frac{f(g(v))-f(g(w))}{g(v)-g(w)}\right)^{-1}.$$

As $v \to w$, continuity of g at w implies $g(v) \to g(w)$, so

$$\frac{f(g(v)) - f(g(w))}{g(v) - g(w)} \longrightarrow f'(g(w)) \neq 0.$$

Taking reciprocals gives the desired limit. $\square$

2.2 Cauchy-Riemann Equations

Let $U \subset \mathbb{C}$ be open. Write $z = x + iy$ and $f = u + iv$ with $u, v : U \to \mathbb{R}$.

Theorem 2.2.1. *Suppose $f : U \to \mathbb{C}$ be complex differentiable at $z = x + iy$ in U. Then the partial derivatives u_x, u_y, v_x, v_y exist at (x, y) and satisfy the* ***Cauchy–Riemann equations***

$$u_x(x, y) = v_y(x, y), \qquad u_y(x, y) = -\, v_x(x, y).$$

In particular,

$$f'(z) = u_x(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y).$$

Proof. Since f is complex differentiable at z, the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists and is independent of the way $h \in \mathbb{C}$ tends to 0. Write $f'(z) = a + ib$ with $a, b \in \mathbb{R}$.

Step 1: approach along the real axis. Take $h \in \mathbb{R}$ and let $h \to 0$. Then

$$\frac{f(z + h) - f(z)}{h} \longrightarrow a + ib.$$

Write $z = x + iy$ and $f = u + iv$. Then

$$f(z + h) - f(z) = \big(u(x + h, y) - u(x, y)\big) + i\big(v(x + h, y) - v(x, y)\big),$$

so

$$\frac{f(z + h) - f(z)}{h} = \frac{u(x + h, y) - u(x, y)}{h} + i\, \frac{v(x + h, y) - v(x, y)}{h}.$$

Taking the limit $h \to 0$ and comparing real and imaginary parts, we get

$$u_x(x, y) = a, \qquad v_x(x, y) = b.$$

Step 2: approach along the imaginary axis. Now take $h = ik$ with $k \in \mathbb{R}$ and $k \to 0$. Then

$$\frac{f(z + ik) - f(z)}{ik} \longrightarrow a + ib.$$

Multiply numerator and denominator by $-i$:

$$\frac{f(z + ik) - f(z)}{ik} = -i\, \frac{f(z + ik) - f(z)}{k}.$$

Thus

$$-i\, \frac{f(z + ik) - f(z)}{k} \longrightarrow a + ib,$$

so
$$\frac{f(z+ik)-f(z)}{k} \longrightarrow i(a+ib).$$
In terms of u, v,
$$f(z+ik)-f(z) = \big(u(x,y+k)-u(x,y)\big) + i\big(v(x,y+k)-v(x,y)\big),$$
hence
$$\frac{f(z+ik)-f(z)}{k} = \frac{u(x,y+k)-u(x,y)}{k} + i\,\frac{v(x,y+k)-v(x,y)}{k}.$$
Letting $k \to 0$ and comparing with the limit $i(a+ib) = -b+ia$, we obtain
$$u_y(x,y) = -b, \qquad v_y(x,y) = a.$$

Step 3: Cauchy–Riemann. From Step 1 and Step 2 we have
$$u_x(x,y) = a = v_y(x,y), \qquad v_x(x,y) = b = -\,u_y(x,y).$$
These are precisely the Cauchy–Riemann equations at z:
$$u_x = v_y, \qquad u_y = -v_x.$$
□

We now recall the notion of real differentiability when we view $\mathbb{C} \cong \mathbb{R}^2$.

Definition 2.2.2 (Real differentiability)**.** *Let $U \subset \mathbb{R}^m$ be open, and let $f : U \to \mathbb{R}^n$ be a map. Then f is called **real differentiable at a point** $x \in U$ if there exists a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ (called the derivative of f at x and often denoted $L = Df(x)$) such that*
$$f(y) - f(x) = L(y-x) + \varepsilon(y)|y-x|,$$
Here $\varepsilon = \varepsilon_x : U \to \mathbb{R}^n$ is a map which satisfies $\varepsilon(y) \to 0$ as $y \to x$. Equivalently, f is real differentiable at x if
$$\lim_{y\to x} \frac{\big|f(y)-f(x)-L(y-x)\big|}{|y-x|} = 0.$$

Proposition 2.2.3. *Let $U \subset \mathbb{R}^2$ be open, and let $f = (u,v) : U \to \mathbb{R}^2$ be a map. Assume that f is differentiable at $z \in U$. Then both u and v are also differentiable at z, their partial derivatives exist at x, and the linear map L is given by the Jacobian matrix $Df(z)$:*
$$L : \mathbb{R}^2 \to \mathbb{R}^2; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto Df(z) \begin{pmatrix} x \\ y \end{pmatrix},$$
where
$$Df(z) = \begin{pmatrix} \partial_x u(z) & \partial_y u(z) \\ \partial_x v(z) & \partial_y v(z) \end{pmatrix}.$$

Definition 2.2.4. *If $w = x+yi, z = a+bi \in \mathbb{C}$, then the multiplication of w by z corresponds to the linear map*
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
*We denote the matrix above by M_z, called the **matrix of multiplication by** z.*

Theorem 2.2.5. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$.*

1. *If f is complex differentiable at $z = x + iy$, then f is real differentiable at z. Moreover, $Df(z) = M_{f'(z)}$.*
2. *Conversely, suppose f is real differentiable at z, and the Cauchy-Riemann equations hold there. Then f is complex differentiable at z, and $f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z)$.*
3. *If all four partial derivatives u_x, u_y, v_x, v_y exist and are continuous in a neighbourhood of z, and they satisfy the Cauchy–Riemann equations there, then f is complex differentiable in that neighbourhood. In particular, if this holds on all of U, then f is analytic on U.*

Proof. 1. Simply compare

$$f(w) - f(z) = f'(z_0)(w - z) + \varepsilon(w)(w - z)$$

with

$$f(w) - f(z) = L(w - z) + \tilde{\varepsilon}(w)|w - z|,$$

and define

$$L = M_{f'(z)}, \quad \tilde{\varepsilon}(w) = \varepsilon(w)\frac{w - z}{|w - z|}.$$

Thus, f is real differentiable at z. By Proposition 2.2.3, $L = Df(z) = M_{f'(z)}$.

2. ByProposition 2.2.3, there is $\varepsilon(w)$ with $\varepsilon(w) \to 0$ as $w \to z$ such that

$$f(w) - f(z) = Df(z)(w - z) + \varepsilon(w)|w - z|.$$

Since Cauchy-Riemann equations hold at z, the Jacobian matrix of f at z is $Df(z) = M_{a+bi}$, where $a = \partial_x u(z) = \partial_y v(z)$ and $b = \partial_x v(z) = -\partial_y u(z)$. If we define

$$\tilde{\varepsilon}(w) = \varepsilon(w)\frac{|w - z|}{w - z},$$

then

$$f(w) - f(z) = (a + bi)(w - z) + \tilde{\varepsilon}(w)|w - z|.$$

Thus, by Proposition 2.1.3, f is complex differentiable at z, and $f'(z) = a + bi = u_x(z) + iv_x(z) = v_y(z) - iu_y(z)$.

3. The last part is because if all four partial derivatives u_x, u_y, v_x, v_y exist and are continuous in a neighbourhood of z_0, then f is real differentiable at z_0. This is a standard result in multivariable calculus. □

The next example shows the continuity assumption in (3) of the above theorem cannot be dropped.

Example 2.2.6. Consider

$$f(z) = \begin{cases} \exp(-z^{-4}), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

We will define the complex exponential e^z precisely in Definition 2.3.4. For this example, it suffices to know the usual exponential e^r for $r \in \mathbb{R}$.

One can show that:

- f is analytic in $\mathbb{C} \setminus \{0\}$;
- the real and imaginary parts of f have partial derivatives everywhere on $\mathbb{C}$, and these satisfy the Cauchy–Riemann equations at every point (including the origin);
- nevertheless, f is not continuous at 0.

Indeed, for real r,

$$f(r) = e^{-r^{-4}} \to 0 \quad \text{and} \quad f(ir) = e^{-(ir)^{-4}} = e^{-r^{-4}} \to 0$$

as $r \to 0$. So f decays rapidly to 0 along the real and imaginary axes. The decay is so fast that the partial derivatives at 0 all exist and equal 0.

On the other hand, along the line spanned by $e^{-i\pi/4}$,

$$f\big(re^{-i\pi/4}\big) = \exp\big(-(re^{-i\pi/4})^{-4}\big) = \exp\big(-r^{-4}e^{i\pi}\big) = \exp(r^{-4}) \xrightarrow{r\to 0} \infty.$$

Thus f blows up near the origin along this line, and hence is not continuous (let alone differentiable) at 0. $\diamond$

We record a few standard corollaries; they are essentially real-variable results once the Cauchy–Riemann equations are known.

Corollary 2.2.7. *Let $U \subset \mathbb{C}$ be open and connected, and let $f : U \to \mathbb{C}$ be analytic with $f'(z) = 0$ for all $z \in U$. Then f is constant on U.*

Proof. Write $f = u + iv$. From Theorem 2.2.1 and $f' \equiv 0$ it follows that all first partial derivatives of u and v vanish on U. Hence $\nabla u = \nabla v = 0$ on U, so both u and v are locally constant. Local constancy plus connectedness implies global constancy by Exercise 2.2.10. $\square$

Corollary 2.2.8. *Let $U \subset \mathbb{C}$ be open and connected, and $f : U \to \mathbb{C}$ analytic. Suppose that one of the following functions is constant on U:*

$$u = \mathrm{Re} f, \qquad v = \mathrm{Im} f, \qquad |f|.$$

Then f is constant on U. The same conclusion holds if $f : U \to \mathbb{C}^*$ and $\mathrm{Arg}(f)$ is constant on U.

Proof. If $\mathrm{Re} f$ or $\mathrm{Im} f$ is constant, then all its partial derivatives vanish; the Cauchy–Riemann equations imply that the derivative f' vanishes, and Corollary 2.2.7 applies. The cases of $|f|$ or $\mathrm{Arg}(f)$ can be reduced to the previous ones by simple algebraic manipulations (details are standard and omitted). $\square$

Exercise 2.2.9. *Let $f : (a,b) \to \mathbb{C}$ be a continuous function such that for every $x \in (a,b)$, there exists $\epsilon = \epsilon(x) > 0$ such that f is constant on $(a,b) \cap (x-\epsilon, x+\epsilon)$. Show that f is constant. As a consequence, deduce that if f is continuous and $f((a,b)) \subset \mathbb{Z}$, then f is constant.*

Solution. Let $x, y \in (a,b)$ be arbitrary. We aim to show $f(x) = f(y)$. Without loss of generality $x < y$. We define

$$I := \{t \in (a,y) : f(t) = f(x)\}.$$

As $x \in I$ we have $I \neq \emptyset$, and thus $s := \sup_{t\in I} t \le y < b$ exists. By the properties of a supremum, there exists $t_k \in I$ so that $t_k \to s$, and so by continuity $f(s) = \lim_{k\to\infty} f(t_k) = \lim_{k\to\infty} f(x) = f(x)$. Aiming for a contradiction, suppose $s < y$. By the locally constant assumption there exists $s < u < y$ so that $f(u) = f(s) = f(x)$. But then $u \in I$ and $u > s$ contradicting that $s = \sup_{t\in I} t$. So we must have $y = s$ and $f(y) = f(s) = f(x)$ as desired. Now suppose that $f : (a,b) \to \mathbb{Z}$ is continuous. By continuity, for each $t \in (a,b)$ there exists $\epsilon > 0$ such that $|f(t) - f(t')| < 1/2$ whenever $|t - t'| < \epsilon$. But as f takes values in $\mathbb{Z}$ we must have $f(t) = f(t')$ for all such t'. But this is the locally constant property of the first part, and so it follows that f is constant. $\blacksquare$

Exercise 2.2.10. *Let $U \subset \mathbb{C}$ be open and connected. Let $f : U \to \mathbb{C}$ be a continuous function which is locally constant: for every $z \in U$ there exists $r > 0$ such that f is constant in $B(z,r)$. Then f is constant in U.*

Solution. Fix $z \in U$. We claim that $f(w) = f(z)$ for all $w \in U$. Fix $w \in U$ and let $\gamma : [0,1] \to U$ be a path with $\gamma(0) = z$ and $\gamma(1) = w$. Then $f \circ \gamma : [0,1] \to U$ is locally constant, and hence constant (exercise 2.2.9). Thus $f(w) = (f\circ\gamma)(1) = (f\circ\gamma)(0) = f(z)$, as claimed. $\blacksquare$

2.3 Branches of Inverse Functions

Many analytic functions are not globally injective on their "natural" domains, and hence do not admit a global analytic inverse. A basic example is $z \mapsto z^n$, $n \geq 2$. The equation

$$z^n = w$$

has multiple solutions in general (Proposition 1.2.12). Nevertheless, on smaller suitable domains one may define analytic inverses. This leads to the following notion.

Definition 2.3.1 (Branch of an inverse). *Let $U \subset \mathbb{C}$ and $f : U \to \mathbb{C}$ be a map (not necessarily injective). Let $V \subset \mathbb{C}$ be a set, and let $g : V \to U$ be a map which is continuous on V and satisfies*

$$f(g(v)) = v, \qquad v \in V.$$

*Then g is called a **branch of f^{-1} in** V.*

By Remark 2.1.10, if g is a branch of f^{-1} in V then $f\big|_{g(V)} : g(V) \to V$ is a bijection with inverse $g : V \to g(V)$.

We now state an analytic version of Proposition 2.1.9.

Theorem 2.3.2 (Analytic branch of an inverse). *Let $U \subset \mathbb{C}$ be open, let $f : U \to \mathbb{C}$ be analytic, and let $V \subset \mathbb{C}$ be open. Suppose $g : V \to U$ is a branch of f^{-1} in V. Assume that*

$$f'(g(w)) \neq 0 \qquad \text{for all } w \in V.$$

Then g is analytic on V, and

$$g'(w) = \frac{1}{f'(g(w))}, \qquad w \in V.$$

2.3.1 Example: Power and Root

Fix $n \geq 2$. Let the n-th power function be $f_n(z) := z^n$. Denote the principal n-th root function as g_n:

$$g_n = \sqrt[n]{\cdot} : \mathbb{C} \to \mathbb{C}, \qquad \sqrt[n]{w} := \begin{cases} 0, & w = 0, \\ \sqrt[n]{|w|}\, e^{i\,\mathrm{Arg}(w)/n}, & w \neq 0. \end{cases}$$

Then for $w \in \mathbb{C}$,

$$f_n\big(g_n(w)\big) = w.$$

Thus the g_n satisfies the identity required in Definition 2.3.1. However, $\sqrt[n]{\cdot}$ is *not* continuous on all of $\mathbb{C}$, due to the jump along $(-\infty, 0]$. Consequently, it is *not* a branch of f_n^{-1} in all of $\mathbb{C}$. However, g_n is continuous in $\mathbb{C}\backslash(-\infty, 0]$, and therefore a branch of f_n^{-1} in $V := \mathbb{C}\backslash(-\infty, 0]$. What does $g_n(V)$ look like? Figure 2.1 shows the image of the principal square root $\sqrt{\cdot}$ restricted to $V = \mathbb{C}\backslash(-\infty, 0]$.

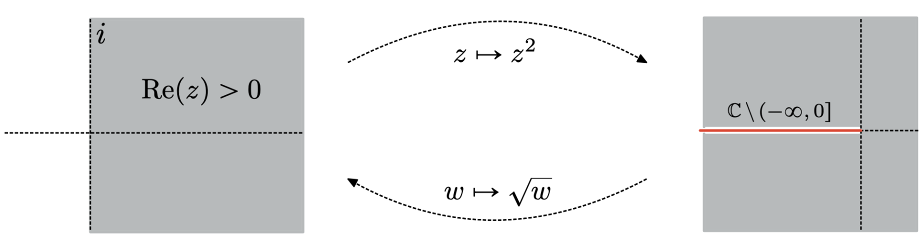


Figure 2.1: Mappings $z \mapsto z^2$ and $w \mapsto \sqrt{w}$.

More generally, the image of $g_n = \sqrt[n]{\cdot}$ of the domain $V := \mathbb{C}\backslash(-\infty, 0]$ is the sector

$$S_n := \left\{re^{i\theta} : r > 0,\ -\tfrac{\pi}{n} < \theta < \tfrac{\pi}{n}\right\},$$

By Theorem 2.3.2, we see the principal n-th root $g_n : \sqrt[n]{\cdot} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is analytic and its derivative is given by

$$\left(\sqrt[n]{\cdot}\right)'(w) = \frac{1}{n\left(\sqrt[n]{w}\right)^{n-1}}, \qquad w \in \mathbb{C} \setminus \{0\}.$$

Exercise 2.3.3. *Show that the square root does not admit a branch on the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Hint: if such a branch existed, you could construct a branch of the argument on S^1, contradicting the earlier exercise about* Arg.

2.3.2 Example: Exponential and Logarithm

Definition 2.3.4 (Complex exponential)**.** *Let $z = x + iy \in \mathbb{C}$. We define*

$$e^z := e^x e^{iy} = e^x(\cos y + i \sin y).$$

This definition agrees with the usual exponential function on the real line and preserves the familiar multiplicative property, i.e., $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for all $z_1, z_2 \in \mathbb{C}$. Another useful property is

$$|e^z| = \left|e^{x+iy}\right| = |e^x|\left|e^{iy}\right| = e^x, \quad z = x + iy \in \mathbb{C}.$$

Remark 2.3.5. *These properties motivate the definition of complex exponential, and the following will be further reassuring: $z \mapsto e^z$ is the unique analytic function on $\mathbb{C}$ which agrees with $x \mapsto e^x$ on $\mathbb{R}$. In fact, we shall later see that if two analytic functions $\mathbb{C} \to \mathbb{C}$ agree on $\mathbb{R}$, then they agree everywhere.*

Proposition 2.3.6. *The map $z \mapsto e^z$ is analytic in $\mathbb{C}$, and its complex derivative is again e^z.*

Proof. Note that the real and imaginary parts of e^z are

$$u(x, y) := \operatorname{Re}\left(e^{x+iy}\right) = e^x \cos y \quad \text{and} \quad v(x, y) := \operatorname{Im}\left(e^{x+iy}\right) = e^x \sin y.$$

The functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ clearly have continuous partial derivatives everywhere. So Theorem 2.2.5 (3) tells us that e^z is analytic in $\mathbb{C}$ if the Cauchy-Riemann equations

$$\begin{cases} \partial_x u(z) = \partial_y v(z) \\ \partial_x u(z) = -\partial_y v(z) \end{cases}$$

hold. Recalling that $(\sin)' = \cos$ and $(\cos)' = -\sin$, we may indeed calculate that

$$\partial_x u(x, y) = e^x \cos y = \partial_y v(x, y)$$

and

$$\partial_x u(x, y) = -e^x \sin y = -\partial_y v(x, y).$$

Therefore e^z is analytic in $\mathbb{C}$. Moreover, its complex derivative can be expressed in terms of the partial derivatives of u and v :

$$\partial_z e^z = \partial_1 u(z) + i\partial_1 v(z) = e^x \cos y + ie^x \sin y = e^z$$

This completes the proof. ☐

The complex exponential has several important mapping properties, which we summarise in the following proposition. See Figure 2.2 for an illustration. For a proof, see e.g. [11, Propositions 4.58-4.62].

Proposition 2.3.7.

(1) The complex exponential is $(2\pi i)$*-periodic:*

$$e^{z+2\pi i} = e^z, \quad z \in \mathbb{C}$$

Moreover, if e^z *is* α*-periodic, then* $\alpha \in 2\pi i\mathbb{Z}$*. In fact,* $e^\alpha = 1 \iff \alpha \in 2\pi i\mathbb{Z}$*, so* $e^z = e^{z+\alpha} \implies e^0 = e^\alpha = 1 \implies \alpha \in 2\pi i\mathbb{Z}$*.*

(2) Let $h \in \mathbb{R}$*. By the* $(2\pi i)$*-periodicity, the complex exponential* e^z *attains all of its values in any belt of the form*

$$S_h := \{z \in \mathbb{C} : \operatorname{Im} z \in (h-\pi, h+\pi]\}, \quad h \in \mathbb{R}. \tag{2.1}$$

Moreover, the complex exponential $z \mapsto e^z$ *is a bijection* $S_h \to \mathbb{C}\backslash\{0\}$*.*

(3) Let $a, b \in \mathbb{R}$*. The complex exponential maps the horizontal line* $\{z : \operatorname{Im} z = a\}$ *to the ray* $\{re^{ia} : r > 0\}$ *and the vertical line* $\{z : \operatorname{Re} z = b\}$ *to the circle* $S\left(0, e^b\right)$*.*

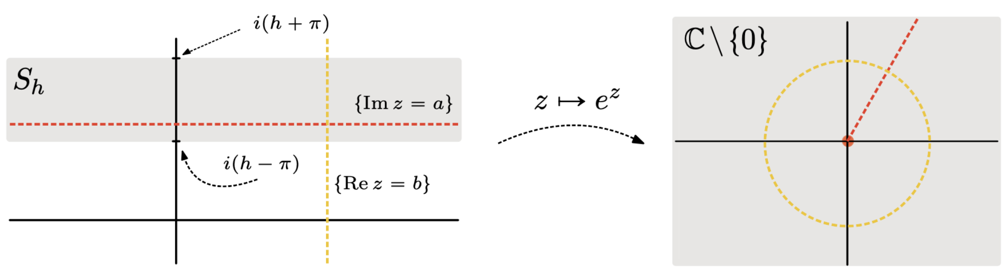


Figure 2.2: The mapping properties of $z \mapsto e^z$. The grey belt S_h maps to $\mathbb{C}\backslash\{0\}$. The red horizontal line $\{\operatorname{Im} z = a\}$ maps to a ray $\{re^{ia} : r > 0\}$ emanating from 0. The beige vertical line $\{\operatorname{Re} z = b\}$ maps to $S\left(0, e^b\right)$.

We now define the complex logarithm. Due to (2) above, the exponential map is not injective on $\mathbb{C}$, so it does not admit a global inverse. However, one can define branches on suitable domains.

Definition 2.3.8 (Logarithm and branches)**.** *Let* $w \in \mathbb{C}\backslash\{0\}$*. A* ***logarithm of*** w *is any complex number* $z \in \mathbb{C}$ *such that* $e^z = w$*. Let* $f(z) = e^z$*. Parallel to the root function, a* ***logarithm function*** $g : X \to \mathbb{C}$ *on* $X \subset \mathbb{C}\backslash\{0\}$ *is a function satisfying* $f \circ g = \mathrm{id}_X$*, i.e.,* $e^{g(w)} = w$ *for all* $w \in X$*. If* g *is continuous in* X*, then by Definition 2.3.1* g *is a branch of* f^{-1} *in* X*. We call it a* ***branch of the logarithm in*** X*.*

Let us then find out what logarithms really look like. Again, the terms are parallel to those for argument/root. The following proposition is straightforward from the definitions.

Proposition 2.3.9.

(1) For $w \in \mathbb{C}\backslash\{0\}$*, define*

$$\operatorname{Log}(w) := \underbrace{\log|w|}_{\textit{real}\ \log} + i\operatorname{Arg}(w),$$

Then Log *is a logarithm function in* $\mathbb{C}\backslash\{0\}$ *in the sense of Definition 2.3.8. Moreover,* Log *is a branch of the logarithm in* $\mathbb{C}\backslash(-\infty, 0]$ *in the sense of Definition 2.3.8.*

(2) Conversely, if $z \in \mathbb{C}$ *is an arbitrary logarithm of* w*, then Parallel to the definition of argument, define the set*

$$z \in \log w := \underbrace{\log|w|}_{\textit{real}\ \log} + i\arg(w) = \operatorname{Log}|w| + i(\operatorname{Arg}(w) + 2\pi\mathbb{Z}) = \operatorname{Log}(w) + 2\pi i\mathbb{Z}.$$

(3) Let $V \subset \mathbb{C}\backslash\{0\}$ be open, and let g be a branch of the logarithm in V. Then g is analytic in V, and

$$g'(w) = w^{-1}, \quad w \in V.$$

The following exercise is analogous to Exercise 1.2.17:

Exercise 2.3.10. *Let $v \in \mathbb{C}\backslash\{0\}$, and let $\ell_v := \{rv : r \geqslant 0\} \subset \mathbb{C}$ be the "ray" spanned by v. Show that there exists a branch of the logarithm in $\mathbb{C}\backslash\ell_v$. Hint: Use the function Arg_v from Example 1.2.6.*

The mapping properties of the principal branch of the logarithm are contained in Figure and verified in the next proposition.

Proposition 2.3.11. *Let $\operatorname{Log} w = \log|w| + i\operatorname{Arg}(w)$ be the principal branch of the logarithm, defined in $\mathbb{C}\backslash(-\infty, 0]$. Then Log maps $\mathbb{C}\backslash(-\infty, 0]$ bijectively onto the open belt*

$$\mathcal{S} = \{z : \operatorname{Im} z \in (-\pi, \pi)\},$$

namely the interior of the belt S_0 defined in (2.1).

Proof. See [11, Proposition 4.76]. □

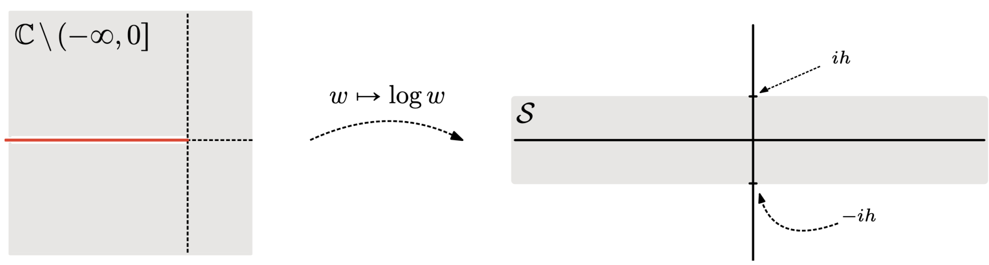


Figure 2.3: The principal branch of the logarithm "Log" maps $\mathbb{C}\backslash(-\infty, 0]$ bijectively onto the open belt $\mathcal{S}$.

Remark 2.3.12. *More generally, if g is a branch of the logarithm in the "slit domain" $\mathbb{C}\backslash\ell_v$ as in Exercise 2.3.10, then the image $g(\mathbb{C}\backslash\ell_v)$ is the interior of one of the belts S_h defined in (2.1). Can you describe the connection between v and h?*

Exercise 2.3.13. *Show an analogous formula to $\arg(zw) = \arg(z) + \arg(w)$ for logarithms: for $z, w \in \mathbb{C}\backslash\{0\}$,*

$$\log(zw) = \log z + \log w.$$

Remark 2.3.14. *The equations above is equation of sets: it is correct as stated, but in general one needs to be quite careful in similar computations. As a very basic example, if $A \subset \mathbb{C}$ is a non-empty set, then $A - A \neq \varnothing$, but instead $A - A = \{a_1 - a_2 : a_1, a_2 \in A\}$. Another counterexample is*

$$2\mathbb{Z} = \mathbb{Z} + \mathbb{Z},$$

where $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$ is the set of even numbers. In general, we only have the inclusion $(z+w)A \subset zA + wA$. On the other hand, $z(A+B) = zA + zB$ for all $z \in \mathbb{C}$ and $A, B \subset \mathbb{C}$.

2.4 Other Functions

2.4.1 Arbitrary Powers

We now extend the notion of powers to arbitrary complex exponents.

Definition 2.4.1 (Complex powers)**.** *Let $w \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$.*

(1) The principal α-power *of w is defined by*

$$w^\alpha := e^{\alpha \mathrm{Log} w}.$$

(2) The set of all α-powers *of w is defined by*

$$\{w^\alpha\} := e^{\alpha \log w} = \left\{ e^{\alpha(\mathrm{Log} w + 2\pi i n)} : n \in \mathbb{Z} \right\}.$$

Remark 2.4.2. *The distinction between w^α and $\{w^\alpha\}$ is essential. The former is single-valued and depends on the chosen branch of the logarithm, whereas the latter is a multi-valued object independent of any branch choice.*

Remark 2.4.3. *If $\alpha \in \mathbb{Z}$, then $\{w^\alpha\}$ consists of a single point and agrees with the usual integer power. If $\alpha = 1/n$ with $n \in \mathbb{N}$, then $\{w^{1/n}\}$ coincides with the set of all nth roots of w.*

Proposition 2.4.4 (Analyticity of principal powers)**.** *Let $\alpha \in \mathbb{C}$. The principal α-power function*

$$w \mapsto w^\alpha = e^{\alpha \mathrm{Log} w}$$

is analytic in $\mathbb{C} \setminus (-\infty, 0]$, and satisfies

$$\partial_w w^\alpha = \alpha w^{\alpha - 1},$$

where the right-hand side is interpreted as the principal $(\alpha - 1)$-power.

Remark 2.4.5. *By choosing a different branch of the logarithm, one can define an analytic α-power function on any slit domain of the form $\mathbb{C} \setminus \ell_v$, where ℓ_v is a ray emanating from the origin.*

Exercise 2.4.6 (Power identities)**.** *Let $z, w \in \mathbb{C} \setminus \{0\}$ and $\alpha, \beta \in \mathbb{C}$. Determine which of the following identities are true and which are false:*

(a) $z^\alpha w^\alpha = (zw)^\alpha$.

(b) $\{z^\alpha\}\{w^\alpha\} = \{(zw)^\alpha\}$.

(c) $w^\alpha w^\beta = w^{\alpha+\beta}$.

(d) $\{w^\alpha\}\{w^\beta\} = \{w^{\alpha+\beta}\}$.

Solution. (a) False. Consider $z = w = -1$ and $\alpha = 1/2$; (b) True; (c) True; (d) False. Consider $w = -1$, $\alpha = \beta = 1/2$. ■

Exercise 2.4.7 (Dependence on the branch)**.** *Let g and $\tilde{g}$ be two branches of the logarithm on a domain $V \subset \mathbb{C} \setminus \{0\}$. Show that the corresponding α-power functions differ by a constant factor of the form $e^{2\pi i k \alpha}$ for some $k \in \mathbb{Z}$.*

Exercise 2.4.8 (Real powers)**.** *Show that if $r > 0$ and $\alpha \in \mathbb{R}$, then the principal complex power r^α agrees with the usual real power.*

2.4.2 Trigonometric Functions

Recall that for $z \in \mathbb{C}$, its real and imaginary parts can be expressed in terms of z and its complex conjugate $\bar{z}$ as

$$\mathrm{Re}\, z = \frac{z + \bar{z}}{2} \quad \text{and} \quad \mathrm{Im}\, z = \frac{z - \bar{z}}{2i}, \quad z \in \mathbb{C}.$$

In particular,

$$\cos\theta = \mathrm{Re}\, e^{i\theta} = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \mathrm{Im}\, e^{i\theta} = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

for all $\theta \in \mathbb{R}$. Now that we have defined e^z for all $z \in \mathbb{C}$, the formulae above suggest a neat way to extend the sine and cosine functions to the whole complex plane:

Definition 2.4.9 (Sine, cosine, tangent). *For $z \in \mathbb{C}$, we define*

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.2}$$

We also define the complex tangent function

$$\tan z := (\sin z)/(\cos z) \quad \textit{whenever } \cos z \neq 0.$$

Since $z \mapsto e^z$ is analytic in $\mathbb{C}$, it follows that that the complex versions of $\cos$ and $\sin$ are also analytic in $\mathbb{C}$. The tangent function is analytic at all points $z \in \mathbb{C}$ where $\cos z \neq 0$. Moreover, the following expressions for the derivatives should look familiar:

$$\partial_z \sin z = \cos z \quad \text{and} \quad \partial_z \cos z = -\sin z.$$

These facts follow readily by differentiating (2.2) and noting that $\partial_z e^{\alpha z} = \alpha e^{\alpha z}$ for all $\alpha \in \mathbb{C}$ by the chain rule. More generally, many of the familiar properties of the cosine and sine functions on the real line have counterparts in $\mathbb{C}$:

Proposition 2.4.10. *Let $z \in \mathbb{C}$. Then,*

$$(\cos z)^2 + (\sin z)^2 = 1, \quad \cos(2z) = (\cos z)^2 - (\sin z)^2, \quad \cos\left(\frac{\pi}{2} - z\right) = \sin z.$$

Also the following addition rules hold for all $z, w \in \mathbb{C}$:

$$\sin(z + w) = \sin z \cos w + \cos z \sin w \quad \textit{and} \quad \cos(z + w) = \cos z \cos w - \sin z \sin w.$$

Proof. Exercise. □

But not everything is so familiar. On the real axis, the functions sin and cos are bounded by 1 in absolute value. This is completely different in $\mathbb{C}$:

Example 2.4.11. Both $\cos$ and $\sin$ are surjective $\mathbb{C} \to \mathbb{C}$ (but not injective). In particular, their moduli are unbounded. The first claim takes a little effort to prove, and we omit it here, but the second one can be observed easily by considering the values of $\cos$ or $\sin$ on the imaginary axis:

$$\lim_{t\to\infty} \cos(it) = \lim_{t\to\infty} \frac{e^{iit} + e^{-iit}}{2} = \lim_{t\to\infty} \frac{e^{-t} + e^{t}}{2} = \infty.$$

Similarly $\sin(it) \to -\infty$ as $t \to \infty$. As a side, the map $t \mapsto \cos(it)$ is real-valued (as we just saw), and is known as the hyperbolic cosine. Similarly, $t \mapsto -i\sin(ix)$ is also real-valued, and is known as the hyperbolic sine. ◇

Example 2.4.12. Starting from the definitions, one can quite easily find that all the solutions to $\sin z = 0$ lie on the real line, and we all know that $\sin x = 0$ for $x \in \pi\mathbb{Z}$. Similarly, all the solutions of $\cos z = 0$ also lie on the real line, more precisely in the set $\pi\mathbb{Z} + \frac{\pi}{2}$. ◇

2.4.3 Hyperbolic Functions

The hyperbolic functions arise naturally from the complex exponential and play an important role in complex analysis, conformal mappings, and hyperbolic geometry.

Definition 2.4.13 (Hyperbolic sine, cosine, and tangent). *For $z \in \mathbb{C}$, we define*

$$\sinh z := \frac{e^z - e^{-z}}{2}, \qquad \cosh z := \frac{e^z + e^{-z}}{2},$$

and, whenever $\cosh z \neq 0$,

$$\tanh z := \frac{\sinh z}{\cosh z}.$$

Proposition 2.4.14. *The hyperbolic functions are related to the trigonometric functions via the identities*

$$\sinh z = -i \sin(iz), \qquad \cosh z = \cos(iz), \qquad \tanh z = -i \tan(iz),$$

Proposition 2.4.15 (Analyticity and derivatives). *The functions $\sinh$ and $\cosh$ are entire, and $\tanh$ is analytic wherever $\cosh z \neq 0$. Their derivatives are given by*

$$\partial_z \sinh z = \cosh z, \qquad \partial_z \cosh z = \sinh z, \qquad \partial_z \tanh z = \frac{1}{\cosh^2 z}.$$

Proposition 2.4.16 (Basic identities). *For all $z \in \mathbb{C}$, the following identities hold:*

$$\cosh^2 z - \sinh^2 z = 1,$$

$$\cosh(2z) = \cosh^2 z + \sinh^2 z, \qquad \sinh(2z) = 2 \sinh z \cosh z,$$

and for all $z, w \in \mathbb{C}$,

$$\sinh(z + w) = \sinh z \cosh w + \cosh z \sinh w,$$

$$\cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w.$$

Remark 2.4.17 (Real axis behaviour). *For real $x \in \mathbb{R}$, the functions $\sinh x$ and $\cosh x$ are real-valued, with $\cosh x \geq 1$. In particular, $\cosh x$ is even, $\sinh x$ is odd, and*

$$\lim_{|x| \to \infty} \cosh x = \infty.$$

Remark 2.4.18 (Zeros and periodicity). *The hyperbolic functions are not periodic. Moreover,*

$$\sinh z = 0 \iff z \in \pi i \mathbb{Z}, \qquad \cosh z = 0 \iff z \in \pi i \left(\mathbb{Z} + \tfrac{1}{2}\right).$$

Remark 2.4.19 (Connection to geometry). *The identity $\cosh^2 z - \sinh^2 z = 1$ is the hyperbolic analogue of the Pythagorean identity for trigonometric functions. In hyperbolic geometry, $\cosh$ and $\sinh$ appear naturally in distance formulas, while $\tanh$ plays a central role in conformal models such as the Poincaré disk and upper half-plane.*

Exercise 2.4.20. *Show that the map $x \mapsto \tanh x$ is a diffeomorphism from $\mathbb{R}$ onto $(-1, 1)$.*

Chapter 3

Complex Integration

3.1 Basic Definitions and Results

A **path** is a continuous map $\gamma : [a,b] \to \mathbb{C}$. The **image of the path** γ is denoted by $\gamma^* := \mathrm{Im}(\gamma) = \gamma([a,b])$. C^1**-path** $\gamma : [a,b] \to \mathbb{C}$ means $\gamma = (\gamma_1, \gamma_2) = (\mathrm{Re}(\gamma), \mathrm{Im}(\gamma))$ as a function from $[a,b] \subseteq \mathbb{R}$ to $\mathbb{R}^2$ is real differentiable and the derivatives of its components $\gamma_1 = \mathrm{Re}(\gamma)$, $\gamma_2 = \mathrm{Im}(\gamma)$ are continuous. Note that a function $f : \mathbb{R} \supseteq U \to R^n$ is real differentiable iff each component function $f_i : \mathbb{R} \supseteq U \to \mathbb{R}$ is real differentiable. Just like vector-valued functions, we write $\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t)$ and treat it as one-sided derivative at the endpoints. It also makes sense to multiply $\gamma'(t)$ with complex numbers, e.g. "$z\gamma'(t)$". A **piecewise** C^1**-path** γ is one with only finitely many points where C^1-property is violated. **Segment** from point z to w is denoted by $[z,w]$. Its image is then $[z,w]^*$, but we sometimes still write it as $[z,w]$. **Reverse path of** $\gamma : [a,b] \to \mathbb{C}$ is defined by $\overline{\gamma} : [a,b] \to \mathbb{C}; t \mapsto \gamma(a+b-t)$. Let $\gamma : [a,b] \to \mathbb{C}$ and $\eta : [c,d] \to \mathbb{C}$ be paths satisfying $\gamma(b) = \eta(c)$. We define the **composite path** $\gamma \star \eta : [a, b+(d-c)] \to \mathbb{C}$ with the formula

$$(\gamma \star \eta)(t) := \begin{cases} \gamma(t), & t \in [a,b], \\ \eta(t-b+c), & t \in [b, b+d-c]. \end{cases}$$

Note that $(\gamma \star \eta)(b) = \eta(b-b+c) = \eta(c) = \gamma(b)$, so the definition at $t = b$ is consistent. It is easy to see that $(\gamma \star \eta)^* = \gamma^* \cup \eta^*$.

A **reparametrization of path** $\gamma : [a,b] \to \mathbb{C}$ is a path $\eta = \rho \circ \gamma$ where $\rho : [c,d] \to [a,b]$ is a continuous bijection such that $\rho(c) = a$ and $\rho(d) = b$. If in addition ρ is (piecewise-)C^1 we say η is a **(piecewise-)**C^1**-reparametrization** of γ. If $\gamma : [a,b] \to \mathbb{C}$ and $\eta : [c,d] \to \mathbb{C}$ are piecewise C^1-paths and $\eta = \gamma \circ \rho$ is a piecewise C^1-reparametrisation of γ, then the chain rule implies that

$$\eta'(t) = \gamma'(\rho(t))\rho'(t) \tag{3.1}$$

for all $t \in [c,d]$ outside the finitely many points where ρ' or γ' do not exist.

We are now ready to define complex integration. Here are three types of complex integrations we will consider.

Definition 3.1.1 (Complex integrations).

1. ***Integration of a complex-valued function over a real interval**: For a continuous function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{C}$, the complex integral of f over $[a,b]$ is defined as*

$$\int_a^b f(x)\,\mathrm{d}x := \int_a^b \mathrm{Re}(f(x))\,\mathrm{d}x + i\int_a^b \mathrm{Im}(f(x))\,\mathrm{d}x, \tag{3.2}$$

2. ***Complex integral along a path**: For a continuous function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ and a piecewise C^1-path $\gamma : [a, b] \to U$, the complex integral of f along γ is defined as*

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &:= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b \mathrm{Re}(f(\gamma(t))\gamma'(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}(f(\gamma(t))\gamma'(t))\,\mathrm{d}t \\ &= \int_a^b [f_1(\gamma(t))\gamma_1'(t) - f_2(\gamma(t))\gamma_2'(t)]\,\mathrm{d}t + i\int_a^b [f_1(\gamma(t))\gamma_2'(t) + f_2(\gamma(t))\gamma_1'(t)]\,\mathrm{d}t\end{aligned} \tag{3.3}$$

where $f_1 = \mathrm{Re}(f)$ and $f_2 = \mathrm{Im}(f)$ and $\gamma_1 = \mathrm{Re}(\gamma)$ and $\gamma_2 = \mathrm{Im}(\gamma)$.

3. ***Arc length integral along a path**: For a continuous function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ and a piecewise C^1-path $\gamma : [a, b] \to U$, the arc length integral of f along γ is defined as*

$$\begin{aligned}\int_\gamma f(z)\,|\mathrm{d}z| &:= \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t = \int_a^b f(\gamma(t))\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}\,\mathrm{d}t \\ &= \int_a^b f_1(\gamma(t))\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}\,\mathrm{d}t + i\int_a^b f_2(\gamma(t))\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}\,\mathrm{d}t,\end{aligned} \tag{3.4}$$

where $f_1 = \mathrm{Re}(f)$ and $f_2 = \mathrm{Im}(f)$.

Remark 3.1.2. *Recall that the line integral of a vector field $\mathbf{F} : \mathbb{R}^2 \supseteq U \to \mathbb{R}^2$ along a path $\gamma : [a, b] \to U$ is defined as*

$$\int_\gamma \mathbf{F}\cdot \mathrm{d}\mathbf{r} := \int_a^b \mathbf{F}(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t.$$

where $\cdot$ denotes the dot product in $\mathbb{R}^2$. Thus, (3.3) can also be written as

$$\int_\gamma \mathbf{G}\cdot\mathrm{d}\mathbf{r} + i\int_\gamma \mathbf{H}\cdot\mathrm{d}\mathbf{r}$$

where $\mathbf{G}(x, y) = (f_1, -f_2)$ and $\mathbf{H}(x, y) = (f_2, f_1)$.

Example 3.1.3. Consider the map $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}, f(z) = 1/z$, and the "circle path" $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) := e^{it}$. Then $\gamma'(t) = (\cos t + i\sin t)' = -\sin t + i\cos t = ie^{it}$, so

$$\int_\gamma f(z)\mathrm{d}z = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\mathrm{d}t = i\int_0^{2\pi}\frac{e^{it}\mathrm{d}t}{e^{it}} = i\int_0^{2\pi}\mathrm{d}t = 2\pi i \tag{3.5}$$

Since $|\gamma'(t)| = |ie^{it}| = 1$, we have

$$\int_\gamma f(z)|\mathrm{d}z| = \int_0^{2\pi} f(\gamma(t))|\gamma'(t)|\mathrm{d}t = \int_0^{2\pi} e^{-it}\mathrm{d}t = \int_0^{2\pi}\cos t\mathrm{d}t - i\int_0^{2\pi}\sin t\mathrm{d}t = 0.$$

More generally, if $n \in \mathbb{Z}, z_0 \in \mathbb{C}$, and $\gamma_n(t) := z_0 + re^{int}$, then $\gamma_n'(t) = inre^{int}$, and hence

$$\int_{\gamma_n}\frac{\mathrm{d}z}{z - z_0} = 2\pi i n, \quad n \in \mathbb{Z},$$

and

$$\int_{\gamma_n}\frac{|\mathrm{d}z|}{z - z_0} = 0 \quad \text{for all } n \in \mathbb{Z}$$

where for $n \neq 0$ it vanishes because $|n|\int_0^{2\pi} e^{-int}\mathrm{d}t = 0$, and for $n = 0$ it vanishes because the path γ_0 is constant with $|\gamma_0'(t)| = 0$.

Note that it is tempting to use Fundamental Theorem of Calculus to say that the integral $\int_\gamma f(z)\,\mathrm{d}z$ evaluates to zero via the antiderivative Log (recall Proposition 2.3.9 (3)). However, Log is not analytic on the image of γ. It is analytic on $\mathbb{C}\backslash\ell_v$, a domain where a whole circle $S^1=\gamma^*$ is not contained. We shall later state and prove the precise complex version of Fundamental Theorem of Calculus. $\diamond$

Some basic properties of complex integrals are summarized in the following proposition whose proof is straightforward from the definitions.

Proposition 3.1.4 (Basic properties of complex integrals)**.**

1. *(Linearity) Let $f,g:U\subseteq\mathbb{C}\to\mathbb{C}$ be continuous functions and $\gamma:[a,b]\to U$ be a piecewise C^1-path. For any $\alpha,\beta\in\mathbb{C}$, we have*
$$\int_\gamma(\alpha f(z)+\beta g(z))\,\mathrm{d}z=\alpha\int_\gamma f(z)\,\mathrm{d}z+\beta\int_\gamma g(z)\,\mathrm{d}z.$$
2. *(Additivity over paths) Let $f:U\subseteq\mathbb{C}\to\mathbb{C}$ be a continuous function and $\gamma:[a,b]\to U$ and $\eta:[c,d]\to U$ be piecewise C^1-paths such that $\gamma(b)=\eta(c)$. Then*
$$\int_{\gamma\star\eta}f(z)\,\mathrm{d}z=\int_\gamma f(z)\,\mathrm{d}z+\int_\eta f(z)\,\mathrm{d}z.$$
3. *(Reparametrization invariance) Let $f:U\subseteq\mathbb{C}\to\mathbb{C}$ be a continuous function and $\gamma:[a,b]\to U$ be a piecewise C^1-path. If $\eta:[c,d]\to U$ is a piecewise C^1-reparametrization of γ, then*
$$\int_\eta f(z)\,\mathrm{d}z=\int_\gamma f(z)\,\mathrm{d}z.$$
4. *(Reversing the path) Let $f:U\subseteq\mathbb{C}\to\mathbb{C}$ be a continuous function and $\gamma:[a,b]\to U$ be a piecewise C^1-path. Then*
$$\int_{\overline{\gamma}}f(z)\,\mathrm{d}z=-\int_\gamma f(z)\,\mathrm{d}z.$$

The arc length integral is useful because it gives an upper bound for the complex path integral (and the upper bound is often easier to compute):

Proposition 3.1.5. *Let $\gamma:[a,b]\to\mathbb{C}$ be piecewise C^1, and let $f:\gamma^*\to\mathbb{C}$ be continuous. Then,*
$$\left|\int_\gamma f(z)\mathrm{d}z\right|\leqslant\int_\gamma|f(z)||\mathrm{d}z|$$

Proof. If $w:=\int_\gamma f(z)\mathrm{d}z=0$, there is nothing to prove. Otherwise, we may write the value of the integral in polar coordinates:
$$w=|w|e^{i\theta},\quad \theta\in\arg(w).$$
Now, by the linearity of the complex path integral, we may write
$$\left|\int_\gamma f(z)\mathrm{d}z\right|=|w|=e^{-i\theta}w=\int_\gamma e^{-i\theta}f(z)\mathrm{d}z.$$
The left hand side is clearly a real number, so also the right hand side is a real number. On the other hand, by the definition of the complex path integral,
$$\int_\gamma e^{-i\theta}f(z)\mathrm{d}z=\int_a^b\mathrm{Re}\left(e^{-i\theta}f(\gamma(t))\gamma'(t)\right)\mathrm{d}t+i\int_a^b\mathrm{Im}\left(e^{-i\theta}f(\gamma(t))\gamma'(t)\right)\mathrm{d}t$$

Since the left hand side is a real number, also the right hand side must be. Therefore

$$\int_a^b \operatorname{Im}\left(e^{-i\theta} f(\gamma(t))\gamma'(t)\right) \mathrm{d}t = 0$$

and we have now shown that

$$\left|\int_\gamma f(z)\mathrm{d}z\right| = \int_a^b \operatorname{Re}\left(e^{-i\theta} f(\gamma(t))\gamma'(t)\right) \mathrm{d}t$$

The right hand side is the Riemann integral of the real function $t \mapsto \operatorname{Re}\left(e^{-i\theta} f(\gamma(t))\gamma'(t)\right)$, so by the basics of real analysis,

$$\int_a^b \operatorname{Re}\left(e^{-i\theta} f(\gamma(t))\gamma'(t)\right) \mathrm{d}t \leqslant \int_a^b \left|\operatorname{Re}\left(e^{-i\theta} f(\gamma(t))\gamma'(t)\right)\right| \mathrm{d}t$$

To conclude the proof, it remains to note that

$$\left|\operatorname{Re}\left(e^{-i\theta} f(\gamma(t))\gamma'(t)\right)\right| \leqslant |f(\gamma(t))|\,|\gamma'(t)|\,, \quad t \in [a,b],$$

so

$$\int_a^b \left|\operatorname{Re}\left(e^{-i\theta} f(\gamma(t))\gamma'(t)\right)\right| \mathrm{d}t \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|\,\mathrm{d}t \overset{\text{defn.}}{=} \int_\gamma |f(z)||\mathrm{d}z|$$

This completes the proof. $\square$

We record the following corollary:

Corollary 3.1.6. *Let $\gamma : [a,b] \to \mathbb{C}$ be piecewise C^1, and let $f : \gamma^* \to \mathbb{C}$ be continuous. Then,*

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leqslant \|f\|_\infty \operatorname{length}(\gamma),$$

where $\|f\|_\infty := \sup\{|f(z)| : z \in \gamma^*\}$, and

$$\operatorname{length}(\gamma) := \int_a^b |\gamma'(t)|\,\mathrm{d}t$$

Proof. We simply note that

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leqslant \int_\gamma |f(z)||\mathrm{d}z| \overset{\text{defn.}}{=} \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t \leqslant \|f\|_\infty \int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

This completes the proof. $\square$

Remark 3.1.7. *It is always true that $\|f\|_{L^\infty(K)} \leq \|f\|_\infty$. A counterexample to the converse inequality is given by the function $f : [0,1] \to \mathbb{C}$ defined as*

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}.$$

Here $\|f\|_\infty = 1$ but $\|f\|_{L^\infty(K)} = 0$ since the rationals have measure zero.

Equality $\|f\|_{L^\infty(K)} = \|f\|_\infty$ holds if $|f|$ does not attain larger values only on null sets. In particular, equality holds if f is continuous on K and K has positive measure (e.g. a domain or compact set with interior). Indeed, continuity prevents isolated "spikes" on measure-zero sets. In our case, γ^* is the image of a continuous function from a closed interval $[a,b]$, so γ^* is compact and has positive measure (it contains a curve of positive length). Thus, for continuous $f : U \to \mathbb{C}$ where $\gamma^* \subseteq U$, we have

$$\|f\|_{L^\infty(\gamma^*)} = \|f\|_\infty.$$

Remark 3.1.8. *Note that* $\operatorname{length}(\gamma) < \infty$ *for every* C^1*-path* $\gamma : [a,b] \to \mathbb{C}$*, since* $t \mapsto |\gamma'(t)|$ *is continuous, and thus uniformly bounded on* $[a,b]$*. In fact, length* $(\gamma) < \infty$ *also remains true for piecewise* C^1*-paths* $\gamma : [a,b] \to \mathbb{C}$*, since* γ *can be written as* $\gamma = \gamma_1 \star \cdots \star \gamma_n$*, where each* γ_j *is* C^1*, and*

$$\operatorname{length}(\gamma) = \operatorname{length}(\gamma_1) + \cdots + \operatorname{length}(\gamma_n) < \infty.$$

Our definition of "$\operatorname{length}(\gamma)$*" agrees with other notions of length the reader may possibly have seen, for example the 1-dimensional Hausdorff measure* $\mathcal{H}^1(\gamma^*)$*. This will not be needed explicitly, and this course does not deal with Hausdorff measures, so we will simply give the following example as justification.*

Example 3.1.9. Let $z, w \in \mathbb{C}$ with $z \neq w$. Consider the path $[z,w] : [0,1] \to \mathbb{C}$ parametrising the segment between z and w, namely

$$[z,w](t) = tw + (1-t)z, \quad t \in [0,1].$$

Then $[z,w]'(t) = w - z$ for all $t \in [0,1]$, and consequently

$$\operatorname{length}([z,w]) \overset{\text{defn.}}{=} \int_0^1 |[z,w]'(t)| \,\mathrm{d}t = \int_0^1 |w-z| \mathrm{d}t = |w-z|$$

So, our definition of "length" coincides with the expected result for (at least) segments. $\diamond$

The following consequence of Corollary 3.1.6 is often useful:

Corollary 3.1.10. *Let* $\gamma : [a,b] \to \mathbb{C}$ *be piecewise* C^1*, and let* $f_k, f : \gamma^* \to \mathbb{C}$ *be continuous functions such that* $f_k \to f$ *uniformly on* γ^**, as* $k \to \infty$*. Then,*

$$\lim_{k \to \infty} \int_\gamma f_k(z) \mathrm{d}z = \int_\gamma f(z) \mathrm{d}z$$

Proof. By Corollary 3.1.6, we have

$$\left| \int_\gamma f_k(z) \mathrm{d}z - \int_\gamma f(z) \mathrm{d}z \right| = \left| \int_\gamma (f_k - f)(z) \mathrm{d}z \right| \leqslant \|f_k - f\|_{L^\infty(\gamma^*)} \cdot \operatorname{length}(\gamma)$$

Here length $(\gamma) < \infty$, and $\|f_k - f\|_{L^\infty(\gamma^*)} \to 0$ by assumption, as $k \to \infty$. $\square$

We end this section with the promised fundamental theorem of calculus for paths. We first need the following definition:

Definition 3.1.11. *Let* $U \subset \mathbb{C}$ *be open, and let* $f : U \to \mathbb{C}$ *be a map. We say that* $F : U \to \mathbb{C}$ *is a primitive of* f *in* U *if* F *is analytic in* U*, and* $F'(z) = f(z)$ *for all* $z \in U$*.*

If F is a primitive of $f : U \to \mathbb{C}$, and $c \in \mathbb{C}$, then $F + c$ is also clearly a primitive of f. The converse is true if U is connected:

Proposition 3.1.12. *Let* $U \subset \mathbb{C}$ *be open and connected, and let* $f : U \to \mathbb{C}$ *be a map. Assume that* $F_1, F_2 : U \to \mathbb{C}$ *are both primitives of* f*. Then there exists a constant* $c \in \mathbb{C}$ *such that* $F_1 = F_2 + c$*.*

Proof. Note that $G := F_1 - F_2 : U \to \mathbb{C}$ is an analytic function with $G'(z) = 0$ for all $z \in U$. Therefore G is a constant on U by Corollary 2.2.7. $\square$

Example 3.1.13. All the primitives of $z \mapsto e^z$ in $\mathbb{C}$ are of the form $F(z) = e^z + c, c \in \mathbb{C}$. All the primitives of $w \mapsto 1/w$ in the set $\mathbb{C} \backslash (-\infty, 0]$ are of the form $G(w) = \operatorname{Log} w + c, c \in \mathbb{C}$. These facts follow from recalling that $z \mapsto e^z$ and $w \mapsto \operatorname{Log} w$ are examples of primitives, and on the other hand the open sets $\mathbb{C}$ and $\mathbb{C} \backslash (-\infty, 0]$ are connected. $\diamond$

Theorem 3.1.14 (Fundamental Theorem of Calculus for Paths)**.** *Let $U \subset \mathbb{C}$ be open, and assume that $f : U \to \mathbb{C}$ is continuous. Let $\gamma : [a, b] \to U$ be a piecewise C^1-path. If F is a primitive of f in U, then*

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

In particular, if γ is a closed path, then $\gamma(a) = \gamma(b)$, so

$$\int_\gamma f(z)\mathrm{d}z = 0$$

Remark 3.1.15. *Recall that the principal branch of the logarithm $\mathrm{Log} : \mathbb{C}\backslash(-\infty, 0]$ is a primitive of $z \mapsto 1/z$ in $\mathbb{C}\backslash(-\infty, 0]$. Therefore, if $\gamma : [a, b] \to \mathbb{C}\backslash(-\infty, 0]$ is a closed piecewise C^1-path, we may deduce from Theorem 3.1.14 that*

$$\int_\gamma \frac{\mathrm{d}z}{z} = \mathrm{Log}\gamma(b) - \mathrm{Log}\gamma(a) = 0$$

As we already observed earlier in Example 3.1.3, this discussion does not apply to the path $\gamma(t) = e^{it}$, for $t \in [0, 2\pi]$, because γ^* is not contained in $\mathbb{C}\backslash(-\infty, 0]$.

On the other hand, there is nothing special about $(-\infty, 0]$. By the same argument, we have $\int_\gamma \mathrm{d}z/z = 0$ for every closed piecewise C^1-path γ with $\gamma^* \subset \mathbb{C}\backslash\ell_v$ (for arbitrary $v \in \mathbb{C}\backslash\{0\}$). Reason: there exists a branch of the logarithm in $\mathbb{C}\backslash\ell_v$.

The following lemma is needed in the proof of Theorem 3.1.14:

Lemma 3.1.16. *Let $F : U \to \mathbb{C}$ be analytic, and let $\gamma : [a, b] \to \mathbb{C}$ be a C^1-path. Then,*

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t), \quad t \in [a, b]$$

where the left hand side refers to the real derivative of the composed map $F \circ \gamma : [a, b] \to \mathbb{R}^2$.

Proof. To calculate $(F \circ \gamma)'(t)$, we apply the real-variable chain rule:

$$(F \circ \gamma)'(t) = DF(\gamma(t))\gamma'(t)$$

using here that complex differentiability implies real differentiability (so $DF(z)$ makes sense for $z \in U$). On the other hand, since F is complex differentiable, the matrix $DF(\gamma(t))$ equals the matrix $M_{F'(\gamma(t))}$ corresponding to complex multiplication by $F'(\gamma(t))$. Therefore $(F \circ \gamma)'(t) = M_{F'(\gamma(t))}\gamma'(t) = F'(\gamma(t))\gamma'(t)$, as claimed. □

Proof of Theorem 3.1.14. Assume first that $\gamma : [a, b] \to \mathbb{C}$ is a C^1-path, and not just piecewise C^1. The primitive $F : U \to \mathbb{C}$ is analytic, so the previous lemma implies that

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t), \quad t \in [a, b].$$

Consequently,

$$\int_\gamma f(z)\mathrm{d}z \overset{\text{def.}}{=} \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t = \int_a^b (F \circ \gamma)'(t)\mathrm{d}t = F(\gamma(b)) - F(\gamma(a)),$$

where the last equation applied the real version of fundamental theorem of calculus to the Riemann integrable (even continuous) function $g := (F \circ \gamma)' : [a, b] \to \mathbb{C}$ and its real primitive $G := F \circ \gamma$ (or to be precise the real and imaginary parts of these functions separately).

Finally, let us deduce the case of piecewise C^1-paths. If $\gamma : [a,b] \to \mathbb{C}$ is piecewise C^1, we may write $\gamma = \gamma_1 \star \cdots \star \gamma_n$ where each $\gamma_j : [t_{j-1}, t_j] \to \mathbb{C}$ is a C^1-path, $a = t_0 < \ldots < t_n = b$, and $\gamma_j(t_{j-1}) = \gamma_{j-1}(t_j)$ for $1 \leqslant j \leqslant n$. Consequently,

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z)\mathrm{d}z = \sum_{j=1}^n \left[F(\gamma_j(t_j)) - F(\gamma_j(t_{j-1}))\right],$$

by the first part of the proof applied separately to the paths γ_j. The sum on the right hand side "telescopes" and its value is

$$F(\gamma_n(t_n)) - F(\gamma_1(t_0)) = F(\gamma(b)) - F(\gamma(a)).$$

This completes the proof. □

Example 3.1.17.

- Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be analytic. If $z, w \in U$, and also the segment connecting z, w is contained in U, then the function f is a primitive of f'. Theorem 3.1.14 yields

$$\int_{[z,w]} f'(\zeta)\mathrm{d}\zeta = f([z,w](1)) - f([z,w](0)) = f(w) - f(z).$$

- Let $\gamma : [a,b] \to \mathbb{C}$ be a closed piecewise C^1-path, and let $p(z) = a_n z^n + \ldots + a_1 z + a_0$ be a polynomial. It has the primitive

$$P(z) := \frac{a_n}{n+1} z^{n+1} + \frac{a_{n-1}}{n} z^n + \ldots + \frac{a_1}{2} z^2 + a_0 z$$

in the whole plane $\mathbb{C}$. Thus, by Theorem 3.1.14,

$$\int_\gamma p(z)\mathrm{d}z = 0$$

◇

Theorem 3.1.14 can be used to show that certain maps do not have primitives:

Example 3.1.18. The map $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $f(z) = 1/z$ does not have a primitive $F : \mathbb{C}\backslash\{0\} \to \mathbb{C}$. To see this, let $\gamma(t) := e^{it}, t \in [0, 2\pi]$. If the primitive F existed, then our calculation (3.5) and Theorem 3.1.14 would contradict each other:

$$2\pi i = \int_\gamma f(z)\mathrm{d}z = F(\gamma(0)) - F(\gamma(2\pi)) = 0$$

Let us emphasise, however, that the existence of primitives is highly dependent on the choice of the domain: for example, $z \mapsto 1/z$ does have a primitive in every slit domain of the form $\mathbb{C}\backslash\ell_v$. Indeed, the primitive is given by a(ny) branch of the logarithm in $\mathbb{C}\backslash\ell_v$. ◇

Exercise 3.1.19. *The function $z \mapsto \bar{z}$ has no primitive in any open set $U \subset \mathbb{C}$.*

Finally, we record the following corollary of Theorem 3.1.14, which is the complex version of integration by parts:

Proposition 3.1.20 (integration by parts)**.** *Let $U \subset \mathbb{C}$ be open, and let $\gamma : [a,b] \to U$ be a piecewise C^1-path. Assume that $f, g : U \to \mathbb{C}$ are analytic. Assume additionally that f', g' are continuous, although we will see soon see that this is an automatic consequence of the analyticity. Then,*

$$\int_\gamma f(z)g'(z)\mathrm{d}z = [f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))] - \int_\gamma g(z)f'(z)\mathrm{d}z$$

Proof. Exercise. □

3.2 Cauchy's Integral Theorem

Theorem 3.2.1 (Cauchy's theorem for triangles)**.** *Let $a, b, c \in \mathbb{C}$. Denote by $\partial\triangle(a, b, c)$ the path parametrizing the boundary of the triangle $\triangle := \triangle(a, b, c)$ spanned by the points $a, b, c \in \mathbb{C}$. That is, $\partial\triangle := [a, b] \star [b, c] \star [c, a]$. The trace $(\partial\triangle)^*$ equals the topological boundary of $\triangle$. Assume that $U \subset \mathbb{C}$ is an open set with $\triangle \subset U$, and let $w_0 \in U$. Assume that f is continuous in U and analytic in $U \setminus \{w_0\}$. Then,*

$$\int_{\partial\triangle} f(z)\mathrm{d}z = 0.$$

Remark 3.2.2. *The "special point" $w_0 \in U$ may seem like an unnecessary technicality, but it will be indispensable when proving the Cauchy integral formula*

Proof. We will assume that the triangle $\triangle$ is non-degenerate, that is, the points $\{a, b, c\}$ are not contained on a common line. The statement also remains true in this case, and is much easier to prove.

We will start with the special case where $w_0 \notin \triangle$. Let a', b', c' be the midpoints of the segments $[a, b]$, $[b, c]$, and $[c, a]$, see Figure 3.1. These give rise to 4 new triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4 \subset \triangle$. Moreover, by the cancellation of interior edges illustrated in Figure 3.1, we have

$$\int_{\partial\triangle} f(z)\mathrm{d}z = \sum_{j=1}^{4} \int_{\partial\triangle_j} f(z)\mathrm{d}z$$

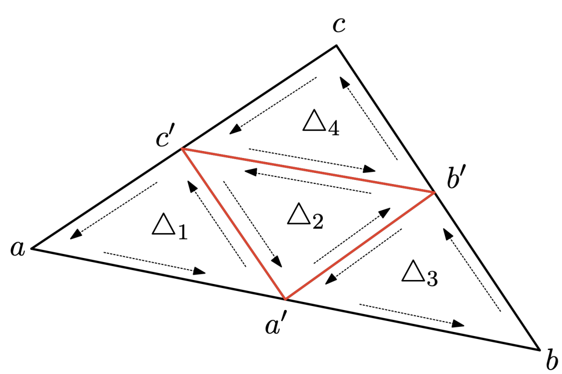


Figure 3.1: Dividing the triangle $\triangle$ into 4 smaller triangles $\triangle_1, \ldots, \triangle_4$, and how the "interior" segments of $\partial\triangle_1, \ldots, \partial\triangle_4$ cancel each other out.

Thus,

$$\left|\int_{\partial\triangle} f(z)\mathrm{d}z\right| \leqslant 4 \max_{1\leqslant j\leqslant 4} \left|\int_{\partial\triangle_j} f(z)\mathrm{d}z\right|.$$

Let $\triangle^1 \in \{\triangle_1, \ldots, \triangle_4\}$ be the triangle which attains the maximum. We focus attention on $\triangle^1$, and repeat the "subdivision" trick inside $\triangle^1$: we divide $\triangle^1$ it into 4 smaller triangles $\triangle^1_1, \ldots, \triangle^1_4$. Repeating the reasoning above, we can find one of them satisfying

$$\left|\int_{\partial\triangle} f(z)\mathrm{d}z\right| \leqslant 4\left|\int_{\partial\triangle^1} f(z)\mathrm{d}z\right| \leqslant 16\left|\int_{\partial\triangle^1_j} f(z)\mathrm{d}z\right|$$

Now, continuing inductively, we can find a sequence of triangles $\triangle =: \triangle^0 \supset \triangle^1 \supset \ldots$ satisfying

$$(*) : \quad \left|\int_{\partial\triangle} f(z)\mathrm{d}z\right| \leqslant 4^n \left|\int_{\partial\triangle^n} f(z)\mathrm{d}z\right|, \quad n \geqslant 0.$$

As $n \to \infty$, the triangles $\triangle_n$ get smaller and smaller: in fact,

$$\operatorname{length}(\partial \triangle^n) = 2^{-n} \operatorname{length}(\partial \triangle), \quad n \geqslant 0.$$

Moreover, the triangles $\triangle^n$ "converge" to a unique point $z_0 \in \triangle$, or more precisely

$$\bigcap_{n \geqslant 0} \triangle^n = \{z_0\} \subset \triangle.$$

The map f is differentiable at z_0, so given $\varepsilon > 0$, there exists a radius $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leqslant \varepsilon |z - z_0|, \quad z \in D(z_0, r).$$

In particular, this estimate holds for all $z \in \triangle^n$ when $n \geqslant n_\varepsilon$ is sufficiently large. For $n \geqslant n_\varepsilon$, we therefore have

$$\begin{aligned}
\left|\int_{\partial \triangle^n} f(z) \mathrm{d}z\right| &= \left|\int_{\partial \triangle^n} [f(z) - f(z_0) - f'(z_0)(z - z_0)] \mathrm{d}z\right| \\
&\leq \int_{\partial \triangle^n} \varepsilon |z - z_0| |\mathrm{d}z| \leqslant \varepsilon (\operatorname{length}(\partial \triangle^n))^2 \\
&= \varepsilon 4^{-n} (\operatorname{length}(\partial \triangle))^2.
\end{aligned}$$

where the first equality used Example 3.1.17 as the linear approximation $-f(z_0) - f'(z_0)(z - z_0)$ is a first degree polynomial in z; the second inequality used the estimate $|\int f \mathrm{d}z| \leq \int |f||\mathrm{d}z|$; and the third inequality used $|z - z_0| \leqslant \operatorname{diam}(\triangle^n) \leqslant \operatorname{length}(\partial \triangle^n), z \in \triangle^n$.

Now, $(*)$ yields, for $n \geqslant n_\varepsilon$,

$$\left|\int_{\partial \triangle} f(z) \mathrm{d}z\right| \leqslant 4^n \left|\int_{\partial \triangle^n} f(z) \mathrm{d}z\right| \leqslant \varepsilon \cdot (\operatorname{length}(\partial \triangle))^2$$

Letting $\varepsilon \to 0$ completes the proof in the special case $w_0 \notin \triangle$. Let us finally consider the case $w_0 \in \triangle$. There are two distinct cases to consider, both shown in Figure 9: either w_0 is a corner of $\triangle$, or then it is not. If w_0 is a corner, we "isolate" it to a small triangle $\triangle_1 \subset \triangle$. Then, we also form two further triangles $\triangle_2, \triangle_3 \subset \triangle$ in such a way that

$$(**) : \quad \int_{\partial \triangle} f(z) \mathrm{d}z = \int_{\partial \triangle_1} f(z) \mathrm{d}z + \int_{\partial \triangle_2} f(z) \mathrm{d}z + \int_{\partial \triangle_3} f(z) \mathrm{d}z$$

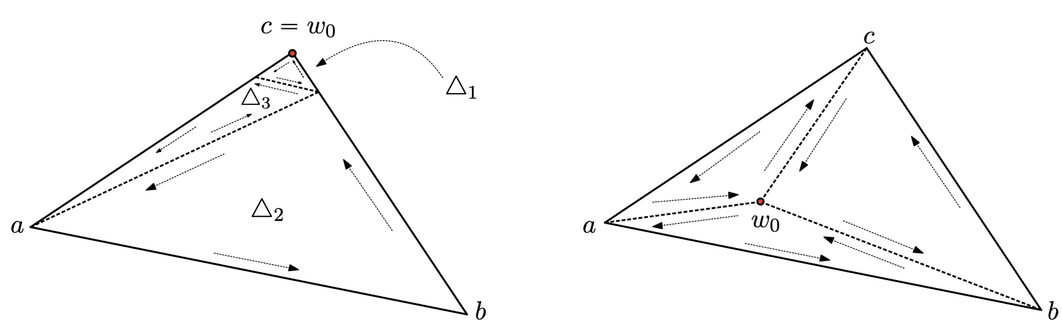


Figure 3.2: The special cases where (i) w_0 is a corner of $\triangle$, and (ii) where $w_0 \in \triangle$ but w_0 is not a corner.

Note that the triangles $\triangle_2, \triangle_3$ have no "special points": f is analytic on a neighbourhood of $\triangle_2 \cup \triangle_3$. Therefore the two latter terms in $(**)$ are zero by the first part of the proof. For the first term, we estimate the path integral from above by the arc length integral:

$$\left|\int_{\partial \triangle_1} f(z) \mathrm{d}z\right| \leqslant \|f\|_{L^\infty(\triangle)} \operatorname{length}(\partial \triangle_1).$$

Since f is continuous on $\triangle$, we have $\|f\|_{L^\infty(\triangle)} < \infty$. On the other hand the length of $\partial\triangle_1$ can be made as small as we like. This implies that $\int_{\partial\Delta} f(z)\mathrm{d}z = 0$.

Finally, consider the case where $w_0 \in \Delta$ is not a corner. We form new triangles $\triangle_1, \triangle_2, \triangle_3 \subset \triangle$ with the properties that w_0 is a common corner of $\triangle_1, \triangle_2, \triangle_3$, and $(**)$ holds (see Figure 3.2, and think what would happen if w_0 lies on an edge of $\triangle$). Now, by the "corner case" we just handled above, all the three terms on the right hand side of $(**)$ are zero. Therefore $\int_{\partial\triangle} f(z)\mathrm{d}z = 0$ also in this case, and the proof is complete. $\square$

We may immediately generalise the previous theorem to all closed piecewise C^1-paths γ, but only under the assumption that f is analytic in a convex open set containing γ^*.

Theorem 3.2.3 (Cauchy's theorem in a convex set)**.** *Assume that $U \subset \mathbb{C}$ is a convex open set, $w_0 \in U$, f is continuous in U, and analytic in $U \setminus \{w_0\}$. Then f has a primitive in U.*

As a consequence (and by Theorem 3.1.14), if $\gamma : [a, b] \to U$ is a closed piecewise C^1-path, then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

Proof. We will define the primitive $F : U \to \mathbb{C}$ with the following explicit formula. Fix $a \in U$ arbitrary, and set

$$F(z) := \int_{[a,z]} f(\zeta)\mathrm{d}\zeta, \quad z \in U$$

This definition is well-posed, because $[a, z]^* \subset U$ by the convexity of U. We also note that

$$F(z) = -\int_{\overline{[a,z]}} f(\zeta)\mathrm{d}\zeta = -\int_{[z,a]} f(\zeta)\mathrm{d}\zeta, \quad z \in U$$

We then claim that F is complex differentiable for all $z \in U$, and indeed $F'(z) = f(z)$. Fix distinct points $z, w \in U$, and consider the triangle

$$\partial\triangle := [a, w] \star [w, z] \star [z, a],$$

whose trace is contained in U by convexity. In fact the entire solid triangle $\triangle$ is also contained in U - as required to apply Theorem 3.2.1. Then,

$$F(w) - F(z) = \int_{[a,w]} f(\zeta)\mathrm{d}\zeta + \int_{[z,a]} f(\zeta)\mathrm{d}\zeta = \int_{\partial\triangle} f(\zeta)\mathrm{d}\zeta - \int_{[w,z]} f(\zeta)\mathrm{d}\zeta.$$

The first term on the right vanishes by Theorem 3.2.1, so we may deduce that

$$\frac{F(w) - F(z)}{w - z} = -\frac{1}{w - z}\int_{[w,z]} f(\zeta)\mathrm{d}\zeta = \frac{1}{z - w}\int_{[w,z]} f(\zeta)\mathrm{d}\zeta$$

On the other hand, applying Example 3.1.17 to the function $\zeta \mapsto c\zeta$ yields

$$\frac{1}{z - w}\int_{[w,z]} c\,\mathrm{d}\zeta = c, \quad c \in \mathbb{C}$$

so in particular with $c := f(z)$ we have

$$\frac{F(w) - F(z)}{w - z} - f(z) = \frac{1}{z - w}\int_{[w,z]} [f(\zeta) - f(z)]\mathrm{d}\zeta$$

Since f is continuous at z, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \varepsilon$ as soon as $|\zeta - z| < \delta$. In particular, this holds for all $\zeta \in [w, z]^*$ if $|w - z| < \delta$. Consequently, for $|w - z| < \delta$ we have the estimate

$$\left|\frac{F(w) - F(z)}{w - z} - f(z)\right| \leqslant \frac{1}{|z - w|}\int_{[w,z]} |f(\zeta) - f(z)||\mathrm{d}\zeta| \leqslant \|f - f(z)\|_{L^\infty([w,z])} \leqslant \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that

$$\lim_{w\to z} \frac{F(w) - F(z)}{w - z} = f(z).$$

This completes the proof. □

Remark 3.2.4.

- *Theorem 3.2.3 implies Theorem 3.2.1 as a special case, because we can find a slightly larger triangle containing $\triangle$ in its interior while staying inside the open set U.*
- *The convexity assumption in Theorem 3.2.3 is quite strong. Later on, we will see that the same conclusion remains true under the weaker assumption that U is simply connected.*

We close this section by recording the following corollary of the proof of Theorem 3.2.3:

Corollary 3.2.5. *Let $U \subset \mathbb{C}$ be open, and assume that $f : U \to \mathbb{C}$ is a continuous function satisfying*

$$\int_{\partial\triangle} f(z)\mathrm{d}z = 0, \quad \triangle \subset U \tag{3.6}$$

Then, f has a primitive in every open disc $D \subset U$ (more generally: every convex open set $V \subset U$).

Proof. Let $D \subset U$ be a disc. Then D is a convex open set, and (3.6) holds for all triangles $\triangle \subset D$. These were all the properties we needed in the proof of Theorem 3.2.3 to conclude that f has a primitive in D. □

Remark 3.2.6. *Corollary 3.2.5 has a converse, which is already familiar to us: if $f : U \to \mathbb{C}$ is continuous, and has a primitive in every convex open set $V \subset U$, then (3.6) holds for all triangles $\Delta \subset U$ by the same reasoning in Remark 3.2.4.*

3.3 Cauchy's Integral Formula

Our first application of Cauchy's theorem will be the following "representation formula" for analytic functions defined on a convex open set $U \subset \mathbb{C}$:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta, \quad z \in U$$

where $r > 0$ is so small that $D(z, r) \subset U$, and $\partial D(z, r)$ refers to the path $\gamma(t) = z + re^{it}$, $t \in [0, 2\pi]$. This formula, and a more general version of it, is known as Cauchy's integral formula. We begin with the definition of winding numbers.

Definition 3.3.1 (Winding number)**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a closed piecewise C^1-path, and let $z \in \mathbb{C}\backslash\gamma^*$. The **winding number of γ around** z is defined by*

$$n_\gamma(z) := \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Example 3.3.2. Let $\gamma_k : [0, 2\pi] \to \mathbb{C}$ be the circle path $\gamma_k(t) := z_0 + re^{ikt}$, where $k \in \mathbb{Z}$. We computed that

$$n_{\gamma_k}(z_0) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{\mathrm{d}z}{z - z_0} = k$$

Thus, the winding number $n_{\gamma_k}(z_0)$ captures the intuition that γ_k "winds $|k|$ times around z_0." The sign of the winding number tells us whether the "winding" happens clockwise (case $k < 0$) or counterclockwise (case $k > 0$). ◇

The following property of winding numbers is elementary but useful:

Proposition 3.3.3. *Let $\gamma : [a, b] \to \mathbb{C}$ and $\eta : [c, d] \to \mathbb{C}$ be closed piecewise C^1-paths such that $\gamma(b) = \eta(c)$. Then,*

$$n_{\gamma \star \eta}(z) = n_\gamma(z) + n_\eta(z), \quad \forall z \in \mathbb{C} \backslash [\gamma^* \cup \eta^*]$$

and

$$n_\gamma(z) = -n_{\bar{\gamma}}(z), \quad \forall z \in \mathbb{C} \backslash \gamma^*.$$

Amazingly, it turns out that the winding number is always an integer:

Theorem 3.3.4. *Let $\gamma : [a, b] \to \mathbb{C}$ be a closed piecewise C^1-path. Then the map $z \mapsto n_\gamma(z)$, defined on $\mathbb{C} \backslash \gamma^*$, is continuous and $\mathbb{Z}$-valued: $n_\gamma(z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \backslash \gamma^*$.*

We need the following lemma in the proof:

Lemma 3.3.5. *Let $\eta : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable at all points $t \in [a, b] \backslash X$, where $X \subset [a, b]$ is finite. If $\eta'(t) = 0$ for all $t \in [a, b] \backslash X$, then η is a constant, and in particular $\eta(b) = \eta(a)$.*

Proof. Since X is finite, we may write $(a, b) \backslash X$ as a finite union of disjoint open intervals $I_1, \ldots, I_n$. It follows from the assumptions that η is a constant on each interval individually, say $\eta \equiv c_j$ on I_j. By continuity, if I_j, I_{j+1} are adjacent intervals with $t \in \bar{I}_j \cap \bar{I}_{j+1}$, then $c_j = \eta(t) = c_{j+1}$. It follows that all the constants c_j are actually the same, and by another appeal to continuity they equal $c_1 = \eta(a)$. □

Proof of Theorem 3.3.4. The continuity of n_γ is fairly clear from the definition: if $\{z_k\}_{k \in \mathbb{N}} \subset \mathbb{C} \backslash \gamma^*$ is a sequence of points converging to a point $z \in \mathbb{C} \backslash \gamma^*$, then

$$\lim_{k \to \infty} n_\gamma(k) = \lim_{k \to \infty} \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z_k} = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z} = n_\gamma(z)$$

It takes a little effort to show that the exchange of limits and integration is legitimate: one has to check that the functions $\zeta \mapsto f_k(\zeta) := (\zeta - z_k)^{-1}$ converge uniformly on γ^* to $\zeta \mapsto f(\zeta) := (\zeta - z)^{-1}$, and then apply Corollary 3.1.10. To check the uniform convergence on γ^*, note that $\operatorname{dist}(z, \gamma^*) =: r > 0$, so also $\operatorname{dist}(z_k, \gamma^*) \geqslant r/2$ for all $k \in \mathbb{N}$ sufficiently large.

