Differential Geometry

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Smooth Manifolds

1.1 Tensor Algebra

1.1.1 Tensors and Their Products

We generalize from linear functionals to multilinear ones. If $V_1, \ldots, V_k$ and W are vector spaces, a map $F : V_1 \times \cdots \times V_k \to W$ is said to be **multilinear** if it is linear as a function of each variable separately, when all the others are held fixed:

$$F\left(v_1, \ldots, a v_i + a' v_i', \ldots, v_k\right) = a F\left(v_1, \ldots, v_i, \ldots, v_k\right) + a' F\left(v_1, \ldots, v_i', \ldots, v_k\right).$$

Given a finite-dimensional vector space V, a **covariant k-tensor on V** is a multilinear map

$$F : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

Similarly, a **contravariant k-tensor on V** is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \to \mathbb{R}.$$

We often need to consider tensors of mixed types as well. A **mixed tensor of type** (k, l), also called a **k-contravariant, l-covariant tensor**, is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to \mathbb{R}.$$

Actually, in many cases it is necessary to consider real-valued multilinear functions whose arguments consist of k covectors and l vectors, but not necessarily in the order implied by the definition above; such an object is still called a tensor of type (k, l). For any given tensor, we will make it clear which arguments are vectors and which are covectors. The spaces of tensors on V of various types are denoted by

$$\begin{aligned} T^k\left(V^*\right) &= \{\text{ covariant } k\text{-tensors on } V\}; \\ T^k(V) &= \{\text{ contravariant } k\text{-tensors on } V\}; \\ T^{(k,l)}(V) = T^k_l(V) &= \{\text{ mixed } (k,l)\text{-tensors on } V\}. \end{aligned}$$

The **rank** of a tensor is the number of arguments (vectors and/or covectors) it takes. By convention, a 0-tensor is just a real number.

There is a natural product, called the **tensor product**, linking the various tensor spaces over V: if $F \in T^{(k,l)}(V)$ and $G \in T^{(p,q)}(V)$, the tensor $F \otimes G \in T^{(k+p,l+q)}(V)$ is defined by

$$\begin{aligned} F \otimes G \left(\omega^1, \ldots, \right. & \left. \omega^{k+p}, v_1, \ldots, v_{l+q}\right) \\ & = F\left(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l\right) G\left(\omega^{k+1}, \ldots, \omega^{k+p}, v_{l+1}, \ldots, v_{l+q}\right) \end{aligned}$$

The tensor product is associative, so we can unambiguously form tensor products of three or more tensors on V. If (b_i) is a basis for V and $\left(\beta^j\right)$ is the associated dual basis, then a basis for $T^{(k,l)}(V)$ is given by the set of all tensors of the form

$$b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l},$$

as the indices i_p, j_q range from 1 to n. These tensors act on basis elements by

$$b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l}\left(\beta^{s_1}, \ldots, \beta^{s_k}, b_{r_1}, \ldots, b_{r_l}\right) = \delta_{i_1}^{s_1} \cdots \delta_{i_k}^{s_k} \delta_{r_1}^{j_1} \cdots \delta_{r_l}^{j_l}.$$

It follows that $T^{(k,l)}(V)$ has dimension n^{k+l}, where $n = \dim V$. Every tensor $F \in T^{(k,l)}(V)$ can be written in terms of this basis (using the summation convention) as

$$F = F_{j_1 \cdots j_l}^{i_1 \cdots j_k} b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l} \tag{1.1}$$

where

$$F_{j_1 \cdots j_l}^{i_1 \cdots i_k} = F\left(\beta^{i_1}, \ldots, \beta^{i_k}, b_{j_1}, \ldots, b_{j_l}\right).$$

If the arguments of a mixed tensor F occur in a nonstandard order, then the horizontal as well as vertical positions of the indices are significant and reflect which arguments are vectors and which are covectors. For example, if A is a $(1,2)$-tensor whose first argument is a vector, second is a covector, and third is a vector, its basis expression would be written

$$A = A_i{}^j{}_k \beta^i \otimes b_j \otimes \beta^k,$$

where

$$A_i{}^j{}_k = A\left(b_i, \beta^j, b_k\right)$$

There are obvious identifications among some of these tensor spaces:

$$\begin{aligned} & T^{(0,0)}(V) = T^0(V) = T^0\left(V^*\right) = \mathbb{R}, \\ & T^{(1,0)}(V) = T^1(V) = V, \\ & T^{(0,1)}(V) = T^1\left(V^*\right) = V^*, \\ & T^{(k,0)}(V) = T^k(V), \\ & T^{(0,k)}(V) = T^k\left(V^*\right). \end{aligned}$$

Due to [11] prop.12.10, we also write

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}},$$

defined as $F(V^{\times k} \times V^{* \times l})/R$, where F is the free vector space on basis $V^{\times k} \times V^{* \times l}$, or the set of all finite formal linear combinations of (k,l)-tuples, and R is the subspace of F spanned by all elements of the following forms

$$
\begin{gathered}
(v_1, \cdots, av_i, \cdots, v_k, \omega_1, \cdots, \omega_l) - a(v_1, \cdots, v_i, \cdots, v_k, \omega_1, \cdots, \omega_l) \\
(v_1, \cdots, v_k, \omega_1, \cdots, a\omega_i \cdots, \omega_l) - a(v_1, \cdots, v_k, \omega_1, \cdots, \omega_i \cdots, \omega_l) \\
(v_1, \cdots, v_i + v_i', \cdots, v_k, \omega_1, \cdots, \omega_l) - (v_1, \cdots, v_i, \cdots, v_k, \omega_1, \cdots, \omega_l) - (v_1, \cdots, v_i', \cdots, v_k, \omega_1, \cdots, \omega_l) \\
(v_1, \cdots, v_k, \omega_1, \cdots, \omega_i + \omega_i' \cdots, \omega_l) - (v_1, \cdots, v_k, \omega_1, \cdots, \omega_i \cdots, \omega_l) - (v_1, \cdots, v_k, \omega_1, \cdots, \omega_i' \cdots, \omega_l)
\end{gathered}
$$

Let $\Pi : F(V^{\times k} \times V^{*\times l}) \to T^{(k,l)} = F(V^{\times k} \times V^{*\times l})/R$ be the natural projection. The equivalence class of an element $(v_1, \cdots, v_k, \omega_1, \cdots, \omega_l)$ in $T^{(k,l)}(V)$ is denoted by

$$v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l = \Pi(v_1, \cdots, v_k, \omega_1, \cdots, \omega_l) = (v_1, \cdots, v_k, \omega_1, \cdots, \omega_l) + R.$$

and is called **(abstract) tensor product of** $v_1, \cdots, v_k, \omega_1, \cdots, \omega_l$. We note that $(v_1, \cdots, v_k, \omega_1, \cdots, \omega_l)$

It folllws from the definition that abstract tensor product satisfy

$$
\begin{aligned}
v_1 \otimes \cdots \otimes av_i \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l &= a(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l) \\
v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes a\omega_i \cdots \otimes \omega_l &= a(v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_i \cdots \otimes \omega_l) \\
v_1 \otimes \cdots \otimes (v_i + v_i') \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l &= v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l \\
&\quad + v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l \\
v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes (\omega_i + \omega_i') \cdots \otimes \omega_l &= v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_i \cdots \otimes \omega_l \\
&\quad + v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_i' \cdots \otimes \omega_l
\end{aligned}
$$

Note that the definition implies that every element of $T^{(k,l)}(V)$ can be expressed as a linear combination of elements of the form $v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l$; but it is not true in general that every element of the tensor product space is of the form $v_1 \otimes \cdots \otimes v_k \otimes \omega_1 \otimes \cdots \otimes \omega_l$.

Proposition 1.1.1 (Characteristic Property of the Tensor Product Space)**.** *Let $V_1, \cdots, V_k$ be finite-dimensional real vector spaces. If $A : V_1 \times \cdots \times V_k \to X$ is any multilinear map into a vector space X, then there is a unique linear map $\widetilde{A} : V_1 \otimes \cdots \otimes V_k \to X$ such that the following diagram commutes:*

$$
\begin{tikzcd}
V_1 \times \cdots \times V_k \arrow[r, "A"] \arrow[d, "h"'] & X \\
V_1 \otimes \cdots \otimes V_k \arrow[ur, dashed, "\widetilde{A}"'] &
\end{tikzcd}
$$

where h is the composition $h = \Pi \circ i$ of the maps $\Pi : F \to F/R$ and $i : V_1 \times \cdots \times V_k \hookrightarrow F$. Explicitly,

$$h(v_1, \cdots, v_k) = v_1 \otimes \cdots \otimes v_k$$

Proof. See [11] Proposition 12.7. ∎

Proposition 1.1.2. *Above characterization of tensor product is unique up to isomorphism.*

Proof. See Rotman's *An Introduction to Homological Algebra* (e2) Proposition 2.44. ∎

Proposition 1.1.3 (Abstract vs. Concrete Tensor Products)**.** *If $V_1, \cdots, V_k$ are finite-dimensional vector spaces, there is a canonical isomorphism*

$$V_1^* \otimes \cdots \otimes V_k^* \cong \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$$

under which the abstract tensor product defined by

$$v_1 \otimes \cdots \otimes v_k = \Pi(v_1, \cdots, v_k) = (v_1, \cdots, v_k) + R$$

corresponds to the tensor product of covectors defined by

$$\omega^1 \otimes \cdots \otimes \omega^k(v_1, \cdots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k).$$

The isomorphism $\widetilde{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \to \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$ *is the map induced by* $\Phi : V_1^* \times \cdots \times V_k^* \to \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$ *defined by* $\Phi(\omega^1, \cdots, \omega^k)(v_1, \cdots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k)$ *through the universal property 1.1.1.*

Proof. See [11] Proposition 12.10. ■

Proposition 1.1.4 (Second Dual Space)**.** *There is a canonical isomorphism between* $V^{**} := (V^*)^*$ *and* V*, namely, the isomorphism sending* v *to its* ***evaluation map*** $\bar{v}$*, defined by*

$$\begin{aligned} \bar{v} : V^* &\to \mathbb{R} \\ \omega &\mapsto \omega(v) \end{aligned}$$

Proof. See [11] Proposition 11.8. ■

We introduce an extremely important identification

$$T^{(1,1)}(V) \cong \mathrm{End}(V),$$

where $\mathrm{End}(V)$ denotes the pace of linear maps from V to itself (also called the **endomorphisms of** V). This is a special case of the following proposition.

Proposition 1.1.5. *Let* V *be a finite-dimensional vector space. There is a natural (basis-independent) isomorphism between* $T^{(k+1,l)}(V)$ *and the space of multilinear maps*

$$\underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to V.$$

Lemma 1.1.6. *Let* $\dim V_j = n_j$ *and* $\dim W = n$ *then*

$$\dim \mathcal{L}(V_1, \cdots, V_k; W) = \sum_{i=1}^{n} \prod_{j=1}^{k} n_j = n n_1 n_2 \cdots n_k$$

Proof. That's because

$$\mathcal{L}(V_1, \cdots, V_k; W) \cong \mathcal{L}(V_1, \cdots, V_k; \mathbb{R}^n) \cong \bigoplus_{i=1}^{n} \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$$

and the fact that $\bigoplus_{i=1}^n X_i$ has dimension $\sum \dim X_i$ and

$$\dim \mathcal{L}(V_1, \cdots, V_k; \mathbb{R}) = \dim V_1 \cdot \cdots \cdot \dim V_k = n_1 \cdots n_k$$

■

Lemma 1.1.7. *Let* V *be a vector space and* $v \neq 0$ *be a vector in it. There exists a linear mapping* $f : V \to \mathbb{R}$ *such that* $f(v) \neq 0$.

Proof. Suppose $V = \mathrm{span}(x_1, \cdots, x_n)$. Let $M = \mathrm{span}(v)$ as in [5] Theorem 1.10.20. Then there is a subspace $H = \mathrm{span}(x_{i_1}, \cdots, x_{i_k})$ such that $V = M \oplus H$. Now define $f(v) = 1$ and $f(x_{i_1}) = \cdots = f(x_{i_k}) = 0$ and extend them linearly to be defined on other vectors in V. ■

First proof of the proposition.
(1) Case $k=0, l=1$:

Proposition 1.1.3 gives $T^{(1,1)}(V)=V\otimes V^*\cong\mathcal{L}(V^*,V;\mathbb{R})$. We then define the mapping

$$\begin{aligned}\Phi:\operatorname{End}(V)&\to\mathcal{L}(V^*,V;\mathbb{R})\\ A&\mapsto\Phi A:=\begin{pmatrix}V^*\times V&\to\mathbb{R}\\(\omega,v)&\mapsto\omega(Av)\end{pmatrix}\end{aligned}$$

Since there is a canonical isomorphism between V and V^{**} by Proposition 1.1.4, we let the isomorphism be denoted by $\tau: V^{**}\to V$. We then define the inverse of Φ as below.

$$\begin{aligned}\Psi:\mathcal{L}(V^*,V;\mathbb{R})&\to\operatorname{End}(V)\\ f&\mapsto\Psi f:=\begin{pmatrix}V&\to V\\v&\mapsto\tau(f(\cdot,v))\end{pmatrix}\end{aligned}$$

where we note that $f(\cdot,v)$ is a map from V^* to $\mathbb{R}$ and thus belongs to V^{**}.

We show that $\Phi(\Psi f)=f$ and $\Psi(\Phi A)=A$:

- $\Phi(\Psi f)=f$. That is, we need to show $\Phi(\Psi f)(\omega,v)=f(\omega,v)$. We compute that

$$\begin{aligned}\Phi(\Psi f)(\omega,v)&=\omega((\Psi f)(v))=\omega(\underbrace{\tau(f(\cdot,v))}_{\xi})\\&=\overline{\xi}(\omega)=[f(\cdot,v)](\omega)=f(\omega,v)\end{aligned}$$

  where we note that $\tau:V^{**}\to V$ and the evaluation $\overline{\cdot}:V\to V^{**}$ are inverse of each other.

- $\Psi(\Phi A)=A$. That is, we need to show that $\Psi(\Phi A)(v)=A(v)$. We compute that

$$\begin{aligned}\Psi(\Phi A)(v)&=\tau((\Phi A)(\cdot,v))\\&=\tau(\cdot(Av))\end{aligned}$$

  Note that $\cdot(Av)$ sends every ω to $\omega(Av)$ and therefore equals to $\overline{Av}$. Thus,

$$\begin{aligned}\Psi(\Phi A)(v)&=\tau(\overline{Av})\\&=Av\end{aligned}$$

  where we again notice that τ and $\overline{\cdot}$ are inverses of each other.

(2) General case:

We similarly consider

$$\begin{aligned}\Phi:\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};V)&\to\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k+1\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};\mathbb{R})\\ A&\mapsto\Phi A:=\begin{pmatrix}V^*\times\cdots\times V^*\times V\times\cdots\times V\to\mathbb{R}\\(\omega^1,\cdots,\omega^{k+1},v_1,\cdots,v_l)\mapsto\omega^{k+1}\left(A\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)\right)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\Psi:\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k+1\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};\mathbb{R})&\to\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};V)\\ f&\mapsto\Psi f:=\begin{pmatrix}V^*\times\cdots\times V^*\times V\times\cdots\times V\to V\\(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l)\mapsto\tau(f\left(\omega^1,\cdots,\omega^k,\cdot,v_1,\cdots,v_l\right))\end{pmatrix}\end{aligned}$$

■

Second proof of the proposition.
(1) Case $k=0, l=1$:

Proposition 1.1.3 gives $T^{(1,1)}(V)=V\otimes V^*\cong\mathcal{L}(V^*,V;\mathbb{R})$. We then define the mapping

$$\begin{aligned}\Phi:\operatorname{End}(V)&\to\mathcal{L}(V^*,V;\mathbb{R})\\ A\mapsto\Phi A&:=\begin{pmatrix}V^*\times V & \to\mathbb{R}\\ (\omega,v) & \mapsto\omega(Av)\end{pmatrix}\end{aligned}$$

It is easy to see that the map Φ is well-defined and linear. Let $\dim V=n$. Notice that $\dim V=\dim V^*=n$. Then by lemma 1.1.6 we see $\dim T^{(1,1)}(V)=\dim\mathcal{L}(V^*,V;\mathbb{R})=n^2$ and $\dim\operatorname{End}(V)=\dim\mathcal{L}(V,V)=n^2$. Thus, it suffices to show that Φ is injective: for $A,B\in\operatorname{End}(V)$ we want to show that $\Phi A=\Phi B\Rightarrow A=$ B. $\Phi A=\Phi B$ implies that for any fixed $v\in V$, the following is true:

$$\begin{aligned}\forall\omega\in V^*,\qquad \Phi A(\omega,v)&=\Phi B(\omega,v)\\ \omega(Av)&=\omega(Bv)\\ \omega(Av-Bv)&=0\end{aligned}$$

Now, $\forall\omega\in V^*$, $\omega(Av-Bv)=0$ implies that $Av-Bv$ cannot be a nonzero vector, because for if it is a nonzero vector, then lemma 1.1.7 implies that we can find some $\omega\in V^*$ such that ω sends it elsewhere. Therefore, for any fixed v, $Av-Bv=(A-B)v=0\implies A-B$, which sends every vector to zero, is a zero mapping. Thus, $A=B$.

(2) General case: consider the mapping

$$\begin{aligned}\Phi:\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};V)&\to\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k+1\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};\mathbb{R})\\ A\mapsto\Phi A&:=\begin{pmatrix}V^*\times\cdots\times V^*\times V\times\cdots\times V\to\mathbb{R}\\ (\omega^1,\cdots,\omega^{k+1},v_1,\cdots,v_l)\mapsto\omega^{k+1}\left(A\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)\right)\end{pmatrix}\end{aligned}$$

We similarly only need to show injectivity: suppose

$$\omega^{k+1}\left(A\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)\right)=\omega^{k+1}\left(B\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)\right)$$

Then by the same argument, $\forall\omega^1,\cdots,\omega^k,v_1,\cdots,v_l$,

$$\begin{aligned}A\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)&=B\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)\\ (A-B)\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)&=0\end{aligned}$$

$A-B$ is then a zero mapping in $\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};V)$, so $A=B$. ■

Third proof of the proposition. We cite [23] to give another argument:

[23] Theorem 2.11: There is a natural isomorphism between $\mathcal{L}(V_1,V_2;W)$ and $\mathcal{L}(V_1,\mathcal{L}(V_2,W))$.

[23] Theorem 4.1:

(i) $V_0^2=V\otimes V\cong\mathcal{L}(V^*,V)$

(ii) $V_1^1=V\otimes V^*\cong V^*\otimes V\cong\mathcal{L}(V,V)\cong\mathcal{L}(V^*,V^*)$

(iii) $V_2^0=V^*\otimes V^*\cong\mathcal{L}(V,V^*)$

[23] Theorem 2.12: There is a natural isomorphism between $\mathcal{L}(V_1, V_2, \cdots, V_p; W)$ and $\mathcal{L}(V_i, \mathcal{L}(V_1, \cdots, \widehat{V_i}, \cdots, V_p; W))$.

For the special case $k = 0, l = 1$, let $V_1 = V^*, V_2 = V, W = \mathbb{R}$ in [23] Theorem 2.11 to get

$$\mathcal{L}\left(V^*, V; \mathbb{R}\right) \cong \mathcal{L}\left(V^*, \mathcal{L}(V, \mathbb{R})\right) = \mathcal{L}\left(V^*, V^*\right) \overset{[23]\ 4.1}{\cong} \mathcal{L}(V, V) = \operatorname{End}(V)$$

For the general case, observe the following **corollary**:

$$\begin{aligned} \mathcal{L}(V_1, \cdots, V_p; W) &\overset{[23]\ 2.12}{\cong} \mathcal{L}(V_i, \mathcal{L}(V_1, \cdots, \widehat{V_i}, \cdots, V_p; W)) \\ &\overset{[23]\ 2.12}{\cong} \mathcal{L}(V_i, \mathcal{L}(V_j, \mathcal{L}(V_1, \cdots, \widehat{V_i}, \cdots, \widehat{V_j} \cdots, V_p; W))) \\ &\overset{\text{use backwards [23] 2.12}}{\cong} \mathcal{L}(V_i, V_j; \mathcal{L}(V_1, \cdots, \widehat{V_i}, \cdots, \widehat{V_j}, \cdots, V_p; W)). \end{aligned}$$

Then

$$\begin{aligned} T^{(k+1,l)}(V) &= \underbrace{V \otimes \cdots \otimes V}_{k+1 \text{ copies}} \times \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}} \\ &\overset{[11]\ 12.10}{\cong} \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k+1 \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; \mathbb{R}) \\ &\overset{[23]\ 2.12}{\cong} \mathcal{L}(\underbrace{V^*}_{i\text{-th}}, \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; \mathbb{R})) \\ &\overset{\text{cor}}{\cong} \cdots \overset{\text{cor}}{\cong} \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}; \mathcal{L}(V^*, \underbrace{V, \cdots, V}_{l \text{ copies}}; \mathbb{R})) \\ &\cong \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; \underbrace{\mathcal{L}\left(V^*, \mathbb{R}\right)}_{=V^{**} \cong V}) \\ &\cong \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; V) \end{aligned}$$

■

Proposition 1.1.8. *For finite-dimensional vector spaces V and W we have*

$$\operatorname{Hom}(V, W) \cong V^* \otimes W.$$

Proof. We build the mappings

$$\Phi : \operatorname{Hom}(V, W) \to V^* \otimes W, \quad T \mapsto \sum_i e^i \otimes T\left(e_i\right)$$

$$\Psi : V^* \otimes W \to \operatorname{Hom}(V, W), \quad \sum_j \phi_j \otimes w_j \mapsto \left[v \mapsto \sum_j \phi_j(v) w_j\right]$$

and check they are inverses of each other:

$$\Phi \circ \Psi : \sum_j \phi_j \otimes w_j \mapsto \sum_i e^i \otimes \left(\sum_j \phi_j(e_i) w_j\right) = \sum_j \underbrace{\sum_i \phi_j(e_i) e^i}_{\phi_j} \otimes w_j = \sum_j \phi_j \otimes w_j$$

and

$$\Psi \circ \Phi : T \mapsto \left(v \mapsto \sum_i v^i T(e_i) = T(\sum_i v^i e_i) = T(v) \right) = T$$

■

Since vector spaces are bimodules, we have $V \otimes W \cong W \otimes V$. Thus,

$$\mathrm{Hom}(V, W) \cong V^* \otimes W \cong W \otimes V^*$$

and letting $W = V$ gives

$$\mathrm{Hom}(V, V) = \mathrm{End}(V) \cong V \otimes V^* = T^{(1,1)}(V).$$

This gives another proof of the Proposition in the base case. One can easily extend the result to the general case.

1.1.2 Contractions

We can use the result of proposition 1.1.5 to define a natural operation called **trace** or **contraction**, which lowers the rank of a tensor by 2. In one special case, it is easy to describe: the operator $\mathrm{tr} : T^{(1,1)}(V) \to \mathbb{R}$ is just the trace of f when it is regarded as an endomorphism of V, or in other words the sum of the diagonal entries of any matrix representation of F.

Recall the following results from basic linear algebra.

Definition 1.1.9. *If T is any linear transformation which maps vector space V of dimension n to vector space W of dimension m, there is always an $m \times n$ matrix A with the property that*

$$Tx = Ax, \qquad \forall x \in V$$

*Let $(E_1, \cdots, E_n)$ be a basis for V and $(\varepsilon^1, \cdot, \varepsilon^m)$ be a basis for W, then the **matrix of linear transformation** A is*

$$A = \begin{bmatrix} | & & | \\ T(E_1) & \cdots & T(E_n) \\ | & & | \end{bmatrix}$$

Proposition 1.1.10. *The sum of the eigenvalues λ_i of the matrix $A \in M_n(\mathbb{R})$ is equal to its trace, i.e., $\sum_{i=1}^n \lambda_i = \mathrm{tr}\, A$. Besides, $\prod_{i=1}^n \lambda_i = \det A$.*

Proposition 1.1.11. *Let $\mathcal{B}$ and $\mathcal{C}$ be any two bases of the vector space V, and let $\tau \in \mathcal{L}(V, V) = \mathrm{End}(V)$ be a linear endomorphism. Then the eigenvalues and eigenvectors are invariant under change of basis:*

$$[\tau]_{\mathcal{B}}[v]_{\mathcal{B}} = \lambda [v]_{\mathcal{B}} \Rightarrow [\tau]_{\mathcal{C}}[v]_{\mathcal{C}} = \lambda [v]_{\mathcal{C}}$$

Proof. Recall the following change of basis formula (see [19] Corollary 2.17 for (2) below for instance):

(1) $[v]_{\mathcal{C}} = \mathcal{M}_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}}$;

(2) $[\tau]_{\mathcal{C}} = \mathcal{M}_{\mathcal{B},\mathcal{C}}[\tau]_{\mathcal{B}}\mathcal{M}_{\mathcal{B},\mathcal{C}}^{-1}$.

Then the assertion directly follows from the computation:

$$\begin{aligned} [\tau]_{\mathcal{C}}[v]_{\mathcal{C}} &= \mathcal{M}_{\mathcal{B},\mathcal{C}}[\tau]_{\mathcal{B}}\mathcal{M}_{\mathcal{B},\mathcal{C}}^{-1}\mathcal{M}_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} \\ &= \mathcal{M}_{\mathcal{B},\mathcal{C}}[\tau]_{\mathcal{B}}[v]_{\mathcal{B}} \\ &= \mathcal{M}_{\mathcal{B},\mathcal{C}}\lambda[v]_{\mathcal{B}} \\ &= \lambda \mathcal{M}\mathcal{B}_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} \\ &= \lambda[v]_{\mathcal{C}} \end{aligned}$$

■

In fact, this invariance can also be seen from the fact that an eigenvalue λ of a linear endomorphism $\tau \in \mathcal{L}(V,V) = \mathrm{End}(V)$ is defined by $\tau x = \lambda x$ for some non-zero vector x and the definition does not involve basis. Now the above proposition combined with the formula of the sum of eigenvalues gives the invariance of trace of a linear endomorphism under change of basis.

Corollary 1.1.12. *The trace of a linear endomorphism is well-defined.*

Proposition 1.1.13. *Let $f \in T^{(1,1)}(V)$. Then under the definition of trace given at the beginning, $\mathrm{tr}(f) := \mathrm{tr}(\Psi f) = \sum f_i^i$, where $f_j^i = f(\varepsilon^i, E_j)$ with respect to the basis $(E_1, \cdots, E_n)$ of V and dual basis $(\varepsilon^1, \cdots, \varepsilon^n)$ of V^*.*

Proof. The linear operator here is

$$\begin{aligned} \Psi f : V &\to V \\ v &\mapsto \tau(f(\cdot, v)) \end{aligned}$$

where Ψ is defined in the second proof of the proposition 1.1.5. We will show that the matrix $[\Psi f]_{(E_k)}$ of Ψf under basis (E_k) is the following, from which we can obtain that the sum of the diagonal elements is $\sum f_i^i$, proving the statement.

$$[\Psi f]_{(E_k)} = \begin{pmatrix} f_1^1 & \cdots & f_n^1 \\ \vdots & \ddots & \vdots \\ f_1^n & \cdots & f_n^n \end{pmatrix}$$

By definition 1.1.9, we want to show that

$$\forall 1 \leq k \leq n : \qquad (\Psi f)(E_k) = \sum_i f_k^i E_i. \tag{1.2}$$

To figure out how Ψf acts on E_k, we need to know what vector ξ has its evaluation map $\bar{\xi}$ equal to $f(\cdot, v)$. Observe that

$$\begin{aligned} \bar{v} : V^* &\to \mathbb{R} \\ \omega &\mapsto \omega(v) = \omega^i \varepsilon^i (v_j E_j) = \omega^i v_i \end{aligned}$$

and that

$$\begin{aligned} f(\cdot, v) : V^* &\to \mathbb{R} \\ \omega &\mapsto f(\omega, v) \xlongequal{(1.1)} f_j^i E_i \otimes \varepsilon^j (\omega, v) \\ &= f_j^i E_i(\omega) \varepsilon^j(v) = f_j^i \omega_i v^j = \omega_i (f_j^i v^j) \end{aligned}$$

Comparing the above two equations to see $\xi_i = \sum_j f_j^i v^j$ and thus $\xi = \sum_i \left(\sum_j f_j^i v^j \right) E_i$. Then, if we let $v = E_k$, we will get

$$\xi = \sum_i (\sum_j f_j^i \delta_{kj}) E_i = \sum_i f_k^i E_i$$

which is just (1.2). ∎

More generally, we can contract a given tensor on any pair of indices as long as one is contravariant, say λ-th $(1 \leq \lambda \leq k+1)$, and one is covariant, say μ-th $(1 \leq \mu \leq l+1)$, and it can be denoted as C_μ^λ, adopted from [23] p.42:

Definition 1.1.14. *Consider the mapping $f : V^{\times(k+1)} \times V^{(l+1)} \to T^{(k,l)}(V)$ defined by*

$$(v_1, \cdots, v_{k+1}, \omega_1, \cdots, \omega_{l+1}) \mapsto \langle \omega_\mu, v_\lambda \rangle\, v_1 \otimes \cdots \otimes \widehat{v}_\lambda \otimes \cdots \otimes v_{k+1} \otimes \omega_1 \otimes \cdots \otimes \widehat{\omega}_\mu \otimes \cdots \otimes \omega_{l+1}$$

The ***contraction****, C^λ_μ, is then the unique linear mapping $\widehat{f} : T^{(k+1,l+1)}(V) \to T^{(k,l)}(V)$ with the property*

$$v_1 \otimes \cdots \otimes v_{k+1} \otimes \omega_1 \otimes \cdots \otimes \omega_{l+1} \mapsto \langle \omega_\mu, v_\lambda \rangle\, v_1 \otimes \cdots \otimes \widehat{v}_\lambda \otimes \cdots \otimes v_{k+1} \otimes \omega_1 \otimes \cdots \otimes \widehat{\omega}_\mu \otimes \cdots \otimes \omega_{l+1}$$

induced by f through the universal property 1.1.1.

As an example $C^2_1 : V^2_1 \to V$ is given by $v \otimes w \otimes \sigma \mapsto \langle \sigma, w \rangle v$, and, in particular, $e_i \otimes e_j \otimes \varepsilon^k \mapsto \langle \varepsilon^k, e_j \rangle e_i = \delta^k_j e_i$. Hence

$$A^{ij}_k e_i \otimes e_j \otimes \varepsilon^k \mapsto A^{ij}_k \delta^k_j e_i = A^{ik}_k e_i.$$

In fact, definition 1.1.14 is equivalent to the following definition.

Definition 1.1.15. *The* ***contraction*** *C^λ_μ can also be defined by*

$$T^{(k+1,l+1)}(V) \to T^{(k,l)}(V) \cong \mathcal{L}(\underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}}; \mathbb{R})$$

$$f \mapsto \left(\begin{array}{c} V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R} \\ (\omega^1, \cdots, \omega^k, v_1, \cdots, v_l) \mapsto \sum_{j=1}^n f\left(\omega^1, \cdots, \omega^{\lambda-1}, \varepsilon^j, \omega^{\lambda+1}, \cdots, \omega^k, v_1, \cdots, v_{\mu-1}, E_j, v_{\mu+1}, v_l\right) \end{array} \right)$$

We also have the following useful result.