We then show that n_γ is $\mathbb{Z}$-valued. Recall from Proposition 2.3.7 (1) that $e^\alpha = 1$ if and only if $\alpha \in 2\pi i\mathbb{Z}$. Thus, to prove that $n_\gamma(z) \in \mathbb{Z}$, it suffices to show that

$$e^{2\pi i n_\gamma(z)} = 1, \quad z \in \mathbb{C} \backslash \gamma^*.$$

Furthermore, if we write down the definition of the path integral appearing in $n_\gamma(z)$, this claim is equivalent to $\varphi(b) = 1$, where

$$\varphi(t) := \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \mathrm{d}s\right), \quad t \in [a, b].$$

(We are suppressing the point "z" from the notation.) To prove $\varphi(b) = 1$, start by noting that

$$\varphi'(t) = \frac{\gamma'(t)}{\gamma(t) - z} \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \mathrm{d}z\right) = \frac{\gamma'(t)}{\gamma(t) - z} \varphi(t),$$

or equivalently

$$\varphi'(t)(\gamma(t)-z)-\gamma'(t)\varphi(t)=0$$

for all $t \in [a,b]$ where γ is differentiable. Since γ is piecewise C^1, this is true for all $t \in [a,b]\backslash X$, where $X \subset [a,b]$ is a finite set. Now, this equation shows that $\eta(t) := \varphi(t)/(\gamma(t)-z)$ defines a continuous map $[a,b] \to \mathbb{C}$, differentiable outside X, and satisfying

$$\eta'(t)=\frac{\varphi'(t)(\gamma(t)-z)-\gamma'(t)\varphi(t)}{(\gamma(t)-z)^2}=0, \quad t \in [a,b]\backslash X.$$

By Lemma 3.3.5 (applied separately to the real and imaginary parts of η), this shows that

$$\frac{\varphi(b)}{\gamma(b)-z}=\eta(b)=\eta(a)=\frac{\varphi(a)}{\gamma(a)-z}=\frac{1}{\gamma(a)-z},$$

noting that $\varphi(a)=e^0=1$ by the definition of φ. Now, it remains to use the assumption that γ is a closed path, so $\gamma(b)=\gamma(a)$:

$$\varphi(b)=\frac{\gamma(b)-z}{\gamma(a)-z}=\frac{\gamma(a)-z}{\gamma(a)-z}=1,$$

as claimed. The proof is complete. □

Remark 3.3.6. *The proof of the theorem was a little magical. The following heuristics may make it more transparent. Assume (for the sake of the discussion) that $z=0$, and there exists a branch of the logarithm "g" (in other words: a primitive of $\zeta \mapsto 1/\zeta$) in some open set U containing γ^*. In this case,*

$$(*): \quad \int_a^t \frac{\gamma'(s)}{\gamma(s)-z}\mathrm{d}s=\int_{\gamma|_{[a,t]}}\frac{\mathrm{d}\zeta}{\zeta} \xlongequal{\text{Thm 3.1.14}} g(\gamma(t))-g(\gamma(a))$$

Consequently, the map "φ" appearing in the proof has the simple expression

$$\varphi(t)=e^{g(\gamma(t))-g(\gamma(a))}=\frac{e^{g(\gamma(t))}}{e^{g(\gamma(a))}}=\frac{\gamma(t)}{\gamma(a)}.$$

In particular, since γ is a closed path, we have $\varphi(b)=1$, as desired. The assumption $z=0$ is innocent, but the existence of "g" is not: indeed, the existence of "g" would actually show that $\int_\gamma \mathrm{d}\zeta/\zeta=0$ in $(*)$, which is generally false. The point of the actual proof is that even though "g" may not exist, the map φ still behaves in the same manner as in the "cheat" proof above.

What do continuous $\mathbb{Z}$-valued functions actually look like?

Proposition 3.3.7. *Let $U \subset \mathbb{C}$ be open and connected, and let $g: U \to \mathbb{R}$ be continuous such that $g(U) \subset \mathbb{Z}$. Then g is constant.*

Proof. Exercise. □

Corollary 3.3.8. *Let $\gamma:[a,b] \to \mathbb{C}$ be a closed piecewise C^1-path, and let $U \subset \mathbb{C}\backslash\gamma^*$ be open and connected. Then n_γ is constant on U. If U is unbounded, this constant is 0.*

Proof. The map $z \mapsto g(z)=n_\gamma(z)$ is continuous and $\mathbb{Z}$-valued in U, so the constancy on U follows immediately from the previous proposition.

Assume then that U is unbounded, and let $\{z_k\}_{k\in\mathbb{N}} \subset U$ be a sequence with $|z_k| \to \infty$ as $k \to \infty$. Let $m \in \mathbb{Z}$ be the constant value of n_ζ on U. Then,

$$m=\lim_{k\to\infty} n_\gamma(z_k)=\lim_{k\to\infty}\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}\zeta}{\zeta-z_k}=0$$

because the functions $\zeta \mapsto (\zeta-z_k)^{-1}$ converge uniformly to 0 on γ^*, as $z_k \to \infty$. □

Example 3.3.9 (Winding numbers of circle paths)**.** Recall that

$$n_{\gamma_k}(z_0) = \frac{1}{2\pi i}\int_{\gamma_k}\frac{\mathrm{d}z}{z-z_0} = k$$

where $z_0 \in \mathbb{C}$, and $\gamma_k(t) = z_0 + re^{ikt}$. From this fact and Corollary 3.3.8, we may deduce that

$$n_{\gamma_k}(w) = \frac{1}{2\pi i}\int_{\gamma_k}\frac{\mathrm{d}z}{z-w} = k, \quad w \in D(z_0, r)$$

since $D(z_0, r) \subset \mathbb{C}\backslash\gamma_k^*$ is connected. Thus, we have rigorously proven the "intuitively clear" fact that the path γ_k winds the same number of times around z_0, and every point $w \in D(z_0, r)$.

For $w \in \mathbb{C}\backslash\bar{D}(z_0, r)$, we have $n_{\gamma_k}(w) = 0$ by Corollary 3.3.8, because $\mathbb{C}\backslash\bar{D}(z_0, r)$ is an unbounded connected set contained in $\mathbb{C}\backslash\gamma_k^*$. We repeat and record these important conclusions:

$$\eta_{\gamma_k}(w) = \begin{cases} k, & w \in D(z_0, r) \\ 0, & w \in \mathbb{C}\backslash\bar{D}(z_0, r) \end{cases}$$

$\diamond$

Proposition 3.3.10. *Let $\gamma, \eta : [0,1] \to \mathbb{C}$ be closed piecewise C^1-paths such that $\gamma(0) = \eta(0)$. Let $z \in \mathbb{C}\backslash[\gamma^* \cup \eta^*]$, and assume that z lies in an unbounded connected open set $U \subset \mathbb{C}\backslash\eta^*$. Then,*

$$n_{\gamma\star\eta}(z) = n_\gamma(z)$$

Proof. We have $n_{\gamma\star\eta}(z) = n_\gamma(z) + n_\eta(z)$, and further $n_\eta(z) = 0$ by Corollary 3.3.8. $\square$

The proposition can for example be applied as follows:

Example 3.3.11. Let $\gamma := [-1,1] \star \sigma^+$ be the "upper semicircle," where $\sigma^+(t) = e^{it}$ for $t \in [0,\pi]$. It is intuitively clear that γ winds exactly once around the point $z = i/2$, thus $n_\gamma(i/2) = 1$. We can deduce this from Proposition 3.3.10 as follows.

Let $\eta := \sigma^- \star \overline{[-1,1]}$ be the "lower semicircle," where $\sigma^-(t) = e^{it}$ for $t \in [\pi, 2\pi]$. Then $\gamma \star \eta$ is the standard circle path $\partial D(0,1)$ (to be precise: a reparametrisation of of $\partial D(0,1)$). Thus,

$$n_{\gamma\star\eta}(i/2) = 1.$$

On the other hand, $i/2$ clearly lies in some unbounded connected set $U \subset \mathbb{C}\backslash\eta^*$, for example $U := \{\mathrm{Im}(z) > 0\}$. Thus $n_\gamma(i/2) = n_{\gamma\star\eta}(i/2) = 1$ by Proposition 3.3.10. $\diamond$

We then arrive at the key result of this section:

Theorem 3.3.12 (Cauchy's integral formula in a convex set)**.** *Let $U \subset \mathbb{C}$ be a convex open set, let $\gamma : [a,b] \to U$ be a closed piecewise C^1-path, and let $f : U \to \mathbb{C}$ be analytic. Then,*

$$f(z) \cdot n_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta, \quad z \in U\backslash\gamma^*$$

Proof. Fix $z \in U\backslash\gamma^*$, and consider the function $g : U \to \mathbb{C}$ defined by

$$g(\zeta) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \in U\backslash\{z\} \\ f'(z), & \zeta = z \end{cases}$$

Clearly g is analytic in $U \backslash \{z\}$ and also continuous in U by Exercise 1.3.13. Consequently, g satisfies the hypotheses of Cauchy's theorem in a convex set. The conclusion is that

$$0=\frac{1}{2\pi i}\int_\gamma g(\zeta)\mathrm{d}\zeta=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)-f(z)}{\zeta-z}\mathrm{d}\zeta$$

Now, it suffices to note that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{\zeta-z}\mathrm{d}\zeta=f(z)\cdot n_\gamma(z)$$

This completes the proof. It is worth remarking that the hypothesis $z\in U\backslash\gamma^*$ was not needed for the first equation, but it was used when passing to second equation (to make sure that the integrals are individually well-defined). □

Corollary 3.3.13 (Cauchy's integral formula in a disc). *Let $U\subset\mathbb{C}$ be open, and assume that $\bar{D}(z_0,r)\subset U$. Then,*

$$f(z)=\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta,\quad z\in D(z_0,r)$$

where ∂D is an abbreviation for the path $\gamma(t)=z_0+re^{it}, t\in[0,2\pi]$.

Example 3.3.14. Let us compute the integral

$$\int_{\partial D(0,2)}\frac{e^z}{z^2-1}\mathrm{d}z$$

This integral is not directly of the form recognisable from Cauchy's integral formula, but it can be easily brought into such a form. We begin by observing that $z^2-1=(z-1)(z+1)$. Once this has been noted, it is well-known that we always have a decomposition

$$\frac{1}{z^2-1}=\frac{A}{z-1}+\frac{B}{z+1}$$

for suitable coefficients $A,B\in\mathbb{C}$. The way to find these coefficients is to write

$$\frac{A}{z-1}+\frac{B}{z+1}=\frac{(z+1)A+(z-1)B}{(z-1)(z+1)}=\frac{(A-B)+z(A+B)}{z^2-1},$$

and see that

$$(A-B)+z(A+B)=1\Longleftrightarrow\left\{\begin{array}{l}A-B=1\\A+B=0\end{array}\right.$$

We get $A=\frac{1}{2}$ and $B=-\frac{1}{2}$. After this, we may decompose the integral as

$$\int_{\partial D(0,2)}\frac{e^z}{z^2-1}\mathrm{d}z=\frac{1}{2}\int_{\partial D(0,2)}\frac{e^z}{z-1}\mathrm{d}z-\frac{1}{2}\int_{\partial D(0,2)}\frac{e^z}{z+1}\mathrm{d}z$$

Since

$$n_{\partial D(0,2)}(1)=n_{\partial D(0,2)}(-1),$$

it follows from Cauchy's integral formula that

$$\frac{1}{2\pi i}\int_{\partial D(0,2)}\frac{e^z}{z-1}\mathrm{d}z=e^1=e\quad\text{and}\quad\frac{1}{2\pi i}\int_{\partial D(0,2)}\frac{e^z}{z-1}\mathrm{d}z=e^{-1}$$

The solution is then

$$\int_{\partial D(0,2)}\frac{e^z}{z^2-1}\mathrm{d}z=\pi i\left(e-e^{-1}\right)$$

◇

In the next example, we compute an indefinite integral on the real line:

Example 3.3.15. Let us compute the integral

$$I=\int_{\mathbb{R}}\frac{\mathrm{d}t}{t^2+1}:=\lim_{r\to\infty}\int_{-r}^{r}\frac{dt}{t^2+1}$$

The correct answer "π" can also be obtained with "real-variable" methods, but now we will see how to use the Cauchy integral formula. Note that $g(z):=\left(z^2+1\right)^{-1}$ defines an analytic function $g:\mathbb{C}\backslash\{-i,i\}\to\mathbb{C}$. Moreover, this function can be factorised as

$$g(z)=\frac{f(z)}{z-i},\quad f(z)=\frac{1}{z+i}$$

and $f:\mathbb{C}\backslash\{-i\}\to\mathbb{C}$ is analytic. Let $U:=\{\operatorname{Im} z>-1\}$. Then U is a convex open set which contains $\mathbb{R}$, the point i, and with the property that f is analytic in U. Now, if $\gamma:[a,b]\to U$ is a closed piecewise C^1-path, we may deduce from Cauchy's theorem that

$$(*):\quad \frac{n_\gamma(i)}{2i}=f(i)\cdot n_\gamma(i)=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-i}\mathrm{d}z=\frac{1}{2\pi i}\int_\gamma\frac{dz}{z^2+1}$$

To wrap up, we need to choose γ suitably.

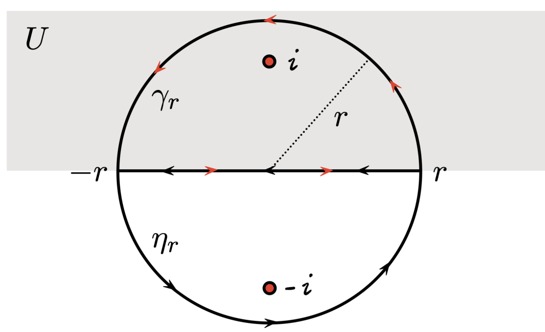


Figure 3.3: The paths γ_r (red arrows) and η_r (black arrows).

Since we are aiming for I, the path $\gamma=\gamma_r$ should at least parametrise the interval $[-r,r]$. However, since γ_r needs to be a closed path, we need to decide some way of "closing" $[-r,r]$. The formula $(*)$ suggests that we might wish to choose γ_r in such a way that $n_{\gamma_r}(i)=1$ (although you are welcome to try other possibilities). A standard choice is the path

$$\gamma_r:=[-r,r]\star\sigma_r,$$

where σ_r parametrises the large semi-circle $S(0,r)\cap\{\operatorname{Im} z\geqslant 0\}$ connecting r to $-r$ in the upper half-plane, see Figure 3.3. Explicitly, $\sigma_r(t)=re^{it}$ for $r\in[0,\pi]$. Clearly $\gamma^*\subset U$, so

$$\frac{1}{2\pi i}\int_{-r}^{r}\frac{dt}{t^2+1}\xlongequal{\beta(t)=t,t\in[-r,r]}\frac{1}{2\pi i}\int_\beta\frac{dz}{z^2+1}=\frac{1}{2\pi i}\int_\gamma\frac{dz}{z^2+1}-\frac{1}{2\pi i}\int_{\sigma_r}\frac{dz}{z^2+1}\overset{(*)}{=}\frac{n_\gamma(i)}{2i}-\frac{1}{2\pi i}\int_{\sigma_r}\frac{dz}{z^2+1}.$$

It is intuitively clear that $n_{\gamma_r}(i)=1$ for $r>1$, and this can be rigorously justified by the trick shown in Example 3.3.11.

Now, we claim that

$$(**):\quad \lim_{r\to\infty}\int_{\sigma_r}\frac{dz}{z^2+1}=0$$

Once this has been established, we find that

$$\int_{\mathbb{R}} \frac{dt}{t^2+1} = \lim_{r\to\infty} \int_{-r}^{r} \frac{dt}{t^2+1} = \lim_{r\to\infty} 2\pi i \cdot \frac{n_{\gamma_r}(i)}{2i} = \pi$$

The claim $(**)$ is obtained with "brute force," using the comparison between path integrals and arc length integrals:

$$\left|\int_{\sigma_r} \frac{dz}{z^2+1}\right| \leqslant \int_0^\pi \frac{|\sigma_r'(t)|}{|\sigma_r(t)^2+1|} dt = \int_0^\pi \frac{r}{|\sigma_r(t)^2+1|} dt$$

Now, note that $\sigma_r(t)$ ranges in the circle $S(0,r)$, so $\left|\sigma_r(t)^2+1\right| \geqslant |\sigma_r(t)|^2 - 1 \geqslant r^2/2$ for every sufficiently large $r>1$. Therefore,

$$\lim_{r\to\infty} \left|\int_{\sigma_r} \frac{dz}{z^2+1}\right| \leqslant \lim_{r\to\infty} \int_0^\pi \frac{2dt}{r} = \lim_{r\to\infty} \frac{2\pi}{r} = 0$$

◇

3.4 Further Consequences

Our first application of Theorem 3.3.12 shows that if f is analytic in an open set U, then f is infinitely differentiable in U.

Theorem 3.4.1. *Let $U \subset \mathbb{C}$ be open, and let $f: U \to \mathbb{C}$ be analytic. Then also $f': U \to \mathbb{C}$ is analytic. As a consequence, the n^{th} complex derivative $f^{(n)}$ exists and is analytic for all $n \geqslant 0$.*

Proof. We first claim the following. Let $z \in U$, and let $r>0$ be so small that $\bar{D}(z,r) \subset U$. Then,

$$f'(w) = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta)}{(\zeta-w)^2} d\zeta, \quad w \in D(z,r) \tag{3.7}$$

Here $\partial D(z,r)$ refers to the circle path $\gamma(t) = z + re^{it}, t \in [0,2\pi]$. We start by applying the Cauchy integral formula in a disc, Corollary 3.3.13, to write

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta-w} d\zeta, \quad w \in D(z,r)$$

Now, morally the formula (3.7) follows by taking ∂_w-derivatives on both sides of the formula above:

$$f'(w) \stackrel{?}{=} \frac{1}{2\pi i} \int_{\partial D(z,r)} \partial_w \left(\frac{1}{\zeta-w}\right) f(\zeta) d\zeta = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta)}{(\zeta-w)^2} d\zeta.$$

We placed a question mark on the first equation, since one needs to make sure that one is allowed to "differentiate under the integral sign": one needs to show that the function $F: D(z,r) \to \mathbb{C}$ defined by

$$F(w) := \int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta-w} d\zeta, \quad w \in D(z,r)$$

is analytic, and its ∂_w-derivative is given by

$$F'(w) = \int_{\partial D(z,r)} \frac{f(\zeta)}{(\zeta-w)^2} d\zeta \tag{3.8}$$

We prove this using the definition of complex differentiability. Abbreviate $\partial D(z,r) =: \partial D$, fix $w \in D$, and start by writing

$$\frac{F(v)-F(w)}{v-w} = \frac{1}{v-w} \int_{\partial D} \left[\frac{1}{\zeta-v} - \frac{1}{\zeta-w}\right] f(\zeta) d\zeta = \int_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta-v)(\zeta-w)}$$

for $v \in D(z,r)$. Consequently,

$$\begin{aligned}\frac{F(v)-F(w)}{v-w}-\int_{\partial D}\frac{f(\zeta)}{(\zeta-w)^2}d\zeta &= \int_{\partial D}\left[\frac{1}{(\zeta-v)(\zeta-w)}-\frac{1}{(\zeta-w)^2}\right]f(\zeta)d\zeta \\ &= (v-w)\int_{\partial D}\frac{f(\zeta)d\zeta}{(\zeta-v)(\zeta-w)^2}\end{aligned}$$

Finally, recall that $w \in D(z,r)$, so in particular $\epsilon := \operatorname{dist}(w,\partial D)>0$. Now, $v \in D\left(w,\frac{\epsilon}{2}\right)$, we have $|\zeta-v| \geqslant \epsilon/2$ for all $\zeta \in \partial D$. Consequently,

$$\left|\frac{F(v)-F(w)}{v-w}-\int_{\partial D}\frac{f(\zeta)}{(\zeta-w)^2}d\zeta\right| \leqslant |v-w|\int_{\partial D}\frac{\|f\|_{L^\infty(D)}|d\zeta|}{|\zeta-v||\zeta-w|^2} \leqslant C_{D,f,\epsilon}|v-w|$$

Here $C_{D,f,\epsilon}>0$ is a constant depending only on D,f,ϵ. An explicit choice which works is $C_{D,f,r}=2\epsilon^{-3}\|f\|_{L^\infty(D)}$ length (∂D). Letting $v \to z$ proves (8.3) at $w=z$. This is not quite the end of the story: we have now proved the nice representation formula (3.8) for f', but this does not immediately say that f' is analytic. However, one can now show that

$$G(w) := \int_{\partial D}\frac{f(\zeta)}{(\zeta-w)^2}d\zeta$$

defines an analytic function in $D(z,r)$, and

$$f''(w)=\frac{1}{2\pi i}G'(w)=\frac{1}{\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta-w)^3}d\zeta \tag{3.9}$$

Thus, the analyticity of f' follows by explicitly differentiating $G=2\pi i f'$. Heuristically, the formula (3.9) for G' follows (again) from differentiation under the integral sign, and the careful justification requires calculations very similar to those we have just seen above. We leave the details as a voluntary exercise. $\square$

Theorem 3.4.2 (Cauchy's integral formula for derivatives (CIFD))**.** *Let $U \subset \mathbb{C}$ be a convex open set, and let $f: U \to \mathbb{C}$ be analytic. Let $\gamma:[a,b] \to U$ be a closed piecewise C^1-path. Then,*

$$f^{(n)}(z)\cdot n_\gamma(z)=\frac{n!}{2\pi i}\int_\gamma\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta, \quad z \in U\backslash\gamma^*, n \geqslant 0 \tag{3.10}$$

Proof. This theorem could be established by a "brute force" approach, using the Cauchy integral formula, and differentiating n times under the integral sign. Ignoring all technical difficulties, the proof can be compressed to the following line:

$$f^{(n)}(z)\cdot n_\gamma(z)=\frac{1}{2\pi i}\int_\gamma \partial_w^{(n)}\left(\frac{1}{\zeta-w}\right)\bigg|_{w=z} f(\zeta)d\zeta=\frac{n!}{2\pi i}\int_\gamma\frac{f(\zeta)}{(\zeta-w)^{n+1}}d\zeta$$

because $\partial_w^{(n)}(\zeta-w)^{-1}=n!(\zeta-w)^{-n-1}$. The main challenge of this approach would to justify the differentiation under the integral sign. This would be a little tedious, but fortunately there is a more elegant path.

We prove the claim by induction on n, the case $n=0$ being Cauchy's integral formula, Theorem 3.3.12. Assume, then, that the formula (3.10) has been established for all analytic functions in U, for some fixed $n \geqslant 0$, and all $z \in U\backslash\gamma^*$. By Theorem 8.1, the derivative $f': U \to \mathbb{C}$ is analytic, so in particular our induction hypothesis applies to f':

$$f^{(n+1)}(z)\cdot n_\gamma(z)=\left(f'\right)^{(n)}(z)\cdot n_\gamma(z)=\frac{n!}{2\pi i}\int_\gamma\frac{f'(\zeta)}{(\zeta-z)^{n+1}}d\zeta, \quad z \in U\backslash\gamma^*$$

The left hand side looks good, but the right hand side still requires processing. We claim that

$$(*): \quad \int_\gamma \frac{f'(\zeta)}{(\zeta-z)^{n+1}} d\zeta = (n+1)\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+2}} d\zeta, \quad z \in U \backslash \gamma^*$$

To prove this, fix $z \in U\backslash\gamma^*$, and consider the function $g : U\backslash\{z\} \to \mathbb{C}$, defined by

$$g(\zeta) := \frac{f(\zeta)}{(\zeta-z)^{n+1}}, \quad \zeta \in U\backslash\{z\}$$

Then g is clearly analytic in $U\backslash\{z\}$, and

$$g'(\zeta) = \frac{f'(\zeta)}{(\zeta-z)^{n+1}} - \frac{(n+1)f(\zeta)}{(\zeta-z)^{n+2}}, \quad \zeta \in U\backslash\{z\}$$

Now g' is continuous in $U\backslash\{z\}$ and has a primitive in $U\backslash\{z\}$ (namely g), so Theorem 3.1.14 implies that

$$0 = \int_\gamma g'(\zeta) d\zeta = \int_\gamma \frac{f'(\zeta)}{(\zeta-z)^{n+1}} d\zeta - (n+1)\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+2}} d\zeta$$

This is equivalent to $(*)$, so the proof is complete. □

Note that the formula (3.10) allows (in principle) you to compute $f^{(n)}$ by integrating f. After seeing this, it is hardly surprising that the size of $f^{(n)}$ can also be estimated by the size of f:

Corollary 3.4.3 (Cauchy's estimates)**.** *Let $D = D(z, r) \subset \mathbb{C}$ be a disc, and let $f : D \to \mathbb{C}$ be analytic. Then,*

$$\left|f^{(n)}(w)\right| \leqslant \frac{n! \cdot r \cdot \|f\|_{L^\infty(D)}}{(r - |w-z|)^{n+1}}, \quad w \in D, n \geqslant 0$$

In particular, $\left|f^{(n)}(z)\right| \leqslant n! \|f\|_{L^\infty(D)}/r^n$.

Proof. Let $0 < s < r$, and let ∂D_s be the usual circle path parametrising the boundary of $D_s := D(z, s)$. Then f is analytic in the convex open set $D = D(z, r)$ containing the trace of ∂D_s, so we may infer from (3.10) that

$$\left|f^{(n)}(w)\right| = \frac{n!}{2\pi}\left|\int_{\partial D_s} \frac{f(\zeta) d\zeta}{(\zeta-w)^{n+1}}\right| \leqslant \frac{n! \cdot \|f\|_{L^\infty(D)}}{2\pi}\int_{\partial D_s} \frac{|d\zeta|}{|\zeta-w|^{n+1}}.$$

Now, note that $|\zeta - w| \geqslant |\zeta - z| - |w - z| = s - |w - z|$ for all $\zeta \in \partial D_s$, so

$$\left|f^{(n)}(w)\right| \leqslant \frac{n! \cdot \|f\|_{L^\infty(D)} \cdot \operatorname{length}(\partial D_s)}{2\pi(s - |w-z|)^{n+1}}.$$

Since length $(\partial D_s) = 2\pi s$, the estimate now follows by letting $s \nearrow r$. □

Sometimes a weaker theorem may appear more surprising than a stronger one. If Cauchy's estimates failed to impress you, perhaps the next corollary of them does:

Corollary 3.4.4 (Liouville's theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and bounded. Then f is constant.*

Proof. Fix $z \in \mathbb{C}$, and apply Cauchy's estimates in a disc $D(z, r) \subset \mathbb{C}$:

$$|f'(z)| \leqslant \frac{\|f\|_{L^\infty(\mathbb{C})}}{r}, \quad r > 0.$$

Letting $r \to \infty$ shows that $f'(z) = 0$, and therefore $f' \equiv 0$. Since $\mathbb{C}$ is connected, f is constant by Corollary 2.2.7. □

Remark 3.4.5. *Liouville's theorem is deep and surprising, but an even stronger result is true: if $f : \mathbb{C} \to \mathbb{C}$ is analytic, then either f is constant, or then f takes all the values in $\mathbb{C}$, except for possibly one. This result is known as Picard's theorem. The example $f(z) = e^z$ shows that Picard's theorem is is sharp, since $e^z \neq 0$ for all $z \in \mathbb{C}$.*

Corollary 3.4.6 (Fundamental theorem of algebra)**.** *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial, where $n \geqslant 1$ and $a_n \neq 0$ (so that p is non-constant). Then there exists $z \in \mathbb{C}$ such that $p(z) = 0$.*

In fact, there exist $z_1, \ldots, z_n \in \mathbb{C}$ that solve the equation $p(z) = 0$.

Proof. We make the counter assumption that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f(z) := p(z)^{-1}$ defines an analytic function on $\mathbb{C}$. We claim that f is bounded. To see this, note that

$$|p(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| \geqslant |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|} \right) \to \infty,$$

as $|z| \to \infty$, using the assumption $a_n \neq 0$. In particular, there exists $R > 0$ (depending on $a_0, \ldots, a_n$) such that we have $|p(z)| \geqslant 1$ for $|z| \geqslant R$. On the other hand, since $p(z) \neq 0$ for $|z| \leqslant R$, there exists $c > 0$ such that $|p(z)| \geqslant c > 0$ for all $|z| \leqslant R$. (Justification: $z \mapsto |p(z)|$ is a continuous function on the compact set $K := \bar{D}(0, R)$, so $K' := \{|p(z)| : z \in K\}$ is also compact set. Since $0 \notin K'$ by assumption, we have $[0, c) \cap K' = \varnothing$ for some $c > 0$. Therefore $|p(z)| \geqslant c$ for all $z \in K$.) Putting these lower bounds for $|p(z)|$ together, we see that $f(z) = p(z)^{-1}$ satisfies

$$\|f\|_{L^\infty(\mathbb{C})} \leqslant \max\left\{c^{-1}, 1\right\}.$$

By Liouville's theorem, we may now deduce that f is constant. This constant is non-zero, since $f(z) = p(z)^{-1} \neq 0$ for all $z \in \mathbb{C}$. So, $f \equiv \alpha \neq 0$. But then $p \equiv \alpha^{-1}$, which contradicts the fact that $|p(z)| \to \infty$ as $|z| \to \infty$.

Assume there exists $z_1 \in \mathbb{C}$ such that $p(z_1) = 0$. Now, it is a fact from abstract algebra we can factorise $p(z) = p_1(z)(z - z_1)$, where p_1 is a polynomial of degree $n - 1$. If $n - 1 \geqslant 1$, we may reapply above process to p_1. Then $p(z) = (z_1 - z)(z_2 - z)p_2(z)$, where p_2 is a polynomial of degree $n - 2$. Repeat the same argument precisely n times to find a factorisation $p(z) = c(z - z_1) \cdots (z - z_n)$, where $p(z_j) = 0$ for all $1 \leqslant j \leqslant n$, and $c \in \mathbb{C} \backslash \{0\}$. Note that there may be repetitions in the roots $z_1, \ldots, z_n$. □

Corollary 3.4.7 (Morera's theorem)**.** *Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be a continuous function satisfying*

$$\int_{\partial \triangle} f(z) dz = 0, \quad \forall \triangle \subset U. \tag{3.11}$$

Then f is analytic in U.

Proof. According to Corollary 3.2.5, the function f has a primitive in every open disc $D \subset U$. In other words, for every $D \subset U$ there exists an analytic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$. Now it follows from Theorem 3.4.1 applied to F that $F' = f$ is analytic in D. Since $D \subset U$ was arbitrary, it follows that f is analytic in U. □

Morera's theorem has a further corollary worth recording:

Corollary 3.4.8 (Analytic continuation to a point)**.** *Let $U \subset \mathbb{C}$ be open, and let $w_0 \in U$. Assume that $f : U \to \mathbb{C}$ is continuous in U and is analytic in $U \backslash \{z_0\}$. Then f is analytic in U.*

Proof. It follows from Cauchy's theorem for triangles (which allowed for one "special point," recall Theorem 3.2.1) that (3.11) holds. The conclusion now follows immediately from Morera's theorem. □

Remark 3.4.9. *Not impressed by the above corollary? To get impressed, consider how terribly a similar result fails in $\mathbb{R}$. For example, the function $f(t) := |t|$ is continuous on $\mathbb{R}$ and infinitely differentiable on $(-\infty, 0) \cup (0, \infty)$, but not differentiable at $t = 0$.*

Corollary 3.4.10 (Mean value principle)**.** *Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be analytic. If $\bar{D}(z, r) \subset U$, then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z + re^{it}\right) dt$$

Proof. This is immediate from Cauchy's integral formula in a disc (Corollary 7.22), and recalling that $\partial D(z, r)$ refers to the circle path $\gamma(t) = z + re^{it}, t \in [0, 2\pi]$:

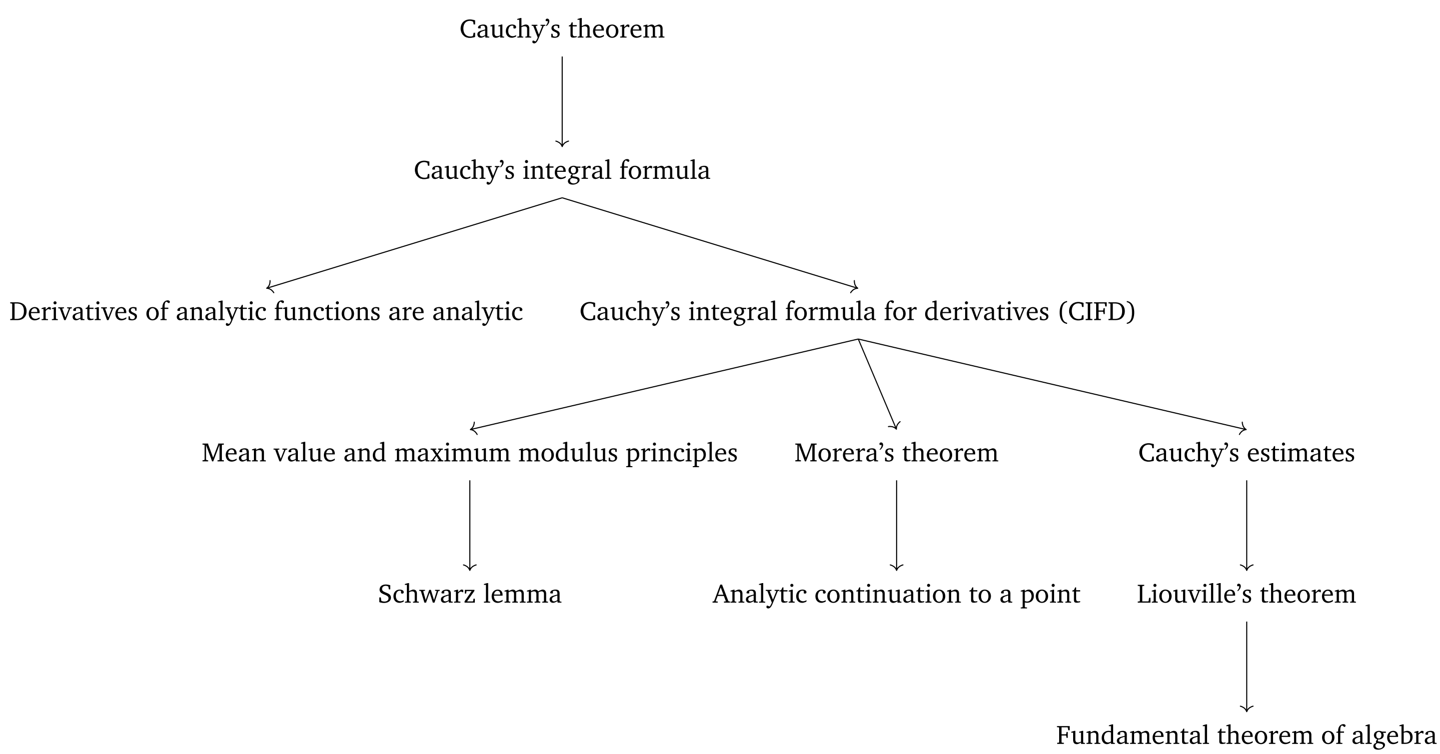
$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t)) i r e^{it}}{re^{it}} dt = \frac{1}{2\pi} \int_0^{2\pi} f\left(z + re^{it}\right) dt$$

This completes the proof. ☐

3.5 A Summary

Let us use $\oint$ to denote the integral over a closed path.

We have seen from fundamental theorem of calculus for paths (Theorem 3.1.14) that the integral $\oint_\gamma f(z) \mathrm{d}z = 0$ for a continuous function $f : U \to \mathbb{C}$ with a primitive in U. The theorem of Cauchy states that the same integral $\oint_\gamma f(z) \mathrm{d}z$ vanishes for every analytic function $f : U \to \mathbb{C}$ in a convex domain $U \subset \mathbb{C}$, even if f does not have a primitive in U. The theorem gives way to Cauchy's integral formula to express $f(z)$ and its derivatives at a point $z \in U$ in terms of integrals over closed paths surrounding z. This in turn leads several powerful consequences such as infinite differentiability of analytic functions, Liouville's theorem, and Morera's theorem. We summarize these results in the chart below:



Let $f : U \to \mathbb{C}$ be continuous, where $U \subset \mathbb{C}$ is convex and open. Consider the following three properties:

(1) f is analytic in U.

(2) $\int_{\partial\triangle} f(z)\mathrm{d}z = 0$ for all triangles $\triangle \subset U$.

(3) f has a primitive in U.

What are the implications between (1), (2), and (3)? Theorem 3.2.3 shows that $(1) \implies (2)$, and Corollary 3.2.5 shows that $(2) \implies (3)$. It turns out that also $(3) \implies (1)$, see Theorem 3.4.1 (applied to the primitive). So, (1) - (3) are equivalent for convex open sets.

The convexity of U was used in both implications $(1) \implies (2)$ and $(2) \implies (3)$, but the converse implications are true for all open $U \subset \mathbb{C}$: namely, $(3) \implies (2)$ by Remark 3.2.6, and the implication $(2) \implies (1)$ is the Morera's theorem.

Chapter 4

Power Series

4.1 Limsup and Liminf

The limit superior of a sequence $\{x_n\}$ in $\mathbb{R}$ is defined by

$$\overline{\lim_{n\to\infty}}\, x_n = \limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right) = \inf_{n\geq 0}\left(\sup_{m\geq n} x_m\right)$$

The limit inferior of a sequence $\{x_n\}$ in $\mathbb{R}$ is defined by

$$\underline{\lim}_{n\to\infty}\, x_n = \liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right) = \sup_{n\geq 0}\left(\inf_{m\geq n} x_m\right)$$

When $\{x_n\}$ has no upper bound, we say $\overline{\lim}_{n\to\infty} x_n = +\infty$; when $\{x_n\}$ has no lower bound, we say $\underline{\lim}_{n\to\infty} x_n = -\infty$.

Theorem 4.1.1. *Let $H = \overline{\lim}\, x_n$. Then*

(a) When H is finite, there are infinitely many x_n falling in the interval $(H-\varepsilon, H+\varepsilon)$ for any $\varepsilon > 0$, while there are only finitely many (or even zero) x_n falling in $(H+\varepsilon, +\infty)$.

(b) When $H = +\infty$, for any $N > 0$, there are infinitely many x_n such that $x_n > N$.

(c) When $H = -\infty$, $\lim x_n = -\infty$.

Proof.

(a) $-\infty < H < +\infty$: the statement will be proved if we show that for any $\varepsilon > 0$ there are infinitely many terms x_n greater than $H-\varepsilon$ and only finitely many terms x_n greater than $H+\varepsilon$. We show the first part: BWOC, suppose there is some $\varepsilon_0 > 0$ s.t. there are only finitely many x_n greater than $H-\varepsilon_0$, say $x_{n_1}, \cdots, x_{n_k}$. Thus, $x_n \leq H-\varepsilon_0$ for all $n > n_k$. Therefore, for all $n > n_k$, the supremums have

$$\beta_n = \sup_{m\geq n} x_m = \sup\{x_n, x_{n+1}, \cdots\} \leq H - \varepsilon_0$$

Thus,

$$H = \overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \beta_n \leq H - \varepsilon_0$$

which is a contradiction. We show the second part: let $\beta_n = \sup_{m\geq n} x_m$. Since $\lim_{n\to\infty}\beta_n = H$, $\forall\varepsilon$, $\exists N\in\mathbb{N}$ s.t. $|\beta_n - H| < \varepsilon$, i.e., $H-\varepsilon < \beta_n < H+\varepsilon$. Since β_n is supremum of $\{x_n, x_{n+1}, \cdots\}$, we see when $n > N$,

$$\forall k\in\mathbb{N},\ x_{n+k} \leq \beta_n \leq H+\varepsilon$$

Thus, those x_n with $x_n > H+\varepsilon$ must have $n \leq N$, which shows that there are only finitely many x_n satisfying $x_n > H+\varepsilon$.

(b) That's because $H = +\infty$ when $\{x_n\}$ has no upper bound by definition.

(c) When $H = -\infty$, for any $G > 0$, there exists n_0, when $n > n_0$, $x_{n+1} \leq \beta_n \leq -G$, so $\lim x_n = -\infty$.

□

We have a liminf counterpart of the above theorem:

Theorem 4.1.2. *Let $h = \underline{\lim}\, x_n$. Then*

(a) When h is finite, there are infinitely many x_n falling in the interval $(h-\varepsilon, h+\varepsilon)$ for any $\varepsilon > 0$, while there are only finitely many (or even zero) x_n falling in $(-\infty, h-\varepsilon)$.

(b) When $h = -\infty$, for any $N > 0$, there are infinitely many x_n such that $x_n < -N$.

(c) When $h = +\infty$, $\lim x_n = +\infty$.

Another useful theorem is

Theorem 4.1.3. *For limsup H and liminf h of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ with limit H and H is the largest among all limits of convergent subsequences of $\{x_n\}$; there also exists a subsequence $\{x_{n_k}\}$ with limit h and h is the smallest among all limits of convergent subsequences of $\{x_n\}$;*

Corollary 4.1.4. $\lim x_n = A$ *(finite or infinite) iff* $\overline{\lim}\, x_n = \underline{\lim}\, x_n = A$.

Example 4.1.5. $a_n = n + (-1)^n n\,(n = 1, 2, 3, \cdots)$. It only has two subsequences with limit (including ∞): a_{2k} and $a_{2k+1}(k = 1, 2, 3, \cdots)$. The limits are respectively $+\infty$ and 0, so

$$\overline{\lim_{n\to\infty}}\, a_n = +\infty,\ \underline{\lim_{n\to\infty}}\, a_n = 0$$

◇

Example 4.1.6. $a_n = \cos\frac{n}{4}\pi\,(n = 0, 1, 2, \cdots)$. Since $-1 \leqslant \cos\frac{n}{4}\pi \leqslant 1$, when $n = 8k(k = 1, 2, \cdots)$, $a_{8k} \to 1(k\to\infty)$; when $n = 4(2k+1)$, $(k = 1, 2, 3, \cdots)$, $a_{4(2k+1)} \to -1(k\to\infty)$. Thus,

$$\overline{\lim_{n\to\infty}}\, a_n = 1,\ \underline{\lim_{n\to\infty}}\, a_n = -1$$

◇

Proposition 4.1.7. *suppose* $\lim_{n\to\infty} x_n = x, -\infty < x < 0$. *Then*

$$\overline{\lim_{n\to\infty}}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \underline{\lim_{n\to\infty}}\, y_n;$$
$$\underline{\lim_{n\to\infty}}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}}\, y_n.$$

suppose $\lim_{n\to\infty} x_n = x, 0 < x < \infty$. *Then*

$$\overline{\lim_{n\to\infty}}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}}\, y_n;$$
$$\underline{\lim_{n\to\infty}}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \underline{\lim_{n\to\infty}}\, y_n.$$

Proof. We prove the first two equations. The others are similar. $\lim x_n = x, -\infty < x < 0$, so for any given $\varepsilon (0 < \varepsilon < -x)$, there exists positive integer N_1 such that for all $n > N_1$,

$$x - \varepsilon < x_n < x + \varepsilon < 0.$$

Let $\overline{\lim}\, y_n = H, \underline{\lim}\, y_n = h$. Then for the above $\varepsilon (0 < \varepsilon < -x)$, there exists N_2 such that for all $n > N_2$,

$$h - \varepsilon < y_n < H + \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$\min\{(x-\varepsilon)(H+\varepsilon), (x+\varepsilon)(H+\varepsilon)\} < x_n y_n < \max\{(x-\varepsilon)(h-\varepsilon), (x+\varepsilon)(h-\varepsilon)\},$$

Thus,

$$\begin{aligned} \underline{\lim}_{n\to\infty} (x_n y_n) &\geqslant \min\{(x-\varepsilon)(H+\varepsilon), (x+\varepsilon)(H+\varepsilon)\}, \\ \overline{\lim}_{n\to\infty} (x_n y_n) &\leqslant \max\{(x-\varepsilon)(h-\varepsilon), (x+\varepsilon)(h-\varepsilon)\}, \end{aligned}$$

By arbitrariness of ε, we get

$$\begin{aligned} \underline{\lim}_{n\to\infty} (x_n y_n) &\geqslant xH = \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n, \\ \overline{\lim}_{n\to\infty} (x_n y_n) &\leqslant xh = \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n. \end{aligned}$$

Since

$$\begin{aligned} \underline{\lim}_{n\to\infty} y_n &= \underline{\lim}_{n\to\infty} \left[\frac{1}{x_n} \cdot (x_n y_n)\right] \geqslant \lim_{n\to\infty} \frac{1}{x_n} \cdot \overline{\lim}_{n\to\infty} (x_n y_n), \\ \overline{\lim}_{n\to\infty} y_n &= \overline{\lim}_{n\to\infty} \left[\frac{1}{x_n} \cdot (x_n y_n)\right] \leqslant \lim_{n\to\infty} \frac{1}{x_n} \cdot \underline{\lim}_{n\to\infty} (x_n y_n), \end{aligned}$$

we have

$$\begin{aligned} \underline{\lim}_{n\to\infty} (x_n y_n) &\leqslant \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n, \\ \overline{\lim}_{n\to\infty} (x_n y_n) &\geqslant \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n. \end{aligned}$$

Combine the last four equations to get

$$\begin{aligned} \overline{\lim}_{n\to\infty} (x_n, y_n) &= \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n \\ \underline{\lim}_{n\to\infty} (x_n y_n) &= \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n. \end{aligned}$$

□

Proposition 4.1.8.

$$\overline{\lim}_{n\to\infty} (cx_n) = \begin{cases} c\,\overline{\lim}_{n\to\infty} x_n & c > 0 \\ c\,\underline{\lim}_{n\to\infty} x_n & c < 0 \end{cases}$$

Similarly,

$$\underline{\lim}_{n\to\infty} (cx_n) = \begin{cases} c\,\underline{\lim}_{n\to\infty} x_n & c > 0 \\ c\,\overline{\lim}_{n\to\infty} x_n & c < 0 \end{cases}$$

In particular, c can be -1.

Proof. This is due to the similar result from supremum and infimum. For example,

$$\overline{\lim_{n\to\infty}}(cx_n) = \lim_{n\to\infty}\sup_{m\geq n}(cx_m) = \begin{cases} \lim_{n\to\infty}(c\cdot\sup_{m\geq n} x_m) = c\lim_{n\to\infty}\sup_{m\geq n} x_m & c>0 \\ \lim_{n\to\infty}(c\cdot\inf_{m\geq n} x_m) = c\lim_{n\to\infty}\inf_{m\geq n} x_m & c<0 \end{cases}$$

□

In fact, once we have $\underline{\lim}\, x_n = -\overline{\lim}(-x_n)$, we only need to state limsup half in the following observations (for 4.1.9 we have $\underline{\lim}(x_n+y_n) \geq \underline{\lim}\, x_n + \underline{\lim}\, y_n$; for 4.1.10 we have $\underline{\lim}(x_n y_n) \leq (\underline{\lim}\, x_n)(\underline{\lim}\, y_n)$). We omit the proofs.

Proposition 4.1.9. *If $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers, then*

$$\limsup_{n\to\infty}(x_n+y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$

provided the sum on the right is well defined (i.e., excluding the case where one summand is ∞ and the other is $-\infty$). If one of the sequences converges then the equality holds (with the same proviso).

Proposition 4.1.10. *If $\{x_n\}$ and $\{y_n\}$ are two sequences of positive real numbers, then*

$$\limsup_{n\to\infty}(x_n y_n) \leq \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right),$$

provided the product on the right is well defined (i.e., excluding the case where one factor is 0 and the other is ∞). If one of the sequences converges then the equality holds (with the same proviso).

4.2 Series

4.2.1 Comparison Test

Suppose we have two *nonngegative* series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$, and they have the following relationship:

$$\exists c>0, \text{ s.t. } u_n \leq cv_n \quad n=k,k+1,k+2,\cdots$$

for some k, i.e., each term of the first series is dominated by the second after $(k-1)$-th term. Since partial sum sequence of nonnegative series converges iff the sequence is bounded, one observes

- $\sum v_n$ converges $\Rightarrow \sum u_n$ converges;
- $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges.

We have the functional version: for real functions $0 \leq f_n(x) \leq g_n(x)$,

- $\sum g_n(x)$ pointwise/uniformly converges $\Rightarrow \sum f_n(x)$ pointwise/uniformly converges;
- $\sum f_n(x)$ diverges $\Rightarrow \sum g_n(x)$ diverges.

where the pointwise one follows immediately from the number series version and the uniform one follows from Cauchy criterion for uniform convergence Let $\varepsilon>0$, then exists $N\in\mathbb{N}$ such that for any $m,n\in\mathbb{N}$ if $N\leq m\leq n$ then

$$\left|\sum_{k=m}^{n} f_k\right| \leq \left|\sum_{k=m}^{n} g_k\right| < \varepsilon$$

Then $\sum f_n$ is uniformly convergent.

4.2.2 Series of Complex Numbers

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$, then the series $\sum_{n=1}^{\infty} z_n$ converges to z iff the sequence of partial sums $\{S_N(z_n)\} = \left\{\sum_{n=1}^{N} z_n\right\}$ converges to z, i.e.,

$$\forall \varepsilon > 0,\ \exists K \in \mathbb{N},\ \text{s.t.}\ \forall N > K:\ |S_N(z_n) - z| < \varepsilon.$$

We say the seires $\sum z_n$ converges absolutely if $\sum |z_n|$ converges. Note that $\{|z_n|\}$ is a nonnegative sequence and $\{S_N(|z_n|)\} = \left\{\sum_{n=1}^{N} |z_n|\right\}$ is a monotone (nonstrictly) increasing sequence, so boundedness of $\{S_N(|z_n|)\}$ is a sufficient condition for convergence of $\sum |z_n|$. Two basic facts are presented first:

Theorem 4.2.1. *If the series $\sum_{n=1}^{\infty} z_n$ converges, then $z_n \to 0$ as $n \to \infty$.*

Proof. Since $\lim_{N\to\infty} S_N(z_n) = z$ for some z, we get $\lim z_n = \lim(S_N(z_n) - S_{N-1}(z_n)) = z - z = 0$. □

Theorem 4.2.2. *If $\sum z_n$ converges absolutely, $\sum z_n$ converges.*

Proof. $\sum z_n$ converges absolutely, so for any $\varepsilon > 0$ there is K s.t. $M, N > K$ (WLOG, $M > N$) implies

$$\varepsilon > |S_M(|z_n|) - S_N(|z_n|)| = ||z_{M+1}| + \cdots + |z_N|| \geq |z_{M+1} + \cdots + z_N| = |S_M(z_n) - S_N(z_n)|$$

so $\{S_N(z_n)\}$ is Cauchy and thus converges by completeness of $\mathbb{C}$. □

Complex series can relate to real series in one way:

Theorem 4.2.3. *Let $z_n = a_n + ib_n$ $(n = 1, 2, \cdots)$, where a_n and b_n are real numbers. Then the series $\sum_{n=1}^{\infty} z_n$ converges to $z = a + ib$ for real numbers a and b iff $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$.*

Proof. Apply [11] Proposition 3.5 to the sequence $S_N(z_n) = A_n + iB_N = \left(\sum_{n=1}^{N} a_n\right) + i\left(\sum_{n=1}^{N} b_n\right)$. □

Example 4.2.4. Consider the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{i}{2^n}\right)$. $\sum \frac{1}{n}$ diverges. Thus, even though $\sum \frac{i}{2^n}$ converges, the whole series diverges. ◇

4.2.3 Sequences of Complex Functions

Consider a sequence of functions $\{f_n(z)\}_{n=1}^{\infty}$ commonly defined on a set $E \subseteq \mathbb{C}$. Several notions of convergence are defined:

1. **Pointwise convergence (PC)**: $\forall \varepsilon > 0, \forall z \in E, \exists N(\varepsilon, z) \in \mathbb{N}$, s.t. $\forall n > N:\ |f_n(z) - f(z)| < \varepsilon$;
2. **Uniform convergence (UC)**: $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$, s.t. $\forall z \in E, \forall n > N:\ |f_n(z) - f(z)| < \varepsilon$;

2′. [Equivalent definition of uniform convergence]: $\sup\{|f_n(z) - f(z)| : z \in E\} \to 0$ as $n \to \infty$.

3. **Absolute convergence (AC)**: pointwise convergence of $\{|f_n(z)|\}_{n=1}^{\infty}$.
4. **Local uniform convergence (LUC)**: $\forall z \in E$, there is a neighborhood U of z in E such that the sequence $\{f_n(z)\}_{n=1}^{\infty}$ converges uniformly on U.
5. **Compact convergence (CC)**: For each compact set $K \subseteq E$, the sequence $\{f_n(z)\}_{n=1}^{\infty}$ converges uniformly on K.

It turns out that for functions on open subset $E \subseteq \mathbb{C}$, the last two notions are equivalent.

Proposition 4.2.5. *Let E be an open set in $\mathbb{C}$. Then the sequence $\{f_n(z)\}_{n=1}^{\infty}$ converges locally uniformly in E iff it converges compactly in E.*

Proof. $\Rightarrow$ is by Heine-Borel theorem. $\Leftarrow$ is because each point has an open disc around it whose closure is compact and contained in E. □

Proposition 4.2.6. *Sequence $\{f_n(z)\}_{n=1}^{\infty}$ converges compactly in $B(a,R)$ iff it converges in $\overline{B}(a,r)$ for every $0<r<R$.*

Proof. Each closed disk is compact. Each compact set is closed and bounded and is thus contained in some closed disk in $B(a,R)$. □

Example 4.2.7. A simple example, with E the open unit disk, is provided by the sequence $f_n(z)=z^n$. We notice that $\sup\{|z^n| : z\in B(0,1)\} = \sup\{|z|^n : |z|\in[0,1)\} = \sup[0,1) = 1$. Thus, $\{z^n\}$ does not uniformly converge. However, if $0<r_0<1$ then this sequence converges uniformly to 0 in the disk $|z|<r_0$, and so it converges locally uniformly to 0 in the disk $|z|<1$. ◇

4.2.4 Series of Complex Functions

Consider a series $\sum_{n=1}^{\infty} f_n(z)$ with its functions commonly defined on a set $E\subseteq\mathbb{C}$. We say the series converges pointwise/absolutely/uniformly/locally uniformly/compactly if the sequence of partial sums $\{S_N(f_n(z))\}$ does so. We restate them:

1. **Pointwise convergence (PC)**: $\forall\varepsilon>0, \forall z\in E, \exists K(\varepsilon,z)\in\mathbb{N}, \text{s.t.} \forall N>K : |S_N(f_n(z))(z)-f(z)|<\varepsilon$;
2. **Uniform convergence (UC)**: $\forall\varepsilon>0, \exists K(\varepsilon)\in\mathbb{N}, \text{s.t.} \forall z\in E, \forall N>K : |S_N(f_n(z))(z)-f(z)|<\varepsilon$;
3. **Absolute convergence (AC)**: pointwise convergence of $\sum_{n=1}^{\infty}|f_n(z)|$, i.e., $\{S_N(|f_n(z)|)\}_{N=1}^{\infty}$.
4. **Local uniform convergence (LUC)**: $\forall z\in E$, there is a neighborhood U of z in E such that the sequence $\{S_N(f_n(z))\}_{N=1}^{\infty}$ converges uniformly on U.
5. **Compact convergence (CC)**: For each compact set $K\subseteq E$, the sequence $\{S_N(f_n(z))\}_{N=1}^{\infty}$ converges uniformly on K.

To prove uniform convergence, one usually strengthens the inequality by finding $P_n(z)$ and Q_n such that

$$|S_N(f_n(z))-f(z)|\le P_n(z)\le Q_n$$

and then finding K for which it is true under $N>K$ by $Q_n<\varepsilon$. To show a series of functions is not uniform convergent, one proves the negation, which is obtained by switching existential and universal quantifiers and negating the statement:

$$\exists\varepsilon>0, \forall K\in\mathbb{N}, \exists z_0\in E, \exists N_0>K \text{ s.t. } |S_{N_0}(f_n(z))-f(z_0)|>\varepsilon$$

The following criterion makes the sup norm a complete metric:

Theorem 4.2.8. *[Cauchy criterion for uniform convergence] The series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on E iff $\forall\varepsilon>0$, $\exists N(\varepsilon)\in\mathbb{N}$, such that $\forall z\in E$,*

$$|f_{n+1}(z)+\cdots+f_{n+p}(z)|<\varepsilon \quad (p=1,2,\cdots)$$

Theorem 4.2.9. *[Weierstrass M-test] Suppose that $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of complex-valued functions defined on a set E, and that there is a sequence of non-negative numbers M_n satisfying the conditions*

- *$|f_n(z)|\le M_n$ for all $n\ge 0$ and all $z\in E$;*
- *$\sum_{n=1}^{\infty} M_n$ converges.*

Then the series $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on E.

Proof. Convergence is absolute by comparison test. It is also uniform by applying Cauchy criterion for uniform convergence to the following

$$|f_n(z) + \cdots + f_{n+p}(z)| \leq |f_n(z)| + \cdots + |f_{n+p}(z)| \leq M_n + \cdots + M_{n+p}$$

□

Weierstrass M-test is often used in combination with the **uniform limit theorem** (see, e.g., [10] Theorem 21.6). Together they say that if, in addition to the above conditions, the functions f_n are continuous on E, then the series converges to a continuous function $f(z)$. A natural question of concern is about the convergence of the termwise differentiation and integration of the sum and that of the limit function.

In Corollary 3.1.10 we see uniform convergence allows us to pass the limit under the integral sign. The condition can be weakened to local uniform convergence (or equivalently compact convergence) since image of curve γ is compact.

Theorem 4.2.10 (Termwize Integration of Series)**.** *Suppose $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of complex-valued functions defined on a set $E \subseteq \mathbb{C}$, and that the series $\sum_{n=1}^{\infty} f_n(z)$ converges locally uniformly to a function $f(z)$ on E. If γ is a piecewise-C^1 curve in E, then*

$$\int_\gamma f(z)\, dz = \lim_{N\to\infty} \int_\gamma S_N(f_n)(z)\, dz.$$

That is,

$$\int_\gamma \sum_{n=1}^{\infty} f_n(z)\, dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)\, dz.$$

Counterpart for differentiation needs theorem 4.2.6, which translates in terms of series as

Proposition 4.2.11. *Series $\sum_{n=1}^{\infty} f_n(z)$ converges compactly in $B(a, R)$, i.e., converges uniformly for each compact set in $B(a, R)$, iff it converges in $\overline{B}(a, r)$ for every $0 < r < R$.*

Example 4.2.12. [Geometric series] $\sum_{n=1}^{\infty} z^n$.

Consider $\sum_{n=1}^{\infty} |z|^n$. Since $1 - |z|^{N+1} = (1 - |z|)(1 + |z| + \cdots + |z|^N)$, we have $\sum_{n=1}^{N} |z|^n = \frac{1-|z|^{N+1}}{1-|z|}$. If $|z| < 1$, then the series converges absolutely. Replacing $|z|$ with z in the above argument, one obtains the limit function $\frac{1}{1-z}$. The convergence is also compact in the disk $|z| < 1$: for every closed disk $\overline{B}(0, r)$ with $0 < r < 1$ the series converges uniformly by applying Weierstrass M-test to $|z^n| \leq r^n$. If $|z| > 1$, then $\lim |z|^n = \infty$ and the series diverges.

However, the series does not converge uniformly on $B(0, 1)$. This is a simple consequence of the fact that each function $S_N(z^n) = \sum_{n=1}^{N} z^n$ is bounded while the limit function $f(z) = \frac{1}{1-z}$ is not. Hence each function $S_N - f$ is unbounded, that is, the sup-norm of $S_k - S$ is infinite, in particular the sequence of the sup-norms does not converge to zero. This last assertion is equivalent to the fact that $\{S_N\}$ does not converge uniformly to f. ⋄

Theorem 4.2.13. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that compactly converges to a function f in region E (connected open subset of $\mathbb{C}$), then f is holomorphic in E.*

Proof. Let D be any disc whose closure is contained in E and $\triangle$ any triangle in that disc. Then, since each f_n is holomorphic, Theorem 3.2.1 implies

$$\int_{\partial\triangle} f_n(z) dz = 0 \quad \text{for all } n$$

By assumption $f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\int_{\partial\triangle} f_n(z)dz \to \int_{\partial\triangle} f(z)dz$$

Thus, we have

$$\int_{\partial\triangle} f(z)dz = 0$$

for every triangle $\triangle$ in D. Therefore, by Morera's theorem, f is holomorphic in D. Since D was arbitrary, f is holomorphic in E. $\square$

This is a striking result that is obviously not true in the case of real variables: the uniform limit of continuously differentiable functions need not be differentiable. For example, we know that every continuous function on $[0, 1]$ can be approximated uniformly by polynomials, by Weierstrass's theorem, yet not every continuous function is differentiable. For a criterion by which the limit function is differentiable, see [13] Theorem 7.17.

Theorem 4.2.14. *Under the hypotheses of the previous theorem, the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ compactly converges to f' in region E. As a consequence, for every $k \geq 0$ the sequence of k^{th} derivatives $\left\{f_n^{(k)}\right\}_{n=1}^{\infty}$ compactly converges to $f^{(k)}$ in E.*

Proof. We may assume without loss of generality that the sequence of functions in the theorem converges uniformly on all of E. Given $r > 0$, let E_r denote the subset of E defined by

$$E_r = \left\{z \in E : \overline{D(z, r)} \subset E\right\}.$$

In other words, E_r consists of all points in E which are at distance $> r$ from its boundary. To prove the theorem, it suffices to show that $\{f_n'\}$ converges uniformly to f' on E_r for each r. This is achieved by proving the following inequality:

$$\sup_{z\in E_r} |F'(z)| \leq \frac{1}{r} \sup_{\zeta\in E} |F(\zeta)|$$

whenever F is holomorphic in E, since it can then be applied to $F = f_n - f$ to prove the desired fact. The inequality follows at once from the Cauchy's estimate: $|F'(z)| \leq \frac{1}{r}\|f\|_{L^\infty(D(z,r))} \leq \frac{1}{r}\sup_{\zeta\in E} |F(\zeta)|$ for each $z \in E_r$. $\square$

The following theorem is a direct consequence of Theorems 4.2.13 and 4.2.14.

Theorem 4.2.15 (Termwize Differentiation of Series)**.** *Suppose we have*

(a) a sequence of functions $f_n(z)$ $(n = 1, 2, \ldots)$ that are analytic in the region E;

(b) the series $\sum_{n=1}^{\infty} f_n(z)$ converges compactly to the function $f(z)$ in region E: $f(z) = \sum_{n=1}^{\infty} f_n(z)$.

Then

(1) the function $f(z)$ is analytic in the region E;

(2) The series $\sum_{n=1}^{\infty} f_n^{(p)}(z)$ converges compactly to $f^{(p)}(z)$ in E for every $p = 1, 2, \cdots$.

In practice, one often uses Theorem 4.2.15 to construct holomorphic functions (say, with a prescribed property) as a series

$$F(z) = \sum_{n=1}^{\infty} f_n(z). \tag{4.1}$$

For instance, various special functions are often expressed in terms of a converging series. A specific example is the Riemann zeta function.

4.3 Power Series

A power series about a is an infinite series of the form $\sum_{n=1}^{\infty} c_n(z-a)^n$. If a power series converges to a function f in a given region, we shall say that the series represents f in that region. It is important, however, to distinguish the series from the function it represents. For example, as shown in the last section, the series $\sum_{n=1}^{\infty} z^n$ represents the function $\frac{1}{1-z}$ in the disk $|z|<1$. However, the series is not "equal" to the function, in a formal sense, even though we can write $\sum_{n=1}^{\infty} z^n = \frac{1}{1-z}$ for $|z|<1$. The same function is represented by other power series; for example, it is represented by the series $\sum_{n=1}^{\infty} 2^{-n-1}(z+1)^n$ in the larger disk $|z+1|<2$, as the reader will easily verify. A power series is best thought of as a formal sum, uniquely determined once its center and its coefficients have been specified.

4.3.1 Power Series Representation of Analytic Function

Our final goal of this subsection is to show that a power series is analytic and that an analytic function can be represented by power series. We first continue studying the convergence of series.

Proposition 4.3.1. *[Abel's Theorem] If the power series $\sum_{n=1}^{\infty} c_n(z-a)^n$ converges on a point $z_1(\neq a)$, then it converges absolutely and compactly in the open disk with center a and radius $|z_1-a|$, i.e., $D: |z-a|<|z_1-a|$. The series diverges for $|z-a|>|z_1-a|$ instead.*

Proof. Let z be any point in D. Since $\sum_0^{\infty} c_n\left(z_1-a\right)^n$ converges, each of its term must be bounded: $\exists M>0$ such that

$$\left|c_n\left(z_1-a\right)^n\right| \leq M \quad (n=0,1,2,\cdots),$$

Therefore,

$$\left|c_n(z-a)^n\right| = \left|c_n\left(z_1-a\right)^n\left(\frac{z-a}{z_1-a}\right)^n\right| \leq M\left|\frac{z-a}{z_1-a}\right|^n,$$

Since $\forall z \in D$, $|z-a|<|z_1-a| \Rightarrow \left|\frac{z-a}{z_1-a}\right|<1 \Rightarrow$ the geometric series

$$\sum_{n=1}^{\infty} M\left|\frac{z-a}{z_1-a}\right|^n$$

converges for each $z \in D$. Thus $\sum c_n(z-a)^n$ converges absolutely in D. Besides, for any closed disk $\overline{D}_\rho$ in D, $\overline{D}_\rho: |z-a| \leq \rho\,(0<\rho<|z_1-a|)$, we have

$$\left|c_n(z-a)^n\right| \leq M\left|\frac{z-a}{z_1-a}\right|^n \leq M\left(\frac{\rho}{|z_1-a|}\right)^n,$$

By convergence of the last geometric series, apply Weierstrass M-test to see $\sum_0^{\infty} c_n(z-a)^n$ converges uniformly in $\overline{D}_\rho$. Then use 4.2.11. The fact that The series diverges for $|z-a|>|z_1-a|$ instead is proved by way of contradiction. □

If a power series has no such $z_1 \neq a$ for which it converges, then the series only converges at $z=a$. For example

$$1+z+2^2z^2+\cdots+n^nz^n+\cdots$$

only converges at $z=0$.

The power series can also converge (pointwise) for all z. For example

$$1+z+\frac{z^2}{2^2}+\cdots+\frac{z^n}{n^n}+\cdots.$$

For any fixed z, after some n, it has $\frac{|z|}{n} < \frac{1}{2}$. Thus, $\left|\frac{z^n}{n^n}\right| < \left(\frac{1}{2}\right)^n$ shows that it is dominated by a convergent geometric series for every z. The convergence is absolute and compact.

By Abel's theorem, if the power series does not fall into the above two cases, then the power series converges for at least $|z-a| < |z_1 - a|$ for each z_1 on which it converges (pointwise). Then we can let R be the supremum of $|z_1 - a|$ ranging over all z_1 on which it converges (pointwise) and see that the series converges for $|z-a| < R$ and diverges for $|z-a| > R$. Obviously, a number R making the series "converge for $|z-a| < R$ and diverge for $|z-a| > R$" is unique.

Definition 4.3.2. *Let $\sum_{n=1}^{\infty} c_n (z-a)^n$ be a power series. We define the* ***radius of convergence*** *R of the series as the unique number such that the series converges in $|z-a| < R$ and diverges in $|z-a| > R$. If $R > 0$ then the series converges absolutely and compactly in the disk $|z-a| < R$; if $R < \infty$ then the series diverges at each point of the region $|z-a| > R$. R does not give information for situation on the circle $|z-a| = R$.*

Presently we shall obtain a general expression for the radius of convergence of a power series in terms of its coefficients.

Theorem 4.3.3 (Cauchy-Hadamard Theorem)**.** *Consider a power series $\sum_{n=1}^{\infty} c_n(z-a)^n$. Then the radius of convergence R is given by*

$$R = \left(\limsup_{n\to\infty} |c_n|^{1/n}\right)^{-1}. \tag{4.2}$$

That is, it satisfies

(a) The series converges absolutely for $|z-a| < R$;

(b) The series diverges for $|z-a| > R$;

(c) The series converges for every closed disk $|z-a| < r$ where $r < R$.

Proof. Assume $a = 0$. Assume $0 < R < \infty$ (the edge cases $R = 0$ and $R = \infty$ are left as an exercise). Due to 4.1.1 (a),

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n > N \Rightarrow \frac{1}{R} - \varepsilon < |c_n|^{1/n} < \frac{1}{R} + \varepsilon. \tag{4.3}$$

So $|c_n| < \left(\frac{1}{R} + \varepsilon\right)^n$ for $n > N$. Let $z \in B(0, R)$, i.e., $|z| < R$, we have $|z|\left(\frac{1}{R} + \varepsilon\right) < 1$ for some fixed $\varepsilon > 0$ chosen small enough. That implies that for $n > N$ (for some large enough N as a function of ε),

$$\sum_{n=N}^{\infty} |c_n z^n| < \sum_{n=N}^{\infty} \left[\left(\frac{1}{R} + \varepsilon\right)|z|\right]^n,$$

so the series is dominated by a convergent geometric series, and hence converges. For (b), when $|z| > R$, we evoke the other side of (4.3): $|c_n| > \left(\frac{1}{R} - \varepsilon\right)^n$ for $n > N$. Besides, $|z|\left(\frac{1}{R} - \varepsilon\right) > 1$ for some small enough fixed $\varepsilon > 0$. Thus

$$\sum_{n=N}^{\infty} |c_n z^n| > \sum_{n=N}^{\infty} \left[\left(\frac{1}{R} - \varepsilon\right)|z|\right]^n$$

Then

$$\left(\frac{1}{R} - \varepsilon\right)^n < |c_n|$$

so the power series diverges as the geometric series diverges.

For (c), one chooses ρ between r and R and then evoke Weierstrass M-test for $|c_n z^n| < \left(\frac{r}{\rho}\right)^n$. □

Theorem 4.3.4 (d'Alambert Test)**.** *If $\sum c_n(z-a)^n$ is a given power series with radius of convergence R, then*

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$$

if this limit exists. If the limit is zero, then $R = 0$; if the limit is infinite, then $R = \infty$.

Proof. See [3] proposition 1.4. □

Exercise 4.3.5. *Find radius of convergence of $\sum_{n=1}^{\infty}(3+4i)^n(z-i)^{2n}$*

Solution. Notice that the coefficients of the odd terms are 0, so we cannot apply the formula directly. Let

$$f_n(z) = (3+4i)^n(z-i)^{2n}$$

Then

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right| = \lim_{n\to\infty}\left|\frac{(3+4i)^{n+1}(z-i)^{2n+2}}{(3+4i)^n(z-i)^{2n}}\right| = \lim_{n\to\infty}|(3+4i)(z-i)^2| = 5|z-i|^2$$

When $5|z-i|^2 < 1$, i.e., $|z-i| < \frac{\sqrt{5}}{5}$, the power series is absolutely convergent. When $5|z-i|^2 > 1$, i.e., $|z-i| > \frac{\sqrt{5}}{5}$, the power series diverges. Thus $R = \frac{\sqrt{5}}{5}$. ■

Definition 4.3.6. *Function $f(z)$ is said to be* ***representable by power series*** *in U if $\forall B(a,r) \subseteq U$, there corresponds some power series $\sum_{n=1}^{\infty} c_n(z-a)^n$ that converges in $B(a,r)$ and equals $f(z)$.*

Theorem 4.3.7 (Power series is analytic and can be differentiated termwize)**.** *Power series $\sum_{n=1}^{\infty} c_n(z-a)^n$, denoted as $f(z)$, is analytic on $B(a,R) = \{z : |z-a| < R\}$, where $R =$ radius of convergence. Besides, for $z \in B(a,R)$,*

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$$

Thus, if f is representable by power series in an open set $U \subseteq \mathbb{C}$, then $f \in H(U) :=$ the set of all analytic functions on U and derivative is given above.

Proof. We can assume $a = 0$ because we can apply chain rule with $g(z) = z - a$ to $f(z) = \sum c_n z^n$ on each $z \in B(0,R)$, and R is defined regardless of a. We write

$$f(z) = \sum_{n=1}^{\infty} c_n z^n = \underbrace{\sum_{n=1}^{N} c_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} c_n z^n}_{E_N(z)}, \quad g(z) = \sum_{n=1}^{\infty} nc_n z^{n-1}.$$

The claim is that f is differentiable on $B(0,R)$ and its derivative is the power series g. Since $\lim n^{1/n} = \lim e^{\log n^{\frac{1}{n}}} = 1$, it is easy to see that $f(z)$ and $g(z)$ have the same radius of convergence by using 4.1.7. Fix z_0 with $|z_0| < r < R$. We wish to show that $\frac{f(z_0+h)-f(z_0)}{h}$ converges to $g(z_0)$ as $h \to 0$. Observe that

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right) \\ &\quad + \left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right) + \left(S_N'(z_0) - g(z_0)\right)\end{aligned}$$

The first term converges to 0 for $h \to 0$ for any fixed N, because $S_N(z)$ is a polynomial. To bound the second term, fix some $\varepsilon > 0$, and note that, if we assume that not only $|z_0| < r$ but also $|z_0 + h| < r$ (an assumption that's clearly satisfied for h close enough to 0) then

$$\begin{aligned}\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| &\leq \sum_{n=N+1}^{\infty}|a_n|\left|\frac{(z_0+h)^n-z_0^n}{h}\right| \\ &= \sum_{n=N+1}^{\infty}|a_n|\left|\frac{h\sum_{k=0}^{n-1}h^k(z_0+h)^{n-1-k}}{h}\right| \\ &\leq \sum_{n=N+1}^{\infty}|a_n|\, n r^{n-1},\end{aligned}$$

where we use the algebraic identity

$$a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1}\right).$$

The last expression in this chain of inequalities is the tail of an absolutely convergent series, so can be made $<\varepsilon$ be taking N large enough (before taking the limit as $h \rightarrow 0$).

Third, when choosing N also make sure it is chosen so that $|S_N'(z_0)-g(z_0)|<\varepsilon$, which of course is possible since $S_N'(z_0) \rightarrow g(z_0)$ as $N \rightarrow \infty$. Finally, having thus chosen N, we get that

$$\limsup_{h \rightarrow 0}\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right| \leq 0+\varepsilon+\varepsilon=2\varepsilon.$$

Since ε was an arbitrary positive number, this shows that $\frac{f(z_0+h)-f(z_0)}{h} \rightarrow g(z_0)$ as $h \rightarrow 0$, as claimed. □

Corollary 4.3.8. *Let $f(z)=\sum_{n=1}^{\infty} c_n(z-a)^n$ have radius of convergence $R>0$. Then by applying the theorem to f', $f''=(f')'$, $\cdots$, the function f is infinitely differentiable on $B(a,R)$ and its k-th derivative is given by a power seires with the same radius of convergence*

$$f^{(k)}(z)=\sum_{n=1}^{\infty} n(n-1)\cdots(n-k+1)c_n(z-a)^{n-k}$$

for all $k \geq 1$ and $|z-a|<R$. In particular, $f^{(n)}(a)=n!c_n$, or $c_n=\frac{1}{n!}f^{(n)}(a)$.

We now show the converse.

Theorem 4.3.9. *Let $f \in H(U)$ where $U \subseteq \mathbb{C}$ is open. Then f is representable by power series in U. That is, for any $\overline{B}(a,r_0) \subseteq U$, f has a power series expansion at a*

$$f(z)=\sum_{n=1}^{\infty} c_n(z-a)^n$$

that is convergent for all $z \in B(a,r_0)$, where $c_n=f^{(n)}(a)/n!$.

Proof. The idea is that Cauchy's integral formula ([11] Corollary 7.22) gives us a representation of $f(z)$ as a weighted "sum" (=an integral, which is a limit of sums) of functions of the form $z \mapsto (\xi-z)^{-1}$. Each such function has a power series expansion since it is, more or less, a geometric series, so the sum also has a power series expansion. Note that analyticity is used in Cauchy's integral formula. Let $r<r_0$. Cauchy's integral formula gives

$$f(z)=\frac{1}{2\pi i}\int_{\partial B(a,r)}\frac{f(\xi)}{\xi-z}d\xi, \quad z \in B(a,r)$$

We write
$$\frac{1}{\xi-z}=\frac{1}{(\xi-a)-(z-a)}=\frac{1}{\xi-a}\cdot\frac{1}{1-\left(\frac{z-a}{\xi-a}\right)}=\frac{1}{\xi-a}\sum_{n=1}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n$$

where $\xi\in\partial B(a,r_0)$. Since $z\in B(a,r)$, we see $\left|\frac{z-a}{\xi-a}\right|=\frac{|z-a|}{r_0}<\frac{r}{r_0}<1$, so the geometric series converges uniformly for $\xi\in\partial B(a,r_0)$. Now,

$$\begin{aligned}
f(z)&=\frac{1}{2\pi i}\int_{\partial B(a,r)} f(\xi)\frac{1}{\xi-a}\sum_{n=1}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n d\xi\\
&=\frac{1}{2\pi i}\int_{\partial B(a,r)}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{f(\xi)}{\xi-a}\left(\frac{z-a}{\xi-a}\right)^n d\xi\\
&\overset{\text{unif conv.+linearity of int}}{=\!=\!=\!=}\lim_{N\to\infty}\frac{1}{2\pi i}\sum_{n=1}^{N}\int_{\partial B(a,r)}\frac{f(\xi)}{\xi-a}\left(\frac{z-a}{\xi-a}\right)^n d\xi\\
&=\sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{\partial B(a,r)}\frac{f(\xi)}{(\xi-a)^{n+1}}d\xi\right)}_{\text{only depends on } a,\text{ called } c_n}(z-a)^n\\
&=\sum_{n=1}^{\infty}c_n(z-a)^n,\quad z\in B(a,r), r<r_0
\end{aligned}$$

We can let $r\to r_0$ since c_n does not depend on r ($c_n=f^{(n)}(a)/n!$ by previous result). □

Remark 4.3.10. *We have a new proof showing that an analytic function is infinitely differentiable due to the corollary 4.3.8. Above theorem also gives a new proof of the n-th derivative of analytic function aside [11] Theorem 8.5.*

4.3.2 Power Series on $|z-a|=R$

We have seen that for given point a in a region D where $f(z)$ is analytic, as long as $\overline{B}(a,r_0)\subseteq D$, we can uniquely expand $f(z)$ on $B(a,r)$, $0<r<r_0$ in terms of power series

$$f(z)=\sum_{n=1}^{\infty}c_n(z-a)^n \tag{4.4}$$

where
$$c_n=\frac{1}{2\pi i}\int_{\partial B(a,r)}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\mathrm{d}\zeta=\frac{f^{(n)}(a)}{n!},\quad n=0,1,2,\cdots \tag{4.5}$$

Hence, the radius can be maximized up to the point where the closure of the ball does not reach the nearest singularity. In fact,

Theorem 4.3.11. *If the power series $\sum_{n=1}^{\infty}c_n(z-a)^n$ has radius of convergence $R>0$ and*

$$f(z)=\sum_{n=1}^{\infty}c_n(z-a)^n,\quad z\in B(a,R),$$

Then $f(z)$ has at least a singularity on the circle $S:|z-a|=R$. Namely, there exists no function $F(z)$ such that it is analytic on S and agrees with $f(z)$ on $B(a,R)$.