Proposition 1.1.16. *For vector space V of dimension n, if $F \in T^{(k+1.l+1)}(V)$ has components $F^{i_1\cdots i_{k+1}}_{j_1\cdots j_{l+1}}$, then $C^\lambda_\mu F$ has components $F^{i_1\cdots i_{\lambda-1} m i_{\lambda+1}\cdots i_{k+1}}_{j_1\cdots j_{\mu-1} m j_{\mu+1}\cdots j_{l+1}}$ (summation on m). Namely,*

$$\left(C^\lambda_\mu F\right)^{i_1\cdots i_k}_{j_1\cdots j_l} = \sum_{m=1}^n F^{i_1\cdots i_{\lambda-1} m i_{\lambda+1}\cdots i_{k+1}}_{j_1\cdots j_{\mu-1} m j_{\mu+1}\cdots j_{l+1}} \tag{1.3}$$

1.1.3 Tensor Bundles and Tensor Fields

On a smooth manifold M with or without boundary, the **bundle of (k,l)-tensors** on M is defined as

$$T^{(k,l)}TM = \coprod_{p\in M} T^{(k,l)}\left(T_pM\right).$$

As special cases, the **bundle of covariant l-tensors** is denoted by $T^kT^*M = T^{(0,l)}TM$, and the **bundle of contravariant k-tensors** is denoted by $T^kTM = T^{(k,0)}TM$. There are natural identifications

$$\begin{aligned} &T^{(0,0)}TM = T^0T^*M = T^0TM = M \times \mathbb{R}, \\ &T^{(0,1)}TM = T^1T^*M = T^*M, \\ &T^{(1,0)}TM = T^1TM = TM, \\ &T^{(0,k)}TM = T^kT^*M, \\ &T^{(k,0)}TM = T^kTM. \end{aligned}$$

Exercise 1.1.17. *Show that each tensor bundle is a smooth vector bundle over M, with a local trivialization over every open subset that admits a smooth local frame for TM.*

A **tensor field on** M is a section of some tensor bundle over M. A section of $T^1T^*M = T^{(0,1)}TM$ (a covariant 1-tensor field) is also called a **covector field**. As we do with vector fields, we write the value of a tensor field F at $p \in M$ as F_p or $\left.F\right|_p$. Because covariant tensor fields are the most common and important tensor fields we work with, we use the following shorthand notation for the space of all smooth covariant k-tensor fields:

$$\mathcal{T}^k(M) = \Gamma\left(T^kT^*M\right).$$

The space of smooth 0-tensor fields is just $C^\infty(M)$. Let $(E_i) = (E_1, \ldots, E_n)$ be any smooth local frame for TM over an open subset $U \subseteq M$. Associated with such a frame is the **dual coframe**, which we typically denote by $\left(\varepsilon^1, \ldots, \varepsilon^n\right)$; these are smooth covector fields satisfying $\varepsilon^i\left(E_j\right) = \delta_j^i$. For example, given a coordinate frame $\left(\partial/\partial x^1, \ldots, \partial/\partial x^n\right)$ over some open subset $U \subseteq M$, the dual coframe is $\left(dx^1, \ldots, dx^n\right)$, where dx^i is the differential of the coordinate function x^i.

smooth local frame (E_i) and its dual coframe $\left(\varepsilon^i\right)$, the tensor fields $E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}$ form a smooth local frame for $T^{(k,l)}\left(T^*M\right)$. In particular, in local coordinates $\left(x^i\right)$, a (k,l)-tensor field F has a coordinate expression of the form

$$F = F_{j_1 \ldots j_l}^{i_1 \ldots i_k} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l},$$

where each coefficient $F_{j_1 \ldots j_l}^{i_1 \ldots i_k}$ is a smooth real-valued function on U.

Exercise 1.1.18. *Suppose $F : M \to T^{(k,l)}TM$ is a rough (k,l)-tensor field. Show that F is smooth on an open set $U \subseteq M$ if and only if whenever $\omega^1, \ldots, \omega^k$ are smooth covector fields and $X_1, \ldots, X_l$ are smooth vector fields defined on U, the real-valued function $F\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)$, defined on U by*

$$F\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)(p) = F_p\left(\left.\omega^1\right|_p, \ldots, \left.\omega^k\right|_p, \left.X_1\right|_p, \ldots, \left.X_l\right|_p\right),$$

is smooth.

An important property of tensor fields is that they are multilinear over the space of smooth functions. Suppose $F \in \Gamma\left(T^{(k,l)}TM\right)$ is a smooth tensor field. Given smooth covector fields $\omega^1, \ldots, \omega^k \in \mathcal{T}^1(M)$ and smooth vector fields $X_1, \ldots, X_l \in \mathfrak{X}(M)$, above exercise shows that the function $F\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)$ is smooth, and thus F induces a map

$$\mathcal{F} : \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to C^\infty(M).$$

It is easy to check that this map is **multilinear over** $C^\infty(M)$, that is, for all functions $u, v \in C^\infty(M)$ and smooth vector or covector fields α, β,

$$\mathcal{F}(\ldots, u\alpha + v\beta, \ldots) = u\widetilde{F}(\ldots, \alpha, \ldots) + v\widetilde{F}(\ldots, \beta, \ldots).$$

Even more important is the converse: as the next lemma shows, every such map that is multilinear over $C^\infty(M)$ defines a tensor field. (This lemma is stated and proved in [11] for covariant tensor fields, but the same argument works in the case of mixed tensors.)

Lemma 1.1.19 (Tensor Characterization Lemma). *[11] Lemma 12.24.*
A map

$$\mathcal{F} : \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to C^\infty(M)$$

is induced by a smooth (k,l)-tensor field as above if and only if it is multilinear over $C^\infty(M)$. Similarly, a map

$$\mathcal{F} : \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to \mathfrak{X}(M)$$

is induced by a smooth $(k+1,l)$-tensor field as in Proposition 1.1.5 if and only if is multilinear over $C^\infty(M)$, where $\mathcal{T}^k(M)=\Gamma\left(T^kT^*M\right)$.

Suppose $F: M \to N$ is a smooth map, then for a covariant k-tensor field on N, we define a rough k-tensor field F^*A on M, called **pullback of A by F**, by

$$(F^*A)_p = \mathrm{d}F_p^*(A_{F(p)}).$$

Proposition 1.1.20 ([11] Proposition 12.25)**.** *Suppose $F: M\to N$ and $G: P\to M$ are smooth maps, A and B are covariant tensor fields on N, and f is a real-valued function on N.*

(a) $F^*(fB)=(f\circ F)F^*B$.

(b) $F^*(A\otimes B)=F^*A\otimes F^*B$.

(c) $F^*(A+B)=F^*A+F^*B$.

(d) *F^*B is a (continuous) tensor field, and is smooth if B is smooth.*

(e) $(F\circ G)^*B=G^*\left(F^*B\right)$.

(f) $\left(\mathrm{Id}_N\right)^*B=B$.

(g) *If $p\in M$ and (y^i) are smooth coordinates for N on a neighborhood of $F(p)$, then F^*B has the following expression in a neighborhood of p:*

$$\begin{aligned}&F^*\left(B_{i_1\ldots i_k}\,\mathrm{d}y^{i_1}\otimes\cdots\otimes\,\mathrm{d}y^{i_k}\right)\\ =&\left(B_{i_1\ldots i_k}\circ F\right)\,\mathrm{d}\left(y^{i_1}\circ F\right)\otimes\cdots\otimes\,\mathrm{d}\left(y^{i_k}\circ F\right)\end{aligned}$$

Remark 1.1.21. To remember the last formula, one can use (a) and the fact that exterior differentiation commutes with pullback $F^*\,\mathrm{d}\omega=\,\mathrm{d}(F^*\omega)$. ♠

1.2 Vector Fields

1.2.1 Lie Bracket

Suppose M and N are smooth manifolds with or without boundary, and $F: M\to N$ is a smooth map. We obtain a smooth map $\mathrm{d}F: TM\to TN$, called the **global differential of** F, whose restriction to each tangent space T_pM is the linear map $\mathrm{d}F_p$ defined above. In general, the global differential does not take vector fields to vector fields. In the special case that $X\in\mathfrak{X}(M)$ and $Y\in\mathfrak{X}(N)$ are vector fields such that $dF\left(X_p\right)=Y_{F(p)}$ for all $p\in M$, we say that the vector fields X and Y are **F-related**.

Lemma 1.2.1. *([11] Prop.8.19 & Cor.8.21) Let $F: M\to N$ be a diffeomorphism between smooth manifolds with or without boundary. For every $X\in\mathfrak{X}(M)$, there is a unique vector field $F_*X\in\mathfrak{X}(N)$, called the **pushforward of** X, that is F-related to X. For every $f\in C^\infty(N)$, it satisfies*

$$\left(\left(F_*X\right)f\right)\circ F=X(f\circ F). \tag{1.4}$$

Suppose $X\in\mathfrak{X}(M)$. Given a real-valued function $f\in C^\infty(M)$, applying X to f yields a new function $Xf\in C^\infty(M)$ by $Xf(p)=X_pf$. The defining equation for tangent vectors translates into the following product rule for vector fields:

$$X(fg)=fXg+gXf. \tag{1.5}$$

A map $X: C^\infty(M)\to C^\infty(M)$ is called a **derivation of** $C^\infty(M)$ (as opposed to a derivation at a point) if it is linear over $\mathbb{R}$ and satisfies (1.5) for all $f,g\in C^\infty(M)$.

Lemma 1.2.2. *([11] Prop.8.15) Let M be a smooth manifold with or without boundary. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some $X \in \mathfrak{X}(M)$.*

Given smooth vector fields $X, Y \in \mathfrak{X}(M)$, define a map $[X, Y] : C^\infty(M) \to C^\infty(M)$ by

$$[X, Y]f = X(Yf) - Y(Xf).$$

The value of the vector field $[X, Y]$ at a point $p \in M$ can be shown to be a deriavtion at p given by the formula $[X, Y]_p f = X_p(Yf) - Y_p(Xf)$. Thus, by Lemma 1.2.2 it defines a smooth vector field, called the **Lie bracket of X and Y**.

Proposition 1.2.3 (Coordinate Formula for the Lie Bracket)**.** *Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \partial/\partial x^i$ and $Y = Y^j \partial/\partial x^j$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X, Y]$ has the following coordinate expression:*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{1.6}$$

or more concisely,

$$[X, Y] = \left(X(Y^j) - Y(X^j) \right) \frac{\partial}{\partial x^j} \tag{1.7}$$

Proposition 1.2.4 (Properties of Lie Brackets)**.** *([11] Prop.8.28) Let M be a smooth manifold with or without boundary and $X, Y, Z \in \mathfrak{X}(M)$.*

(a) BILINEARITY: $[X, Y]$ is bilinear over $\mathbb{R}$ as a function of X and Y.

(b) ANTISYMMETRY: $[X, Y] = -[Y, X]$.

(c) JACOBI IDENTITY: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

(d) For $f, g \in C^\infty(M), [fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$.

Proposition 1.2.5 (Naturality of Lie Brackets)**.** *([11] Prop.8.30 & Cor.8.31) Let $F : M \to N$ be a smooth map between manifolds with or without boundary, and let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be vector fields such that X_i is F-related to Y_i for $i = 1, 2$. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$. In particular, if F is a diffeomorphism, then $F_* [X_1, X_2] = [F_* X_1, F_* X_2]$.*

Now suppose $\widetilde{M}$ is a smooth manifold with or without boundary and $M \subseteq \widetilde{M}$ is an immersed or embedded submanifold with or without boundary. The bundle $T\widetilde{M}\big|_M$, obtained by restricting $T\widetilde{M}$ to M, is called the **ambient tangent bundle**. It is a smooth bundle over M whose rank is equal to the dimension of $\widetilde{M}$. The tangent bundle TM is naturally viewed as a smooth subbundle of $T\widetilde{M}\big|_M$, and smooth vector fields on M can also be viewed as smooth sections of $T\widetilde{M}\big|_M$. A vector field $X \in \mathfrak{X}(\widetilde{M})$ always restricts to a smooth section of $T\widetilde{M}\big|_M$, and it restricts to a smooth section of TM if and only if it is **tangent to** M, meaning that $X_p \in T_pM \subseteq T_p\widetilde{M}$ for each $p \in M$.

Corollary 1.2.6 (Brackets of Vector Fields Tangent to Submanifolds)**.** *([11] Cor.8.32) Let $\widetilde{M}$ be a smooth manifold and let M be an immersed submanifold with or without boundary in $\widetilde{M}$. If Y_1 and Y_2 are smooth vector fields on $\widetilde{M}$ that are tangent to M, then $[Y_1, Y_2]$ is also tangent to M.*

Exercise 1.2.7. *Let $\widetilde{M}$ be a smooth manifold with or without boundary and let $M \subseteq \widetilde{M}$ be an embedded submanifold with or without boundary. Show that a vector field $X \in \mathfrak{X}(\widetilde{M})$ is tangent to M if and only if $(Xf)|_M = 0$ whenever $f \in C^\infty(\widetilde{M})$ is a function that vanishes on M.*

1.2.2 Integral Curves and Flows

A **curve** in a smooth manifold M (with or without boundary) is a continuous map $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is some interval. If γ is smooth, then for each $t_0 \in I$ we obtain a vector $\gamma'(t_0) = d\gamma_{t_0}\left(d/\, dt|_{t_0}\right)$, called the **velocity of** γ **at time** t_0. It acts on functions by

$$\gamma'(t_0) f = (f \circ \gamma)'(t_0) .$$

In any smooth local coordinates, the coordinate expression for $\gamma'(t_0)$ is exactly the same as it would be in $\mathbb{R}^n$: the components of $\gamma'(t_0)$ are the ordinary t-derivatives of the components of γ.

If $X \in \mathfrak{X}(M)$, then a smooth curve $\gamma : I \to M$ is called an **integral curve of** X if its velocity at each point is equal to the value of X there: $\gamma'(t) = X_{\gamma(t)}$ for each $t \in I$.

The fundamental fact about vector fields (at least in the case of manifolds without boundary) is that there exists a unique maximal integral curve starting at each point, varying smoothly as the point varies. These integral curves are all encoded into a global object called a flow, which we now define.

Given a smooth manifold M (without boundary), a **flow domain for** M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$$

is an open interval containing 0. Given a flow domain $\mathcal{D}$ and a map $\theta : \mathcal{D} \to M$, for each $t \in \mathbb{R}$ we let

$$M_t = \{p \in M : (t, p) \in \mathcal{D}\},$$

and we define maps

$$\theta_t : M_t \to M$$

and

$$\theta^{(p)} : \mathcal{D}^{(p)} \to M$$

by $\theta_t(p) = \theta^{(p)}(t) = \theta(t, p)$. A **flow on** M is a continuous map $\theta : \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies

$$\begin{aligned} \theta_0 &= \mathrm{Id}_M, \\ \theta_t \circ \theta_s(p) &= \theta_{t+s}(p) \quad \text{wherever both sides are defined.} \end{aligned}$$

If θ is a smooth flow, we obtain a smooth vector field $X \in \mathfrak{X}(M)$ defined by $X_p = \left(\theta^{(p)}\right)'(0)$, called the **infinitesimal generator of** θ.

Theorem 1.2.8 (Fundamental Theorem on Flows). *([11] Thm.9.12) Let X be a smooth vector field on a smooth manifold M (without boundary). There is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ whose infinitesimal generator is X. This flow has the following properties:*

(a) For each $p \in M$, the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of X starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \left\{t - s : t \in \mathcal{D}^{(p)}\right\}$.

(c) For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

Although the fundamental theorem guarantees only that each point lies on an integral curve that exists for a short time, the next lemma can often be used to prove that a particular integral curve exists for all time.

Lemma 1.2.9 (Escape Lemma). *Suppose M is a smooth manifold and $X \in \mathfrak{X}(M)$. If $\gamma : I \to M$ is a maximal integral curve of X whose domain I has a finite least upper bound b, then for every $t_0 \in I, \gamma([t_0, b))$ is not contained in any compact subset of M.*

Proposition 1.2.10 (Canonical Form for a Vector Field)**.** *([11] Thm.9.22) Let X be a smooth vector field on a smooth manifold M, and suppose $p \in M$ is a point where $X_p \neq 0$. There exist smooth coordinates $\left(x^i\right)$ on some neighborhood of p in which X has the coordinate representation $\partial/\partial x^1$.*

Suppose M is a smooth manifold, V is a smooth vector field of M, and θ is the flow of V. For any smooth vector field W on M, define a vector field $\mathcal{L}_V W$ (which is smooth by [11, Lemma 9.36]) and call it **Lie derivative of W with respect to V**:

$$\left(\mathcal{L}_V W\right)_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{d}(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = \lim_{t\to 0} \frac{\mathrm{d}(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t},$$

provided the derivative exists. For a smooth k-tensor field A on M, we define a k-tensor field $\mathcal{L}_V A$ (which is smooth by [11, Lemma 12.30]) and call it **Lie derivative of A with respect to V**:

$$\left(\mathcal{L}_V A\right)_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\theta_t^* A\right)_p = \lim_{t\to 0} \frac{\left(\theta_t^* A\right)_p - A_p}{t}, \tag{1.8}$$

provided the derivative exists. Because the expression being differentiated lies in vector space $T^k\left(T_p^* M\right)$ for all t, $\left(\mathcal{L}_V A\right)_p$ makes sense as an element of $T^k\left(T_p^* M\right)$. We say A is invariant under θ if for each t, $\theta_t^* A = A$. We give a way to reconclie the two definitions using [11, Problem 12-10]:

Definition 1.2.11. *For diffeomorphism $F : M \to N$ and nonnegative integers k, l, we define **pushforward isomorphism***

$$F_* : \Gamma\left(T^{(k,l)} TM\right) \to \Gamma\left(T^{(k,l)} TN\right)$$

*and **pushback isomorphism***

$$F^* : \Gamma\left(T^{(k,l)} TN\right) \to \Gamma\left(T^{(k,l)} TM\right)$$

via the following steps: supposing $F(p) = q$,

(1) When $k = 0, l = 0$, we define $F_* f = f \circ F^{-1}$ for $f \in C^\infty(M)$; $F^* g = g \circ F$ for $g \in C^\infty(N)$.