Proof. Aiming for a contradiction, we suppose such $F(z)$ exists. Then each point on S is the center of some circle O, and $F(z)$ is analytic in circle O. We by compactness choose finite number of circles to cover S. These chosen circles form a region G. Denote $\rho > 0$ the distance between the boundary of G and C. Then, $F(z)$ is analytic in $B(a, R+\rho)$. Thus, F can be represented as a series in $B(a, R+\rho)$. However, $F(z) \equiv f(z)$ in $B(a, R)$, so their derivatives are the same on $z = a$. Therefore, $\sum c_n(z-a)^n$ is just the series for $F(z)$, and their radius of convergence is no smaller than $R+\rho$. Contradiction. □

Corollary 4.3.12. *If $f(z)$ is analytic on point a, and point b is the closest singularity of f to a, and if we define our series by (4.4) with coefficients given by (4.5), then $R = |b-a|$ is the radius of convergence for the series.*

Proof. Suppose not. Then by definition of radius of convergence, it should be the case that there is a radius $R' < R = |b-a|$ serving as the radius of convergence. We can choose the ball $B(a, (R+R')/2)$, whose closure is inside the open set $U = B(a, R)$. Since $\overline{B}(a, (R+R')/2) \subseteq B(a, R)$, then 4.3.9 implies the series converge to $f(z)$ for every $z \in B(a, (R+R')/2)$ including every $z \in \partial B(a, R') = S(a, R')$. This is a contradiction to above theorem, because there should be at least one singularity on $S(a, R')$ for R' to be the radius of convergence. □

Remark 4.3.13. *Let R be the radius of convergence of a power series $\sum c_n(z-a)^n$. Even if the power series converges for every point on $|z-a| = R$, its pointwise-defined sum function $f(z) := \sum c_n(z-a)^n$ still has at least one singularity on $|z-a| = R$. See the following example.*

Example 4.3.14. Let

$$f(z) = \frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \cdots + \frac{z^n}{n^2} + \cdots.$$

Then

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = 1 > 0.$$

On circle $S : |z| = 1$, the series

$$\sum_{n=1}^{\infty}\left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

is convergent, so the original series $\sum z^n/n^2$ absolutely converges everywhere on $|z| = 1$. Thus, $\sum z^n/n^2$ converges absolutely and uniformly on $\overline{B}(0, 1)$. However,

$$f'(z) = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots + \frac{z^{n-1}}{n} + \cdots, \quad (|z| \le 1).$$

When z approaches 1 along the real axis in the unit circle, $f'(z)$ goes to infinity. Therefore, $z = 1$ is a singularity for $f(z)$. ◇

4.3.3 Operations of Power Series

Addition

Let $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=1}^{\infty} b_n (z-z_0)^n$ be two power series with the same center. Suppose the series $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ has a positive radius of convergence R_1 and the series $\sum_{n=1}^{\infty} b_n (z-z_0)^n$ has a positive radius of convergence R_2.

Exercise 4.3.15. *Show that $\sum_{n=1}^{\infty}(a_n+b_n)(z-z_0)^n$ has $R \ge \min\{R_1, R_2\}$.*

Multiplication

The **Cauchy product** of the two series is by definition the power series $\sum_{n=1}^{\infty} c_n(z-z_0)^n$ whose n-th coefficient is given by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. It arises when one forms all products $a_j (z-z_0)^j\, b_k (z-z_0)^k$, adds for each n the ones with $j+k = n$, and sums the resulting terms.

Proposition 4.3.16. *Their Cauchy product converges in the disk $|z-z_0|<\min\{R_1,R_2\}$ to the product of the functions represented by the two original series.*

Proof. We can assume without loss of generality that $z_0=0$. Suppose $|z|<\min\{R_1,R_2\}$. For N a positive integer we have

$$\begin{aligned}&\left(\sum_{j=0}^{N} a_j z^j\right)\left(\sum_{k=0}^{N} b_k z^k\right)-\sum_{n=1}^{N} c_n z^n\\ &=\sum_{0\leq j,k\leq N} a_j b_k z^{j+k}-\sum_{n=1}^{N}\sum_{j+k=n} a_j b_k z^{j+k}\\ &=\sum_{\substack{0\leq j,k\leq N\\ j+k>N}} a_j b_k z^{j+k}.\end{aligned}$$

It follows that

$$\begin{aligned}&\left|\left(\sum_{j=0}^{N} a_j z^j\right)\left(\sum_{k=0}^{N} b_k z^k\right)-\sum_{n=1}^{N} c_n z^n\right|\\ &\leq \sum_{\substack{j\leq j,k\leq N\\ j+k>N}}\left|a_j b_k z^{j+k}\right|\\ &\leq \sum_{\frac{N}{2}<\max\{j,k\}\leq N}\left|a_j b_k z^{j+k}\right|\\ &\leq\left(\sum_{j>\frac{N}{2}}\left|a_j z^j\right|\right)\left(\sum_{k=0}^{N}\left|b_k z^k\right|\right)+\left(\sum_{j=0}^{N}\left|a_j z^j\right|\right)\left(\sum_{k>\frac{N}{2}}\left|b_k z^k\right|\right)\\ &\leq\left(\sum_{j>\frac{N}{2}}\left|a_j z^j\right|\right)\left(\sum_{k=0}^{\infty}\left|b_k z^k\right|\right)+\left(\sum_{j=0}^{\infty}\left|a_j z^j\right|\right)\left(\sum_{k>\frac{N}{2}}\left|b_k z^k\right|\right).\end{aligned}$$

The last expression tends to 0 as $N\longrightarrow\infty$, because both series $\sum_{j=0}^{\infty}\left|a_j z^j\right|$ and $\sum_{k=0}^{\infty}\left|b_k z^k\right|$ converge. In view of the preceding inequality, therefore, we can conclude that

$$\sum_{n=1}^{\infty} c_n z^n=\left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{k=0}^{\infty} b_k z^k\right)$$

as desired. □

Division

Suppose the power series $\sum_{n=1}^{\infty} b_n (z-z_0)^n$ and $\sum_{n=1}^{\infty} c_n (z-z_0)^n$ have positive radii of convergence and so represent holomorphic functions g and h, respectively, in disks with center z_0. Suppose also that $g(z_0)=b_0\neq 0$. The quotient $f=h/g$ is then holomorphic in some disk with center z_0. Then f is represented in that disk by a power series $\sum_{n=1}^{\infty} a_n (z-z_0)^n$, how does one find the coefficients a_n in terms of the coefficients b_n and c_n?

A method that always works in principle uses the Cauchy product, according to which

$$c_n=\sum_{k=0}^{n} a_k b_{n-k},\quad n=0,1,\ldots$$

From this we can conclude that $a_0 = c_0/b_0$ and

$$a_n = \frac{1}{b_0}\left(c_n - \sum_{k=0}^{n-1} a_k b_{n-k}\right), \quad n = 1, 2, \ldots$$

The last equality expresses a_n in terms of $c_n, b_0, \ldots, b_n$ and $a_0, \ldots, a_{n-1}$, enabling one to determine the coefficients a_n recursively starting from the initial value $a_0 = c_0/b_0$.

Exercise 4.3.17. *Use the scheme above to determine the power series with center 0 representing the function* $f(z) = 1/\left(1+z+z^2\right)$ *near 0 . What is the radius of convergence of the series?*

Chapter 5

Zeros and Residues

The fact that every holomorphic function is locally the sum of a convergent power series has a large number of interesting consequences. A few of these are developed in this chapter.

5.1 Isolated Zeros

We will see that zeros of non-vanishing analytic functions are isolated and analytic functions that agree locally actually agree globally. We recall that $x \in X$ is a limit point of A if for $\forall \varepsilon > 0,\ B(x,\varepsilon) \cap (A - \{x\}) \neq \varnothing$, and it is easy to show that for metric space (X,d), x is a limit point of A if and only if there exists a sequence $\{x_i\}$ in A such that $x_i \to x$ ($\Leftarrow$ is because each ball $B(x,\varepsilon)$ of x contain infinitely many x_i as along as $i > N(\varepsilon)$; for $\Rightarrow$, see [11] Remark 3.39 to find such sequence). We call a point x in A an **isolated point** if it is not a limit point of A. Thus, $x \in A$ is an isolated point if there is some $B(x,\varepsilon)$ intersecting no other points in A, or equivalently, there exists no sequence $\{x_i\}$ in A converging to x.

Theorem 5.1.1. *Suppose U is a region (an open and connected subset), $f \in H(U)$, and*

$$Z(f) = \{a \in U : f(a) = 0\}$$

Then either

(i) $Z(f) = U$, or

(ii) $Z(f)$ has no limit point in U.

In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z-a)^m g(z) \quad (z \in U), \tag{5.1}$$

where $g \in H(U)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

Definition 5.1.2. *The integer m is called the **order** or **multiplicity** of the zero which f has at the point a. Clearly, $Z(f) = U$ if and only if f is identically 0 in U. We call $Z(f)$ the **zero set** of f. Analogous results hold of course for the set of α-points of f (α-level set), i.e., the zero set of $f - \alpha$, where α is any complex number.*

Proof. Let $A = (Z(f))^{\mathrm{acc}} \cap U$ be the set of all limit points of $Z(f)$ in U. Since f is continuous and for each $a \in A$, $\exists \{z_i\} \in Z(f)$ s.t. $z_i \to a$ we have $f(a) = f(\lim z_i) = \lim f(z_i) = 0$. Thus, $A \subset Z(f)$.

Fix $a \in Z(f)$, and choose $r > 0$ so that $B(a,r) \subset U$. By Theorem 4.3.9,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad (z \in B(a,r)) \tag{5.2}$$

There are now two possibilities. Either

(a) all c_n are 0, in which case $B(a,r) \subset A$; or

(b) there is a smallest integer m (necessarily positive, since $f(a) = c_0 = 0$) such that $c_m \neq 0$.

In case (b), define

$$g(z) = \begin{cases} (z-a)^{-m} f(z), & z \in U - \{a\}, \\ c_m, & z = a. \end{cases}$$

Then 5.1 holds and $g(a) = c_m \neq 0$.

We claim that $g \in H(U)$. It is clear that $g \in H(U - \{a\})$, so we need to show it is complex differentiable at a. In fact,

$$\begin{aligned} g(z) = (z-a)^{-m} f(z) &= (z-a)^{-m} \sum_{n=m}^{\infty} c_n (z-a)^n \\ &= \sum_{n=m}^{\infty} c_n (z-a)^{n-m} = \sum_{n=0}^{\infty} c_{n+m} (z-a)^n \quad (z \in B(a,r) \setminus \{a\}) \end{aligned} \tag{5.3}$$

and

$$g(a) = c_m = \sum_{n=0}^{\infty} c_{n+m} (a-a)^n.$$

Thus,

$$g(z) = \sum_{n=0}^{\infty} c_{n+m} (z-a)^n \quad (z \in B(a,r)).$$

It follows that $g \in H(B(a,r))$. In particular, g is analytic at $z = a$, so $g \in H(U)$. It is important to note that integer m must be smallest with $c_m \neq 0$ to write the second equality in (5.3). The uniqueness of m in (ii) follows from this as well.

Moreover, since $g(a) \neq 0$, the continuity of g shows that there is a neighborhood of a in which g has no zeros. (5.1) shows that f and g have the same zeros except for a, so f is nonzero in the same neighborhood except for a. Thus, a is an isolated point of $Z(f)$.

In case (a), we show that once there is *some* $a \in Z(f)$ with case (a) holding, then $Z(f) = U$. Define

$$W := \{z \in U : \exists r > 0 \text{ with } B(z,r) \subset U \text{ and } f \equiv 0 \text{ on } B(z,r)\}.$$

This is "the set of points having a neighborhood on which f vanishes identically."

W is open in U: If $z \in W$, then $f \equiv 0$ on some ball $B(z,r)$. For any $w \in B(z,r)$, take a smaller ball $B(w,r') \subset B(z,r)$. Then $f \equiv 0$ on $B(w,r')$, so $w \in W$. Hence $B(z,r) \subset W$.

W is closed in U: Let $z_k \to z \in U$ with $z_k \in W$. Then $f(z_k) = 0$ for all k. Since f is continuous, $f(z) = \lim f(z_k) = 0$. Thus, $z \in Z(f)$. Either (a) or (b) happens at z. But $z_k \to z$ in $Z(f)$, so z is not an isolated zero and (b) cannot happen at z. Thus (a) holds at z. (a) says $B(z,r) \subset A$. Since $A \subset Z(f)$, we see $z \in W$. Thus, W is closed.

Thus, if there is any $a \in Z(f)$ with case (a) holding, then W is a nonempty clopen subset of the connected set U, so $W = U$. Hence $Z(f) = U$ and case (i) holds.

If case (i) does not hold, then (a) fails for *every* $a \in Z(f)$, so case (b) holds for *every* $a \in Z(f)$, and we have shown that each such a is an isolated point of $Z(f)$. Thus, $Z(f)$ has no limit point in U, and case (ii) holds. □

Note: The theorem fails if we drop the assumption that U is connected: If $U = U_0 \cup U_1$, and U_0 and U_1 are disjoint open sets, put $f = 0$ in U_0 and $f = 1$ in U_1. Then $Z(f) = U_0 \neq U$. Each $z \in U_0$ is a limit point of U_0 and is in U.

Corollary 5.1.3 (Identity Theorem). *f and g are holomorphic functions in a region U. If there is some sequence $\{x_i\}$ in U s.t. $f(x_i) = g(x_i)$ and $x_i \to x$ for a point $x \in U$, then $f = g$ on U. Thus, if $f(z) = g(z)$ for all z in some set A which has a limit point x in U, then $f = g$ on U. In particular, A can be an open set or the trace of a path in U.*

Proof. Apply previous theorem to $f - g$. Note that $(f-g)(x) = (f-g)(\lim x_i) = \lim(f-g)(x_i) = \lim 0 = 0$ implies that $x \in Z(f-g)$. Since $x_i \to x$, x is a limit point. Previous theorem then says it has to be the case that $Z(f-g) = U$. □

[2, Section III.3 p.125-131] has a beautiful exposition of consequences of the Identity Theorem. Essentially, knowing only a little local information about a holomorphic function (e.g. the values of the function along a path) allows us to deduce a great deal of global information about it. From this, we can uniquely extend real functions $\sin$, $\cos$, e^x, etc. to holomorphic functions on $\mathbb{C}$. We know $H(U)$ with U open is a commutative ring with 1 and we can show that it is in fact an integral domain (no zero divisors) by the Identity Theorem if U is connected. It gives a way to show the Open Mapping Theorem, which Maximum Modulus Principle and Schwarz Lemma both follow from.

5.2 Isolated Singularities

5.2.1 Classification of Isolated Singularities

Definition 5.2.1. *If $a \in U$ and $f \in H(U - \{a\})$, then f is said to have an **isolated singularity** at the point a. If f can be defined at a so that the extended function is holomorphic in U, the singularity is said to be **removable**.*

Theorem 5.2.2. *[Criterion for removable singularity] Suppose $f \in H(U - \{a\})$ and f is bounded in $B'(a, r) := \{z : 0 < |z - a| < r\} \subseteq U$ for some $r > 0$. Then f has a removable singularity at a.*

Remark 5.2.3. *We previously had a similar result, but there we assumed f is continuous in U instead of being bounded.*

Proof. Define

$$h(z) = \begin{cases} (z-a)^2 f(z), & z \in U - \{a\}, \\ 0, & z = a. \end{cases}$$

h is evidently differentiable at $U - \{a\}$, and

$$\lim_{\substack{z \to a \\ z \in U \setminus \{a\}}} \frac{h(z) - h(a)}{z - a} = \lim_{\substack{z \to a \\ z \in U \setminus \{a\}}} (z-a) f(z) = 0$$

due to boundedness of f near a. Thus $h \in H(U)$ with $h'(a) = 0$. Thus we can represent h by a power series in $B(a, r) \subset U$:

$$h(z) = \sum_{n=2}^{\infty} c_n (z-a)^n \quad (z \in B(a, r)).$$

Notice that the first two coefficients are zero because

$$c_n = \frac{h^{(n)}(a)}{n!}, \text{ and } h(a) = h'(a) = 0$$

We obtain the desired holomorphic extension of f by setting $f(a) = c_2$, for then

$$\sum_{n=0}^{\infty} c_{n+2}(z-a)^n \quad (z \in B(a,r))$$

is a power series representation of f at a: 1. the power series has the same radius of convergence as the one representing h; 2. the power series equals f for $z \in B(a,r)$ because $(z-a)^{-2}h(z)$ agrees with this for $z \neq a$, and both sides equal c_2 for $z = a$ after setting $f(a) = c_2$. ☐

We note that boundedness of f is only used for showing that

$$\lim_{\substack{z \to a \\ z \in U \setminus \{a\}}} (z-a)f(z) = 0.$$

Therefore, we have the following criterion:

Theorem 5.2.4 (Riemann's Criterion on Removable Singularity). *Let $U \subset \mathbb{C}$ be an open subset of the complex plane, $a \in U$ a point of U and f holomorphic on $U \setminus \{a\}$. The following are equivalent:*

(a) f has a removable singularity at a, i.e., f is holomorphically extendable over a.

(b) f is continuously extendable over a.

(c) There exists a neighborhood of a on which f is bounded.

(d) The limit

$$\lim_{\substack{z \to a \\ z \in U \setminus \{a\}}} (z-a)f(z)$$

is zero.

(e) The limit

$$\lim_{\substack{z \to a \\ z \in U \setminus \{a\}}} f(z)$$

exists in $\mathbb{C}$.

Proof. The direction $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ is clear. The proof of Theorem 5.2.2 gives $(d) \Rightarrow (c) \Rightarrow (a)$. Corollary 3.4.8 also gives $(b) \Rightarrow (a)$. Thus, the first four conditions are equivalent. If (e) is true, i.e., $\lim_{\substack{z \to a \\ z \in U \setminus \{a\}}} f(z) = L \in \mathbb{C}$, then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ for $0 < |z-a| < \delta$. This implies $f(z)$ is bounded in $B'(a,\delta)$, so (c) holds. Conversely, we show (b) implies (e). Indeed, if the extension of f is continuous at a, then $\lim_{\substack{z \to a \\ z \in U \setminus \{a\}}} f(z) = f(a)$. ☐

We introduce two other isolated singularities and claim that together with removable singularity they exhaust all kinds of isolated singularities.

Theorem 5.2.5. *Let $U \subseteq \mathbb{C}$ be open. If $f : U \to \mathbb{C}$ has a singularity at a, i.e., $a \in U$ and $f \in H(U - \{a\})$, then one of the following three cases must occur:*

(a) f has a removable singularity at a.

(b) There are complex numbers $c_{-1}, \ldots, c_{-m}$, where m is a positive integer and $c_{-m} \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_{-k}}{(z-a)^k}$$

has a removable singularity at a.

(c) If $r>0$ and $B(a,r)\subset U$, then $f\left(B'(a,r)\right)$ is dense in the plane.

Proof. Suppose (c) fails. Then we must have some $w\in\mathbb{C}$ and $r>0$ such that $w\notin\overline{f(B'(a,r))}$; so there is a neighborhood $B(w,\delta)$ of w such that $B(w,\delta)\cap f(B'(a,r))=\varnothing$, i.e., $|f(z)-w|>\delta$ for $z\in B'(a,r)$. Write $B=B(a,r)$, $B'=B'(a,r)$, and define

$$g(z)=\frac{1}{f(z)-w},\quad z\in B'$$

Clearly, $g\in H(B')$ and $|g(z)|<\frac{1}{\delta}$. By Theorem 5.2.2, g has a removable singularity at $z=a$, so g extends to $\widetilde{g}\in H(B)$. If $\widetilde{g}(a)\neq 0$, then this means

$$0\neq|\widetilde{g}(a)|=\lim_{\substack{z\to a\\ z\in B'}}|\widetilde{g}(z)|\overset{\widetilde{g}|_{B'}=g}{=\!=\!=}\lim_{\substack{z\to a\\ z\in B'}}|g(z)|=\lim_{\substack{z\to a\\ z\in B'}}\frac{1}{|f(z)-w|}$$

and thus $|f(z)-w|$ has to stay bounded in B', and so does $|f|$. Thus again by Theorem 5.2.2, f has a removable singularity at a. (a) holds.

The other case is $\widetilde{g}(a)=0$. We will show that this implies (b). Note that $\widetilde{g}|_{B'}(z)=g(z)=\frac{1}{f(z)-w}$ has no zeros, so $\widetilde{g}$ is not identically zero in the connected open set B. Theorem 5.1.1 then gives a unique integer $m\geq 1$ such that

$$\widetilde{g}(z)=(z-a)^m g_1(z),\quad z\in B,$$

for some $g_1\in H(B)$ with $g_1(a)\neq 0$. Also, g_1 has no zeros in B' as $\widetilde{g}$ is so. Thus, g_1 has no zeros in B. We then define $h=1/g_1$ in B. It is holomorphic in B and has no zeros in B. Now

$$f(z)-w=\frac{1}{g(z)}=(z-a)^{-m}h(z),\quad z\in B' \tag{5.4}$$

We expand the holomorphic h into power series: $h(z)=\sum_{n=0}^{\infty}b_n(z-a)^n$, $z\in B$ with $0\neq h(a)=b_0$. Thus we get

$$\begin{aligned}
f(z)-w&=(z-a)^{-m}h(z)=\sum_{n=0}^{\infty}b_n(z-a)^{n-m}\\
&=\frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\cdots+\frac{b_{m-1}}{z-a}+\sum_{n=0}^{\infty}b_{n+m}(z-a)^n\\
\Rightarrow f(z)&-\frac{b_0}{(z-a)^m}-\frac{b_1}{(z-a)^{m-1}}-\cdots-\frac{b_{m-1}}{z-a}=G(z):=w+\sum_{n=0}^{\infty}b_{n+m}(z-a)^n,\quad z\in B'.
\end{aligned}$$

Simply define $G(a)=w+b_m$ and we show $G(z)$ is analytic in B: h is holomorphic in B so its radius of convergence is at least r. Now, $\limsup_{n\to\infty}|b_{n+m}|^{1/n}=\limsup_{n\to\infty}|b_n|^{1/n}$ shows that $\sum_{n=0}^{\infty}b_{n+m}(z-a)^n$ has the same radius of convergence as h, so this power series converges in B as well. Thus $G\in H(B)$ by Theorem 4.3.7. Thus (b) holds with $c_{-m}=b_0,\cdots,c_{-1}=b_{m-1}$. ☐

In case (b), f is said to have a **pole of order** m at a. Note that the integer m in (ii) is unique as it comes from Theorem 5.1.1 in above proof. The function

$$\sum_{k=1}^{m}c_{-k}(z-a)^{-k}$$

is a polynomial in $(z-a)^{-1}$ and is called the **principal part of** f **at** a. Coefficient c_{-1} is called **residue** of f at a, denoted as $\operatorname{Res}(f;a)$.

In case (c), f is said to have an **essential singularity at** a.

We state the following equivalent characterizations without proof. We may omit $z \in U \setminus \{a\}$ in the limit $\lim_{\substack{z\to a\\ z\in U\setminus\{a\}}}$ as it is clear from the context.

Proposition 5.2.6. *Let $U \subseteq \mathbb{C}$ be an open set, $a \in U$, and $f \in H(U - \{a\})$. Then the following are equivalent:*

(a) f has a pole of order m at a. That is, there are complex numbers $c_{-1}, \ldots, c_{-m}$, where m is a positive integer and $c_{-m} \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_{-k}}{(z-a)^k}$$

has a removable singularity at a.

(b) $(z-a)^m f(z)$ has a removable singularity at a and the extended value at a is $\neq 0$. That is, there is a holomorphic function h on U such that $f(z) = (z-a)^{-m} h(z)$ for $z \in U \setminus \{a\}$ and $h(a) \neq 0$.

(c) The limit

$$\lim_{\substack{z\to a\\ z\in U\setminus\{a\}}} (z-a)^m f(z)$$

exists and is nonzero. (Moreover, under (a) *this limit equals c_{-m}.)*

(d) $1/f$ has zero of order m at a. This exactly means $1/f$ extends to a holomorphic function g such that $g(a) = 0$ and there exists a holomorphic function $\varphi \in H(U)$ with $\varphi(a) \neq 0$ such that

$$g(z) = (z-a)^m \varphi(z) \quad \text{for all } z \in U.$$

Proposition 5.2.7. *Let $U \subseteq \mathbb{C}$ be an open set, $a \in U$, and $f \in H(U - \{a\})$. Then the following are equivalent:*

(a) a is a pole of f (of some finite order),

(b) $\lim_{z\to a} |f(z)| = \infty$ (equivalently $\lim_{z\to a} f(z) = \infty$),

(c) $1/f$ has a zero at a, i.e. $1/f$ has a removable singularity at a whose extended value is 0.

Proposition 5.2.8. *Let $U \subseteq \mathbb{C}$ be an open set, $a \in U$, and $f \in H(U - \{a\})$. f has an essential singularity at a if and only if for any complex number $w \in \mathbb{C}$ there exists a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$.*

Proposition 5.2.9. *Let $f \in H(U \setminus \{a\})$. Then exactly one holds:*

- *Removable:* $\lim_{z\to a} f(z) \in \mathbb{C}$.
- *Pole:* $\lim_{z\to a} |f(z)| = \infty$ *(equivalently* $\lim_{z\to a} f(z) = \infty$*).*
- *Essential:* $\lim_{z\to a} f(z)$ *does not exist in Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (i.e., neither a finite limit nor ∞).*

And you have the reciprocity:

(a) a is removable for f with $f(a) \neq 0 \iff a$ is removable for $1/f$ with $(1/f)(a) \neq 0$.

(b) a is a zero of f of order $m \iff a$ is a pole of $1/f$ of order m.

(c) a is a pole of f of order $m \iff a$ is a zero of $1/f$ of order m.

(d) a is essential for $f \iff a$ is essential for $1/f$.

Example 5.2.10 (Removable singularity by cancellation)**.**

$$f(z) = \frac{z^2 - 1}{z - 1}, \quad z \neq 1.$$

For $z \neq 1$, $f(z) = z + 1$, hence
$$\lim_{z\to 1} f(z) = 2.$$
Thus $z = 1$ is a removable singularity. Defining $f(1) = 2$ makes f entire. The reciprocal
$$\frac{1}{f(z)} = \frac{z-1}{z^2-1}$$
is also removable at $z = 1$ with nonzero value. ◇

Example 5.2.11 (Removable singularity and reciprocity)**.**
$$f(z) = \frac{\sin z}{z}, \quad z \neq 0.$$
Using the Taylor expansion
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots ,$$
we obtain
$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots ,$$
and hence
$$\lim_{z\to 0} \frac{\sin z}{z} = 1.$$
Therefore $z = 0$ is a removable singularity. Defining $f(0) = 1$ makes f holomorphic in a neighborhood of 0. In particular, f is bounded and nonvanishing near 0.

The reciprocal
$$\frac{1}{f(z)} = \frac{z}{\sin z}$$
is holomorphic on a punctured neighborhood of 0: since $f(0) = 1 \neq 0$, there exists $r > 0$ such that $f(z) \neq 0$ for all $|z| < r$, and hence the reciprocal is well-defined and holomorphic there. We show $1/f$ has a removable singularity at 0 with extended value nonzero. This verifies Proposition 5.2.9 (a).

Recall the geometric series
$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad (|w| < 1).$$
Set $w = -z$. Then $|z| < 1$ gives $|w| < 1$, and
$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = 1 - z + z^2 - z^3 + \cdots .$$
Replace z with $a(z)$ to see
$$\frac{1}{1+a(z)} = 1 - a(z) + a(z)^2 - \cdots ,$$
where $|a(z)| < 1$ for z sufficiently close to 0.

Using the above observation,
$$\frac{z}{\sin z} = \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = 1 + \frac{z^2}{3!} + O(z^4),$$
which shows that
$$\lim_{z\to 0} \frac{z}{\sin z} = 1.$$
Thus the reciprocal also has a removable singularity at 0, and extending it by the value 1 yields a holomorphic function near 0. ◇

Example 5.2.12 (Zero-pole reciprocity)**.** We give an example to illustrate (b) and (c) of Proposition 5.2.9.

$$f(z) = \frac{1}{z} + \frac{1}{z^2} = \frac{z+1}{z^2}.$$

has a pole $z = 0$ of order 2 and a simple zero at $z = -1$. The reciprocal

$$\frac{1}{f(z)} = \frac{z^2}{z+1}$$

has a zero of order 2 at 0 and a simple pole at $z = -1$, illustrating that reciprocation preserves the local order at a point but may introduce new singularities elsewhere. ◇

Example 5.2.13 (Essential singularity)**.**

$$f(z) = e^{1/z}.$$

Consider the sequences $z_n = \frac{1}{n}$ and $w_n = -\frac{1}{n}$. Both satisfy $z_n \to 0$ and $w_n \to 0$, but

$$f(z_n) = e^n \to \infty, \qquad f(w_n) = e^{-n} \to 0.$$

Hence $\lim_{z\to 0} f(z)$ does not exist, neither as a finite complex number nor as ∞. Therefore $z = 0$ is neither a removable singularity nor a pole, and so it is an essential singularity.

The reciprocal

$$\frac{1}{f(z)} = e^{-1/z}$$

satisfies the same property and also has an essential singularity at 0. This verifies Proposition 5.2.9 (d). ◇

5.2.2 Residue Theorem for One Pole

Suppose now U is open and convex and $f \in H(U - \{a\})$ has a pole at $z = a$. Then we can write

$$f(z) = \sum_{k=1}^{m} c_{-k}(z-a)^{-k} + g(z)$$

for some $g \in H(U)$. Let γ be a closed piecewise C^1 curve in $U - \{a\}$. Each $z \mapsto (z-a)^{-k}$, $k > 1$ has a primitive $\frac{1}{1-k}(z-a)^{1-k}$ in a neighborhood of γ^* (since $\operatorname{dist}(a, \gamma^*) > 0$) and their integrals over γ vanish. However, $(z-a)^{-1}$ does not have primitive in $U - \{a\}$ (the only candidate $\operatorname{Log}(z-a)$ cannot be analytic in any subset with a full circle contained in it). We note that Cauchy's integral theorem does not apply to either case to conclude anything, as $(z-a)^{-n}$ is not continuous at a for all $n \geq 1$. Cauchy's integral theorem applies to holomorphic function g, so its integration over γ vanishes. We have,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma \frac{c_{-1}}{z-a}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{\operatorname{Res}(f;a)}{z-a}\,dz = \operatorname{Res}(f;a)n_\gamma(a) \tag{5.5}$$

We will generalize this residue theorem later after we get to the "global Cauchy theorem."

Example 5.2.14. We calculate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx \quad 0 < a < 1$$

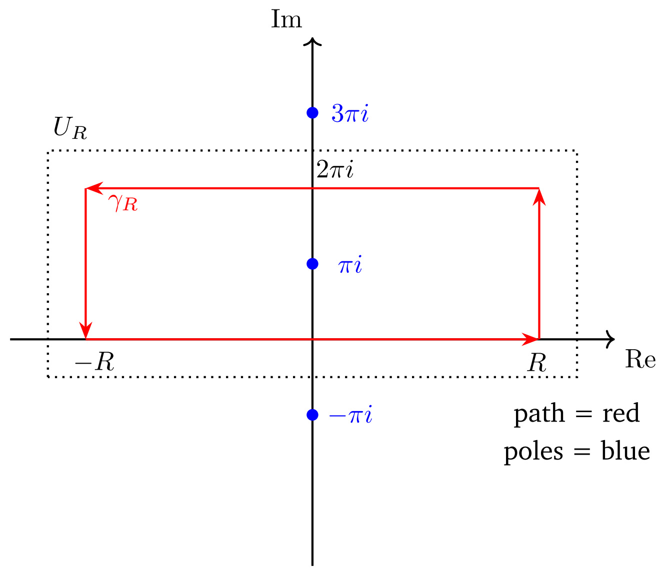
We will show that its value is $\frac{\pi}{\sin(\pi a)}$.

Let

$$f(z) = \frac{e^{az}}{1+e^z}$$

Step 1 (find poles): Notice that e^{az} is entire and that $1+e^z=0 \Leftrightarrow e^{i\theta}=e^{i\pi} \Leftrightarrow \theta=2k'\pi+\pi\,(k=0,\pm1,\cdots)=k\pi\,(k=\pm1,\cdots)$, so the (candidates for) poles are $z=i\theta=k\pi i=\cdots,-3\pi i,-\pi i,\pi i,3\pi i,\cdots$

Step 2 (choose path and U): We consider the path γ_R in the picture, and we choose an open convex set U_R with $\gamma_R^*\subseteq U$ (for instance, U_R can be a small flattening of the box).



Step 3 (apply residue theorem): Now, f is analytic in $U_R\setminus\{\pi i\}$. We apply our toy residue theorem given just before this example to see

$$\int_{\gamma_R} f(z)\,dz = 2\pi i\mathrm{Res}(f;\pi i)$$

where the winding number $n_{\gamma_R}(\pi i)$ is arguably just 1 as shown in the picture (we will develop tools to systematically justify computation of winding numbers later). Heuristically, we guess the order of the pole πi is 1. We calculate $\lim_{z\to\pi i}(z-\pi i)f(z)$. If it exists then Then Proposition 5.2.6 (c) implies πi is indeed a simple pole for f and the limit is $\mathrm{Res}(f;\pi i)$.

$$\begin{aligned}(z-\pi i)f(z)=(z-\pi i)\frac{e^{az}}{1+e^z}&=e^{az}\left(\frac{e^z-e^{\pi i}}{z-\pi i}\right)^{-1}\\ &\xrightarrow{z\to\pi i} e^{a\pi i}\left(\frac{d}{dz}e^z\right)\bigg|_{z=\pi i}=e^{a\pi i}e^{\pi i}=-e^{a\pi i}\end{aligned}$$

Thus, indeed, as the limit exists, we must have

$$\mathrm{Res}(f;\pi i)=-e^{a\pi i}$$

Thus,

$$\int_{\gamma_R} f(z)\,dz=-2\pi i e^{a\pi i}$$

Step 4 (calculate the original integral): We then relate this result to the original real integral. Let

$$I=\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx=\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx=:\lim_{R\to\infty}I_R$$

The integral of f over the top line of the rectangle with orientation from right to left is along $\eta_R(t) = -t + 2\pi i,\ t \in [-R, R]$, so

$$\begin{aligned}
\int_{\eta_R} f(z)\,dz &= \int_{-R}^{R} f(\eta_R(t))\eta_R'(t)dt \\
&= -\int_{-R}^{R} \frac{e^{a(-t+2\pi i)}}{1+e^{-t+2\pi i}}dt \\
&= -e^{2\pi i a}\int_{-R}^{R} \frac{e^{-at}}{1+e^{-t}}dt \\
&\overset{\substack{x=-t\\dx=-dt}}{=\!=\!=} -e^{2\pi i a}\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}dx \qquad \left(-\int_{R}^{-R} = \int_{-R}^{R}\right) \\
&= -e^{2\pi i a} I_R
\end{aligned}$$

Thus,

$$\int_{\gamma_R} f = (1-e^{2\pi i a})I_R + \int_{\text{left vertical}} f + \int_{\text{right vertical}} f$$

Notice, for instance, right vertical is parametrized by $t \to R + it,\ t \in [0, 2\pi]$. Then,

$$\begin{aligned}
\left|\int_{\text{right vertical}} f(z)\,dz\right| = \left|\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{R+it}} i dt\right| &\le \int_0^{2\pi} \frac{|e^{iat}||e^{R}|}{\underbrace{|1+e^{R}e^{it}|}_{\ge |e^Re^{it}|-1 = e^R-1}}dt \\
&\le \int_0^{2\pi} \frac{e^{aR}}{e^R-1}dt = 2\pi\frac{e^{aR}}{e^R-1} \le \pi e^{(a-1)R} \qquad \left(e^R - 1 \ge \frac{e^R}{2} \text{ for large } R\right) \\
&\xrightarrow{R\to\infty} 0 \qquad (0 < a < 1 \text{ so } a - 1 < 0)
\end{aligned}$$

Similarly,

$$\left|\int_{\text{left vertical}} f(z)\,dz\right| \xrightarrow{R\to\infty} 0$$

Therefore, by noticing that $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, we have

$$\begin{aligned}
-2\pi i e^{a\pi i} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = (1-e^{2\pi i a})I \\
\Rightarrow I &= -2\pi i\frac{e^{a\pi i}}{1-e^{2\pi i a}} = \frac{2\pi i}{e^{\pi i a}-e^{-\pi i a}} = \frac{\pi}{\sin(\pi a)}
\end{aligned}$$

◇

We prove a useful formula to calculate the residue of f at a pole.

Proposition 5.2.15. *If $f \in H(U \setminus \{a\})$ has a pole of order n at a, then*

$$\operatorname{Res}(f; a) = \lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}[(z-a)^n f(z)].$$

Proof. For $f \in H(U \setminus \{a\})$ with a pole a of order n, we can write

$$f(z) = \sum_{k=1}^{n} c_{-k}(z-a)^{-k} + g(z)$$

for some $g(z) \in H(U)$. Then g is representable by a power series in U. That is, for any $\overline{B}(a,r) \subseteq U$, g has a power series expansion at a

$$g(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$$

that is convergent for all $z \in B(a,r)$. Thus,

$$f(z) = \sum_{k=1}^{n} c_{-k}(z-a)^{-k} + g(z) = \sum_{k=1}^{n} c_{-k}(z-a)^{-k} + \sum_{k=0}^{\infty} c_k (z-a)^k = \frac{\overbrace{\sum_{k=0}^{\infty} c_{k-n} (z-a)^k}^{\varphi(z)}}{(z-a)^n} \tag{5.6}$$

where $\varphi(z)$ is a power series with the same radius of convergence as g, i.e., r, since changing finitely many elements of a sequence does not affect its limsup. Thus $\varphi(z)$ is analytic on $B(a,r)$ as shown in class, and its $(n-1)$-th derivative at a is given by

$$\varphi^{(n-1)}(a) = \frac{(n-1)!}{2\pi i} \int_{\partial B} \frac{\varphi(z)}{(z-a)^n}\, dz \tag{5.7}$$

by noting that the winding number of $\partial B(a,r)$ around a is 1. The equation is deduced by (3.10). Now we note that equation 5.6 gives $\varphi(z) = f(z)(z-a)^n$ whenever $z \neq a$. Therefore,

$$\varphi^{(n-1)}(a) = \lim_{z\to a} \varphi^{(n-1)}(z) = \lim_{z\to a} \left(\frac{d}{dz}\right)^{n-1} (z-a)^n f(z) \tag{5.8}$$

The residue is computed as

$$\begin{aligned}
\operatorname{Res}(f;a) &= \frac{1}{2\pi i}\int_{\partial B} f(z)\, dz = \frac{1}{2\pi i}\int_{\partial B} \frac{\varphi(z)}{(z-a)^n}\, dz \\
&\overset{(5.7)}{=\!=} \frac{1}{2\pi i}\left(\frac{2\pi i}{(n-1)!}\varphi^{(n-1)}(a)\right) \\
&\overset{(5.8)}{=\!=} \lim_{z\to a} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1} (z-a)^n f(z)
\end{aligned}$$

$\square$

Example 5.2.16. Calculate $\operatorname{Res}(f;0)$ for

$$f(z) := \frac{\sinh(z)e^z}{z^5},$$

where $\sinh(z) := (e^z - e^{-z})/2$. $\diamond$

Solution. We now apply this formula to $f(z) = \sinh(z)e^z/z^5$. Since e^z is entire, we see $\sinh(z) = \frac{e^z - e^{-z}}{2}$ is also entire. Therefore, $a = 0$ is a pole of order 5 for f. Thus,

$$\begin{aligned}
\operatorname{Res}(f;0) &= \lim_{z\to 0}\frac{1}{4!}\left(\frac{d}{dz}\right)^4 z^5 f(z) = \frac{1}{24}\lim_{z\to 0}\left(\frac{d}{dz}\right)^4 \sinh(z)e^z \\
&= \frac{1}{24}\lim_{z\to 0}\frac{d^4\left(\frac{e^{2z}}{2} - \frac{1}{2}\right)}{dz^4} = \frac{1}{24}\lim_{z\to 0} 8e^{2z} = \frac{1}{3}\cdot 1 = \frac{1}{3}
\end{aligned}$$

There is a rather elementary way to see this, if we recall the power series expansions of $\sinh(z)$ and e^z at 0:

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

Then,

$$\operatorname{Res}(f;0) = c_{-1} = \text{coeff}([z(z^3/3!) + z(z^3/6)]/z^5) = \frac{1}{3}$$

■

5.3 Global Cauchy Theorem

Let $\gamma : [a,b] \to U$ be a piecewise C^1 closed path, where U is convex and open. Suppose $z_0 \in \mathbb{C} \backslash U$. Now $f(z) := \frac{1}{z-z_0}$ is analytic in U, and so by Cauchy theorem on convex set,

$$n_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_\gamma f(z) dz = 0.$$

Thus,

$$n_\gamma(z_0) \quad \forall z_0 \in \mathbb{C} \setminus U$$

Could it be that the validity of Cauchy is not so much about the convexity of U, but rather about this property of γ itself? Is it true that if $\gamma : [a,b] \to U$ is a closed piecewise C^1 path with $n_\gamma(z) = 0$, $\forall z \in \mathbb{C} \backslash U$, then for all $f \in H(U)$ we have $\int_\gamma f = 0$ even if U is just assumed open? Yes!

And even more is true: we can use formal sums of paths (called cycles) and not just individual paths. To this end, we first quickly study the following concepts of chains and cycles.

5.3.1 Chains and Cycles

Suppose $\gamma_1, \ldots, \gamma_n$ are paths in the plane, and put $K = \gamma_1^* \cup \cdots \cup \gamma_n^*$. Each γ_i induces a linear functional $\widetilde{\gamma}_i$ on the vector space $C(K)$, by the formula

$$\widetilde{\gamma}_i(f) = \int_{\gamma_i} f(z) dz$$

Define

$$\widetilde{\Gamma} = \widetilde{\gamma}_1 + \cdots + \widetilde{\gamma}_n.$$

Explicitly, $\widetilde{\Gamma}(f) = \widetilde{\gamma}_1(f) + \cdots + \widetilde{\gamma}_n(f)$ for all $f \in C(K)$. The above relation suggests that we introduce a "formal sum"

$$\Gamma = \gamma_1 + \cdots + \gamma_n = \sum_{i=1}^{n} \gamma_i$$

and define

$$\int_\Gamma f(z) dz = \widetilde{\Gamma}(f).$$

Then $\Gamma = \gamma_1 + \cdots + \gamma_n$ is merely an abbreviation for the statement

$$\int_\Gamma f(z) dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z) dz \quad (f \in C(K)).$$

Note that this equation serves as the definition of its left side. The objects Γ so defined are called **chains**. If each γ_j in $\Gamma = \sum_{i=1}^n \gamma_i$ is a closed path, then Γ is called a **cycle**. If each γ_j in $\Gamma = \sum_{i=1}^n \gamma_i$ is a path in some open set U, we say that Γ is a **chain** in U. If $\Gamma = \sum_{i=1}^n \gamma_i$ holds, we define

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*$$

If Γ is a cycle and $\alpha \notin \Gamma^*$, we define the index of α with respect to Γ by

$$\operatorname{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-\alpha},$$

Obviously, $\Gamma = \sum_{i=1}^n \gamma_i$ implies

$$\operatorname{Ind}_\Gamma(\alpha) = \sum_{i=1}^n \operatorname{Ind}_{\gamma_i}(\alpha).$$

If each γ_i in $\Gamma = \sum_{i=1}^n \gamma_i$ is replaced by its opposite path $\overline{\gamma_i}$, the resulting chain will be denoted by $-\Gamma$. Then

$$\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz \quad \left(f \in C\left(\Gamma^*\right)\right).$$

In particular, $\operatorname{Ind}_{-\Gamma}(\alpha) = -\operatorname{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$. Chains can be added and subtracted in the obvious way, by adding or subtracting the corresponding functionals: The statement $\Gamma = \Gamma_1 + \Gamma_2$ means

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

for every $f \in C\left(\Gamma_1^* \cup \Gamma_2^*\right)$. Finally, note that a chain may be represented as a sum of paths in many ways. To say that

$$\gamma_1 + \cdots + \gamma_n = \delta_1 + \cdots + \delta_k$$

means simply that

$$\sum_i \int_{\gamma_i} f(z)dz = \sum_j \int_{\delta_j} f(z)dz$$

for every f that is continuous on $\gamma_1^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_k^*$. In particular, a cycle may very well be represented as a sum of paths that are not closed.

5.3.2 Global Cauchy Theorem

We will use the following lemma for proof of the global Cauchy theorem.

Lemma 5.3.1. *If $f \in H(U)$ and g is defined in $U \times U$ by*

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

then g is continuous in $U \times U$.

Proof. The only points $(z,w) \in U \times U$ at which the continuity of g is possibly in doubt have $z = w$.

Fix $a \in U$. Fix $\varepsilon > 0$. There exists $r > 0$ such that $B(a,r) \subset U$ and $|f'(\xi) - f'(a)| < \varepsilon$ for all $\xi \in B(a,r)$. If z and w are in $B(a,r)$ and if

$$\gamma(t) = [z,w](t) = (1-t)z + tw,$$

then $\gamma(t) \in B(a,r)$ for $0 \le t \le 1$. When $(w,z) \neq (a,a)$, by Example 3.1.17 (b),

$$\begin{aligned} g(z,w) - g(a,a) &= \frac{f(z)-f(w)}{z-w} - f'(a) = -\frac{1}{z-w}\int_\gamma f'(\xi)d\xi - f'(a) \\ &= \frac{1}{w-z}\int_0^1 f'(\gamma(t))\gamma'(t)dt - f'(a) = \int_0^1 [f'(\gamma(t)) - f'(a)]dt \end{aligned}$$

and this equation is also true for the case $(w,z) = (a,a)$ where $\gamma(t) = a$ for $0 \le t \le 1$. The absolute value of the integrand is $< \varepsilon$, for every t. Thus $|g(z,w) - g(a,a)| < \varepsilon$. This proves that g is continuous at (a,a). □

Theorem 5.3.2 (Global Cauchy Theorem). *Suppose $f \in H(U)$, where $U \subseteq \mathbb{C}$ is an open set. If Γ is a cycle in U that satisfies*

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \quad \textit{for every } \alpha \textit{ not in } U, \tag{5.9}$$

then

$$f(z)\cdot \operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw \quad \textit{for } z \in U - \Gamma^* \tag{5.10}$$

and

$$\int_\Gamma f(z)dz = 0. \tag{5.11}$$

If Γ_0 and Γ_1 are cycles in U such that

$$\operatorname{Ind}_{\Gamma_0}(\alpha) = \operatorname{Ind}_{\Gamma_1}(\alpha) \quad \textit{for every } \alpha \textit{ not in } U, \tag{5.12}$$

then

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz \tag{5.13}$$

Proof. The function g defined in $U \times U$ by

$$g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

is continuous in $U \times U$ by the lemma, so $h(z)$ defined by the integral

$$h(z) = \frac{1}{2\pi i}\int_\Gamma g(z,w)dw \quad (z \in U)$$

is well-defined. Since $\int_\Gamma f'(z)dw = 0$ and $\frac{1}{2\pi i}\int_\Gamma \frac{f(w)-f(z)}{w-z}dw = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)dw}{w-z} - f(z)\operatorname{Ind}_\Gamma(z)$, the formula (5.10) is equivalent to the assertion that

$$h(z) = 0 \quad (z \in U - \Gamma^*) \tag{5.14}$$

To prove equation (5.14), let us first show h is continuous on U. Fix $z_0 \in U$ and an arbitrary sequence $\{z_n\} \in U, z_n \to z_0$. As $g: U \times U \to \mathbb{C}$ is continuous, it is uniformly continues on compact sets of $U \times U$. Choose r s.t. $\bar{B}(z_0,r) \subset U$. Fix $\varepsilon > 0$. Now, y is uniformly continuous on $\bar{B}(z_0,r) \times \Gamma^* \Rightarrow \exists \delta < r$ s.t. whenever $(z_1,w_1),(z_2,w_2) \in \bar{B}(z_0,r) \times \Gamma^*$ satisfy $|(z_1,w_1) - (z_2,w_2)| < \delta$ then $|g(z_1,w_1) - g(z_2,w_2)| < \varepsilon$. Now, choose N s.t. $z_n \in B(z_0,\delta)\ \forall n > N$. Then $\forall w \in \Gamma^*$ we have for all $n > N$ that

$$|g(z_n,w) - g(z_0,w)| < \varepsilon$$

as $(z_n,w),(z_0,w) \in B(z_0,\delta) \times \Gamma^* \subset \overline{B}(z,r) \times \Gamma^*$ and $|(z_n,w) - (z_0,w)| = |z_n - z_0| < \delta$. Therefore,

$$g_n(w) := g(z_n,w) \rightrightarrows g(z_0,w)$$

uniformly for $w \in \Gamma^*$. Then

$$\lim_{n\to\infty} h(z_n) = \frac{1}{2\pi i}\lim_{n\to\infty}\int_\Gamma g_n(w)dw = \frac{1}{2\pi i}\int_\Gamma \lim_{n\to\infty} g_n(w)dw = \frac{1}{2\pi i}\int_\Gamma g(z_0,w)dw = h(z_0)$$

This shows that h is continuous at z_0. Since z_0 is an arbitrary point in U, h is continuous on U.

To further show that $h \in H(U)$, we recall Morera's theorem that a function $U \to \mathbb{C}$ is analytic on an open set U if its integral over any triangle $\partial\triangle$ in U is zero. Now,

$$\int_{\partial\triangle} h(z)\,dz = \int_{\partial\triangle}\left(\frac{1}{2\pi i}\int_\Gamma g(z,w)dw\right)dz \overset{\text{Fubini}}{=\!=} \frac{1}{2\pi i}\int_\Gamma\left(\int_{\partial\triangle} g(z,w)dw\right)dw$$

Now, with $w \in \Gamma^*$ fixed, the function $z \to g(z,w)$ is obviously analytic in $U \setminus \{w\}$ and continuous in U. Hence, we can either use Cauchy for triangles that allow a special point (Theorem 3.2.1), or conclude that $z \to g(z,w)$ is in $H(U)$ as the singularity at $z = w$ must be removable by continuity. In either way, we have $\int_{\partial\triangle} g(z,w)dw = 0$. Thus, $\int_{\partial\triangle} h(z)\,dz = 0$ and Morera's theorem gives $h \in H(U)$.

Next, we define function

$$p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi \quad (z \in \mathbb{C}\setminus\Gamma^*)$$

and show its analyticity. This is more straightforward than h. In fact, recall how we proved that an analytic function can be developed into a power series by writing it via Cauchy's integral formula and expanding $\frac{1}{\xi - z}$ into a power series (Theorem 4.3.9). The same strategy can be applied to p.

Fix $z_0 \in \mathbb{C}\setminus\Gamma^*$ and choose $\delta > 0$ such that $B(z_0, 2\delta) \subseteq \mathbb{C}\setminus\Gamma^*$. Write

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0}\frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\xi - z_0)^{n+1}}.$$

which uniformly converges for $\xi \in \Gamma^*$ and $z \in B(z_0,\delta)$ because $B(z_0, 2\delta) \subseteq \mathbb{C}\setminus\Gamma^* \Rightarrow |\xi - z_0| > 2\delta$ and $z \in B(z_0,\delta) \Rightarrow |z - z_0| < \delta$, so $\left|\frac{z-z_0}{\xi - z_0}\right| \le \frac{\delta}{2\delta} = \frac{1}{2}$. Then

$$p(z) = \sum_{n=0}^{\infty}\underbrace{\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi}_{c_n}(z-z_0)^n \quad (z \in B(z_0,\delta))$$

Thus, $p \in H(B(z_0,\delta))$. Since $z_0 \in \mathbb{C}\setminus\Gamma^*$ was arbitrary, p is analytic in $\mathbb{C}\setminus\Gamma^*$.

Now, we have analytic functions

$$h(z) = \frac{1}{2\pi i}\int_\Gamma g(z,w)dw \quad (z \in U)$$
$$p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi \quad (z \in \mathbb{C}\setminus\Gamma^*)$$

We glue them to get an entire function, which will exploit the assumption on Γ we still didn't use, i.e., $n_\Gamma(\alpha) = 0$, $\forall \alpha \in \mathbb{C}\setminus U$.

Let $\Omega := \{z \in \mathbb{C} \mid \Gamma^* : n_\Gamma(z) = 0\}$. Our assumption on Γ implies that $\mathbb{C}\setminus U \subset \Omega$, and so $\mathbb{C} = U \cup \Omega$.

If $z \in U \cap \Omega$, then both h and p are defined, and in fact, we have

$$h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi - f(z)\overbrace{n_\Gamma(z)}^{=0} = p(z).$$

So we can define $\varphi \in H(\mathbb{C})$ by setting

$$\varphi(z) := \begin{cases} h(z), & z \in U, \\ p(z), & z \in \Omega. \end{cases}$$

This is well-defined, since in $U \cap \Omega$ we have $h(z) = p(z)$.

Notice that Ω is open: $\Omega = n_\Gamma^{-1} B(0,1)$ as $n_\gamma : \mathbb{C} \setminus \Gamma^* \to \mathbb{C}$ is $\mathbb{Z}$-valued and continuous. It is then clear that $\varphi \in H(\mathbb{C})$, as for each $z \in \mathbb{C}$ there is an open neighborhood in which φ equals to h or p.

Our final step is to show $h \equiv 0$.

We will apply Liouville's theorem to φ. Notice that Γ^* is bounded and so $\Gamma^* \subset D$ for some closed disk D, Thus $\mathbb{C} \backslash D \subset \mathbb{C} \backslash \Gamma^*$ and for big $|z|$ we must be in the unbounded connected open set $\mathbb{C} \setminus D$ where $n_\Gamma(z) = 0$. So for big $|z|$ we have $z \in \Omega$ and so

$$|\varphi(z)| = |p(z)| = \left| \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi \right| \leq \frac{1}{2\pi} \frac{\|f\|_\infty}{\operatorname{dist}(z, \Gamma^*)}$$

$$\left(\xi \in \Gamma \Rightarrow \operatorname{dist}(z, \Gamma^*) \leq |\xi - z| \Rightarrow \sup_{\xi \in \Gamma^*} \left| \frac{f(\xi)}{\xi - z} \right| \leq \frac{\sup_{\xi \in \Gamma^* |f(\xi)|}}{\operatorname{dist}(z, \Gamma^*)} = \frac{\|f\|_\infty}{\operatorname{dist}(z, \Gamma^*)} \right)$$

This implies φ is bounded in $\mathbb{C}$ (compare to the argument in the proof of the fundamental theorem of algebra) and

$$\varphi(z) \to 0 \text{ as } |z| \to \infty$$

By Liouville's theorem, φ is constant and the constant must be 0. So $h(z) = \varphi(z) = 0$ for $z \in U$, and $h(z) = 0$ for $z \in U \backslash \Gamma^*$, so (5.10) is verified (notice that $n_\Gamma(z)$ only makes sense in $U \setminus \Gamma^*$, not in whole U). To deduce (5.11) from (5.10), we pick $z_0 \in U \backslash \Gamma^*$ and define $F(z) = (z - z_0) f(z)$. Then $F \in H(U)$ and they apply to F to give

$$\frac{1}{2\pi i} \int_\Gamma f(z) dz = \frac{1}{2\pi i} \int_\Gamma \frac{F(z)}{z - z_0} dz = \overbrace{F(z_0)}^{=0} n_\Gamma(z_0) = 0.$$

Finally, the path deformation claim follows from applying (5.11) to $\Gamma := \Gamma_1 - \Gamma_0$. □

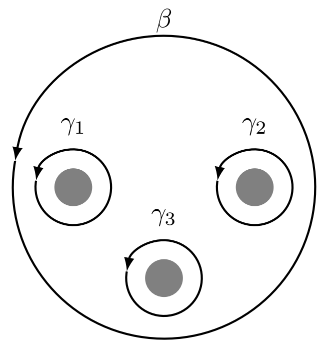
Remark 5.3.3.
(a) If γ is a closed path in a convex region Ω and if $\alpha \notin \Omega$, an application of Cauchy's theorem on convex set to $f(z) = (z - \alpha)^{-1}$ shows that $\operatorname{Ind}_\gamma(\alpha) = 0$. Assumption on Γ in global version is therefore satisfied by every cycle in Ω if Ω is convex. This shows that global version generalizes Cauchy's theorem and formula on convex set.

(b) The path deformation part of the above theorem shows under what circumstances integration over one cycle can be replaced by integration over another, without changing the value of the integral. For example, let U be the plane with three disjoint closed discs D_i removed, i.e., $U = \mathbb{C} \setminus (D_1 \cup D_2 \cup D_3)$. If $\beta, \gamma_1, \gamma_2, \gamma_3$ are positively oriented circles in Ω such that β surrounds $D_1 \cup D_2 \cup D_3$ and γ_i surrounds D_i but not D_j for $j \neq i$, then

$$\forall \alpha \in \mathbb{C} \setminus U, \operatorname{Ind}_\beta(\alpha) = \operatorname{Ind}_{\gamma_1 + \gamma_2 + \gamma_3}(\alpha).$$

Then the global Cauchy theorem implies that for every $f \in H(U)$,

$$\int_\beta f(z) dz = \sum_{i=1}^{3} \int_{\gamma_i} f(z)\, dz$$



(c) In order to apply global Cauchy, it is desirable to have a reasonably efficient method of finding the index of a point with respect to a closed path. The following theorem does this for all paths that occur in practice. It says, essentially, that the index increases by 1 when the path is crossed "from right to left." Recall $\mathrm{Ind}_\gamma(\alpha) = 0$ *for* α *in the unbounded component of the complement* W *of* γ^*. *We can then successively determine* $\mathrm{Ind}_\gamma(\alpha)$ *in the other components of* W, *provided that* W *has only finitely many components and that* γ *traverses no arc more than once.*

Theorem 5.3.4. *Suppose* γ *is a closed path in the plane, with parameter interval* $[\alpha, \beta]$. *Suppose* $\alpha < u < v < \beta$, a *and* b *are complex numbers,* $|b| = r > 0$, *and*

(i) $\gamma(u) = a - b, \gamma(v) = a + b$,

(ii) $|\gamma(s) - a| < r$ *if and only if* $u < s < v$,

(iii) $|\gamma(s) - a| = r$ *if and only if* $s = u$ *or* $s = v$.

Assume furthermore that $D(a; r) - \gamma^*$ *is the union of two regions,* D_+ *and* D_-, *labeled so that* $a + bi \in \bar{D}_+$ *and* $a - bi \in \bar{D}_-$. *Then*

$$\mathrm{Ind}_\gamma(z) = 1 + \mathrm{Ind}_\gamma(w)$$

if $x \in D_+$ *and* $w \in D_-$. *As* $\gamma(t)$ *traverses* $D(a; r)$ *from* $a - b$ *to* $a + b, D_-$ *is "on the right" and* D_+ *is "on the left" of the path.*

Proof. See [14] Theorem 10.37 □

5.3.3 Homotopy

We introduce a concept in algebraic topology that is also related to Cauchy's theorem. First, to be clearer on the terminology, we would say *curves* are continuous but not necessarily differentiable while *paths* are assumed to be piecewise C^1.

Suppose $\gamma_0, \gamma_1 : I \to U$ are closed curves in a topological space X. We say that γ_0 and γ_1 are U**-homotopic** if there is a continuous map $H : I \times I \to U$ such that

$$H(s, 0) = \gamma_0(s), \quad H(s, 1) = \gamma_1(s), \quad H(0, t) = H(1, t)$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s, t)$. Then H defines a one-parameter family of closed curves γ_t in X, which connects γ_0 and γ_1. Intuitively, this means that γ_0 can be continuously deformed to γ_1, within X.

If γ_0 is U-homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is null-homotopic in U. If U is connected and if every closed curve in U is null-homotopic, U is said to be simply connected.

For example, every convex region U is simply connected. To see this, let γ_0 be a closed curve in U, fix $z_1 \in U$, and define **straight-line homotopy**

$$H(s,t) = (1-t)\gamma_0(s) + tz_1 \quad (0 \le s \le 1,\, 0 \le t \le 1)$$

Theorem 5.3.6 will show that (5.12) in the global Cauchy holds whenever Γ_0 and Γ_1 are U-homotopic closed paths. As a special case of this, note that condition (5.9) holds for every closed path Γ in U if U is simply connected, since constant paths necessarily have zero index.

Lemma 5.3.5. *Let γ_0 and γ_1 be closed paths with parameter interval $[0,1]$ and let α be a complex number. If*

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \quad (0 \le s \le 1) \tag{5.15}$$

then

$$\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_0}(\alpha).$$

Proof. Condition (5.15) implies that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$. Hence one can define $\gamma(t) = (\gamma_1(t) - \alpha) / (\gamma_0(t) - \alpha)$. Since

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \Rightarrow \left|\frac{\gamma_0 - \gamma_1}{\gamma_0 - \alpha}\right| = \left|\frac{\gamma_0 - \alpha - (\gamma_1 - \alpha)}{\gamma_0 - \alpha}\right| = |1-\gamma| < 1$$

we see $\gamma^* \in B(1,1)$, which implies that $\mathrm{Ind}_\gamma(0) = 0$. The following relationship between the paths and their derivatives can also be easily calculated.

$$\frac{\gamma'}{\gamma} = \frac{\gamma_1'}{\gamma_1 - \alpha} - \frac{\gamma_0'}{\gamma_0 - \alpha}$$

Integrating the above identity over $[0,1]$ and writing them in path integral forms will give the desired result:

$$\underbrace{\int_\gamma \frac{1}{z}\,dz}_{\mathrm{Ind}_\gamma(0)=0} = \underbrace{\int_{\gamma_1} \frac{1}{z-\alpha}\,dz}_{\mathrm{Ind}_{\gamma_1}(\alpha)} - \underbrace{\int_{\gamma_0} \frac{1}{z-\alpha}\,dz}_{\mathrm{Ind}_{\gamma_0}(\alpha)}$$

□

Theorem 5.3.6. *If Γ_0 and Γ_1 are U-homotopic closed paths in an open connected set U, and if $\alpha \notin U$, then*

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha)$$

Proof. By definition, there is a continuous $H : I^2 \to U$ such that

$$H(s,0) = \Gamma_0(s), \quad H(s,1) = \Gamma_1(s), \quad H(0,t) = H(1,t).$$

Since I^2 is compact, so is $H\left(I^2\right)$. Since α is not in the closed set $H(I^2)$, there exists $\varepsilon > 0$ such that $B(\alpha, 2\varepsilon) \cap H(I^2) = \varnothing$, i.e.,

$$|\alpha - H(s,t)| \ge 2\varepsilon \quad \forall (s,t) \in I^2. \tag{5.16}$$

Since H is continuous on compact set and is thus uniformly continuous, there is a positive integer n such that:

$$|H(s,t) - H(s',t')| < \varepsilon \quad \text{if} \quad |s-s'| + |t-t'| \le 1/n. \tag{5.17}$$

(Note that $\sqrt{|s-s'|^{1/2} + |t-t'|^{1/2}} \le |s-s'| + |t-t'|$)

Define polygonal closed paths $\gamma_0, \ldots, \gamma_n$ by

$$k = 0,1,\cdots,n: \quad \gamma_k(s) = H\left(\frac{i}{n}, \frac{k}{n}\right)(1-(i-ns)) + H\left(\frac{i-1}{n}, \frac{k}{n}\right)(i-ns) \tag{5.18}$$

if $0 \le i - ns \le 1$ $\left(\text{i.e., } \frac{i-1}{n} \le s \le \frac{i}{n}\right)$ and $i = 1, \ldots, n$. By (5.17) and (5.18), for $k \in [n], s \in [0,1]$,

$$(*): \left|\gamma_k(s) - H\left(s, \frac{k}{n}\right)\right| \le (ns+1-i)\underbrace{\left|H\left(\frac{i}{n}, \frac{k}{n}\right) - H\left(s, \frac{k}{n}\right)\right|}_{<\varepsilon} + (i-ns)\underbrace{\left|H\left(\frac{i-1}{n}, \frac{k}{n}\right) - H\left(s, \frac{k}{n}\right)\right|}_{<\varepsilon} < \varepsilon$$

(Note that $(ns+1-i) + (i-ns) = 1$ when using the triangle inequality)

In particular, taking $k = 0$ and $k = n$,

$$|\gamma_0(s) - \Gamma_0(s)| < \varepsilon, \quad |\gamma_n(s) - \Gamma_1(s)| < \varepsilon.$$

By $(*)$ and (5.16),

$$|\alpha - \gamma_k(s)| > 2\varepsilon - \varepsilon = \varepsilon \quad (k \in [n]; s \in [0,1]).$$

On the other hand, (5.17) and (5.18) also imply that for $k \in [n]; s \in [0,1]$,

$$|\gamma_{k-1}(s) - \gamma_k(s)| \le (ns+1-i)\underbrace{\left|H\left(\frac{i}{n}, \frac{k-1}{n}\right) - H\left(\frac{i}{n}, \frac{k}{n}\right)\right|}_{<\varepsilon} + (i-ns)\underbrace{\left|H\left(\frac{i-1}{n}, \frac{k-1}{n}\right) - H\left(\frac{i-1}{n}, \frac{k}{n}\right)\right|}_{<\varepsilon} < \varepsilon.$$

Now it follows from the last three inequalities, and $n+2$ applications of Lemma 5.3.5 that α has the same index with respect to each of the paths $\Gamma_0, \gamma_0, \gamma_1, \ldots, \gamma_n, \Gamma_1$. This proves the theorem. □

Remark 5.3.7.

1. *If $\Gamma_t(s) = H(s,t)$ in the preceding proof, then each Γ_t is a closed curve, but not necessarily a path, since H is not assumed to be differentiable. The paths γ_k were introduced for this reason. Another (and perhaps more satisfactory) way to circumvent this difficulty is to extend the definition of index to closed curves.*
2. *As promised, this theorem of sufficiency of path deformation and global cauchy show that any integral of analytic function along γ in a simply-connected set U is zero. We shall see the converse is also true, i.e., this property can be used as definition of simply-connectedness (of a set in complex plane).*

Theorem 5.3.8. *Let $U \subset \mathbb{C}$ be open and connected. Then the following are equivalent.*

(a) *U is homeomorphic to $B(0,1)$ (i.e. there is a continuous bijection $\psi : U \to B(0,1)$ s.t. ψ^{-1} is also continuous).*

(b) *U is simply connected;*

(c) *$\int_\gamma f(z)dz = 0 \quad \forall f \in H(U)$ and every closed path $\gamma : [a,b] \to U$;*

(d) *Every $f \in H(U)$ has a primitive;*

(e) *If $f, 1/f \in H(U)$ (i.e., f is analytic and non-vanishing), then $f = e^g$ for some $g \in H(U)$ ("f has a holomorphic logarithm g in U");*

(f) *If f in a non-vanishing analytic function, then $f = \varphi^2$ for some $\varphi \in H(U)$ ("f has a holomorphic square root φ in U").*

Proof.

$(a) \Rightarrow (b)$: that's basically because $B(0,1)$ is simply-connected and homeomorphism preserves null-homotopy (in fact the whole fundamental group). Let $\psi : U \to B(0,1)$ be the homeomorphism. For the closed curve γ in U, the map $H(s,t) = \psi^{-1}((1-t)0 + t\psi(\gamma(s))) = \psi^{-1}(t\psi(\gamma(s)))$ defines a homotopy between constant map $c_{\psi^{-1}(0)}$ and curve $\gamma(s)$.

$(b) \Rightarrow (c)$: consequence of 5.3.6 and 5.3.2.

$(c) \Rightarrow (d)$: The proof resembles that for Cauchy's theorem in a convex set (Theorem 3.2.3). Fix $a \in U$ arbitrary, and set

$$g(z) := \int_{\gamma_z} f(w)\mathrm{d}w$$

where γ_z is any path (i.e., piecewise C^1 curve) connecting a to z inside of U. Note: the fact that we can always select a piecewise C^1 curve is not guaranteed by the definition of connectness (which only gives a continuous curve) - however, it is a well-known result from basic topology that in an open, connected set we can always connect two points with a finite union of line segments (a polygonal path).

Also note that g is well-defined: if $\tilde{\gamma}_z$ is some other path connecting a to z in U, then $\gamma_z - \tilde{\gamma}_z$ is a closed path in U and by the given assumption (c),

$$\int_{\gamma_z - \tilde{\gamma}_z} f(w)\mathrm{d}w = 0 \Rightarrow \int_{\gamma_z} f(w)\mathrm{d}w = \int_{\tilde{\gamma}_z} f(w)\mathrm{d}w$$

Choose now $r > 0$ such that $B(z, r) \subset U$. Consider $h \in \mathbb{C}$ with $|h| < r$ so that $z + h \in B(z, r)$. Let $\eta = [z, z+h]$ be the line segment path connecting z to $z+h$ inside of $B(z,r) \subset U$. Then $\gamma + \eta$ is a path connecting a to $z+h$, and due to invariance of path in the definition of $g(z+h)$ as we just showed, we have

$$g(z+h) = \int_{\gamma+\eta} f(w)\mathrm{d}w = \int_{\gamma} f(w)\mathrm{d}w + \int_{\eta} f(w)\mathrm{d}w$$

and so

$$g(z+h) - g(z) = \int_{\eta} f(w)\mathrm{d}w.$$

Since

$$\frac{1}{h}\int_{\eta} f(z)\mathrm{d}w = \frac{1}{(z+h)-z}\int_{[z,z+h]} f(z)\mathrm{d}w = f(z),$$

and $\mathrm{length}(\eta) = |z+h-z| = |h|$, we apply Corollary 3.1.6 to see

$$\left|\frac{g(z+h)-g(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{\eta} f(w) - f(z)\mathrm{d}w\right| \leq \|f - f(z)\|_{L^\infty([z,z+h])} \leq \varepsilon$$

where the last inequality holds for sufficiently small $|h|$ due to continuity of f. Thus,

$$g'(z) = \lim_{h\to 0} \frac{g(z+h)-g(z)}{h} = f(z)$$

$(d) \Rightarrow (e)$: The identity $f = e^g$, if it were to hold for some $g \in H(U)$, implies $f'(z) = g'(z)e^{g(z)} = g'(z)f(z)$ so that $g'(z) = f'(z)/f(z)$. So we want a primitive of the analytic function f'/f (recall f has no zeros in U by assumption). Let $z_0 \in U$ be a fixed point and c_0 is a complex number with $e^{c_0} = f(z_0)$ (recall e^z obtains all values except 0 and $f(z_0) \neq 0$). Since $f' \in H(U)$ and $1/f \in H(U)$, assumption (d) gives a primitive g of f'/f. We can assume $g(z_0) = c_0$ (otherwise take $\tilde{g} = g + c_0 - g(z_0)$).

We claim that g in turn satisfies $f = e^g$. Motivated by the midterm Q2, we study $G(z) := e^{g(z)}/f(z)$. Now, $g'(z) = f'(z)/f(z)$ gives

$$G'(z) = g'(z)e^{g(z)}/f(z) - e^{g(z)}f(z)^{-2}f'(z) = e^{g(z)}f(z)^{-2}f'(z) - e^{g(z)}f(z)^{-2}f'(z) = 0.$$

This implies, as U is connected, that $G(z) = e^{g(z)}/f(z) = C$ for some constant C, and so $e^{g(z)} = Cf(z)$ in U. Now, as $e^{g(z_0)} = e^{c_0} = f(z_0)$ we must have $C = 1$, and so we are done.

$(e) \Rightarrow (f)$: Use (e) to write $f = e^g$ with $g \in H(U)$. Define $\varphi = e^{g/2}$. Then $\varphi^2 = e^g = f$.

$(f) \Rightarrow (a)$: If $U = \mathbb{C}$, the homeomorphism is just directly given (without using (f)) by $z \mapsto \frac{z}{1+|z|}$. If the open connected set U is not the whole $\mathbb{C}$, there actually exists a holomorphic homeomorphism $U \to B(0,1)$ (a conformal mapping). This is the Riemann Mapping Theorem. This implication is thus proved as soon as we later prove Riemann mapping theorem (using only (f) and nothing else about simply connected domains).

□

5.4 Laurent Series

Taylor series describe holomorphic functions locally near a point where the function is regular. When a point is removed from the domain, however, negative powers may appear and the correct local expansion is a Laurent series. Laurent series are the natural analytic tool for understanding isolated singularities.

Let

$$A := \{z \in \mathbb{C} : r < |z-a| < R\}$$

be an annulus and suppose $f \in H(A)$. Let γ_r and γ_R be positively oriented circles centered at a of radii r and R respectively.

By the global Cauchy theorem applied to the cycle $\Gamma = \gamma_R - \gamma_r$, we obtain

$$\int_{\gamma_R} f(z)\,dz = \int_{\gamma_r} f(z)\,dz. \tag{5.19}$$

More generally, for any w with $r < |w-a| < R$, we have $\mathrm{Ind}_\Gamma(w) = 1$ and

$$f(w) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z-w}\,dz - \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z-w}\,dz. \tag{5.20}$$

This is the Cauchy integral formula adapted to an annulus.

Theorem 5.4.1 (Laurent decomposition). *Let $f \in H(A)$, where $A = \{z : r < |z-a| < R\}$. Then there exist unique holomorphic functions*

$$g \in H(B(a,R)), \qquad h \in H(B(0,1/r))$$

such that

$$f(z) = g(z) + h\left(\frac{1}{z-a}\right), \qquad r < |z-a| < R. \tag{5.21}$$

Proof. Define

$$g(w) := \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z-w}\,dz, \qquad |w-a| < R,$$

and

$$h(\zeta) := -\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z-(a+1/\zeta)}\,dz, \qquad |\zeta| < 1/r.$$

Both integrals define holomorphic functions in the stated domains by the Cauchy integral theorem. Equation (5.20) implies (5.21). Uniqueness follows from analytic continuation. □

Theorem 5.4.2 (Laurent series). *Let $f \in H(A)$ with $A = \{z : r < |z-a| < R\}$. Then f admits a unique expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad r < |z-a| < R, \tag{5.22}$$

where the coefficients are given by

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz, \tag{5.23}$$

for any positively oriented circle γ with radius $\rho \in (r, R)$.

Proof. Expand the functions g and h from Theorem 5.4.1 into power series and combine them. Formula (5.23) follows from Cauchy's integral formula. □

Let $f \in H(U \setminus \{a\})$. The Laurent expansion of f at a determines the nature of the singularity.

Theorem 5.4.3 (Classification via Laurent series)**.** *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

be the Laurent expansion of f in a punctured neighborhood of a.

(a) $c_n = 0$ for all $n < 0 \iff a$ is removable.

(b) finitely many c_n with $n < 0$ are nonzero $\iff a$ is a pole.

(c) infinitely many c_n with $n < 0$ are nonzero $\iff a$ is an essential singularity.

This provides a direct analytic criterion for singularity classification.

When a function is expressed using known Taylor series, the most efficient way to determine the type of a singularity is often to compute the Laurent series explicitly.

Example 5.4.4. (1) The function

$$f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \quad \text{with } f(z) = \frac{\sin z}{z}$$

has at $a = 0$ a removable singularity, because from

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots$$

we find for all $z \in \mathbb{C} \setminus \{0\}$,

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} \pm \cdots$$

(2) The function

$$f(z) = \frac{\exp z}{z^3} \quad (z \neq 0)$$

has at $a = 0$ a pole of order 3, because we have:

$$\begin{aligned} f(z) &= \frac{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots}{z^3} \\ &= \underbrace{\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z}}_{h(1/z)} + \underbrace{\frac{1}{3!} + \frac{1}{4!}z + \frac{1}{5!}z^2 + \cdots}_{g(z)} \end{aligned}$$

(3) The function

$$f(z) = \exp\left(-\frac{1}{z^2}\right) \quad (z \neq 0)$$

has at $a=0$ an essential singularity, because the Laurent series is of the form

$$f(z)=1-\frac{1}{z^2}+\frac{1}{2!}\frac{1}{z^4}-\frac{1}{3!}\frac{1}{z^6}\pm\cdots=1+h(1/z)$$

The principal part contains infinitely many coefficients $\neq 0$. ◇

Changing annuli can change Laurent series. Specifying the center a for the punctured disk is not enough to pin down the Laurent series.

Example 5.4.5 (Different Laurent expansions in different annuli). Consider

$$f(z):=\frac{2}{z^2-4z+3}=\frac{1}{1-z}+\frac{1}{z-3},\qquad z\in\mathbb{C}\setminus\{1,3\}.$$

(a) $0<|z|<1$. Using the geometric series,

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\qquad \frac{1}{3-z}=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n,$$

we obtain

$$f(z)=\sum_{n=0}^{\infty}\left(1-\frac{1}{3^{n+1}}\right)z^n.$$

Here the Laurent series is a Taylor series and $z=0$ is a removable singularity.

(b) $1<|z|<3$. We write

$$\frac{1}{z-1}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}},\qquad \frac{1}{3-z}=\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}},$$

which yields

$$f(z)=\sum_{n=1}^{\infty}\frac{-1}{z^n}+\sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}z^n.$$

This expansion has both positive and negative powers.

(c) $|z|>3$. Using

$$\frac{1}{z-3}=\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}},$$

we obtain

$$f(z)=\sum_{n=1}^{\infty}\left(3^{n-1}-1\right)\frac{1}{z^n}.$$

The same function admits different Laurent expansions in different annuli centered at the same point. Uniqueness of the Laurent series holds only after the annulus is fixed. ◇

5.5 Residue Theorem

Definition 5.5.1. *A function f is said to be **meromorphic** in an open set U if there is a set $A\subset U$ such that*

(a) A has no limit point in U,

(b) $f\in H(U-A)$,

(c) f has a pole at each point of A.

Note that the possibility $A = \varnothing$ is not excluded. Thus every $f \in H(U)$ is meromorphic in U.

Note also that (a) implies that no compact subset of U contains infinitely many points of A, and that A is therefore at most countable.

If f and A are as above, if $a \in A$, and if

$$Q(z) = \sum_{k=1}^{m} c_{-k}(z-a)^{-k}$$

is the principal part of f at a, as defined in Theorem 5.2.5 (i.e., if $f - Q$ has a removable singularity at a), then the number c_{-1} is called the **residue** of f at a:

$$c_{-1} = \operatorname{Res}(f; a).$$

Theorem 5.5.2 (The Residue Theorem). *Suppose f is a meromorphic function in U. Let A be the set of points in U at which f has poles. If Γ is a cycle in $U - A$ such that*

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \quad \textit{for all} \quad \alpha \notin U$$

then

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{a\in A} \operatorname{Res}(f; a) \operatorname{Ind}_\Gamma(a)$$

Proof. We will argue the sum on the RHS, though formally infinite, is actually finite. Let

$$B := \{a \in A : n_\Gamma(a) \neq 0\}.$$

Let $\Omega := \mathbb{C}\backslash\Gamma^*$. Denote the components of Ω by $\mathcal{F} = \cup V$. Each V is connected. The components are disjoint. Every connected set $E \subset \Omega$ is containted in exactly one $V \in \mathcal{F}$. Choose a disk D s.t. $\Gamma^* \subset D$ (possible as Γ^* is compact) and notice $\mathbb{C} \setminus D \subset \Omega$. So there exists a unique $V_0 \in \mathcal{F}$ s.t. $\mathbb{C} \setminus D \subset V_0$ (as $\mathbb{C} \setminus D \subset \Omega$ and $\mathbb{C} \setminus D$ is connected). So $\forall V \in \mathcal{F}^* = \mathcal{F} - \{V_0\}$, we have $V \subset \mathbb{C} \setminus V_0 \subset D$. Notice that $n_\Gamma(z) = 0$ in V_0 as $V_0 \subset \Omega$ unbounded, connected. Thus, $B \subset \bigcup_{V\in\mathcal{F}^*} V \subset D$ is bounded.