(2) When $k = 0, l = 1$, we define $(F_*\alpha)_q = \alpha_p \circ (\mathrm{d}F_p)^{-1} = \alpha_p \circ \mathrm{d}(F^{-1})_q$ for $\alpha \in \Gamma(T^*M)$; $(F^*\beta)_p = \beta_q \circ \mathrm{d}F_p$ for $\beta \in \Gamma(T^*N)$.

(3) When $k = 1, l = 0$, we define $(F_* X)_q = (\mathrm{d}F_p) X_p$ for $X \in \Gamma(TM)$; $(F^* Y)_p = (\mathrm{d}F_p)^{-1} Y_q = \mathrm{d}(F^{-1})_q Y_q$ for $Y \in \Gamma(TN)$.

(4) When $k, l \geq 1$, we define, using case (2) and (3), for $A \in \Gamma\left(T^{(k,l)} TM\right)$ and $B \in \Gamma\left(T^{(k,l)} TN\right)$,

$$\begin{aligned}
(F_* A)_q(\beta^1(p), \ldots, \beta^k(p), Y_1(p), \ldots, Y_l(p)) &= A_p\left(F^*\beta^1(p), \cdots, F^*\beta^k(p), F^* Y_1(p), \cdots, F^* Y_l(p)\right) \\
(F^* B)_p(\alpha_1(q), \ldots, \alpha_k(q), X_1(q), \ldots, X_l(q)) &= B_q\left(F_*\alpha_1(q), \cdots, F_*\alpha_k(q), F_* X_1(q), \cdots, F_* X_l(q)\right)
\end{aligned}$$

By these notations (specifically case (3)) and the fact that inverse of the diffeomorphism θ_t is θ_{-t}, we can also write the Lie derivative of W with respect to V as

$$\left(\mathcal{L}_V W\right)_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\theta_t^* W\right)_p.$$

The pushforward and pushback isomorphisms enjoy the following properties (some don't require F to be a diffeomorphism, as we have seen in Proposition 1.1.20):

(a) $F_* = (F^*)^{-1}$.

(b) $F^*(A \otimes B) = F^* A \otimes F^* B$.

(c) $(F \circ G)_* = F_* \circ G_*$.

(d) $(F \circ G)^* = G^* \circ F^*$.

(e) $(\mathrm{Id}_M)^* = (\mathrm{Id}_M)_* = \mathrm{Id} : \Gamma\left(T^{(k,l)}TM\right) \to \Gamma\left(T^{(k,l)}TM\right)$.

(f) $F^*\left(A\left(X_1, \ldots, X_k\right)\right) = F^*A\left(F_*^{-1}\left(X_1\right), \ldots, F_*^{-1}\left(X_k\right)\right)$ for $A \in \mathcal{T}^k(N)$ and $X_1, \ldots, X_k \in \mathfrak{X}(N)$.

Proposition 1.2.12 ([11] Prop.12.32-36)**.** *Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. Suppose f is a smooth real-valued function (regarded as a 0-tensor field) on M, and A, B are smooth covariant tensor fields on M.*

(a) $\mathcal{L}_V f = Vf$.

(b) $\mathcal{L}_V(fA) = (\mathcal{L}_V f)\, A + f\mathcal{L}_V A$.

(c) $\mathcal{L}_V(A \otimes B) = (\mathcal{L}_V A) \otimes B + A \otimes \mathcal{L}_V B$.

(d) If $X_1, \ldots, X_k$ are smooth vector fields and A is a smooth k-tensor field,

$$\begin{aligned}\mathcal{L}_V\left(A\left(X_1, \ldots, X_k\right)\right) = &\left(\mathcal{L}_V A\right)\left(X_1, \ldots, X_k\right) + A\left(\mathcal{L}_V X_1, \ldots, X_k\right) \\ &+ \cdots + A\left(X_1, \ldots, \mathcal{L}_V X_k\right).\end{aligned}$$

$$\begin{aligned}\left(\mathcal{L}_V A\right)\left(X_1, \ldots, X_k\right) = &V\left(A\left(X_1, \ldots, X_k\right)\right) - A\left(\left[V, X_1\right], X_2, \ldots, X_k\right) \\ &- \cdots - A\left(X_1, \ldots, X_{k-1}, \left[V, X_k\right]\right).\end{aligned}$$

(f) If $f \in C^\infty(M)$, then $\mathcal{L}_V(df) = d\left(\mathcal{L}_V f\right)$.

(g) For any smooth covariant tensor field A and any (t_0, p) in the domain of θ,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}\left(\theta_t^* A\right)_p = \left(\theta_{t_0}^*\left(\mathcal{L}_V A\right)\right)_p$$

Thus, A is invariant under $\theta \iff \mathcal{L}_V A = 0$.

Proposition 1.2.13. *([11] Thmeorem 9.38) Suppose M is a smooth manifold and $X, Y \in \mathfrak{X}(M)$. The Lie derivative of Y with respect to X is equal to the Lie bracket $[X, Y]$.*

One of the most important applications of the Lie derivative is as an obstruction to invariance under a flow. If θ is a smooth flow, we say that a vector field Y is **invariant under** θ if $\left(\theta_t\right)_* Y = Y$ wherever the left-hand side is defined.

Proposition 1.2.14. *([11] Thmeorem 9.42) Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. A smooth vector field is invariant under the flow of X if and only if its Lie derivative with respect to X is identically zero.*

A k-tuple of vector fields $X_1, \ldots, X_k$ is said to **commute** if $[X_i, X_j] = 0$ for each i and j.

1.3 Smooth Covering Maps

A **covering map** is a surjective continuous map $\pi : \widetilde{M} \to M$ between connected and locally path-connected topological spaces, for which each point of M has connected neighborhood U that is **evenly covered**, meaning that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π. It is called a **smooth covering map** if $\widetilde{M}$ and M are smooth manifolds with or without boundary and each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U. For every evenly covered open set $U \subseteq M$, the components of $\pi^{-1}(U)$ are called the **sheets of the covering over** U.

Here are the main properties of covering maps that we need.

Proposition 1.3.1 (Elementary Properties of Smooth Covering Maps)**.**

(a) *Every smooth covering map is a local diffeomorphism, a smooth submersion, an open map, and a quotient map.*

(b) *An injective smooth covering map is a diffeomorphism.*

(c) *A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.*

Proof. See [11] Prop. 4.33. ■

Proposition 1.3.2. *A covering map is a proper map if and only if it is finite-sheeted.*

Exercise 1.3.3. *Prove the preceding proposition.*

If $\pi : \widetilde{M} \to M$ is a covering map and $F : B \to M$ is a continuous map from a topological space B into M, then a lift of $\boldsymbol{F}$ is a continuous map $\widetilde{F} : B \to \widetilde{M}$ such that $\pi \circ \widetilde{F} = F$.

Proposition 1.3.4 (Lifts of Smooth Maps are Smooth)**.** *If $\pi : \widetilde{M} \to M$ is a smooth covering map, B is a smooth manifold with or without boundary, and $F : B \to M$ is a smooth map, then every lift of F is smooth.*

Proof. Since π is a smooth submersion, every lift $\widetilde{F} : B \to \widetilde{M}$ can be written locally as $\widetilde{F} = \sigma \circ F$, where σ is a smooth local section of π (see [11] Thm. 4.26). ■

Proposition 1.3.5 (Lifting Properties of Covering Maps)**.** *Suppose $\pi : \widetilde{M} \to M$ is a covering map.*

(a) *UNIQUE LIFTING PROPERTY ([10] Thm. 11.12): If B is a connected topological space and $F : B \to M$ is a continuous map, then any two lifts of F that agree at one point are identical.*

(b) *PATH LIFTING PROPERTY ([10] Cor. 11.14): Suppose $f : [0,1] \to M$ is a continuous path. For every $\tilde{p} \in \pi^{-1}(f(0))$, there exists a unique lift $\tilde{f} : [0,1] \to \widetilde{M}$ of f such that $\tilde{f}(0) = \tilde{p}$.*

(c) *MONODROMY THEOREM ([10] Thm. 11.15): Suppose $f, g : [0,1] \to M$ are path-homotopic paths and $\widetilde{f}, \widetilde{g} : [0,1] \to \widetilde{M}$ are their lifts starting at the same point. Then $\tilde{f}$ and $\tilde{g}$ are path-homotopic and $\widetilde{f}(1) = \widetilde{g}(1)$.*

Theorem 1.3.6 (Injectivity Theorem)**.** *([10] Thm. 11.16) If $\pi : \widetilde{M} \to M$ is a covering map, then for each point $\tilde{x} \in \widetilde{M}$, the induced fundamental group homomorphism $\pi_* : \pi_1(\widetilde{M}, \widetilde{x}) \to \pi_1(M, \pi(\widetilde{x}))$ is injective.*

Theorem 1.3.7 (Lifting Criterion)**.** *([10] Thm. 11.18) Suppose $\pi : \widetilde{M} \to M$ is a covering map, B is a connected and locally path-connected topological space, and $F : B \to M$ is a continuous map. Given $b \in B$ and $\widetilde{x} \in \widetilde{M}$ such that $\pi(\widetilde{x}) = F(b)$, the map F has a lift to $\widetilde{M}$ if and only if $F_* \left(\pi_1(B, b)\right) \subseteq \pi_* \left(\pi_1(\widetilde{M}, \widetilde{x})\right)$.*

Corollary 1.3.8 (Lifting Maps from Simply Connected Spaces)**.** *([10] Cor. 11.19) Suppose $\pi : \widetilde{M} \to M$ and $F : B \to M$ satisfy the hypotheses of Theorem A.56, and in addition B is simply connected. Then every continuous map $F : B \to M$ has a lift to $\widetilde{M}$. Given any $b \in B$, the lift can be chosen to take b to any point in the fiber over $F(b)$.*

Corollary 1.3.9 (Covering Map Homeomorphism Criterion)**.** *A covering map $\pi : \widetilde{M} \to M$ is a homeomorphism if and only if the induced homomorphism $\pi_* : \pi_1(\widetilde{M}, \widetilde{x}) \to \pi_1(M, \pi(\widetilde{x}))$ is surjective for some (hence every) $\tilde{x} \in \widetilde{M}$. A smooth covering map is a diffeomorphism if and only if the induced homomorphism is surjective.*

Proof. By Theorem 1.3.7, the hypothesis implies that the identity map Id: $M \to M$ has a lift $\widetilde{\mathrm{Id}} : M \to \widetilde{M}$, which in this case is a continuous inverse for π. If π is a smooth covering map, then the lift is also smooth. ■

Corollary 1.3.10 (Coverings of Simply Connected Spaces)**.** *([10] Cor. 11.33) If M is a simply connected manifold with or without boundary, then every covering of M is a homeomorphism, and if M is smooth, every smooth covering is a diffeomorphism.*

Proposition 1.3.11 (Existence of a Universal Covering Manifold)**.** *([11] Cor. 4.43) If M is a connected smooth manifold, then there exist a simply connected smooth manifold $\widetilde{M}$, called the universal covering manifold of M, and a smooth covering map $\pi : \widetilde{M} \to M$. It is unique in the sense that if $\widetilde{M}'$ is any other simply connected smooth manifold that admits a smooth covering map $\pi' : \widetilde{M}' \to M$, then there exists a diffeomorphism $\Phi : \widetilde{M} \to \widetilde{M}'$ such that $\pi' \circ \Phi = \pi$.*

Proposition 1.3.12. *([10] Cor. 11.31)With $\pi : \widetilde{M} \to M$ as in the previous proposition, each fiber of π has the same cardinality as the fundamental group of M.*

Exercise 1.3.13. *Suppose $\pi : \widetilde{M} \to M$ is a covering map. Show that $\widetilde{M}$ is compact if and only if M is compact and π is a finite-sheeted covering.*

1.4 Vector Spaces $T^k(V^*), \Sigma^k(V^*), \Lambda^k(V^*)$

Let V be a f.d. vector space. The vector spaces of all **covariant k-tensor**, **contravariant l-tensor**, **(k,l)-mixed type tensor** are

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ factors}}, \quad T^l(V) = \underbrace{V \otimes \cdots \otimes V}_{l \text{ factors}}, \quad T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ factors}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ factors}}$$

It (E_i) is a basis for V and (ε^j) is the dual basis for V^*, then their bases are

$$\begin{aligned} &\left\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \le i_1, \ldots, i_k \le n\right\} \quad \text{for } T^k\left(V^*\right); \\ &\left\{E_{i_1} \otimes \cdots \otimes E_{i_k} : 1 \le i_1, \ldots, i_k \le n\right\} \quad \text{for } T^k(V); \\ &\left\{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \le i_1, \ldots, i_k, j_1, \ldots, j_l \le n\right\} \quad \text{for } T^{(k,l)}(V). \end{aligned}$$

Therefore, $\dim T^k\left(V^*\right) = \dim T^k(V) = n^k$ and $\dim T^{(k,l)}(V) = n^{k+l}$.

Subspace $\Sigma^k(V^*)$

A covariant k-tensor α on V is said to be **symmetric** if its value is unchanged by interchanging any pair of arguments:

$$\alpha\left(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k\right) = \alpha\left(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k\right)$$

whenever $1 \le i < j \le k$. These symmetric covariant k-tensors form linear subspace $\Sigma^k\left(V^*\right)$ in $T^k\left(V^*\right)$. Given a k-tensor α and a permutation $\sigma \in S_k$, we define a new k-tensor ${}^\sigma\alpha$ by

$${}^\sigma\alpha\left(v_1, \ldots, v_k\right) = \alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)$$

Note that ${}^\tau\left({}^\sigma\alpha\right) = {}^{\tau\sigma}\alpha$. We define a projection $\operatorname{Sym} : T^k\left(V^*\right) \to \Sigma^k\left(V^*\right)$ called **symmetrization** by

$$\operatorname{Sym}\alpha = \frac{1}{k!} \sum_{\sigma \in S_k} {}^\sigma\alpha$$

[11] Proposition 12.14 shows that $\operatorname{Sym}\alpha$ is indeed symmetric and a form α is symmetric if and only if $\operatorname{Sym}\alpha = \alpha$. If $\alpha \in \Sigma^k\left(V^*\right)$ and $\beta \in \Sigma^l\left(V^*\right)$, we define their **symmetric product** to be the $(k+l)$ tensor $\alpha\beta$ (denoted by juxtaposition) given by

$$\alpha\beta = \operatorname{Sym}(\alpha \otimes \beta)$$

Example 1.4.1. By [11] p.315 Proposition 12.15, if α and β are covectors, then

$$\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha). \tag{1.9}$$

♣

A basis of $\Sigma^k(V^*)$ is given by

$$\left\{\operatorname{Sym}\left(\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}\right), 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n\right\}$$

so that

$$\dim\left(\Sigma^k(V^*)\right) = \binom{n+k-1}{k}$$

For an attempt to write the basis in form $\{\alpha \otimes \cdots \otimes \alpha'\}$, see this post.

Subspace $\Lambda^k(V^*)$

A covariant k-tensor α on V is said to be **alternating** (or **antisymmetric** or **skew-symmetric**) if

$$\alpha\left(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k\right) = -\alpha\left(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k\right)$$

Alternating covariant k-tensors are also variously called **exterior forms**, **multicovectors**, or k-**covectors**. The subspace of all alternating covariant k-tensors on V is denoted by $\Lambda^k\left(V^*\right) \subseteq T^k\left(V^*\right)$.

Recall that for any permutation $\sigma \in S_k$, the sign of σ, denoted by $\operatorname{sgn}\sigma$, is equal to $+1$ if σ is even (i.e., can be written as a composition of an even number of transpositions), and -1 if σ is odd.