If $|B|$ is not finite, we can choose $a_1, a_2, \ldots \in B$ s.t, $a_i \neq a_j$ $(i \neq j)$. As $\bar{B}$ is compact, $\exists$ subseq. $a_{i_k} \to a \in \bar{B}$. Now $a \in A^{\text{acc}}$ clearly, but also $a \in U$. Indeed, if $a \notin U$ then $n_\Gamma(a) = 0$ bs assumption. But as n_Γ is continuous and $\mathbb{Z}$-valued, this forces $n_\Gamma(a_{i_k}) = 0$ for k large, contradicting to $a_{ik} \in B$. So $a \in U \cap A^{\text{acc}}$, contradicting to our assumption that $U \cap A^{\text{acc}} = \varnothing$. Thus $|B|$ is finite and

$$\sum_{a\in A} \operatorname{Res}(f; a) n_\Gamma(a) = \sum_{a\in B} \operatorname{Res}(f; a) n_\Gamma(a) < \infty$$

Write $B = \{a_1, \cdots, a_n\}$. Let $Q_1, \cdots, Q_n$ be the principal parts of f at $a_1, \cdots, a_n$. Set

$$g := f - \sum_{i=1}^{n} Q_i.$$

Put $U_0 := U\backslash(A\backslash B)$. As A has no accumulation point in U, this means $\forall z \in U_0$, $\exists r$ s.t. $B(z, r) \in U$ contains no other points of A than possibly z. Thus $B(z, r) \subset U_0$ and U_0 is open.

If $z \in U_0 \cap A$, then $z \in B = \{a_1, \ldots, a_n\}$, and g has a removable singularity at z. If $z \in U_0\backslash A, g$ is obvious complex differentiable, so $g \in H(U_0)$. We apply the global Cauchy theorem to $g \in H(U_0)$:

$$\int_\Gamma g(z)\, dz = 0$$

as Γ is a cycle in U_0 with the property that

$$n_\Gamma(z) = 0 \quad \forall z \in \mathbb{C} \setminus U_0$$

Indeed, if $z \in \mathbb{C} \setminus U_0$, then either $z \in \mathbb{C} \setminus U$ and $n_\Gamma(z) = 0$ by the assumption of the theorem, or $z \in A \setminus B$ where also $n_\Gamma(z) = 0$ by the definition of B. Hence, by (5.5),

$$0 = \int_\Gamma f(z)dz - \sum_{i=1}^n \int_\Gamma Q_i(z)dz = \int_\Gamma f(z)dz - \sum_{i=1}^n \operatorname{Res}(f; a_i)\, 2\pi i\, n_\Gamma(a_i)$$

yielding

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{i=1}^n \operatorname{Res}(f; a_i)\, n_\Gamma(a_i) = \sum_{a\in A} \operatorname{Res}(f; a) n_\Gamma(a)$$

as desired. □

We will use L'Hôpital's rule in the following example:

Proposition 5.5.3. *[L'Hôpital's rule] Let $U \subset \mathbb{C}$ be open, let $z \in \mathbb{C}$, and let $f, g : U \to \mathbb{C}$ be complex differentiable at z, with $g'(z) \neq 0$. Assume moreover that $f(z) = 0 = g(z)$. Then*

$$\lim_{\substack{w\to z\\ w\in\mathbb{C}\setminus\{z\}}} \frac{f(w)}{g(w)} = \frac{f'(z)}{g'(z)}$$

Proof. By Proposition 2.1.3, we may write

$$\lim_{\substack{w\to z\\ w\in\mathbb{C}\setminus\{z\}}} \frac{f(w)}{g(w)} = \lim_{\substack{w\to z\\ w\in\mathbb{C}\setminus\{z\}}} \frac{f(z) + (f'(z) + \varepsilon_f(w))(w-z)}{g(z) + (g'(z) + \varepsilon_g(w))(w-z)},$$

where $\varepsilon_f(w) \to 0$ and $\varepsilon_g(w) \to 0$ as $w \to z$. Moreover, recalling that $f(z) = 0 = g(z)$, the above limit simplifies to

$$\lim_{\substack{w\to z\\ w\in\mathbb{C}\setminus\{z\}}} \frac{f(w)}{g(w)} = \lim_{\substack{w\to z\\ w\in\mathbb{C}\setminus\{z\}}} \frac{f'(z) + \varepsilon_f(w)}{g'(z) + \varepsilon_g(w)} = \frac{f'(z)}{g'(z)},$$

as claimed. Notice that here diving with $g'(z)$ makes sense as $g'(z) \neq 0$. Moreover, notice that by (b) this implies that in a small neighborhood of z we have $g(w) \neq g(z) = 0$ for $w \neq z$, so also the expression $f(w)/g(w)$ above makes sense for $w \neq z$ close to z. □

Example 5.5.4. We calculate the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

We want to find a path γ_R such that above equals to

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz$$

where $f(z) = 1/(1+z^4)$. We let $\gamma_R = [-R, R] \star \sigma_R$, $\sigma_R(t) = Re^{it}$, $t \in [0, \pi]$. It works simply because of the usual estimate

$$\left| \int_{\sigma_R} f(z)\,dz \right| \leq \frac{CR}{R^4} = \frac{C}{R^3} \xrightarrow{R\to\infty} 0.$$

Now $1+z^4=0$ has solutions $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$, all with order 1. Only poles $z_1=e^{i\pi/4}$ and $z_2=e^{i3\pi/4}$ are inside γ_R. By the residue theorem and formula 5.2.15,

$$\begin{aligned}\int_{\gamma_R} f(z)\,dz &= 2\pi i(\operatorname{Res}(f;z_1)+\operatorname{Res}(f;z_2))\\ &= 2\pi i\left(\lim_{z\to z_1}(z-z_1)f(z)+\lim_{z\to z_2}(z-z_2)f(z)\right)\end{aligned}$$

Instead of writing $z^4+1=(z-z_1)\cdots(z-z_4)$, it is convenient to use L'Hôpital's rule instead:

$$\lim_{z\to z_1}(z-z_1)f(z)=\lim_{z\to z_1}\frac{z-z_1}{1+z^4}\overset{\text{L'Hôpital}}{=\!=\!=}\lim_{z\to z_1}\frac{1}{4z^3}=\frac{1}{4(e^{i\pi/4})^3}=\frac{1}{4}e^{-i3\pi/4}.$$

Similarly,

$$\lim_{z\to z_2}(z-z_2)f(z)=\frac{1}{4}e^{-i\pi/4}.$$

Thus,

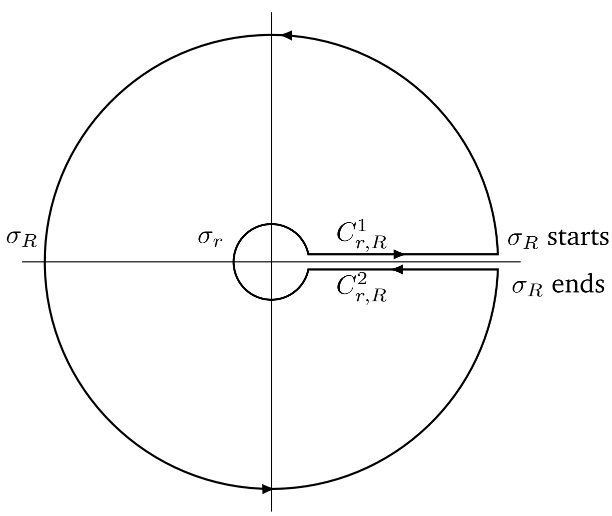
$$\int_{-\infty}^{\infty}\frac{dx}{1+x^4}=\frac{2\pi i}{4}\left(e^{-i3\pi/4}+e^{-i\pi/4}\right)=\frac{\pi i}{2}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=-\frac{\pi i}{2}\frac{2i}{\sqrt{2}}=\frac{\pi}{\sqrt{2}}$$

◇

Example 5.5.5. Calculate the integral

$$I:=\int_0^{\infty}\frac{x^{1/3}}{1+x^2}dx.$$

We need a keyhole contour line $\gamma_{r,R}:=C^1_{r,R}\star\sigma_R\star C^2_{r,R}\star\sigma_r$



We define a branch of argument $\widetilde{\operatorname{Arg}}z\in(0,2\pi)$ on $\mathbb{C}\setminus[0,\infty)=\mathbb{C}\setminus\ell_v$ where $v=1$ (Exercise 2.3.10). Then we define

$$g(z)=|z|^{1/3}e^{i\widetilde{\operatorname{Arg}}z/3}$$

and define

$$f(z)=\frac{g(z)}{1+z^2},\quad z\in\mathbb{C}\backslash[0,\infty),z\neq\pm i.$$

we first calculate $\int_\gamma f(z)dz$ using residues.

Clearly, g is analytic in a neighborhood (fattening) of the area enclosed by $\gamma_{r,R}$, and $n_{\gamma_{r,R}}(z) = 0$ for all z outside of this fattening (as z will be in the unbounded component of $\mathbb{C} \setminus \gamma^*_{r,R}$). Residue theorem gives

$$\int_{\gamma_{r,R}} f(z)\, dz = 2\pi i(\operatorname{Res}(f;i) + \operatorname{Res}(f;-i))$$

where it is easy to see by an argument of homotopy that $n_{\gamma_{r,R}}(i) = n_{\gamma_{r,R}}(-i) = 1$. Now,

$$\begin{aligned}
\operatorname{Res}(f;i) &= \lim_{z\to i} \frac{g(z)}{z+i} = \frac{e^{i\frac{\pi}{6}}}{2i} = \frac{1}{2}e^{i\left(\frac{\pi}{6}-\frac{\pi}{2}\right)} = \frac{e^{-i\frac{\pi}{3}}}{2} \\
\operatorname{Res}(f;-i) &= \lim_{2\to -i} \frac{g(z)}{z-i} = \frac{e^{i\frac{\pi}{2}}}{-2i} = -\frac{1}{2}
\end{aligned}$$

Then,

$$\begin{aligned}
\int_{\gamma_{r,R}} f(z)\, dz &= 2\pi i \cdot \frac{1}{2}\left(e^{-i\frac{\pi}{3}} - 1\right) \\
&= \pi i \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i - 1\right) \\
&= \pi i \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\
&= -\pi i e^{i\frac{\pi}{3}}
\end{aligned}$$

Now, we relate this complex integral to the original real integral

$$\begin{aligned}
&\left|\int_{\sigma_R} f(z)dz\right| \leqslant C\frac{R^{1/3}R}{R^2} = CR^{-2/3} \to 0, \quad R\to\infty \\
&\left|\int_{\sigma_r} f(z)dz\right| \leqslant Cr \to 0, \quad r\to 0 \\
&\int_{c^1_{r,R}} f(z)dz \to I \text{ as } r\to 0, \quad R\to\infty \\
&\int_{c^2_{r,R}} f(z)dz \to -e^{i\frac{2\pi}{3}}I, \quad r\to 0, R\to\infty
\end{aligned}$$

(Since $\widetilde{\operatorname{Arg}} z \to 2\pi$ here, and we travel with opposite direction.)

Then

$$\left(1 - e^{i\frac{2\pi}{3}}\right) I = \lim_{\substack{r\to 0\\ R\to\infty}} \int_{\gamma_{r,R}} f(z)dz = -\pi i e^{i\frac{\pi}{3}}$$

Thus,

$$\begin{aligned}
I &= \frac{-\pi i e^{i\frac{\pi}{3}}}{1 - e^{i\frac{i\pi}{3}}} = \frac{\pi i}{e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}} \\
&= \frac{\pi i}{2i \sin\frac{\pi}{3}} = \frac{\pi}{2}\frac{1}{\sin\frac{\pi}{3}} \\
&= \frac{\pi}{2}\frac{2}{\sqrt{3}} = \frac{\pi}{\sqrt{3}}.
\end{aligned}$$

$\diamond$

5.6 Residue Calculus and Applications

We have used the term order for both zeros and poles separately. They can be unified in the following way.

Definition 5.6.1 (Order of a zero or pole). *Let $U \subset \mathbb{C}$ be a domain, $a \in U$, and let f be meromorphic on U. Then there exists a unique integer $m \in \mathbb{Z}$ and a holomorphic function u defined in a neighborhood of a such that*

$$f(z) = (z-a)^m u(z), \qquad u(a) \neq 0.$$

The integer m is called the ***order*** *of f at a and is denoted $\operatorname{ord}(f;a)$. we have:*

- *$m > 0$: a is a zero of order m of f.*
- *$m = 0$: f is holomorphic and nonvanishing at a.*
- *$m < 0$: a is a pole of order $-m$ of f.*

We record a proposition for calculation of the order.

Proposition 5.6.2 (Residue formulas for non-essential singularities). *Let $U \subset \mathbb{C}$ be a domain, $a \in U$, and let $f, g : U \setminus \{a\} \to \mathbb{C}$ be analytic functions having non-essential singularities at a (i.e. removable singularities or poles). Then the following hold.*

(a) If $\operatorname{ord}(f;a) \geq -1$ (equivalently, a is removable for f or a pole of order 1), then

$$\operatorname{Res}(f;a) = \lim_{z\to a}(z-a)f(z).$$

More generally, if a is a pole of order $k \geq 1$ for f, set $\widetilde{f}(z) := (z-a)^k f(z)$. Then $\widetilde{f}$ extends holomorphically to a with $\widetilde{f}(a) \neq 0$, and

$$\operatorname{Res}(f;a) = \frac{\widetilde{f}^{(k-1)}(a)}{(k-1)!}.$$

(b) Assume $\operatorname{ord}(f;a) \geq 0$ (so f is holomorphic at a) and $\operatorname{ord}(g;a) = 1$ (so g has a simple zero at a). Then f/g has a simple pole at a and

$$\operatorname{Res}\left(\frac{f}{g};a\right) = \frac{f(a)}{g'(a)}.$$

(c) If $f \not\equiv 0$, then for every $a \in U$ one has

$$\operatorname{Res}\left(\frac{f'}{f};a\right) = \operatorname{ord}(f;a),$$

where $\operatorname{ord}(f;a) > 0$ for a zero, $\operatorname{ord}(f;a) < 0$ for a pole, and $\operatorname{ord}(f;a) = 0$ if $f(a) \neq 0$.

(d) If h is holomorphic on U, then for every $a \in U$,

$$\operatorname{Res}\left(h\,\frac{f'}{f};a\right) = h(a)\,\operatorname{ord}(f;a).$$

Proof. (a) This is Proposition 5.2.15.

(b) Since $\operatorname{ord}(g;a) = 1$, we can write $g(z) = (z-a)u(z)$ where u is holomorphic near a and $u(a) \neq 0$. Since $\operatorname{ord}(f;a) \geq 0$, f is holomorphic at a. Then

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)u(z)} = \frac{1}{z-a}\cdot\frac{f(z)}{u(z)}.$$

The function $\phi(z) := f(z)/u(z)$ is holomorphic near a, and thus

$$\operatorname{Res}\left(\frac{f}{g};a\right) = \phi(a) = \frac{f(a)}{u(a)}.$$

Finally, differentiating $g(z) = (z-a)u(z)$ gives $g'(a) = u(a)$, hence

$$\operatorname{Res}\left(\frac{f}{g};a\right) = \frac{f(a)}{g'(a)}.$$

(c) Fix $a \in U$ and assume $f \not\equiv 0$. By Theorem 5.1.1 and Theorem 5.2.5, there exists an integer $m = \operatorname{ord}(f;a)$ and a holomorphic function u near a with $u(a) \neq 0$ such that

$$f(z) = (z-a)^m u(z)$$

(where $m > 0$ means a zero, $m < 0$ a pole, and $m = 0$ means $f(a) \neq 0$). Then for $z \neq a$,

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z) = \frac{m}{z-a} + \frac{u'(z)}{u(z)}.$$

Since $u'(z)/u(z)$ is holomorphic near a, its residue at a is 0, and hence

$$\operatorname{Res}\left(\frac{f'}{f};a\right) = m = \operatorname{ord}(f;a).$$

(d) With the same factorization $f(z) = (z-a)^m u(z)$ as above,

$$h(z)\frac{f'(z)}{f(z)} = h(z)\left(\frac{m}{z-a} + \frac{u'(z)}{u(z)}\right) = \frac{mh(z)}{z-a} + h(z)\frac{u'(z)}{u(z)}.$$

The second term is holomorphic near a, hence contributes no residue. For the first term, write $h(z) = h(a) + (z-a)h_1(z)$ with h_1 holomorphic near a; then

$$\frac{mh(z)}{z-a} = \frac{mh(a)}{z-a} + mh_1(z),$$

so the residue is $mh(a) = h(a)\operatorname{ord}(f;a)$. □

5.6.1 Argument Principle

One important application of the residue theorem is the argument principle.

Theorem 5.6.3 (Argument Principle). *Let $U \subset \mathbb{C}$ be a domain. Let f be a meromorphic function on U with the zeros $a_1, \ldots, a_n \in U$, and the poles $b_1, \ldots, b_p \in U$. Let γ be a closed piecewise-C^1 path γ in U such that it avoids in its image all zeros and poles and it is null-homotopic. Then one has:*

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f'}{f}(\zeta)d\zeta = \sum_{i=1}^{n} \operatorname{ord}(f;a_i)\, n_\gamma(a_i) + \sum_{j=1}^{p} \operatorname{ord}(f;b_j)\, n_\gamma(b_j).$$

In particular, if γ is also simple, i.e., non-intersecting, then the winding numbers are either zero (when the point is in the unbounded component of $U \setminus \gamma^*$) or one (otherwise). Then,

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f'}{f}(\zeta)d\zeta = N - P$$

where $N = \sum_{i=1}^{n} \operatorname{ord}(f;a_i)$ is the total number of zeros of f counted with multiplicity; and $P = -\sum_{j=1}^{p} \operatorname{ord}(f;b_j) = \sum_{j=1}^{p}$(order of pole b_j) is the total number of poles of f counted with multiplicity.

Proof. The poles of f'/f are just poles of f and zeros of f; and the residue at each of them is given by Proposition 5.6.2 (c). The null-homotopy condition is to ensure $n_\gamma(\alpha) = 0$ for $\alpha \notin U$. Now, use Theorem 5.5.2. □

An application of the argument principle is the following result of A. Hurwitz:

Theorem 5.6.4. *Let $U \subset \mathbb{C}$ be a domain. Let $f_0, f_1, f_2, \cdots : U \to \mathbb{C}$ be a sequence of analytic functions that converges locally uniformly to (the analytic function) $f : U \to \mathbb{C}$. Assume that none of the functions f_n has a zero in U. Then f is either identically zero, or it has no zero in U.*

Proof. We assume the contrary, i.e. that $f \not\equiv 0$ but there exists a point a with $f(a) = 0$, and derive a contradiction. We can choose an $\varepsilon > 0$ small enough such that the disk centered at a of radius 2ε is contained in U, and there are no zeros of f in this disk, excepting a. One easily sees that the sequence f_n'/f_n converges locally uniformly in $B(a, 2\varepsilon)\backslash\{a\}$ to f'/f, which gives

$$0 = \frac{1}{2\pi \mathrm{i}} \int_{\partial B(a.\varepsilon)} \frac{f_n'}{f_n} \to \frac{1}{2\pi \mathrm{i}} \int_{\partial B(a,\varepsilon)} \frac{f'}{f},$$

in contradiction to the assumption $f(a) = 0$. □

Corollary 5.6.5. *Let $U \subset \mathbb{C}$ be a domain, and $f_0, f_1, f_2, \ldots$ a sequence of injective analytic functions $f_n : U \to \mathbb{C}$, which converges locally uniformly to the (analytic) function $f : U \to \mathbb{C}$. Then f is either constant, or injective.*

Proof. We assume f to be non-constant, and pick an arbitrary $a \in U$. Because of the injectivity, each function $z \to f_n(z) - f_n(a)$ does not vanish in $U\backslash\{a\}$. By above theorem, this is also the case for the limit function

$$z \mapsto f(z) - f(a)$$

and thus $f(z) \neq f(a)$ for all $z \in U\backslash\{a\}$. □

5.6.2 Rouché's Theorem

We conclude this chapter with Rouché's theorem.

Theorem 5.6.6. *Suppose γ is a closed path in an open connected set U, such that $\mathrm{Ind}_\gamma(\alpha) = 0$ for every α not in U. Suppose also that $\mathrm{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in U - \gamma^*$. Let $U_1 = \{z \in \mathbb{C} \setminus \gamma^* : \mathrm{Ind}_\gamma(z) = 1\} \subset U$. For any $f \in H(U)$ let N_f be the number of zeros of f in U_1, counted with multiplicities.*

(a) If $f \in H(U)$ and f has no zeros on γ^* then

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \mathrm{Ind}_\Gamma(0) \tag{5.24}$$

where $\Gamma = f \circ \gamma$.

(b) (Rouché's theorem) If also $g \in H(U)$ and

$$|f(z) - g(z)| < |f(z)| \quad \text{for all } z \in \gamma^* \tag{5.25}$$

then $N_g = N_f$.

Part (b) is usually called Rouché's theorem. It says that two holomorphic functions have the same number of zeros in U_1 if they are close together on the boundary of U_1, as specified by (5.25).

Proof. Since $f \in H(U)$, it has no poles in U. Thus the first half of (5.24) is immediate from the argument principle. The other half is a matter of direct computation: supposing $\gamma : [a, b] \to \mathbb{C}$,

$$\begin{aligned}\operatorname{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_a^b \frac{\Gamma'(s)}{\Gamma(s)}ds \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)ds = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.\end{aligned}$$

Next, (5.25) gives $|g(z)| \geq |f(z)| - |f(z) - g(z)| > 0, \forall z \in \gamma^*$, so g has no zeros on γ^*. Hence (5.24) holds with g in place of f. Put $\Gamma_0 = g \circ \gamma$. In order to apply Lemma 5.3.5 to Γ_0 and $\Gamma_1 = f \circ \gamma$ to get $\operatorname{Ind}_{\Gamma_0}(0) = \operatorname{Ind}_{\Gamma_1}(0)$, we need verify that

$$|\Gamma_0(s) - \Gamma_1(s)| = |g(\gamma(s)) - f(\gamma(s))| < |f(\gamma(s))| = |\underbrace{0}_{\alpha \text{ in the lemma}} - \Gamma_1(s)|.$$

Then it follows from (5.24) that

$$N_g = \operatorname{Ind}_{\Gamma_0}(0) = \operatorname{Ind}_{\Gamma_1}(0) = N_f$$

□

Example 5.6.7. How many roots does the polynomial $z \mapsto z^5 + 3z^2 + 1$ have in the annulus $A = \{1 < |z| < 2\}$? ◇

Solution. Choose U to be a larger region than the annulus, say $B(0, 3)$. The strategy is to let g be the original function $g(z) = z^5 + 3z^2 + 1$ and choose f to be the dominant part on the circle $|z| = 1$ and circle $|z| = 2$.

Define $f_1(z) = 3z^2$. Then, on the circle $|z| = 1$, we have

$$|f_1(z) - g(z)| = |z^5 + 1| < |z|^5 + 1 = 2 < 3 = 3|z|^2 = |f_1(z)|$$

By Rouché's theorem, we have

$$|\{a \in B(0, 1) : g(a) = 0\}| = |\{a \in B(0, 1) : f_1(a) = 0\}| = 2$$

since $z \mapsto 3z^2$ has a zero of order 2 at the origin.

Define $f_2(z) = z^5$. For $|z| = 2$ we have

$$|f_2(z) - g(z)| = |3z^2 + 1| \leq 3 \cdot 2^2 + 1 = 13 < 32 = 2^5 = |z|^5 = |f_2(z)|$$

By Rouché's theorem, we have

$$|\{a \in B(0, 2) : g(a) = 0\}| = |\{a \in B(0, 2) : f_2(a) = 0\}| = 5$$

since $z \mapsto z^5$ has a zero of order 5 at the origin.

Therefore, there are $5 - 2 = 3$ zeros in annulus A. ■

Example 5.6.8. Use Rouche's theorem to prove that all the zeros of the polynomial

$$z^n + c_{n-1}z^{n-1} + \cdots + c_0$$

lie in the open ball with center 0 and radius

$$\sqrt{1 + |c_{n-1}|^2 + \cdots + |c_0|^2}.$$

◇

Solution. Let
$$P(z) := z^n + c_{n-1}z^{n-1} + \cdots + c_0$$
and
$$R := \sqrt{1 + |c_{n-1}|^2 + \cdots + |c_0|^2}.$$
If all of the coefficients c_i are zero, then $R = 1$, and the result is trivial as the only zero of $P(z) = z^n$ is at $z = 0 \in B(0,1)$. So we may assume $R > 1$. We will apply Rouche, and the dominating term we choose is $f(z) := z^n$. In the notation of Rouche, let $g(z) := P(z)$. For $|z| = R$ by Cauchy-Schwarz (i.e. $|w_1 \cdot w_2| \leq |w_1| |w_2|, w_1, w_2 \in \mathbb{R}^d$, where $w_1 \cdot w_2$ is the dot product in $\mathbb{R}^d$) we have
$$\begin{aligned}
|f(z) - g(z)| &= \left|c_{n-1}z^{n-1} + \cdots + c_0\right| \\
&\leq \sum_{k=0}^{n-1} |c_k| R^k \\
&\leq \left(\sum_{k=0}^{n-1} |c_k|^2\right)^{1/2} \left(\sum_{k=0}^{n-1} R^{2k}\right)^{1/2} = \left(R^2 - 1\right)^{1/2} \left(\sum_{k=0}^{n-1} R^{2k}\right)^{1/2}.
\end{aligned}$$
In the last identity we used the definition of R. By the formula for a geometric sum
$$\sum_{k=0}^{n-1} t^k = \frac{1 - t^n}{1 - t}, \quad t \neq 1,$$
we have (as $R^2 > 1$) that
$$\sum_{k=0}^{n-1} R^{2k} = \frac{1 - R^{2n}}{1 - R^2} = \frac{R^{2n} - 1}{R^2 - 1}.$$
Thus, we obtain
$$|f(z) - g(z)| \leq \left(R^{2n} - 1\right)^{1/2} < R^n = |f(z)|^n$$
on $|z| = R$. By Rouche f and $g = P$ have the same number of zeros inside $B(0,R)$, which is clearly n for $f(z) = z^n$ (a zero of multiplicity n at $z = 0$). But as P is a polynomial of order n, it has exactly n roots, and so all of its roots are in $B(0,R)$. We are done. ■

Example 5.6.9. We use Rouché's theorem to prove the open mapping theorem:
Suppose $f \in H(U)$ is non-constant and $U \subset \mathbb{C}$ is open and connected. Then f is open, that is, it maps open sets to open sets. ◇

Proof. Let $V \subset U$ be open. Consider $w_0 \in fV$, that is, $w_0 = f(z_0)$ for $z_0 \in V$. As $f - w_0$ is an analytic function that is not identically zero (as f is non-constant) in the open, connected U, we know that its zero set in U, that is, the set $\{z \in U : f(z) = w_0\}$, does not have accumulation points in U. Using this we choose $r > 0$ so that $\bar{B}(z_0, r) \subset V$ and $f(z) \neq w_0$ for $z \in \bar{B}(z_0, r) \setminus \{z_0\}$, in particular, if $|z - z_0| = r$. As the continuous function $|f - w_0|$ attains its minimum on the compact set $|z - z_0| = r$ we find $\epsilon > 0$ so that $|f(z) - w_0| \geq \epsilon$ for all z with $|z - z_0| = r$.

Consider w with $|w - w_0| < \epsilon$. We show that then $w \in fV$, showing that fV is open as desired. We do this by showing that $f - w$ has a zero in $B(z_0, r) \subset V$. Indeed, we have
$$|f(z) - w - (f(z) - w_0)| = |w - w_0| < \epsilon \leq |f(z) - w_0|$$
on the circle $|z - z_0| = r$. By Rouche's theorem $f - w$ and $f - w_0$ have the same number of zeros in $B(z_0, r)$, and we know that $f - w_0$ has exactly one zero in that ball, namely z_0. So, indeed, $f - w$ has a zero in $B(z_0, r)$ and we are done. □

Chapter 6

Analytic Functions

6.1 Riemann Sphere

Let the symbol ∞ be called the point at infinity and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the extended complex plane and is put with the topology of one-point compactification. We for $r > 0$ define

$$B'(\infty, r) = \{z \in \mathbb{C} : |z| > 1/r\}$$
$$B(\infty, r) = B'(\infty, r) \cup \{\infty\}$$

The topology is then defined as this: we declare $U \subseteq \overline{\mathbb{C}}$ to be open iff U is a union of some $B(a, r)$ with $a \in \overline{\mathbb{C}}$ and $r > 0$ on $\overline{\mathbb{C}} \setminus \{\infty\}$. This gives the usual topology on $\mathbb{C}$. We will show that $\overline{\mathbb{C}}$ is homeomorphic to the unit sphere $\mathbb{S}^2$, so $\overline{\mathbb{C}}$ is also called **Riemann sphere**. First, think of $\mathbb{C}$ as the embedded set $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\} = \{x \in \mathbb{R}^3 : x_3 = 0\} \subseteq \mathbb{R}^3$. Let S be $\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - 1)^2 = 1\} = \partial B((0, 0, 1), 1)$. Then $\mathbb{S}^2 \approx \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where the homeomorphism is **stereographic projection**: in Cartesian coordinates (x, y, z) on the sphere and (X, Y) on the plane, the projection and its inverse are given by the formulas

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$
$$(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}\right)$$

The name "projection" comes from this: if one picks a point on the sphere and draw a line passing through the point and the North pole, then the line will intersect with a point on the plane. When the point one picks is the North pole then the line intersects with the extended plane $\overline{\mathbb{C}}$ at the infinity point ∞.

Behavior of functions at ∞: if f is holomorphic in $B'(\infty, r)$, we say f has an **isolated singularity at** ∞. The type of this singularity (removable/pole/essential) is *by definition* the same as that of

$$z \mapsto f(1/z) \quad z \in B'(0, 1/r)$$

at $z = 0$. In particular, if f is bounded in $B'(\infty, r)$, then $\exists \lim_{z\to\infty} f(z) \in \mathbb{C}$ and setting $f(\infty) = \lim_{z\to\infty} f(z)$ gives a function defined on $B(\infty, r)$, which we call holomorphic (apply known result to $z \mapsto f(1/z)$). Similarly for poles and essential singularities: f has a pole of order m at ∞ if $z \mapsto f(1/z)$ has a pole of order m at 0.

6.2 Conformal Mappings

Stated loosely, a function is **conformal** at a point $P \in \mathbb{C}$ if the function "preserves angles" at p and "stretches equally in all directions" at p. Both of these statements must be interpreted infinitesimally; we shall learn to do so in the discussion below. Holomorphic functions enjoy both properties: Let f be holomorphic in a neighborhood of $p \in \mathbb{C}$. Let w_1, w_2 be complex numbers of unit modulus. Consider the directional derivatives

$$D_{w_1} f(p) \equiv \lim_{t \to 0} \frac{f(P + t w_1) - f(P)}{t}$$

and

$$D_{w_2} f(p) \equiv \lim_{t \to 0} \frac{f(p + t w_2) - f(p)}{t}.$$

Then

(I) : $|D_{w_1} f(p)| = |D_{w_2} f(p)|$.

(II) : If $|f'(p)| \neq 0$, then the directed angle from w_1 to w_2 equals the directed angle from $D_{w_1} f(p)$ to $D_{w_2} f(p)$.

Statement (I) is the analytical formulation of “stretching equally in all directions.” Statement (II) is the analytical formulation of “preserves angles.”

In fact let us now give a discursive description of why conformality works. Either of these two properties actually characterizes holomorphic functions.

It is worthwhile to picture the matter in the following manner: Let f be holomorphic on the open set $U \subseteq \mathbb{C}$. Fix a point $p \in U$. Write $f = u + iv$ as usual. Thus we may write the mapping f as $(x, y) \mapsto (u, v)$. Then the (real) Jacobian matrix of the mapping is

$$J(p) = \begin{pmatrix} u_x(p) & u_y(p) \\ v_x(p) & v_y(p) \end{pmatrix},$$

where subscripts denote derivatives. We may use the Cauchy-Riemann equations to rewrite this matrix as

$$J(p) = \begin{pmatrix} u_x(p) & u_y(p) \\ -u_y(p) & u_x(p) \end{pmatrix}$$

Factoring out a numerical coefficient, we finally write this two-dimensional derivative as

$$\begin{aligned} J(P) &= \sqrt{u_x(P)^2 + u_y(p)^2} \cdot \begin{pmatrix} \frac{u_x(p)}{\sqrt{u_x(p)^2 + u_y(p)^2}} & \frac{u_y(p)}{\sqrt{u_x(p)^2 + u_y(p)^2}} \\ \frac{-u_y(p)}{\sqrt{u_x(P)^2 + u_y(p)^2}} & \frac{u_x(p)}{\sqrt{u_x(p)^2 + u_y(p)^2}} \end{pmatrix} \\ &\equiv h(P) \cdot \mathcal{J}(p). \end{aligned}$$

The matrix $\mathcal{J}(p)$ is of course a special orthogonal matrix (that is, its rows form an orthonormal basis of $\mathbb{R}^2$, and it is oriented positively-so it has determinant 1). Of course a special orthogonal matrix represents a rotation. Thus we see that the derivative of our mapping is a rotation $\mathcal{J}(p)$ (which preserves angles) followed by a positive ”stretching factor” $h(p)$ (which also preserves angles). Of course a rotation stretches equally in all directions (in fact it does not stretch at all); and our stretching factor, or dilation, stretches equally in all directions (it simply multiplies by a positive factor). So we have established (I) and (II).

In fact the second characterization of conformality (in terms of preservation of directed angles) has an important converse: If (II) holds at points near p, then f has a complex derivative at p. If (I) holds at points near p, then either f or $\bar{f}$ has a complex derivative at p. Thus a function that is conformal (in either sense) at all points of an open set U must possess the complex derivative at each point of U.

Due to (II), we see a holomorphic function with nonzero derivative is conformal in the angle-preserving and stretching sense, and we will stick to holomorphicity plus nonzero derivative as the definition of conformality (as a stronger version than previous sense) from now on.

Definition 6.2.1. *Let $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}$ open be a holomorphic function. We say f is* ***conformal at*** $z \in U$ *if $f'(z) = 0$, we say f is* ***conformal in*** U *if $f'(z) = 0\ \forall z \in U$.*

We have discussed analytic formulation of angle-preservation and now look for a geometric interpretation (in fact, directional derivative is just the velocity of a curve in differential-geometric sense).

Let $w = f(z)$ be analytic in a region U, $z_0 \in U$, and $f'(z_0) \neq 0$, i.e., f is conformal at point z_0. Let $\gamma(t)$, $t \in [t_0, t_1]$, be a smooth curve passing through z_0 with $\gamma(t_0) = z_0$. Then $\gamma'(t_0) \neq 0$ (why?). γ has tangent line at z_0 and $\gamma'(z_0)$ is the tangent vector with angle $\psi = \operatorname{Arg}(\gamma'(t_0))$. The image curve under $w = f(z)$ is $\tilde{\gamma}(t) = f(\gamma(t))$, $t \in [t_0, t_1]$. Since $\tilde{\gamma}'(t_0) = f'(z_0)\gamma'(t_0) \neq 0$, $\tilde{\gamma}$ has tangent line at $f(z_0)$ as well, with tangent vector $\tilde{\gamma}'(t_0)$. Its angle is

$$\phi = \operatorname{Arg}(\tilde{\gamma}'(t_0)) = \operatorname{Arg}(f'(z_0)) + \operatorname{Arg}(\gamma'(t_0)) = \psi + \operatorname{Arg}(f'(z_0)).$$

Suppose $f'(z_0) = re^{i\theta}$, then $|f'(z_0)| = r$, $\operatorname{Arg}(f'(z_0)) = \theta$. Then

$$\theta = \phi - \psi, \quad r = |f'(z_0)|.$$

We continue our discussion of geometric interpretation of conformality. Let γ_0 and γ_1 be two paths that go through $z_0 = \gamma_0(t_0) = \gamma_1(t_1)$ with $\gamma_0'(t_0) \neq 0$ and $\gamma_1'(t_1) \neq 0$.

Suppose

$$\gamma_0'(t_0) = r_0 e^{i\varphi_0} \gamma_1'(t_0) = r_1 e^{i\varphi_1}$$

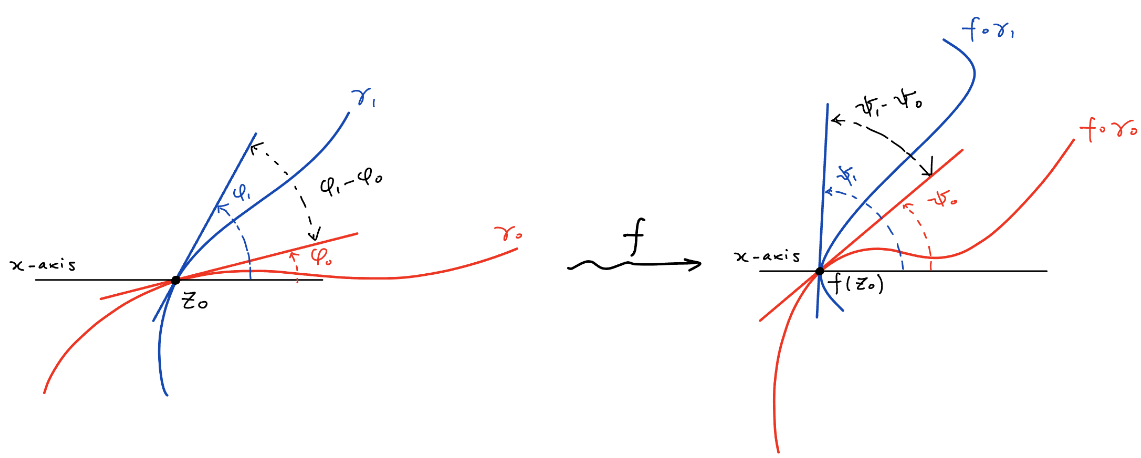


Figure 6.1: Conformal mapping preserves the angle.

Let f be conformal, then by [11] Lemma 5.44 we see

$$(f \circ \gamma_0)'(t_0) = f'(z_0)\gamma_0'(t_0) = \rho e^{i\omega} r_0 e^{i\varphi_0} = \rho r_0 e^{i(\omega+\varphi_0)} = \rho r_0 e^{i\psi_0}$$
$$(f \circ \gamma_1)'(t_1) = f'(z_0)\gamma_1'(t_1) = \rho e^{i\omega} r_1 e^{i\varphi_1} = \rho r_1 e^{i(\omega+\varphi_1)} = \rho r_1 e^{i\psi_1}$$

Then we observe that multiplication by $f'(z_0) = \rho e^{i\omega}$ stretches and rotates but preserves the original angle between $\gamma_0'(t_0)$ and $\gamma_1'(t_1)$, that is $\varphi_1 - \varphi_0 = \psi_1 - \psi_0$. In particular, for any rays L', L'' starting at z_0, the angle between the images fL', fL'' at $f(z_0)$ is the same as that made by L' and L''.

The **conformal mapping problem** between two regions $U, V \subseteq \mathbb{C}$ is the problem on the *existence* and *explicit construction* of a conformal bijection $f : U \to V$. Riemann Mapping Theorem will tell us about the *existence* and will be proved soon. First, however, we look at the relationship between $f' \neq 0$ and injectivity and then at the *explicit constructions* of cnformal bijections between geometrically simple regions (balls, sectors, half-planes, $\cdots$).

6.2.1 Relationship between nonzero derivative and injectivity

We will show that an analytic injection automatically has nonzero derivative. Thus, looking for a conformal bijection $f : U \to V$ between regions U and V is the same as finding an analytic bijection $f : U \to V$. We also see that $f^{-1} : V \to U$ is automatically analytic too (but the converse of "nonzero derivative $\implies$ injectivity" is not true: $f(z) = e^z$ satisfies $f'(z) = e^z \neq 0$ for any $z \in \mathbb{C}$, but f is not injective in the whole $\mathbb{C}$). Nonetheless, $f'(z) \neq 0$ implies local injectivity.

Theorem 6.2.2. *Suppose $U \subseteq \mathbb{C}$ is open. $z_0 \in U$, and $f \in H(U)$, $f'(z_0) \neq 0$. Then there is a neighborhood $V \subseteq U$ of z_0 such that*

(a) f is injective in V. (local injectivity)

(b) $W = f(V)$ is open. (Open Mapping Theorem (as a consequence))

(c) Due to (a) we can define $f^{-1} : W \to V$, and this is analytic. (holomorphic inverse)

Proof. Recall from the Lemma 5.3.1 where we built a function g continuous on $U \times U$,

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

So there is a neighborhood V of z_0 such that for $z_1, z_2 \in V$,

$$g(z_1,z_2) - \underbrace{g(z_0,z_0)}_{f'(z_0)} < \frac{1}{2}|f'(z_0)|$$

Thus, for $z_1, z_2 \in V$,

$$\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \geq |f'(z_0)| - \left|\underbrace{\frac{f(z_1)-f(z_2)}{z_1-z_2}}_{g(z_1,z_2)} - \underbrace{f'(z_0)}_{g(z_0,z_0)}\right| \geq \frac{1}{2}|f'(z_0)|$$

so

$$(*): \quad |f(z_1) - f(z_2)| \geq \frac{1}{2}|f'(z_0)||z_1 - z_2| \quad \forall z_1, z_2 \in V$$

In particular, $z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2)$, so (a) holds. We prove (b) in a manner similar to Example 5.6.9. We note that $(*)$ implies $f'(z) \neq 0$ for any $z \in V$. Pick an arbitrary $w_0 = f(z) \in fV$, $a \in V$. Then pick $v > 0$ such that $B(a, 2r) \subset V$ (V open). Then $(*)$ gives that

$$|f(z) - \underbrace{w_0}_{f(a)}| \geq \frac{1}{2}|f'(z_0)|r =: \varepsilon \neq 0 \quad \forall z \in \partial B(a,r)$$

We claim that $B(w_0, \varepsilon) \subset fV \Rightarrow fV$ is open. Indeed, for any $w \in B(w_0, \varepsilon)$, we have

$$|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < \varepsilon \leq |f(z) - w_0| \quad \forall z \in \partial B(a,r)$$

Then Rouché's theorem shows that $f - w$ and $f - w_0$ have the same number of zeros inside $B(a,r)$. As $f - w_0$ has one, there is some $b \in B(a,r)$ such that $w = f(b) \in f(\underbrace{B(a,r)}_{\subset V}) \subset fV$. So fV is open. (b) is proved.

For (c), we consider $f^{-1} : W \to V$ where $W := fV$. Let $w_1 \in W$ where $w_1 = f(z_1)$, $z_1 \in V$. For $w = f(z) \in w$ we notice that

$$\frac{f^{-1}(w) - f^{-1}(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{\frac{f(z)-f(z_1)}{z-z_1}}$$

Here,

- $(*) \Rightarrow |z - z_1| \leq \frac{2}{f'(z_0)|}|f(z) - f(z_1)| = \frac{2}{f'(z_0)}|w - w_1| \Rightarrow w \to w_1$ implies $z \to z_1$.
- $f'(z_1) \neq 0$ as $z_1 \in V$. So

$$(f^{-1})'(w_1) = \lim_{w \to w_1} \frac{f'(w) - f'(w_1)}{w - w_1} = \frac{1}{f'(z_1)} = \frac{1}{f'(f^{-1}(w_1))}$$

□

The **Open Mapping Theorem** states that if Ω is a region and $f \in H(\Omega)$, then $f(\Omega)$ is either a region or a point. We will prove a sharper form below. After that we show the converse of Theorem 6.2.2 that injectivity implies $f' \neq 0$.

Theorem 6.2.3. *Suppose U is a region, $f \in H(U), f$ is not constant, $z_0 \in U$, and $w_0 = f(z_0)$. Let m be the order of the zero the function $f - w_0$ has at z_0. Then there exists a neighborhood V of z_0, $V \subset U$, and there exists $\varphi \in H(V)$, such that*

(a) $f(z) = w_0 + [\varphi(z)]^m$ *for all* $z \in V$,

(b) φ' *has no zero in* V *and* φ *is an invertible mapping of* V *onto a disc* $B(0, r)$.

Proof. Let $V_0 = B(z_0, r_0) \subset U$ such that $f(z) \neq w_0$ for any $z \in V_0 \setminus \{z_0\}$ (we can do this because zeros of $f - w_0$ are isolated). We write

$$f(z) - w_0 = (z - z_0)^m g(z)$$

for $g \in H(V_0)$, g non-vanishing. Since $V_0 = B(z_0, r_0)$ is obviously homeomorphic to $B(0, 1)$, we by Theorem 5.3.8 see that g has a holomorphic logarithm h in V_0, that is, we can write $g = e^h$ for some $h \in H(V_0)$. We define $\varphi(z) = (z - z_0)e^{h(z)/m},\ z \in V_0$. Then

$$\begin{aligned} z \in V_0: \quad w_0 + [\varphi(z)]^m &= w_0 + (z - z_0)^m e^{h(z)} \\ &= w_0 + (z - z_0)^m g(z) \\ &= f(z) \end{aligned}$$

Thus, (a) holds.

Now $\varphi(z_0) = 0$, $\varphi'(z) = e^{h(z)/m} + (z - z_0)e^{h(z)/m} \cdot h'(z)/m$, so $\varphi'(z_0) = e^{h(z_0)/m} \neq 0$. By Theorem 6.2.2, φ is injective on some neighborhood V_1 of z_0 contained in V_0, $\varphi' \neq 0$ on V_1, and $0 = \varphi(z_0) \in \varphi(V_1)$, where $\varphi(V_1)$ is open. Thus, we choose $r > 0$ such that $B(0, r) \subset \varphi(V_1)$. Define $V = \varphi^{-1}(B(0, r)) \subset V_1 \subset V_0$. Then $\varphi : V \to B(0, r)$ in invertible. □

Corollary 6.2.4. *Suppose U is a region, $f \in H(U)$, and f is injective on U. Then $f'(z) \neq 0$ for every $z \in U$, and the inverse f^{-1} is holomorphic.*

Proof. If $f'(z_0) \neq 0$ for some $z_0 \in U$, write $f(z) = f(z_0) + [\varphi(z)]^m$ for $z \in V$, where m, φ, V are the same as in previous theorem. If $m = 1$, we would get $f'(z_0) = \varphi'(z)$ and thus $0 = f'(z_0) = \varphi'(z_0)$ (contradiction since $\varphi'(z_0) \neq 0$ as a result of previous theorem). So $m > 1$, but then f is not injective (since each $w \neq 0$ equals z^m for precisely m distinct z). Therefore, $f' \neq 0$. Analyticity of f^{-1} follows Theorem 6.2.2. □

6.2.2 Möbius Mappings

Definition 6.2.5. *Rational functions of the form*

$$f(z) = \frac{az + b}{cz + d},$$

*where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$, are called **Möbius mappings**, or **linear fractional transformations**.*

It is convenient to regard f as a mapping from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ with

$$f\left(-\frac{d}{c}\right) := \infty$$

(notice that $ad-bc \neq 0$ guarantees that $z \mapsto az+b$ does not vanish at $z=-\frac{d}{c}$) and

$$f(\infty)=\begin{cases}\frac{a}{c}, & \text{if } c \neq 0 \\ \infty, & \text{if } c=0\end{cases}$$

Thus, f is meromorphic in $\overline{\mathbb{C}}$ with a pole at $z=-\frac{d}{c}$. Notice that f is analytic at ∞ if $c \neq 0$.

Lemma 6.2.6. *A Möbius mapping is a bijection $\overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ whose inverse is a Möbius mapping.*

Proof. One can directly verify that the inverse is given by the following Möbius mapping:

$$g(z)=\frac{dz-b}{-cz+a}$$

□

Lemma 6.2.7. *The composition of Möbius mappings is Möbius. Möbius mappings are conformal in $\overline{\mathbb{C}}$. Here,*

*(1) When $f(\infty) \in \mathbb{C}$ (i.e., $f(\infty) \neq \infty$), we say f is **conformal at** ∞ if $z \mapsto f(1/z)$ (with $0 \mapsto f(\infty)$) is conformal at 0.*

*(2) When $f(\infty)=\infty$, we say f is **conformal at** ∞ if $z \mapsto \frac{1}{f(1/z)}$ (with $0 \mapsto 0$) is conformal at 0.*

*(3) When $f(z_0)=\infty$, we say f is **conformal at** z_0 if $z \mapsto \frac{1}{f(z)}$ (with $z_0 \mapsto 0$) is conformal at z_0.*

Proof. See Math 5022 Homework 1. □

Our first explicit construction problem is to find a Möbius mapping that maps given three points as we want.

There is a particular invariance that all Möbius mappings have (they preserve the so-called cross ratio). Let

$$f(z)=\frac{az+b}{cz+d}$$

We choose first $z_2 \in \overline{\mathbb{C}}$. Now,

$$f(z)-f(z_2)=\frac{az+b}{cz+d}-\frac{az_2+b}{cz_2+d}=\frac{(ad-bc)(z-z_2)}{(cz+d)(cz_2+d)}$$

Let then $z_3 \in \overline{\mathbb{C}} \backslash \{z_2\}$. Similarly,

$$f(z)-f(z_3)=\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}$$

So,

$$(*): \quad \frac{f(z)-f(z_2)}{f(z)-f(z_3)}=\lambda \frac{z-z_2}{z-z_3}$$

with

$$\lambda=\lambda(z_2, z_3)=\frac{cz_3+d}{cz_2+d}$$

Notic that λ is independent of z. Apply $(*)$ twith $z = z_4$ to get

$$\frac{f(z_4) - f(z_2)}{f(z_4) - f(z_3)} = \lambda \frac{z_4 - z_2}{z_4 - z_3}.$$

Therefore,

$$\frac{(f(z) - f(z_2))(f(z_3) - f(z_4))}{(f(z) - f(z_3))(f(z_2) - f(z_4))} = \frac{(z - z_2)(z_3 - z_4)}{(z - z_3)(z_2 - z_4)}$$

This gives an invariance (preservation of cross ratio, defined below) for all Möbius mappings.

Definition 6.2.8. *For complex numbers $a_1, \cdots, a_4$, where $a_i \neq a_j$ if $i \neq j$, their cross ratio is*

$$(a_1, a_2, a_3, a_4) = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_3)(a_2 - a_4)}$$

(To help memorize: numerator have indices in order and in denominator the leftmost and rightmost indices are the same as those of the numerator).

We thus have proved that

Lemma 6.2.9. *f Möbius, then $(f(z), f(z_2), f(z_3), f(z_4)) = (z, z_2, z_3, z_4)$.*

Remark 6.2.10. *If $a_i = \infty$ is in the cross ratio, we interpret it as a limit $a_i \to \infty$. For example,*

$$(\infty, a_2, a_3, a_4) := \lim_{a_1 \to \infty} \frac{(1 - a_2/a_1)(a_3 - a_4)}{(1 - a_3/a_1)(a_2 - a_4)} = \frac{a_3 - a_4}{a_2 - a_4}$$

The cross ratio is a practical tool to find a Möbius mapping that maps distinct points z_2, z_3, z_4 to given w_2, w_3, w_4.

Example 6.2.11. Find the Möbius map that satisfies $f(0) = 1, f(1) = 2, f(2) = 3$. The answer is obvious, $f(z) = z + 1$, but let's still see how to obtain it via the cross ratio. We want to find f such that

$$(z, 0, 1, 2) = (f(z), 1, 2, 3)$$

This is the same as

$$\frac{-z}{(z-1)(-2)} = \frac{(f(z)-1)(-1)}{(f(z)-2)(-2)} \iff \frac{z}{2(z-1)} = \frac{f(z)-1}{2(f(z)-2)} \iff f(z) = z + 1$$

$\diamond$

Our next explicit construction problem concerns finding the Möbius mapping that converts circles and lines.

Lemma 6.2.12. *Let $a > 0$ and $w_1 \in \mathbb{C} \backslash \{0\}$, and define*

$$F := \left\{ w \in \mathbb{C} : \frac{|w|}{|w - w_1|} = a \right\}.$$

Then F is a line if $a = 1$, and is a circle otherwise. Conversely, every circle not centered at 0 and not going through 0, or any line not going through 0, can be written in above form.

Proof. Exercise. $\square$

Corollary 6.2.13. *Let $w_1, w_2 \in \mathbb{C}$, $w_1 \neq w_2$, and $a > 0$. Let*

$$F := \left\{ w \in \mathbb{C} : \frac{|w - w_1|}{|w - w_2|} = a \right\}.$$

Then F is a circle if $a \neq 1$, and a line if $a = 1$. Conversely, every line and every circle is of this form.

Remark 6.2.14. *In our language, lines L contain ∞.*

Corollary 6.2.15. *Every Möbius mapping f maps a circle to a circle or a line. The same holds for lines, i.e., Möbius mapping f maps a line to a line or a circle.*

Proof. All circles and lines are of the form

$$F := \left\{ z \in \mathbb{C} : \frac{|z - z_1|}{|z - z_2|} = a \right\}$$

for some $z_1 \neq z_2$, $a > 0$. Recall that

$$(*) : \quad \frac{f(z) - f(z_1)}{f(z) - f(z_2)} = \lambda \frac{z - z_1}{z - z_2}$$

for a constant λ depending on z_1, z_2. Thus,

$$f(F) = \left\{ f(z) : \frac{|z - z_1|}{|z - z_2|} = a \right\} \overset{(*)}{=\!=} \left\{ f(z) : \frac{|f(z) - f(z_1)|}{|f(z) - f(z_2)|} = a|\lambda| \right\} = \left\{ w : \frac{|w - f(z_1)|}{|w - f(z_2)|} = a|\lambda| \right\}.$$

is a line or a circle. □

Remark 6.2.16. *Note that a line in $\overline{\mathbb{C}}$ can be seen as a circle since the "ends" of the line meet at infinity on the ball from which the plane is stereographically projected.*

Now, every line L can be mapped with a Moöbius mapping into a given circle S. Simply choose $z_1, z_2, z_3 \in L$ (one can be ∞) and map them to some $w_1, w_2, w_3 \in S$. Then f^{-1} maps S to L, where f^{-1} is still Möbius. One finds f with the cross ratio.

Recall the following topological fact:

Lemma 6.2.17. *Let E be connected in a topological space X and let $A \subset X$. If E meets A and $X \backslash A$, then it must meet ∂A.*

We ask how Möbius mapping f maps balls and planes. For instance, suppose it maps $|z| = 1$ to the real axis. Then $f(B(0,1))$ is connected, and by above topological fact lies completely in $H_1 = \{\mathrm{Im}(z) > 0\}$ or $H_2 = \{\mathrm{Im}(z) < 0\}$. Same is true for $f(\mathbb{C}\backslash\overline{B}(0,1))$. Thus, as f is bijective, $f(B(0,1))$ is either H_1 or H_2 (to figure out which of them, just check where $f(0)$ is mapped to). If one gets an f that maps to H_2 and wants to get H_1 instead then just apply a rotation (multiplicaiton with -1).

If we are asked to find a map from half-plane to ball, we can do the inverse problem and then find inverse of mobius mapping. Note that applying inversion (composition with $1/z$) can map outside of the circle to its inside and vice versa.

To solve more complicated problems, combine with e^z, $\log z$, z^n, $z^{1/n}$, $\cdots$ For instance, e^z maps horizontal strips to sectors or half planes (see [11]). z^α maps sectors to sectors.

Example 6.2.18. Find a conformal bijection $\mathbb{C}\backslash(-\infty, 0] \to B(0,1)$. ⋄

Solution. First use $z = re^{i\theta} \mapsto z^{1/2} = r^{1/2}e^{i\theta/2}$ to map $\mathbb{C}\backslash(-\infty, 0]$ to $H = \{\mathrm{Re}(z) > 0\}$. Then find Möbius mapping g for $H \to B(0,1)$.

Choose for example $i, 0, -i$ from $\{x = 0\}$ and map them to $1, i, -1$. We compute the cross ratio

$$(g(z), 1, i, -1) = (z, i, 0, -i)$$

$$\frac{(g(z) - 1)(i + 1)}{(g(z) - i)(1 + 1)} = \frac{(z - i)(0 + 1)}{(z - 0)(i + i)}$$

$$\iff g(z) = i\frac{1-z}{1+z}$$

As $g(1) = 0$, it turns out that we are lucky and g works (it maps to the inside of the circle; if that's not the case, we need to rotate $\{\mathrm{Re}(z) > 0\}$ to $\{\mathrm{Re}(z) < 0\}$).

Therefore, the composition is the desired conformal mapping

$$f(z) = g(e^{1/2}) = i\frac{1-\sqrt{z}}{1+\sqrt{z}}$$

■

Example 6.2.19. Let

$$U := B(0,1) \cap \{\mathrm{Re}\, z > 0\}.$$

Find a conformal bijection from U to to $B(0,1)$. ◇

Solution. Consider the points $\pm i$ that are at the intersection of the arc and the line that form ∂U. Using a Möbius mapping we want to map both the line part and the arc part to rays that go from 0 to ∞ (and these image lines must be perpendicular by conformality as the arc and the line are) so that hopefully U gets mapped to a quadrant. It should be easy from there.

So lets map $i \mapsto 0$ and $-i \mapsto \infty$, say. We also choose to map $1 \mapsto i$ (so that the arc part will go the positive imaginary axis). We find the Möbius mapping f with the cross ratio

$$(\underbrace{f(z)}_{=w}, 0, \infty, i) = (z, i, -i, 1).$$

This says

$$\lim_{a\to\infty} \frac{(w-0)\left(1-\frac{i}{a}\right)}{\left(\frac{w}{a}-1\right)(0-i)} = \frac{w}{i} = \frac{(z-i)(-i-1)}{(z+i)(i-1)} = i\frac{z-i}{z+i}$$

from which we solve

$$f(z) = w = -\frac{z-i}{z+i}.$$

So from the theory we know that the unit circle $|z| = 1$ gets mapped to either a circle or a line, and that it must get mapped to a line as the image contains ∞. And this line must go through 0 and i so it is the imaginary axis. Thus, the arc part of U gets mapped to the positive imaginary axis (as $1 \mapsto i$). Similarly, as $f(0) = 1$ we argue that the line part of ∂U maps to the positive real axis. Notice that $f(1/2) = \frac{3}{5} + \frac{4}{5}i$ belongs to the upper right quadrant and, thus, $f(U)$ is the upper right quadrant. We use $z \mapsto z^2$ to map this to the upper half-space. So

$$z \mapsto \frac{(z-i)^2}{(z+i)^2}$$

maps U to the upper half-space $H := \{\mathrm{Im}\, z > 0\}$. We can then use the inverse of the mapping from Q2 to map H to $B(0,1)$. The inverse of the said mapping, namely $z \mapsto \frac{1-iz}{z-i}$, is

$$z \mapsto \frac{-iz-1}{-z-i} = \frac{iz+1}{z+i}$$

Thus, our final mapping is

$$z \mapsto \frac{i\frac{(z-i)^2}{(z+i)^2}+1}{\frac{(z-i)^2}{(z+i)^2}+i} = \frac{z^2+2z-1}{z^2-2z-1}.$$

■

6.3 Maximum Modulus Principles and Schwarz Lemma

Let Ω be any subset of $\mathbb{C}$ and suppose α is in the interior of Ω. We can, therefore, choose a positive number ρ such that $B(\alpha;\rho) \subset \Omega$; it readily follows that there is a point ξ in Ω with $|\xi| > |\alpha|$. To state this another way, if α is a point in Ω with $|\alpha| \geq |\xi|$ for each ξ in the set Ω ($\neg(\exists \xi \in \Omega)$ such that $|\xi| > |\alpha|$), then α is not an interior point and belongs to $\partial\Omega$.

Theorem 6.3.1 (Maximum Modulus Theorem-First Version). *If f is analytic in a region U and a is a point in U with $|f(a)| \geq |f(z)|$ for all z in U then f must be a constant function.*

Proof. [11, Theorem 8.19] gives a proof using the Mean Value Property. Here is an alternative proof using the Open Mapping Theorem.

Let $\Omega = f(U)$ and put $\alpha = f(a)$. From the hypothesis we have that $|\alpha| \geq |\xi|$ for each ξ in Ω; as in the discussion preceding the theorem α is in $\partial\Omega \cap \Omega$. In particular, the set Ω cannot be open (because then $\Omega \cap \partial\Omega = \varnothing$). Hence the Open Mapping Theorem says that f must be constant. □

Theorem 6.3.2 (Maximum Modulus Theorem-Second Version). *Let U be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on $\overline{U}$ which is analytic in U. Then*

$$\max\left\{|f(z)| : z \in \overline{U}\right\} = \max\{|f(z)| : z \in \partial U\}.$$

Proof. Since U is bounded there is a point $a \in \overline{U}$ such that $|f(a)| \geq |f(z)|$ for all z in $\overline{U}$. If f is a constant function the conclusion is trivial; if f is not constant then the result follows from first version. □

Note that in the second version we did not assume that U is connected as in the first version. Do you understand how first version puts the finishing touches on the proof of the second? Or, could the assumption of connectedness in first version be dropped?

Let $U = \left\{z = x + iy : -\frac{1}{2}\pi < y < \frac{1}{2}\pi\right\}$ and put $f(z) = \exp[\exp z]$. Then f is continuous on $\overline{U}$ and analytic on U. If $z \in \partial U$ then $z = x \pm \frac{1}{2}\pi i$ so $|f(z)| = |\exp(\pm i e^x)| = 1$. However, as x goes to infinity through the real numbers, $f(x) \to \infty$. This does not contradict the Maximum Modulus Theorem because U is not bounded.

In light of the above example it is impossible to drop the assumption of the boundedness of U in the second version.

Theorem 6.3.3 (Schwarz Lemma). *Let $\mathbb{D} = \{z \mid |z| < 1\}$ be the unit disk and let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function on it, i.e., $\forall z \in \mathbb{D}$, $|f(z)| \leq 1$. Also suppose $f(0) = 0$. Then*

(i) $\forall z \in \mathbb{D}$, $|f(z)| \leq |z|$.

(ii) If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ then f is a rotation, i.e., $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

(iii) $|f'(0)| \leq 1$, and if the equality holds, then f is a rotation, i.e., $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Proof. Let

$$g(z) := \begin{cases} f(z)/z, & z \neq 0, \\ f'(0), & z = 0, \end{cases}$$

Notice that this is analytic in $\mathbb{D}$, since

$$\lim_{z\to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - \overbrace{f(0)}^{=0}}{z - 0} = f'(0).$$

so $z = 0$ must be a removable singularity of $\frac{f(z)}{z}$ and assigning the value $f'(0)$ at $z = 0$ makes $\frac{f(z)}{z}$ analytic.

Now if $r < 1$ we have for $|z| = r$ that

$$|g(z)| = \frac{\overbrace{|f(z)|}^{<1}}{r} \leq \frac{1}{r}.$$

Applying the maximum modulus principle to $g \in H(\bar{B}(0,r))$, where $B(0,r)$ is open, connected and bounded, we get

$$\max_{z \in \bar{B}(0,r)} |g(z)| = \max_{|z|=r} |g(z)| \leqslant \frac{1}{r}.$$

Letting $r \to 1$ gives $|g(z)| \leqslant 1$ and so $\forall z \in \mathbb{D}$, $|f(z)| \leqslant |z|$. If we have equality for same $0 \neq z_0 \in \mathbb{D}$, then g attains its maximum in the interior of $\mathbb{D}$. Maximum modulus principle implies that g is constant, so $f(z) = cz$ for some constant c. Since $|z_0| = |f(z_0)| = |c||z_0|$ we see $|c| = 1 \Rightarrow c = e^{i\theta}$. (i) and (ii) are then proved.

Finally, if $|f'(0)| = 1$ this means $|g(0)| = 1$ so again g reaches its maximum at the interior point $0 \in \mathbb{D}$, and is so a constant. This proves (iii). $\square$

6.4 Automorphisms of the Unit Disk

A conformal bijection $U \to U$ is called an **automorphism** of U.

What is in $\mathrm{Aut}(\mathbb{D})$ - the automorphism group of $\mathbb{D}$? Obviously $z \mapsto z$ and all rotations $z \mapsto e^{i\theta} z$. But recall also Q4 of 5021/HW1 that for $\alpha \in \mathbb{C}$, $|\alpha| < 1$, the mapping

$$z \mapsto \frac{\alpha - z}{1 - \bar{\alpha} z}$$

maps $\mathbb{D}$ to $\mathbb{D}$. It is clearly Möbius, so we can write its inverse

$$z \mapsto \frac{z - \alpha}{\bar{\alpha} z - 1} = \frac{\alpha - z}{1 - \bar{\alpha} z}$$

so it is its own inverse. Also, from formula of derivative in Q1 of 5022/HW1, one can easily check that

$$(\psi_\alpha)'(0) = \frac{|\alpha|^2 - 1}{(-\bar{\alpha} z + 1)^2}(0) = |\alpha|^2 - 1$$

$$(\psi_\alpha)'(\alpha) = \frac{|\alpha|^2 - 1}{(-\bar{\alpha} z + 1)^2}(\alpha) = (|\alpha|^2 - 1)^{-1}$$

These mappings $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$ thus satisfy

- $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$,
- $\psi_\alpha(\alpha) = 0, \psi_\alpha(0) = \alpha$,
- $\psi_\alpha^{-1} = \psi_\alpha$.
- $(\psi_\alpha)'(0) = |\alpha|^2 - 1,\ (\psi_\alpha)'(\alpha) = (|\alpha|^2 - 1)^{-1}$.

What's also interesting is that rotations of these Möbius mappings exhaust all of $\mathbb{D}$:

Theorem 6.4.1. *If $f \in \mathrm{Aut}(\mathbb{D})$, then $f(z) = e^{i\theta} \psi_\alpha(z)$ for $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$.*

Proof. $\exists! \alpha \in \mathbb{D}$ s.t. $f(\alpha) = 0$ as $f \in \mathrm{Aut}(\mathbb{D})$. Define $g := f \circ \psi_\alpha \in \mathrm{Aut}(\mathbb{D})$. Then $g(0) = f(\alpha) = 0$ and Schwarz lemma gives $|g(z)| \leq |z|\ \forall z \in \mathbb{D}$, Also $g^{-1} = \psi_\alpha^{-1} \circ f^{-1} \in \mathrm{Aut}(\mathbb{D})$ satisfies $g^{-1}(0) = 0$ so again $\left|g^{-1}(z)\right| \leq |z|\ \forall z \in \mathbb{D}$. Thus

$$|z| = |g^{-1}(g(z))| \leq |g(z)| \leq |z|$$

which implies that $|g(z)| = |z|\ \forall z \in \mathbb{D}$. Schwarz lemma $\Rightarrow g(z) = e^{i\theta} z$. Then

$$f(z) = g\left(\psi_\alpha^{-1}(z)\right) = g\left(\psi_\alpha(z)\right) = e^{i\theta}\psi_\alpha(z)$$

□

Corollary 6.4.2. *If $f \in \operatorname{Aut}(\mathbb{D})$ with $f(0) = 0$, then f is a rotation.*

Proof. $f(z) = e^{i\theta}\psi_\alpha(z)$ and $0 = f(0) = e^{i\theta}\alpha \Rightarrow \alpha = 0$, so $f(z) = e^{i\theta}\psi_0(z) = -e^{i\theta} z$. □

Remark 6.4.3. *Notice that if you want $f \in \operatorname{Aut}(\mathbb{D})$ with $f(\alpha) = \beta$ for given $\alpha, \beta \in \mathbb{D}$, just set $f = \psi_\beta \circ \psi_\alpha$.*

Using this one can calculate

$$\operatorname{Aut}(\mathbb{H}^+) = \left\{ \frac{az+b}{cz+d} \middle| ad - bc = 1 \right\}$$

for $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ by using a conformal mapping $f : H \to \mathbb{D}$. See other references for more details.

6.5 Space of Continuous and Analytic Functions

This section belongs to classical results in a real analysis course. We proceed by [3].

If U is an open set in $\mathbb{C}$ and (Ω, d) is a complete metric space then designate by $C(U, \Omega)$ the set of all continuous functions from U to Ω.

The set $C(U, \Omega)$ is never empty since it always contains the constant functions. However, it is possible that $C(U, \Omega)$ contains only the constant functions. For example, suppose that U is connected and $\Omega = \mathbb{N} = \{1, 2, \ldots\}$. If f is in $C(U, \Omega)$ then $f(U)$ must be connected in Ω and, hence, must reduce to a point. However, our principal concern will be when Ω is either $\mathbb{C}$ or $\overline{\mathbb{C}}$.

To put a metric on $C(U, \Omega)$ we record a fact about open subsets of $\mathbb{C}$.

Proposition 6.5.1. *If U is open in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of U such that $U = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions:*

(a) $K_n \subset$ int K_{n+1};

(b) $K \subset U$ and K compact implies $K \subset K_n$ for some n;

(c) Every component of $\overline{\mathbb{C}} - K_n$ contains a component of $\overline{\mathbb{C}} - U$.

If $U = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact and $K_n \subset \operatorname{int} K_{n+1}$, define

$$\rho_n(f, g) = \sup\left\{d(f(z), g(z)) : z \in K_n\right\}$$

for all functions f and g in $C(U, \Omega)$. Also define

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}; \tag{6.1}$$

since $t(1+t)^{-1} \leq 1$ for all $t \geq 0$, the series above is dominated by $\sum \left(\frac{1}{2}\right)^n$ and must converge. It will be shown that ρ is a metric for $C(U, \Omega)$. To do this the following lemma, whose proof is left as an exercise, is needed.

Lemma 6.5.2. *If (S, d) is a metric space then*

$$\mu(s,t) = \frac{d(s,t)}{1 + d(s,t)}$$

is also a metric on S. A set is open in (S, d) iff it is open in (S, μ); a sequence is a Cauchy sequence in (S, d) iff it is a Cauchy sequence in (S, μ).

Proposition 6.5.3. *$(C(U, \Omega), \rho)$ is a metric space.*

Proof. It is clear that $\rho(f, g) = \rho(g, f)$. Also, since each ρ_n satisfies the triangle inequality, the preceding lemma can be used to show that ρ satisfies the triangle inequality. Finally, the fact that $U = \bigcup_{n=1}^{\infty} K_n$ gives that $f = g$ whenever $\rho(f, g) = 0$ □

The next lemma concerns subsets of $C(U, \Omega) \times C(U, \Omega)$ and is very useful because it gives insight into the behavior of the metric ρ. Those who know the appropriate definitions will recognize that this lemma says that two uniformities are equivalent.

Lemma 6.5.4. *Let the metric ρ be defined as eq. (6.1). If $\varepsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subset U$ such that for f and g in $C(U, \Omega)$,*

$$\sup\{d(f(z), g(z)) : z \in K\} < \delta \Rightarrow \rho(f, g) < \varepsilon.$$

Conversely, if $\delta > 0$ and a compact set K are given, there is an $\varepsilon > 0$ such that for f and g in $C(U, \Omega)$,

$$\rho(f, g) < \varepsilon \Rightarrow \sup\{d(f(z), g(z)) : z \in K\} < \delta.$$

Proposition 6.5.5.
(a) A set $\mathcal{O} \subset (C(U, \Omega), \rho)$ is open iff for each f in $\mathcal{O}$ there is a compact set K and a$\delta > 0$ such that

$$\mathcal{O} \supset \{g : d(f(z), g(z)) < \delta, z \in K\}$$

(b) A sequence $\{f_n\}$ in $(C(U, \Omega), \rho)$ converges to f iff $\{f_n\}$ converges to f uniformly on all compact subsets of U.

Henceforward, whenever we consider $C(U, \Omega)$ as a metric space it will be assumed that the metric ρ is given by formula (6.1) for some sequence $\{K_n\}$ of compact sets such that $K_n \subset \text{int } K_{n+1}$ and $U = \bigcup_{n=1}^{\infty} K_n$. Actually, the requirement that $K_n \subset \text{int } K_{n+1}$ can be dropped and the above results will remain valid. However, to show this requires some extra effort (e.g., the Baire Category Theorem) which, though interesting, would be a detour.

Nothing done so far has used the assumption that Ω is complete. However, if Ω is not complete then $C(U, \Omega)$ is not complete. In fact, if $\{\omega_n\}$ is a non-convergent Cauchy sequence in Ω and $f_n(z) = \omega_n$ for all z in U, then $\{f_n\}$ is a non-convergent Cauchy sequence in $C(U, \Omega)$. However, we are assuming that Ω is complete and this gives the following.

Proposition 6.5.6. *$C(U, \Omega)$ is a complete metric space.*

Definition 6.5.7. *A set $\mathcal{F} \subset C(U, \Omega)$ is* ***normal*** *if each sequence in $\mathcal{F}$ has a subsequence which converges to a function f in $C(U, \Omega)$.*

This of course looks like the definition of sequentially compact subsets, but the limit of the subsequence is not required to be in the set $\mathcal{F}$. The next proof is left to the reader.

Proposition 6.5.8. *A set $\mathcal{F} \subset C(U, \Omega)$ is normal iff its closure is compact.*

Proposition 6.5.9. *A set $\mathcal{F} \subset C(U, \Omega)$ is normal iff for every compact set $K \subset U$ and $\delta > 0$ there are functions $f_1, \ldots, f_n$ in $\mathcal{F}$ such that for f in $\mathcal{F}$ there is at least one $k, 1 \leq k \leq n$, with*

$$\sup \{d(f(z), f_k(z)) : z \in K\} < \delta.$$

This section concludes by presenting the Arzela-Ascoli Theorem. Although its proof is not overly complicated it is a deep result which has proved extremely useful in many areas of analysis. Before stating the theorem a few results of a more general nature are needed.

Let (X_n, d_n) be a metric space for each $n \geq 1$ and let $X = \prod_{n=1}^{\infty} X_n$ be their cartesian product. That is, $X = \{\xi = \{x_n\} : x_n \in X_n \text{ for each } n \geq 1\}$. For $\xi = \{x_n\}$ and $\eta = \{y_n\}$ in X define

$$d(\xi, \eta) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Proposition 6.5.10. *$(\prod_{n=1}^{\infty} X_n, d)$, where d is defined above, is a metric space. If $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in $X = \prod_{n=1}^{\infty} X_n$ then $\xi^k \to \xi = \{x_n\}$ iff $x_n^k \to x_n$ for each n. Also, if each (X_n, d_n) is compact then X is compact.*

The following definition plays a central role in the Arzela-Ascoli Theorem.

Definition 6.5.11. *A set $\mathcal{F} \subset C(U, \Omega)$ is **equicontinuous** at a point z_0 in U iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that for $|z - z_0| < \delta$,*

$$d(f(z), f(z_0)) < \varepsilon$$

*for every f in $\mathcal{F}$.$\mathcal{F}$ is **equicontinuous** over a set $E \subset U$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for z and z' in E and $|z - z'| < \delta$,*

$$d(f(z), f(z')) < \varepsilon$$

for all f in $\mathcal{F}$.

Notice that if $\mathcal{F}$ consists of a single function f then the statement that $\mathcal{F}$ is equicontinuous at z_0 is only the statement that f is continuous at z_0. The important thing about equicontinuity is that the same δ will work for all the functions in $\mathcal{F}$. Also, for $\mathcal{F} = \{f\}$ to be equicontinuous over E is to require that f is uniformly continuous on E. For a larger family $\mathcal{F}$ to be equicontinuous there must be uniform uniform continuity.

Because of this analogy with continuity and uniform continuity the following proposition should not come as a surprise.

Proposition 6.5.12. *Suppose $\mathcal{F} \subset C(U, \Omega)$ is equicontinuous at each point of U; then $\mathcal{F}$ is equicontinuous over each compact subset of U.*

Proof. Let $K \subset U$ be compact and fix $\varepsilon > 0$. Then for each w in K there is a $\delta_w > 0$ such that

$$d(f(w'), f(w)) < \frac{1}{2}\varepsilon$$

for all f in $\mathcal{F}$ whenever $|w - w'| < \delta_w$. Now $\{B(w; \delta_w) : w \in K\}$ forms an open cover of K; by Lebesgue's Covering Lemma there is a $\delta > 0$ such that for each z in $K, B(z; \delta)$ is contained in one of the sets of this cover. So if z and z' are in K and $|z - z'| < \delta$ there is a w in K with $z' \in B(z; \delta) \subset B(w; \delta_w)$. That is, $|z - w| < \delta_w$ and $|z' - w| < \delta_w$. This gives $d(f(z), f(w)) < \frac{1}{2}\varepsilon$ and $d(f(z'), f(w)) < \frac{1}{2}\varepsilon$; so that $d(f(z), f(z')) < \varepsilon$ and $\mathcal{F}$ is equicontinuous over K. $\square$

Theorem 6.5.13 (Arzela-Ascoli Theorem). *A set $\mathcal{F} \subset C(U, \Omega)$ is normal iff the following two conditions are satisfied:*

(a) for each z in $U, \{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω;

(b) $\mathcal{F}$ is equicontinuous at each point of U.

Proof. First assume that $\mathcal{F}$ is normal. Notice that for each z in U the map of $C(U,\Omega) \rightarrow \Omega$ defined by $f \rightarrow f(z)$ is continuous; since $\mathcal{F}^{-}$is compact its image is compact in Ω and (a) follows. To show (b) fix a point z_0 in U and let $\varepsilon > 0$. If $R > 0$ is chosen so that $K = \bar{B}\left(z_0; R\right) \subset U$ then K is compact and Proposition 6.5.9 implies there are functions $f_1, \ldots, f_n$ in $\mathcal{F}$ such that for each f in $\mathcal{F}$ there is at least one f_k with

$$(*): \qquad \sup \left\{d\left(f(z), f_k(z)\right): z \in K\right\}<\frac{\varepsilon}{3}.$$

But since each f_k is continuous there is a $\delta, 0<\delta<R$, such that $\left|z-z_0\right|<\delta$ implies that

$$d\left(f_k(z), f_k\left(z_0\right)\right)<\frac{\varepsilon}{3}$$

for $1 \leq k \leq n$. Therefore, if $\left|z-z_0\right|<\delta, f \in \mathcal{F}$, and k is chosen so that $(*)$ holds, then

$$d\left(f(z), f\left(z_0\right)\right) \leq d\left(f(z), f_k(z)\right)+d\left(f_k(z), f_k\left(z_0\right)\right)+d\left(f_k\left(z_0\right), f\left(z_0\right)\right)<\varepsilon$$

That is, $\mathcal{F}$ is equicontinuous at z_0. Now suppose $\mathcal{F}$ satisfies conditions (a) and (b); it must be shown that $\mathcal{F}$ is normal. Let $\left\{z_n\right\}$ be the sequence of all points in U with rational real and imaginary parts (so for z in U and $\delta>0$ there is a z_n with $\left|z-z_n\right|<\delta$). For each $n \geq 1$ let

$$X_n=\left\{f\left(z_n\right): f \in \mathcal{F}\right\}^{-} \subset \Omega;$$

from part (a), $\left(X_n, d\right)$ is a compact metric space. Thus, by Proposition 6.5.10, $X=\prod_{n=1}^{\infty} X_n$ is a compact metric space. For f in $\mathcal{F}$ define $\tilde{f}$ in X by

$$\tilde{f}=\left\{f\left(z_1\right), f\left(z_2\right), \ldots\right\}.$$

Let $\left\{f_k\right\}$ be a sequence in $\mathcal{F}$; so $\left\{\tilde{f}_k\right\}$ is a sequence in the compact metric space X. Thus there is a ξ in X and a subsequence of $\left\{\tilde{f}_k\right\}$ which converges to ξ. For the sake of convenient notation, assume that $\xi=\lim \tilde{f}_k$. Again from Proposition 6.5.10,

$$(**): \qquad \lim_{k \rightarrow \infty} f_k\left(z_n\right)=\omega_n$$

where $\xi=\left\{\omega_n\right\}$. It will be shown that $\left\{f_k\right\}$ converges to a function f in $C(U, \Omega)$. By $(**)$ this function f will have to satisfy $f\left(z_n\right)=\omega_n$. The importance of $(**)$ is that it imposes control over the behavior of $\left\{f_k\right\}$ on a dense subset of U. We will use the fact that $\left\{f_k\right\}$ is equicontinuous to spread this control to the rest of U.

To find the function f and show that $\left\{f_k\right\}$ converges to f it suffices to show that $\left\{f_k\right\}$ is a Cauchy sequence. So let K be a compact set in U and let $\varepsilon>0$; by Lemma 6.5.5(b) it suffices to find an integer J such that for $k, j \geq J$,

$$(***): \qquad \sup \left\{d\left(f_k(z), f_j(z)\right): z \in K\right\}<\varepsilon.$$

Since K is compact $R=d(K, \partial U)>0$. Let $K_1=\left\{z: d(z, K) \leq \frac{1}{2} R\right\}$; then K_1 is compact and $K \subset$ int $K_1 \subset K_1 \subset U$. Since $\mathcal{F}$ is equicontinuous at each point of U it is equicontinuous on K_1 by Proposition 6.5.11. So choose δ, $0<\delta<\frac{1}{2} R$, such that

$$(1): \qquad d\left(f(z), f\left(z^{\prime}\right)\right)<\frac{\varepsilon}{3}$$

for all f in $\mathcal{F}$ whenever z and $z^{\prime}$ are in K_1 with $\left|z-z^{\prime}\right|<\delta$. Now let D be the collection of points in $\left\{z_n\right\}$ which are also points in K_1; that is

$$D=\left\{z_n: z_n \in K_1\right\}$$

If $z \in K$ then there is a z_n with $|z - z_n| < \delta$; but $\delta < \frac{1}{2}R$ gives that $d(z_n, K) < \frac{1}{2}R$, or that $z_n \in K_1$. Hence $\{B(w;\delta) : w \in D\}$ is an open cover of K. Let $w_1, \ldots, w_n \in D$ such that

$$K \subset \bigcup_{i=1}^{n} B(w_i;\delta).$$

Since $\lim_{k\to\infty} f_k(w_i)$ exists for $1 \leq i \leq n$ (by $(**)$) there is an integer J such that for $j, k \geq J$

$$(2): \qquad d\left(f_k(w_i), f_j(w_i)\right) < \frac{\varepsilon}{3}$$

for $i = 1, \ldots, n$. Let z be an arbitrary point in K and let w_i be such that $|w_i - z| < \delta$. If k and j are larger than J then (1) and (2) give

$$d\left(f_k(z), f_j(z)\right) \leq d\left(f_k(z), f_k(w_i)\right) + d\left(f_k(w_i), f_j(w_i)\right) + d\left(f_j(w_i), f_j(z)\right) < \varepsilon.$$

Since z was arbitrary this establishes $(***)$. □

Let U be an open subset of the complex plane. If $H(U)$ is the collection of analytic functions on U, we can consider $H(U)$ as a subset of $C(U,\mathbb{C})$. We use $H(U)$ to denote the analytic functions on U rather than $A(U)$ because it is a universal practice to let $A(U)$ denote the collection of continuous functions $f : \overline{U} \to \mathbb{C}$ that are analytic in U. Thus $A(U) \neq H(U)$.

Let $H(U)$ inherits the metric from $C(U,\mathbb{C})$. Theorem 4.2.13 and Theorem 4.2.14 show that $H(U)$ is closed in $C(U,\mathbb{C})$ and the function $f \to f'$ is continuous from $H(U)$ into $H(U)$. Thus, $H(U)$ as a closed subspace of complete space $C(U,\mathbb{C})$ is complete.

6.6 Riemann Mapping Theorem

Definition 6.6.1. *A set $\mathcal{F} \subset H(U)$ is* ***locally bounded*** *if for each point a in U there are constants M and $r > 0$ such that for all f in $\mathcal{F}$,*

$$|f(z)| \leq M, \text{ for } |z - a| < r.$$

Alternately, $\mathcal{F}$ is locally bounded if there is an $r > 0$ such that

$$\sup\{|f(z)| : |z - a| < r, f \in \mathcal{F}\} < \infty.$$

That is, $\mathcal{F}$ is locally bounded if about each point a in U there is a disk on which $\mathcal{F}$ is uniformly bounded. This immediately extends to the requirement that $\mathcal{F}$ be uniformly bounded on compact sets in U.

Lemma 6.6.2. *A set $\mathcal{F}$ in $H(U)$ is locally bounded iff for each compact set $K \subset U$ there is a constant M such that*

$$|f(z)| \leq M$$

for all f in $\mathcal{F}$ and z in K.

Theorem 6.6.3 (Montel's Theorem)**.** *A family $\mathcal{F}$ in $H(U)$ is normal iff $\mathcal{F}$ is locally bounded.*

Proof. Suppose $\mathcal{F}$ is normal but fails to be locally bounded; then there is a compact set $K \subset U$ such that $\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} = \infty$. That is, there is a sequence $\{f_n\}$ in $\mathcal{F}$ such that $\sup\{|f_n(z)| : z \in K\} \geq n$. Since $\mathcal{F}$ is normal there is a function f in $H(U)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$. But this gives that $\sup\{|f_{n_k}(z) - f(z)| : z \in K\} \to 0$ as $k \to \infty$. If $|f(z)| \leq M$ for z in K,

$$n_k \leq \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + M$$

since the right hand side converges to M, this is a contradiction. Now suppose $\mathcal{F}$ is locally bounded; the Arzela-Ascoli Theorem will be used to show that $\mathcal{F}$ is normal. Since condition (a) of Arzela-Ascoli Theorem is clearly satisfied, we must show that $\mathcal{F}$ is equicontinuous at each point of U. Fix a point a in U and $\varepsilon > 0$; from the hypothesis there is an $r > 0$ and $M > 0$ such that $\bar{B}(a;r) \subset U$ and $|f(z)| \leq M$ for all z in $\bar{B}(a;r)$ and for all f in $\mathcal{F}$. Let $|z-a| < \frac{1}{2}r$ and $f \in \mathcal{F}$; then using Cauchy's Formula with $\gamma(t) = a + re^{it}, 0 \leq t \leq 2\pi$,

$$\begin{aligned} |f(a) - f(z)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}dw\right| \\ &\leq \frac{2M}{r}|a-z| \end{aligned}$$

Letting $\delta < \min\left\{\frac{1}{2}r, \frac{r}{4M}\varepsilon\right\}$ it follows that $|a-z| < \delta$ gives $|f(a) - f(z)| < \varepsilon$ for all f in $\mathcal{F}$. □

Corollary 6.6.4. *A set $\mathcal{F} \subset H(U)$ is compact iff it is closed and locally bounded.*

We're now ready to show the main theorem.

Theorem 6.6.5 (Riemann Mapping Theorem). *$U \subset \mathbb{C}$ a simply connected set, $U \neq \mathbb{C}$. Then there is a conformal bijeciton $U \to \mathbb{D} := B(0,1)$.*

Proof. Let
$$\Sigma = \{\psi : U \to B(0,1) \mid \psi \in H(U) \text{ injective}\}$$
We need to show that there is a surjective $\psi \in \Sigma$.

Step I: we claim that $\sigma \neq \varnothing$.
Let $w_0 \in \mathbb{C} \setminus U$ (As $U \neq \mathbb{C}$). Then $z \mapsto z - w_0$, $z \in U$, is a non-vanishing element of $H(U)$, so by simply-connectedness (Theorem 5.3.8) there is a $\psi \in H(U)$ such that $\varphi^2(z) = z - w_0,\ \forall z \in U$. Notice that φ is injective. Indeed, $\varphi(z_1) = \varphi(z_2) \to z_1 - w_0 = \varphi(z_1)^2 = \varphi(z_2)^2 = z_2 - w_0$. Also,

$$(*): \quad \varphi(z_1) = -\varphi(z_2) \Rightarrow z_1 - w_0 = \varphi(z_1)^2 = (-\varphi(z_2))^2 = \varphi(z_2)^2 = z_2 - w_0.$$

By open mapping theorem, φ is open, and so $\varphi(U) \subset \mathbb{C}$ is open. Choose a ball $\overline{B(a,r)} \subset \varphi(U)$, $0 < r < |a|$. Now $\varphi(U) \cap \overline{B(-a,r)} = \varnothing$: if there were some $w = \varphi(z)$, $z \in U$, $|w+a| \leq r$, then we would have $|-w-a| = |w+a| \leq r$. This implies that $-w \in \overline{B(a,r)} \subset \varphi(U)$. Thus, $-w = \varphi(\widetilde{z})$ for some $\widetilde{z} \in U$. But then $\varphi(\widetilde{z}) = -\varphi(z) \Rightarrow \widetilde{z} = z$ (by $(*)$) $\Rightarrow w = -w \Rightarrow w = 0$. Contradiction (as $0 \notin \overline{B(-a,r)}$ by condition $r < |a|$.)

Define
$$\psi(z) := \frac{r}{\varphi(z) + a}, \quad z \in U.$$
By $\varphi(U) \cap \overline{B(-a,r)} = \varnothing$, we see that $|\varphi(z) + a| > r$ for all $a \in U$, so ψ mapsto $\mathbb{D}$. Also, ψ is injective: $\psi(z_1) = \psi(z_2) \Rightarrow \varphi(z_1) = \varphi(z_2) \Rightarrow z_1 = z_2$. Therefore, $\psi \in \Sigma$.

Step II: Our second claim is that if we let $z_0 \in U$ be fixed and define

$$\eta := \sup_{\psi \in \Sigma} |\psi'(z_0)|$$

Then $\eta < \infty$ and there exsits $\psi \in \Sigma$ such that $|\psi'(z_0)| = \eta$.
To see $\eta < \infty$, notice that by Cauchy estimate, if we choose some ball $\overline{B(z_0,r)} \subset U$, we get

$$|\psi'(z_0)| \leq \frac{\|\psi\|_{L^\infty(\overline{B(z_0,r)})}}{r} \leq \frac{1}{r} \quad \forall \psi \in \Sigma$$

simply because $\psi : \mathbb{D} \to \mathbb{D}$. Thus, $\eta \leq \frac{1}{r} \leq \infty$. By properties of supremum, there is $\psi_n \in \Sigma$ such that $|\psi_n'(z_0)| \to \eta$ as $n \to \infty$. We hope that $\{\psi_n\}$ has a limit in some sense. We use a normal family argument. By

Montel's theorem, as $\{\psi_n\}$ is clearly uniformly bounded (they map to $\mathbb{D}$ after all), $\{\psi_n\}$ is a normal family. For simplicity, define the subsequence that converges uniformly in compact subsets of U still by $\{\psi_n\}$. Let $h \in H(U)$ be the limit. Also, $\psi_n' \to h'$ uniformly in compact subsets, so

$$|h'(z_0)| = \lim_{n\to\infty} |\psi_n'(z_0)| = \eta$$

h is injective by Corollary 5.6.5, as it is not constant: $\eta > 0$ ($\Sigma \neq \varnothing$ and injectivity and analiticity imply nonzero derivative) and $|h'(z_0)| = \eta$. Now, h also maps to $\mathbb{D}$:

(1) $|h(z)| = \lim \underbrace{|\psi_n(z)|}_{\leq 1} \leq 1$.

(2) if $|h(z)| = 1$ for some $z \in U$, then the fact that $|h|$ obtains its maximum on the open connected U implies h is a constant.

Therefore, $h \in \Sigma$ and $|h'(z_0)| = \eta$. Step II is then complete.

Step III: any $h \in \Sigma$ with $|h'(z_0)| = \eta$ is always a surjection (This step will finish the proof due to Step II.) We will show this by showing that if $\psi \in \Sigma$ is not subjective (ie. $\psi U \neq B(0,1)$), then $\exists \psi_1 \in \Sigma$ with $|\psi_1'(z_0)| > |\psi'(z_0)|$. This obviously implies step III. So fix $\psi \in \Sigma$ with $\psi U \neq B(0,1)$. Choose $\alpha \in B(0,1) \backslash \psi U$. We use the automorphism of the disk $\mathbb{D} = B(0,1)$

$$\varphi_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha} z},$$

t which has $\varphi_\alpha(\alpha) = 0$, $\varphi_\alpha(0) = \alpha$, and $\varphi_\alpha^{-1} = \varphi_\alpha$. Notice that $\varphi_\alpha \circ \psi \in \Sigma$ has no zero, so $\varphi_\alpha \circ \psi = g^2$ for some $g \in H(U)$ due to simply-connectedness. Note that g is injecitve:

$$\begin{aligned} g(z_1) = g(z_2) &\implies \varphi_\alpha \circ \psi(z_1) = g^2(z_1) = g^2(z_2) = \varphi_\alpha \circ \psi(z_2) \\ &\implies z_1 = z_2 \text{ (because } \varphi_\alpha \circ \psi \text{ injective)} \end{aligned}$$

In fact, $g \in \Sigma$ as

$$|g(z)| = |g(z)^2|^{1/2} = \underbrace{|\varphi_\alpha \circ \psi(z)|}_{<1}{}^{1/2} < 1.$$

Let $\beta = g(z_0) \in \mathbb{D}$, and let $\psi_1 := \varphi_\beta \circ g \in \Sigma$. We use Schwarz's lemma to prove that

$$|\psi_1'(z_0)| > |\psi'(z_0)|.$$

We figure out how to write the original ψ in terms of ψ_1. Define $s(z) = z^2$. Now,

$$\psi = \varphi_\alpha \circ \overbrace{s \circ g}^{g^2 = \varphi_\alpha \circ \psi} = \underbrace{\varphi_\alpha \circ s \circ \varphi_\beta}_{=:F} \circ \underbrace{\varphi_\beta \circ g}_{=\psi_1} = F \circ \psi_1.$$

where $F : \mathbb{D} \to \mathbb{D}$. We will show $|F'(0)| < 1$ using Schwartz. Notice $F : \mathbb{D} \to \mathbb{D}$ is analytic with

$$\begin{aligned} F(0) &= \varphi_\alpha\left(\beta^2\right) \\ &= \varphi_\alpha\left(g(z_0)^2\right) \\ &= \varphi_\alpha(\varphi_\alpha(\psi(z_0))) \\ &= \psi(z_0) =: \gamma \end{aligned}$$

As we want $0 \mapsto 0$ to be able to use Schwarz, we will use $\varphi_\alpha \circ F$, since $\varphi_\gamma(F(0)) = \varphi_\gamma(\gamma) = 0$. Apply Schwarz to $\varphi_\gamma \circ F$ to obtain

$$|(\varphi_\gamma \circ F)'(0)| < 1$$

as otherwise it would be a rotation, which is not possible as it is not injecitve due to s. Chain rule gives

$$(\varphi_\gamma \circ F)'(0) = \varphi_\gamma'(\gamma)F'(0)$$

where $|\varphi_\gamma'(\gamma)| = \frac{1}{1-|\gamma|^2} \geq 1$ and so $|F'(0)| < \frac{1}{|\varphi_\gamma'(\gamma)|} \leq 1$. So this Schwartz trickery gave us $|F'(0)| < 1$. Recall $\psi = F \circ \psi_1$, and so

$$\begin{aligned}\psi'(z_0) &= F'(\psi_1(z_0))\psi_1'(z_0) \\ &= F'(0)\psi_1'(z_0),\end{aligned}$$

since $\psi_1(z_0) = \varphi_\beta(\beta) = 0$. So

$$|\psi'(z_0)| = \underbrace{|F'(0)|}_{<1}|\psi_1'(z_0)| < |\psi_1'(z_0)|.$$

Notice this requires us to know $\psi_1'(z_0) \neq 0$, but this follows as $\psi \in \Sigma$ is an analytic injection. □

Notice that the proof only used that

In U we have: if $f \in H(U)$ non-vanishing, $f = g^2$ for some $g \in H(U)$.

This then completes the proof of $(f) \implies (a)$ part of the Theorem 5.3.8. But for $U \neq \mathbb{C}$, Riemann mapping theorem is much stronger than $(f) \implies (a)$.

6.7 The Phragmén-Lindelöff Type Results

We recall that the maximum modulus principle fails without assuming U to be bounded.

Example 6.7.1.
(1) $U = \{\mathrm{Re}(z) > 0\}$, $f(z) = e^z$. Then $|f(z)| = e^{\mathrm{Re}(z)} = e^0 = 1$ on ∂U but $|f(x)| = e^x \to \infty$ as $x \to \infty$ inside the half-plane.

(2) $U = \left\{-\frac{\pi}{4} < \operatorname{Arg} z < \frac{\pi}{4}\right\}$, $f(z) = e^{z^2}$. Then $f\left(re^{\pm i\frac{\pi}{4}}\right) = e^{r^2(\pm i)}$ so $|f| = 0$ on the boundary ∂U of the sector U. still $f(x) = e^{x^2} \to \infty$ as $x \to \infty$ inside the sector. ◇

What if we impose some growth restriction on f?

Theorem 6.7.2. *Let $\alpha \geq \frac{1}{2}$ and put*

$$U = \left\{z : |\operatorname{Arg} z| < \frac{\pi}{2\alpha}\right\}.$$

Suppose that f is analytic on U and continuous on $\overline{U}$ and there is a constant M such that $|f(z)| \leq M$ on ∂U. If there are positive constants C, c and $\beta < \alpha$ such that

$$(*): \quad |f(z)| \leq C\exp\left(c|z|^\beta\right)$$

for all $z \in U$, then $|f(z)| \leq M$ for all z in U.

Remark 6.7.3. *If we suppose some reasonable growth condition inside U, then maximum modulus holds even in the unbounded sector U. $\beta < \alpha$ cannot be dropped: in the examples above, $\alpha = \beta = 1$ and $\alpha = \beta = 2$ make them fail.*

proof of the theorem. Fix $\varepsilon > 0$ and $\gamma \in (\beta, \alpha)$. Define $F_\varepsilon(z) = F(z) = \exp(-\varepsilon z^\gamma) f(z)$. Here, $z^\gamma = r^\gamma e^{i\gamma\theta}$ when $z = re^{i\theta} \in \overline{U} \iff r \geq 0$ and $|\theta| \leq \frac{\pi}{2\alpha}$. Notice that $|\gamma\theta| \leq \frac{\gamma}{\alpha}\frac{\pi}{2} < \frac{\pi}{2}$, so z^γ is analytic, and also $\cos(\gamma\theta) > 0$. Now

$$|F(z)| = \exp(\mathrm{Re}(-\varepsilon z^\gamma))|f(z)| = \exp(-\varepsilon r^\gamma\cos(\gamma\theta))|f(z)| \leq |f(z)| \quad \forall z \in \overline{U}$$

In particular,

$$(*): \quad |F(z)| \le M \text{ on } \partial U.$$

For $z \in U$, we use the growth assumption $(*)$ and above equation to see

$$\begin{aligned}|F(z)| \le C\exp(cr^\beta - \varepsilon r^\gamma \cos(\gamma\theta)) &\le C\exp\left(cr^\beta - \varepsilon r^\gamma \cos\left(\frac{\gamma\pi}{2\alpha}\right)\right)\\ &= C\exp\left(r^\gamma\left(\underbrace{\frac{c}{r^{\gamma-\beta}}}_{\xrightarrow{r\to\infty}0} \underbrace{-\varepsilon\cos\left(\frac{\gamma\pi}{2\alpha}\right)}_{>0}\right)\right).\end{aligned}$$

As $\gamma > \beta$, clearly, for $|z| = r$ large enough uniformly on θ the negative power dominates and the whole term goes very small. Thus, there is $R > 0$ such that

$$(**): \quad |F(z)| \le M \text{ for all } z \in U \text{ and } |z| \ge R.$$

Define compact subset (green area in Fig. 6.2)

$$K := \{z \in \overline{U} : |z| \le R\}.$$

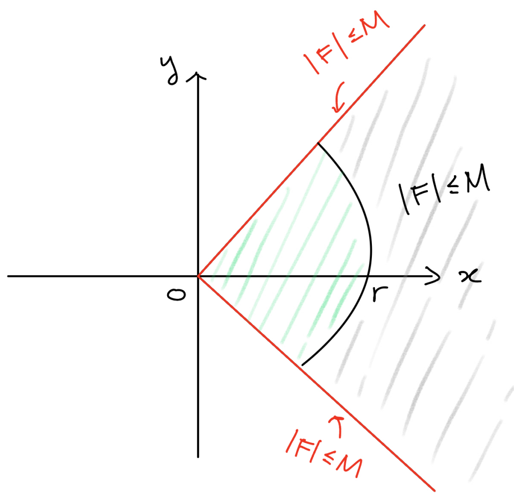


Figure 6.2: A compact part of the sector.

On ∂K, we have:

- $|F(z)| \le M$ on $\partial U \cap \partial K$ by $(*)$;
- $|F(z)| \le M$ on $\{z \in \overline{U} : |z| = R\}$ by $(**)$.

Hence $|F(z)| \le M$ for all $z \in \partial K$.

Since F is analytic in U and continuous on $\overline{U}$, the maximum modulus principle applied to K yields

$$|F(z)| \le M \quad \text{for all } z \in K.$$

Combining this with the estimate $(**)$ for $|z| \ge R$, we conclude that

$$|F(z)| \le M \quad \text{for all } z \in U.$$

Finally, for $z \in U$,

$$|f(z)| = |\exp(\varepsilon z^\gamma) F(z)| \leq M \exp(\varepsilon r^\gamma \cos(\gamma\theta)) \xrightarrow{\varepsilon \to 0} M$$

□

The idea of this Phragmén-Lindelöff type proof is to modify f by some h_ε, that is, to form $F_\varepsilon = h_\varepsilon f$. And h_ε is chosen so that,

(i) The boundary behavior moves to F_ε, e.g., $|F_\varepsilon| \leq |f|$.

(ii) Function F_ε vanishes fast enough as we go towards the unbounded parts of U. This allows one to apply maximum modulus principle in a bounded set.

(iii) We can move the result obtained from F_ε to f, e.g., $1/|h_\varepsilon(z)| \to 1$ as $\varepsilon \to 0$

Remark 6.7.4. *The sector in the Phragmén-Lindelöff theorem can be rotated and $U = \left\{z : |\operatorname{Arg} z| < \frac{\pi}{2\alpha}\right\}$ with $\alpha \geq \frac{1}{2}$ is the maximal possible sector for which the theorem applies since we need the cosine of $\alpha\theta$ to be positive (see the proof). In practice, one choose particular α or a range of α so that the associated sector is no larger than this theorem assumes (see 6.7.5 for example). We will later in the next subsection apply to the quadrants too.*

The following version is used to show the Riesze-Thorin interpolation theorem later.

Proposition 6.7.5. *Let $U := \left\{ -\frac{\pi}{2\alpha} < \operatorname{Arg} z < \frac{\pi}{2\alpha}\right\}$, $\alpha > 1/2$. Suppose $f \in H(U) \cap C(\overline{U})$ and $|f| \leq M$ on ∂U. Suppose we have the following a priori estimate: for all $\delta > 0$ we have for all $z \in U$ that*

$$|f(z)| \lesssim_\delta e^{\delta|z|^\alpha}$$

with the notation meaning that for some $C_\delta < \infty$ we have $|f(z)| \leq C_\delta e^{\delta|z|^\alpha}$, $z \in U$. Then

$$|f(z)| \leq M, \qquad z \in U.$$

Proof. We define $F : U \to \mathbb{C}$ as

$$F(z) = F_\varepsilon(z) := e^{-\varepsilon z^\alpha} f(z)$$

Suppose $t \geq 0$ and we choose δ such that $0 < \delta < \varepsilon$. Then there is a constant C_δ with

$$|F(t)| = |f(t)| e^{-\varepsilon t^\alpha} \leq C_\delta e^{\delta t^\alpha} e^{-\varepsilon t^\alpha} = C_\delta e^{t^\alpha(\delta - \varepsilon)} \xrightarrow{t \to 0} 0.$$

because $\alpha > 1/2$ and $\delta - \varepsilon < 0$. Thus there exists some point k to form a compact set $[0, k]$ s.t. the continuous function $|F(t)|,\ t \geq 0$ assumes its maximum $M_1 = \max_{t \geq 0} |F(t)|$ on some point in K. Thus,

$$|F(z)| \leq M_1, z \in [0, \infty) \tag{6.2}$$

Let $M_2 = \max\{M_1, M\}$ and we split the sector into halfs:

$$U^+ = \{z \in U : 0 < \operatorname{Arg} z < \pi/2a\},\ U^- = \{z \in U : 0 > \operatorname{Arg} z > -\pi/2a.\}$$

For $z = re^{i\theta} \in U$ (so $-\frac{\pi}{2\alpha} < \theta < \frac{\pi}{2\alpha}$ and $\cos\alpha\theta \in (0, 1)$), we have

$$|F(z)| = |f(z)||e^{-\varepsilon z^\alpha}| = |f(z)| e^{\operatorname{Re}(-\varepsilon z^\alpha)} = |f(z)| e^{\overbrace{-\varepsilon r^\alpha \cos(\alpha\theta)}^{<0}} < |f(z)| \leq C_\delta e^{\delta|z|^\alpha}$$

If $z \in \partial U$, we by above equaton also have

$$|F(z)| < |f(z)| < M \tag{6.3}$$

Therefore, by (6.2) and (6.3), we have $|F(z)| \leq M_2$ for all z in ∂U^+ and ∂U^-.

To use Phragmén-Lindelöff theorem, we can for example get a looser bound on $|F(z)|$ by picking $\delta = \varepsilon$ with

$$|F(z)| \leq |f(z)| \lesssim e^{\varepsilon|z|^\alpha} e^{\varepsilon|z|^\alpha} = e^{2\varepsilon|z|^\alpha}$$

Remark 6.7.4 now applies to U^+ and U^-, which both have size within $\frac{\pi}{4\alpha}$, or $\frac{\pi}{2\beta}$ with $\beta = 2\alpha$, and therefore the power of $|z|$ is $\alpha < \beta = 2\alpha$, satisfying the condition $(*)$ in the Phragmén-Lindelöff theorem (where the roles of α and β are switched in the theorem). The theorem then says $|f(z)| \leq M_2$ on both U^+ and U^- and thus on the whole U.

Lastly, we show that it is not possible that $M_1 > M$. In fact, if $M_1 > M$ then $M_2 = M_1$. Since $0 \in \partial U$, we see $|F(0)| = |f(0)| \leq M < M_1 = \max_{t\geq 0}|F(t)|$, which implies that the point by which $|F|$ reaches it maximum is not 0 but rather a point $x \in \mathbb{R}_+$ which thus lies inside U. Also note that for any $z = re^{i\theta} \in U$, we have $t < r(\cos(\alpha\theta))^{1/\alpha}$ so that $e^{-\varepsilon t^\alpha} > e^{-\varepsilon r^\alpha \cos(\alpha\theta)}$, which indicate that $\max_{z\in U}|F(z)| = \max_{z\in\mathbb{R}_+}|F(z)|$. Therefore, the point x makes $|F|$ reach the max on whole U. Applying the Maximum modulus principle to this open connected U gives us a constant F. Thus $M = M_1 = M_2$. Contradiction. Thus, $M_1 < M$ and $M_2 = M$. Then $|F(z)| \leq M$ for all z in U. Then

$$|F(z)| = |f(z)||e^{-\varepsilon r^\alpha \cos(\alpha\theta)}| \leq M \Rightarrow |f(z)| \leq Me^{\varepsilon r^\alpha \cos(\alpha\theta)}, \quad \forall z \in U.$$

Since M is independent of ε, we can let $\varepsilon \to 0$ and get $|f(z)| \leq M$ for all z in U. □

There are versions in other type of regions as well - in particular, in some strips.

Theorem 6.7.6 (Hadamard three-lines theorem)**.** *Let $f(z)$ be a function on the strip*

$$U = \{x + iy : a < x < b\},$$

holomorphic in the strip and continuous on the closure of the strip. Suppose $|f(z)| \leq B\,\forall z \in \overline{U}$. If

$$M(x) = \sup_y |f(x+iy)|$$

then $\log M(x)$ is a convex function on $[a, b]$. In other words, if $x = (1-t)a + tb$ with $0 \leq t \leq 1$, then

$$\log M(x) \leq (1-t)\log M(a) + t\log M(b),$$

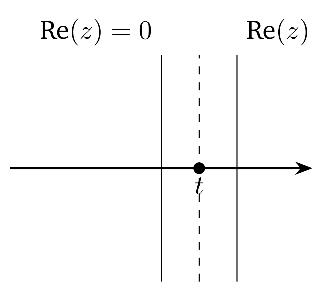
or

$$M(x) \leq M(a)^{1-t} M(b)^t.$$

or

$$M(x) \leq M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}}$$

In particular, $|f| \leq B$ can be replaced by $|f| \leq \max(M(a), M(b))$



Proof. After an affine transformation in the coordinate z, we can assume that $a = 0$, $b = 1$. Then $x = (1-t)a + tb = t \in [0,1]$ and we need to show

$$M(x) \leq M(0)^{1-x} M(1)^x,\ 0 < x < 1.$$

We do the special case $B = 1$ first: $|f(z)| \leq 1$ on $\partial U \implies |f| \leq 1$ on $\overline{U}$.

This is the usual Phragmén-Lindelöff strategy. Uiven $\varepsilon > 0$, define

$$F_\varepsilon(z) = F(z) = \frac{f(z)}{1+\varepsilon z},\ z \in \overline{U}$$

As $|\mathrm{Re}(z)| \leq |z|$, we have $|1+\varepsilon z| \geq |\mathrm{Re}(1+\varepsilon z)| = 1 + \varepsilon x \geq 1$, so $|F(z)| \leq |f(z)|$ for $z \in \overline{U}$. In particular, $|F(z)| \leq 1$ on ∂U. As $|1+\varepsilon z| \geq |\operatorname{lm}(1+\varepsilon z)| = \varepsilon|y|$, we see

$$|F(z)| \leq \frac{|f(z)|}{\varepsilon|y|} \leqslant \frac{1}{\varepsilon|y|}.$$

Thus now $|F| \leq 1$ for $|y|$ larger then some number c uniformly on x. By maximum modulus principle applied to the rectangular region bounded by $\partial\{|y| \geq c\}$ and $\partial\{x \in [0,1]\}$, we see $|F| \leq 1$ on $\overline{U}$ after letting c go to infinity. Let $\varepsilon \to 0$ to get $|f| \leq 1$ on $\overline{U}$.

The general case reduces to this, but the reduction is non-trivial. First observe that we can assume $M(0), M(1) > 0$ (note that $M(x) = \sup_y |f(x+yi)| \geq 0$) because suppose we have proved the statement for the case $M(0), M(1) > 0$ and now for some f we have say $M(0) = 0$, then we define $g = f + \varepsilon$ and apply the statment to g:

$$\sup_y |f| - \varepsilon \leq \sup_y |g| \overset{\text{statement}}{\leq} (M(0)+\varepsilon)^{1-x}(M(1)+\varepsilon)^x$$

where the last step is by noticing that for any $x \in [0,1]$,

$$\sup_y |g| = \sup_y |f(x+yi)+\varepsilon| \leq \sup_y |f(x+yi)| + \varepsilon = M(x) + \varepsilon.$$

Now $M(0) = 0$, so

$$\sup_y |f| \leq \varepsilon + \varepsilon^{1-x}(M(1)+\varepsilon)^x \overset{\varepsilon \to 0}{\longrightarrow} 0$$

implying that $f = 0$, which means the statement is trivially true.

Assume $M(0), M(1) > 0$. Define $g(z) = M(0)^{1-z} M(1)^z$ for $z \in \mathbb{C}$. Here for any $M \neq 0$ a function of the type $z \mapsto M^z := e^{z \log M}$ is clearly entire, and so g is entire. We have

$$\begin{aligned}
|g(z)| &= |M(0)^{1-z}||M(1)^z| \\
&= e^{(1-x)\log M(0)} e^{x \log M(1)} \\
&= M(0)^{1-x} M(1)^x \\
&= |g(x)|
\end{aligned}$$

is independent of y. In particular, $|g| \geq \min\{M(0), M(1)\} > 0$, implying $1/g$ is bounded. So $f/g \in C(\overline{U}) \cap H(U)$ is bounded and when $x = 0$ we have $|g(iy)| = g(0) = M(0)$ and when $x = 1$ we have $|g(1+iy)| = g(1) = M(1)$. This implies $|f/g| \leq 1$ on ∂U, and then our special case $B = 1$ above implies $|f/g| \leq 1$ on $\overline{U}$. Thus,

$$|f(z)| \leq |g(z)| = M(0)^{1-x} M(1)^x, \quad z \in \overline{U}.$$

□

6.8 The Riesz interpolation theorem

We copy the section 1 and 2 from second chapter of [16].

6.8.1 Motivation from Fourier Analysis

An initial problem considered was that of formulating an L^p analog of the basic L^2 Parseval relation for functions on $[0, 2\pi]$. This theorem states that if $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$ denotes the Fourier coefficients of a function f in $L^2([0, 2\pi])$, usually written as

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \tag{6.4}$$

then the following fundamental identity holds:

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta. \tag{6.5}$$

Conversely, if $\{a_n\}$ is a sequence for which the left-hand side of (6.5) is finite, then there exists a unique f in $L^2([0, 2\pi])$ so that both (6.4) and (6.5) hold. Notice, in particular, if $f \in L^2([0, 2\pi])$, then its Fourier coefficients $\{a_n\}$ belong to $L^2(\mathbb{Z}) = \ell^2(\mathbb{Z})$. The question that arose was: is there an analog of this result for L^p when $p \neq 2$?

Here an important dichotomy between the case $p > 2$ and $p < 2$ occurs. In the first case, when $f \in L^p([0, 2\pi])$, although f is automatically in $L^2([0, 2\pi])$, examples show that no better conclusion than $\sum |a_n|^2 < \infty$ is possible. On the other hand, when $p < 2$ one can see that essentially there can be no better conclusion than $\sum |a_n|^q < \infty$, with q the dual exponent of p. Analogous restrictions must be envisaged when the roles of f and $\{a_n\}$ are reversed. In fact, what does hold is the Hausdorff-Young inequality:

$$\left(\sum |a_n|^q\right)^{1/q} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^p d\theta\right)^{1/p}, \tag{6.6}$$

and its ”dual”

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^q d\theta\right)^{1/q} \leq \left(\sum |a_n|^p\right)^{1/p} \tag{6.7}$$

both valid when $1 \leq p \leq 2$ and $1/p + 1/q = 1$. (The case $q = \infty$ corresponds to the usual L^∞ norm.) These may be viewed as intermediate results, between the case $p = 2$ corresponding to Parseval's theorem, and its ”trivial” case $p = 1$ and $q = \infty$.

A few words about how the inequalities (6.6) and (6.7) were first attacked are in order, because they contain a useful insight about L^p spaces: often, the simplest case arises when p (or its dual) is an even integer. Indeed, when, for example $q = 4$, a function belonging to L^4 is the same as its square belonging to L^2, and this sometimes allows reduction to the easier situation when $p = 2$. To see how this works in the present situation, let us take $q = 4$ (and $p = 4/3$) in (6.6). With f given in L^p, we denote by $\mathcal{F}$ the convolution of f with itself,

$$\mathcal{F}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta - \varphi)f(\varphi)d\varphi$$

By the multiplicative property of Fourier coefficients of convolutions we have

$$\mathcal{F}(\theta) \sim \sum_{n=-\infty}^{\infty} a_n^2 e^{in\theta},$$

with $\{a_n\}$ the Fourier coefficients of f. Parseval's identity applied to $\mathcal{F}$ then yields

$$\sum |a_n|^4 = \frac{1}{2\pi}\int_0^{2\pi} |\mathcal{F}(\theta)|^2 d\theta,$$

and Young's inequality for convolutions gives

$$\|\mathcal{F}\|_{L^2} \leq \|f\|^2_{L^{4/3}},$$

proving (6.6) when $p = 4/3$ and $q = 4$. Once the case $q = 4$ has been established, the cases corresponding to $q = 2k$, where k is a positive integer, can be handled in a similar way. However the general situation, $2 \leq q \leq \infty$, corresponding to $1 \leq p \leq 2$, involves further ideas.

In contrast to the above ingenious but special argument, in turns out that there is a general principle of great interest that underlies such inequalities, which in fact leads to direct and abstract proofs of both (6.6) and (6.7). This is the M. Riesz interpolation theorem. Stated succinctly, it asserts that whenever a linear operator satisfies a pair of inequalities (like (6.6) for $p = 2$ and $p = 1$), then automatically the operator satisfies the corresponding inequalities for the intermediate exponents: here all p for $1 \leq p \leq 2$, and q with $1/p+1/q = 1$. The formulation and proof of this general theorem will be our first task in the next section.

6.8.2 The Riesz interpolation theorem

Suppose (p_0, q_0) and (p_1, q_1) are two pairs of indices with $1 \leq p_j, q_j \leq \infty$, and assume that

$$\begin{cases} \|T(f)\|_{L^{q_0}} \leq M_0\|f\|_{L^{p_0}}, & \forall f \in L^{p_0} \\ \|T(f)\|_{L^{q_1}} \leq M_1\|f\|_{L^{p_1}}, & \forall f \in L^{q_0} \end{cases}$$

where T is a linear operator. That is, T maps $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$ boundedly in the above sense. Does it follow that $T : L^p \to L^q$ for some intermediate p, q, i.e.,

$$\|T(f)\|_{L^q} \leq M(M_0, M_1)\|f\|_{L^p}, \quad \text{for other pairs } (p, q)?$$

Yes, and the Marcinkiewicz interpolation theorem (see Math5051) applies but in the case where
(1) $p_0 = q_0$ and $p_1 = q_1$.
(2) T is sublinear instead of linear.
(3) there is an even weaker assumption than T mapping $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$ boundedly to conclude $T : L^p \to L^q$.
(4) there isn't a good bound resulted for $M(M_0, M_1)$.

Thus, (2) and (3) are positive things while (1) and (4) are restrictions. We will give Riesz's answer to the question.

The precise statement of the theorem requires that we fix some notation. Let (X, μ) and (Y, ν) be a pair of measure spaces. We shall abbreviate the L^p norm on (X, μ) by writing $\|f\|_{L^p} = \|f\|_{L^p(X,\mu)}$, and similarly for the L^q norm for functions on (Y, ν). We will also consider the space $L^{p_0} + L^{p_1}$ that consists of functions on (X, μ) that can be written as $f_0 + f_1$, with $f_j \in L^{p_j}(X, \mu)$, with a similar definition for $L^{q_0} + L^{q_1}$.

Theorem 6.8.1 (Riesz interpolation theorem). *Suppose T is a linear mapping from $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$. Assume that T is bounded from L^{p_0} to L^{q_0} and from L^{p_1} to L^{q_1}*

$$\begin{cases} \|T(f)\|_{L^{q_0}} \leq M_0\|f\|_{L^{p_0}}, \\ \|T(f)\|_{L^{q_1}} \leq M_1\|f\|_{L^{p_1}}. \end{cases}$$

Then T is bounded from L^p to L^q,

$$(*): \quad \|T(f)\|_{L^q} \leq M\|f\|_{L^p},$$

whenever the pair (p,q) can be written as

$$\frac{1}{p}=\frac{1-t}{p_0}+\frac{t}{p_1} \quad \text{and} \quad \frac{1}{q}=\frac{1-t}{q_0}+\frac{t}{q_1}$$

for some t with $0\le t\le 1$. Moreover, the bound M satisfies $M\le M_0^{1-t}M_1^t$.

Proof. We begin by establishing the inequality when f is a simple function, $f=\sum a_k\chi_{E_k}$ where sets E_k are disjoint and of finite measure. We can assume $\|f\|_{L^p}=1$ because if we proved $(*)$ for this case, i.e.,

$$\|T(f)\|_{L^q}\le M,$$

we then apply the result to the function $f/\|f\|_{L^p}$ for any $f\in L^p$ to get

$$\|T(f/\|f\|_{L^p})=\|_{L^q}\le M,$$

Lemma 10.0.16 and remark 10.0.17 assert that

$$\|Tf\|_{L^q}=\sup_{\substack{\|g\|_{L^{q'}}=1\\ g\text{ simple}}}\left|\int(Tf)gd\nu\right|$$

so we only need to show for any g simple and $\|g\|_{L^{q'}}=1$,

$$\left|\int(Tf)gd\nu\right|\le M\|f\|_{L^p}\|g\|_{L^{q'}}$$

For now, we also assume that $p<\infty$ and $q>1$. Suppose $f\in L^p$ is simple with $\|f\|_{L^p}=1$, and define

$$f_z=|f|^{\gamma(z)}\frac{f}{|f|} \quad \text{where} \quad \gamma(z)=p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right),$$

and

$$g_z=|g|^{\delta(z)}\frac{g}{|g|} \quad \text{where} \quad \delta(z)=q'\left(\frac{1-z}{q_0'}+\frac{z}{q_1'}\right),$$

with q',q_0' and q_1' denoting the duals of q,q_0, and q_1 respectively. Then, we note that $f_t=f$. We also observe that if $\mathrm{Re}(z)=0$, i.e., $z=yi$, then

$$\begin{aligned}|f_z|^{p_0}\left||f|^{\gamma(z)}\right|^{p_0}&=\left|e^{\gamma(yi)\log|f|}\right|^{p_0}\\&=\left|e^{p\left(\frac{1-yi}{p_0}+\frac{yi}{p_1}\right)\log|f|}\right|^{p_0}\\&=\left|e^{p\left(\frac{1}{p_0}+i\left(\frac{yi}{p_1}-\frac{y}{p_0}\right)\right)\log|f|}\right|^{p_0}\\&=e^{p\cdot\frac{1}{p_0}\cdot\log|f|\cdot p_0}=|f|^p\end{aligned}$$

and consequently $\|f\|_{L^p}=1$ implies that

$$\|f_z\|_{L^{p_0}}=\left(\int|f_z|^{p_0}\right)^{1/p_0}=\left(\int|f|^p\right)^{1/p_0}=(\|f\|_{L^p})^{p/p_0}=1$$

One can also compute that $\mathrm{Re}(z)=1$ results in $\|f_z\|_{L^{p_1}}=1$, and there are analogous results for g_z. We summarize them below.
- $f_t=f$ and

$$\begin{cases}\|f_z\|_{L^{p_0}}=1 & \text{if } \mathrm{Re}(z)=0\\ \|f_z\|_{L^{p_1}}=1 & \text{if } \mathrm{Re}(z)=1.\end{cases}$$

- $g_t = g$ and

$$\begin{cases} \|g_z\|_{L^{p_0'}} = 1 & \text{if } \operatorname{Re}(z) = 0 \\ \|g_z\|_{L^{p_1'}} = 1 & \text{if } \operatorname{Re}(z) = 1. \end{cases}$$

The trick now is to consider

$$\Phi(z) = \int (Tf_z)\, g_z d\nu$$

Since f is a finite sum, $f = \sum a_k \chi_{E_k}$ where the sets E_k are disjoint and of finite measure, then f_z is also simple with

$$f_z = \sum |a_k|^{\gamma(z)} \frac{a_k}{|a_k|} \chi_{E_k}.$$

Since $g = \sum b_j \chi_{F_j}$ is also simple, then

$$g_z = \sum |b_j|^{\delta(z)} \frac{b_j}{|b_j|} \chi_{F_j}.$$

With the above notation, we find

$$\Phi(z) = \sum_{j,k} |a_k|^{\gamma(z)} |b_j|^{\delta(z)} \frac{a_k}{|a_k|} \frac{b_j}{|b_j|} \left(\int T(\chi_{E_k}) \chi_{F_j} d\nu \right),$$

so that the function Φ is a holomorphic function in the strip $0 < \operatorname{Re}(z) < 1$ that is bounded and continuous in its closure. After an application of Hölder's inequality and using the fact that T is bounded on L^{p_0} with bound M_0, we find that if $\operatorname{Re}(z) = 0$, then

$$|\Phi(z)| \leq \|Tf_z\|_{L^{q_0}} \|g_z\|_{L^{q_0'}} \leq M_0 \|f_z\|_{L^{p_0}} = M_0.$$

Similarly we find $|\Phi(z)| \leq M_1$ on the line $\operatorname{Re}(z) = 1$. Therefore, by the Hadamard three-lines theorem 6.7.6, we conclude that Φ is bounded by $M_0^{1-t} M_1^t$ on the line $\operatorname{Re}(z) = t$. Since $\Phi(t) = \int (Tf) g d\nu$, this gives the desired result, at least when f is simple.

In general, when $f \in L^p$ with $1 \leq p < \infty$, we may choose a sequence $\{f_n\}$ of simple functions in L^p so that $\|f_n - f\|_{L^p} \to 0$ (as in Exercise 6 in Chapter 1 of [16]). Since $\|T(f_n)\|_{L^q} \leq M \|f_n\|_{L^p}$, we find that $T(f_n)$ is a Cauchy sequence in L^q and if we can show that $\lim_{n\to\infty} T(f_n) = T(f)$ almost everywhere, it would follow that we also have $\|T(f)\|_{L^q} \leq M \|f\|_{L^p}$.

To do this, write $f = f^U + f^L$, where $f^U(x) = f(x)$ if $|f(x)| \geq 1$ and 0 elsewhere, while $f^L(x) = f(x)$ if $|f(x)| < 1$ and 0 elsewhere. Similarly, set $f_n = f_n^U + f_n^L$. Now assume that $p_0 \leq p_1$ (the case $p_0 \geq p_1$ is parallel). Then $p_0 \leq p \leq p_1$, and since $f \in L^p$, it follows that $f^U \in L^{p_0}$ and $f^L \in L^{p_1}$. Moreover, since $f_n \to f$ in the L^p norm, then $f_n^U \to f^U$ in the L^{p_0} norm and $f_n^L \to f^L$ in the L^{p_1} norm. By hypothesis, then $T(f_n^U) \to T(f^U)$ in L^{q_0} and $T(f_n^L) \to T(f^L)$ in L^{q_1}, and selecting appropriate subsequences we see that $T(f_n) = T(f_n^U) + T(f_n^L)$ converges to $T(f)$ almost everywhere, which establishes the claim.

It remains to consider the cases $q = 1$ and $p = \infty$. In the latter case then necessarily $p_0 = p_1 = \infty$, and the hypotheses $\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^\infty}$ and $\|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^\infty}$ imply the conclusion

$$\|T(f)\|_{L^q} \leq M_0^{1-t} M_1^t \|f\|_{L^\infty}$$

by Hölder's inequality (as in Exercise 20 in Chapter 1 of [16]). Finally if $p < \infty$ and $q = 1$, then $q_0 = q_1 = 1$, then we may take $g_z = g$ for all z, and argue as in the case when $q > 1$. This completes the proof of the theorem. $\square$

We shall now describe a slightly different but useful way of stating the essence of the theorem. Here we assume that our linear operator T is initially defined on simple functions of X, mapping these to functions on Y that are integrable on sets of finite measure. We then ask: for which (p,q) is the operator of **type** (p,q), in the sense that there is a bound M so that

$$\|T(f)\|_{L^q} \leq M\|f\|_{L^p}, \quad \text{whenever } f \text{ is simple?} \tag{6.8}$$

In this formulation of the question, the useful role of simple functions is that they are at once common to all the L^p spaces. Moreover, if (6.8) holds then T has a unique extension to all of L^p, with the same bound M in (6.8), as long as either $p < \infty$; or $p = \infty$ in the case X has finite measure. This is a consequence of the density of the simple functions in L^p, and the extension argument in Proposition 5.4 of Chapter 1 of [16].

With these remarks in mind, we define the **Riesz diagram** of T to consist of all all points in the unit square $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ that arise when we set $x = 1/p$ and $y = 1/q$ whenever T is of type (p,q). We then also define $M_{x,y}$ as the least M for which (8) holds when $x = 1/p$ and $y = 1/q$.

Corollary 6.8.2. *With T as before:*
(a) The Riesz diagram of T is a convex set.
(b) $\log M_{x,y}$ is a convex function on this set.

Conclusion (a) means that if $(x_0,y_0) = (1/p_0, 1/q_0)$ and $(x_1,y_1) = (1/p_1, 1/q_1)$ are points in the Riesz diagram of T, then so is the line segment joining them. This is an immediate consequence of Riesz interpolation. Similarly the convexity of the function $\log M_{x,y}$ is its convexity on each line segment, and this follows from the conclusion $M \leq M_0^{1-t} M_1^t$ guaranteed also by Riesz interpolation.

In view of this corollary, the theorem is often referred to as the "Riesz convexity theorem."

Example 6.8.3. The first application of Riesz interpolation is the Hausdorff-Young inequality (6.6). Here X is $[0,2\pi]$ with the normalized Lebesgue measure $d\theta/(2\pi)$, and $Y = \mathbb{Z}$ with its usual counting measure. The mapping T is defined by $T(f) = \{a_n\}$, with

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta.$$

◇

Corollary 6.8.4. *If $1 \leq p \leq 2$ and $1/p + 1/q = 1$, then*

$$\|T(f)\|_{L^q(\mathbb{Z})} \leq \|f\|_{L^p([0,2\pi])}.$$

Note that since $L^2([0,2\pi]) \subset L^1([0,2\pi])$ and $L^2(\mathbb{Z}) \subset L^\infty(\mathbb{Z})$ we have $L^2([0,2\pi]) + L^1([0,2\pi]) = L^1([0,2\pi])$, and also $L^2(\mathbb{Z}) + L^\infty(\mathbb{Z}) = L^\infty(\mathbb{Z})$.

The inequality for $p_0 = q_0 = 2$ is a consequence of Parseval's identity, while the one for $p_1 = 1, q_1 = \infty$ follows from the observation that for all n,

$$|a_n| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|d\theta$$

Thus Riesz's theorem guarantees the conclusion when $1/p = \frac{(1-t)}{2} + t$, $1/q = \frac{(1-t)}{2}$ for any t with $0 \leq t \leq 1$. This gives all p with $1 \leq p \leq 2$, and q related to p by $1/p + 1/q = 1$.

Example 6.8.5. We next come to the dual Hausdorff-Young inequality (6.7). Here we define the operator T' mapping functions on $\mathbb{Z}$ to functions on $[0,2\pi]$ by

$$T'(\{a_n\}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

Notice that since $L^p(\mathbb{Z}) \subset L^2(\mathbb{Z})$ when $p \leq 2$, then the above is a welldefined function on $L^2([0, 2\pi])$ when $\{a_n\} \in L^p(\mathbb{Z})$, by the unitary character of Parseval's identity. $\diamond$

Corollary 6.8.6. *If $1 \leq p \leq 2$ and $1/p + 1/q = 1$, then*

$$\|T'(\{a_n\})\|_{L^q([0,2\pi])} \leq \|\{a_n\}\|_{L^p(\mathbb{Z})}.$$

The proof is parallel to that of the previous corollary. The case $p_0 = q_0 = 2$ is, as has already been mentioned, a consequence of Parseval's identity, while the case $p_1 = 1$ and $q_1 = \infty$ follows directly from the fact that

$$\left|\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right| \leq \sum_{n=-\infty}^{\infty} |a_n|$$

6.8.3 Fourier Transform and Paley-Wiener Theorem

We recall that the **Fourier transform (FT)** $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ of a function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx, \ \xi \in \mathbb{R}^d. \tag{6.9}$$

The **inversion** is

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi, \ x \in \mathbb{R}^d. \tag{6.10}$$

the most elegant and useful formulations of Fourier inversion are in terms of the L^2 theory, or in its greatest generality stated in the language of distributions. We are satisfied by the following results. See Stein's third book p.86 for proofs.

Proposition 6.8.7. *Suppose $f \in L^1(\mathbb{R}^d)$. Then $\widehat{f}$ defined by (6.9) is continuous and bounded in $\mathbb{R}^d$.*

Proposition 6.8.8. *Suppose $f \in L^1(\mathbb{R}^d)$ and assume also that $\widehat{f} \in L^1(\mathbb{R}^d)$. Then the inversion formula (6.10) holds for almost every x.*

Corollary 6.8.9. *Suppose $\widehat{f}(\xi) = 0$ for all ξ. Then $f = 0$ a.e.*

We consider the analog of Hausdorff-Young for the Fourier transform. Here the setting is $\mathbb{R}^d$ and the L^p spaces are taken with respect to the usual Lebesgue measure. We initially define the Fourier transform (denoted here by T) on simple functions by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Then clearly, $\|\mathcal{F}(f)\|_{L^\infty} \leq \|f\|_{L^1}$, and $\mathcal{F}$ has an extension (by Proposition 5.4 in Chapter 1 for instance) to $L^1(\mathbb{R}^d)$ for which this inequality continues to hold. Also, $\mathcal{F}$ has an extension to $L^2(\mathbb{R}^d)$ as a unitary mapping. (This is essentially the content of Plancherel's theorem. See Section 1, Chapter 5 in Book III.) Thus in particular $\|\mathcal{F}(f)\|_{L^2} \leq \|f\|_{L^2}$, for f simple. The same arguments as before then prove:

Corollary 6.8.10 (Hausdorff-Young). *If $1 \leq p \leq 2$ and $1/p + 1/q = 1$, then the Fourier transform $\mathcal{F}$ defines a linear mapping $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ has a unique extension to a bounded map $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$, i.e., with $\|T(f)\|_{L^q} \leq \|f\|_{L^p}$.*

Proof. $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ with $\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}$ because $\forall \xi$,

$$|\widehat{f}(\xi)| = \left|\int_{\mathbb{R}^d} f(x) e^{2\pi i x \cdot \xi} dx\right| \leq \int |f(x) e^{2\pi i x \cdot \xi}| dx = \|f\|_{L^1(\mathbb{R}^d)}$$

Also, $\mathcal{F}: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with $\|\widehat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R}^d)}$ by Plancherel. We will use Riesz interpolation theorem with $p_0 = 1$, $q_0 = \infty$, $M_0 = 1$ and $p_1 = q_1 = 2$, $M_1 = 1$.

Fix $1 \le p \le 2$. Notice that then $q \ge 2$ implying that $1/q \le 1/2$ and thus $t := 2/q \le 1$. We deifne

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

as in Riesz's theorem. With the t we defined, we have

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} = \frac{1-2/q}{1} + \frac{2/q}{2} = 1 - \frac{2}{q} + \frac{1}{q} = 1 - \frac{1}{q} = \frac{1}{p}$$

so $p_t = p$. Similar computation gives $q_t = q$.

Riesz's theorem gives

$$\|\widehat{f}\|_{L^q} = \|\widehat{f}\|_{L^{q_t}} \le M_0^{1-t} M_1^t \|f\|_{L^{p_t}} = \|f\|_{L^p}.$$

□

Finishing the applications of Riesz's interpolation theorem to Fourier transform, we now go to the main theme of this subsection.

Lemma 6.8.11. *Suppose a function f is entire and is bounded, i.e., $f \in H(\mathbb{C})$ and $|f(x)| \le A\ \forall x \in R$ for some $A > 0$. If for any $z \in \mathbb{C}$, $|f(z)| \le Be^{2\pi M|z|}$ for some $B > 0$ and $M \in \mathbb{R}$, then we have*

$$|f(z)| \le Ce^{2\pi M|y|}, \quad \forall z = x + yi \in \mathbb{C}$$

for some constant C.

Proof. Let $U_1 = \{z = x + yi : x > 0, y > 0\}$ be the first quadrant. Define

$$F(z) = e^{2\pi iMz} f(z).$$

Then $|F(x)| = |f(x)| \le A$ for $x \ge 0$ and $|F(iy)| = e^{-2\pi My}|f(yi)| \le B$ for $y \ge 0$. Thus, F is bounded on ∂U_1 and satisfy the growth condition $|F(z)| \le |f(z)| \le Be^{2\pi M|z|}$ on U_1. This growth restriction is sufficient, Since U_1 is a sector with angle $\pi/2$, $\alpha = 2$ in first version of Phragmén-Lindelöff theorem, which gives

$$\forall z \in U_1 : \ e^{-2\pi My}|f(z)| = |F(z)| \lesssim 1 \implies |f(z)| \lesssim e^{2\pi My}$$

The same argument works in the second quadrant $U_2 = \{x + yi : x < 0, y > 0\}$. In the remaining quadrants, where $y < 0$, this argument is applied to $\widetilde{F}(z) := e^{-2\pi iMz} f(z)$. □

Lemma 6.8.12. *Let $F: U \times [a, b] \to \mathbb{C}$ be continuous, where $U \subset \mathbb{C}$ is open and $a < b$. Assume also that $z \mapsto F(z, s)$ is analytic for all $s \in [a, b]$. Then*

$$f(z) := \int_a^b F(z, s) \mathrm{d}s, \quad z \in U$$

is analytic in U (i.e. $f \in H(U)$).

Proof. Without loss of generality $[a, b] = [0, 1]$. Define f_n as in the hint. Notice that these are analytic as finite sums of analytic functions. It suffices to show that $f_n \to f$ uniformly in all compact $K \subset U$. Let $\varepsilon > 0$. Now F is uniformly continuous in the compact product set $K \times [0, 1]$. Thus, there exists $\delta > 0$ so that

$$|F(z, s_1) - F(z, s_2)| < \varepsilon$$

for all $z \in K$ whenever $s_1, s_2 \in [0,1]$ satisfy $|s_1 - s_2| < \delta$. So if $n > 1/\delta$, then for all $z \in K$ we have

$$|f_n(z) - f(z)| \le \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} |F(z,k/n) - F(z,s)| ds \le \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon,$$

and we are done. □

The following characterizes, via complex analysis, when the support of $\hat{f}$ is bounded.

Theorem 6.8.13 (Paley-Wiener Theorem)**.** *Let $f \in L^1(\mathbb{R})$ be bounded. Then*

$$\operatorname{spt}(\hat{f}) = \{\xi \in \mathbb{R} : \hat{f} \neq 0\} \subset [-M, M]$$

for some $M > 0$, if and only if f can be extended to an entire function satisfying $|f(z)| \lesssim e^{2\pi M|z|}$ for $z \in \mathbb{C}$.

Proof. Suppose first that $\operatorname{spt}(\hat{f}) \subset [-M, M]$. Then clearly, $\hat{f} \in L^1$ and by Fourier inversion formula,

$$f(x) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad x \in \mathbb{R}.$$

We can actually now define f for $z \in \mathbb{C}$ via

$$f(z) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i z \xi} d\xi$$

which extends $f(x)$ on $\mathbb{R}$ to $\mathbb{C}$. Now lemma 6.8.12 implies that $f \in H(\mathbb{C})$, since $F(z,\xi) := \hat{f}(\xi) e^{2\pi i z \xi}$ is continuous (by DCT) and with fixed ξ the function $z \mapsto F(z,\xi)$ is clearly analytic. Moreover, notice that $\hat{f} \in L^\infty$ because $|\hat{f}(\xi)| \le \int |f| = \|f\|_1$. Then

$$|f(z)| \le \int_{-M}^{M} \|\hat{f}\|_\infty e^{-2\pi y \xi} d\xi \le 2M \|\hat{f}\|_\infty e^{2\pi M|y|} \le 2M \|\hat{f}\|_\infty e^{2\pi M|z|}.$$

So it remains to prove the converse. We are assuming $f \in L^1(\mathbb{R})$ bounded has an extnesion to $f \in H(\mathbb{C})$ with $|f(z)| \lesssim e^{2\pi M|z|}$.

Lemma 6.8.11 implies the improved bound $|f(z)| \lesssim e^{2\pi M|y|}$ for $z = x + yi \in \mathbb{C}$. We still need some extra control on the x-direction. Fix $\xi > M$. We want $\hat{f}(\xi) = 0$. To prove this, define $\forall \varepsilon > 0$ the helper function

$$f_\varepsilon(z) := \frac{f(z)}{(1 + i\varepsilon z)^2}, \quad z \neq \frac{i}{\varepsilon}$$

Notice that $|1 + i\varepsilon(x + yi)|^2 = (1 - \varepsilon y)^2 + (\varepsilon x)^2 \ge 1 + \varepsilon^2 x^2 \ge 1$ if $x \in \mathbb{R}$ and $y \le 0$. Thus, $|f_\varepsilon(z)| \le |f(z)|$ in $\{z = x + yi : y \le 0\}$. In particular, this is true on $\mathbb{R}$, so $f_\varepsilon \in L^1(\mathbb{R})$ and $\hat{f}_\varepsilon$ is defined. We now have

$$|\hat{f}(\xi) - \hat{f}_\varepsilon(\xi)| \le \int_{-\infty}^{\infty} |f(x)| \left| \frac{1}{(1 + i\varepsilon x)^2} - 1 \right| dx.$$

As $\frac{1}{1+i\varepsilon x} - 1 \to 0$, we see that $\hat{f}_\varepsilon \to \hat{f}(\xi)$. To be rigorous, one needs to show that

$$\lim_{\varepsilon \to 0} \int |f(x)| \left| \frac{1}{(1 + i\varepsilon x)^2} - 1 \right| dx = \int |f(x)| \lim_{\varepsilon \to 0} \left| \frac{1}{(1 + i\varepsilon x)^2} - 1 \right| dx.$$

by DCT and the fact that $|f(x)| \left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| \le 2|f(x)|$.

Our goal is to show that $\hat{f}_\varepsilon(\xi) = 0$ for all ε (where $\xi > M$ is fixed). This then implies $\hat{f}(\xi) = 0$ as desired. The good thing is that f_ε behaves better than f in x-direction:

$$|f_\varepsilon(x + yi)| \lesssim \frac{2^{2\pi M|y|}}{1 + \varepsilon^2 x^2} \lesssim_\varepsilon \frac{e^{2\pi M|y|}}{1 + x^2}, \quad x \in \mathbb{R}, y \leq 0.$$

We will "move" integration in $\hat{f}_\varepsilon$ into some other horizontal line than $\mathbb{R}$. To this end, define for $y > 0$ and $R > 0$ then path

$$\gamma := [-R, R] \star [R, R - yi] \star [R - yi, -R - yi] \star [-R - yi, -R.]$$

As $f_\varepsilon \in H(\{\mathrm{Im}(z) < \frac{1}{\varepsilon}\})$ and γ is a path in the convex set $\{\mathrm{Im}(z) < \frac{1}{\varepsilon}\}$, Cauchy's theorem says that

$$\int_\gamma f_\varepsilon(z) e^{-2\pi i z\xi} dz = 0$$

The integral over the vertical lines vanishes by, for instance,

$$\begin{aligned}
&\left| \int_R^{R-iy} f_\varepsilon(z) e^{-2\pi i z\xi} dz \right| \\
&= \left| \int_0^y f_\varepsilon(R - it) e^{-2\pi i(R-it)\xi} (-i) dt \right| \\
&\lesssim_\varepsilon \frac{y e^{2\pi M y}}{1 + R^2} \overset{R\to\infty}{\longrightarrow} 0
\end{aligned}$$

(notice how we critically needed the extra x-direction decay and the improved estimate of $e^{2\pi M|z|}$!)Thus, we have

$$\begin{aligned}
\hat{f}_\varepsilon(\xi) &= \lim_{R\to\infty} \int_{-R}^{R} f_\varepsilon(x) e^{-2\pi i x\xi} dx \\
&= \lim_{R\to\infty} \int_{[-R-iy, R+iy]} f_\varepsilon(z) e^{-2\pi i z\xi} dz \\
&= \lim_{R\to\infty} \int_{-R}^{R} f_\varepsilon(x - iy) e^{-2\pi i(x-iy)\xi} dx \\
&= \int_{-\infty}^{\infty} f_\varepsilon(x - iy) e^{-2\pi i(x-iy)\xi} dx \\
\implies \left|\hat{f}_\varepsilon(\xi)\right| &\lesssim_\varepsilon \int_{-\infty}^{\infty} \frac{e^{2\pi M y}}{1 + x^2} e^{-2\pi y\xi} dx = e^{2\pi y(M-\xi)} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}}_{<\infty} \lesssim e^{2\pi y(M-\xi)}
\end{aligned}$$

Recall we fixed $\xi > M$, so $M - \xi < 0$ above. The parameter $y > 0$ was arbitrary here, and we can let $y \to \infty$ to get $f_\varepsilon(\xi) = 0$. This holds with any $\varepsilon > 0$, and so $f(\xi) = 0$. To prove $\hat{f}(\xi) = 0$ for $\xi < -M$ we do almost the same things: modify f as $\frac{1}{(1-i\varepsilon z)^2}$ and work in the upper half-plane instead. $\square$

Remark 6.8.14. *This is a more refined version of the principle that if f and $\hat{f}$ both have compact support, then $f = 0$. Indeed, if also f would have compact support on $\mathbb{R}$, then the zeros of the extended version $f \in H(\mathbb{C})$ would accumulate and $f = 0$.*

Fourier analysis is a rich subject. we end our treatment here, however. The idea was to simply demonstrate that many problems of Fourier analysis can be fruitfully be attacked using complex techniques.

Chapter 7

Harmonic Functions

Let U be a region (open connected set in $\mathbb{C}$). A real-valued function $f : U \to \mathbb{R}$ is said to be **harmonic** if, considered as $\mathbb{R}^2 \to \mathbb{R}$, it is **twice continuously differentiable** (i.e., its second derivatives are continuous) and satisfies the **Laplace equation**

$$\Delta f = \nabla^2 f := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = f_{xx} + f_{yy} = 0$$

A complex-valued function $f : U \to \mathbb{C}$ is said to be **harmonic** if and only if $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$ are real harmonic. The study of harmonic functions is called **potential theory**.

7.1 Harmonicity and Analyticity

Suppose $f = u + iv$ is an analytic function on region U. Then [11] Corollary 4.45 claims that

$$f'(z) = \partial_x u(z) + i\partial_x v(z) = \partial_y v(z) - i\partial_y u(z). \tag{7.1}$$

[11] Theorem 8.1 claims that f is infinitely differentiable. Thus, with $\mathrm{Re} f' = \partial_x u$, $\mathrm{Im} f' = \partial_x v$, we apply above equation again to get

$$\begin{aligned} f'' &= \partial_x(\partial_x u) + i\partial_x(\partial_x v) = \partial_y(\partial_x v) - i\partial_y(\partial_x u) \\ &= \partial_{xx} u + i\partial_{xx} v = \partial_{yx} v - i\partial_{yx} u \end{aligned}$$

With $\mathrm{Re} f' = \partial_y v$, $\mathrm{Im} f' = -\partial_y u$, we have

$$\begin{aligned} f'' &= \partial_x(\partial_y v) + i\partial_x(-\partial_y u) = \partial_y(-\partial_y u) - i\partial_y(\partial_y v) \\ &= \partial_{xy} v - i\partial_{xy} u = -\partial_{yy} u - i\partial_{yy} v \end{aligned}$$

Apply equation (7.1) repetitively to see that u and v are smooth functions on U, i.e., $\in C^\infty(U)$. Therefore, f being analytic makes u and v automatically twice continuously differentiable, and we shall also see they are harmonic conjugates of each other as well.

Definition 7.1.1. *Let U be a region. Two harmonic functions $u : U \to \mathbb{R}$, $v : U \to \mathbb{R}$ are said to be harmonic conjugates of each other if in U they satisfy the Cauchy Riemann equation (C.R. eq)*

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Proposition 7.1.2. *Suppose $f = u + iv$ is an analytic function on region U. Then u and v are harmonic conjugates of each other.*

Proof. Analytic function f satisfies the C.R. eq.

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

so u and v satisfy C.R. eq. u and v are twice continuously differentiable as we noted above. Thus, we have their mixed second partial derivatives equal, that is,

$$(*): \qquad \partial_{xy} u = \partial_{yx} u, \quad \partial_{xy} v = \partial_{yx} v.$$

Now we take x-derivatives of C.R. eq. to get

$$\begin{cases} \partial_{xx} u = \partial_{xy} v \\ \partial_{yy} u = -\partial_{yx} v \end{cases}$$

By $(*)$, we see

$$\begin{aligned} \partial_{xx} u &= \partial_{yx} v = -\partial_{yy} u \\ \implies \Delta u &= \partial_{xx} u + \partial_{yy} u = 0. \end{aligned}$$

Thus u is harmonic. Similarly, taking y-derivatives of C.R. eq. will show that v is also harmonic

$$\Delta v = \partial_{xx} v + \partial_{yy} v = 0.$$

□

We show that the converse is also true.

Proposition 7.1.3. *Let U be a region. u and v are harmonic conjugates on U. Then $f = u + iv$ is analytic on U.*

Proof. Let $z = x + iy \in U$ and let $B(z; r) \subset U$. If $h = s + it \in B(0, r)$ then

$$u(x+s, y+t) - u(x, y) = [u(x+s, y+t) - u(x, y+t)] + [u(x, y+t) - u(x, y)]$$

Applying the mean value theorem for the derivative of a function of one variable to each of these bracketed expressions, yields for each $s + it$ in $B(0, r)$ numbers s_1 and t_1 such that $|s_1| < |s|$ and $|t_1| < |t|$ and

$$\begin{cases} u(x+s, y+t) - u(x, y+t) = u_x\left(x+s_1, y+t\right) s \\ u(x, y+t) - u(x, y) = u_y\left(x, y+t_1\right) t \end{cases} \tag{7.2}$$

Letting

$$\varphi(s, t) := [u(x+s, y+t) - u(x, y)] - [u_x(x, y)s + u_y(x, y)t]$$

(7.2) gives that

$$\frac{\varphi(s, t)}{s+it} = \frac{s}{s+it}\left[u_x\left(x+s_1, y+t\right) - u_x(x, y)\right] + \frac{t}{s+it}\left[u_y\left(x, y+t_1\right) - u_y(x, y)\right]$$

But $|s| \le |s+it|, |t| \le |s+it|, |s_1| < |s|, |t_1| < |t|$, and the fact that u_x and u_y are continuous gives that

$$\lim_{s+it \to 0} \frac{\varphi(s, t)}{s+it} = 0 \tag{7.3}$$

Hence
$$u(x+s,y+t)-u(x,y)=u_x(x,y)s+u_y(x,y)t+\varphi(s,t)$$
where φ satisfies (7.3). Similarly
$$v(x+s,y+t)-v(x,y)=v_x(x,y)s+v_y(x,y)t+\psi(s,t)$$
where ψ satisfies
$$\lim_{s+it\to 0}\frac{\psi(s,t)}{s+it}=0 \tag{7.4}$$
Using the fact that u and v satisfy the Cauchy-Riemann equations it is easy to see that
$$\frac{f(z+s+it)-f(z)}{s+it}=u_x(z)+iv_x(z)+\frac{\varphi(s,t)+i\psi(x,t)}{s+it}$$
In light of (7.3) and (7.4), f is differentiable and $f'(z)=u_x(z)+iv_x(z)$. Since u_x and v_x are continuous, f' is continuous and f is analytic. □

Due to above two propositions, we see

Corollary 7.1.4. *Two real-valued functions u and v are harmonic conjugates of each other on region U if and only if $f=u+iv$ is analytic on U.*

Now we ask: if we have a harmonic function u, how to find its harmonic conjugate v? If v_1 and v_2 are two harmonic conjugates of u then $i\left(v_1-v_2\right)=\left(u+iv_1\right)-\left(u+iv_2\right)$ is analytic on U and only takes on purely imaginary values. It follows that two harmonic conjugates of a harmonic function differ by a constant (exercise).

Theorem 7.1.5. *Let U be a simply connected region. If $u:U\to\mathbb{R}$ is a harmonic function then u has a harmonic conjugate.*

Proof. Since u is a harmonic function, we have $u\in C^2(U)$ and
$$\partial_{xx}u+\partial_{yy}u=0.$$
Thus, $-\partial_y u$ and $\partial_x u$ have continuous partial derivatives in U and
$$\partial_y(-\partial_y u)=\partial_x(\partial_x u).$$
From necessary and sufficient condition of exact equation (see ode note), we know that $-\partial_y u\mathrm{d}x+\partial_x u\mathrm{d}y$ is the total derivative of some function v. In fact, it is given by
$$v(x,y)=\int_0^y u_x(x,t)\mathrm{d}t-\int_0^x u_y(s,0)\mathrm{d}s.$$
(See [3] p.43 Theorem 2.30). It can be checked that u and v satisfy C.R. eq. Simply-connectedness is used for independence of path in integration. □

Corollary 7.1.6. *If $u:G\to\mathbb{R}$ is harmonic then u is infinitely differentiable.*

Proof. Fix $z_0=x_0+iy_0$ in U and let δ be chosen such that $B\left(z_0;\delta\right)\subset U$. Then u has a harmonic conjugate v on $B\left(z_0,\delta\right)$. That is, $f=u+iv$ is analytic and hence infinitely differentiable on $B\left(z_0,\delta\right)$. It now follows that u is infinitely differentiable by our discussion right before definition 7.1.1. □

In fact, we can show that the converse of the theorem is also true. Therefore, the existence of a harmonic conjugate is another equivalent characterization of simply-connectedness apart from Theorem 5.3.8.

Theorem 7.1.7. *For a harmonic u we can find v s.t. $f = u + iv$ is analytic on region U iff the region U is simply-connected*

Proof. We only need to show $\Rightarrow$ due to previous theorem. Due to Theorem 5.3.8, it suffices to show that for any $0 \neq f \in H(U)$ there is a function $g \in H(U)$ such that $f(z) = e^{g(z)}$.

Let $u = \mathrm{Re} f$, $v = \mathrm{Im} f$. If $\phi : U \to \mathbb{R}$ is defined by $\phi(x, y) = \log |f(x + iy)| = \log \left[u(x, y)^2 + v(x, y)^2\right]^{\frac{1}{2}}$ then a computation shows that ϕ is harmonic. Let φ be a harmonic function on U such that $g = \phi + i\varphi$ is analytic on U and let $h(z) = \exp g(z)$. Then h is analytic, never vanishes, and $\left|\frac{f(z)}{h(z)}\right| = 1$ for all z in U. That is, f/h is an analytic function whose range is not open. It follows that there is a constant c such that $f(z) = c\, h(z) = c \exp g(z) = \exp\left[g(z) + c_1\right]$. Thus, $g(z) + c_1$ is a branch of $\log f(z)$. $\square$

7.2 Dirichlet Problem

Given some $g \in C(\partial\mathbb{D})$, consider the **Dirichlet problem (Dir)**

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ u = g & \text{on } \partial\mathbb{D} \end{cases}$$

we want to find $u = u_g \in C^2(\mathbb{D}) \cap C(\mathbb{D})$ satisfying (Dir). We do it in a reverse way. Suppose u solves (Dir), what does it look like? Two approaches will be presented to show that functions u satisfying (Dir) will be of **Poisson integral formula** (where the P_r is Poisson kernel that will be defined)

$$(P): \qquad u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{it}) P_r(\theta - t) \mathrm{d}t \tag{7.5}$$

7.2.1 (Dir) $\implies$ (P) by separation of variables.

We first present the polar representation of (Dir): let $u(r, \theta) := u(r\cos\theta, r\sin\theta)$. Then the boundary condition becomes

$$u(1, \theta) = u(\cos\theta, \sin\theta) = g(\cos\theta, \sin\theta) =: G(\theta).$$

The Laplacian is specified by the following proposition.

Proposition 7.2.1. *In $\mathbb{C}$ the Laplacian in the polar coordinates $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, $r \geq 0$, $\theta \in [-\pi, \pi)$ takes the form*

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}, \qquad r > 0,\ \theta \in [-\pi, \pi)$$

if u is nice enough. More carefully stated, we are claiming that $u(r, \theta) := u(re^{i\theta}) = u(r\cos\theta, r\sin\theta)$ satisfies

$$\begin{aligned} U_{rr}(r, \theta) + \frac{1}{r} U_r(r, \theta) + \frac{1}{r^2} U_{\theta\theta}(r, \theta) \\ = u_{xx}(r\cos\theta, r\sin\theta) + u_{yy}(r\cos\theta, r\sin\theta). \end{aligned}$$

Proof. See Math5022 HW4 Q1 (the same solution is hide as comment in latex.) $\square$

In (Dir), we seek $u \in C(\mathbb{D})$ and $g \in \partial\mathbb{D}$. In polar coordinates, they become $u \in C([0, 1] \times [-\pi, \pi])$ and $G \in C[-\pi, \pi]$, both 2π-periodic on $\theta \in \mathbb{R}$.

We use method of **separation of variables**: look first for solutions to $\Delta U = 0$ ot the special form $u(r, \theta) = v(r)w(\theta)$. By continuity, we look for bounded v, w with w 2π-periodic. Substituting this form into the polar

Laplacian to get

$$\begin{aligned} v''(r)w(\theta) + \frac{1}{r}v'(r)w(\theta) + \frac{1}{r^2}v(r)w''(\theta) = 0 \\ \Longrightarrow \underbrace{\frac{r^2v''(r) + rv'(r)}{v(r)}}_{\text{independent of } \theta} = \underbrace{-\frac{w''(\theta)}{w(\theta)}}_{\text{independent of } r} \end{aligned}$$

So both sides need to equal some common constant λ:

$$\begin{gathered} \frac{r^2v''(r) + rv'(r)}{v(r)} = -\frac{w''(\theta)}{w(\theta)} = \lambda \\ \Rightarrow \begin{cases} r^2v''(r) + rv'(r) - \lambda v(r) = 0, \\ w''(\theta) + \lambda w(\theta) = 0. \end{cases} \end{gathered}$$

We analyze first for which $\lambda \in \mathbb{R}$ the problem $\omega'' = -\lambda\omega$ has solutions. Here we use some basic results from the theory of ODEs without proofs.

(1) $\lambda < 0$:
Solution would take the form $w(\theta) = C_1e^{\sqrt{-\lambda}\theta} + C_2e^{-\sqrt{-\lambda}\theta}$. The only possible 2π-periodic solution is $w = 0$.

(2) $\lambda = 0$:
Then $w(\theta) = C_1 + C_2\theta$. 2π-periodicity demands $C_2 = 0$ and we get the solution $w(\theta) = K$. However, for $\lambda = 0$, the corresponding ODE for v is

$$r^2v''(r) + rv'(r) = 0.$$

This ODE is an example of Euler's equation. Notice that one solution is a constant function. Another is $\ln r$, since

$$r^2 \cdot \left(-r^{-2}\right) + r \cdot \frac{1}{r} = -1 + 1 = 0.$$

Thus, the general solution is $v(r) = C_1 \ln r + C_2$. For this to stay bounded, we need $C_1 = 0$. Thus, $\lambda = 0$ gives $u(r,\theta) = C$.