Proposition 1.4.2. *Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:*

(a) α *is alternating.*

(b) For any vectors $v_1, \ldots, v_k$ and any permutation $\sigma \in S_k$,

$$\alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = (\operatorname{sgn}\sigma)\alpha\left(v_1, \ldots, v_k\right)$$

(c) $\alpha\left(v_1, \ldots, v_k\right) = 0$ *whenever the k-tuple $\left(v_1, \ldots, v_k\right)$ is linearly dependent.*

(d) α *gives the value zero whenever two of its arguments are equal:*

$$\alpha\left(v_1, \ldots, w, \ldots, w, \ldots, v_k\right) = 0$$

Proof. See [11] Exercise 12.17 and Lemma 14.1. ■

Example 1.4.3. Every 0-tensor (which is just a real number) is both symmetric and alternating, because there are no arguments to interchange. Similarly, every 1-tensor is both symmetric and alternating. An alternating 2-tensor on V is a skew-symmetric bilinear form. It is interesting to note that every covariant 2-tensor β can be expressed as a sum of an alternating tensor and a symmetric one, because

$$\beta(v,w) = \frac{1}{2}(\beta(v,w) - \beta(w,v)) + \frac{1}{2}(\beta(v,w) + \beta(w,v)) = \alpha(v,w) + \sigma(v,w)$$

where $\alpha(v,w) = \frac{1}{2}(\beta(v,w) - \beta(w,v))$ is an alternating tensor, and $\sigma(v,w) = \frac{1}{2}(\beta(v,w) + \beta(w,v))$ is symmetric. This is not true for tensors of higher rank, as [11] Problem 12-7 shows. ♣

We define **alteration**, an analogue of symmetrization as the projection $\mathrm{Alt} : T^k(V^*) \to \Lambda^k(V^*)$, as follows:

$$\mathrm{Alt}\,\alpha = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\left({}^\sigma \alpha\right)$$

More explicitly, this means

$$(\mathrm{Alt}\,\alpha)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)$$

Example 1.4.4. If α is any 1-tensor, then $\mathrm{Alt}\,\alpha = \alpha$. If β is a 2 -tensor, then

$$(\mathrm{Alt}\,\beta)(v, w) = \frac{1}{2}(\beta(v, w) - \beta(w, v))$$

♣

Similar to the properties of symmetrization operator, we have $\mathrm{Alt}\,\alpha$ is alternating; and that $\mathrm{Alt}\,\alpha = \alpha \iff \alpha$ is alternating.

To describe the basis of $\Lambda^k(V^*)$, we introduce some notations. For multi-index $I = (i_1, \cdots, i_k)$, we let

$$I_\sigma = \left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right).$$

Note that $I_{\sigma\tau} = (I_\sigma)_\tau$ for $\sigma, \tau \in S_k$.

For a multi-index $I = (i_1, \ldots, i_k)$ with non-decreasing components $i_1 \leq \cdots \leq i_k$, define a covariant k-tensor $\varepsilon^I = \varepsilon^{i_1 \ldots i_k}$ by

$$\varepsilon^I(v_1, \ldots, v_k) = \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \ldots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \ldots & \varepsilon_k^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \ldots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \ldots & v_k^{i_k} \end{pmatrix}.$$

In other words, if $\mathbb{V}$ denotes the $n \times k$ matrix whose columns are the components of the vectors $v_1, \ldots, v_k$ with respect to the basis (E_i) dual to $\left(\varepsilon^i\right)$, then $\varepsilon^I(v_1, \ldots, v_k)$ is the determinant of the $k \times k$ submatrix consisting of rows $i_1, \ldots, i_k$ of $\mathbb{V}$. Because the determinant changes sign whenever two columns are interchanged, it is clear that ε^I is an alternating k-tensor. We call ε^I an **elementary alternating tensor** or **elementary k-covector**.

We adopt the following notation

$$\sum_I{}' \alpha_I \varepsilon^I = \sum_{\{I : i_1 < \cdots < i_k\}} \alpha_I \varepsilon^I.$$

Proposition 1.4.5. *Let V be an n-dimensional vector space. If $\left(\varepsilon^i\right)$ is any basis for V^*, then for each positive integer $k \leq n$, the collection of k-covectors*

$$\mathcal{E} = \left\{\varepsilon^I : I \text{ is an increasing multi-index of length } k\right\}$$

is a basis for $\Lambda^k(V^*)$. Therefore,

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If $k > n$, then $\dim \Lambda^k(V^*) = 0$.

In particular, for an n-dimensional vector space V, this proposition implies that $\Lambda^n(V^*)$ is 1-dimensional and is spanned by $\varepsilon^{1 \ldots n}$. By definition, this elementary n covector acts on vectors $(v_1, \ldots, v_n)$ by taking the determinant of their component matrix $\mathbb{V} = \left(v_j^i\right)$. For example, on $\mathbb{R}^n$ with the standard basis, $\varepsilon^{1 \ldots n}$ is precisely the determinant function.

Proposition 1.4.6. *Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n\left(V^*\right)$. If $T: V \rightarrow V$ is any linear map and $v_1, \ldots, v_n$ are arbitrary vectors in V, then*

$$\omega\left(T v_1, \ldots, T v_n\right)=(\det T) \omega\left(v_1, \ldots, v_n\right)$$

Given $\omega \in \Lambda^k\left(V^*\right)$ and $\eta \in \Lambda^l\left(V^*\right)$, we define their wedge product or exterior product to be the following $(k+l)$-covector:

$$\omega \wedge \eta:=\frac{(k+l)!}{k!l!} \omega \barwedge \eta:=\frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta)$$

Proposition 1.4.7 (Properties of Wedge Product; [11] Lemma 14.10 and Proposition 14.11)**.** *Suppose $\omega, \omega^{\prime}, \eta, \eta^{\prime}$, and ξ are multicovectors on a finite-dimensional vector space V.*

(a) For any multi-indices I and J of lengths k and l, we have $\varepsilon^I \wedge \varepsilon^J=\varepsilon^{I J}$ where $I J$ is the concatenation.

(a) BILINEARITY: For $a, a^{\prime} \in \mathbb{R}$,

$$\begin{gathered}\left(a \omega+a^{\prime} \omega^{\prime}\right) \wedge \eta=a(\omega \wedge \eta)+a^{\prime}\left(\omega^{\prime} \wedge \eta\right) \\ \eta \wedge\left(a \omega+a^{\prime} \omega^{\prime}\right)=a(\eta \wedge \omega)+a^{\prime}\left(\eta \wedge \omega^{\prime}\right)\end{gathered}$$

(b) ASSOCIATIVITY:

$$\omega \wedge(\eta \wedge \xi)=(\omega \wedge \eta) \wedge \xi$$

(c) ANTICOMMUTATIVITY: For $\omega \in \Lambda^k\left(V^*\right)$ and $\eta \in \Lambda^l\left(V^*\right)$,

$$\omega \wedge \eta=(-1)^{k l} \eta \wedge \omega$$

(d) If $\left(\varepsilon^i\right)$ is any basis for V^* and $I=\left(i_1, \ldots, i_k\right)$ is any multi-index, then

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}=\varepsilon^I$$

(e) For any covectors $\omega^1, \ldots, \omega^k$ and vectors $v_1, \ldots, v_k$,

$$\omega^1 \wedge \cdots \wedge \omega^k\left(v_1, \ldots, v_k\right)=\det\left(\omega^j\left(v_i\right)\right)$$

There is an important operation that relates vectors with alternating tensors. Let V be a finite-dimensional vector space. For each $v \in V$, we define a linear map $i_v: \Lambda^k\left(V^*\right) \rightarrow \Lambda^{k-1}\left(V^*\right)$, called **interior multiplication by** v, as follows:

$$i_v \omega\left(w_1, \ldots, w_{k-1}\right)=\omega\left(v, w_1, \ldots, w_{k-1}\right)$$

In other words, $i_v \omega$ is obtained from ω by inserting v into the first slot. By convention, we interpret $i_v \omega$ to be zero when ω is a 0-covector (i.e., a number). Another common notation is

$$v \lrcorner \omega=i_v \omega$$

This is often read "v into ω."

Proposition 1.4.8 ([11] Lemma 14.13.)**.** *Let V be a finite-dimensional vector space and $v \in V$.*

(a) $i_v \circ i_v=0$.

(b) If $\omega \in \Lambda^k\left(V^*\right)$ and $\eta \in \Lambda^l\left(V^*\right)$,

$$i_v(\omega \wedge \eta)=\left(i_v \omega\right) \wedge \eta+(-1)^k \omega \wedge\left(i_v \eta\right)$$

(c) More generally, if $\omega^1, \cdots, \omega^k$ are k covectors, we have

$$v \lrcorner(\omega^1 \wedge \cdots \wedge \omega^k)=\sum_{i=1}^k(-1)^{i-1} \omega^i(v) \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k .$$

We make a brief summary.

	spaces	projection	product of k form & l form
symmetric k-tensor	$\Sigma^k(V^*)$	symmetrization Sym	symmetric product $\alpha\beta = \operatorname{Sym}(\alpha \otimes \beta)$
alternating k-tensor	$\Lambda^k(V^*)$	alternation Alt	wedge product $\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)$

1.5 Differential Forms

We record the following notations

$$T^k T^* M = \coprod_{p \in M} T^k(T_p^* M), \quad \Lambda^k T^* M = \coprod_{p \in M} \Lambda^k(T_p^* M), \quad \Sigma^k T^* M = \coprod_{p \in M} \Sigma^k(T_p^* M).$$

They are the bundle of all k-tensors on M, the bundle of all alternating k-tensors on M, and the bundle of all symmetric k-tensors on M. A section of each of these three is called a **k-tensor field**, a **differential k-form**, and a **symmetric k-tensor field**. We denote that vector space of all smooth k-forms by

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M).$$

In a smooth coordinate $(U, (x^i))$, we have basis $\{\frac{\partial}{\partial x^i}|_p\}$ for T_pM and basis $\{dx^i|_p\}$ for T_p^*M. A 0-form is just a continuous real-valued function, and a 1-form is a covector field.

If $F : M \to N$ is a smooth map and ω is a differential form on N, the **pullback** $F^*\omega$ is a differential form on M, defined iin the same wat as for any covariant tensor field:

$$(F^*\omega)_p (v_1, \ldots, v_k) = \omega_{F(p)} \left(\mathrm{d}F_p (v_1), \ldots, \mathrm{d}F_p (v_k)\right)$$

Similar to Proposition 1.1.20, we have

Lemma 1.5.1 (Pullback; [11] Lemma 14.16)**.** *Suppose $F : M \to N$ is smooth.*

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart (y^i) in N,

$$F^*\left(\sum_I{}' \omega_I \, \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}\right) = \sum_I{}' \left(\omega_I \circ F\right) \, \mathrm{d}\left(y^{i_1} \circ F\right) \wedge \cdots \wedge \mathrm{d}\left(y^{i_k} \circ F\right).$$

Proposition 1.5.2 (Pullback Formula for Top-Degree Forms; [11] Proposition 14.20)**.** *Let $F : M \to N$ be a smooth map between n-manifolds with or without boundary. If $\left(x^i\right)$ and $\left(y^j\right)$ are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:*

$$F^*\left(u \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n\right) = (u \circ F)(\det DF) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

where DF represents the Jacobian matrix of F in these coordinates.

Corollary 1.5.3. *If $\left(U, \left(x^i\right)\right)$ and $\left(\tilde{U}, \left(\tilde{x}^j\right)\right)$ are overlapping smooth coordinate charts on M, then the following identity holds on $U \cap \tilde{U}$:*

$$\mathrm{d}\tilde{x}^1 \wedge \cdots \wedge \mathrm{d}\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

Theorem 1.5.4 (Exterior Differentiation)**.** *Suppose M is a smooth manifold with or without boundary. There are unique operators $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k, called **exterior differentiation**, satisfying the following four properties:*

(i) d *is linear over* $\mathbb{R}$.

(ii) If $\omega \in \Omega^k(M)$ *and* $\eta \in \Omega^l(M)$*, then*

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$$

(iii) $\mathrm{d} \circ \mathrm{d} \equiv 0$.

(iv) For $f \in \Omega^0(M) = C^\infty(M)$, $\mathrm{d}f$ *is the differential of* f*, given by* $\mathrm{d}f(X) = Xf$.

In any smooth coordinate chart, d *is given by*

$$\mathrm{d}\left(\sum_J{}' \omega_J \, \mathrm{d}x^J\right) = \sum_J{}' \, \mathrm{d}\omega_J \wedge \mathrm{d}x^J \tag{1.10}$$

where $\mathrm{d}\omega_J = \sum_i \frac{\partial \omega_J}{\partial x^i} \mathrm{d}x^i$ *is the differential of the function* ω_J.

For example, suppose (U, φ) is a local chart for smooth manifold M and f is a smooth function on M. Then

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i} \mathrm{d}x^i$$

is interpreted as follow:

$$\mathrm{d}f_p = \sum_i \frac{\partial \hat{f}}{\partial x^i}(\hat{p}) \, \mathrm{d}x^i|_p = \sum_i \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) \, \mathrm{d}x^i|_p.$$

Proposition 1.5.5 (Naturality of the Exterior Derivative). *If* $F : M \to N$ *is a smooth map, then for each* k *the pullback map* $F^* : \Omega^k(N) \to \Omega^k(M)$ *commutes with d: for all* $\omega \in \Omega^k(N)$*,*

$$F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega)$$

Proposition 1.5.6 (Exterior Derivative of a 1-Form). *For any smooth 1-form* ω *and smooth vector fields* X *and* Y*,*

$$\mathrm{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Proposition 1.5.7 (Invariant Formula for the Exterior Derivative). *Let* M *be a smooth manifold with or without boundary, and* $\omega \in \Omega^k(M)$*. For any smooth vector fields* $X_1, \ldots, X_{k+1}$ *on* M*,*

$$\begin{aligned} &\mathrm{d}\omega\left(X_1, \ldots, X_{k+1}\right) \\ &= \sum_{1 \le i \le k+1} (-1)^{i-1} X_i \left(\omega\left(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}\right)\right) \\ &\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left(\left[X_i, X_j\right], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\right), \end{aligned}$$

where the hats indicate omitted arguments.

Remark 1.5.8. As noted on page 372 of [LeeSM], we can use the formula above to define $d\omega$ and derive all the properties of exterior differentiation too. ♠

Proposition 1.5.9 (Some formulas for Lie Derivative and Differential Forms). *Let* M *be a smooth manifold,* $V \in \mathfrak{X}(M)$*, and* $\omega, \eta \in \Omega^*(M) = \bigoplus_k \Omega^k(M)$*. Then*

$$\begin{aligned} &\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V\omega) \wedge \eta + \omega \wedge (\mathcal{L}_V\eta) \\ &\mathcal{L}_V\omega = i_V(\mathrm{d}\omega) + \mathrm{d}(i_V\omega) \quad (\textit{Cartan's magic formula}) \\ &\mathcal{L}_V(\mathrm{d}\omega) = \mathrm{d}(\mathcal{L}_V\omega). \end{aligned}$$

1.6 Orientation on Manifolds

For every finite dimensional vector space V, two ordered bases (E_i), $(\widetilde{E}_j)$ are **consistently oriented** if the transition matrix B_i^j such that $E_i = B_i^j \widetilde{E}_j$ has positive determinant. This is an equivalence relation among the collection of all ordered bases of V. Since transition matrix is invertible, the determinant is either positive or negative, so there are exactly two equivalence classes $[E_1, \cdots, E_n]$ and $-[E_1, \cdots, E_n]$. A choice of one of them gives an **orientation** for V. A vector space with a choice of orientation is said to be **oriented**. When $n = 0$ we define orientation as a choice of a number.

Proposition 1.6.1. *Let V be a vector space of dimension n. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then $\mathcal{O}_\omega$ is the set of ordered bases $(E_1, \ldots, E_n)$ such that $\omega(E_1, \ldots, E_n) > 0$; while if $n = 0$, then $\mathcal{O}_\omega$ is +1 if $\omega > 0$, and -1 if $\omega < 0$. Two nonzero n-covectors determine the same orientation if and only if each is a positive multiple of the other.*

If V is an oriented n-dimensional vector space and ω is an n-covector that determines the orientation of V as described in this proposition, we say that ω is a **(positively) oriented n-covector**.

For any n-dimensional vector space V, the space $\Lambda^n(V^*)$ is 1-dimensional. The proposition shows that choosing an orientation for V is equivalent to choosing one of the two components of $\Lambda^n(V^*) \setminus \{0\}$. This formulation also works for 0-dimensional vector spaces, and explains why we have defined an orientation of a 0 -dimensional space in the way we did.

Now suppose M is a manifold. We define a **pointwise orientation** on M to be a choice of orientation of each tangent space.

Let M be a smooth n-manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM, we say that (E_i) is **(positively) oriented** if $\left(\left.E_1\right|_p, \ldots, \left.E_n\right|_p\right)$ is a positively oriented basis for T_pM at each point $p \in U$. A **negatively oriented frame** is defined analogously.

A pointwise orientation is said to be **continuous** if every point of M is in the domain of some oriented local frame. An **orientation of** M is a continuous pointwise orientation. We say that M is **orientable** if there exists an orientation for it, and **nonorientable** if not. An **oriented** manifold is an ordered pair $(M, \mathcal{O})$, where M is an orientable smooth manifold and $\mathcal{O}$ is a choice of orientation for M; an oriented manifold with boundary is defined similarly. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$.

If M is zero-dimensional, this definition just means that an orientation of M is a choice of ± 1 attached to each of its points. The continuity condition is vacuous in this case, and the notion of oriented frames is not useful. Clearly, every 0-manifold is orientable.

Proposition 1.6.2 (The Orientation Determined by an n-Form; [11] Proposition 15.5)**.** *Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*

Remark 1.6.3. Because of this proposition, if M is a smooth n-manifold with or without boundary, any nonvanishing n-form on M is called an **orientation form**. If M is oriented and ω is an orientation form determining the given orientation, we also say that ω is **(positively) oriented**. It is easy to check that if ω and $\widetilde{\omega}$ are two positively oriented smooth forms on M, then $\widetilde{\omega} = f\omega$ for some strictly positive smooth realvalued function f. If M is a 0-manifold, a nonvanishing 0 -form (i.e., real-valued function) assigns the orientation +1 to points where it is positive and -1 to points where it is negative. ♠

A smooth coordinate chart on an oriented smooth manifold with or without boundary is said to be **(positively) oriented** if the coordinate frame $\left(\partial/\partial x^i\right)$ is positively oriented, and **negatively oriented** if the coordinate frame is negatively oriented. A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is said to be **consistently oriented** if for each α, β, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Proposition 1.6.4 (The Orientation Determined by a Coordinate Atlas; [11] Proposition 15.6.)**.** *Let M be a smooth positive-dimensional manifold with or without boundary. Given any consistently oriented smooth atlas for M, there is a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \varnothing$ or $\dim M > 1$, then the collection of all oriented smooth charts is a consistently oriented atlas for M.*

We have seen

A consistently oriented C^∞ atlas
$\Updownarrow$ [LeeSM] 15.6.
A continuous pointwise orientation
$\Updownarrow$ [LeeSM] 15.5.
An orientation form, i.e., a nonvanishing top-degree form

Proposition 1.6.5. *Let M be a connected, orientable, smooth manifold with or without boundary. Then M has exactly two orientations. If two orientations of M agree at one point, then they are equal.*

Let M and N be oriented smooth manifolds with or without boundary, and suppose $F : M \to N$ is a local diffeomorphism. If M and N are positive-dimensional, we say that F is **orientation-preserving** if for each $p \in M$, the isomorphism dF_p takes oriented bases of T_pM to oriented bases of $T_{F(p)}N$, and **orientation-reversing** if it takes oriented bases of T_pM to negatively oriented bases of $T_{F(p)}N$. If M and N are 0-manifolds, then F is orientation-preserving if for every $p \in M$, the points p and $F(p)$ have the same orientation; and it is orientation-reversing if they have the opposite orientation.

Exercise 1.6.6. *Suppose M and N are oriented positive-dimensional smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Show that the following are equivalent.*

(a) F is orientation-preserving.

(b) With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.

(c) For any positively oriented orientation form ω for N, the form $F^*\omega$ is positively oriented for M.

Exercise 1.6.7. *Show that a composition of orientation-preserving maps is orientation-preserving.*

Suppose M and N are smooth manifolds with or without boundary. Suppose $F : M \to N$ be a local diffeomorphism and N is given with orientation $\mathcal{O}$. Then we can naturally define an orientation $F^*\mathcal{O}$, called **pullback orientation induced by** F, making F orientation-preserving: for each $p \in M$, choose the orientation of T_pM such that $dF_p : T_pM \to T_{F(p)}N$ preserves orientation. Another orientation arising naturally from constructed manifolds is the **product orientation**: it's the unique orientation on the product of orientatable manifolds $M_1 \times \cdots \times M_k$ with the property that if for each $i = 1, \ldots, k, \omega_i$ is an orientation form for the given orientation on M_i, then $\pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k$ is an orientation form for the product orientation.

Example 1.6.8. Every parallelizable smooth manifold is orientable via declaring the oritentation (the equivalence class) containing $(E_i|_p)$ as the positive one, given the global smooth frame (E_i) on M. This pointwise assignment is continuous as each $p \in M$ is in a positively oriented frame containing it. Examples for such manifolds include Euclidean space $\mathbb{R}^n$, the n-torus $\mathbb{T}^n$, the spheres $\mathbb{S}^1, \mathbb{S}^3$, and $\mathbb{S}^7$, Lie groups and products of them. ♣

1.6.1 Orientation on Submanifolds

Proposition 1.6.9 (Orientations of Codimension-0 Submanifolds)**.** *Suppose M is an oriented smooth manifold with or without boundary, and $D \subseteq M$ is a smooth codimension-0 submanifold with or without boundary. Then the orientation of M restricts to an orientation of D. If ω is an orientation form for M, then $\iota_D^*\omega$ is an orientation form for D.*

Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is a smooth submanifold (immersed or embedded, with or without boundary). Recall that a vector field along S is a section of the ambient tangent bundle $TM|_S$, i.e., a continuous map $N : S \to TM$ with the property that $N_p \in T_pM$ for each $p \in S$. For example, any vector field on M restricts to a vector field along S, but in general, not every vector field along S is of this form (see [11, Problem 10-9]).

Proposition 1.6.10 (Orientation induced by nowhere tangent vector field on submanifold)**.** *Suppose M is an oriented smooth n-manifold with or without boundary, S is an immersed hypersurface (i.e., codimension-1 submanifold) with or without boundary in M, and N is a vector field along S that is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \ldots, E_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \ldots, E_{n-1})$ is an oriented basis for T_pM. If ω is an orientation form for M, then $\iota_S^*(N \lrcorner \omega)$ is an orientation form for S with respect to this orientation, where $\iota_S : S \hookrightarrow M$ is inclusion.*

Remark 1.6.11. The reason for us to put N_p in $(N_p, E_1, \ldots, E_{n-1})$ in the first place is that we want to use the notation $N \lrcorner \omega$ in the above proposition. If we remove the last sentence, the position of N_p in the tuple does not matter. ♠

Proof. Let ω be an orientation form for M. Then $\sigma = \iota_S^*(N \lrcorner \omega)$ is an $(n-1)$-form on S. (Recall that the pullback ι_S^* is really just restriction to vectors tangent to S.) It will follow that σ is an orientation form for S if we can show that it never vanishes. Given any basis $(E_1, \ldots, E_{n-1})$ for T_pS, the fact that N is nowhere tangent to S implies that $(N_p, E_1, \ldots, E_{n-1})$ is a basis for T_pM. The fact that ω is nonvanishing implies that

$$\sigma_p(E_1, \ldots, E_{n-1}) = \omega_p(N_p, E_1, \ldots, E_{n-1}) \neq 0$$

Since $\sigma_p(E_1, \ldots, E_{n-1}) > 0$ if and only if $\omega_p(N_p, E_1, \ldots, E_{n-1}) > 0$, the orientation determined by σ is the one defined in the statement of the proposition. ■

Example 1.6.12. The sphere $\mathbb{S}^n$ is a hypersurface in $\mathbb{R}^{n+1}$, to which the vector field $N = x^i \partial/\partial x^i$ is nowhere tangent, so this vector field induces an orientation on $\mathbb{S}^n$. This shows that all spheres are orientable. We define the standard orientation of $\mathbb{S}^n$ to be the orientation determined by N. Unless otherwise specified, we always use this orientation. (The standard orientation on $\mathbb{S}^0$ is the one that assigns the orientation $+1$ to the point $+1 \in \mathbb{S}^0$ and -1 to $-1 \in \mathbb{S}^0$.) ♣

Not every hypersurface admits a nowhere tangent vector field. (See [11, Problem 15-6].) However, the following proposition gives a sufficient condition that holds in many cases.

Proposition 1.6.13. *Let M be an oriented smooth manifold, and suppose $S \subseteq M$ is a regular level set of a smooth function $f : M \to \mathbb{R}$. Then S is orientable.*

Proof. Choose any Riemannian metric on M, and let $N = \operatorname{grad} f|_S$. $S = f^{-1}(c)$ being a regular level set, so $T_pS = \ker \mathrm{d}f_p$ by [11, Theorem 5.38]. By definition of gradient, $\mathrm{d}f_p(N_p) = g(N_p, N_p) = \|N_p\|^2 > 0$ (note that $\operatorname{grad} f(p) = 0 \iff \mathrm{d}f_p = 0$ while for any $p \in S$, $\mathrm{d}f_p$ is surjective and thus nonzero). But any tangent vector $X \in T_pS$ must satisfy $\mathrm{d}f_p(X) = 0$ (since f is constant on S). So $N_p \notin \ker \mathrm{d}f_p = T_pS$, i.e., $N_p \notin T_pS$ – it's transverse to S. $N := \operatorname{grad} f|_S$ is a nowhere tangent vector field along S. The result then follows from Proposition 1.6.10. ■

An important case of the oritentation of a submanifold is the orientation of the boundary of a smooth manifold. This will be used for Stokes's theorem later.

Theorem 1.6.14. *For an oriented smooth n-manifold M with boundary $S = \partial M \neq \varnothing$ and $n \geq 1$, [11, Theorem 5.11] shows that S with subspace topology has a smooth structure making it an embedded hypersurface (i.e., $\operatorname{codim} = 1$) in M. [11, Problem 8-4] claims that there is a smooth outward-pointing vector field along S, and Proposition 2.2.6 claims that after giving a metric for M, this outward-pointing vector field can be chosen to be the normal vector field along S. Proposition 1.6.10 says the outward-pointing vector field determines an* *orientation on $S = \partial M$. In fact, [11, Proposition 15.24] shows that all outward-pointing vector fields along it determine the same orientation, called the **induced orientation** or the **Stokes orientation on** ∂M.*

Example 1.6.15. This Theorem gives a simpler proof that $\mathbb{S}^n$ is orientable, because it is the boundary of the closed unit ball. The orientation thus induced on $\mathbb{S}^n$ is the standard one. ♣

Example 1.6.16. Let us determine the induced orientation on $\partial\mathbb{H}^n$ when $\mathbb{H}^n$ itself has the standard orientation inherited from $\mathbb{R}^n$. We can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ under the correspondence $\left(x^1, \ldots, x^{n-1}, 0\right) \leftrightarrow \left(x^1, \ldots, x^{n-1}\right)$. Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\partial\mathbb{H}^n$, the standard coordinate frame for $\mathbb{R}^{n-1}$ is positively oriented for $\partial\mathbb{H}^n$ if and only if $\left[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}\right]$ is the standard orientation for $\mathbb{R}^n$. This orientation satisfies

$$\begin{aligned}\left[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}\right] &= -\left[\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}\right] \\ &= (-1)^n \left[\partial/\partial x^1, \ldots, \partial/\partial x^{n-1}, \partial/\partial x^n\right]\end{aligned}$$

Thus, the induced orientation on $\partial\mathbb{H}^n$ is equal to the standard orientation on $\mathbb{R}^{n-1}$ when n is even, but it is opposite to the standard orientation when n is odd. In particular, the standard coordinates on $\partial\mathbb{H}^n \approx \mathbb{R}^{n-1}$ are positively oriented if and only if n is even. ♣

1.7 Integration on Manifolds

A **domain of integration** in $\mathbb{R}^n$ is a bounded subset whose boundary has measure zero. Open and closed rectangles are examples of domains of integration. Finite union of domains of integration is also a domain of integration. Obviously, a domain of integration should be a domain on which we can integrate:

Proposition 1.7.1. *If $D \subseteq \mathbb{R}^n$ is a domain of integration, then every bounded continuous real-valued function on D is integrable over D.*

If D is compact, then the assumption of boundedness in the above proposition is redundant.

In general, the integration of a real-valued function f over a manifold M is not coordinate-free. We need to multiply f by some additional number to account for the effect of the change of coordinate:

Theorem 1.7.2 (Change of Variables)**.** *Suppose D and E are open domains of integration in $\mathbb{R}^n$, and $G : \bar{D} \to \bar{E}$ is smooth map that restricts to a diffeomorphism from D to E. For every continuous function $f : \bar{E} \to \mathbb{R}$,*

$$\int_E f \,\mathrm{d}V = \int_D (f \circ G) |\det DG| \,\mathrm{d}V$$

I. Integration of an n-form on a domain of integration $D \subseteq \mathbb{R}^n$

Let us start with a domain of integration $D \subseteq \mathbb{R}^n$ and a continuous n-form $\omega = f \,\mathrm{d}x^1 \wedge \cdots \wedge \,\mathrm{d}x^n$ over $\overline{D}$ where $f : \overline{D} \to \mathbb{R}$ is a continuous function. Note that $\mathbb{R}^n$ has Heine-Borel property, i.e., every closed and bounded subset of $\mathbb{R}^n$ is compact. Thus, $\overline{D}$ is compact and then f is integrable over D. We then define **integral of** ω **over** D as

$$\int_D \omega = \int_D f \,\mathrm{d}V.$$

This can be written more suggestively as

$$\int_D f \,\mathrm{d}x^1 \wedge \cdots \wedge \,\mathrm{d}x^n = \int_D f \,\mathrm{d}x^1 \cdots \,\mathrm{d}x^n$$

In simple terms, to compute the integral of a form such as $f dx^1 \wedge \cdots \wedge dx^n$, just "erase the wedges"!

Somewhat more generally, let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and suppose ω is a compactly supported n-form on U. We define

$$\int_U \omega = \int_D \omega,$$

where $D \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is any domain of integration (such as a rectangle) containing $\operatorname{supp} \omega$, and ω is extended to be zero on the complement of its support. Of course, this definition does not depend on what domain D is chosen.

Like the definition of the integral of a 1-form over an interval, our definition of the integral of an n-form might look like a trick of notation. The next proposition shows why it is natural.

Proposition 1.7.3. *Suppose D and E are open domains of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G : \bar{D} \to \bar{E}$ is a smooth map that restricts to an orientation-preserving or orientation-reversing diffeomorphism from D to E. If ω is an n-form on $\bar{E}$, then*

$$\int_D G^*\omega = \begin{cases} \displaystyle\int_E \omega \text{ if } G \text{ is orientation-preserving} \\ \displaystyle -\int_E \omega \text{ if } G \text{ is orientation-reversing} \end{cases}$$

Proof. Let us use $\left(y^1, \ldots, y^n\right)$ to denote standard coordinates on E, and $\left(x^1, \ldots, x^n\right)$ to denote those on D. Suppose first that G is orientation-preserving. With $\omega = f \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$, the change of variables formula (Theorem 1.7.2) together with pullback formula for top-degree form (Proposition 1.5.2) yields

$$\begin{aligned} \int_E \omega &\xlongequal{\text{definition of integration}} \int_E f \, \mathrm{d}V \\ &\xlongequal{\text{change of variables}} \int_D (f \circ G)|\det DG| \, \mathrm{d}V \\ &\xlongequal{G \text{ orientation-preserving; see [LeeSM] Exercise 15.13(b)}} \int_D (f \circ G)(\det DG) \, \mathrm{d}V \\ &\xlongequal{\text{definition of integration}} \int_D (f \circ G)(\det DG) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &\xlongequal{\text{Proposition 1.5.2}} \int_D G^*\omega. \end{aligned}$$

If G is orientation-reversing, the same computation holds except that a negative sign. ■

II. Integration of an n-form on an open set $U \subseteq \mathbb{R}^n$

We would like to extend this theorem to compactly supported n-forms defined on open subsets. However, since we cannot guarantee that arbitrary open subsets or arbitrary compact subsets are domains of integration, we need the following lemma.

Lemma 1.7.4. *Suppose U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and K is a compact subset of U. Then there is an open domain of integration D such that $K \subseteq D \subseteq \bar{D} \subseteq U$.*

Proposition 1.7.5. *Suppose U, V are open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G : U \to V$ is an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n-form on V, then*

$$\int_V \omega = \pm \int_U G^*\omega$$

with the positive sign if G is orientation-preserving, and the negative sign otherwise.

Proof. Let E be an open domain of integration such that supp $\omega = K \subseteq E \subseteq \bar{E} \subseteq V$. Since diffeomorphisms take interiors to interiors, boundaries to boundaries, and sets of measure zero to sets of measure zero (see Munkres's Analysis on Manifolds [14] Theorem 18.1-18.2), $D = G^{-1}(E) \subseteq U$ is an open domain of integration containing supp $G^*\omega$. † The result then follows from Proposition 1.7.3. ■

1.7.1 Integration on Oriented Manifolds

III. Integration of an n-form on an oriented smooth n-manifold M

Now let M be an oriented smooth n-manifold with or without boundary, and let ω be an n-form on M.