(3) $\lambda > 0$:
Now $\omega(\theta) = C_1e^{i\sqrt{\lambda}\theta} + C_2e^{-i\sqrt{\lambda}\theta}$. For this to be 2π-periodic, we need $\sqrt{\lambda}$ to be an integer, i.e., $\sqrt{\lambda} = m$, $m = 1, 2, \cdots$. That is, $\lambda = m^2,\ m = 1, 2\cdots$. We get corresponding solutions

$$w_m(\theta) = C_1e^{im\theta} + C_2e^{-im\theta}.$$

For $\lambda = m^2$, the corresponding problem for v is $r^2v''(r) + rv'(r) - m^2v(r) = 0$. Again, this is an Euler's equation. By noticing that $r^{\pm m}$ are solutions, we have

$$v(r) = c_1r^m + c_2r^{-m}.$$

For this to be bounded, we need $c_2 = 0$. Thus, we have the solution

$$u_m(r,\theta) = r^m(c_1e^{im\theta} + c_2e^{-im\theta}).$$

Previously, we fouond the constant solution. Of course, these do not have U as the boundary value. But the following superposition of the solutions could

$$u(r,\theta) := \sum_{m\in\mathbb{Z}} a_m r^{|m|} e^{im\theta}.$$

Now, for this to satisfy $u(1,\theta) = G(\theta)$, it must be the case that a_m satisfy

$$G(\theta) = \sum_{m\in\mathbb{Z}} a_m e^{im\theta}.$$

Integration on both sides gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(t)e^{-imt}\mathrm{d}t = \sum_{k\in\mathbb{Z}} a_k \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-m)\theta}\mathrm{d}\theta}_{=\delta_{k,m}} = a_m.$$

(the Fourier coefficients of Fourier series). Mere continuity of U is insufficient for a Fourier series representation

$$G(\theta) = \sum_{m\in\mathbb{Z}} a_m e^{im\theta}, \quad a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(t)e^{-imt},$$

while C^1 would suffice. We don't worry about this subtlety: all we want at this point is to arrive at some formula how u should look like if everything is nice, and then later prove this formula actually solves (Dir) even for just a continuous U. We now simplify

$$\begin{aligned}
u(r,\theta) &= \sum_{m\in\mathbb{Z}} a_m r^{|m|} e^{im\theta} \\
&= \sum_{m\in\mathbb{Z}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} G(t)e^{-imt}\mathrm{d}t\right) r^{|m|} e^{im\theta} \\
&= \lim_{N\to\infty} \sum_{|m|\le N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} G(t)e^{-int}\mathrm{d}t\right) r^{|m|} e^{im\theta} \\
&= \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} G(t) \sum_{|m|\le N} r^{|m|} e^{im(\theta-t)}\mathrm{d}t
\end{aligned}$$

The finite sum inside the integral is bounded by

$$\left|\sum_{|m|\le N} r^{|m|} e^{im(\theta-t)}\right| \leqslant \sum_{m=-\infty}^{\infty} r^{|m|} < \infty$$

where the latter is a geometric series with $r < 1$. We can interchange the limit and the integral (follows Dominated Convergence Theorem). Thus,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(t) \sum_{m\in\mathbb{Z}} r^{|m|} e^{im(\theta-t)} dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(t) P_r(\theta-t) dt,$$

where $P_r(\theta) := \sum_{m\in\mathbb{Z}} r^{|m|} e^{im\theta}$ is the **Poisson kernel**, $0 \leqslant r < 1, \theta \in \mathbb{R}$. Now we have arrived at the **Poisson integral formula**:

$$\begin{aligned}
u_g\left(re^{i\theta}\right) &= u\left(re^{i\theta}\right) = u(r,\theta) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(t) P_r(\theta-t) dt. \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g\left(e^{it}\right) P_r(\theta-t) dt.
\end{aligned} \tag{7.6}$$

7.2.2 (Dir) $\Longrightarrow$ (P) by mean value property

We introduce the second approach to show solution of (Dir) is of the form (P). Let $u(z)$ solve (Dir), then $u(z)$ has the mean value property

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g\left(e^{i\theta}\right) d\theta. \tag{7.7}$$

Let $z_0 = re^{i\theta_0}$ be a point in D. Then there is a similar representation formula for $u(z_0)$, obtained by changing variables through a Möbius transformation. Let $\tau(z) = (z - z_0) / (1 - \bar{z}_0 z)$. The unit circle $\partial\mathbb{D}$ is invariant under τ, and we may write $\tau\left(e^{i\theta}\right) = e^{i\varphi}$. From

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\tau(e^{i\theta}) = \frac{\mathrm{d}e^{i\varphi(\theta)}}{\mathrm{d}\theta} = ie^{i\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}\theta} = i\tau(e^{i\theta})\frac{\mathrm{d}\varphi}{\mathrm{d}\theta},$$

we see that

$$\begin{aligned}
\frac{\mathrm{d}\varphi}{\mathrm{d}\theta} &= \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}\tau(e^{i\theta})}{i\tau(e^{i\theta})} = \frac{\mathrm{d}}{\mathrm{d}\theta}\tau(e^{i\theta}) \cdot \frac{1 - ie^{i\theta}\bar{z}_0}{i(e^{i\theta} - z_0)} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{e^{i\theta} - z_0}{1 - \bar{z}_0 e^{i\theta}}\right) \cdot \frac{1 - ie^{i\theta}\bar{z}_0}{i(e^{i\theta} - z_0)} \\
&= \frac{ie^{i\theta} - ie^{2i\theta}\bar{z}_0 + ie^{2i\theta}\bar{z}_0 - ie^{i\theta}|z_0|^2}{(1 - e^{i\theta}\bar{z}_0)^2} \cdot \frac{1 - ie^{i\theta}\bar{z}_0}{i(e^{i\theta} - z_0)} \\
&= \frac{ie^{i\theta}(1 - |z_0|^2)}{e^{i\theta}(e^{-i\theta} - \bar{z}_0)(1 - e^{i\theta}\bar{z}_0)} \cdot \frac{1 - ie^{i\theta}\bar{z}_0}{i(e^{i\theta} - z_0)} \\
&= \frac{1 - |z_0|^2}{|e^{i\theta} - z_0|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta_0 - \theta) + r^2} = P_{z_0}(\theta).
\end{aligned}$$

This function $P_{z_0}(\theta)$ is called the Poisson kernel for the point $z_0 \in \mathbb{D}$. Since $u\left(\tau^{-1}(z)\right)$ is another function continuous on $\bar{\mathbb{D}}$ and harmonic on $\mathbb{D}$, the function $u \circ \tau^{-1}$ solves (Dir) with $h := u \circ \tau^{-1}$ on $\partial\mathbb{D}$. The change of variables yields

$$\begin{aligned}
u(z_0) = u\left(\tau^{-1}(0)\right) &\xlongequal{(7.7)} \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\varphi})\mathrm{d}\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau^{-1})\left(e^{i\varphi}\right)\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{i\theta}\right) P_{z_0}(\theta)\mathrm{d}\theta \xlongequal{u=g \text{ on } \partial\mathbb{D}} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\theta})P_{z_0}(\theta)\mathrm{d}\theta.
\end{aligned}$$

This is the same as the Poisson integral formula (7.6).

Remark 7.2.2. *If we are given any u that is harmonic on $\mathbb{D}$ and continuous on $\partial\mathbb{D}$ (notice that harmonic on $\mathbb{D}$ implies continuity on $\mathbb{D}$), the formula*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})P_r(\theta - t)dt$$

holds for any $r < 1$ and should make no confusion with Proposition 7.2.5. Poisson integral formula is obtained when assuming u is harmonic on $\mathbb{D}$ and continuous on $\partial\mathbb{D}$, i.e., solves $(\mathrm{Dir})_u$. If this is the case, we have the Poisson integral formula and Proposition 7.2.5 reads as

$$\lim_{r \to 1^-} u(re^{i\theta}) = u(e^{i\theta}).$$

The Poisson integral formula should be understood in the way that the values of a function harmonic on $\mathbb{D}$ continuous on $\partial\mathbb{D}$ inside the unit disk is determined by its values on the unit circle.

7.2.3 Poisson Expression $\implies$ (Dir)

Several properties of the Poisson kernel are given first.

Proposition 7.2.3 (Properties of Poisson kernel)**.** *Let $r \in [0, 1)$. Then*

(1)

$$P_r(\theta) = \frac{1 - r^2}{\sin^2\theta + (\cos\theta - r)^2} \geq 0.$$

(2)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1.$$

(3)

$$\lim_{r\to 1^-}\int_{\delta\le|t|\le\pi} P_r(t)dt = 0 \quad \forall\delta\in(0,\pi).$$

Remark 7.2.4. *Due to (2) and (3) and the fact that $P_r(\theta)$ is positive, Poisson kernel is a **good kernel**.*

proof of the proposition.

$$\begin{aligned}
&\sum_{m=-N}^{N} r^{|m|}e^{im\theta} = 1+\sum_{m=1}^{N} r^m e^{im\theta} + \sum_{m=1}^{N} r^m e^{-im\theta} \\
=&1+\sum_{m=1}^{N}(re^{i\theta})^m + \sum_{m=1}^{N}(re^{-i\theta})^m = 1+\frac{\omega-\omega^{N+1}}{1-\omega}+\frac{\overline{\omega}-\overline{\omega}^{N+1}}{1-\overline{\omega}}, \quad \omega = re^{i\theta}
\end{aligned}$$

By letting $N\to\infty$, we have

$$\begin{aligned}
P_r(\theta) =&1+\frac{\omega}{1-\omega}+\frac{\overline{\omega}}{1-\overline{\omega}}, \qquad \text{since } 0\le r<1 \\
=&1+\frac{\omega(1-\overline{\omega}+\overline{}(1-\omega))}{(1-\omega)(1-\overline{\omega})} = 1+\frac{-2r^2+r(e^{i\theta}+e^{-i\theta})}{1-r(e^{i\theta}+e^{-i\theta})+r^2} \\
=&1+\frac{-2r^2+2r\cos\theta}{1-2r\cos\theta+r^2} = \frac{1+r^2}{1-2r\cos\theta+r^2}.
\end{aligned}$$

This proves (1).

To show (2), we note that the convergence for the series $f_N(\theta)=\sum_{m=-N}^{N} r^{|m|}e^{im\theta}$ is absolute and uniform (observe that $|f_N|\le\sum_{m=-N}^{N}\left|r^{|m|}e^{im\theta}\right| = \sum_{m=-N}^{N} r^{|m|}$ which is a convergent geometric series since $0\le r<1$. Then Weierstrass M-test concludes the absolute and uniform convergence). Uniform convergence ensures the termwize integrations of f_N converge to the integration of the limit function of the series. Namely,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta)d\theta = \frac{1}{2\pi}\lim_{N\to\infty}\sum_{m=-N}^{N}\int_{-\pi}^{\pi} r^{|m|}e^{im\theta}d\theta = \frac{1}{2\pi}\lim_{N\to\infty}\sum_{m=-N}^{N} r^{|m|}\underbrace{\int_{-\pi}^{\pi}e^{im\theta}d\theta}_{\begin{cases}=0, & m\ne 0\\ =2\pi, & m=0\end{cases}} = \frac{1}{2\pi}2\pi = 1.$$

To show (3), we note that

$$1-2r\cos\theta+r^2 = (1-r)^2+2r(1-\cos\theta).$$

Thus, if $\frac{1}{2}\le r\le 1$ and $\delta\le|\theta|\le\pi$, then

$$1-2r\cos\theta+r^2\ge c_\delta>0 \implies 0\le P_r(\theta)\le(1-r^2)/c_\delta$$

Therefore,

$$0\le\lim_{r\to 1^-}\int_{\delta\le|t|\le\pi}P_r(t)dt\le\lim_{r\to 1^-}\int_{\delta\le|t|\le\pi}\frac{1-r^2}{c_\delta}dt = 0,$$

which proves (3). □

Proposition 7.2.5. *The boundary value g is obtained in the sense that*

$$\lim_{r\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi} g\left(e^{it}\right)P_r(\theta-t)\mathrm{d}t = g(e^{i\theta}),\quad -\pi\le\theta\le\pi.$$

In fact, one can show the convergence is uniform.

Proof. See Math5022 HW4 Q3 and the remark following the solution. □

We define the **Poisson expression** associated with $g\in C(\partial\mathbb{D})$ as the function

$$(P):\qquad u(re^{i\theta}) = u_g(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})P_r(\theta-t)\mathrm{d}t,\qquad 0\le r<1 \tag{7.8}$$

It is natural to ask the following question: if we directly define u as in (P) by the given $g\in C(\partial\mathbb{D})$ can we get $\Delta u=0$? The answer is yes by realizing it as the real part of an analytic function, as we shall see. One step further, our final goal in this subsection is to show the function

$$z\mapsto\begin{cases}u_g(z), & z\in\mathbb{D}\\ g(z), & z\in\partial\mathbb{D}\end{cases}$$

solves (Dir) uniquely.

Note that $P_r(\theta-t)$ in (P) can be written as

$$P_r(\theta-t)=\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\;\underset{e^{it}\in\partial\mathbb{D}}{\overset{z=re^{i\theta}\in\mathbb{D}}{=\!=\!=}}\;\frac{1-|z|^2}{|e^{it}-z|^2} \tag{7.9}$$

by noting that

$$|e^{it}-z|^2=(e^{it}-z)(e^{-it}-\overline{z})=1-\overline{z}e^{it}-ze^{-it}+|z|^2=1-\overline{ze^{-it}}-ze^{-it}+|z|^2=1-2\mathrm{Re}(ze^{-it})+|z|^2$$

So $P_r(\theta-t)$ can be also regarded as a function of $z=re^{it}\in\mathbb{D}$ and $e^{i\theta}\in\partial\mathbb{D}$. Then

$$P_r(\theta-t)=P\left(z,e^{it}\right)=\frac{1-|z|^2}{\left|e^{it}-z\right|^2}=\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)$$

Indeed,

$$\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)=\frac{1}{2}\left(\frac{e^{it}+2}{e^{it}-2}+\frac{e^{-it}+\overline{z}}{e^{-it}-\overline{z}}\right)=\frac{1}{2}\frac{1-e^{it}\overline{z}+ze^{-it}-|z|^2+1+e^{it}\overline{z}-ze^{-it}-|z|^2}{|e^{it}-z|^2}=\frac{1-|z|^2}{|e^{it}-z|^2}.$$

Suppose $g\in C(\partial\mathbb{D})$ is real-valued. Then

$$u(z)=u\left(re^{i\theta}\right)=\mathrm{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}g\left(e^{it}\right)\frac{e^{it}+z}{e^{it}-z}dt\right), \tag{7.10}$$

where the function inside $\mathrm{Re}(\cdot)$ is analytic in $z\in\mathbb{D}$ by Lemma 6.8.12. So $\Delta u=0$ as the real part of an analytic function is always harmonic.

Note: Of course, u is then still harmonic even if g is $\mathbb{C}$-valued.

Theorem 7.2.6. *Let $g\in C(\partial\mathbb{D})$. Then*

$$u_g(z)=u(z)=u\left(re^{i\theta}\right):=\frac{1}{2\pi}\int_{-\pi}^{\pi}g\left(e^{it}\right)P_r(\theta-t)dt$$

is the real part of a function $f\in H(\mathbb{D})$ given by (7.10) and is thus harmonic in $\mathbb{D}$. Besides, $u_g=g$ on $\partial\mathbb{D}$ in the sense that

$$\lim_{r\to 1^-}u\left(re^{i\theta}\right)=g\left(e^{i\theta}\right)$$

Remark 7.2.7. *The function*

$$z \mapsto \begin{cases} u_g(z), & z \in \mathbb{D} \\ g(z), & z \in \partial\mathbb{D} \end{cases}$$

is continuous on $\overline{\mathbb{D}}$, harmonic on $\mathbb{D}$. We show the continuity now.

Proof. Define operator H as

$$(Hg)(re^{i\theta}) = \begin{cases} u_g(re^{i\theta}), & 0 \le r < 1 \\ g(e^{i\theta}), & r = 1 \end{cases}$$

Our task is to show $Hg \in C(\overline{\mathbb{D}})$. Notice, for $0 \le r < 1$,

$$\begin{aligned} \left|(Hg)\left(re^{i\theta}\right)\right| &= \left|u_g\left(re^{i\theta}\right)\right| \\ &\overset{P_r \ge 0}{\le} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g\left(e^{it}\right)\right| P_r(\theta - t)\mathrm{d}t \\ &\le \|g\|_{L^\infty(\partial\mathbb{D})}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\mathrm{d}t \\ &= \|g\|_{L^\infty(\partial\mathbb{D})}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} P_r(u)du \\ &= \|g\|_{L^\infty(\partial\mathbb{D})}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(u)du \\ &= \|g\|_{L^\infty(\partial\mathbb{D})} \end{aligned}$$

Thus

$$\|Hg\|_{L^\infty(\overline{\mathbb{D}})} = \|g\|_{L^\infty(\partial\mathbb{D})}.$$

It is known by Fourier analysis that every continuous function on $\partial\mathbb{D}$ is a uniform limit of trigonometric polynomials:

$$\exists g_k(e^{i\theta}) = \sum_{|n| \le N_k} c_{n,k}e^{in\theta} \text{ s.t. } \|g_k - g\|_{L^\infty(\partial\mathbb{D})} \overset{k\to\infty}{\longrightarrow} 0.$$

But it is obvious that $Hg_k \in C(\overline{\mathbb{D}})$, since

$$\begin{aligned} u_{g_k}\left(r^{i\theta}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g_k\left(e^{it}\right) P_r(\theta - t)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{|n| \le N_k} c_{n,k}e^{int}\right)\sum_{m\in\mathbb{Z}} r^{|m|}e^{im(\theta - t)}dt \\ &= \sum_{n=-N_k}^{N_k} c_{n,k}\sum_{n=-\infty}^{\infty} r^{|m|}e^{in\theta}\cdot\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}dt}_{=\delta_{n,m}} \\ &= \sum_{n=-N_k}^{N_k} c_{n,k}r^{|n|}e^{in\theta} \end{aligned}$$

meaning that $Hg_k(re^{i\theta}) = \sum_{n=-N_k}^{N_k} c_{n,k}r^{|n|}e^{in\theta}$, $0 \le r \le 1$. But

$$\|Hg - Hg_k\|_{L^\infty(\overline{\mathbb{D}})} = \|H(g - g_k)\|_{L^\infty(\overline{\mathbb{D}})} = \|g - g_k\|_{L^\infty(\overline{\mathbb{D}})} \to 0$$

so $Hg \in C(\overline{\mathbb{D}})$ as a uniform limit of $Hg_k \in C(\overline{\mathbb{D}})$. □

We have showed that

$$Hg = \begin{cases} u_g & \text{on } \mathbb{D} \\ g & \text{on } \partial\mathbb{D} \end{cases}$$

is a solution to (Dir). Let's show the uniqueness.

proof of uniqueness. Suppose we have two functions solving (Dir). Then their difference is still harmonic and thus on $\partial\mathbb{D}$ their difference is zero. So the uniqueness problem can be alternatively formulated by the following:

Suppose $h \in C(\overline{\mathbb{D}})$ is real-valued and

$$\begin{cases} \Delta h = 0 & \text{on } \mathbb{D} \\ h = 0 & \text{on } \partial\mathbb{D} \end{cases}$$

We show that this implies $h = 0$. Suppose $h(z_0) > 0$ for some $z_0 \in \mathbb{D}$. Fix $\varepsilon \in (0, h(z_0))$. Then define $g(z) := h(z) + \varepsilon|z|^2, z \in \overline{\mathbb{D}}$. Then $g \in C(\overline{\mathbb{D}})$, $g(z_0) \geq h(z_0) > \varepsilon$ and $g(z) = \varepsilon$ on $\partial\mathbb{D}$. This implies $\max_{\overline{\mathbb{D}}} |g|$ must be obtained at some $z_1 \in \mathbb{D}$. But a local maximum at an interior point, essentially by the elementary second derivative test implies, second derivatives at this point ≤ 0; in particular $\Delta g(z_1) \leq 0$. But $\Delta g(z) = \Delta h(z) + 4\varepsilon = 4\varepsilon$ in $\mathbb{D}$. Then $\Delta g(z_1) = 4\varepsilon > 0$. Contradiction.

Thus, such z_0 cannot exist and $h \leq 0$. Similarly, $h \geq 0$, so $h = 0$. □

Theorem 7.2.8. *Given $g \in C(\partial\mathbb{D})$, $\exists!\, u \in C(\overline{\mathbb{D}})$ such that $\Delta u = 0$ on $\mathbb{D}$ and $u = g$ on $\partial\mathbb{D}$.*

We can extend the theorem to arbitrarily disks. If $R > 0$ then substituting r/R for r in the middle of (7.9) gives

$$\frac{R^2 - r^2}{R^2 - 2rR\cos\theta + r^2} \tag{7.11}$$

for $0 \leq r < R$ and all θ. So if u is continuous on $\overline{B}(a; R)$ and harmonic in $B(a, R)$ then

$$u\left(a + re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r^2}\right] u\left(a + Re^{it}\right) dt \tag{7.12}$$

Now (7.11) can also be written

$$\frac{R^2 - r^2}{|R - re^{i\theta}|^2}$$

and $R - r \leq \left|Re^{it} - re^{i\theta}\right| \leq R + r$. Therefore

$$\frac{R - r}{R + r} \leq \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r^2} \leq \frac{R + r}{R - r}.$$

If $u \geq 0$ then equation (7.12) yields the following.

Theorem 7.2.9 (Harnack's Inequality). *If $u : \overline{B}(a; R) \to \mathbb{R}$ is continuous, harmonic in $B(a, R)$, and $u \geq 0$ then for $0 \leq r < R$ and all θ*

$$\frac{R - r}{R + r} u(a) \leq u\left(a + re^{i\theta}\right) \leq \frac{R + r}{R - r} u(a)$$

Remark 7.2.10. *With conformal mappings, we can also explicitly solve in half-planes and strips. We skip calculatioins for now. For more on this, see elliptic pde theory, and subharmonic functions and Perron's method in [3].*

Let $u : U \to \mathbb{R}$ be a harmonic function on an open set $U \subseteq \mathbb{C}$. Then we can choose a ball B so that $\overline{B} \subset U$. Since u is harmonic, clearly $u \in C(\partial B)$, and thus we can consider the problem

$$\begin{cases} \Delta v = 0 & \text{on } B \\ v = u & \text{on } \partial B \end{cases}$$

From Theorem 7.2.6, we know solution v is given inside of B by the Poisson expression and is the real part of a function $f_B \in H(B)$. By uniquness,

$$u = v = \operatorname{Re}(f_B).$$

Since the ball is arbitrary, u is locally the real part of an analytic function. In particular, $u \in C^\infty(U)$.

7.3 More Properties of Harmonic Functions

Another thing that follows from the Poisson integral is

Theorem 7.3.1 (Harnack's theorem)**.** *Let U be a region.*
(a) The ***space of harmonic functions on*** *U, $\operatorname{Har}(U)$, inheriting the metric from $C(U, \mathbb{R})$, is complete.*
(b) If $\{u_n\}$ is a sequence in $\operatorname{Har}(U)$ such that $u_1 \leq u_2 \leq \ldots$ then either $u_n(z) \to \infty$ uniformly on compact subsets of U or $\{u_n\}$ converges in $\operatorname{Har}(U)$ to a harmonic function,

Proof.
(a) We notice that to show a subspace of a complete metric space is complete, it suffices to show it is closed (that's because we already have every Cauchy sequence in that subspace converges to some limit point in any complete metric space, but now closedness makes that limit lies inside the subspace). Therefore, we let $\{u_n\}$ be a sequence in $\operatorname{Har}(U)$ such that $u_n \to u$ for some $u \in C(U, \mathbb{R})$. By Proposition 6.5.5 (b), we see $u_n \to u$ uniformly on all compact subsets of U.

Fix a ball $B = B(a, R)$ such that $\overline{B} \subset U$. Write, using what we derive just before proving Harnack's inequality,

$$u_n(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta - t) + r^2} \underbrace{u_n(a + Re^{it})}_{\substack{\longrightarrow u(a+Re^{it}) \text{ uni-} \\ \text{formly in } t \in [-\pi,\pi]}} \mathrm{d}t, \quad \text{when } r < R.$$

Thus, for $r < R$, letting $n \to \infty$ gives

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta - t) + r^2} u_n(a + Re^{it}) \mathrm{d}t.$$

But this implies u is harmonic: if u is $\mathbb{R}$-valued, it is the real part of an analytic function in B and is thus harmonic.

(b) The proof is not central to the course, but for completeness, we copy it from [3].

We may assume that $u_1 \geq 0$ (if not, consider $\{u_n - u_1\}$). Let $u(z) = \sup\{u_n(z) : n \geq 1\}$ for each z in U. So for each z in U one of two possibilities occurs: $u(z) = \infty$ or $u(z) \in \mathbb{R}$ and $u_n(z) \to u(z)$. Define

$$A = \{z \in U : u(z) = \infty\}$$
$$B = \{z \in U : u(z) < \infty\}$$

then $U = A \cup B$ and $A \cap B = \varnothing$. We will show that both A and B are open. If $a \in U$, let R be chosen such that $\overline{B}(a, R) \subset U$. By Harnack's inequality

$$\frac{R - |z - a|}{R + |z - a|} u_n(a) \leq u_n(z) \leq \frac{R + |z - a|}{R - |z - a|} u_n(a) \tag{7.13}$$

for all z in $B(a, R)$ and all $n \geq 1$. If $a \in A$ then $u_n(a) \to \infty$ so that the left half of (7.13) gives that $u_n(z) \to \infty$ for all z in $B(a, R)$. That is, $B(a, R) \subset A$ and so A is open. In a similar fashion, if $a \in B$ then the right half of (7.13) gives that $u(z) < \infty$ for $|z-a| < R$. That is B is open.

Since U is connected, either $A = U$ or $B = U$. Suppose $A = U$; that is $u \equiv \infty$. Again if $\overline{B}(a, R) \subset U$ and $0 < \rho < R$ then $M = (R-\rho)(R+\rho)^{-1} > 0$ and (7.13) gives that $Mu_n(a) \leq u_n(z)$ for $|z-a| \leq \rho$. Hence $u_n(z) \to \infty$ uniformly for z in $\overline{B}(a; \rho)$. In other words, we have shown that for each a in U there is a $\rho > 0$ such that $u_n(z) \to \infty$ uniformly for $|z-a| \leq \rho$. From this it is easy to deduce that $u_n(z) \to \infty$ uniformly for z in any compact set.

Now suppose $B = U$, or that $u(z) < \infty$ for all z in U. If $\rho < R$, then for $m \leq n$ Harnack's Inequality applied to the positive harmonic function $u_n - u_m$ implies there is a constant C depending only on ρ and R such that

$$0 \leq u_n(z) - u_m(z) \leq C\left[u_n(a) - u_m(a)\right]$$

for $|z-a| \leq \rho$. Thus, $\{u_n(z)\}$ is a uniformly Cauchy sequence on $\overline{B}(a; \rho)$. It follows that $\{u_n\}$ is a Cauchy sequence in $\operatorname{Har}(G)$ and so, by part (a), must converge to a harmonic function. Since $u_n(z) \to u(z)$, u is this harmonic function. □

We have seen that harmonic function $u : U \to \mathbb{R}$ on region U is infinitely differentiable, a propoerty shared with analytic functions as well. The next result is the analogue of the Cauchy integral formula.

Theorem 7.3.2 (Mean Value Property of harmonic function)**.** *If $u : U \to \mathbb{R}$ is a harmonic function and $\overline{B}(a, r)$ is a closed disk contained in U, then*

$$u(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta}) d\theta.$$

Proof. Let D be a disk such that $\overline{B}(a, r) \subset D \subset U$ and let f be an analytic function on D such that $u = \operatorname{Re} f$. [11] Corollary 8.18 (mean value property of analytic function) states that

$$f(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(a + re^{i\theta}\right) d\theta.$$

By taking the real part of each side of this equation we complete the proof. □

In order to study this property of harmonic functions we isolate it.

Definition 7.3.3. *A continuous function $u : G \to \mathbb{R}$ has the **Mean Value Property (MVP)** if $\forall z \in U$, $\exists r_n > 0$, $r_n \stackrel{n\to\infty}{\longrightarrow} 0$ such that*

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(z + r_n e^{i\theta}\right) d\theta.$$

Remark 7.3.4. *We note that this is weaker than the definition given in [3], where the above averaging effect holds for every circles inside U. Our definition assumes for each $z \in U$, there is some sequence of circles $B(z, r_n)$ where the averaging effect holds. Showing the converse of theorem 7.3.2 only needs our weaker definition. [3] uses the stronger one to prove it in theorem 2.11, while isolating parts of the proof as maximum principle for harmonic function.*

Theorem 7.3.5 (MVP implies harmonicity)**.** *If a continuous function u has MVP in U, then u is harmonic in U.*

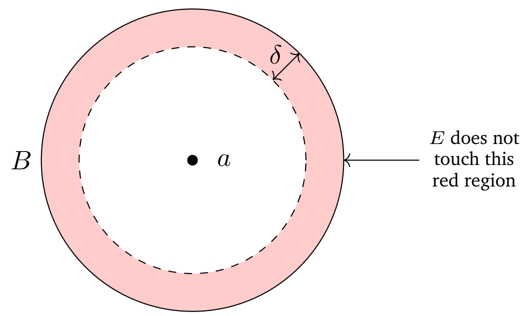
Proof. WLOG, we assume u is real-valued. Fix $B = B(a, R)$ with $\overline{B} \subset U$. Since $u \in C(\partial B)$, we let $h \in C(\overline{U})$ be the unique solution to the Dirichlet problem

$$\begin{cases} \Delta h = 0, & \text{on } B \\ h = u, & \text{on } \partial B \end{cases}.$$

Let $v = u - h$. We will show that $v \equiv 0$ in B, which ends the proof due to uniqueness of solution of (Dir). Since h is harmonic, it by theorem 7.3.2 has MVP. Thus, u has MVP by assumption and v has MVP by linearity of integral. Also, $v = 0$ on ∂B.

By compactness we let $m := \max\{v(z) : z \in \overline{B}\}$. Aiming for a contradiction, we assume $m > 0$. Notice that v is uniformly continuous on $\overline{B}$, and so $\exists \delta > 0$ such that $|v(w_1) - v(w_2)| < m/2$ whenever $w_1, w_2 \in \overline{B}$ with $|w_1 - w_2| < \delta$. It follows that if $w \in \partial B$ and $z \in \overline{B}$ such that $|z - w| < \delta$, we have $|v(z)| = |v(z) - \underbrace{v(w)}_{=0}| < m/2$.

This means the set $E := \{z \in \overline{B} : v(z) = m\} = v^{-1}(m)$ stays a positive distance away from ∂B. Define $f := \operatorname{dist}(\cdot, \partial B) : E \to \mathbb{R}$, which is continuous on compact set $E = \underbrace{v^{-1}(m)}_{\text{closed,bounded}}$ and thus attains its minimum at some $z_0 \in E$, i.e., $\operatorname{dist}(E, \partial B) = \operatorname{dist}(z_0, \partial B)$.



By definition of MVP, we let r be a small radius such that $\overline{B}(z_0, r) \subset B$ and the averaging formula holds for this r:

$$
\begin{aligned}
& v(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(z_0 + re^{it}) \mathrm{d}t \\
\implies & \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{(v(z_0) - v(z_0 + re^{it}))}_{m - v(z_0 + re^{it}) \geq 0} \mathrm{d}t = 0.
\end{aligned}
$$

By continuity we must have that the non-negative function $t \mapsto v(z_0) - v(z_0 + re^{it})$ is in fact zero for $\forall t \in [-\pi, \pi]$. But we arranged so that even at least half of $\partial B(z_0, r)$ does not intersect E, and so $v(z_0 + re^{it}) < m = v(z_0)$ for some (in fact, many) t. Contradiction.

Therefore, we must have $m \leq 0$, i.e., $v \leq 0$ in B. The same argument applies to $-v$, resulting in $-v \leq 0$ in B. Thus, $v = 0$ in B. $\square$

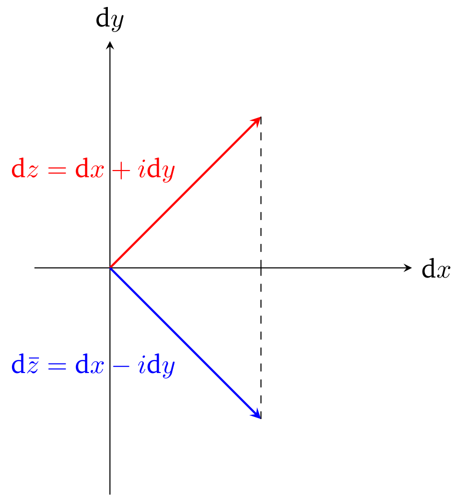
7.4 Harmonic and Holomorphic Functions Via Differential Forms

Recall that the **differential** of a function $u(z)$, considered as a function of $(x, y) \in \mathbb{R}^2$ (via $z = x + iy$), is a differential one-form given by

$$
\mathrm{d}u = \frac{\partial u}{\partial x} \mathrm{d}x + \frac{\partial u}{\partial y} \mathrm{d}y.
$$

We define the complex-valued one-forms

$$
\begin{cases} dz = dx + idy \\ d\overline{z} = dx - idy \end{cases}
$$



Then by this change of basis, we can write $\mathrm{d}u = F\mathrm{d}z + G\mathrm{d}\bar{z}$ with

$$\begin{aligned} F &= \frac{\partial u}{\partial z} := \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \\ G &= \frac{\partial u}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \end{aligned}$$

They are called **Wirtinger derivatives**.

Let $f(z) = u(z) + iv(z)$ be a complex-valued, real differentiable function on a planar domain Ω. Then, Notice that

$$\begin{aligned} 0 =& \frac{\partial}{\partial \bar{z}} f(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)[u(x,y) + iv(x,y)] \\ =& \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \text{ on } \Omega. \end{aligned}$$

if and only if

$$\frac{\partial}{\partial x} u = \frac{\partial}{\partial y} v \quad \text{and} \quad \frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} u.$$

But these are just the Cauchy-Riemann equations. We conclude that

Theorem 7.4.1. *Let Ω be a domain. Then,*

$$\frac{\partial}{\partial \bar{z}} f = 0 \text{ on } \Omega \quad \textit{iff} \quad f \text{ is holomorphic on } \Omega.$$

Further notice that

$$\begin{aligned} \frac{\partial}{\partial z} z = 1; \quad \frac{\partial}{\partial z} \bar{z} = 0 \\ \frac{\partial}{\partial \bar{z}} z = 0; \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1. \end{aligned}$$

Thus $\partial/\partial z$ and $\partial/\partial \bar{z}$ naturally fit into complex function theory. Besides, when $\partial_{\bar{z}} f(z) = 0\, \forall z \in \Omega$, i.e., C.R.eq is satisfied, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, we have

$$\partial_z f = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

which is the expression in Theorem 2.2.5. Alternatively, one observes that if f is holomorphic then

$$f'(z) = \lim_{\mathbb{C}\ni h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{\mathbb{R}\ni s\to 0} \frac{f(z+s)-f(z)}{s} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}.$$

To translate to Dirichlet problem, observe that the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

may now be written as

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}.$$

The following properties of these operators will be used routinely. The complex conjugate of $\partial u/\partial z$ is $\partial \bar{u}/\partial \bar{z}$ and the complex conjugate of $\partial u/\partial \bar{z}$ is $\partial \bar{u}/\partial z$.

Proposition 7.4.2. *If f and g are continuously differentiable functions, and if $f \circ g$ is well defined on some open set $U \subseteq \mathbb{C}$, then we have*

$$\frac{\partial}{\partial z}(f\circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\frac{\partial \bar{g}}{\partial z}(z)$$

and

$$\frac{\partial}{\partial \bar{z}}(f\circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial \bar{z}}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\frac{\partial \bar{g}}{\partial \bar{z}}(z).$$

Proof. We will sketch the proof of the first identity and leave the second as an exercise. We have

$$\frac{\partial}{\partial z}(f\circ g) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f\circ g).$$

We write $g(z) = \alpha(z) + i\beta(z)$, with α and β real-valued functions, and apply the usual calculus chain rule for $\partial/\partial x$ and $\partial/\partial y$. We obtain that the last line equals

$$\frac{1}{2}\left(\frac{\partial f}{\partial x}\frac{\partial \alpha}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \beta}{\partial x} - i\frac{\partial f}{\partial x}\frac{\partial \alpha}{\partial y} - i\frac{\partial f}{\partial y}\frac{\partial \beta}{\partial y}\right).$$

Now, with the aid of the identities

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right),$$

we may reduce the expression $(*)$ (after some tedious calculations) to the desired formula. □

Corollary 7.4.3. *If either f or g is holomorphic, then*

$$\frac{\partial}{\partial z}(f\circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z).$$

Proof. Exercise. □

Corollary 7.4.4. *If f is harmonic and g is holomorphic, then $f \circ g$ is harmonic.*

Proof. Exercise. □

Let $\zeta = x + iy$ be the complex variable and let $u, v \in C^1(\Omega)$, where $\Omega \subseteq \mathbb{C}$ is a domain with piecewise C^1 boundary $\partial\Omega$. Consider the complex-valued 1-form

$$\omega := u\, d\zeta + v\, d\bar{\zeta}.$$

Using the Wirtinger decomposition

$$du = u_\zeta\, d\zeta + u_{\bar{\zeta}}\, d\bar{\zeta}, \qquad dv = v_\zeta\, d\zeta + v_{\bar{\zeta}}\, d\bar{\zeta},$$

we compute

$$\begin{aligned} d\omega &= d(u\, d\zeta) + d(v\, d\bar{\zeta}) \\ &= du \wedge d\zeta + dv \wedge d\bar{\zeta} \\ &= (u_\zeta\, d\zeta + u_{\bar{\zeta}}\, d\bar{\zeta}) \wedge d\zeta + (v_\zeta\, d\zeta + v_{\bar{\zeta}}\, d\bar{\zeta}) \wedge d\bar{\zeta} \\ &= u_{\bar{\zeta}}\, d\bar{\zeta} \wedge d\zeta + v_\zeta\, d\zeta \wedge d\bar{\zeta} \\ &= (v_\zeta - u_{\bar{\zeta}})\, d\zeta \wedge d\bar{\zeta}. \end{aligned}$$

Therefore, by Stokes' theorem,

$$\int_{\partial\Omega} \left(u\, d\zeta + v\, d\bar{\zeta}\right) = \iint_\Omega (v_\zeta - u_{\bar{\zeta}})\, d\zeta \wedge d\bar{\zeta}. \tag{7.14}$$

In particular, setting $v \equiv 0$ or $u \equiv 0$ gives the two **complex Green's identities**:

$$\int_{\partial\Omega} u\, d\zeta = \iint_\Omega u_{\bar{\zeta}}\, d\zeta \wedge d\bar{\zeta}, \tag{7.15}$$

$$\int_{\partial\Omega} u\, d\bar{\zeta} = -\iint_\Omega u_\zeta\, d\zeta \wedge d\bar{\zeta}. \tag{7.16}$$

Finally, note that

$$d\zeta \wedge d\bar{\zeta} = (dx + idy) \wedge (dx - idy) = -2i\, dx \wedge dy,$$

so these double integrals are (up to the constant factor $-2i$) ordinary area integrals. This is the differential-forms bridge between $\bar{\partial}$-calculations and the classical integral identities used to study harmonic and holomorphic functions.

7.5 Reflection Principles

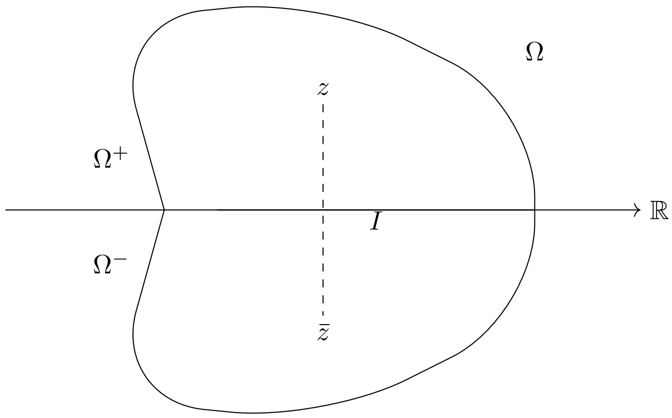
We will study a way to extend analyticity: the Schwarz reflection principle. The proof consists of two parts. First we define the extension, and then check that the resulting function is still holomorphic. We begin with the second point.

Let Ω be an open subset of $\mathbb{C}$ that is symmetric with respect to the real line, that is

$$z \in \Omega \quad \text{if and only if} \quad \bar{z} \in \Omega.$$

Let Ω^+denote the part of Ω that lies in the upper half-plane and Ω^- that part that lies in the lower half-plane. Also, let $I = \Omega \cap \mathbb{R}$ so that I denotes the interior of that part of the boundary of $\Omega^+$and Ω^-that lies on the real axis. Then we have

$$\Omega^+ \cup I \cup \Omega^- = \Omega$$



and the only interesting case occurs, of course, when I is non-empty.

Theorem 7.5.1. *Let u be harmonic in Ω^+ with $\lim_{n\to\infty} u(z_n) = 0$ for all sequences $\{z_n\}$ in Ω^+ converging to a point of I. Then the extended function*

$$\widetilde{u}(z) = \begin{cases} u(z) & \text{if } z \in \Omega^+ \\ 0 & \text{if } z \in I \\ -u(\bar{z}) & \text{if } z \in \Omega^- \end{cases}$$

is harmonic in Ω.

Proof. $\widetilde{u}$ is continuous by pasting lemma. We then observe that $\widetilde{u}$ is harmonic on Ω^+ and Ω^-. For $(x, y) \in \Omega^-$, we have $(\Delta u)(x, -y) = 0$ and thus

$$\Delta(-u(x, -y)) = -\Delta(u(x, -y)) = -(u_{xx}(x, -y) + (-(-u_{yy}(x, -y)))) = -(\Delta u)(x, -y) = 0.$$

By theorem 7.3.2, we see $\widetilde{u}$ is harmonic on $\Omega^+ \cup \Omega^-$ and thus has MVP. We note that $\widetilde{u}$ may not be able to be twice continuously differetiable on I even if the laplace equation is formally satisfied.

MVP also holds in points $t \in I$ as $\widetilde{u}(z) = -\widetilde{u}(\bar{z}) \Rightarrow \widetilde{u}(t + re^{i\theta}) = -\widetilde{u}(t + re^{i(-\theta)})$ guarantees the integral of $\widetilde{u}$ over small circles centered at t vanish. To be more precise,

$$\int_0^\pi \widetilde{u}(t + re^{i\theta})\mathrm{d}\theta = -\int_0^\pi \widetilde{u}(t + re^{i(-\theta)})\mathrm{d}\theta = \int_0^{-\pi} \widetilde{u}(t + re^{iu})\mathrm{d}u = -\int_{-\pi}^0 \widetilde{u}(t + re^{iu})\mathrm{d}u.$$

(Note that $\widetilde{u}$ is zero on I but we don't even need this as $I \cap \partial B(t, r)$ has measure zero.) Then we invoke theorem 7.3.5 to conclude that $\widetilde{u}$ is harmonic. ☐

Theorem 7.5.2 (Symmetry principle)**.** *If $f^+$and $f^-$are holomorphic functions in $\Omega^+$and Ω^-respectively, that extend continuously to I and*

$$f^+(x) = f^-(x) \quad \text{for all } x \in I,$$

then the function f defined on Ω by

$$f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^+(z) = f^-(z) & \text{if } z \in I \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}$$

is holomorphic on all of Ω.

Proof. One notes first that f is continuous throughout Ω due to pasting lemma. The only difficulty is to prove that f is holomorphic at points of I. Suppose D is a disc centered at a point on I and entirely contained in Ω. We prove that f is holomorphic in D by Morera's theorem. Suppose T is a triangle in D (for a triangle we mean both the boundary and its interior). If T does not intersect I, then

$$\int_{\partial T} f(z) dz = 0$$

since f is holomorphic in the upper and lower half-discs.

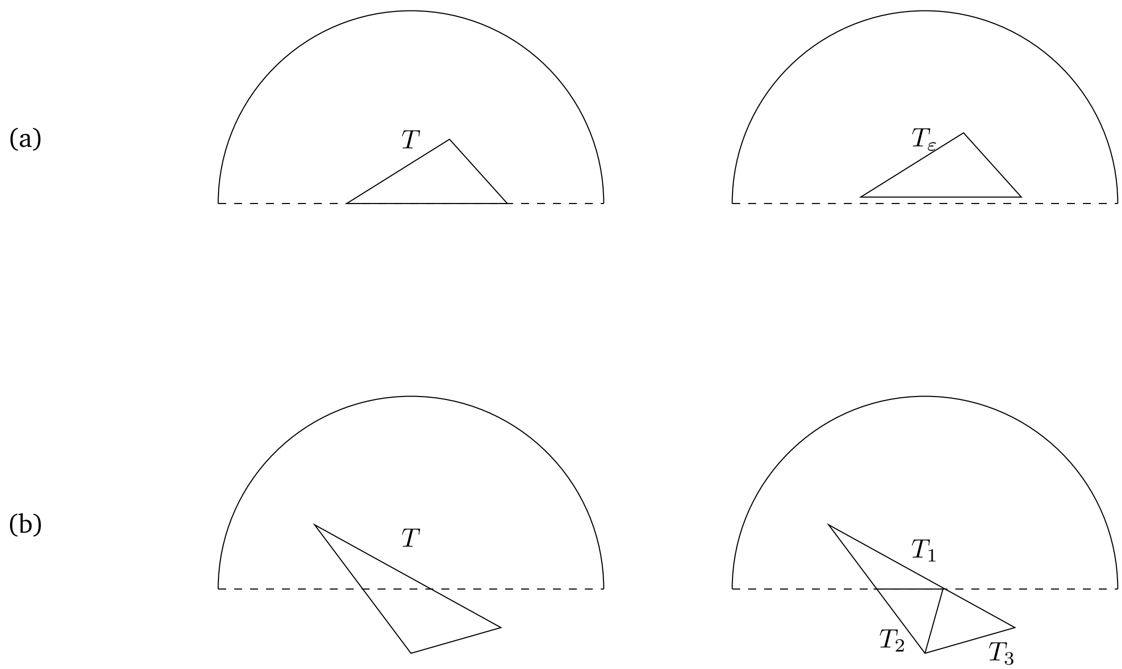


Figure 7.1: (a) Raising a vertex; (b) splitting a triangle

Suppose now that one side or vertex of T is contained in I, and the rest of T is in, say, the upper half-disc. If T_ε is the triangle obtained from T by slightly raising the edge or vertex which lies on I, we have $\int_{\partial T_\varepsilon} f = 0$ since T_ε is entirely contained in the upper half-disc (an illustration of the case when an edge lies on I is given in Figure ?? (a)). Intuitively, we then let $\varepsilon \to 0$, and by continuity we conclude that

$$\int_{\partial T} f(z) \mathrm{d}z = 0.$$

A complete argument of showing $\int_{\partial T} f dz = 0$ where T touches I is shown below.

Since f is continuous on $\Omega^+ \cup I$, f is uniformly continuous on T. So if $\varepsilon > 0$ there is a $\delta > 0$ such that when z and $z' \in T$ and $|z - z'| < \delta$ then $|f(z) - f(z')| < \varepsilon$. We label the vertices of T as a, b, c as in Figure ??. Now choose α and β on the line segments $[c, a]$ and $[b, c]$ respectively, so that $|\alpha - a| < \delta$ and $|\beta - b| < \delta$. Let T_1 be the triangle bounded by $\partial T_1 = [\alpha, \beta, c, \alpha]$ and Q be the trapezoid bounded by $\partial Q = [a, b, \beta, \alpha, a]$. Then $\int_{\partial T} f = \int_{\partial T_1} f + \int_{\partial Q} f$, but T_1 is contained in Ω^+ and f is analytic there; hence

$$\int_{\partial T} f \mathrm{d}z = \int_{\partial Q} f \mathrm{d}z. \tag{7.17}$$

If $0 \leq t \leq 1$ then

$$|[t\beta + (1-t)\alpha] - [tb + (1-t)a]| < \delta$$

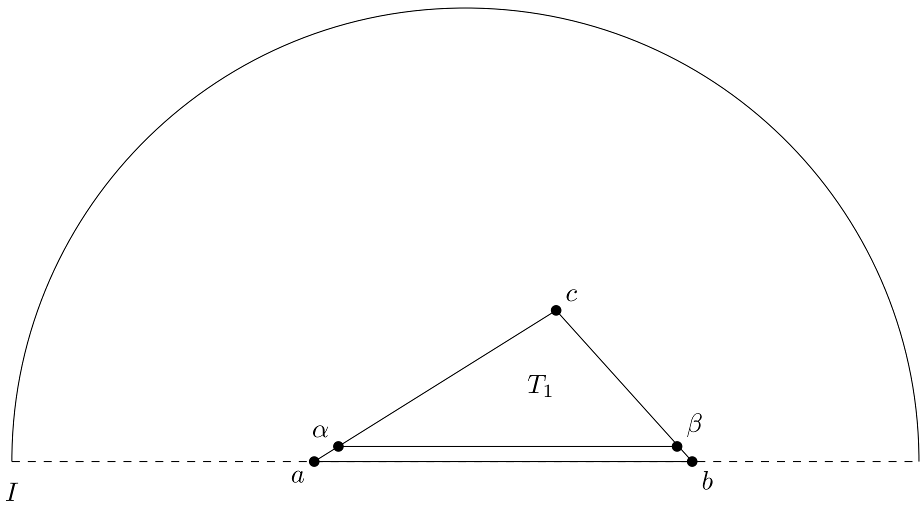


Figure 7.2: Triangle T

so that

$$|f(t\beta + (1-t)\alpha) - f(tb + (1-t)a)| < \varepsilon.$$

If $M = \max\{|f(z)| : z \in T\}$ and $\ell =$ the perimeter of T then

$$\begin{aligned}
\left| \int_{[a,b]} f + \int_{[\beta,\alpha]} f \right| &= \left| (b-a) \int_0^1 f(tb + (1-t)a)dt - (\beta - \alpha) \int_0^1 f(t\beta + (1-t)\alpha)dt \right| \\
&\leq |b-a| \left| \int_0^1 [f(tb + (1-t)a) - f(t\beta + (1-t)\alpha)]\mathrm{d}t \right| \\
&\quad + |(b-a) - (\beta - \alpha)| \left| \int_0^1 f(t\beta + (1-t)\alpha)dt \right| \\
&\leq \varepsilon|b-a| + M|(b-\beta) + (\alpha - a)| \\
&\leq \varepsilon\ell + 2M\delta
\end{aligned}$$

Also

$$\left| \int_{[\alpha,a]} f \right| \leq M|a - \alpha| \leq M\delta$$

and

$$\left| \int_{[b,\beta]} f \right| \leq M\delta$$

Combining these last two inequalities with (7.17) gives that

$$\left| \int_{\partial T} f \right| \leq \varepsilon\ell + 4M\delta$$

Since it is possible to choose $\delta < \varepsilon$ and since ε is arbitrary, it follows that $\int_{\partial T} f = 0$.

If the interior of T intersects I, we can reduce the situation to the previous one by writing T as the union of triangles each of which has an edge or vertex on I as shown in Figure ??(b). By Morera's theorem we conclude that f is holomorphic in D, as was to be shown. $\square$

Theorem 7.5.3 (Schwarz reflection principle; version I). *Suppose that f is a holomorphic function in Ω^+that extends continuously to I and such that f is real-valued on I. Then*

$$F(z)=\begin{cases} f(z) & \text{if } z\in\Omega^+\cup I \\ \overline{f(\bar{z})} & \text{if } z\in\Omega^- \end{cases}$$

is analytic on Ω.

Proof. To prove that F is holomorphic in Ω^-we note that if $z, z_0 \in \Omega^-$, then $\bar{z}, \overline{z_0} \in \Omega^+$and hence, the power series expansion of f near $\overline{z_0}$ gives

$$f(\bar{z})=\sum a_n\left(\bar{z}-\overline{z_0}\right)^n .$$

As a consequence we see that

$$F(z)=\sum \overline{a_n}\left(z-z_0\right)^n$$

and F is holomorphic in Ω^-. Since f is real valued on I we have $\overline{f(x)}=f(x)$ whenever $x\in I$ and hence F extends continuously up to I. The proof is complete once we invoke the symmetry principle. □

There is a weaker version of this, where $f=u+iv\in H(\Omega^+)$ and we only assume $v(z_n)\to 0$ whenever $z_n\in\Omega^+$ is a sequence coverging to a point on I without assuming continuity of $u=\operatorname{Re}(f)$ on I. This form is harder to prove: we do not right away even know how to assign real values to f on I. We will need harmonic function theory.

Theorem 7.5.4 (Schwarz reflection principle; version II). *Suppose $f=u+iv\in H(\Omega^+)$ with $\lim_{n\to\infty} v(z_n)$ for all $\{z_n\}\in\Omega^+$ converging to a point in I. Then there exists $F\in H(\Omega)$ such that $F(z)=f(z)$ in Ω^+ and $F(\bar{z})=\overline{F(\bar{z})}\ \forall z\in\Omega$.*

Proof. Extend v into a harmonic function in Ω by theorem 7.5.1 snd still denote it by v, so $v(z)=0$ for $z\in I$ and $v(z)=-v(\bar{z})$ for $z\in\Omega^-$.

For each $t\in I$ consider a ball B_t that is centered at t and satisfies $B_t\cap H^+\subseteq\Omega^+$, $\overline{B_t}\subseteq\Omega$. As v is harmonic and thus locally real part of an analytic function, then there is $g_t\in H(B_t)$ such that $g_t=v_t+iu_t$ for some u_t and $v=v_t$ on B_t. Define $f_t=ig_t=-u_t+iv_t$. Then $f_t\in H(B_t)$ by [11] 4.39 and 4.47. Then $v=\operatorname{Im}(f_t)$. Notice that if there is some other $\widetilde{f_t}\in H(B_t)$ for which $v=\operatorname{Im}(\widetilde{f_t})$, then $\operatorname{Im}(f_t-\widetilde{f_t})=0$ implies that $f_t-\widetilde{f_t}\equiv c\in\mathbb{R}$ due to [11] Corollary 4.50. Thus, f_t is uniquely determined up to a real additive constant.

We shall strategically choose this constant. Fix $z_0\in B_t\cap H^+\subset\Omega^+$. Then $\operatorname{Im}(f_t(z_0))=v_t(z_0)=v(z_0)=\operatorname{Im}(f(z_0))$ and we adjust f_t with a real constant such that $\operatorname{Re}(f_t(z_0))=\operatorname{Re}(f(z_0))$, i.e., $f_t(z_0)=f(z_0)$. This is z_0-specific. However, $\operatorname{Im}(f-f_t)=0$ in the region $B_t\cap H^+$ implies that $f-f_t$ is a real constant in $B_t\cap H^+$ by above argument again. This constant must be zero as $f(z_0)-f_t(z_0)=0$. Therefore, this actaully guarantees $f_t(z)=f(z)$ for every $z\in B_t\cap H^+$. In what follows, we assume f_t is adjusted by the procedure just mentioned.

Now, f_t gives a natural way to define our extension on B_t, as long as we have $f_t(z)=f_s(z)$ for $z\in B_t\cap B_s$ for $t,s\in I$. We show it now: Define function f_t-f_s on $B_t\cap B_s$ (assumed to be nonempty). Then it have zeros on the whole $B_t\cap B_s\cap H^+$ because $f_t=f=f_s\ \forall z\in B_t\cap B_s\cap H^+$. Then connectedness of $B_t\cap B_s$ plus theorem 5.1.1 show that $f_t-f_s\equiv 0$ on $B_t\cap B_s$.

We claim that $f_t(\bar{z})=\overline{f_t(z)}$ for $z\in B_t$. Indeed, write

$$f_t(z)=\sum_{n=0}^{\infty} c_n(z-t)^n,\quad z\in B_t,$$

where $c_n = f_t^{(n)}(t)/n!$. We know $f_t^{(n)}(t) \in \mathbb{R}$, since $\operatorname{Im}(f_t) = 0$ on $I \cap B_t$. For instance,

$$f_t'(t) = \lim_{\theta \to t; \theta \in I} \frac{f(\theta) - f(t)}{\theta - t} \in \mathbb{R}.$$

Thus $c_n \in \mathbb{R}$, and so

$$f_t(\bar{z}) = \sum_{n=0}^{\infty} c_n (\bar{z} - t)^n = \overline{\sum_{n=0}^{\infty} c_n (z - t)^n} = \overline{f(z)}, \quad z \in B_t.$$

We are now ready to define our extension F via

$$F(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \\ f_t(z) & \text{if } z \in B_t \\ \overline{f(\bar{z})} & \text{if } z \in \Omega^- \end{cases}$$

This is now a well-defined extension of f with the property that $F(\bar{z}) = \overline{F(z)}$. We also know that $F \in H(\Omega)$ by the same proof of $F \in H(\Omega^-)$ in verison I of Schwarz reflection principle. $\square$

Chapter 8

Entire Functions

In this chapter, we will study functions that are holomorphic in the whole complex plane; these are called **entire functions**. Our presentation will be organized around the following three questions:

1. What is the range of entire functions? We will do a review of Liouvelle's theorem and talk about even stronger results. For more on the range of analytic functions, see [3] Chapter XII.

2. Where can such functions vanish? We shall see that the obvious necessary condition is also sufficient: if $\{z_n\}$ is any sequence of complex numbers having no limit point in $\mathbb{C}$, then there exists an entire function vanishing exactly at the points of this sequence. The construction of the desired function is inspired by Euler's product formula for $\sin \pi z$ (the prototypical case when $\{z_n\}$ is $\mathbb{Z}$), but requires an additional refinement: the Weierstrass canonical factors.

3. How do these functions grow at infinity? Here, matters are controlled by an important principle: the larger a function is, the more zeros it can have. This principle already manifests itself in the simple case of polynomials. By the fundamental theorem of algebra, the number of zeros of a polynomial P of degree d is precisely d, which is also the exponent in the order of (polynomial) growth of P, namely $\sup_{|z|=R} |P(z)| \approx R^d$ as $R \to \infty$. A precise version of this general principle is contained in Jensen's formula. This formula, central to much of the theory developed in this chapter, exhibits a deep connection between the number of zeros of a function in a disc and the (logarithmic) average of the function over the circle.

4. To what extent are these functions determined by their zeros? It turns out that if an entire function has a finite (exponential) order of growth, then it can be specified by its zeros up to multiplication by a simple factor. The precise version of this assertion is the Hadamard factorization theorem.

Before we start, here is a table of pronunciations of some symbols we may encounter.

- sinh - Sinch (sɪntʃ) (Others say "shine" (ʃaɪn) according to Olivier Bégassat et al.)
- cosh - Kosh (kɒʃ or koʊʃ)
- tanh - Tanch (tæntʃ) (Others say "tsan" (tsæn) or "tank" (teɪnk) according to André Nicolas)
- coth - Koth (kɒθ) according to J. M.
- csch - Kisch (kɪʃ) according to J. M.
- sech - Seech (siːtʃ)

Figure 8.1: Provided by Argon.

8.1 More on Liouville's Theorem

Liouville's theorem states that a bounded entire function is constant.

Before showing this we mention some preliminary results.

- [11] Corollary 4.49. Assume that $U \subset \mathbb{R}^2$ is open and connected, and $f : U \to \mathbb{C}$ is an analytic function such that $f'(z) = 0$ for all $z \in U$. Then f is constant on U.
- [11] Corollary 4.50. Let $U \subset \mathbb{C}$ be open and connected, and assume that $f : U \to \mathbb{C}$ is analytic. Assume that one of the following three functions is constant on U :

  $$u = \operatorname{Re} f, \quad v = \operatorname{Im} f, \quad \text{or} \quad |f|.$$

  Then f is constant on U. The same conclusion is also true if $f : U \to \mathbb{C}\backslash\{0\}$, and $\operatorname{Arg}(f)$ is constant on U.

Now we recall [11] Theorem 8.5.

Theorem 8.1.1 (Cauchy's integral formula for derivatives)**.** *Let $U \subset \mathbb{C}$ be a convex open set, and let $f : U \to \mathbb{C}$ be analytic. Let $\gamma : [a, b] \to U$ be a closed piecewise C^1-path. Then,*

$$f^{(n)}(z) \cdot n_\gamma(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad z \in U\backslash\gamma^*, n \geqslant 0$$

By which we obtain [11] Corollary 8.8.

Corollary 8.1.2 (Cauchy's estimates)**.** *Let $D = D(z, r) \subset \mathbb{C}$ be a disc, and let $f : D \to \mathbb{C}$ be analytic. Then,*

$$\left|f^{(n)}(w)\right| \leqslant \frac{n! \cdot r \cdot \|f\|_{L^\infty(\partial D)}}{(r - |w - z|)^{n+1}}, \quad w \in D, n \geqslant 0.$$

In particular, $\left|f^{(n)}(z)\right| \leqslant n!\|f\|_{L^\infty(\partial D)}/r^n$ by letting $w = z$. Note that $\|f\|_{L^\infty(\partial D)} = \sup\{|f(z)| : z \in \partial D\}$.

[11] uses this to show Liouville's theorem (Corollary 8.10).

Corollary 8.1.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and bounded. Then f is constant.*

Proof. Fix $z \in \mathbb{C}$, and apply Cauchy's estimates in a disc $D(z, r) \subset \mathbb{C}$:

$$|f'(z)| \leqslant \frac{\|f\|_{L^\infty(\mathbb{C})}}{r}, \quad r > 0.$$

Letting $r \to \infty$ (as f's domain is $\mathbb{C}$) shows that $f'(z) = 0$, and therefore $f' \equiv 0$. Since $\mathbb{C}$ is connected, f is constant by [11] Corollary 4.49. $\square$

Remark 8.1.4. *Geometrically, Liouville's theorem is saying that the values of a non-constant entire function cannot be entirely contained within a single circle. In fact, the values of a non-constant entire function cannot be entirely contained outside a single circle either.*

Liouville's theorem is deep and surprising, but an even stronger result is true: if $f : \mathbb{C} \to \mathbb{C}$ is analytic, then either f is constant, or then f takes all the values in $\mathbb{C}$, except for possibly one. In other words, if an entire function omits two values, then it is constant. This result is known as ***Picard's little theorem****. The example $f(z) = e^z$ shows that Picard's theorem is sharp, since $e^z \neq 0$ for all $z \in \mathbb{C}$. There is also* ***Picard's great theorem****: if a holomorphic function f on open set $U = B'(a, r)$ has an essential singularity at a, then for any $0 < s < r$ we have $\{f(z) : 0 < |z - a| < s\}$ contains all complex numbers except possibly one value.*

It should be clear that the great theorem is a generalization of the little theorem, since an entire function that is not a polynomial has an essential singularity at infinity. If $f(z)$ is a non-constant polynomial and α is a complex *number, then the equation $f(z) = \alpha$ has a solution by fundamental theorem of algebra. Since α is arbitrary, $f(z)$ can take all complex values.*

We don't prove these results. One can consult [3]. We mention that for functions of finite growth (see the section "Functions of Finite Order") it is proved in Math5022 HW6, recorded below.

Exercise 8.1.5. *Suppose an entire function f has finite growth. Show that if f omits two values, then f is a constant.*

Solution. We prove its contrapositive, i.e., that a nonconstant entire function f of finite growth can at most omit one value in $\mathbb{C}$. That is, we show that if such a function f omits one value a, then it has range $\mathbb{C} \setminus \{a\}$, i.e., there won't be another value $b \neq a \in \mathbb{C}$ that is also omitted.

We define $g(z) = f(z) - a$ and observe that g is also nonconstant, entire, and of finite growth. g has no zero, so Hadamard's factorization theorem implies that g is of the form $e^{p(z)}$ for some polynomial $p(z)$. Now, g being nonconstant implies that $p(z)$ has degree at least 1. In particular,

$$p(z) = a_k z^k + \cdots + a_1 z + a_0, \quad k \geq 1, a_k \neq 0.$$

Now, for any $\omega \in \mathbb{C}$, we by Fundamental Theorem of Algebra see that there is a solution z_0 in $\mathbb{C}$ such that

$$\omega = p(z_0).$$

Hence, $p(z)$ obtains all values in $\mathbb{C}$, and $g(z) = f(z) - a = e^{p(z)}$ then obtains all values in $\mathbb{C} \setminus \{0\}$. Therefore, $f(z)$ obtains all values in $\mathbb{C} \setminus \{a\}$. ■

We give another proof of the Liouvelle's theorem.

second proof. Let a, b be any two points in $\mathbb{C}$, and within a circle of sufficiently large radius R, we have

$$\begin{aligned} f(a) - f(b) &= \frac{1}{2\pi i} \int_{|z|=R} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) f(z) dz \\ &= \frac{a-b}{2\pi i} \int_{|z|=R} \frac{f(z) dz}{(z-a)(z-b)} \end{aligned}$$

Let the upper bound of f be M, we can estimate the above expression as

$$\begin{aligned} |f(a) - f(b)| &\leq \frac{|a-b| M}{2\pi} \int_{|z|=R} \frac{|dz|}{|z-a||z-b|} \\ &\leq \frac{MR|a-b|}{(R-|a|)(R-|b|)} \end{aligned}$$

As $R \to +\infty$, the above expression tends to zero, thus f is a constant function. □

Some more results are given.

Proposition 8.1.6. *If $f(z)$ is an entire function, and $\operatorname{Re}(f(z)) < M$ for some M, then $f(z)$ is a constant.*

Proof. This observation is trivial by Picard's little theorem. Since we didn't show that, we use Liouvelle's theorem instead. Let $F(z) = e^{f(z)}$. Then

$$|F(z)| = e^{\operatorname{Re}(f(z))} < e^M \implies \text{ entire function } F(z) \text{ is bounded}$$

Liouvelle's theorem then concludes. In the geometrical sense as Remark 8.1.4, values of the entire function $e^{f(z)}$ is encircled as the half-plane $\{z : \operatorname{Re}(z) < M\}$ is mapped to the disc $D(0, e^M)$ by the exponential function. □

Proposition 8.1.7. *Let $f(z)$ be an entire function. Suppose*

$$\lim_{z\to\infty}\frac{f(z)}{z^n}=0.$$

Then $f(z)$ is a polynomial of at most degree $n-1$.

Proof. By [11] Corollary 4.49, we only need to show $\forall z\in\mathbb{C}$, $f^{(n)}(z)=0$. Since

$$\lim_{z\to\infty}\frac{f(z)}{z^n}=0,$$

We know, $\forall\varepsilon>0$, $\exists R>0$ such that

$$|z|>R\implies|f(z)|<\varepsilon|z|^n.$$

Fix $z_0\in\mathbb{C}$ and let $C:|z-z_0|=r$, $C_1:|z|=R$ be contained inside C. Then $|z_0|<r$ and $|\zeta|>R$ for $\zeta\in C$. Thus,

$$|f(\zeta)|<\varepsilon|\zeta|^n\le\varepsilon(|z_0|+r)^n.$$

Cauchy's estimates gives

$$|f(^{(n)}z)|\le\frac{n!}{r^n}\varepsilon(|z_0|+r)^n=n!\varepsilon\left(1+\frac{|z|}{r}\right)^n\le n!2^n\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, $f^{(n)}(z)=0$. Then note that the disk in theorem 4.3.9 can be chosen to be $\mathbb{C}$ and the coefficients $c_n=f^{(n)}/n!$ after n are all zero. ☐

Proposition 8.1.8. *If the function $f(z)$ is entire, and $\lim_{r\to+\infty}\frac{\max_{|z|=r}|f(z)|}{r^n}<M$, then $f(z)$ is at most a polynomial of degree n.*

Proof. The Taylor series of $f(z)$ at any $a\in\mathbb{C}$ is $\sum_{n=0}^{\infty}c_n(z-a)^n$ where $c_n=f^{(n)}(a)/n!$. Then choose a circle $B(a,r)$ and by Cauchy's estimates,

$$|c_n|=\left|\frac{f^{(n)}(a)}{n!}\right|\le\frac{\|f\|_{L^\infty(\partial B(a,r))}}{r^n}=\frac{\max_{|z-a|=r}|f(z)|}{r^n}$$

Letting $a=0$ and $r\to\infty$ to get $|c_n|<M$. Then $|c_{n+k}|<\lim_{r\to\infty}\frac{\|f\|_\infty}{r^{n+k}}=0$ for any $k=1,2,\cdots$. This shows f, which agrees with the series that vanishes after n-th term on the disk whose radius goes to infinity, is at most a polynomial of degree n on $\mathbb{C}$. ☐

Exercise 8.1.9. *If $f(z)$ is entire and does not take values on a simple arc, then $f(z)$ is constant. [Hint: Riemann Mapping Theorem.]*

8.2 Jensen's Formula

In this section, we denote by D_R and C_R the open disc and circle of radius R centered at the origin. We shall also, in the rest of this chapter, exclude the trivial case of the function that vanishes identically.

Theorem 8.2.1 (Jensen's formula). *Let Ω be an open set that contains the closure of a disc D_R and suppose that f is holomorphic in Ω, $f(0)\neq0$, and f vanishes nowhere on the circle C_R. If $z_1,\dots,z_N$ denote the zeros of f inside the disc (counted with multiplicities), then*

$$\log|f(0)|=\sum_{k=1}^{N}\log\left(\frac{|z_k|}{R}\right)+\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(Re^{i\theta}\right)\right|d\theta. \tag{8.1}$$

Proof. The proof of the theorem consists of several steps.

Step 1. First, we observe that if f_1 and f_2 are two functions satisfying the hypotheses and the conclusion of the theorem, then the product $f_1 f_2$ also satisfies the hypothesis of the theorem and formula (8.1). This observation is a simple consequence of the fact that $\log xy = \log x + \log y$ whenever x and y are positive numbers, and that the set of zeros of $f_1 f_2$ is the union of the sets of zeros of f_1 and f_2.

Step 2. The function

$$g(z) = \frac{f(z)}{(z - z_1) \cdots (z - z_N)}$$

initially defined on $\Omega - \{z_1, \ldots, z_N\}$, is bounded near each z_j. Therefore each z_j is a removable singularity, and hence we can write

$$f(z) = (z - z_1) \cdots (z - z_N)\, g(z)$$

where g is holomorphic in Ω and nowhere vanishing in the closure of D_R in the sense that we can now define $g(z_j)$'s properly so that g is holomorphic. By Step 1, it suffices to prove Jensen's formula for functions like g that vanish nowhere, and for functions of the form $z - z_j$.

Step 3. We first prove (8.1) for a function g that vanishes nowhere in the closure of D_R. More precisely, we must establish the following identity:

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| g\left(Re^{i\theta}\right) \right| \mathrm{d}\theta$$

In a slightly larger disc, we can write $g(z) = e^{h(z)}$ where h is holomorphic in that disc. This is possible since discs are simply connected, and we can define $h = \log g$ (see Theorem 5.3.8 (e)). Now we observe that

$$|g(z)| = \left| e^{h(z)} \right| = \left| e^{\operatorname{Re}(h(z)) + i \operatorname{Im}(h(z))} \right| = e^{\operatorname{Re}(h(z))},$$

so that $\log |g(z)| = \operatorname{Re}(h(z))$. The taking real part of both sides of the MVP formula in [11] Corollary 8.18 immediately implies the desired formula for g.

Step 4. The last step is to prove the formula for functions of the form $f(z) = z - w$, where $w \in D_R$. That is, we must show that

$$\log |w| = \log \left(\frac{|w|}{R} \right) + \frac{1}{2\pi} \int_0^{2\pi} \log \left| Re^{i\theta} - w \right| \mathrm{d}\theta.$$

Since $\log(|w|/R) = \log |w| - \log R$ and $\log \left| Re^{i\theta} - w \right| = \log R + \log \left| e^{i\theta} - w/R \right|$, it suffices to prove that

$$\int_0^{2\pi} \log \left| e^{i\theta} - a \right| d\theta = 0, \quad \text{whenever } |a| < 1.$$

This in turn is equivalent (after the change of variables $\theta \mapsto -\theta$) to

$$\int_0^{2\pi} \log \left| 1 - ae^{i\theta} \right| d\theta = 0, \quad \text{whenever } |a| < 1.$$

To prove this, we use the function $F(z) = 1 - az$, which vanishes nowhere in the closure of the unit disc. As a consequence, there exists a holomorphic function G in a disc of radius greater than 1 such that $F(z) = e^{G(z)}$. Then $|F| = e^{\operatorname{Re}(G)}$, and therefore $\log |F| = \operatorname{Re}(G)$. Since $F(0) = 1$ we have $\log |F(0)| = 0$, and an application of the mean value property (again, by taking the real parts) to the harmonic function $\log |F(z)|$ concludes the proof of the theorem. □

From Jensen's formula we can derive an identity linking the growth of a holomorphic function with its number of zeros inside a disc. If f is a holomorphic function on the closure of a disc D_R, we denote by $\mathfrak{n}(r)$ (or $\mathfrak{n}_f(r)$ when it is necessary to keep track of the function in question) the number of zeros of f (counted with their multiplicities) inside the disc D_r, with $0<r<R$. A simple but useful observation is that $\mathfrak{n}(r)$ is a non-decreasing function of r.

We claim that if $f(0) \neq 0$, and f does not vanish on the circle C_R, then

$$\int_0^R \mathfrak{n}(r) \frac{\mathrm{d} r}{r}=\frac{1}{2 \pi} \int_0^{2 \pi} \log \left|f\left(R e^{i \theta}\right)\right| d \theta-\log |f(0)| . \tag{8.2}$$

This formula is immediate from Jensen's equality and the next lemma.

Lemma 8.2.2. *If $z_1, \ldots, z_N$ are the zeros of f inside the disc D_R, then*

$$\int_0^R \mathfrak{n}(r) \frac{\mathrm{d} r}{r}=\sum_{k=1}^N \log \left|\frac{R}{z_k}\right| .$$

Proof. First we have

$$\sum_{k=1}^N \log \left|\frac{R}{z_k}\right|=\sum_{k=1}^N \int_{\left|z_k\right|}^R \frac{\mathrm{d} r}{r} .$$

If we define the characteristic function

$$\eta_k(r)= \begin{cases}1 & \text { if } r>\left|z_k\right| \\ 0 & \text { if } r \leq\left|z_k\right|\end{cases}$$

then $\sum_{k=1}^N \eta_k(r)=\mathfrak{n}(r)$, and the lemma is proved using

$$\sum_{k=1}^N \int_{\left|z_k\right|}^R \frac{\mathrm{d} r}{r}=\sum_{k=1}^N \int_0^R \eta_k(r) \frac{\mathrm{d} r}{r}=\int_0^R\left(\sum_{k=1}^N \eta_k(r)\right) \frac{\mathrm{d} r}{r}=\int_0^R \mathfrak{n}(r) \frac{\mathrm{d} r}{r}$$

□

8.3 Functions of Finite Order

Let f be an entire function. If there exist a positive number ρ and constants $A, B>0$ such that

$$|f(z)| \leq A e^{B|z|^\rho} \quad \text { for all } z \in \mathbb{C},$$

then we say that f has an **order of growth** $\leq \rho$. We define the **order of growth of** f as

$$\rho_f=\inf \rho,$$

where the infimum is over all $\rho>0$ such that f has an order of growth $\leq \rho$. For example, the order of growth of the function e^{z^2} is 2 .

Theorem 8.3.1. *If f is an entire function that has an order of growth $\leq \rho$, then:*

(i) $\mathfrak{n}(r) \leq C r^\rho$ *for some $C>0$ and all sufficiently large r.*

(ii) *If $z_1, z_2, \ldots$ denote the zeros of f, with $z_k \neq 0$, then for all $s>\rho$ we have*

$$\sum_{k=1}^{\infty} \frac{1}{\left|z_k\right|^s}<\infty$$

Proof. It suffices to prove the estimate for $\mathfrak{n}(r)$ when $f(0) \neq 0$. Indeed, consider the function $F(z) = f(z)/z^{\ell}$ where ℓ is the order of the zero of f at the origin (so locally $f = z^{\ell} g$ for holomorphic g nonzero at 0). Then $\mathfrak{n}_f(r)$ and $\mathfrak{n}_F(r)$ differ only by a constant, and F also has an order of growth $\leq \rho$.

If $f(0) \neq 0$ we may use formula (8.2), namely

$$\int_0^R \mathfrak{n}(x) \frac{\mathrm{d}x}{x} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(Re^{i\theta}\right)\right| \mathrm{d}\theta - \log |f(0)|.$$

Choosing $R = 2r$, this formula implies

$$\int_r^{2r} \mathfrak{n}(x) \frac{\mathrm{d}x}{x} \leq \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(Re^{i\theta}\right)\right| \mathrm{d}\theta - \log |f(0)|.$$

On the one hand, since $\mathfrak{n}(r)$ is increasing, we have

$$\int_r^{2r} \mathfrak{n}(x) \frac{\mathrm{d}x}{x} \geq \mathfrak{n}(r) \int_r^{2r} \frac{\mathrm{d}x}{x} = \mathfrak{n}(r)[\log 2r - \log r] = \mathfrak{n}(r) \log 2,$$

and on the other hand, the growth condition on f gives

$$\int_0^{2\pi} \log \left| f\left(Re^{i\theta}\right)\right| \mathrm{d}\theta \leq \int_0^{2\pi} \log \left| Ae^{BR^{\rho}} \right| \mathrm{d}\theta \leq C' r^{\rho}$$

for all large r. Consequently, $\mathfrak{n}(r) \leq Cr^{\rho}$ for an appropriate $C > 0$ and all sufficiently large r. The following estimates prove the second part of the theorem:

$$\begin{aligned}
\sum_{|z_k| \geq 1} |z_k|^{-s} &= \sum_{j=0}^{\infty} \left(\sum_{2^j \leq |z_k| < 2^{j+1}} |z_k|^{-s} \right) \leq \sum_{j=0}^{\infty} \left(\sum_{2^j \leq |z_k| < 2^{j+1}} 2^{-js} \right) \\
&\leq \sum_{j=0}^{\infty} 2^{-js} \mathfrak{n}\left(2^{j+1}\right) \leq c \sum_{j=0}^{\infty} 2^{-js} 2^{(j+1)\rho} \\
&\leq c' \sum_{j=0}^{\infty} \left(2^{\rho - s}\right)^j < \infty.
\end{aligned}$$

The last series converges because $s > \rho$. $\square$

Part (ii) of the theorem is a noteworthy fact, which we shall use in a later part of this chapter.

We give two simple examples of the theorem; each of these shows that the condition $s > \rho$ cannot be improved.

Example 8.3.2. Consider $f(z) = \sin \pi z$. Recall Euler's identity, namely

$$f(z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

which implies that $|f(z)| \leq e^{\pi |z|}$, and f has an order of growth ≤ 1. By taking $z = ix$, where $x \in \mathbb{R}$, it is clear that the order of growth of f is actually equal to 1 . However, f vanishes to order 1 at $z = n$ for each $n \in \mathbb{Z}$, and $\sum_{n \neq 0} 1/|n|^s < \infty$ precisely when $s > 1$. $\diamond$

Example 8.3.3. Consider $f(z)=\cos z^{1/2}$, which we define by

$$\cos z^{1/2}=\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(2n)!}.$$

Then f is entire, and it is easy to see that

$$|f(z)|\leq e^{|z|^{1/2}},$$

and the order of growth of f is $1/2$. Moreover, $f(z)$ vanishes when $z_n=((n+1/2)\pi)^2$, while $\sum_n 1/|z_n|^s<\infty$ exactly when $s>1/2$. ◇

8.4 Infinite Products

The notion of convergence in $H(G)$ can be used to solve the following problem. Given any sequence of complex numbers $z_1,z_2,\ldots$, whether or not there exists an entire function f with zeros precisely at the points of this sequence. A necessary condition is that $z_1,z_2,\ldots$ do not accumulate, in other words we must have

$$\lim_{k\to\infty}|z_k|=\infty,$$

otherwise f would vanish identically. Weierstrass proved that this condition is also sufficient by explicitly constructing a function with these prescribed zeros. A first guess is of course the product

$$\left(z-z_1\right)\left(z-z_2\right)\cdots,$$

which provides a solution in the special case when the sequence of zeros is finite. In general, Weierstrass showed how to insert factors in this product so that the convergence is guaranteed, yet no new zeros are introduced. To study this, we need first look at the theory of infinite products.