Case 1 (Single chart): If ω is compactly supported in the domain of a single smooth chart (U, φ) that is either positively or negatively oriented. We define the **integral of** ω **over** M to be

$$\int_M \omega = \pm \int_{\varphi(U)} \left(\varphi^{-1}\right)^* \omega$$

with the positive sign for a positively oriented chart, and the negative sign otherwise. Since $\left(\varphi^{-1}\right)^* \omega$ is a compactly supported n-form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, its integral is defined as discussed above.

Proposition 1.7.6 ([11] Proposition 16.4). *With ω as above, $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains supp ω.*

Case 2 (Multiple charts): If ω is not just compactly supported in a single chart but nonzero over many charts or even the entire manifold M, we need to use partition of unity to define the integral. Suppose M is an oriented smooth n-manifold with or without boundary, and ω is a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of supp ω by domains of positively or negatively oriented smooth charts, and let $\{\psi_i\}$ be a subordinate smooth partition of unity. Define the **integral of** ω **over** M to be

$$\int_M \omega = \sum_i \int_M \psi_i \omega$$

Since for each i, the n-form $\psi_i \omega$ is compactly supported in U_i, each of the terms in this sum is well defined according to Case 1. To show that the integral is well defined, we need only examine the dependence on the open cover and the partition of unity.

Proposition 1.7.7. *The definition of integration above does not depend on the choice of open cover or partition of unity.*

Case 2': As usual, we have a special definition in the zero-dimensional case. The integral of a compactly supported 0-form (i.e., a real-valued function) f over an oriented 0-manifold M is defined to be the sum

$$\int_M f = \sum_{p \in M} \pm f(p)$$

where we take the positive sign at points where the orientation is positive and the negative sign at points where it is negative. The assumption that f is compactly supported implies that there are only finitely many nonzero terms in this sum.

Case 3 (Oriented Submanifolds): If $S \subseteq M$ is an oriented immersed k-dimensional submanifold (with or without boundary), and ω is a k-form on M whose restriction to S is compactly supported, we interpret $\int_S \omega$ to mean $\int_S \iota_S^* \omega$, where $\iota_S : S \hookrightarrow M$ is inclusion.

†In fact, if $\omega = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, then $\{f \circ G = 0\} \supseteq G^{-1}\{f \neq 0\} \implies [\{f \circ G = 0\}]^c \subseteq [G^{-1}\{f \neq 0\}]^c \implies$ supp $G^*\omega = \overline{\{f \circ G \neq 0\}} = \overline{[\{f \circ G = 0\}]^c} \subseteq \overline{[G^{-1}\{f \neq 0\}]^c} \overset{G \text{ homeo}}{=\!=\!=} G^{-1}\overline{\{f \neq 0\}} = G^{-1}K \subseteq G^{-1}(E) = D$.

Case 3': In particular, if M is a compact, oriented, smooth n-manifold with boundary $S = \partial M$ an embedded hypersurface with Stokes orientation, and ω is an $(n-1)$-form on M, [†] we can interpret $\int_{\partial M} \omega$ as the integral of $\iota_{\partial M}^* \omega$ over ∂M.

Remark 1.7.8. It is worth remarking that it is possible to extend the definition of the integral to some noncompactly supported forms, and such integrals are important in many applications. However, in such cases the resulting multiple integrals are improper, so one must pay close attention to convergence issues. ♠

Proposition 1.7.9 (Properties of Integrals of Forms)**.** *Suppose M and N are nonempty oriented smooth n-manifolds with or without boundary, and ω, η are compactly supported n-forms on M.*

(a) LINEARITY: If $a, b \in \mathbb{R}$, then

$$\int_M a\omega + b\eta = a \int_M \omega + b \int_M \eta.$$

(b) ORIENTATION REVERSAL: If $-M$ denotes M with the opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega$$

Alternatively, this is also recorded as

$$\int_{M,\circlearrowleft} \omega = -\int_{M,\circlearrowright} \omega$$

(c) POSITIVITY: If ω is a positively oriented orientation form, then $\int_M \omega > 0$.

(d) DIFFEOMORPHISM INVARIANCE: If $F : N \to M$ is an orientation-preserving or orientation-reversing diffeomorphism, then

$$\int_M \omega = \begin{cases} \displaystyle\int_N F^*\omega \text{ if } F \text{ is orientation-preserving} \\ \displaystyle -\int_N F^*\omega \text{ if } F \text{ is orientation-reversing} \end{cases}$$

Although the definition of the integral of a form based on partitions of unity is very convenient for theoretical purposes, it is useless for doing actual computations. It is generally quite difficult to write down a smooth partition of unity explicitly, and even when one can be written down, one would have to be exceptionally lucky to be able to compute the resulting integrals (think of trying to integrate $e^{-1/x}$).

For computational purposes, it is much more convenient to "chop up" the manifold into a finite number of pieces (including just a single piece;as we shall see in Example 1.7.12) whose boundaries are sets of measure zero, and compute the integral on each piece separately by means of local parametrizations. One way to do this is described below.

Proposition 1.7.10 (Integration Over Parametrizations)**.** *Let M be an oriented smooth n-manifold with or without boundary, and let ω be a compactly supported n-form on M. Suppose $D_1, \ldots, D_k$ are open domains of integration in $\mathbb{R}^n$, and for $i = 1, \ldots, k$, we are given smooth maps $F_i : \bar{D}_i \to M$ satisfying*

(i) F_i restricts to an orientation-preserving diffeomorphism from D_i onto an open subset $W_i \subseteq M$;

(ii) $W_i \cap W_j = \varnothing$ when $i \neq j$;

(iii) $\operatorname{supp} \omega \subseteq \bar{W}_1 \cup \cdots \cup \bar{W}_k$.

[†] ω is compactly supported in M because supp ω is a closed subset of a compact space and is thus compact. Besides, ω is compactly supported within S as S is closed ([11] Proposition 1.38) and thus compact as well. supp $\iota_S^* \omega$ as a closed subset of compact space S is thus also compact.

Then

$$\int_M \omega = \sum_{i=1}^k \int_{D_i} F_i^* \omega$$

Remark 1.7.11. The conditions above can be loosened; see [11] p.410. ♠

Example 1.7.12. Let us use this technique to compute the integral of a 2-form over $\mathbb{S}^2$, oriented as the boundary of $\overline{\mathbb{B}}^3$. It is an embedded submanifold of $\mathbb{R}^3$ (as a regular level set of the distance function; see [11] Example 5.15) and is thus immersed. Embedded submanifold has subspace topology, so the sphere as a compact subset of $\mathbb{R}^3$ is compact with respect to its subspace topology. A form ω defined on $\mathbb{R}^3$ has its restriction $\iota_{\mathbb{S}^2}^* \omega$ compactly supported on S. For example, consider 2-form ω on $\mathbb{R}^3$:

$$\omega = x \,\mathrm{d}y \wedge \,\mathrm{d}z + y \,\mathrm{d}z \wedge \,\mathrm{d}x + z \,\mathrm{d}x \wedge \,\mathrm{d}y.$$

Note that the support is all of $\mathbb{S}^2$, which cannot be covered up by a single chart. Proposition 1.7.10 is thus needed here. The the open rectangle $D = (0, \pi) \times (0, 2\pi)$ will work as the open domain of integration (open, bounded, and has a null-set-boundary), and the spherical coordinate parametrization $F(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ will work as the smooth map $F : \bar{D} \to \mathbb{R}^3$. The restriction $F|_D : D \twoheadrightarrow \mathbb{S}^2 \backslash$ a closed half-meridian $=: W$ is an orientation-preserving diffeomorphism due to [11] Example 15.28. Note that $\overline{W} = \mathbb{S}^2 = \operatorname{supp} \iota_{\mathbb{S}^2}^* \omega$. The conditions in the Proposition 1.7.10 are now all satisfied. Then

$$\int_{\mathbb{S}^2} \omega := \int_{\mathbb{R}^3} \iota_{\mathbb{S}^2}^* \omega = \int_D (\iota_{\mathbb{S}^2} \circ F)^* \omega = \int_D F^* \omega,$$

where we still use F to denote the map $\iota_{\mathbb{S}^2} \circ F$. Then apply Lemma 1.5.1(c) and notice that

$$\begin{aligned}
F^* \,\mathrm{d}x &= \,\mathrm{d}(x \circ F) = \cos\varphi\cos\theta \,\mathrm{d}\varphi - \sin\varphi\sin\theta \,\mathrm{d}\theta \\
F^* \,\mathrm{d}y &= \,\mathrm{d}(y \circ F) = \cos\varphi\sin\theta \,\mathrm{d}\varphi + \sin\varphi\cos\theta \,\mathrm{d}\theta \\
F^* \,\mathrm{d}z &= \,\mathrm{d}(z \circ F) = -\sin\varphi \,\mathrm{d}\varphi
\end{aligned}$$

We compute:

$$\begin{aligned}
\int_{\mathbb{S}^2} \omega &= \int_D \big(-\sin^3\varphi\cos^2\theta \,\mathrm{d}\theta \wedge \,\mathrm{d}\varphi + \sin^3\varphi\sin^2\theta \,\mathrm{d}\varphi \wedge \,\mathrm{d}\theta \\
&\quad + \cos^2\varphi\sin\varphi\cos^2\theta \,\mathrm{d}\varphi \wedge \,\mathrm{d}\theta - \cos^2\varphi\sin\varphi\sin^2\theta \,\mathrm{d}\theta \wedge \,\mathrm{d}\varphi \big) \\
&= \int_D \sin\varphi \,\mathrm{d}\varphi \wedge \,\mathrm{d}\theta = \int_0^{2\pi} \int_0^{\pi} \sin\varphi \,\mathrm{d}\varphi \,\mathrm{d}\theta = 4\pi
\end{aligned}$$

♣

Theorem 1.7.13 (Stokes's Theorem)**.** *Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M \,\mathrm{d}\omega = \int_{\partial M} \omega$$

Remark 1.7.14. The statement of this theorem is concise and elegant, but it requires a bit of interpretation. First, as usual, ∂M is understood to have the induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\iota_{\partial M}^* \omega$. If $\partial M = \varnothing$, then the right-hand side is to be interpreted as zero. When M is 1-dimensional, the right-hand integral is really just a finite sum. ♠

Proof. See [11] Theorem 16.11. ∎

Corollary 1.7.15 (Integrals of Exact Forms). *If M is a compact oriented smooth manifold without boundary, then the integral of every exact form over M is zero:*

$$\int_M \mathrm{d}\omega = 0 \quad \text{if } \partial M = \varnothing$$

Corollary 1.7.16 (Integrals of Closed Forms over Boundaries). *Suppose M is a compact oriented smooth manifold with boundary. If ω is a closed form on M, then the integral of ω over ∂M is zero:*

$$\int_{\partial M} \omega = 0 \quad \text{if } \mathrm{d}\omega = 0 \text{ on } M.$$

1.7.2 Integration on Nonorientable Manifolds

On an oriented n-manifold with or without boundary, n-forms are the natural objects to integrate. But in order to integrate on a nonorientable manifold, we need closely related objects called densities.

If V is an n-dimensional real vector space, a **density on** V is a function

$$\mu : \underbrace{V \times \cdots \times V}_{n \text{ copies}} \to \mathbb{R}$$

satisfying the following formula for every linear map $T : V \to V$:

$$\mu\left(Tv_1, \ldots, Tv_n\right) = |\det T| \mu\left(v_1, \ldots, v_n\right). \tag{1.11}$$

A density μ is said to be **positive** if $\mu\left(v_1, \ldots, v_n\right) > 0$ whenever $\left(v_1, \ldots, v_n\right)$ is a basis of V; it is clear from (1.11) that if this is true for some basis, then it is true for every one. Every nonzero alternating n-tensor μ determines a positive density $|\mu|$ by the formula

$$|\mu|\left(v_1, \ldots, v_n\right) = \left|\mu\left(v_1, \ldots, v_n\right)\right|.$$

The set $\mathcal{D}(V)$ of all densities on V is a 1-dimensional vector space, spanned by $|\mu|$ for any nonzero alternating n-tensor μ. When M is a smooth manifold with or without boundary, the set

$$\mathcal{D}M = \coprod_{p \in M} \mathcal{D}\left(T_p M\right)$$

is called the **density bundle of** M. $\mathcal{D}M$ is a smooth rank-1 vector bundle by [11, Proposition 16.36], with $\left|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right|$ as a smooth local frame over any smooth coordinate chart. A **density on** M is a (smooth) section μ of $\mathcal{D}M$; in any local coordinates, it can thus be written as

$$\mu = u\left|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right|$$

for some locally defined smooth function u. Under smooth maps, densities pull back in the same way as differential forms: Suppose $F : M \to N$ is a smooth map between n-manifolds with or without boundary, and μ is a density on N. The **pullback of** μ is the density $F^*\mu$ on M defined by

$$\left(F^*\mu\right)_p\left(v_1, \ldots, v_n\right) = \mu_{F(p)}\left(\mathrm{d}F_p\left(v_1\right), \ldots, \mathrm{d}F_p\left(v_n\right)\right)$$

In coordinates, [11, Proposition 16.40] says that μ satisfies

$$F^*\left(u\left|\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n\right|\right) = (u \circ F)|\det DF|\left|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right|$$

where DF represents the matrix of partial derivatives of F in these coordinates.

Now we turn to integration.

I. Integration of μ on domain of integration $D \subseteq \mathbb{R}^n$

If $D \subseteq \mathbb{R}^n$ is a domain of integration and μ is a density on $\bar{D}$, we can write $\mu = f \left| \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \right|$ for some uniquely determined continuous function $f : \bar{D} \to \mathbb{R}$. We define the integral of μ over D by

$$\int_D f \left| \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \right| = \int_D f \, \mathrm{d}x^1 \cdots \mathrm{d}x^n \tag{1.12}$$

II. Integration of μ on an open subset $U \subseteq \mathbb{R}^n$

If U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and μ is compactly supported in U, we define

$$\int_U \mu = \int_D \mu$$

where D is any domain of integration containing the support of μ. The key fact is that this is diffeomorphism-invariant by [11, Proposition 16.41].

III. Integration of μ on smooth n-manifold M

Now let M be a smooth n-manifold (with or without boundary). If μ is a density on M whose support is contained in the domain of a single smooth chart (U, φ), the integral of μ over M is defined as

$$\int_M \mu = \int_{\varphi(U)} \left(\varphi^{-1}\right)^* \mu$$

This is extended to arbitrary densities μ by setting

$$\int_M \mu = \sum_i \int_M \psi_i \mu$$

where $\{\psi_i\}$ is a smooth partition of unity subordinate to an open cover of M by smooth charts. The fact that this is independent of the choices of coordinates or partition of unity follows just as in the case of forms. The following proposition is proved in the same way as Proposition 1.7.9.

Proposition 1.7.17 (Properties of Integrals of Densities)**.** *Suppose M and N are smooth n-manifolds with or without boundary, and μ, η are compactly supported densities on M.*

(a) LINEARITY: If $a, b \in \mathbb{R}$, then
$$\int_M a\mu + b\eta = a \int_M \mu + b \int_M \eta$$

(b) POSITIVITY: If μ is a positive density, then $\int_M \mu > 0$.

(c) DIFFEOMORPHISM INVARIANCE: If $F : N \to M$ is a diffeomorphism, then $\int_M \mu = \int_N F^* \mu$

Just as for forms, integrals of densities are usually computed by cutting the manifold into pieces and parametrizing each piece, just as in Proposition 1.7.10.

1.7.3 Integration on Lie Groups

We know that parallelizable smooth manifolds are all orientable and Lie groups are parallelizable because we can translate basis vectors on T_eG to every other tangent spaces via $\mathrm{d}(L_g)_e, \forall g \in G$. The vector field $g \mapsto \mathrm{d}(L_g)_e v$ is denoted by v^L. See [11] Corollary 8.39 for more detail. We call an orientation on G **left-invariant** if the diffeomorphisms L_g's are all orientation-preserving. We call a vector field X on G

left-invariant if $(L_g)_*X = X$ for every g, while a covariant tensor field A on G is **left-invariant** if $L_g^*A = A$ for all $g \in G$, i.e.,

$$\forall g, g' \in G, \quad A_{g'}(u_1, \cdots, u_k) = A_{gg'}(\,\mathrm{d}(L_g)_{g'}(u_1), \cdots, \,\mathrm{d}(L_g)_{g'}(u_k)).$$

When defining left-invariance for the vector fields, we used pushforward. $(L_g)_*X = X$ means that the vectors $X_{g'}$ pushed by the isomorphisms $d(L_g)_{g'}$'s still belong to that vector field X, i.e., $d(L_g)_{g'}(X_{g'}) = X_{L_g(g')}$. We didn't use pullback because we did not introduce the notion of pullback for vector fields yet. See [11] p.326 Problem 12-10 for a broader definition of pullback and pushforward mapping where $F : M \to N$ is a diffeomorphism.