Clearly one should define an infinite product of numbers z_n (denoted by $\prod_{n=1}^{\infty}z_n$) as the limit of the finite products. Observe, however, that if one of the numbers z_n is zero, then the limit is zero, regardless of the behavior of the remaining terms of the sequence. This does not present a difficulty, but it shows that when zeros appear, the existence of an infinite product is trivial. However, the limit of finite products being zero does not imply one of the factors is evaluated as zero.

8.4.1 Generalities

Definition 8.4.1. *If $\{z_n\}$ is a sequence of complex numbers and if $z=\lim\prod_{k=1}^{n}z_k$ exists, then z is the **infinite product** of the numbers z_n and it is denoted by*

$$z=\prod_{n=1}^{\infty}z_n.$$

Suppose that none of the numbers z_n is zero, and that $z=\prod_{n=1}^{\infty}z_n$ exists and is also not zero. Let $p_n=\prod_{k=1}^{n}z_k$ for $n\geq 1$; then no p_n is zero and $\frac{p_n}{p_{n-1}}=z_n$. Since $z\neq 0$ and $p_n\to z$ we have that $\lim z_n=1$. So that except for the cases where zero appears, a necessary condition for the convergence of an infinite product is that the n-th term must go to 1. On the other hand, note that for $z_n=a$ for all n and $|a|<1,\prod z_n=0$ although $\lim z_n=a\neq 0$.

Because of the fact that the exponential of a sum is the product of the exponentials of the individual terms, it is possible to discuss the convergence of an infinite product (when zero is not involved) by discussing the convergence of the series $\sum\log z_n$, where $\log$ is the principal branch of the logarithm. However, before this can be made meaningful the z_n must be restricted so that $\log z_n$ is meaningful. If the product is to be non-zero, then $z_n \to 1$. So it is no restriction to suppose that $\operatorname{Re} z_n > 0$ for all n. Now suppose that the series $\sum \log z_n$ converges. If $s_n = \sum_{k=1}^n \log z_k$ and $s_n \to s$ then $\exp s_n \to \exp s$. But $\exp s_n = \prod_{k=1}^n z_k$ so that $\prod_{n=1}^\infty z_n$ is convergent to $z = e^s \neq 0$.

Proposition 8.4.2. *Let $\operatorname{Re} z_n > 0$ for all $n \geq 1$. Then $\prod_{n=1}^\infty z_n$ converges to a non zero number iff the series $\sum_{n=1}^\infty \log z_n$ converges.*

Proof. Let $p_n = (z_1 \cdots z_n)$, $z = re^{i\theta}, -\pi < \theta \leq \pi$, and $\ell(p_n) = \log|p_n| + i\theta_n$ where $\theta - \pi < \theta_n \leq \theta + \pi$. If $s_n = \log z_1 + \cdots + \log z_n$ then $\exp(s_n) = p_n$ so that $s_n = \ell(p_n) + 2\pi i k_n$ for some integer k_n. Now suppose that $p_n \to z$. Then $s_n - s_{n-1} = \log z_n \to 0$; also $\ell(p_n) - \ell(p_{n-1}) \to 0$, Hence, $(k_n - k_{n-1}) \to 0$ as $n \to \infty$. Since each k_n is an integer this gives that there is an n_0 and a k such that $k_m = k_n = k$ for $m, n \geq n_0$. So $s_n \to \ell(z) + 2\pi i k$; that is, the series $\sum \log z_n$ converges. Since the converse was proved above, this completes the proof. □

Consider the power series expansion of $\log(1+z)$ about $z=0$:

$$\log(1+z) = \sum_{n=1}^\infty (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \ldots,$$

which has radius of convergence 1. If $|z| < 1$ then

$$\begin{aligned} \left|1 - \frac{\log(1+z)}{z}\right| &= \left|\frac{1}{2}z - \frac{1}{3}z^2 + \ldots\right| \\ &\leq \frac{1}{2}\left(|z| + |z|^2 + \ldots\right) \\ &= \frac{1}{2}\frac{|z|}{1-|z|}. \end{aligned}$$

If we further require $|z| < \frac{1}{2}$ then

$$\left|1 - \frac{\log(1+z)}{z}\right| \leq \frac{1}{2}.$$

This gives that for $|z| < \frac{1}{2}$

$$\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z|. \tag{8.3}$$

This will be used to prove the following result.

Proposition 8.4.3. *Let $\operatorname{Re} z_n > -1$; then the series $\sum \log(1+z_n)$ converges absolutely iff the series $\sum z_n$ converges absolutely.*

Proof. If $\sum |z_n|$ converges then $z_n \to 0$; so eventually $|z_n| < \frac{1}{2}$. By (8.3) $\sum |\log(1+z_n)|$ is dominated by a convergent series, and it must converge also. If, conversely, $\sum |\log(1+z_n)|$ converges, then it follows that $|z_n| < \frac{1}{2}$ for sufficiently large n (why?). Again (8.3) allows us to conclude that $\sum |z_n|$ converges. □

There is also a useful necessary condition that guarantees the existence of a product.

Proposition 8.4.4. *If $(a_n) \in \ell^1$, i.e., $\|(a_n)\|_{\ell^1} = \sum_n |a_n| < \infty$, then the product $\prod_{n=1}^\infty (1+a_n)$ converges. Moreover, the product is 0 if and only if one of its factors is 0.*

Remark 8.4.5. *The first part of this proposition can be easily seen as a corollary of the last proposition, as $\sum \log(1+a_n)$ conv. abs. implies $\sum \log(1+a_n)$ conv. and then one uses proposition 8.4.2. But we still record this proposition, along with its second part, from [16].*

Proof. If $\sum |a_n|$ converges, then for all large n we must have $|a_n| < 1/2$. Disregarding if necessary finitely many terms, we may assume that this inequality holds for all n. Recall the principal branch of the logarithm $z \mapsto \log z = \log|z| + i\operatorname{Arg}(z)$, $z \in \mathbb{C} \setminus (-\infty, 0]$ has the power series expansion

$$\log(1+z) = -\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n}$$

for $|z| < 1$. The logarithm satisfies the property that $1 + z = e^{\log(1+z)}$ whenever $|z| < 1$. Hence we may write the partial products as follows:

$$\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{B_N},$$

where $B_N = \sum_{n=1}^{N} b_n$ with $b_n = \log(1+a_n)$. By the power series expansion we see that $|\log(1+z)| \le 2|z|$, if $|z| < 1/2$. Hence $|b_n| \le 2|a_n|$, so B_N converges as $N \to \infty$ to a complex number, say B. Since the exponential function is continuous, we conclude that e^{B_N} converges to e^B as $N \to \infty$, proving the first assertion of the proposition.

To show the second statement, suppose $(a_n) \in \ell^1$ and none of the factors is zero, $1 + a_n \neq 0\ \forall n$. Let N_0 be such that $|a_n| \le 1/2$ for $n \ge N_0$. Then

$$\prod_{n=1}^{\infty}(1+a_n) = \underbrace{\prod_{n=1}^{N_0-1}(1+a_n)}_{\neq 0}\ \underbrace{\prod_{n=N_0}^{\infty}(1+a_n)}_{=e^B \neq 0 \text{ for some } B} \neq 0.$$

If one of the factor is zero, the product is of course zero. □

We wish to define the absolute convergence of an infinite product. The first temptation should be avoided. That is, we do not want to say that $\prod |z_n|$ converges. Why? If $\prod |z_n|$ converges it does not follow that $\prod z_n$ converges. In fact, let $z_n = -1$ for all n; then $|z_n| = 1$ for all n so that $\prod |z_n|$ converges to 1 . However $\prod_{k=1}^{n} z_k$ is ± 1 depending on whether n is even or odd, so that $\prod z_n$ does not converge. Thus, if absolute convergence is to imply convergence, we must seek a different definition. On the basis of Proposition 8.4.2 the following definition is justified.

Definition 8.4.6. *If* $\operatorname{Re} z_n > 0$ *for all* n *then the infinite product* $\prod z_n$ *is said to converge absolutely if the series* $\sum \log z_n$ *converges absolutely.*

According to Proposition 8.4.2 and the fact that absolute convergence of a series implies convergence, we have that absolute convergence of a product implies the convergence of the product. Similarly, if a product converges absolutely then any rearrangement of the terms of the product results in a product which is still absolutely convergent. If we combine Propositions 8.4.2 and 8.4.3 with the definition, the following fundamental criterion for convergence of a infinite product is obtained.

Corollary 8.4.7. *If* $\operatorname{Re} z_n > 0$ *then the product* $\prod z_n$ *converges absolutely iff the series* $\sum (z_n - 1)$ *converges absolutely.*

Although the preceding corollary gives a necessary and sufficient condition for the absolute convergence of an infinite product phrased in terms with which we are familiar, it does not give a method for evaluating infinite products in terms of the corresponding infinite series. To evaluate a particular product one must often resort to trickery.

We now apply these results to the convergence of products of functions. A fundamental question to be answered is the following. Suppose $\{f_n\}$ is a sequence of functions on a set X and $f_n(x) \to f(x)$ uniformly for x in X; when will $\exp(f_n(x)) \to \exp(f(x))$ uniformly for x in X? Below is a partial answer which is sufficient to meet our needs.

Lemma 8.4.8. *Let X be a set and let $f, f_1, f_2, \ldots$ be functions from X into $\mathbb{C}$ such that $f_n(x) \to f(x)$ uniformly for x in X. If there is a constant a such that $\operatorname{Re} f(x) \leq a$ for all x in X then $\exp f_n(x) \to \exp f(x)$ uniformly for x in X.*

Proof. If $\varepsilon > 0$ is given then choose $\delta > 0$ such that $|e^z - 1| < \varepsilon e^{-a}$ whenever $|z| < \delta$. Now choose n_0 such that $|f_n(x) - f(x)| < \delta$ for all x in X whenever $n \geq n_0$. Thus

$$\begin{aligned}\varepsilon e^{-a} &> |\exp[f_n(x) - f(x)] - 1| \\ &= \left|\frac{\exp f_n(x)}{\exp f(x)} - 1\right|\end{aligned}$$

It follows that for any x in X and for $n \geq n_0$,

$$|\exp f_n(x) - \exp f(x)| < \varepsilon e^{-a} |\exp f(x)| \leq \varepsilon$$

□

Lemma 8.4.9. *Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X into $\mathbb{C}$ such that $\sum g_n(x)$ converges absolutely and uniformly for x in X. Then the product*

$$f(x) = \prod_{n=1}^{\infty} (1 + g_n(x))$$

converges absolutely and uniformly for x in X. Also there is an integer n_0 such that $f(x) = 0$ iff $g_n(x) = -1$ for some $n, 1 \leq n \leq n_0$.

Proof. Since $\sum g_n(x)$ converges uniformly for x in X there is an integer n_0 such that $|g_n(x)| < \frac{1}{2}$ for all x in X and $n > n_0$. This implies that $\operatorname{Re}[1 + g_n(x)] > 0$ and also, according to inequality (8.3), $|\log(1 + g_n(x))| \leq \frac{3}{2}|g_n(x)|$ for all $n > n_0$ and x in X. Thus

$$h(x) = \sum_{n=n_0+1}^{\infty} \log(1 + g_n(x))$$

converges uniformly for x in X. Since h is continuous and X is compact it follows that h must be bounded; in particular, there is a constant a such that $\operatorname{Re} h(x) < a$ for all x in X. Thus, Lemma 8.4.8 applies and gives that

$$\exp h(x) = \prod_{n=n_0+1}^{\infty} (1 + g_n(x))$$

converges uniformly for x in X. Finally,

$$f(x) = [1 + g_1(x)] \cdots [1 + g_{n_0}(x)] \exp h(x)$$

and $\exp h(x) \neq 0$ for any x in X. So if $f(x) = 0$ it must be that $g_n(x) = -1$ for some n with $1 \leq n \leq n_0$. □

We now leave this general situation to discuss analytic functions.

Theorem 8.4.10. *Let G be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence in $H(G)$ such that no f_n is identically zero. If $\sum [f_n(z) - 1]$ converges absolutely and uniformly on compact subsets of G then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a.*

Proof. Since $\sum[f_n(z)-1]$ converges uniformly and absolutely on compact subsets of G, it follows from the preceding lemma that $f(z)=\prod f_n(z)$ converges uniformly and absolutely on compact subsets of G. That is, the infinite product converges in $H(G)$.

Suppose $f(a)=0$ and let $r>0$ be chosen such that $\bar{B}(a;r)\subset G$. By hypothesis, $\sum[f_n(z)-1]$ converges uniformly on $\bar{B}(a;r)$. According to Lemma 8.4.9 there is an integer n such that $f(z)=f_1(z)\ldots f_n(z)g(z)$ where g does not vanish in $\bar{B}(a;r)$. The proof of the remainder of the theorem now follows. □

Proposition 8.4.11. *Suppose $\{F_n\}$ is a sequence of holomorphic functions on the open set Ω. If there exist constants $c_n>0$ such that*

$$\sum c_n<\infty \quad \textit{and} \quad |F_n(z)-1|\leq c_n \quad \textit{for all } z\in\Omega,$$

then:

(i) The product $\prod_{n=1}^{\infty}F_n(z)$ converges uniformly in Ω to a holomorphic function $F(z)$.

(ii) If $F_n(z)$ does not vanish for any n, then

$$\frac{F'(z)}{F(z)}=\sum_{n=1}^{\infty}\frac{F_n'(z)}{F_n(z)}$$

Proof. To prove the first statement, note that for each z we may argue as in the proposition 8.4.4 if we write $F_n(z)=1+a_n(z)$, with $|a_n(z)|\leq c_n$. Then, we observe that the estimates are actually uniform in z because the c_n 's are constants. It follows from 4.2.13 that the product converges uniformly to a holomorphic function, which we denote by $F(z)$.

To establish the second part of the theorem, suppose that K is a compact subset of Ω, and let

$$G_N(z)=\prod_{n=1}^{N}F_n(z).$$

be the partial products. We have just proved that $G_N\to F$ uniformly in Ω, so by Theorem 4.2.14, the sequence $\{G_N'\}$ converges uniformly to F' in K. Besides, F is non-vanishing as is each F_n assumed to be (see previous lemma). We claim that $G_N'/G_N\to F'/F$ uniformly on K. That's because

$$\left|\frac{G_N'}{G_N}-\frac{F'}{F}\right|=\frac{|FG_N'-F'G_N|}{|G_N||F|}$$

and $\min_K|F|>0$ implies $(|G_N|)$ is uniformly bounded from below. Notice that

$$\frac{(F_1F_2)'}{F_1F_2}=\frac{F_1'F_2+F_1F_2'}{F_1F_2}=\frac{F_1'}{F_1}+\frac{F_2'}{F_2}$$

which generalizes to

$$\frac{G_N'}{G_N}=\sum_{n=1}^{N}\frac{F_n'}{F_n},$$

so part (ii) of the proposition is also proved. □

Before proceeding with the general theory of Weierstrass products, we first consider some explicit product formulae and a related technique to calculate series using residues.

8.4.2 Examples

In this section, we will prove that following product formula for the sine function.

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right). \tag{8.4}$$

We need to first show that

$$\pi \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} = \lim_{N\to\infty} \sum_{|n|\le N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}. \tag{8.5}$$

The first formula holds for all complex numbers z, and the second whenever z is not an integer. The sum $\sum_{n=-\infty}^{\infty} 1/(z+n)$ needs to be properly understood, because the separate halves corresponding to positive and negative values of n do not converge. Only when interpreted symmetrically, as $\lim_{N\to\infty} \sum_{|n|\le N} 1/(z+n)$, does the cancellation of terms lead to a convergent series as in (8.5) above.

The evalutation of the series (8.5) will be proved using residue theorem.

Remark 8.4.12. *Recall two things:*

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} \tag{1}$$

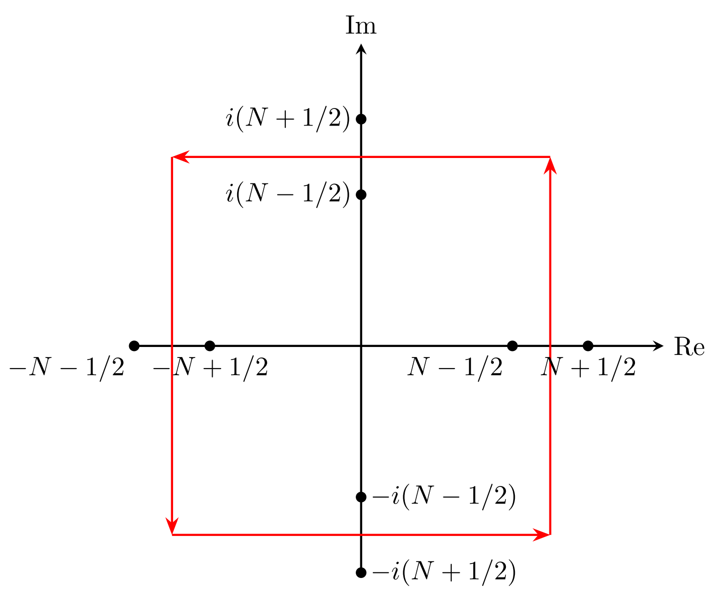
$$\sin \pi z = 0 \iff z = n \in \mathbb{Z} \tag{2}$$

The function $F(z) := \pi \cot \pi z$ has simple poles at $z = n$, $n \in \mathbb{Z}$, with

$$\begin{aligned}\operatorname{Res}(F; n) &= \lim_{z\to n} \pi(z-n)\frac{\cos \pi z}{\sin \pi z} = \lim_{z\to n} \pi \frac{\cos \pi z}{\frac{\sin \pi z - \sin \pi n}{z-n}} \\ &= \pi \frac{\cos \pi n}{(\sin \pi z)'|_{z=n}} = \pi \frac{\cos \pi n}{\pi \cos \pi n} = 1\end{aligned}$$

This will be a reason why this function can be used to calculate some series.

Let $N \in \mathbb{N}$ and γ_N be a rectangular contour going through $\pm(N+1/2)$ and $\pm i(N+1/2)$:



We need some estimates.

Lemma 8.4.13.

$$|\cot \pi z| \leq \coth \frac{\pi}{2} \quad \forall z \in \gamma_N,$$

where

$$\coth z := \frac{\cosh z}{\sinh z} = \frac{e^{2z}+1}{e^{2z}-1}.$$

Proof. This is a direct calculation using the identity

$$|\cot \pi z|^2 = \frac{\cos^2 \pi x + \sinh^2 \pi y}{\cosh^2 \pi y - \cos^2 \pi x}, \quad z = x + iy.$$

For instance, on the vertical strips $\cos\left(\frac{\pi}{2} + N\pi\right) = 0$ and so $|\cot \pi z| = |\tanh \pi y| \leq 1$. □

Remark 8.4.14. *The important part is not the explicit bound but the uniform estimate $|F(z)| \lesssim 1$ $\forall z \in \gamma_N$ and $\forall N$.*

Example 8.4.15. To get an idea of this strategy, we prove the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Notice

$$\left| \int_{\gamma_N} \frac{F(z)}{z^2} \mathrm{d}z \right| \lesssim N \cdot \frac{1}{N^2} = \frac{1}{N} \overset{N\to\infty}{\longrightarrow} 0.$$

where N comes from the length of γ_N, $= \Theta(N)$, and $\lesssim$ comes from $|F(z)| \lesssim 1$. On the other hand, by residue formula and remark 8.4.12,

$$\int_{\gamma_N} \frac{F(z)}{z^2} \mathrm{d}z = \int_{\gamma_N} \frac{\pi \cos \pi z}{z^2 \sin \pi z} \mathrm{d}z = 2\pi i \left(2 \sum_{n=1}^{N} \frac{1}{n^2} + \operatorname{Res}\left(\frac{\pi \cos \pi z}{z^2 \sin \pi z}; 0 \right) \right).$$

Here, we used

$$\operatorname{Res}\left(\frac{F(z)}{z^2}; n \right) = \frac{1}{n^2}$$

for $n \in \mathbb{Z} \setminus \{0\}$ since $\operatorname{Res}(F; n) = 1$. We need to calculate this residue at 0 to continue. Recall for g with a pole of order n at a, one has

$$\operatorname{Res}(g; a) = \lim_{z \to a} \frac{1}{(n-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{n-1} (z-a)^n g(z)$$

here $n = 3$, $a = 0$, so we have

$$\operatorname{Res}\left(\frac{\pi \cos \pi z}{z^2 \sin \pi z}; 0 \right) = \lim_{z \to 0} \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \frac{\pi z \cos \pi z}{\sin \pi z}.$$

Recall that

$$\sin \pi z = \pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots$$

$$\cos \pi z = 1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} + \cdots$$

Thus

$$\frac{\pi z\cos\pi z}{\sin\pi z}=\frac{\cos\pi z}{\frac{\sin\pi z}{\pi z}}=(\cos\pi z)/\left(\frac{\pi z-\frac{(\pi z)^3}{3!}+\frac{(\pi z)^5}{5!}-\cdots}{\pi z}\right)=\frac{\cos\pi z}{1-\frac{(\pi z)^2}{3!}+\frac{(\pi z)^4}{5!}-\cdots}.$$

Now we use the geometric series

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\quad |z|<1$$

to write

$$\begin{aligned}\frac{1}{1-\underbrace{\left(\frac{(\pi z)^2}{3!}-\frac{(\pi z)^4}{5!}+\cdots\right)}_{=:\omega}}&=1+\omega+\omega^2+\cdots\\&=1+\frac{\pi^2}{6}z^2+\left(\underbrace{\frac{\pi^4}{(3!)^2}}_{\text{from }\omega^2}-\underbrace{\frac{\pi^4}{5!}}_{\text{from }\omega}\right)z^4+\cdots\\&=1+\frac{\pi^2}{6}z^2+\frac{7\pi^4}{360}+\cdots\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\pi z\cos\pi z}{\sin\pi z}&=\overbrace{\left(1-\frac{\pi^2}{2}z^2+\frac{\pi^4}{24}z^4+\cdots\right)}^{\cos\pi z}\cdot\left(1+\frac{\pi^2}{6}z^2+\frac{7\pi^4}{360}z^4+\cdots\right)\\&=1-\frac{\pi^2}{2}z^2+\frac{\pi^2}{6}z^2+\cdots=1-\frac{\pi^2}{3}z^2+O\left(z^4\right).\end{aligned}$$

It follows that

$$\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{\pi z\cos\pi z}{\sin\pi z}=\frac{-2\pi^2}{3}.$$

Then

$$\operatorname{Res}\left(\frac{\pi\cos\pi z}{z^2\sin\pi z};0\right)=\lim_{z\to 0}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{\pi z\cos\pi z}{\sin\pi z}=\frac{-\pi^2}{3}.$$

and

$$0=\lim_{N\to\infty}\int_{\gamma_N}\frac{F(z)}{z^2}\mathrm{d}z=\lim_{N\to\infty}2\pi i\left(2\sum_{n=1}^{N}\frac{1}{n^2}-\frac{\pi^2}{3}\right)=2\pi i\left(2\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{\pi^2}{3}\right).$$

We conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

This particular series can be calculated in other ways, but this method is of interest to us. ◇

Example 8.4.16. We will consider the same path γ_N above to show first that

$$\sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2}=\frac{\pi^2}{\sin^2(\pi z)},\qquad z\notin\mathbb{Z}.$$

We consider the function

$$F(\xi)=\frac{\pi\cot\pi\xi}{(\xi+z)^2}$$

It has simple poles at $z = n$, $n \in \mathbb{Z}$, with

$$\begin{aligned}\operatorname{Res}(F;n) &= \lim_{\xi\to n} \pi(\xi - n)\frac{\cos\pi\xi}{(\xi+z)^2\sin\pi\xi} \\ &= \frac{1}{(z+n)^2}\lim_{\xi\to n}\pi\frac{\cos\pi z}{\frac{\sin\pi\xi - \sin\pi n}{\xi - n}} \\ &= \frac{1}{(z+n)^2}\pi\frac{\cos\pi n}{(\sin\pi\xi)'|_{\xi=n}} \\ &= \frac{1}{(z+n)^2}\end{aligned}$$

Thus, the path γ_N does not go through the poles, including $-z$, when N goes to ∞.

Reusing the estimate we talked about in class that $|\pi\cot\pi\xi| \lesssim 1$ $\forall\xi\in\gamma_N$ and $\forall N$, we see $|\pi\cot\pi\xi| \lesssim \frac{1}{N^2}$ $\forall\xi\in\gamma_N$ and $\forall N$. Thus,

$$\left|\int_{\Gamma_N} F(z)dz\right| \lesssim N\cdot\frac{1}{N^2} = \frac{1}{N} \xrightarrow{N\to\infty} 0.$$

where N comes from the length of γ_N. Residue theorem then tells us

$$0 = \int_{\gamma_N} F(z)\,dz = 2\pi i\left(\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2} + \operatorname{Res}\left(\frac{\pi\cot\pi\xi}{(\xi+z)^2}; -z\right)\right).$$

Recall for g with a pole of order n at a, one has

$$\operatorname{Res}(g;a) = \lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-a)^n g(z)$$

here $n=2$, $a=-z$, so we have

$$\begin{aligned}\operatorname{Res}(F;-z) &= \lim_{\xi\to -z}\frac{d}{d\xi}(\xi+z)^2\frac{\pi\cot\pi\xi}{(\xi+z)^2} \\ &= \pi\lim_{\xi\to -z}\frac{d}{d\xi}\cot\pi\xi \\ &= [-\pi^2\csc^2\pi\xi]_{\xi=-z} \\ &= -\frac{\pi^2}{\sin(\pi z)^2}.\end{aligned}$$

Therefore, $\forall z\notin\mathbb{Z}$,

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2} = \frac{\pi^2}{\sin(\pi z)^2} \tag{8.6}$$

To use it to derive the formula

$$z\notin\mathbb{Z},\quad \pi\cot(\pi z) = \sum_{n=-\infty}^{\infty}\frac{1}{z+n},$$

we want to prove

Claim:

$$\forall z\notin\mathbb{Z},\quad \frac{d}{dz}\left(\sum_{n=-\infty}^{\infty}\frac{1}{z+n}\right) = \sum_{n=-\infty}^{\infty}\frac{d}{dz}\frac{1}{z+n} = \sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2}.$$

The second equality is a direct computation. To show the first equality, we recall that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω, and the sequence of derivatives $\{f_n'\}_{n=1}^\infty$ converges uniformly to f' on every compact subset of Ω. It thus suffices to show that the partial sum $S_N = \sum_{n=-N}^{N} \frac{1}{z+n}$ uniformly converges for any compact set $K \subseteq \Omega := \mathbb{C} \setminus \mathbb{Z}$. $|z|$ as a continuous function on K attains its maximum M somewhere. Now we note that

$$S_N = \sum_{n=-N}^{N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{N} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) = \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2-n^2}.$$

Note that

$$\left| \frac{2z}{z^2-n^2} \right| \leq \frac{2|z|}{|z|^2-n^2} \leq \frac{C}{n^2+1}$$

for some constnat $C > 0$, because for $|\mathrm{Im} z| \leq 1$, holomorphicity ensures bounedness, and for $|\mathrm{Im} z| > 1$, one can choose $C > M$. Since $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \geq 1$, the positive series

$$\sum_{n=1}^{\infty} \frac{C}{n^2+1} = C \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

is convergent. Thus, the series S_∞ majorized by a series getting rid of z uniformly converges. The claim is then proved.

By the claim and equation (8.6), we have, for $z \in \Omega = \mathbb{C} \setminus \mathbb{Z}$,

$$\begin{aligned} \frac{d}{dz} \left(\sum_{n=-\infty}^{\infty} \frac{1}{z+n} \right) &= \frac{\pi^2}{\sin(\pi z)^2} = \frac{d}{dz} (\pi \cot \pi z) \\ \implies \sum_{n=-\infty}^{\infty} \frac{1}{z+n} &= \pi \cot \pi z + C' \quad \text{for some constant } C' \end{aligned}$$

where we evoked the fact that primitives only differ by a constant (See Math5021 Orponen lecture note prop.8.4.30). Plugging in $z = \frac{1}{2}$ results in

$$\sum_{n=-\infty}^{\infty} \frac{1}{\frac{1}{2}+n} = C'$$

We will explicitly show its partial sum is $S_N = \frac{1}{2N+1}$ by induction. For $N = 0$ and 1, the formula can be easily verified. We then observe that

$$\begin{aligned} S_{N+1} &= \frac{1}{-2(N+1)+1} + S_N + \frac{1}{2(N+1)+1} \\ &= \frac{1}{-2N-1} + \frac{1}{2N+1} + \frac{1}{2(N+1)+1} \\ &= \frac{1}{2(N+1)+1}. \end{aligned}$$

Now

$$C' = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2N+1} = 0.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{1}{z+n} = \pi \cot \pi z.$$

$\diamond$

Remark 8.4.17. *[16] uses another way to show (8.5).*

To prove (8.4), we now let

$$G(z) = \frac{\sin \pi z}{\pi} \quad \text{and} \quad P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Proposition 8.4.11 and the fact that $\sum 1/n^2 < \infty$ guarantee that the product $P(z)$ converges, and that away from the integers we have

$$\frac{P'(z)}{P(z)} = \frac{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) + z\left(\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right)'}{z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1-z^2/n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}.$$

Since $G'(z)/G(z) = \pi \cot \pi z$, the cotangent formula (8.5) gives

$$\left(\frac{P(z)}{G(z)}\right)' = \frac{P(z)}{G(z)}\left[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right] = 0,$$

and so $P(z) = cG(z)$ for some constant c. Dividing this identity by z, and taking the limit as $z \to 0$, we find $c = 1$.

Remark 8.4.18. *Above example of pole expansion of meromorphic function belongs to a broader class of result called **Mittag-Leffler theorem**, which asserts the existence of a meromorphic function with prescribed sequence of poles and principal parts. For example*

$$\begin{aligned}
\tan(z) &= \sum_{n=0}^{\infty} \frac{8z}{(2n+1)^2\pi^2 - 4z^2} \\
\csc^2(z) &= \sum_{n\in\mathbb{Z}} \frac{1}{(z-n\pi)^2} \\
\csc(z) &= \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{z-n\pi} = \frac{1}{z} + 2z\sum_{n=1}^{\infty}(-1)^n \frac{1}{z^2-(n\pi)^2}
\end{aligned}$$

One can find the proof of Mittag-Leffler theorem in [3] p.205 Theorem 3.2. It needs Runge's theorem. For more remarks on that, see the post.

Mittag-Leffler theorem is the sister of Weierstrass factorization theorem, which asserts existence of holomorphic functions with prescribed zeros. We then talk about it in next section.

8.5 Weierstrass infinite products

Given a sequence (a_n) for which $\{|a_n|\} \to \infty$, we construct an entire function that vanishes at all $z = a_n$ and nowhere else. The product formula

$$\frac{\sin(\pi z)}{\pi} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = z\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right)$$

suggests to try something like the product $\prod_n (1 - z/a_n)$. The obvious problem is that this product converges only for suitable sequences (a_n). To fix this issue, we will insert certain exponential factors, which will make this product converge without adding new zeros.

Definition 8.5.1. *The **canonical factors** $E_k, k \geq 0$, are defined by setting*

$$E_0(z) = 1 - z \quad \text{and} \quad E_k(z) = (1-z)e^{z+z^2/2+\cdots+z^k/k}, \quad k \geq 1.$$

*The integer k is called the **degree** of the canonical factor.*

Notice that these factors vanish when $z = 1$ and nowhere else. We will soon use them in the form $E_n(z/a)$ – such a factor vanishes for $z = a$. The decay given by the next lemma is the key to the usefulness of these factors.

Lemma 8.5.2. *If $|z| \leq 1/2$, then $|1 - E_k(z)| \leq 2e|z|^{k+1}$.*

Proof. Recall that

$$\log(1+\xi) = -\sum_{n=1}^{\infty}(-1)^n \frac{\xi^n}{n}, \quad |\xi| < 1$$

so that here $\log(1-z) = -\sum_{n=1}^{\infty} z^n/n$ (as $|z| \leq 1/2$ by assumption). Thus, we have

$$E_k(z) = e^{\log(1-z)+z+z^2/2+\cdots+z^k/k} = e^w,$$

where $w = -\sum_{n=k+1}^{\infty} z^n/n$. We now have

$$|w| \leq |z|^{k+1} \sum_{n=k+1}^{\infty} \frac{|z|^{n-k-1}}{n} \leq |z|^{k+1} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2|z|^{k+1}.$$

In particular, $|w| \leq 1$. Therefore, (using the power series $e^\xi = \sum_{n=0}^{\infty} \xi^n/n\,!$) we have

$$|1 - E_k(z)| = |1 - e^w| \leq \sum_{n=1}^{\infty} \frac{|w|^n}{n!} = |w| \sum_{n=0}^{\infty} \frac{|w|^n}{(n+1)!} \leq e|w| \leq 2e|z|^{k+1}$$

□

We are ready for the main construction of this section.

Theorem 8.5.3 (Weierstrass Infinite Product)**.** *Given any sequence (a_n) of complex numbers with $|a_n| \to \infty$, there exists an entire function f with zeros $z = a_n$ (counted with multiplicities, i.e., we assume the sequence lists the repeated zeros repeatedly). Any other such entire function is of the form fe^g for some entire g.*

Proof. We begin by proving the second assertion first. Note that since we allow for repetitions in the sequence (a_n), this theorem actually guarantees the existence of entire functions with prescribed zeros and with desired multiplicities. Let now f_1 and f_2 be two entire functions that vanish at all $z = a_n$ and nowhere else. Then given one of their shared zeros a there exists $m > 0$ so that $f_i(z) = (z-a)^m g_i(z)$ for some g_i that is holomorphic and non-vanishing in a neighborhood of a. This implies that in that neighborhood we have

$$\frac{f_1(z)}{f_2(z)} = \frac{g_1(z)}{g_2(z)}$$

so that the singularity at $z = a$ is removable. Therefore, f_1/f_2 is a non-vanishing entire function, which implies that $f_1/f_2 = e^g$ for some entire g (as $\mathbb{C}$ is simply connected). This ends the proof of the second assertion of the theorem.

To prove the first assertion, we relabel the setting so that we assume that we have a zero of order m at the origin, and that $a_1, a_2, \ldots$ are all non-zero. Then, we define the Weierstrass product by setting

$$f(z) := z^m \prod_{n=1}^{\infty} E_n\left(z/a_n\right).$$

Fix an arbitrary $R > 0$. We shall prove that f has the desired properties in the disc $D_R := D(0, R)$, and because R is arbitrary, this will prove the theorem. We split

$$f(z) := z^m \prod_{n:|a_n|\leq 2R} E_n\left(z/a_n\right) \prod_{n:|a_n|>2R} E_n\left(z/a_n\right)$$

There are only finitely many n for which $|a_n| \leq 2R$ because $|a_n| \to \infty$. In the ball D_R the finite product $z^m \prod_{|a_n| \leq 2R} E_n(z/a_n)$ vanishes at all $z = a_n$ with $|a_n| < R$ and at $z = 0$, and nowhere else. Then we need to argue that the remaining infinite product $\prod_{n:|a_n|>2R} E_n(z/a_n)$ defines a non-vanishing holomorphic function in D_R. To see this, notice first that if $|a_n| > 2R$, we have $|z/a_n| < 1/2$ in D_R. Thus, Lemma 8.5.2 implies that

$$|1 - E_n(z/a_n)| \leq 2e \left|\frac{z}{a_n}\right|^{n+1} \leq \frac{2e}{2^{n+1}} = \frac{e}{2^n}, \quad z \in D_R, |a_n| > 2R.$$

By Proposition 8.4.11 this implies that the product

$$\prod_{|a_n|>2R} E_n(z/a_n)$$

defines a holomorphic function in D_R - it is also non-vanishing in D_R by Proposition 8.4.4. □

Weierstrass' theorem states that if a non-trivial entire function f has non-zero zeros $a_1, a_2, \ldots$ (and a zero of order m at the origin), it takes the form

$$e^{g(z)} z^m \prod_{n=1}^{\infty} E_n(z/a_n).$$

This is not always convenient to work with – the index n of the canonical factors E_n involved are unbounded, and no information is provided about the entire function g. We aim to further refine this construction by proving Hadamard's factorization theorem, which basically states that in the case of functions of finite order (we define this notion in the next section), the degree n of the canonical factors E_n can be taken to be a constant, and that g is then a polynomial.

8.6 Hadamard's Factorization Theorem

We will need several lemmas before being able to prove Hadamard's factorization theorem. First, to have a finer understanding of the canonical factors, we will bound them from below – such bounds are a critical component of the proof of Hadamard's factorization as we will later see.

Lemma 8.6.1. *The canonical factors satisfy*

$$|E_k(z)| \geq e^{-2|z|^{k+1}} \quad \text{if } |z| \leq 1/2$$

and

$$|E_k(z)| \geq |1 - z| e^{(1-2^k)|z|^k} \quad \text{if } |z| \geq 1/2.$$

Proof. Assume first that $|z| \leq 1/2$. Recall from the proof of Lemma 8.5.2 that we can write $E_k(z) = e^w$, where $|w| \leq 2|z|^{k+1}$. Note that

$$|\omega| = \sqrt{(\text{Re}\omega)^2 + (\text{Im}\omega)^2} \geq |\text{Re}\omega| \implies -|\omega| \leq -|\text{Re}\omega| \leq \text{Re}\omega, \tag{8.7}$$

so

$$|E_k(z)| = e^{\text{Re}\omega} \geq e^{-|w|} \geq e^{-2|z|^{k+1}}.$$

(8.7) also implies

$$\left|e^{z+z^2/2+\cdots+z^k/k}\right| = e^{\text{Re}\left(z+z^2/2+\cdots+z^k/k\right)} \geq e^{-|z+z^2/2+\cdots+z^k/k|}.$$

If $|z| \geq 1/2$, we see

$$|E_k(z)| = |1-z|\left|e^{z+z^2/2+\cdots+z^k/k}\right| \geq |1-z|e^{-\left|z+z^2/2+\cdots+z^k/k\right|},$$

where

$$\left|z+z^2/2+\cdots+z^k/k\right| \leq \left(\sum_{n=1}^{k}\frac{1}{2^{n-k}}\right)|z|^k = \left(2^k-1\right)|z|^k.$$

We thus completed the proof. □

We now state the setting under which we will be working for the rest of this section. Suppose f is entire and has an order of growth ρ_0. Let k be the unique integer so that $k \leq \rho_0 < k+1$. Let $a_1, a_2, \ldots$ denote the non-zero zeros of f in $\mathbb{C}$.

Lemma 8.6.2. *For any s with $\rho_0 < s < k+1$, we have*

$$\left|\prod_{n=1}^{\infty} E_k\left(z/a_n\right)\right| \geq e^{-c|z|^s}$$

except possibly when z belongs to the union of the discs centered at a_n of radius $|a_n|^{-k-1}$, for $n = 1, 2, \ldots$

Proof. Write

$$\prod_{n=1}^{\infty} E_k\left(z/a_n\right) = \prod_{|a_n|\leq 2|z|} E_k\left(z/a_n\right) \prod_{|a_n|>2|z|} E_k\left(z/a_n\right).$$

For the second product, we have by the first part of Lemma 8.6.1 that

$$\left|\prod_{|a_n|>2|z|} E_k\left(z/a_n\right)\right| \geq \prod_{|a_n|>2|z|} e^{-2|z/a_n|^{k+1}} = e^{-2|z|^{k+1}\sum_{|a_n|>2|z|}|a_n|^{-k-1}}.$$

As $|a_n| > 2|z|$ and $s < k+1$, we have $s-k-1 < 0$ and

$$|a_n|^{-k-1} = |a_n|^{-s}|a_n|^{s-k-1} \leq 2^{s-k-1}|a_n|^{-s}|z|^{s-k-1}.$$

Therefore, we have

$$\left|\prod_{|a_n|>2|z|} E_k\left(z/a_n\right)\right| \geq e^{-2^{s-k}|z|^s\sum_{|a_n|>2|z|}|a_n|^{-s}} \geq e^{-2^{s-k}|z|^s\sum_n|a_n|^{-s}}$$

Recall that by Theorem 8.3.1 we have $\sum_n |a_n|^{-s} < \infty$, since $s > \rho_0$. Thus, we have proved that for some $c > 0$ it holds

$$\left|\prod_{|a_n|>2|z|} E_k\left(z/a_n\right)\right| \geq e^{-c|z|^s}.$$

We now estimate the first product, where the product is over the finitely many n for which $|a_n| \leq 2|z|$. Notice first that using the second part of Lemma 8.6.1 we have

$$\left|\prod_{|a_n|\leq 2|z|} E_k\left(z/a_n\right)\right| \geq \prod_{|a_n|\leq 2|z|}\frac{|a_n-z|}{|a_n|}\prod_{|a_n|\leq 2|z|} e^{-c|z/a_n|^k}.$$

Now note that

$$\left| \prod_{|a_n| \le 2|z|} e^{-c|z/a_n|^k} \right| = e^{-c|z|^k \sum_{|a_n| \le 2|z|} |a_n|^{-k}},$$

and $|a_n|^{-k} = |a_n|^{-s} |a_n|^{s-k} \le 2^{s-k} |a_n|^{-s} |z|^{s-k}$. Thus, it holds (by Theorem 8.3.1 again)

$$\left| \prod_{|a_n| \le 2|z|} e^{-c|z/a_n|^k} \right| \ge e^{-c|z|^s}$$

Finally, we need to estimate

$$\prod_{|a_n| \le 2|z|} \frac{|a_n - z|}{|a_n|}.$$

Here we will, for the first time, employ the condition imposed on z. Indeed, whenever z does not belong to a disc of radius $|a_n|^{-k-1}$ centered at a_n, we have $|a_n - z| \ge |a_n|^{-k-1}$. Therefore, for such z we have

$$\prod_{|a_n| \le 2|z|} \frac{|a_n - z|}{|a_n|} \ge \prod_{|a_n| \le 2|z|} |a_n|^{-k-2}.$$

We take a log of the RHS to convert it into a sum (this is not strictly necessary but we find it convenient):

$$\log \left(\prod_{|a_n| \le 2|z|} |a_n|^{-k-2} \right) = -(k+2) \sum_{|a_n| \le 2|z|} \log |a_n|.$$

If we show

$$\sum_{|a_n| \le 2|z|} \log |a_n| \lesssim |z|^s \tag{8.8}$$

it then follows that

$$\log \left(\prod_{|a_n| \le 2|z|} |a_n|^{-k-2} \right) \gtrsim -|z|^s$$

and so

$$\prod_{|a_n| \le 2|z|} |a_n|^{-k-2} \ge e^{-c|z|^s}$$

as desired. To end the proof, we then only need to show (8.8). Let s' be such that $\rho_0 < s' < s$. By Theorem 8.3.1 we have $n(r) \lesssim r^{s'}$ for all $r \gtrsim 1$. We first estimate

$$\sum_{|a_n| \le 2|z|} \log |a_n| \le n(2|z|) \log(2|z|).$$

If $\log(2|z|) < 0$, obviously this is $\le |z|^s$. Otherwise, $|z| \gtrsim 1$ and we use the size estimate on $n(r)$ to get

$$n(2|z|) \log(2|z|) \lesssim |z|^{s'} \log(2|z|) \lesssim |z|^s.$$

□

Corollary 8.6.3. *There exists a sequence of radii, $r_1, r_2, \ldots$, with $r_m \to \infty$, such that*

$$\left|\prod_{n=1}^{\infty} E_k\left(z/a_n\right)\right| \geq e^{-c|z|^s}$$

for $|z| = r_m$.

Proof. Since $k+1 > \rho_0$, we have by Theorem 8.3.1 that $\sum_{n=N}^{\infty} |a_n|^{-k-1} < 1/2$ for some integer N (as a tail of a converging series). Consider any large integer L for which we have

$$[L, L+1] \cap \left(|a_n| - |a_n|^{-k-1}, |a_n| + |a_n|^{-k-1}\right) = \varnothing$$

for all $n < N$. We claim that there exists $r \in [L, L+1]$ so that $\{|z| = r\}$ does not intersect any of the disks $B\left(a_n, |a_n|^{-k-1}\right), n = 1, 2, \ldots$ For otherwise, for every $r \in [L, L+1]$ there exists $n_r \in \mathbb{N}$ so that we have an element $z_r \in \{|z| = r\} \cap B\left(a_{n_r}, |a_{n_r}|^{-k-1}\right)$. Now, we have

$$|r - |a_{n_r}|| = ||z_r| - |a_{n_r}|| \leq |z_r - a_{n_r}| < |a_{n_r}|^{-k-1}.$$

Thus, it holds

$$[L, L+1] \subset \bigcup_{n=N}^{\infty} \left(|a_n| - |a_n|^{-k-1}, |a_n| + |a_n|^{-k-1}\right),$$

and so

$$2\sum_{n=N}^{\infty} |a_n|^{-k-1} \geq 1$$

which is a contradiction. Employ the previous lemma to finish the proof. □

One final lemma is needed before we can prove Hadamard's factorization theorem.

Lemma 8.6.4. *Suppose g is entire and $u = \mathrm{Re}(g)$ satisfies*

$$u(z) \leq Cr^s \quad \textit{whenever } |z| = r$$

for a sequence of positive real numbers r that tends to infinity. Then g is a polynomial of degree $\leq s$.

Proof. Write $g(z) = \sum_{n=0}^{\infty} a_n z^n$, and deduce that for all $r > 0$ we have (by a simple application of Cauchy's integral formula; see Math5022 HW6 Ex2)

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(re^{i\theta}\right) e^{-in\theta} \mathrm{d}\theta = \begin{cases} a_n r^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

By taking complex conjugates we have

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{g\left(re^{i\theta}\right)} e^{in\theta} \mathrm{d}\theta = \begin{cases} \overline{a_n} r^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Thus, for $n > 0$, or $k = -n < 0$

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{g\left(re^{i\theta}\right)} e^{ik\theta} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \overline{g\left(re^{i\theta}\right)} e^{-in\theta} \mathrm{d}\theta = 0$$

Recall that $u=(g+\bar{g})/2$, and sum the previous identities to obtain that for $n>0$ we have

$$\frac{1}{\pi}\int_0^{2\pi} u\left(re^{i\theta}\right)e^{-in\theta}\mathrm{d}\theta = a_n r^n.$$

Also, we take real parts from both sides of the $n=0$ formula to see

$$2\operatorname{Re}\left(a_0\right)=\frac{1}{\pi}\int_0^{2\pi} u\left(re^{i\theta}\right)\mathrm{d}\theta$$

Using the fact that $\int_0^{2\pi} e^{-in\theta}=0$ for all $n\neq 0$, we infer that

$$a_n=\frac{1}{\pi r^n}\int_0^{2\pi}\left[u\left(re^{i\theta}\right)-Cr^s\right]e^{-in\theta}\mathrm{d}\theta \quad \text{when } n>0,$$

and consequently, for an r as in the hypothesis, we have

$$\left|a_n\right|\leq\frac{1}{\pi r^n}\int_0^{2\pi}\left[Cr^s-u\left(re^{i\theta}\right)\right]\mathrm{d}\theta\leq 2Cr^{s-n}-2\operatorname{Re}\left(a_0\right)r^{-n}.$$

Let $r\to\infty$ along the sequence given in the hypothesis of the lemma to obtain that $a_n=0$ for $n>s$. □

Remark 8.6.5. *This lemma generalizes proposition 8.1.6.*

We are now ready to prove Hadamard's factorization theorem. We state it carefully with all of the underlying assumptions on f.

Theorem 8.6.6 (Hadamard Factorization Theorem)**.** *Suppose f is entire and has growth order $\leq\rho_0$. Let k be the integer so that $k\leq\rho_0<k+1$. If $a_1,a_2,\ldots$ denote the non-zero zeros of f and f has a zero of order m at the origin, then*

$$f(z)=e^{p(z)}z^m\prod_{n=1}^{\infty}E_k\left(z/a_n\right),$$

where p is a polynomial of degree $\leq k$.

Proof. The beginning of the proof is quite similar to that of Theorem 8.5.3. We set

$$E(z):=z^m\prod_{n=1}^{\infty}E_k\left(z/a_n\right),\quad z\in\mathbb{C}.$$

Consider an arbitrary $R>0$. We utilize Lemma 8.5.2 to note that

$$\left|1-E_k\left(z/a_n\right)\right|\leq 2e\left|\frac{z}{a_n}\right|^{k+1}\leq 2eR^{k+1}\left|a_n\right|^{-k-1}$$

for all large n (must have $|a_n|>2R$) and for all $z\in D_R$. Because the series $\sum\left|a_n\right|^{-k-1}$ converges (by Theorem 8.3.1), we have by Theorem 8.4.11 that $z\mapsto E(z)$ is entire (as R was arbitrary), and has the zeros of f (see Proposition 8.4.4). Therefore, as we have seen before, f/E is entire and nowhere vanishing. Consequently, there exists an entire g such that $f/E=e^g$. Since f has a growth order $\leq\rho_0$, we have by Corollary 8.6.3 that for any s with $\rho_0<s<k+1$ it holds

$$e^{\operatorname{Re}(g(z))}=\left|e^{g(z)}\right|=\left|\frac{f(z)}{E(z)}\right|\leq Ce^{c|z|^s}$$

for $|z|=r_m$, and for some $C,c>0$. Lemma 8.6.4 finishes the proof. □

Example 8.6.7. Let $f(z) := \sin(\pi z)$. We derive the already known product formula of sine from Hadamard. We have already previously noted that the growth order of f is 1 , and so Hadamard tells us that

$$f(z) = e^{p(z)} z \prod_{n \in \mathbb{Z} \setminus \{0\}} E_1(z/n) = e^{p(z)} z \prod_{n \in \mathbb{Z} \setminus \{0\}} (1 - z/n) e^{z/n} = e^{p(z)} z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right),$$

where $p(z) = az + b$ is some polynomial of degree at most 1 . We first show that $a = 0$. This follows from the fact that

$$e^{p(z)} z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right) = f(z) = -f(-z) = e^{p(-z)} z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right)$$

implying that $e^{p(z)} = e^{p(-z)}$ for $z \notin \mathbb{Z}$. Thus, for those z we have

$$1 = e^{p(z) - p(-z)} = e^{az + b - (-az + b)} = e^{2az}$$

forcing $a = 0$. It follows that

$$f(z) = e^b z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right).$$

Divide both sides by πz and let $z \to 0$ to conclude that $e^b/\pi = 1$, i.e., $e^b = \pi$. Thus, we have the formula for sine:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right).$$

$\diamond$

8.7 Blaschke Products

Blaschke products are bounded anologues in the disc $\mathbb{D}$ of the Weierstrass products for the entire functions. In this section we quickly record, omitting some details of the proofs, their basic theory.

Definition 8.7.1. *The space $H^\infty = H^\infty(\mathbb{D})$ consists of all bounded analytic functions f on $\mathbb{D}$:*

$$H^\infty = H^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) \mid \|f\|_\infty < \infty\},$$

where $\|f\|_\infty$ is the sup-norm

$$\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}.$$

Theorem 8.7.2. *Assume that $f \in H^\infty$ is not identically zero and $a_1, a_2, \ldots \in \mathbb{D}$ are its zeros. Then we have*

$$\sum_n (1 - |a_n|) < \infty.$$

Proof. We tacitly assume f has infinitely many zeros, otherwise there is nothing to prove. If f has a zero of order m at the origin, set $F(z) = f(z)/z^m$. Then $F \in H^\infty$ but now $F(0) \neq 0$. Thus, without loss of generality, we may assume $f \in H^\infty$ does not vanish at zero (so $a_n \neq 0$ for all n).

Let $p_N := \prod_{i=1}^N |a_n|$. Notice that as $0 < |a_n| < 1$ we have $p_1 > p_2 > \cdots$, and so the infinite product $\prod_{n=1}^\infty |a_n|$ exists (a bounded decreasing sequence has a limit). It is non-trivial that $\prod_{n=1}^\infty |a_n| > 0$, but a short argument using Jensen's formula (Math5022 HW7 Ex3) shows that

$$\prod_{n=1}^{\infty} |a_n| \geq \frac{|f(0)|}{\|f\|_\infty} > 0$$

This then implies (see Math5022 HW7 Ex1) that

$$\sum_n (1-|a_n|) < \infty.$$

□

So if $f \in H^\infty$ with $\sum_n (1-|a_n|) = \infty$, then $f = 0$ identically. A Blaschke product allows to do what the Weirstrass product did in the entire case.

Theorem 8.7.3. *Let (a_n) satisfy $a_n \in \mathbb{D}\backslash\{0\}$. Define the **Blaschke product***

$$B(z) := \prod_{n=1}^{\infty} \psi_{a_n}(z) \frac{|a_n|}{a_n},$$

*where $\psi_\alpha, \alpha \in \mathbb{D}$, is the following familiar automorphism of $\mathbb{D}$ (sometimes called a **Blaschke factor**):*

$$\psi_\alpha(z) := \frac{\alpha - z}{1-\bar{\alpha}z}.$$

Then a necessary and sufficient condition for B to converge locally uniformly (i.e., uniformly for each $\overline{D}(0,r) = \{z : |z| \leq r\}$ (as the estimates above are uniform on such z) for any $0 \leq r < 1$) is that $\prod_n |a_n| < \infty$, or equivalently, $\sum_n (1-|a_n|) < \infty$ (due to proposition 8.4.3)

If this is the case, then $B \in H^\infty$ with $|B(z)| < 1$ and B vanishes precisely at the points $z = a_n$.

Proof. Necessity is immediate; simply substitute $z = 0$. Conversely, assume that

$$\sum_{n=1}^{\infty} (1-|a_n|) < \infty, \tag{8.9}$$

which implies $\lim_{n\to\infty}(1-|a_n|) = 0 \implies \lim_{n\to\infty} |a_n| = 1$. Thus, there exists some integer N such that $|a_n| \geq \frac{1}{2}$, or $1/|a_n| \leq 2$, for $n \geq N$. Then for $z \in \overline{D}(0,r)$ $(0 \leq r < 1)$, we have

$$\begin{aligned}
\left| 1 - \overbrace{\frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1-\bar{a}_n z}}^{=b_n(z)} \right| &= \left| \frac{(a_n + |a_n|z)(1-|a_n|)}{a_n(1-\bar{a}_n z)} \right| \\
&\leq \frac{1+r}{1-r} \frac{|1-|a_n||}{|a_n|} < \frac{2}{1-r} \frac{1-|a_n|}{|a_n|} \\
&\leq \frac{4}{1-r}(1-|a_n|) = c_n \qquad \text{for all } n \geq N.
\end{aligned}$$

and our assumption (8.9) gives that $\sum_N^\infty c_n < \infty$. Thus, Theorem 8.4.10 concludes that the product

$$\prod_{n=1}^{\infty} b_n(z) = \prod_{n=1}^{N-1} b_n(z) \prod_{n=N}^{\infty} b_n(z)$$

converges uniformly to a holomorphic function $B(z)$ for $z \in \overline{D}(0,r)$ $(0 \leq r < 1)$ and has exactly zeros $a_1, a_2, \cdots$ since each Blaschke factor ψ_{a_n} has precisely one zero a_n.

Since Blaschke factors $\psi_{a_n}(z)$ are automorphisms on $\mathbb{D}$, we have $\forall z \in \mathbb{D}$, $|\psi_{a_n}(z)| < 1$ and

$$\sup_{z\in\mathbb{D}} |P_K(z)| = \sup_{z\in\mathbb{D}} \left| \prod_{n=1}^{K} \psi_{a_n}(z) \frac{|a_n|}{a_n} \right| = \sup_{z\in\mathbb{D}} \prod_{n=1}^{K} |\psi_{a_n}(z)| \frac{|a_n|}{|a_n|} < 1.$$

Letting $K \to \infty$ shows $|B(z)| < 1$ and hence $B(z) \in H^\infty = H^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) \mid \|f\|_\infty < \infty\}$. □

Chapter 9

Hardy Space on $\mathbb{D}$

9.1 Poisson Integrals Revisited

We recall that given $z = re^{i\theta_0} \in \mathbb{D}$ (so $0 \le r < 1$), the **Poisson kernel for unit disk** $\mathbb{D}$ is defined for $\theta \in \mathbb{R}$,

$$P_z(\theta) = \sum_{m\in\mathbb{Z}} r^{|m|} e^{im(\theta_0-\theta)} = \frac{1-r^2}{1-2r\cos(\theta_0-\theta)+r^2} = \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right).$$

When $z = 0$, i.e., $\theta_0 = 0$, we often write $P_z(\theta)$ as $P_r(\theta)$. Note that $P_z(\theta) = P_r(\theta_0 - \theta)$. For fixed $e^{i\theta}$, $P_z(\theta)$ as the real part of an analytic function is a harmonic function of $z \in \mathbb{D}$. Hence the function defined by

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_z(\theta) f(\theta)\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta_0-\theta) f(\theta)\mathrm{d}\theta = (P_r * f)(\theta_0) \tag{9.1}$$

is harmonic on $\mathbb{D}$ whenever $f(\theta) \in L^1(\partial\mathbb{D})$ (notice that $C(\partial\mathbb{D}) \subseteq L^1(\partial\mathbb{D})$). We have shown that the function H_f which is u on $\mathbb{D}$ and f on $\partial\mathbb{D}$ uniquely solves $(\mathrm{Dir})_f$ for $f \in C(\partial\mathbb{D})$. Since $P_z(\theta)$ is also a continuous function of θ, we get a harmonic function from above equation if we replace $f(\theta)\mathrm{d}\theta$ by a finite measure $d\mu(\theta)$ on $\partial\mathbb{D}$.

We have seen in proposition 7.2.5 that for $g \in C([-\pi,\pi])$ we have $G(\theta) = g(e^{i\theta}) \in C([-\pi,\pi])$ and uniformly,

$$\lim_{r\to 1^-} |(P_r * G)(\theta) - G(\theta)| = 0$$

The uniformity gives

$$\lim_{r\to 1^-} \|(P_r * G)(\theta) - G(\theta)\|_{L^\infty(\partial\mathbb{D})} = 0$$

We now replace L^∞ norm by L^p norm with $1 \le p < \infty$ and relax the class of g.

Proposition 9.1.1. *Let $1 \le p \le \infty$, $f \in L^p(\partial\mathbb{D})$, and define the function on unit circle*

$$\begin{aligned} u_r &: \partial\mathbb{D} \to \mathbb{C} \\ e^{i\theta} &\mapsto \mathcal{P}[f](\theta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) dt. \end{aligned}$$

Then

$$\|u_r\|_{L^p(\partial\mathbb{D})} \le \|f\|_{L^p(\partial\mathbb{D})}.$$

If $1 \le p < \infty$, then

$$\lim_{r\to 1^-} \|u_r - f\|_{L^p(\partial\mathbb{D})} = 0$$

(recall $f \in C(\partial\mathbb{D})$ is required for $p = \infty$ case.)

Proof. Using the notation $⨍_{-\pi}^{\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi}$ and freely employing the properties of the Poisson kernel, we note that

$$\begin{aligned}|u_r(e^{i\theta})| &\leq ⨍_{-\pi}^{\pi} |f(e^{it})|P_r(\theta-t)\mathrm{d}t \\ &= ⨍_{-\pi}^{\pi} |f(e^{it})|P_r(\theta-t)^{1/p}P_r(\theta-t)^{1/p'}\mathrm{d}t \\ &\overset{\text{Hölder}}{\leq} \left(⨍_{-\pi}^{\pi} |f(e^{it})|^p P_r(\theta-t)\mathrm{d}t\right)^{1/p}\left(\underbrace{⨍_{-\pi}^{\pi} P_r(\theta-t)\mathrm{d}t}_{=1}\right)^{1/p'}\end{aligned}$$

Thus,

$$|u_r(e^{i\theta})|^p \leq ⨍_{-\pi}^{\pi} |f(e^{it})|^p P_r(\theta-t)\mathrm{d}t$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} |u_r(e^{i\theta})|^p\mathrm{d}\theta &\leq \int_{-\pi}^{\pi}⨍_{-\pi}^{\pi} |f(e^{it})|^p P_r(\theta-t)\mathrm{d}t\mathrm{d}\theta \\ &\overset{\text{Fubini}}{=\!=} ⨍_{-\pi}^{\pi} |f(e^{it})|^p \int_{-\pi}^{\pi} P_r(\theta-t)\mathrm{d}\theta\mathrm{d}t \\ &= 2\pi⨍_{-\pi}^{\pi} |f(e^{it})|^p\mathrm{d}t = \int_{-\pi}^{\pi} |f(e^{it})|^p\mathrm{d}t\end{aligned}$$

This implies that $\|u_r\|_{L^p(\partial\mathbb{D})} \leq \|f\|_{L^p(\partial\mathbb{D})}$ (the $p=\infty$ case is trivial.) We now show the second statement:

Since $C(\partial\mathbb{D})$ is dense in $L^p(\partial\mathbb{D})$ for $1 \leq p < \infty$ (see e.g. [8] p.245.), we can find $g \in C(\partial\mathbb{D})$ s.t. $\|f - g\|_{L^p(\partial\mathbb{D})} < \varepsilon$. Let $v := \mathcal{P}[g](\theta)$. Then

$$\|u_r - f\|_{L^p(\partial\mathbb{D})} \leq \|u_r - v_r\|_{L^p(\partial\mathbb{D})} + \|v_r - g\|_{L^p(\partial\mathbb{D})} + \|g - f\|_{L^p(\partial\mathbb{D})}$$

where $\|u_r - v_r\|_{L^p(\partial\mathbb{D})} = \|(u-v)_r\|_{L^p(\partial\mathbb{D})} \leq \|f - g\|_{L^p(\partial\mathbb{D})} < \varepsilon$ by the first part of the theorem. so

$$\|u_r - f\|_{L^p(\partial\mathbb{D})} < 2\varepsilon + \|v_r - g\|_{L^p(\partial\mathbb{D})}.$$

Now,

$$\|v_r - g\|^p_{L^p(\partial\mathbb{D})} = \int_{-\pi}^{\pi} \underbrace{|v_r(e^{i\theta}) - g(e^{i\theta})|^p}_{\leq \|v_r - g\|^p_{L^p(\partial\mathbb{D})}}\mathrm{d}\theta \leq 2\pi\|v_r - g\|^p_{L^p(\partial\mathbb{D})}$$

implying that $\|v_r - g\|_{L^p(\partial\mathbb{D})} < \varepsilon$ for all $r \to 1^-$, as $v = \mathcal{P}[g]$ and $g \in C(\partial\mathbb{D})$. $\square$

The more difficult question is the pointwise convergence $u_r(e^{i\theta}) \xrightarrow{r\to 1^-} f(e^{i\theta})$ for $u = \mathcal{P}[f]$ with mere assumption $f \in L^1(\partial\mathbb{D})$ (notice that $L^p(\partial\mathbb{D}) \subset L^1(\partial\mathbb{D})$ by Hölder's inequality if $p > 1$; be careful to distinguish the three forms of convergence, convergence in metric space L^p, pointwise convergence, and uniform convergence (this last one is equivalent to convergence in L^∞))

9.2 Approach Regions

Let $0 < \alpha < 1$. We first defne a non-tangential approach region related to $z = 1$. Let

$$\Omega_\alpha = B(0,\alpha) \cup \bigcup_{z\in B(0,\alpha)} [z, 1].$$

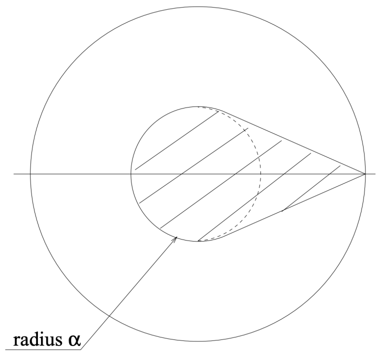


Figure 9.1: Region Ω_α

This is the smallest convex set containing $B(0,\alpha)$ and having $z=1$ in its boundary. Near $z=1$, Ω_α is a sector bisected by the radius of $\mathbb{D}$ terminating at 1. Curves that approach 1 within Ω_α cannot be tangent to $\partial\mathbb{D}$, and so Ω_α is called a **non-tangential approach region with vertex** 1. Rotated copies with vertex at $e^{it}\in\partial\mathbb{D}$ are denoted by $e^{it}\Omega_\alpha$.

Definition 9.2.1. *A function F, defined in $\mathbb{D}$, is said to have a **non-tangential limit** λ at $e^{i\theta}\in\partial\mathbb{D}$ if, for each $0<\alpha<1$,*

$$\lim_{j\to\infty} F(z_j)=\lambda$$

for all sequences $\{z_j\}\in e^{i\theta}\Omega_\alpha$ with $\{z_j\}\to e^{i\theta}$.

We want to show the following result.

Theorem 9.2.2. *If $F=\mathcal{P}[f]$ for $f\in L^1(\partial\mathbb{D})$, then F has non-tangential limit $f(e^{i\theta})$ at every Lebesgue point $e^{i\theta}$ of f.*

Recall that for a locally Lebesgue integrable function f on $\mathbb{R}^d$, a point x in the domain of f is a **Lebesgue point** if

$$\lim_{r\to 0^+}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|\mathrm{d}y=0.$$

where μ is the Lebesgue measure. The Lebesgue points of f are thus points where f does not oscillate too much, in an average sense. The **Lebesgue differentiation theorem** states that, given any $f\in L^1_{\text{Loc}}\left(\mathbb{R}^d\right)$ (i.e., has finite L^p norm on every compact subset K of $\mathbb{R}^d$), almost every x is a Lebesgue point of f. In our case, it can also be written as

$$\lim_{I\to e^{i\theta}}\frac{1}{\sigma(I)}\int_I|f(y)-f(e^{i\theta})|\mathrm{d}\sigma(y)=0$$

where $I\to e^{i\theta}$ stands for the open arcs centered at $e^{i\theta}$ shrinking to $e^{i\theta}$ and σ is the normalized surface measure on $\partial\mathbb{D}$. That is, we essentially have $\mathrm{d}t/2\pi$ on $[-\pi,\pi]$ if we think of f as a function on $[-\pi,\pi]$ via $\theta\mapsto f(e^{i\theta})$. So $\sigma(\partial\mathbb{D})=1$, $\sigma(I)=$length of the arc I.

If f is continuous, trivially every point is a Lebesgue point:

$$\frac{1}{\sigma(I)}\int_I \overbrace{|f-f(e^{i\theta})|}^{<\varepsilon \text{ on } I \text{ for sufficiently small } I} <\varepsilon\frac{\sigma(I)}{\sigma(I)}=\varepsilon.$$

Lebesgue differentiation theorem says that $\sigma\{e^{i\theta}\in\partial\mathbb{D}\mid e^{i\theta}\text{ is not a Lebesgue point of }f\}=0$. Thus, if we show that $f\in L^1(\partial\mathbb{D})\implies F=\mathcal{P}[f]$ has non-tangential limit $f(e^{i\theta})$ at each Leb pt., then it has non-tangential limit $f(e^{i\theta})$ at a.e. $e^{i\theta}\in\partial\mathbb{D}$.

proof of Theorem 9.2.2. Fix a Lebesgue point $e^{i\theta} \in \partial\mathbb{D}$ of f. Let $g(z) := f(z) - f(e^{i\theta})$. Then by definition of Lebesgue point, with our notation of limit for $I \to e^{i\theta}$, we have

$$\lim_{I \to e^{i\theta}} \frac{1}{\sigma(I)} \int_I |g| \mathrm{d}\sigma = 0.$$

Pick $\varepsilon > 0$. Thus, we can pick a small enough arc I_0 centered at $e^{i\theta}$ such that

$$\frac{1}{\sigma(I)} \int_I |g| \mathrm{d}\sigma < \varepsilon \tag{9.2}$$

for all arcs $I \subseteq I_0$ centered at $e^{i\theta}$.

We decompose $g = g_0 + g_1$, where $g_0 := g|_{I_0}$ and $g_1 := g - g_0 = g|_{\partial\mathbb{D} \setminus I_0}$. Let $F_0 := \mathcal{P}[g_0]$ and $F_1 := \mathcal{P}[g_1]$. We want to show $\mathcal{P}[g](e^{i\theta}) \xrightarrow{\text{NT}} 0$ for the Lebesgue point $e^{i\theta}$ (NT means "nontangentially"; the notation reads as function $\mathcal{P}[g]$ has nontangential limit 0 at point $e^{i\theta}$), because this gives $\mathcal{P}[f](e^{i\theta}) \xrightarrow{\text{NT}} f(e^{i\theta})$. To do so, we will first show $F_1 \xrightarrow{\text{NT}} 0$.

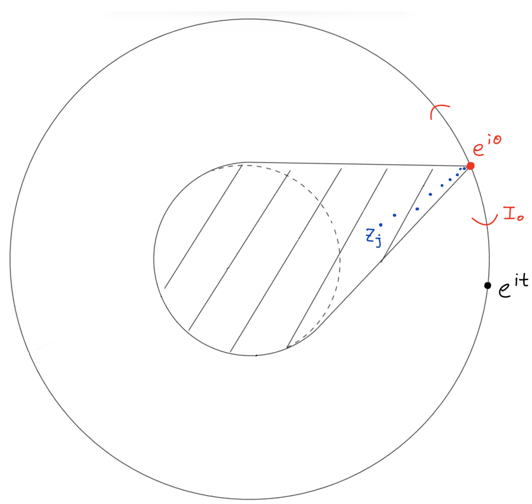
Fix $\{z_j\}$ converging to $e^{i\theta}$, where $z_j \in e^{i\theta}\Omega_\alpha$ for some fixed $0 < \alpha < 1$. We recall from (7.8) and (7.9) that We can write $\mathcal{P}[h]$ for a function $h \in C(\partial\mathbb{D})$ as

$$\begin{aligned} \forall z = re^{i\theta} \in \mathbb{D}: \qquad \mathcal{P}[h](z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \mathrm{d}t \\ &= \int_{\partial\mathbb{D}} h(e^{it}) P(z, e^{it}) \mathrm{d}\sigma(e^{it}). \end{aligned}$$

where $P(z, e^{it}) = P_r(\theta - t) = (1 - |z|^2)/|e^{it} - z|^2$ for $z = re^{i\theta}$ (the Poisson kernel can be seen as a function of r, θ, and t) and recall that σ is the normalized surface measure on $\partial\mathbb{D}$. Now

$$F_1(z_j) = \int_{\partial\mathbb{D} \setminus I_0} P(z_j, e^{it}) g(e^{it}) \mathrm{d}\sigma(e^{it}) = \int_{\partial\mathbb{D} \setminus I_0} \frac{1 - |z_j|^2}{|e^{it} - z_j|^2} g(e^{it}) \mathrm{d}\sigma(e^{it})$$

In this integral, z_j stays away a positive distance from each $e^{it} \in \partial\mathbb{D} \setminus I_0$ (convince yourself pictorially), so $|e^{it} - z_j| \gtrsim 1$ (uniformly on $e^{it} \in \partial\mathbb{D} \setminus I_0$ and z_j).



Thus,

$$\begin{aligned} &\left|\frac{1-|z_j|^2}{|e^{it}-z_j|^2}g(e^{it})\right| \lesssim (1-|z_j|)|g(e^{it})| \\ \implies &|F_1(z_j)| \lesssim (1-|z_j|)\underbrace{\int |g|\mathrm{d}\sigma}_{<\infty} \xrightarrow{j\to\infty} 0 \\ \implies &F_1(z_j) \xrightarrow{j\to\infty} 0. \end{aligned}$$

We will show $|F_0(z_j)| \lesssim_\alpha \varepsilon,\ \forall j$. This implies $|\mathcal{P}[g](z_j)| \le |F_0(z_j)| + |F_1(z_j)| \le C_\alpha\varepsilon + \varepsilon$ for all large j. Since ε is arbitrary, this implies $\mathcal{P}[g](z_j) \xrightarrow{j\to\infty} 0$. But

$$\mathcal{P}[g](z_j) = \mathcal{P}[f - f(e^{i\theta})](z_j) = \mathcal{P}[f](z_j) - f(e^{i\theta})$$

implies that $\mathcal{P}[f](z_j) \xrightarrow{j\to\infty} f(e^{i\theta})$, as desired.

However, the proof of the fact $|F_0(z_j)| \lesssim_\alpha \varepsilon,\ \forall j$ needs us to introduce new tools utilizing the nontangential approach of $\{z_j\}$ to $e^{i\theta}$, i.e., $z_j \in e^{i\theta}\Omega_\alpha,\ \forall j$. The idea is as follows: recall the **the Hardy-Littlewood maximal function** of g_0 is

$$Mg_0(e^{it}) := \sup_{I \text{ arc centered at } e^{it}} \frac{1}{\sigma(I)}\int_I |g_0|\mathrm{d}\sigma$$

where $\partial\mathbb{D}$ is understood as an arc here as well. We want to use the fact that $Mg_0(e^{i\theta}) \le \varepsilon$, since

$$\int_I |g_0|\mathrm{d}\sigma = \int_{I\cap I_0} |g|\mathrm{d}\sigma \overset{(9.2)}{<} \varepsilon\sigma(I\cap I_0) \le \varepsilon\sigma(I)$$

for all arcs $I \subseteq \partial\mathbb{D}$ centered at $e^{i\theta}$.

We also define the **nontangential maximal function** $N_\alpha h$ for any function h in $\mathbb{D}$ by setting

$$N_\alpha h(e^{it}) = \sup\left\{|h(z)| : z \in e^{it}\Omega_\alpha\right\}$$

for all $e^{it} \in \partial\mathbb{D}$ (so h defined in $\mathbb{D}$ but $N_\alpha h$ in $\partial\mathbb{D}$.) Then $\forall z \in e^{i\theta}\Omega_\alpha$, $|F_0(z)| \le N_\alpha F_0\left(e^{i\theta}\right)$. In particular, $|F_0(z_j)| \le N_\alpha F_0\left(e^{i\theta}\right)$. Now if we assume the claim $N_\alpha\left(\mathcal{P}[g_0]\right)(e^{i\theta}) \lesssim_\alpha Mg_0(e^{i\theta})$, then

$$\begin{aligned} &N_\alpha F_0(e^{i\theta}) = N_\alpha\left(\mathcal{P}[g_0]\right)(e^{i\theta}) \lesssim_\alpha Mg_0(e^{i\theta}) \le \varepsilon \\ \implies &\forall j,\ |F_0(z_j)| \le C_\alpha\varepsilon \end{aligned}$$

ending the proof. We shall prove the strengthened form of the claim we assumed in theorem 9.3.2. □

9.3 Maximal Functions

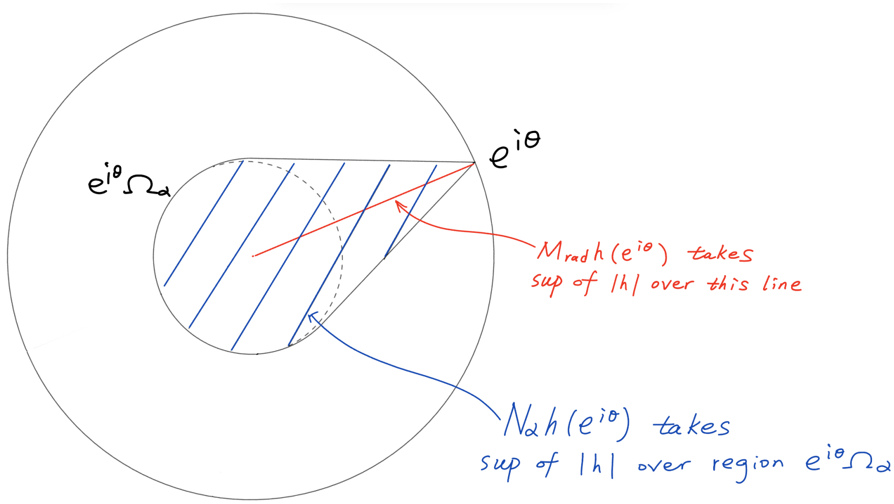
We first introduce a new variant of maximal function.

Definition 9.3.1. *We define the* ***radial maximal function*** *as*

$$M_{\mathrm{rad}}h(e^{i\theta}) = \sup\{|h(re^{i\theta})| : 0 \le r < 1\}$$

for a function h on $\mathbb{D}$.

We will then prove the promised estimates of maximal functions.



Theorem 9.3.2. *Assume* $0 < \alpha < 1$. *Then there is* $C_\alpha > 0$ *with the following property: if* $f \in L^1(\partial\mathbb{D})$, $f \geq 0$, $F := \mathcal{P}[f]$, *then*

$$\forall e^{i\theta} \in \partial\mathbb{D}, \quad C_\alpha N_\alpha F(e^{i\theta}) \leq M_{\text{rad}}F(e^{i\theta}) \leq Mf(e^{i\theta}),$$

Remark 9.3.3. *The set over which* $M_{\text{rad}}h(e^{i\theta})$ *takes the sup of* $|h|$ *is much smaller than that for* $N_\alpha h(e^{i\theta})$, *as shown in the figure.*

Therefore, we always have

$$N_\alpha h(e^{i\theta}) \geq M_{\text{rad}}h(e^{i\theta}).$$

The non-trivial direction $N_\alpha h(e^{i\theta}) \lesssim_\alpha M_{\text{rad}}h(e^{i\theta})$ *works for functions* h *of the special form* $h = \mathcal{P}[f]$, $f \in L^1(\partial\mathbb{D})$. *Also the second inequality connecting* $M_{\text{rad}}(\mathcal{P}[f])$ *to* Mf *needs some work as well.*

proof of the theorem. We will prove $\exists C_\alpha$ such that $C_\alpha N_\alpha F(1) \leq M_{\text{rad}}F(1) \leq Mf(1)$; this is the case $\theta = 0$. The general case follows by a rotation.

1. $N_\alpha F(1) \lesssim M_{\text{rad}}F(1)$.

We compute that

$$\begin{aligned}
N_\alpha F(1) &= \sup_{z\in\Omega_\alpha} \left| \int_{\partial\mathbb{D}} \underbrace{P(z, e^{it})}_{\geq 0} \underbrace{f(e^{it})}_{\geq 0} \mathrm{d}\sigma(e^{it}) \right| \\
&= \sup_{z\in\Omega_\alpha} \int_{\partial\mathbb{D}} \underbrace{P(z, e^{it})}_{\text{will prove: } \lesssim_\alpha P(|z|, e^{it}),\ \forall z \in \Omega_\alpha, \forall e^{it} \in \partial\mathbb{D}} f(e^{it}) \mathrm{d}\sigma(e^{it}) \\
&\lesssim \sup_{z\in\Omega_\alpha} \int_{\partial\mathbb{D}} P(|z|, e^{it}) f(e^{it}) \mathrm{d}\sigma(e^{it}) \\
&= \sup_{0\leq r<1} \underbrace{\int_{\partial\mathbb{D}} P(r, e^{it}) f(e^{it}) \mathrm{d}\sigma(e^{it})}_{=\mathcal{P}[f](r)} \\
&= M_{\text{rad}}F(1),
\end{aligned}$$

provided we prove the pointwise estimate that for some C_α

$$C_\alpha P(z,e^{it}) \le P(|z|,e^{it}), \quad \forall z\in\Omega_\alpha, \forall e^{it}\in\partial\mathbb{D}.$$

As

$$P\left(z,e^{it}\right)=\frac{1-|z|^2}{\left|e^{it}-z\right|^2}$$
$$P\left(|z|,e^{it}\right)=\frac{1-|z|^2}{\left|e^{it}-|z|\right|^2}$$

we must prove for $r:=|z|$,

$$(*): \qquad C_\alpha\left|e^{it}-r\right|\le\left|e^{it}-z\right|$$

Now, we observe a geometric feature of Ω_α that

$$\frac{|z-r|}{1-r}=\frac{|z-|z||}{1-|z|}\lesssim_\alpha 1, \quad \forall z\in\Omega_\alpha$$

Thus,

$$\begin{aligned}\left|e^{it}-r\right| &\le \left|e^{it}-z\right|+|z-r|\\ &\le \underbrace{\left|e^{it}-z\right|}_{\ge|e^{it}|-|z|=1-r} +C_\alpha(1-r)\\ &\le(1+C_\alpha)\left|e^{it}-z\right|\end{aligned}$$

which shows $(*)$. We thus proved $N_\alpha F(1)\lesssim_\alpha M_{\mathrm{rad}}F(1)$.

2. $M_{\mathrm{rad}}F(1)\le Mf(1)$.

This requires us to show

$$\int_{\partial\mathbb{D}}P(r,e^{it})f(e^{it})\mathrm{d}\sigma(e^{it})\le \sup_{I\text{ arcs centered at }1}\fint_I f\mathrm{d}\sigma, \quad \forall 0\le r<1,$$

Recall $P(r,e^{it})=\frac{1-r^2}{|e^{it}-r|^2}$. We fix r and choose open arcs $I_1\subset I_2\subset\cdots\subset I_{n-1}$ centered at 1 and set $I_n=\partial\mathbb{D}$ (we will eventually choose thhis s.t. the end points of the arcs form a fine partition of $\partial\mathbb{D}$.) Notice that $|e^{it}-r|$ becomes larger as $e^{it}\in\partial\mathbb{D}$ moves away from $1\in\partial\mathbb{D}$. So the bigger the arc I_j, the bigger $|e^{it}-r|$ can be as $e^{it}\in I_j$; so $P(r,e^{it})$ obtains smaller values on the bigger I_j ($t\mapsto P(r,e^{it})$ decreases as t ranges from 0 to π.) Define $\lambda_j:=\inf_{e^{it}\in I_j}P(r,e^{it})>0$ (recall r is fixed). Then $\lambda_1>\lambda_2>\cdots>\lambda_n$ and $P(r,e^{it})\ge\lambda_j\mathbb{1}_{I_j}$, $\forall e^{it}\in\partial\mathbb{D}$. We will approximate $P(r,e^{it})$ with the step function

$$K_r=K:=\sum_{j=1}^n(\lambda_j-\lambda_{j+1})\mathbb{1}_{I_j}.$$

Notice that if $e^{it}\in I_j\setminus I_{j-1}$ with $I_0=\varnothing$ we have

$$\begin{aligned}K(e^{it}) &= \sum_{i=1}^n(\lambda_i-\lambda_{i+1})\mathbb{1}_{I_i}(e^{it})\\ &= \sum_{i=j}^n(\lambda_i-\lambda_{i+1})\\ &\text{bc. } e^{it}\in I_j\setminus I_{j-1}\implies e^{it}\notin I_i\ \forall i\le j-1 \text{ and } e^{it}\in I_i\ \forall i\ge j\\ &= (\lambda_j-\lambda_{j+1})+(\lambda_{j+1}-\lambda_{j+2})+\cdots+(\lambda_{n-1}-\lambda_n)+(\lambda_n-\lambda_{n+1})\\ &= \lambda_j-\underbrace{\lambda_{n+1}}_{=0\text{ by defn.}}\\ &= \lambda_j\end{aligned}$$

Since $\forall e^{it} \in \partial\mathbb{D},\ P(r,e^{it}) \geq \lambda_j \mathbb{1}_{I_j}$ and we just showed $\forall e^{it} \in I_j \setminus I_{j-1},\ K(e^{it}) = \lambda_j$, we see $P\left(r, e^{it}\right) \geq K\left(e^{it}\right)$ for $e^{it} \in \partial\mathbb{D}$. Clearly, the fact that on $I_j \backslash I_{j-1}$ we have

$$K\left(e^{it}\right) = \lambda_j = \inf_{I_j} P\left(r, e^{it}\right) = \inf_{I_j \backslash I_{j-1}} P\left(r, e^{it}\right)$$

means that $K = K_r \to P(r,\cdot)$ uniformly on $\mathbb{D}$ if we make the partition $I_j \backslash I_{j-1}$ finer and finer (the last equality comes from the fact that $P(r,e^{it})$ is lower bounded by a larger value λ_{j-1} on I_{j-1}, compared to λ_j.) So we essentially now estimate

$$\begin{aligned}
&\int_{\partial\mathbb{D}} K\left(e^{it}\right) f\left(e^{it}\right) \mathrm{d}\sigma\left(e^{it}\right) \\
=&\sum_{j=1}^{n} \left(\lambda_j - \lambda_{j+1}\right) \underbrace{\int_{I_j} f\left(e^{it}\right) \mathrm{d}\sigma\left(e^{it}\right)}_{\leq \sigma(I_j) Mf(1)} \\
\leq& Mf(1) \sum_{j=1}^{n} \left(\lambda_j - \lambda_{j+1}\right) \sigma\left(I_j\right) \\
=& Mf(1) \int_{\partial D} \left(\sum_{j=1}^{n} \left(\lambda_j - \lambda_{j+1}\right) \mathbb{1}_{I_j}\left(e^{it}\right)\right) \mathrm{d}\sigma\left(e^{it}\right) \\
=& Mf(1) \int_{\partial\mathbb{D}} \underbrace{K\left(e^{it}\right)}_{\leq P(r,e^{it})} \mathrm{d}\sigma\left(e^{it}\right) \\
\leq& Mf(1) \qquad \underbrace{\int_{\partial D} P\left(r, e^{it}\right) d\sigma\left(e^{it}\right)}_{=1 \text{ as Poisson kernel has } \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\mathrm{d}\theta = 1,\ \forall r<1} \\
=& Mf(1).
\end{aligned}$$

The limit $K = K_r \to P(r,\cdot)$ gives

$$\int_{\partial\mathbb{D}} P\left(r, e^{it}\right) f\left(e^{it}\right) \mathrm{d}\sigma\left(e^{it}\right) \leq Mf(1), \qquad \forall 0 \leq r < 1,$$

and so $M_{\mathrm{rad}} F(1) \leq Mf(1)$. □

Recall that we proved in proposition 9.1.1 that for $F = \mathcal{P}[f]$, one has $\|F_r\|_{L^p(\partial\mathbb{D})} \leq \|f\|_{L^p(\partial\mathbb{D})}$, $1 \leq p \leq \infty$, $0 \leq r < 1$. The above gives a stronger result: if $1 < p \leq \infty$ (we emphasize that $p \neq 1$), then

$$\|N_\alpha F\|_{L^p(\partial\mathbb{D})} \lesssim_\alpha \|Mf\|_{L^p(\partial\mathbb{D})} \lesssim \|f\|_{L^p(\partial\mathbb{D})} \tag{9.3}$$

using the fundamental L^p-to-L^p, $1 < p \leq \infty$, estimate of M:

Theorem 9.3.4. *We have that $M : L^1\left(\mathbb{R}^d\right) \to L^{1,\infty}\left(\mathbb{R}^d\right)$ boundedly - i.e.,*

$$\|Mf\|_{L^{1,\infty}} \lesssim \|f\|_1.$$

Besides, for all $1 < p \leq \infty$ and $f \in L^p$ we have

$$\|Mf\|_p \lesssim \|f\|_p.$$

Proof. See Henri's Math5051 note. □

Notice that (9.3) is a much improved L^p estimate. Indeed,

$$\left|F_r\left(e^{it}\right)\right| = \left|F\left(re^{it}\right)\right| \le M_{\mathrm{rad}}F\left(e^{it}\right) \le N_\alpha F\left(e^{it}\right).$$

Remark 9.3.5. *For $p=1$ there is a $L^1 \to L^{1,\infty}$ type estimate for $N_\alpha F$*

$$\sigma\left(N_\alpha F > \lambda\right) \le \sigma\left(Mf > C_\alpha \lambda\right) \lesssim_\alpha \frac{1}{\lambda}\int |f| d\sigma, \quad \forall \lambda > 0.$$

Recall that given $f \in L^p(\partial\mathbb{D})$ the function $\mathcal{P}[f]$ is harmonic in $\mathbb{D}$. How about the converse: given a harmonic function F in $\mathbb{D}$, how to tell if it is the case $F = \mathcal{P}[f]$ for some $f \in L^p(\partial\mathbb{D})$ (we call this existence of boundary values)? Recall that if it is, then $\|F_r\|_{L^p(\partial\mathbb{D})} \le \|f\|_{L^P(\partial\mathbb{D})}$ so $\sup_{0\le r<1}\|F_r\|_{L^p(\partial\mathbb{D})} < \infty$. The following theorem claims that the converse is also true.

Theorem 9.3.6. *Suppose F is harmonic in $\mathbb{D}$, $1 < p \le \infty$. If $\sup_{0\le r<1}\|F_r\|_{L^p(\partial\mathbb{D})} < \infty$, then $\exists! f \in L^p(\partial\mathbb{D})$ s.t. $F = \mathcal{P}[f]$.*

Remark 9.3.7. *For $p=1$ there is a version where $F = \mathcal{P}[d\mu]$ for a unique "complex Borel measure" μ on $\partial\mathbb{D}$. See [14] p.239-p.245.*

To get a taste of the $p=\infty$ case on what can be said about existence of boundary values, we prove the following result.

Theorem 9.3.8. *To every $f \in H^\infty$ there corresponds a function $f^* \in L^\infty(\partial\mathbb{D})$ defined at every Lebesgue point of f by*

$$f^*(e^{i\theta}) = \lim_{r\to 1} f_r(e^{i\theta}).$$

The equality $\|f^*\|_{L^\infty(\partial\mathbb{D})} = \|f\|_\infty := \|f\|_{L^\infty(\mathbb{D})}$ holds. Moreover, if $f^*(e^{i\theta}) = 0$ a.e. on some arc $I \subseteq \partial\mathbb{D}$, then $f \equiv 0$ in $\mathbb{D}$.

Proof. As $f \in H^\infty$, we have f harmonic as analytic, and $\sup_{0\le r<1}\|f_r\|_{L^\infty(\partial\mathbb{D})} = \|f_r\|_{\mathbb{D}} < \infty$, so by previous theorem, there is a unique function g in $L^\infty(\partial\mathbb{D})$ such that $f = \mathcal{P}[g]$. Then if we define $f^* = g$, Theorem 9.2.2 and Lebesgue differentiation theorem imply that $f = \mathcal{P}[g]$ has nontangential limit $g(e^{i\theta}) = f^*(e^{i\theta})$ at a.e. $e^{i\theta} \in \partial\mathbb{D}$. To prove the first claim $\|f^*\|_{L^\infty(\partial\mathbb{D})} = \|f\|_\infty$, notice that $|f^*| = \lim_{r\to1}|f_r| \le \lim_{r\to1}\|f\|_\infty = \|f\|_\infty$. Thus, $\|f^*\|_{L^\infty(\partial D)} \le \|f\|_\infty$. But we also know

$$\begin{aligned}\|f\|_\infty = \sup_{0\le r<1}\|f_r\|_{L^\infty(\partial\mathbb{D})} = \sup_{0\le r<1} & \underbrace{\|(\mathcal{P}[g])_r\|_{L^\infty(\partial\mathbb{D})}}_{\le\|g\|_{L^\infty(\partial\mathbb{D})}=\|f^*\|_{L^\infty(\partial\mathbb{D})},\ \forall r} \\ \le \|f^*\|_{L^\infty(\partial\mathbb{D})} \quad & (\text{recall } \|(\mathcal{P}[h])_r\|_{L^p(\partial\mathbb{D})} \le \|h\|_{L^p(\partial\mathbb{D})},\ 1 \le p \le \infty)\end{aligned}$$

Thus, $\|f\|_\infty = \|f^*\|_{L^\infty(\partial\mathbb{D})}$.

For the second claim, we note that if $f^*(e^{i\theta}) = 0$ a.e. on the whole circle $\partial\mathbb{D}$, then $\|f\|_\infty = \|f^*\|_{L^\infty(\partial\mathbb{D})} = 0$, so $f \equiv 0$ (see [8] p.236 defn. of L^∞ norm and Problem 12; continuous function has its infinity norm equivalent to the one defined supremum but the original one applied to f^* is defined by least a.e. upper bound.) We can reduce the case I being a general arc on which f^* vanishes a.e. to this $\partial\mathbb{D}$ case by a rotation trick. Let n be so large that $\ell(I) := \mathrm{length}(I) > 2\pi/n$. Define the following modified version F of f by setting for $z \in \mathbb{D}$ that

$$F(z) := \prod_{k=1}^{n} f\left(\alpha^k z\right), \quad \alpha := e^{2\pi i/n}.$$

Clearly this is a bounded analytic function in $\mathbb{D}$ so that $F \in H^\infty$. In particular, F^∞ exists and we have

$$F^*\left(e^{i\theta}\right) = \lim_{r\to1}\prod_{k=1}^{n} f\left(\alpha^k re^{i\theta}\right) = \prod_{k=1}^{n} f^*\left(\alpha^k e^{i\theta}\right) = 0,$$

since it must be the case that for some $k \in \{1, \ldots, n\}$ we have $\alpha^k e^{i\theta} \in I$. This is because $\ell(I) > 2\pi/n$ and multiplying by $\alpha = e^{2\pi i/n}$ rotates by $2\pi/n$. Thus, using the known special case $I = \partial\mathbb{D}$ to F gives that $F = 0$ in $\mathbb{D}$. It remains to conclude that this implies that $f = 0$ in $\mathbb{D}$. If f would not vanish identically, then the zero set of f must be at most countable. But clearly then also the zero set of F would be at most countable (it is obtained as a union of n sets obtained from the zero set of f by rotations) - but the zero set of F is $\mathbb{D}$. So f must vanish identically. □

9.4 Hardy Space on Disc

Recall if $f \in H(\mathbb{D})$ we have used the notation $f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$ to have a family of functions $f_r, 0 \leq r < 1$, defined on the boundary $\partial\mathbb{D}$. As previously, we use the measure σ (normalized surface measure on $\partial\mathbb{D} \sim \mathrm{d}t/2\pi$ on $[0, 2\pi]$) on $\partial\mathbb{D}$, and define

$$\begin{aligned} \|f_r\|_p &:= \|f_r\|_{L^p(\sigma,\partial\mathbb{D})} \\ &= \begin{cases} \left(\int_{\partial\mathbb{D}} |f_r|^p \, d\sigma\right)^{1/p}, & 0 < p < \infty \\ \sup_\theta \left|f\left(re^{i\theta}\right)\right|, & p = \infty. \end{cases} \end{aligned}$$

In view of the Nevanlinna space N (see Math5022 HW7Q4), we also set

$$\|f_r\|_0 = \exp\left(\int_{\partial\mathbb{D}} \log^+ |f_r| \mathrm{d}\sigma\right),$$

where

$$\begin{aligned} \log^+ s = \log s, &\quad s \geq 1 \\ \log^+ s = 0, &\quad s < 1. \end{aligned}$$

Definition 9.4.1. *If $f \in H(\mathbb{D})$ and $0 \leq p \leq \infty$, we set*

$$\|f\|_p := \sup_{0 \leq r < 1} \|f_r\|_p$$

Then the ***Hardy space*** *for $0 < p \leq \infty$ is*

$$H^p := \{f \in H(\mathbb{D}) : \|f\|_p < \infty\}.$$

(notice that in the case $p = \infty$ this agrees trivially with our previous definition of H^∞.) Also, let

$$N := \{f \in H(\mathbb{D}) : \|f\|_0 < \infty\}$$

Lemma 9.4.2. *We have $H^\infty \subset H^p \subset H^q \subset N$ if $0 < q < p < \infty$. Also, H^p is a complete normed space for $1 \leq p \leq \infty$.*

Proof. Notice first that

$$\left(\int_{\partial\mathbb{D}} |f_r|^p \, \mathrm{d}\sigma\right)^{1/p} \leq \left(\int_{\partial\mathbb{D}} \|f\|_\infty^p \, \mathrm{d}\sigma\right)^{1/p} = \|f\|_\infty \sigma(\partial\mathbb{D})^{1/p} = \|f\|_\infty.$$

This implies $H^\infty \subset H^p$ as (by taking sup over $0 \leq r < 1$) we proved $\|f\|_p \leq \|f\|_\infty$. Next, using Hölder with $s := p/q > 1$ we get

$$\left(\int_{\partial\mathbb{D}} |f_r|^q \, \mathrm{d}\sigma\right)^{1/q} \leq \left(\int_{\partial\mathbb{D}} |f_r|^{sq} \, \mathrm{d}\sigma\right)^{1/qs} \left(\int_{\partial\mathbb{D}} |1|^{s'} \mathrm{d}\sigma\right)^{1/qs'} \leq \|f\|_p$$

using that $sq = p$ and that $\sigma(\partial\mathbb{D}) = 1$. This implies $\|f\|_q \le \|f\|_p$ and so $H^p \subset H^q$. Finally, the last inclusion follows from $\log x \lesssim x^q$ for $x \ge 1$.

We then prove completeness - so fix a Cauchy sequence (f_j) from H^p. This means that given $\varepsilon > 0$ there is N so that $\|f_i - f_j\|_p < \varepsilon$ for $i, j > N$. Let $r < 1$ and consider $|z| \le r$. Choose some $R = R(r) \in (r, 1)$. We use Cauchy's integral formula over the circle $S_R := \partial B(0, R)$ as follows:

$$\begin{aligned}
|f_i(z) - f_j(z)| &= \left|\frac{1}{2\pi i}\int_{S_R} \frac{f_i(z) - f_j(z)}{\xi - z}\,\mathrm{d}\xi\right| \\
&\le \frac{1}{2\pi}\int_{S_R} \frac{|f_i(z) - f_j(z)|}{R - r} |\,\mathrm{d}\xi| \\
&= (R-r)^{-1}\frac{R}{2\pi}\int_0^{2\pi} \left|(f_i - f_j)_R\left(e^{i\theta}\right)\right| \mathrm{d}\theta \lesssim r\,\|f_i - f_j\|_1 \le \|f_i - f_j\|_p\,.
\end{aligned}$$

It follows that $(f_i(z))$ is Cauchy, and by completeness of $\mathbb{C}$ converges to some $f(z)$. By the above quantitative estimate (which is uniform in $|z| \le r$), the convergence in uniform in $\bar{B}(0, r)$ with any $r < 1$, and so $f_i \to f$ uniformly in compact subsets implying that $f \in H(\mathbb{D})$.

It remains to show that $f \in H^p$ and $f_i \to f$ in H^p. Indeed, we have for any $r < 1$ by uniform convergence that

$$\|(f_i - f)_r\|_p = \lim_{j\to\infty}\left\|(f_i - f_j)_r\right\|_p \le \limsup_{j\to\infty}\|f_i - f_j\|_p\,.$$

It follows that

$$\|f_i - f\|_p \le \limsup_{j\to\infty}\|f_i - f_j\|_p < \varepsilon$$

for all large i. This proves that $f = (f - f_i) + f_i \in H^p$ and that $f_i \to f$ in H^p. We are done. □

Remark 9.4.3. *We want to record that actually*

$$\|f\|_p = \lim_{r\to 1}\|f_r\|_p\,, \quad 0 \le p \le \infty.$$

*This follows since $r \mapsto \|f_r\|_p$ is a nondecreasing function of r for every $f \in H(\mathbb{D})$. Indeed, for $p = \infty$, this follows from the maximum modulus principle. For $p < \infty$ we must use the fact that $\log^+|f|$ (for $p = 0$) and $|f|^p$ (for $0 < p < \infty$) are so-culled **subharmonic functions**, weaker variant of harmonic functions, and for such functions MVP holds in the following sense (see [14] Theorem 17.5.)*

Theorem 9.4.4. *Suppose u is a continuous subharmonic function in U, and*

$$m(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta \quad (0 \le r < 1).$$

If $r_1 < r_2$, then $m(r_1) \le m(r_2)$.

This theory of subharmonic functions is straightforward but we omit it for now. Just need to know that it gives a nondecreasing function $r \mapsto \|f_r\|_p$ for $p < \infty$ as well.

Notice that the definition of Hardy spaces are tailored to directly give that if $f \in H^p,\ 1 < p \le \infty$, then $f = \mathcal{P}[f^*]$ for some $f^* \in L^p(\partial\mathbb{D})$ (by theorem 9.3.6 since analyticity implies harmonicity) and $f \to f^*\left(e^{i\theta}\right)$ non-tangentially for a.e. $e^{i\theta} \in \partial\mathbb{D}$. We also know that then $\|N_\alpha f\|_{L^p(\partial\mathbb{D})} \lesssim \|f^*\|_{L^p(\partial\mathbb{D})}$, so $N_\alpha f \in L^p(\partial\mathbb{D})$.

But what about $0 < p \le 1$? Turns out that still $\exists f^* \in L^p(\partial\mathbb{D})$ such that $f \to f^*$ non-tangentially and $N_\alpha f \in L^p(\partial\mathbb{D})$. But this is harder to prove and requires factorizations. We do this now.

Theorem 9.4.5. *If $f \in H^\infty$, f not identically zero, define for $0 < r < 1$,*

$$\mu_r(f) := \fint_{-\pi}^{\pi} \log \left| f_r\left(e^{i\theta}\right)\right| d\theta;$$
$$\mu^*(f) := \fint_{-\pi}^{\pi} \log \left| f^*\left(e^{i\theta}\right)\right| d\theta$$

Then,

(1) $\mu_r(f) \le \mu_s(f), \quad 0 < r < s < 1$*;*

(2) $\mu_r(f) \to \log|f(0)|$ *as* $r \to 0$*;*

(3) $\mu_r(f) \le \mu^*(f), \quad 0 < r < 1$.

Remark 9.4.6. *The first inequality is true philosophically because also $\log|f|$ is subharmonic. However, often the flow of logic goes so that this theorem is first proved independently (without any mention of sobharmonicity) and then this theorem is used to show that $\log|f|$ is sobharmonic, which actually implies the fact used previously that so are then $\log^+|f|$ and $|f|^p, 0 < p < \infty$. Thus, we prove this the directly.*

Proof. We use Jensen's formula. Put $g(z) = \frac{f(z)}{z^m}$ where $m \ge 0$ is the order of the zero of f at $z = 0$. Then

$$(*): \qquad |g(0)| \prod_{n=1}^{N} \frac{r}{|a_n|} = \exp\left(\mu_r(g)\right),$$

where $a_1, \ldots, a_n$ are the zeros of g in $B(0, r)$ and we set r s.t. $g \ne 0$ on ∂B_r. Obviously the LHS of $(*)$ can only increase as r increases, showing (1) for g. But $\mu_r(f) = \mu_r(g) + m \log r$, so (1) is true for f as well.

For what follows we assume WLOG $|f| \le 1$. Notice that $f_r \to f(0)$ as $r \to 0$ and $f_r \to f^*$ as $r \to 1$. Now,

$$\fint_{-\pi}^{\pi} \lim_{r\to 1} \log\left|f_r\left(e^{i\theta}\right)\right| \mathrm{d}\theta = -\fint_{-\pi}^{\pi} \lim_{r\to 1} \underbrace{\log \frac{1}{\left|f_r\left(e^{i\theta}\right)\right|}}_{\substack{\text{non-negative function} \\ \text{as } 1/|f_r| \ge 1 \text{ (can be } \infty)}} \mathrm{d}\theta$$

Recall Fatou's lemma that for non-negative sequence of functions h_n, $n = 1, 2, \cdots$, we have

$$\int \liminf h_n \le \liminf \int h_n$$

Thus,

$$\begin{aligned}
\mu^*(f) &= \fint_{-\pi}^{\pi} \log\left|f^*\left(e^{i\theta}\right)\right| \mathrm{d}\theta = \fint_{-\pi}^{\pi} \lim_{r\to 1} \log\left|f_r\left(e^{i\theta}\right)\right| \mathrm{d}\theta \\
&= -\fint_{-\pi}^{\pi} \lim_{r\to 1} \underbrace{\log \frac{1}{\left|f_r\left(e^{i\theta}\right)\right|}}_{\ge 0} \mathrm{d}\theta \overset{\text{Fatou}}{\ge} -\lim_{r\to 1} \fint_{-\pi}^{\pi} \log \frac{1}{\left|f_r\left(e^{i\theta}\right)\right|} \mathrm{d}\theta \\
&= \lim_{r\to 1} \mu_r(f) \overset{(1)}{=\!=} \sup_{r<1} \mu_r(f),
\end{aligned}$$

This shows (3). Similarly,

$$\begin{aligned}
\inf_{r>0} \mu_r(f) &= \lim_{r\to 0} \mu_r(f) = \lim_{r\to 0} \fint_{-\pi}^{\pi} \log|f_r(e^{i\theta})| \mathrm{d}\theta \\
&= -\lim_{r\to 0} \fint_{-\pi}^{\pi} \log \frac{1}{|f_r(e^{i\theta})|} \mathrm{d}\theta \overset{\text{Fatou}}{\le} -\fint_{-\pi}^{\pi} \lim_{r\to 0} \log \frac{1}{|f_r(e^{i\theta})|} \mathrm{d}\theta \\
&= \log|f(0)|
\end{aligned}$$

How to see $\mu_r(f) \geq \log|f(0)|\ \forall r$? If $f(0)=0$ then trivial. If $f(0) \neq 0$, then Jensen's formula gives

$$\log|f(0)| = \underbrace{\sum_{k=1}^{N} \log\frac{|a_n|}{r}}_{\leq 0} + \mu_r(f) \leq \mu_r(f).$$

Thus, (2) holds. $\square$

We now have a much stronger uniqueness statement than the "moreover,..." part in Theorem 9.3.8. Indeed, notice that now there already is some (many) r_0 such that $|f|$ does not vanish on $\partial B(0, r_0)$. Then,

$$\mu^*(f) \geq \sup_{r>0} \mu_r(f) \geq \mu_{r_0}(f) > -\infty$$

implying $\log|f^*(e^{i\theta})| \neq -\infty$ and thus $f^*(e^{i\theta}) \neq 0$ for a.e. $e^{i\theta} \in \partial\mathbb{D}$. That is, for a non-trivial $f \in H^\infty$ we must in fact have $f^*\left(e^{i\theta}\right) \neq 0$ a.e. while previously we only knew f^* cannot vanish a.e. on some arc!

We end this section with a theorem on Blaschke product for which we left as an exercise its proof (see Math5022 HW8; using previous theorem).

Theorem 9.4.7. *Consider the Blaschke product*

$$B(z) = z^m \prod_{n=1}^{\infty} \psi_{a_n}(z)\frac{|a_n|}{a_n}$$

where $a_n \in \mathbb{D}\backslash\{0\}$ *satisfy* $\sum(1-|a_n|) < \infty$. *Then* $\left|B^*\left(e^{i\theta}\right)\right| = 1$ *(*$B^* \in L^\infty(\partial\mathbb{D})$ *exists as* $B \in H^\infty$*) almost everywhere and*

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log\left|B\left(re^{i\theta}\right)\right| d\theta = 0.$$

Proof. We know that $\|B^*\|_{L^\infty(\partial\mathbb{D})} = \|B\|_\infty \leq 1$ so $|B^*| \leq 1$ (almost everywhere). It follows that $\log|B^*| \leq 0$ and, therefore, if we show $\int_{\partial\mathbb{D}} \log|B^*| \, \mathrm{d}\sigma = 0$ then it must be that $\log|B^*| = 0$ (almost everywhere), and so $|B^*| = 1$ (almost everywhere). So the first claim will follow from this integral claim. It is also connected to the second claim as we now explain. First, recall that

$$\mu_r(B) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|B\left(re^{i\theta}\right)\right| \mathrm{d}\theta$$

satisfy $\mu_r(B) \leq \mu_s(B)$ ir $r < s$, and so, in particular, the limit

$$\lim_{r\to 1} \mu_r(B)$$

exists. We also know that $\mu_r(B) \leq \mu\left(B^*\right)$ so

$$\lim_{r\to 1} \mu_r(B) \leq \mu\left(B^*\right) \leq 0$$

where the last inequality is just because $\log|B^*| \leq 0$ as deduced above. So to complete the whole proof, we must only show that

$$\lim_{r\to 1} \mu_r(B) \geq 0$$

since then $\lim_{r\to 1} \mu_r(B) = \mu_r\left(B^*\right) = 0$ as desired. Given any N write

$$B(z) = z^m \prod_{n=1}^{N-1} \psi_{a_n}(z)\frac{|a_n|}{a_n} \cdot \prod_{n=N}^{\infty} \psi_{a_n}(z)\frac{|a_n|}{a_n} =: B_N(z) \cdot H_N(z).$$

Write

$$\mu_r(B)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|B_N\left(re^{i\theta}\right)\right|\mathrm{d}\theta+\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|H_N\left(re^{i\theta}\right)\right|\mathrm{d}\theta=\mu_r\left(B_N\right)+\mu_r\left(H_N\right).$$

We fix N and let $r_0<1$ be so large, depending on the fixed N, that $a_n\in B\left(0,r_0\right)$ for $n=1,\ldots,N-1$. Then for r with $r_0\leq r\leq 1$ we have that $B_N\left(re^{i\theta}\right)\neq 0$ and so by continuity and compactness $|\log|B_N\left(re^{i\theta}\right)||\lesssim_N 1$ for such r. Thus, by dominated convergence theorem we have

$$\lim_{r\to 1}\mu_r\left(B_N\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lim_{r\to 1}\log\left|B_N\left(re^{i\theta}\right)\right|\mathrm{d}\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log 1\ \mathrm{d}\theta=0.$$

Here we used that

$$|B_N(z)|=|z|^m\prod_{n=1}^{N}\left|\psi_{a_n}(z)\right|=1$$

if $|z|=1$, since all of the Blaschke factors satisfy by direct calculation that $\left|\psi_{a_n}(z)\right|=1$ for $|z|=1$ (see for instance the calculations of Math5021 HW1 Ex4). We have shown that with any N we have

$$\lim_{r\to 1}\mu_r(B)=\lim_{r\to 1}\mu_r\left(H_N\right).$$

Next, notice that

$$H_N(0)=\prod_{n=N}^{\infty}\left|a_n\right|$$

simply because $\psi_{a_n}(0)=a_n$. Then we have the simple fact that

$$\lim_{N\to\infty}\prod_{n=N}^{\infty}\left|a_n\right|=\lim_{N\to\infty}\frac{\prod_{n=1}^{\infty}\left|a_n\right|}{\prod_{n=1}^{N-1}\left|a_n\right|}=\frac{\prod_{n=1}^{\infty}\left|a_n\right|}{\prod_{n=1}^{\infty}\left|a_n\right|}=1,$$

where we used that $\prod_{n=1}^{\infty}\left|a_n\right|>0$, since $\sum\left(1-\left|a_n\right|\right)<\infty$ (see Math5022 HW7 Ex1). This means that

$$\lim_{N\to\infty}\log H_N(0)=\log 1=0.$$

We should now have the pieces to tie the estimates together. Recall that

$$\mu_r(f)\geq\log|f(0)|$$

for $f\in H^{\infty}$ - apply this to $f=H_N$ yielding

$$\mu_r\left(H_N\right)\geq\log\left|H_N(0)\right|.$$

Let $\varepsilon>0$. First, choose N so that $\log\left|H_N(0)\right|\geq-\varepsilon$. Then we have

$$\lim_{r\to 1}\mu_r(B)=\lim_{r\to 1}\mu_r\left(H_N\right)\geq\log\left|H_N(0)\right|\geq-\varepsilon.$$

As $\varepsilon>0$ was arbitrary this shows $\lim_{r\to 1}\mu_r(B)\geq 0$ as desired and we are done. $\square$

9.5 Factorizations of Functions in $H^p(\mathbb{D})$

The factorizations of functions in Hardy space on disc uses results on Blaschke products in a critical way. In turn, these factorizations are absolutely fundamental for Hardy space theory.

Recall the Nevanlinna space

$$N := \{f \in H(\mathbb{D}) : \|f\|_0 < \infty\}$$

where $\|f_r\|_0 = \exp\left(\int_{\partial\mathbb{D}} \log^+ |f_r| \mathrm{d}\sigma\right)$ and $\|f\|_p := \sup_{0\le r<1} \|f_r\|_p$ for $0 \le p \le \infty$. In Math5022 HW7Q4, we showed that

$$f \in N \implies \sum(1-|a_n|) < \infty \quad (*)$$

(where $(*)$ is a necessary and sufficient condition for the Blaschke product to converge; see Theorem 8.7.3.) Since $H^\infty \subset H^p \subset N \quad \forall p$, the condition $(*)$ is also satisfied in all Hardy spaces. In particular, we can always construct, given $f \in N$, the Blaschke product B associated with the zeros (a_n) of f satisfying $(*)$. The following shows we can divide out the zeros without increasing norm.

Theorem 9.5.1. *Let $f \in N$, $f \not\equiv 0$, and let B be the Blaschke product with zeros of f. Put $g := f/B$. Then $g \in N$ with $\|g\|_0 = \|f\|_0$. Also, if $f \in H^p$ then $g \in H^p$ with $\|g\|_p = \|f\|_p$.*

Proof. As $|B(z)| \le 1$ we have $|g(z)| \ge |f(z)|$ pointwise, so $\|g\|_p \ge \|f\|_p$ is always clear. For reverse inequality, let $p = 0$ first. Notice that $\log^+ xy \le \log^+ x + \log^+ y$ (the inequality comes from the case that x or y is smaller than 1; otherwise equality holds). Thus,

$$\log^+ |g| \le \log^+ |f| + \log^+ \frac{1}{|B|}$$

and it implies

$$\exp\left(\fint_{-\pi}^{\pi} \log^+ \left|g\left(re^{i\theta}\right)\right| \mathrm{d}\theta\right) \leqslant \exp\left(\fint_{-\pi}^{\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| \mathrm{d}\theta\right) \cdot \exp\left(\int_{-\pi}^{\pi} \log^+ \frac{1}{|B(re^{i\theta})|} \mathrm{d}\theta\right), 0 \le r < 1.$$

Recall from Theorem 9.4.7 that

$$\lim_{r\to 1} \fint_{-\pi}^{\pi} \log |B(re^{i\theta})| \mathrm{d}\theta = 0$$

Since the Blaschke product has $|B| \le 1$ we see $\log^+(1/|B|) = \log(1/|B|)$ and

$$\fint_{-\pi}^{\pi} \log^+ \frac{1}{|B\left(r^{i\theta}\right)|} \mathrm{d}\theta = -\fint_{-\pi}^{\pi} \log\left|B\left(re^{i\theta}\right)\right| \mathrm{d}\theta \xrightarrow{r\to 1} 0.$$

Then Remark 9.4.3 gives

$$\|g\|_0 = \lim_{r\to 1} \|g_r\|_0 \le \lim_{r\to 1} \|f_r\|_0 = \|f\|_0.$$

Let then $p > 0$. We let B_n be a finite Blaschke product formed with the first zeros in the sequence $a_1, a_2, \ldots$ Put $g_n := f/B_n$. Recall that $|\psi_\alpha(z)| = 1$ if $|z| = 1$ and one can then check $|B_n(re^{i\theta})| \to 1$ uniformly as $r \to 1$. Thus if $p < \infty$

$$\|g_n\|_p = \lim_{r\to 1} \left(\fint_{\partial\mathbb{D}} \left|\frac{f\left(re^{i\theta}\right)}{B_n\left(re^{i\theta}\right)}\right|^p \mathrm{d}\sigma\right)^{1/p} = \|f\|_p$$

and, for $p = \infty$,

$$\|g_n\|_\infty = \lim_{r\to 1} \sup_\theta \left|\frac{f\left(re^{i\theta}\right)}{B_n\left(re^{i\theta}\right)}\right| = \|f\|_\infty.$$

So $\|g_n\|_p = \|f\|_p \quad \forall n, 0 < p \le \infty.$

As $|B_n|$ can only decrease as n increases, $|g_1| \leq |g_2| \leq \cdots$. Recall $g = f/B$. Then, with fixed r, we use monotone convergence theorem to see

$$\|g_r\|_p = \lim_{n\to\infty} \|(g_n)_r\|_p$$

But $\|(g_n)_r\|_p \leqslant \|g_n\|_p = \|f\|_p$, $\forall r$, so $\|g_r\|_p \leq \|f\|_p$, which implies $\|g\|_p \leq \|f\|_p$. □

The factorization is the key for $p \leq 1$ since it allows to use the theory of H^2 (e.g. existence of h^* for $h \in H^2$) when proving results in H^p.

Theorem 9.5.2. *Suppose $0 < p < \infty$, $f \in H^p$, $f \not\equiv 0$, and B is the Blaschke product formed with the zeros of f. Then there is a nonvanishing function $h \in H^2$ such that*

$$f = B \cdot h^{2/p}.$$

In particular, every $f \in H^1$ is a product

$$f = gh$$

in which both factors are in H^2.

Proof. By Theorem 9.5.1, $f/B \in H^p$; in fact, $\|f/B\|_p = \|f\|_p$. Since f/B has no zero in U (dividing B rules out all zeros of f) and U is simply connected, there exists $\varphi \in H(U)$ so that $\exp(\varphi) = f/B$ (Theorem 5.3.8). Put $h = \exp(p\varphi/2)$. Then $h \in H(U)$ and $|h|^2 = |f/B|^p$, hence $h \in H^2$. In fact, $\|h\|_2^2 = \|f\|_p^p$. Also, $f = B\frac{f}{B} = Be^{\varphi} = B\left(e^{p\varphi/2}\right)^{2/p} = Bh^{2/p}$. For $p = 1$, we can write this in the form $f = (Bh) \cdot h$, where $h \in H^2$ and $Bh \in H^2$. □

Even if we would not care about H^p for $p < 1$, we still want to understand H^1 and for instance, existence of boundary values in H^1. Luckily this now works, since we can use $\exists B^*$ (as $B \in H^\infty$ and we even know $|B^*| = 1$ a.e.) and $\exists h^*$ for $h \in H^2$. These are some of the most important properties of H^p functions.

Theorem 9.5.3. *If $0 < p < \infty$ and $f \in H^p$, then*

(a) the nontangential maximal functions $N_\alpha f$ are in $L^p(\partial\mathbb{D})$, for all $\alpha < 1$;

(b) the nontangential limits $f^*\left(e^{i\theta}\right)$ exist a.e. on $\partial\mathbb{D}$, and $f^* \in L^p(\partial\mathbb{D})$;

(c) $\lim_{r\to 1} \|f^* - f_r\|_p = 0$, and

(d) $\|f^*\|_p = \|f\|_p$

Proof. Notice (a)-(b) hold clearly if $p > 1$ (see theorem 9.3.6; $f = \mathcal{P}[f^*]$ for $f^* \in L^p(\partial\mathbb{D})$, $f \to f^*$ nontangentially and $\|N_\alpha f\|_{L^p(\partial\mathbb{D})} = \|N_\alpha(\mathcal{P}[f^*])\|_{L^p(\partial\mathbb{D})} \lesssim \|f^*\|_{L^p(\partial\mathbb{D})}$.)

Now, if $0 < p \leq 1$, we use the factorization $f = Bh^{2/p}$ provided by theorem 9.5.2, where B is a Blaschke product and $h \in H^2$ is non-vanishing. Since $|f| \leq |h|^{2/p}$ we get

$$\left[N_\alpha f\left(e^{i\theta}\right)\right]^p = \sup_{z\in e^{i\theta}\Omega_\alpha} \overbrace{|f(z)|^p}^{\leq |h(z)|^2} \leq \left[N_\alpha h\left(e^{i\theta}\right)\right]^2$$

and so

$$\|N_\alpha f\|_{L^p(\partial\mathbb{D})} \leq \|N_\alpha h\|_{L^2(\partial\mathbb{D})}^{2/p} < \infty$$

as $N_\alpha h \in L^2(\partial\mathbb{D})$ for $h \in H^2$. Thus, $N_\alpha f \in L^p(\partial\mathbb{D})$ and (a) holds. Similarly, $\exists B^*$ (as $B \in H^\infty$ and we even know $|B^*| = 1$ a.e.) and $\exists h^* \in L^2$ (as $h \in H^2$). So nontangential limits of f, say, $f^* = B^* {h^*}^{2/p}$ exist a.e.

Also $f^* \in L^p(\partial\mathbb{D})$, since obviously $|f^*| \leq N_\alpha f \in L^p(\partial\mathbb{D})$, so (b) holds. Thus (a)-(b) hold $\forall 0 < p < \infty$. For (c), we compute

$$\begin{aligned}\lim_{r\to 1} \|f^* - f_r\|_p^p &= \lim_{r\to 1} \int_{\partial\mathbb{D}} \left|f^*\left(e^{i\theta}\right) - f_r\left(e^{i\theta}\right)\right|^p \mathrm{d}\theta \\ &\stackrel{?}{=} \int_{\partial\mathbb{D}} \underbrace{\lim_{r\to 1} \left|f^*\left(e^{i\theta}\right) - f_r\left(e^{i\theta}\right)\right|^p}_{=0 \text{ by (b)}} \mathrm{d}\theta \\ &= 0.\end{aligned}$$

"?" follows by dominated convergence theorem: the sequence $(f^* - f_r)_r$ has an integrable dominant:

$$|f^* - f_r| \leq f^* + N_\alpha f \in L^p \quad \forall r.$$

(c) holds. For $p \geq 1$, (d) follows from (c) by triangle inequality:

$$|\|f^*\|_p - \|f_r\|_p| \leq \|f^* - f_r\|_p \xrightarrow[\text{by (c)}]{r\to 1} 0$$

For $p < 1$ it is easy to see that $\|h_1 - h_2\|_p^p \leq \|h_1\|_p^p + \|h_2\|_p^p,\ \forall h_1, h_2$. Thus, $0 \leftarrow \|f^* - f_r\|_p^p \geq \left|\|f^*\|_p^p - \|f_r\|_p^p\right|$. $\square$

Finally, for $p > 1$ we had the nice representation $f = \mathcal{P}[f^*]$. What kind of representation for f holds if just $f \in H^1$? Let $f \in H^1, r < 1$, and define $\tilde{f}(z) = f(rz)$. Then $\tilde{f}$ is analytic in the larger disc $B(0, 1/r) \supset \overline{\mathbb{D}}$. So $\tilde{f}$ can be represented by the Cauchy formula, for $z \in \mathbb{D}$, as

$$\begin{aligned}f(rz) = \tilde{f}(z) &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\tilde{f}(\xi)}{\xi - z} \mathrm{d}\xi \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\tilde{f}\left(e^{it}\right)}{e^{it} - z} i e^{it} \mathrm{d}t \\ &= \fint_{-\pi}^{\pi} \frac{f_r\left(e^{it}\right)}{1 - e^{-it} z} \mathrm{d}t\end{aligned}$$

Recall from prev. thm. that $f_r \to f^*$ in L^1. Thus

$$\begin{aligned}&\left|\fint_{-\pi}^{\pi} \frac{f_r\left(e^{it}\right)}{1 - e^{-it} z} \mathrm{d}t - \fint_{-\pi}^{\pi} \frac{f^*\left(e^{it}\right)}{1 - e^{-it} z} \mathrm{d}t\right| \\ &\leq \frac{1}{1 - |z|} \fint_{-\pi}^{\pi} \left|f_r\left(e^{it}\right) - f^*\left(e^{it}\right)\right| \mathrm{d}t \\ &\longrightarrow 0 \text{ as } r \to 1.\end{aligned}$$

Also $f(rz) \to f(z)$. So,

$$\begin{aligned}f(z) &= \fint_{-\pi}^{\pi} \frac{f^*\left(e^{it}\right)}{1 - e^{-it} z} \mathrm{d}t \\ &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f^*(\xi)}{\xi - z} \mathrm{d}\xi.\end{aligned}$$

Thus this Cauchy formula for f in terms of f^* holds for $f \in H^1$.

Is there a Poisson formula? Yes, actually, this still works in H^1. We go through $\tilde{f}$ again.

By the uniqueness of the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ u = \tilde{f} & \text{on } \partial\mathbb{D} \end{cases}$$

we must have, for $z \in \mathbb{D}$,

$$\begin{aligned} f(rz) &= \tilde{f}(z) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P\left(z, e^{it}\right) \tilde{f}\left(e^{it}\right) dt \\ &= \fint_{-\pi}^{\pi} P\left(z, e^{it}\right) f_r\left(e^{it}\right) dt, \end{aligned}$$

Again, use that with $z \in \mathbb{D}$ fixed, $t \mapsto P(z, e^{it})$ bounded, and $f_r \to f^*$ in L^1 to get $f(z) = \fint_{-\pi}^{\pi} P(z, e^{it}) f^*(e^{it}) \mathrm{d}t$.

Chapter 10

Appendix

To smoothen our discussion, we copy some sections from [16] on L^p Spaces and Banach Spaces.

Let $(X, \mathcal{A})$ be a measurable space and μ a measure on it. The measure μ is called a σ**-finite measure**, if it satisfies one of the four following equivalent criteria:
1. the set X can be covered with at most countably many measurable sets with finite measure. This means that there are sets $A_1, A_2, \ldots \in \mathcal{A}$ with $\mu\left(A_n\right)<\infty$ for all $n \in \mathbb{N}$ that satisfy $\bigcup_{n \in \mathbb{N}} A_n=X$.
2. the set X can be covered with at most countably many measurable disjoint sets with finite measure. This means that there are sets $B_1, B_2, \ldots \in \mathcal{A}$ with $\mu\left(B_n\right)<\infty$ for all $n \in \mathbb{N}$ and $B_i \cap B_j=\varnothing$ for $i \neq j$ that satisfy $\bigcup_{n \in \mathbb{N}} B_n=X$.
3. the set X can be covered with a monotone sequence of measurable sets with finite measure. This means that there are sets $C_1, C_2, \ldots \in \mathcal{A}$ with $C_1 \subset C_2 \subset \cdots$ and $\mu\left(C_n\right)<\infty$ for all $n \in \mathbb{N}$ that satisfy $\bigcup_{n \in \mathbb{N}} C_n=X$.
4. there exists a strictly positive measurable function f whose integral is finite. This means that $f(x)>0$ for all $x \in X$ and $\int f(x) \mu(\mathrm{d} x)<\infty$.

If μ is a σ-finite measure, the measure space $(X, \mathcal{A}, \mu)$ is called a σ**-finite measure space**.

Throughout this section $(X, \mathcal{F}, \mu)$ denotes a σ-finite measure space: X denotes the underlying space, $\mathcal{F}$ the σ-algebra of measurable sets, and μ the measure. If $1 \leq p<\infty$, the space $L^p(X, \mathcal{F}, \mu)$ consists of all comple-xvalued measurable functions on X that satisfy

$$\int_X|f(x)|^p d \mu(x)<\infty. \tag{10.1}$$

To simplify the notation, we write $L^p(X, \mu)$, or $L^p(X)$, or simply L^p when the underlying measure space has been specified. Then, if $f \in L^p(X, \mathcal{F}, \mu)$ we define the L^p norm of f by

$$\|f\|_{L^p(X, \mathcal{F}, \mu)}=\left(\int_X|f(x)|^p d \mu(x)\right)^{1 / p} .$$

We also abbreviate this to $\|f\|_{L^p(X)},\|f\|_{L^p}$, or $\|f\|_p$. When $p=1$ the space $L^1(X, \mathcal{F}, \mu)$ consists of all integrable functions on X, and we have shown in Chapter 6 of Book III, that L^1 together with $\|\cdot\|_{L^1}$ is a complete normed vector space. Also, the case $p=2$ warrants special attention: it is a Hilbert space.

We note here that we encounter the same technical point that we already discussed in Book III. The problem is that $\|f\|_{L^p}=0$ does not imply that $f=0$, but merely $f=0$ almost everywhere (for the measure μ). Therefore, the precise definition of L^p requires introducing the equivalence relation, in which f and g are equivalent if $f = g$ a.e. Then, L^p consists of all equivalence classes of functions which satisfy (10.1). However, in practice there is little risk of error by thinking of elements in L^p as functions rather than equivalence classes of functions.

The following are some common examples of L^p spaces.
(a) The case $X = \mathbb{R}^d$ and μ equals Lebesgue measure is often used in practice. There, we have

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}.$$

(b) Also, one can take $X = \mathbb{Z}$, and μ equal to the counting measure. Then, we get the "discrete" version of the L^p spaces. Measurable functions are simply sequences $f = \{a_n\}_{n\in\mathbb{Z}}$ of complex numbers, and

$$\|f\|_{L^p} = \left(\sum_{n=-\infty}^{\infty} |a_n|^p\right)^{1/p}.$$

When $p = 2$, we recover the familiar sequence space $\ell^2(\mathbb{Z})$. The spaces L^p are examples of normed vector spaces. The basic property satisfied by the norm is the triangle inequality, which we shall prove shortly.

The range of p which is of interest in most applications is $1 \leq p < \infty$, and later also $p = \infty$. There are at least two reasons why we restrict our attention to these values of $p:$ when $0 < p < 1$, the function $\|\cdot\|_{L^p}$ does not satisfy the triangle inequality, and moreover, for such p, the space L^p has no non-trivial bounded linear functionals.

When $p = 1$ the norm $\|\cdot\|_{L^1}$ satisfies the triangle inequality, and L^1 is a complete normed vector space. When $p = 2$, this result continues to hold, although one needs the Cauchy-Schwarz inequality to prove it. In the same way, for $1 \leq p < \infty$ the proof of the triangle inequality relies on a generalized version of the Cauchy-Schwarz inequality. This is Hölder's inequality, which is also the key in the duality of the L^p spaces, as we will see in subsection 4.

The Hölder and Minkowski inequalities

If the two exponents p and q satisfy $1 \leq p, q \leq \infty$, and the relation

$$\frac{1}{p} + \frac{1}{q} = 1$$

holds, we say that p and q are conjugate or dual exponents. Here, we use the convention $1/\infty = 0$. Later, we shall sometimes use p' to denote the conjugate exponent of p. Note that $p = 2$ is self-dual, that is, $p = q = 2$; also $p = 1, \infty$ corresponds to $q = \infty, 1$ respectively.

Theorem 10.0.1 (Hölder)**.** *Suppose $1 < p < \infty$ and $1 < q < \infty$ are conjugate exponents. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Note. Once we have defined L^∞ the corresponding inequality for the exponents 1 and ∞ will be seen to be essentially trivial.

The proof of the theorem relies on a simple generalized form of the arithmetic-geometric mean inequality: if $A, B \geq 0$, and $0 \leq \theta \leq 1$, then

$$A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B. \tag{10.2}$$

Note that when $\theta = 1/2$, the inequality (10.2) states the familiar fact that the geometric mean of two numbers is majorized by their arithmetic mean.

To see (10.2), we observe that we may assume $B \neq 0$, and replacing A by AB, we see that it suffices to prove that $A^\theta \leq \theta A + (1-\theta)$. If we let $f(x) = x^\theta - \theta x - (1-\theta)$, then $f'(x) = \theta\left(x^{\theta-1} - 1\right)$. Thus $f(x)$ increases when $0 \leq x \leq 1$ and decreases when $1 \leq x$, and we see that the continuous function f attains a maximum at $x = 1$, where $f(1) = 0$. Therefore $f(A) \leq 0$, as desired.

To prove Hölder's inequality we argue as follows. If either $\|f\|_{L^p} = 0$ or $\|f\|_{L^q} = 0$, then $fg = 0$ a.e. and the inequality is obviously verified. Therefore, we may assume that neither of these norms vanish, and after replacing f by $f/\|f\|_{L^p}$ and g by $g/\|g\|_{L^q}$, we may further assume that $\|f\|_{L^p} = \|g\|_{L^q} = 1$. We now need to prove that $\|fg\|_{L^1} \leq 1$.

If we set $A = |f(x)|^p, B = |g(x)|^q$, and $\theta = 1/p$ so that $1 - \theta = 1/q$, then (10.2) gives

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q.$$

Integrating this inequality yields $\|fg\|_{L^1} \leq 1$, and the proof of the Hölder inequality is complete.

We are now ready to prove the triangle inequality for the L^p norm.

Theorem 10.0.2 (Minkowski). *If $1 \leq p < \infty$ and $f, g \in L^p$, then $f + g \in L^p$ and $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.*

Proof. The case $p = 1$ is obtained by integrating $|f(x) + g(x)| \leq |f(x)| + |g(x)|$. When $p > 1$, we may begin by verifying that $f + g \in L^p$, when both f and g belong to L^p. Indeed,

$$|f(x) + g(x)|^p \leq 2^p \left(|f(x)|^p + |g(x)|^p\right),$$

as can be seen by considering separately the cases $|f(x)| \leq |g(x)|$ and $|g(x)| \leq |f(x)|$. Next we note that

$$|f(x) + g(x)|^p \leq |f(x)||f(x) + g(x)|^{p-1} + |g(x)||f(x) + g(x)|^{p-1}.$$

If q denotes the conjugate exponent of p, then $(p-1)q = p$, so we see that $(f+g)^{p-1}$ belongs to L^q, and therefore Hölder's inequality applied to the two terms on the right-hand side of the above inequality gives

$$\|f+g\|_{L^p}^p \leq \|f\|_{L^p} \left\|(f+g)^{p-1}\right\|_{L^q} + \|g\|_{L^p} \left\|(f+g)^{p-1}\right\|_{L^q}. \tag{10.3}$$

However, using once again $(p-1)q = p$, we get

$$\left\|(f+g)^{p-1}\right\|_{L^q} = \|f+g\|_{L^p}^{p/q}.$$

From (10.3), since $p - p/q = 1$, and because we may suppose that $\|f+g\|_{L^p} > 0$, we find

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

so the proof is finished. □

We also see a simple corollary of Hölder's inequality.

Corollary 10.0.3. *Suppose $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ (so $p_1 > p$ and $p_2 > p$ if neither of them is ∞, and thus $1 < \frac{p_1}{p}, \frac{p_2}{p} < \infty$). Then $1 = \frac{p}{p_1} + \frac{p}{p_2} = \frac{1}{p_1/p} + \frac{1}{p_2/p}$ and Hölder's inequality gives*

$$\|fg\|_{L^p} = \||f|^p|g|^p\|_{L^1}^{1/p} \leq \||f|^p\|_{L^{p_1/p}}^{1/p}\||g|^p\|_{L^{p_2/p}}^{1/p} = \|f\|_{L^{p_1}}\|g\|_{L^{p_2}},$$

if $f \in L^{p_1}$ and $g \in L^{p_2}$.

Remark 10.0.4. *The L^p and L^q spaces need in general not be contained in one another in any particular way. There is one exception, where we have a clear rule. If $\mu(X) < \infty$ and $p < q$ we have by Hölder's inequality that with $s = q/p > 1$ that*

$$\begin{aligned}\int_X |f|^p \, \mathrm{d}\mu &\leq \left(\int_X |f|^{ps} \, \mathrm{d}\mu\right)^{\frac{1}{s}} \left(\int_X 1^{s'} \mathrm{d}\mu\right)^{\frac{1}{s'}} \\ &= \left(\int_X |f|^q \, \mathrm{d}\mu\right)^{\frac{p}{q}} \mu(X)^{1-\frac{1}{s}} \\ &= \|f\|_{L^q(\mu)}^p \mu(X)^{1-\frac{p}{q}}\end{aligned}$$

and so

$$\|f\|_{L^p(\mu)} \leq \|f\|_{L^q(\mu)} \mu(X)^{\frac{1}{p}-\frac{1}{q}}.$$

So we have the quantitative estimate from above - in particular, we have $L^q(\mu) \subset L^p(\mu)$. It would be possible to establish the inclusion with a more elementary argument as well.

Completeness of L^p

The triangle inequality makes L^p into a metric space with distance $d(f,g) = \|f - g\|_{L^p}$. The basic analytic fact is that L^p is complete in the sense that every Cauchy sequence in the norm $\|\cdot\|_{L^p}$ converges to an element in L^p. Taking limits is a necessity in many problems, and the L^p spaces would be of little use if they were not complete. Fortunately, like L^1 and L^2, the general L^p space does satisfy this desirable property.

Theorem 10.0.5. *The space $L^p(X, \mathcal{F}, \mu)$ is complete in the norm $\|\cdot\|_{L^p}$.*

Proof. The argument is essentially the same as for L^1 (or L^2); see Section 2, Chapter 2 and Section 1, Chapter 4 in Book III. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in L^p, and consider a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ with the following property $\left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^p} \leq 2^{-k}$ for all $k \geq 1$. We now consider the series whose convergence will be seen below

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|,$$

and the corresponding partial sums

$$S_K(f)(x) = f_{n_1}(x) + \sum_{k=1}^{K} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and

$$S_K(g)(x) = |f_{n_1}(x)| + \sum_{k=1}^{K} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|.$$

The triangle inequality for L^p implies

$$\begin{aligned}\|S_K(g)\|_{L^p} &\leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{K} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^p} \\ &\leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{K} 2^{-k}.\end{aligned}$$

Letting K tend to infinity, and applying the monotone convergence theorem proves that $\int g^p < \infty$, and therefore the series defining g, and hence the series defining f converges almost everywhere, and $f \in L^p$.

We now show that f is the desired limit of the sequence $\{f_n\}$. Since (by construction of the telescopic series) the $(K-1)^{\text{th}}$ partial sum of this series is precisely f_{n_K}, we find that

$$f_{n_K}(x) \to f(x) \quad \text{a.e. } x.$$

To prove that $f_{n_K} \to f$ in L^p as well, we first observe that

$$\begin{aligned} |f(x) - S_K(f)(x)|^p &\leq [2\max(|f(x)|, |S_K(f)(x)|)]^p \\ &\leq 2^p |f(x)|^p + 2^p\, |S_K(f)(x)|^p \\ &\leq 2^{p+1}|g(x)|^p \end{aligned}$$

for all K. Then, we may apply the dominated convergence theorem to get $\|f_{n_K} - f\|_{L^p} \to 0$ as K tends to infinity.

Finally, the last step of the proof consists of recalling that $\{f_n\}$ is Cauchy. Given $\epsilon > 0$, there exists N so that for all $n, m > N$ we have $\|f_n - f_m\|_{L^p} < \epsilon/2$. If n_K is chosen so that $n_K > N$, and $\|f_{n_K} - f\|_{L^p} < \epsilon/2$, then the triangle inequality implies

$$\|f_n - f\|_{L^p} \leq \|f_n - f_{n_K}\|_{L^p} + \|f_{n_K} - f\|_{L^p} < \epsilon$$

whenever $n > N$. This concludes the proof of the theorem. □

Further remarks

We begin by looking at some possible inclusion relations between the various L^p spaces. The matter is simple if the underlying space has finite measure.

Proposition 10.0.6. *If X has finite positive measure, and $p_0 \leq p_1$, then $L^{p_1}(X) \subset L^{p_0}(X)$ and*

$$\frac{1}{\mu(X)^{1/p_0}} \|f\|_{L^{p_0}} \leq \frac{1}{\mu(X)^{1/p_1}} \|f\|_{L^{p_1}}.$$

We may assume that $p_1 > p_0$. Suppose $f \in L^{p_1}$, and set $F = |f|^{p_0}$, $G = 1, p = p_1/p_0 > 1$, and $1/p + 1/q = 1$, in Hölder's inequality applied to F and G. This yields

$$\|f\|_{L^{p_0}}^{p_0} \leq \left(\int |f|^{p_1}\right)^{p_0/p_1} \cdot \mu(X)^{1-p_0/p_1}.$$

In particular, we find that $\|f\|_{L^{p_0}} < \infty$. Moreover, by taking the p_0^{th} root of both sides of the above equation, we find that the inequality in the proposition holds.

However, as is easily seen, such inclusion does not hold when X has infinite measure. Yet, in an interesting special case the opposite inclusion does hold.

Proposition 10.0.7. *If $X = \mathbb{Z}$ is equipped with counting measure, then the reverse inclusion holds, namely $L^{p_0}(\mathbb{Z}) \subset L^{p_1}(\mathbb{Z})$ if $p_0 \leq p_1$. Moreover, $\|f\|_{L^{p_1}} \leq \|f\|_{L^{p_0}}$.*

Indeed, if $f = \{f(n)\}_{n\in\mathbb{Z}}$, then $\sum |f(n)|^{p_0} = \|f\|_{L^{p_0}}^{p_0}, \text{and} \sup_n |f(n)| \leq \|f\|_{L^{p_0}}$. However

$$\begin{aligned} \sum |f(n)|^{p_1} &= \sum |f(n)|^{p_0} |f(n)|^{p_1-p_0} \\ &\left(\leq \sup_n |f(n)|\right)^{p_1-p_0} \|f\|_{L^{p_0}}^{p_0} \\ &\leq \|f\|_{L^{p_0}}^{p_1} \end{aligned}$$

Thus $\|f\|_{L^{p_1}} \leq \|f\|_{L^{p_0}}$.

The case $p = \infty$

Finally, we also consider the limiting case $p = \infty$. The space L^∞ will be defined as all functions that are "essentially bounded" in the following sense. We take the space $L^\infty(X, \mathcal{F}, \mu)$ to consist of all (equivalence classes of) measurable functions on X, so that there exists a positive number $0 < M < \infty$, with

$$|f(x)| \leq M \quad \text{a.e. } x.$$

Then, we define $\|f\|_{L^\infty(X,\mathcal{F},\mu)}$ to be the infimum of all possible values M satisfying the above inequality. The quantity $\|f\|_{L^\infty}$ is sometimes called the **essential-supremum** of f.

We note that with this definition, we have $|f(x)| \leq \|f\|_{L^\infty}$ for a.e. x. Indeed, if $E = \{x : |f(x)| > \|f\|_{L^\infty}\}$, and $E_n = \{x : |f(x)| > \|f\|_{L^\infty} + 1/n\}$, then we have $\mu(E_n) = 0$, and $E = \bigcup E_n$, hence $\mu(E) = 0$.

Theorem 10.0.8. *The vector space L^∞ equipped with $\|\cdot\|_{L^\infty}$ is a complete vector space.*

This assertion is easy to verify and is left to the reader. Moreover, Hölder's inequality continues to hold for values of p and q in the larger range $1 \leq p, q \leq \infty$, once we take $p = 1$ and $q = \infty$ as conjugate exponents, as we mentioned before.

The fact that L^∞ is a limiting case of L^p when p tends to ∞ can be understood as follows.

Proposition 10.0.9. *Suppose $f \in L^\infty$ is supported on a set of finite measure. Then $f \in L^p$ for all $p < \infty$, and*

$$\|f\|_{L^p} \to \|f\|_{L^\infty} \quad \text{as } p \to \infty.$$

Proof. Let E be a measurable subset of X with $\mu(E) < \infty$, and so that f vanishes in the complement of E. If $\mu(E) = 0$, then $\|f\|_{L^\infty} = \|f\|_{L^p} = 0$ and there is nothing to prove. Otherwise

$$\|f\|_{L^p} = \left(\int_E |f(x)|^p d\mu\right)^{1/p} \leq \left(\int_E \|f\|_{L^\infty}^p d\mu\right)^{1/p} \leq \|f\|_{L^\infty}\mu(E)^{1/p}.$$

Since $\mu(E)^{1/p} \to 1$ as $p \to \infty$, we find that $\limsup_{p\to\infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}$. On the other hand, given $\epsilon > 0$, we have

$$\mu\left(\{x : |f(x)| \geq \|f\|_{L^\infty} - \epsilon\}\right) \geq \delta \quad \text{for some } \delta > 0,$$

hence

$$\int_X |f|^p d\mu \geq \delta\left(\|f\|_{L^\infty} - \epsilon\right)^p.$$

Therefore $\liminf \inf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty} - \epsilon$, and since ϵ is arbitrary, we have $\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}$. Hence the limit $\lim_{p\to\infty} \|f\|_{L^p}$ exists, and equals $\|f\|_{L^\infty}$. $\square$

Banach spaces

We introduce here a general notion which encompasses the L^p spaces as specific examples.

First, a **normed vector space** consists of an underlying vector space V over a field of scalars (the real or complex numbers), together with a norm $\|\cdot\| : V \to \mathbb{R}^+$that satisfies:
- $\|v\| = 0$ if and only if $v = 0$.
- $\|\alpha v\| = |\alpha|\|v\|$, whenever α is a scalar and $v \in V$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

The space V is said to be **complete** if whenever $\{v_n\}$ is a Cauchy sequence in V, that is, $\|v_n - v_m\| \to 0$ as $n, m \to \infty$, then there exists a $v \in V$ such that $\|v_n - v\| \to 0$ as $n \to \infty$.

A complete normed vector space is called a **Banach space**. Here again, we stress the importance of the fact that Cauchy sequences converge to a limit in the space itself, hence the space is "closed" under limiting operations.

Examples

The real numbers $\mathbb{R}$ with the usual absolute value form an initial example of a Banach space. Other easy examples are $\mathbb{R}^d$, with the Euclidean norm, and more generally a Hilbert space with its norm given in terms of its inner product. Several further relevant examples are as follows:

Example 10.0.10. The family of L^p spaces with $1 \leq p \leq \infty$ which we have just introduced are also important examples of Banach spaces. Incidentally, L^2 is the only Hilbert space in the family L^p, where $1 \leq p \leq \infty$ (Exercise 25) and this in part accounts for the special flavor of the analysis carried out in L^2 as opposed to L^1 or more generally L^p for $p \neq 2$.

Finally, observe that since the triangle inequality fails in general when $0 < p < 1$, $\|\cdot\|_{L^p}$ is not a norm on L^p for this range of p, hence it is not a Banach space. ◇

Example 10.0.11. Another example of a Banach space is $C([0,1])$, or more generally $C(X)$ with X a compact set in a metric space. By definition, $C(X)$ is the vector space of continuous functions on X equipped with the sup-norm $\|f\| = \sup_{x \in X} |f(x)|$. Completeness is guaranteed by the fact that the uniform limit of a sequence of continuous functions is also continuous. ◇

Example 10.0.12. The space $\Lambda^\alpha\left(\mathbb{R}^d\right)$ of all continuous functions on $\mathbb{R}^d$ with the norm

$$\|f\|_{\Lambda^\alpha(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is a Banach space.

The space $L_k^p\left(\mathbb{R}^d\right)$ is the subspace of $L^p\left(\mathbb{R}^d\right)$ of all functions that have weak derivatives up to order k. This space is usually referred to as a Sobolev space. A norm that turns $L_k^p\left(\mathbb{R}^d\right)$ into a Banach space is

$$\|f\|_{L_k^p(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_{L^p(\mathbb{R}^d)}$$

◇

Linear functionals and the dual of a Banach space

For the sake of simplicity, we restrict ourselves in this and the following two sections to Banach spaces over $\mathbb{R}$; the reader will find in Section 6 the slight modifications necessary to extend the results to Banach spaces over $\mathbb{C}$.

Suppose that $\mathcal{B}$ is a Banach space over $\mathbb{R}$ equipped with a norm $\|\cdot\|$. A **linear functional** is a linear mapping ℓ from $\mathcal{B}$ to $\mathbb{R}$, that is, $\ell : \mathcal{B} \to \mathbb{R}$, which satisfies

$$\ell(\alpha f + \beta g) = \alpha \ell(f) + \beta \ell(g), \quad \text{for all } \alpha, \beta \in \mathbb{R}, \text{ and } f, g \in \mathcal{B}.$$

A linear functional ℓ is **continuous** if given $\epsilon > 0$ there exists $\delta > 0$ so that $|\ell(f) - \ell(g)| \leq \epsilon$ whenever $\|f - g\| \leq \delta$. Also we say that a linear functional is **bounded** if there is $M > 0$ with $|\ell(f)| \leq M\|f\|$ for all $f \in \mathcal{B}$. The linearity of ℓ shows that these two notions are in fact equivalent.

Proposition 10.0.13. *A linear functional on a Banach space is continuous, if and only if it is bounded.*

Proof. The key is to observe that ℓ is continuous if and only if ℓ is continuous at the origin.

Indeed, if ℓ is continuous, we choose $\epsilon = 1$ and $g = 0$ in the above definition so that $|\ell(f)| \leq 1$ whenever $\|f\| \leq \delta$, for some $\delta > 0$. Hence, given any non-zero h, an element of $\mathcal{B}$, we see that $\delta h / \|h\|$ has norm equal to δ, and hence $|\ell(\delta h / \|h\|)| \leq 1$. Thus $|\ell(h)| \leq M\|h\|$ with $M = 1/\delta$.

Conversely, if ℓ is bounded it is clearly continuous at the origin, hence continuous. □

The significance of continuous linear functionals in terms of closed hyperplanes in $\mathcal{B}$ is a noteworthy geometric point to which we return later on. Now we take up analytic aspects of linear functionals.

The set of all continuous linear functionals over $\mathcal{B}$ is a vector space since we may add linear functionals and multiply them by scalars:

$$(\ell_1 + \ell_2)(f) = \ell_1(f) + \ell_2(f) \quad \text{and} \quad (\alpha\ell)(f) = \alpha\ell(f).$$

This vector space may be equipped with a norm as follows. The norm $\|\ell\|$ of a continuous linear functional ℓ is the infimum of all values M for which $|\ell(f)| \leq M\|f\|$ for all $f \in \mathcal{B}$. From this definition and the linearity of ℓ it is clear that

$$\|\ell\| = \sup_{\|f\|\leq 1} |\ell(f)| = \sup_{\|f\|=1} |\ell(f)| = \sup_{f\neq 0} \frac{|\ell(f)|}{\|f\|}.$$

The vector space of all continuous linear functionals on $\mathcal{B}$ equipped with $\|\cdot\|$ is called the **dual space** of $\mathcal{B}$, and is denoted by $\mathcal{B}^*$.

Theorem 10.0.14. *The vector space $\mathcal{B}^*$ is a Banach space with the norm $\|\cdot\|$.*

In general, given a Banach space $\mathcal{B}$, it is interesting and very useful to be able to describe its dual $\mathcal{B}^*$. This problem has an essentially complete answer in the case of the L^p spaces introduced before.

The dual space of L^p when $1 \leq p < \infty$

Suppose that $1 \leq p \leq \infty$ and q is the conjugate exponent of p, that is, $1/p + 1/q = 1$. The key observation to make is the following: Hölder's inequality shows that every function $g \in L^q$ gives rise to a bounded linear functional on L^p by

$$\ell(f) = \int_X f(x)g(x)d\mu(x), \tag{10.4}$$

and that $\|\ell\| \leq \|g\|_{L^q}$. Therefore, if we associate g to ℓ above, then we find that $L^q \subset (L^p)^*$ when $1 \leq p \leq \infty$. The main result in this section is to prove that when $1 \leq p < \infty$, every linear functional on L^p is of the form (10.4) for some $g \in L^q$. This implies that $(L^p)^* = L^q$ whenever $1 \leq p < \infty$. We remark that this result is in general not true when $p = \infty$; the dual of L^∞ contains L^1, but it is larger.

Theorem 10.0.15. *Suppose $1 \leq p < \infty$, and $1/p + 1/q = 1$. Then, with $\mathcal{B} = L^p$ we have*

$$\mathcal{B}^* = L^q,$$

in the following sense: For every bounded linear functional ℓ on L^p there is a unique $g \in L^q$ so that

$$\ell(f) = \int_X f(x)g(x)d\mu(x), \quad \textit{for all } f \in L^p.$$

Moreover, $\|\ell\|_{\mathcal{B}^*} = \|g\|_{L^q}$.

This theorem justifies the terminology whereby q is usually called the dual exponent of p.

The proof of the theorem is based on two ideas. The first, as already seen, is Hölder's inequality; to which a converse is also needed. The second is the fact that a linear functional ℓ on $L^p, 1 \leq p < \infty$, leads naturally to a (signed) measure ν. Because of the continuity of ℓ the measure ν is absolutely continuous with respect to the underlying measure μ, and our desired function g is then the density function of ν in terms of μ. We begin with:

Lemma 10.0.16. *Suppose $1 \leq p, q \leq \infty$, are conjugate exponents.*

(i) If $g \in L^q$, then $\|g\|_{L^q} = \sup_{\|f\|_{L^p}\leq 1} \left|\int fg\right|$.

(ii) Suppose g is integrable on all sets of finite measure, and

$$\sup_{\substack{\|f\|_{L^p}\le 1\\ f\ \text{simple}}}\left|\int fg\right| = M < \infty.$$

Then $g \in L^q$, and $\|g\|_{L^q} = M$.

For the proof of the lemma, we recall the ***signum*** *of a real number defined by*

$$\operatorname{sign}(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Proof. We start with (i). If $g = 0$, there is nothing to prove, so we may assume that g is not 0 a.e., and hence $\|g\|_{L^q} \neq 0$. By Hölder's inequality, we have that

$$\|g\|_{L^q} \geq \sup_{\|f\|_{L^p}\le 1}\left|\int fg\right|.$$

To prove the reverse inequality we consider several cases.

- First, if $q = 1$ and $p = \infty$, we may take $f(x) = \operatorname{sign} g(x)$. Then, we have $\|f\|_{L^\infty} = 1$, and clearly, $\int fg = \|g\|_{L^1}$.
- If $1 < p, q < \infty$, then we set $f(x) = |g(x)|^{q-1}\operatorname{sign} g(x)/\|g\|_{L^q}^{q-1}$. We observe that

$$\|f\|_{L^p}^p = \int |g(x)|^{p(q-1)} d\mu / \|g\|_{L^q}^{p(q-1)} = 1$$

  since $p(q-1) = q$, and that

$$\int fg = \|g\|_{L^q}.$$

- Finally, if $q = \infty$ and $p = 1$, let $\epsilon > 0$, and E a set of finite positive measure, where $|g(x)| \geq \|g\|_{L^\infty} - \epsilon$. (Such a set exists by the definition of $\|g\|_{L^\infty}$ and the fact that the measure μ is σ-finite.) Then, if we take $f(x) = \chi_E(x)\operatorname{sign} g(x)/\mu(E)$, where χ_E denotes the characteristic function of the set E, we see that $\|f\|_{L^1} = 1$, and also

$$\left|\int fg\right| = \frac{1}{\mu(E)}\int_E |g| \geq \|g\|_\infty - \epsilon.$$

This completes the proof of part (i). To prove (ii) we recall, e.g. see Section 2 in Chapter 6 of Book III, that we can find a sequence $\{g_n\}$ of simple functions so that $|g_n(x)| \le |g(x)|$ while $g_n(x) \to g(x)$ for each x. When $p > 1$ (so $q < \infty$), we take $f_n(x) = |g_n(x)|^{q-1}\operatorname{sign} g(x)/\left\|g_n\right\|_{L^q}^{q-1}$. As before, $\|f_n\|_{L^p} = 1$. However

$$\int f_n g = \frac{\int |g_n(x)|^q}{\|g_n\|_{L^q}^{q-1}} = \|g_n\|_{L^q},$$

and this does not exceed M. By Fatou's lemma it follows that $\int |g|^q \le M^q$, so $g \in L^q$ with $\|g\|_{L^q} \le M$. The direction $\|g\|_{L^q} \geq M$ is of course implied by Hölder's inequality.

When $p = 1$ the argument is parallel with the above but simpler. Here we take $f_n(x) = (\operatorname{sign} g(x))\chi_{E_n}(x)$, where E_n is an increasing sequence of sets of finite measure whose union is X. The details may be left to the reader.

With the lemma established we turn to the proof of the theorem. It is simpler to consider first the case when the underlying space has finite measure. In this case, with ℓ the given functional on L^p, we can then define a set function ν by

$$\nu(E)=\ell\left(\chi_E\right),$$

where E is any measurable set. This definition makes sense because χ_E is now automatically in L^p since the space has finite measure. We observe that

$$|\nu(E)| \leq c(\mu(E))^{1/p}, \tag{10.5}$$

where c is the norm of the linear functional, taking into account the fact that $\left\|\chi_E\right\|_{L^p}=(\mu(E))^{1/p}$.

Now the linearity of ℓ clearly implies that ν is finitely-additive. Moreover, if $\{E_n\}$ is a countable collection of disjoint measurable sets, and we put $E=\bigcup_{n=1}^{\infty} E_n, E_N^*=\bigcup_{n=N+1}^{\infty} E_n$, then obviously

$$\chi_E=\chi_{E_N^*}+\sum_{n=1}^N \chi_{E_n}.$$

Thus $\nu(E)=\nu\left(E_N^*\right)+\sum_{n=1}^N \nu\left(E_n\right)$. However $\nu\left(E_N^*\right) \rightarrow 0$, as $N \rightarrow \infty$, because of (10.5) and the assumption $p<\infty$. This shows that ν is countably additive and, moreover, (10.5) also shows us that ν is absolutely continuous with respect to μ.

We can now invoke the key result about absolutely continuous measures, the Lebesgue-Radon-Nykodim theorem. (See for example Theorem 4.3, Chapter 6 in Book III.) It guarantees the existence of an integrable function g so that $\nu(E)=\int_E g d\mu$ for every measurable set E. Thus we have $\ell\left(\chi_E\right)=\int \chi_E g d\mu$. The representation $\ell(f)=\int f g d\mu$ then extends immediately to simple functions f, and by a passage to the limit, to all $f \in L^p$ since the simple functions are dense in $L^p, 1 \leq p<\infty$. Also by Lemma 4.2, we see that $\|g\|_{L^q}=\|\ell\|$. □

To pass from the situation where the measure of X is finite to the general case, we use an increasing sequence $\{E_n\}$ of sets of finite measure that exhaust X, that is, $X=\bigcup_{n=1}^{\infty} E_n$. According to what we have just proved, for each n there is an integrable function g_n on E_n (which we can set to be zero in E_n^c) so that

$$\ell(f)=\int f g_n d\mu$$

whenever f is supported in E_n and $f \in L^p$. Moreover by conclusion (ii) of the lemma $\left\|g_n\right\|_{L^q} \leq\|\ell\|$.

Now it is easy to see because of above displayed equation that $g_n=g_m$ a.e. on E_m, whenever $n \geq m$. Thus $\lim_{n \rightarrow \infty} g_n(x)=g(x)$ exists for almost every x, and by Fatou's lemma, $\|g\|_{L^q} \leq\|\ell\|$. As a result we have that $\ell(f)=\int f g d\mu$ for each $f \in L^p$ supported in E_n, and then by a simple limiting argument, for all $f \in L^p$. The fact that $\|\ell\| \leq\|g\|_{L^q}$, is already contained in Hölder's inequality, and therefore the proof of the theorem is complete.

Remark 10.0.17. *For Lemma above, we note that*

$$\sup \left\{\left|\int f g\right|:\|f\|_{L^p} \leq 1\right\}=\sup \left\{\left|\int f g\right|:\|f\|_{L^p}=1\right\}$$

and similarly,

$$\sup \left\{\left|\int f g\right|:\|f\|_{L^p} \leq 1,\ f \textit{ simple}\right\}=\sup \left\{\left|\int f g\right|:\|f\|_{L^p}=1,\ f \textit{ simple}\right\}$$

The $\geq$ direction is trivial. The other direction is because for any g with $\|f\|_{L^p} \leq 1$, we have

$$(*): \quad \frac{1}{\|f\|_{L^p}} \geq 1 \Rightarrow\left|\int\left(\frac{f}{\|f\|_{L^p}}\right) g\right|=\frac{\left|\int f g\right|}{\|f\|_{L^p}} \geq\left|\int f g\right|.$$

Since $\|\frac{f}{\|f\|_{L^p}}\|_{L^p} = 1$*, we see*

$$\sup\left\{\left|\int fg\right| : \|f\|_{L^p} = 1\right\} \geq \left|\int \frac{f}{\|f\|_{L^p}} g\right| \overset{(*)}{\geq} \left|\int fg\right|,$$

establishing the reverse direction.

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