Proposition 1.7.18. *Every Lie group G has precisely two left-invariant orientations, corresponding to the two orientations of its Lie algebra T_eG.*

Proof. T_eG is a vector space and it has two orientations $\mathcal{O}_e^+, \mathcal{O}_e^-$, each consisting of the vector bases equivalent to each other, i.e., determinant of change of basis matrix is > 0. Isomorphisms $d(L_g)_e$ send orientation to orientation, i.e., two bases of the same orientation are sent to two bases of the same orientation. Define pointwise orientations on G by $g \mapsto \,\mathrm{d}(L_g)_e(\mathcal{O}_e^+)$ and $g \mapsto \,\mathrm{d}(L_g)_e(\mathcal{O}_e^-)$. Each point $p \in G$ is in the smooth oriented global frame $\{v_i^L\}$ where $\{v_i\}$ is a basis of the Lie algebra, so the two pointwise orientations are continuous. They are left-invariant by construction. ∎

Proposition 1.7.19. *Let G be a compact Lie group endowed with a left-invariant orientation. Then G has a unique positively oriented left-invariant n-form ω_G with the property that $\int_G \omega_G = 1$.*

Proof. If $\dim G = 0$, we just let ω_G be the constant function $1/k$, where k is the cardinality of G. Otherwise, let $E_1 = v_1^L, \ldots, E_n = v_n^L$ be a left-invariant global frame on G (where $\{v_i\}$ is a basis for the Lie algebra T_eG). By replacing E_1 with $-E_1$ if necessary, we may assume that this frame is positively oriented. Let $\varepsilon^1, \ldots, \varepsilon^n$ be the dual coframe. Left invariance of E_j implies that

$$\left(L_g^*\varepsilon^i\right)(E_j) = \varepsilon^i\left(L_{g*}E_j\right) = \varepsilon^i\left(E_j\right) = \delta_j^i$$

which shows that $L_g^*\varepsilon^i = \varepsilon^i$, so ε^i is left-invariant. Let $\omega_G = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$. Then by [LeeSM] Lemma 14.16(b),

$$L_g^*\left(\omega_G\right) = L_g^*\varepsilon^1 \wedge \cdots \wedge L_g^*\varepsilon^n = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n = \omega_G$$

so ω_G is left-invariant as well. Because $\omega_G\left(E_1, \ldots, E_n\right) = 1 > 0, \omega_G$ is an orientation form for the given orientation. Clearly, any positive constant multiple of ω_G is also a left-invariant orientation form. Conversely, if $\widetilde{\omega}_G$ is any other left-invariant orientation form, we can write $\left.\widetilde{\omega}_G\right|_e = \left.c\omega_G\right|_e$ for some positive number c. Note that the left-invariance of a tensor field A gives $L_g^*A = A \implies A_{gg'}(\,\mathrm{d}(L_g)_{g'}) = A_{g'} \implies A_e(d(L_{g^{-1}})_g) = A_g$, so we have

$$\left.\tilde{\omega}_G\right|_g = \left.L_{g^{-1}}^*\tilde{\omega}_G\right|_e = \left.cL_{g^{-1}}^*\omega_G\right|_e = \left.c\omega_G\right|_g$$

which proves that $\tilde{\omega}_G$ is a positive constant multiple of ω_G. Since G is compact and oriented and ω_G is an orientation form for the given orientation, Proposition 1.7.9 (c) implies that $\int_G \omega_G$ is a positive real number. We then define $\tilde{\omega}_G = \left(\int_G \omega_G\right)^{-1} \omega_G$. Clearly, $\tilde{\omega}_G$ is the unique positively oriented left-invariant orientation form with integral 1. ∎

Remark 1.7.20. The orientation form whose existence is asserted in this proposition is called the **Haar volume form on** G. Similarly, the map $f \mapsto \int_G f\omega_G$ is called the **Haar integral**. Observe that the proof above did not use the fact that G was compact until the last paragraph; thus every Lie group has a left-invariant orientation form that is uniquely defined up to a constant multiple. It is only in the compact case, however, that we can use the volume normalization to single out a unique one. ♠

1.7.4 Integration on Riemannian Manifolds*

After we talk about Riemannian manifold, we can define integration using Riemannian volume form (for the orientable case) and Riemannian density (for the nonorientable case); see subsections 2.3.3, 2.3.4, and 2.3.5.

1.8 Homology and Cohomology

This section is a brief review of the concepts and results of homology and cohomology.

1.8.1 Singular Homology and Relative Homology

Let X be a topological space. Let $A_n := \{\phi : \Delta^n \to X \mid \phi \text{ continuous}\}$ denote the set of all singular n-simplices in X, and define the **singular chain group** $S_n(X) := F(A_n)$, the free abelian group generated by A_n. The **boundary operator** $\partial : S_n(X) \to S_{n-1}(X)$ is defined via **face operators** ∂_i as

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i,$$

extended linearly using the **universal property of free abelian groups**. From a continuous map $f : X \to Y$ we can define the **pushforward operator** $f_\#$ over basis A_n of X and that of Y. universal property again induces a chain map $f_\# : S_n(X) \to S_n(Y)$ satisfying $f_\# \circ \partial = \partial \circ f_\#$, which in turn induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ on homology groups.

Proposition 1.8.1 (Homotopical Invariance of Homology)**.** *Suppose we have two chain maps $f, g : C_* \to D_*$. A **chain homotopy** is a homomorphism $\Phi : C_* \to D_*$ of degree -1 such that $\partial \circ \Phi + \Phi \circ \partial = f - g$.*

(a) If there is a chain homotopy Φ between f and g, then the induced homomorphisms f_*, g_* of the chain maps f, g are equal.

(b) Let $f, g : X \xrightarrow{C^0} Y$ and $f_\#, g_\# : S_*(X) \to S_*(Y)$. If f and g are homotopic, then there is a chain homotopy Φ between f_* and g_*.

(c) As a corollary of (b), the topological spaces of the same homotopy type have the same homology groups.

Given a short exact sequence of chain complexes

$$0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0,$$

a **diagram chasing** yields the **connecting homomorphism** $\Delta : H_n(E) \to H_{n-1}(C)$. This gives a long exact sequence in homology called **Zig-Zag LES**:

$$\cdots \to H_n(C) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\Delta} H_{n-1}(C) \to \cdots.$$

For space of the form $X = U^o \cup V^o$, we have the following SES

$$0 \to S_n(U \cap V) \xrightarrow{f=(i,-j)} S_n(U) \oplus S_n(V) \xrightarrow{g=k+l} S_n(U + V) \to 0$$

where i, j, k, l are inclusions of the respective groups. $S_n(U + V)$ refers to the free abelian group generated by the set of all singular simplices that are either in U or in V. Note that U and V have their interiors forming a covering $\mathcal{U}$ of X. In general, for any covering $\mathcal{U}$, the **locality principle** states that the honology groups $H_n^{\mathcal{U}}(X)$ of the associated chain complex $S_n^{\mathcal{U}}(X)$ is isomorphic to the ordinary homology groups $H_n(X)$.

From Zig-Zag LES and the locality principle we have the **Mayer–Vietoris Sequence**:

$$\cdots \to H_n(U \cap V) \xrightarrow{f_*} H_n(U) \oplus H_n(V) \xrightarrow{g_*} H_n(X) \xrightarrow{\Delta} H_{n-1}(U \cap V) \to \cdots$$

where $f_*(\alpha) = (i_*\alpha, -j_*\alpha)$ and $g_*(\alpha, \beta) = k_*\alpha + l_*\beta$.

For a pair (X, A), the relative chain group is $S_n(X, A) := S_n(X)/S_n(A)$, and the boundary operator descends from $S_n(X)$. This gives rise to a natural long exact sequence:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\Delta} H_{n-1}(A) \to \cdots$$

A useful result is the **Excision theorem**, which states that if $U \subset A \subset X$, and the closure of U is contained in the interior of A, then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(X \setminus U, A \setminus U) \cong H_n(X, A).$$

Some examples of homology groups include:

- $H_n(\mathrm{pt}) = \mathbb{Z}$ if $n = 0$, and 0 otherwise.
- $H_0(X) \cong \mathbb{Z}^r$, where r is the number of path components of X.
- $H_n(\mathbb{S}^n) = \mathbb{Z}$, $H_0(\mathbb{S}^n) = \mathbb{Z}$, and $H_0(\mathbb{S}^0) = \mathbb{Z} \oplus \mathbb{Z}$.

1.8.2 Homology with Coefficients and Cohomology

Let $\mathbf{Comp}$ denote the category of chain complexes and chain maps, and let $\mathbf{Ab}$ be the category of abelian groups and homomorphisms. Fix an abelian group $G \in \mathbf{Ab}$ and consider the chain complex $C_* = S_*(X, A)$:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

We have two ways to generate new structures:

- The tensor functor $- \otimes G : \mathbf{Ab} \to \mathbf{Ab}$ is right exact, and extends to a functor $\mathbf{Comp} \to \mathbf{Comp}$. Applying this to a chain complex $C_* \in \mathbf{Comp}$, we obtain a new chain complex $C_* \otimes G$:

  $$\cdots \longrightarrow C_{n+1} \otimes G \xrightarrow{\partial_{n+1} \otimes \mathrm{id}} C_n \otimes G \xrightarrow{\partial_n \otimes \mathrm{id}} C_{n-1} \otimes G \longrightarrow \cdots$$

  and define its homology to be the homology with coefficients:

  $$H_n(X, A; G) := H_n(S_*(X, A) \otimes G).$$

- The contravariant Hom functor $\mathrm{Hom}(-, G) : \mathbf{Ab}^{op} \to \mathbf{Ab}$ is left exact, and extends to a contravariant functor $\mathbf{Comp}^{op} \to \mathbf{Comp}$, producing a cochain complex $\mathrm{Hom}(C_*, G)$:

  $$\cdots \longleftarrow \mathrm{Hom}(C_{n+1}, G) \xleftarrow{\delta^n} \mathrm{Hom}(C_n, G) \xleftarrow{\delta^{n-1}} \mathrm{Hom}(C_{n-1}, G) \longleftarrow \cdots$$

  where $\delta^n = \partial_{n+1}^{\#}$ is the **pullback** (precomposition) of ∂, i.e., $\delta^n(f) = f \circ \partial_{n+1}$. We define the **cohomology groups**

  $$H^n(X, A; G) := H^n(\mathrm{Hom}(S_*(X, A), G)) = \frac{\mathrm{Ker}\, \delta^n}{\mathrm{Im}\, \delta^{n-1}}$$

There is a series of analogous results for homology with coefficients and cohomology as in homology with coefficient $\mathbb{Z}$.

Theorem 1.8.2 (Universal Coefficient Theorem for Homology)**.** *There is a split short exact sequence:*

$$0 \to H_n(X, A) \otimes G \to H_n(X, A; G) \to \mathrm{Tor}(H_{n-1}(X, A), G) \to 0.$$

Theorem 1.8.3 (Universal Coefficient Theorem for Cohomology)**.** *There is a split short exact sequence:*

$$0 \to \mathrm{Ext}(H_{n-1}(X, A), G) \to H^n(X, A; G) \to \mathrm{Hom}(H_n(X, A), G) \to 0.$$

Theorem 1.8.4 (Künneth Formula)**.** *Let X and Y be topological spaces. Then there is a split short exact sequence:*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(X), H_q(Y)) \to 0.$$

1.8.3 Computational Tools: Simplicial, Cellular, and Smooth Homology

Let $H_n^\Delta(X)$ denote the **simplicial homology group** of a simplicial complex X, defined using the chain complex $C_n^\Delta(X) :=$ the free group of all standard n-simplices Δ^n in X. The boundary operator is the same as in singular homology.

Theorem 1.8.5 (Equivalence between Singular and Simplicial Homology)**.** *There is a canonical homomorphism $H_n^\Delta(X, A) \to H_n(X, A)$ induced by the chain map $\Delta_n(X, A) \to C_n(X, A)$ sending each n-simplex of X to its characteristic map $\sigma : \Delta^n \to X$. The possibility $A = \varnothing$ is not excluded, in which case the relative groups reduce to absolute groups. In fact, the homomorphisms constitue isomorphisms*

$$H_n^\Delta(X, A) \cong H_n(X, A)$$

for all n and all Δ-complex pairs (X, A).

Proof. See [4, Theorem 2.27]. ■

Let $H_n^\infty(M)$ denote the **smooth singular homology group** of a smooth manifold M, defined using the chain complex $C_n^\infty(M)$ of smooth n-simplicies $\phi : \Delta^n \xrightarrow{C^\infty} M$ in M. Smoothness is interpreted in the sense that it has a smooth extension to a neighborhood of each point. The boundary operator is the same as in the singular homology and note that the boundary of a smooth simplex is a smooth chain. The inclusion map $\iota : C_n^\infty(M) \hookrightarrow C_n(M)$ is a chain map, i.e., commutes with the boundary operator, and so induces a map on homology: $\iota_* : H_n^\infty(M) \to H_n(M)$ by $\iota_*[c] = [\iota(c)]$.

Theorem 1.8.6 (Equivalence between Singular and Smooth Singular Homology)**.** *For any smooth manifold M, the map ι_* induced by inclusion gives an isomorphism*

$$H_n^\infty(M) \cong H_n(M).$$

Proof. The proof is involved; see [11, Theorem 18.7].

The basic idea of the proof is to construct, with the help of the Whitney approximation theorem, two operators: first, a smoothing operator $s : C_n(M) \to C_n^\infty(M)$ such that $s \circ \partial = \partial \circ s$ and $s \circ \iota$ is the identity on $C_p^\infty(M)$; and second, a homotopy operator that shows that $\iota \circ s$ induces the identity map on $H_n(M)$. The key to the proof is a systematic construction of a homotopy from each continuous simplex to a smooth one, in a way that respects the restriction to each boundary face of Δ^n. ■

Let $H_n^{\mathrm{CW}}(X)$ denote the **cellular homology group** of a (finite) CW complex X, defined using the chain complex $C_n^{\mathrm{CW}}(X) := H_n(X^n, X^{n-1})$, where X^n denotes the n-skeleton of X. The boundary maps $d_n : C_n^{\mathrm{CW}}(X) \to C_{n-1}^{\mathrm{CW}}(X)$ are defined via the connecting homomorphisms in the Zig-Zag LES of the triple (X^n, X^{n-1}, X^{n-2}):

$$\cdots \to H_n(X^{n-1}, X^{n-2}) \to H_n(X^n, X^{n-2}) \to H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

Theorem 1.8.7 (Equivalence between Singular and Cellular Homology)**.** *Cellular homology groups are isomorphic to singular homology groups:*

$$H_n^{\mathrm{CW}}(X) \cong H_n(X).$$

Proof. See [4, Theorem 2.35] or [22, Theorem 2.21]. ■

1.8.4 Computational Tools: Products and Dualities

When the coefficient group is also a ring, the cohomology of a space may be given a natural ring structure that homology groups do not have. Moreover, there are some operations only for the cohomology, which lead to the Poincaré duality.

- **Natural pairing** $\langle \cdot\,, \cdot \rangle : S^n(X, A; G) \otimes S_n(X, A) \to G$ induced by the bilinear mapping $\langle \phi, c\rangle = \phi(c)$.
- **Kronecker index** $\langle \cdot\,, \cdot \rangle : H^n(X, A; G) \otimes H_n(X, A) \to G$ induced by the bilinear mapping $\langle [\phi], [c]\rangle = \langle \phi, c\rangle$.
- **Cup product** is a mapping
$$\smile: H^p(X; R) \otimes H^q(X; R) \to H^{p+q}(X; R)$$
defined at the cochain level as follows: for $\alpha \in C^p(X; R)$, $\beta \in C^q(X; R)$, and $\sigma : \Delta^{p+q} \to X$,
$$(\alpha \smile \beta)(\sigma) := \alpha(\sigma|[v_0, \dots, v_p]) \cdot \beta(\sigma|[v_p, \dots, v_{p+q}]),$$
where $\sigma|[v_0, \dots, v_p]$ and $\sigma|[v_p, \dots, v_{p+q}]$ are the front and back faces of σ, respectively. It satisfies the following properties:
  - **Coboundary rule**: $\delta(\alpha \smile \beta) = \delta\alpha \smile \beta + (-1)^p \alpha \smile \delta\beta$.
  - **Graded commutativity**: $\alpha \smile \beta = (-1)^{pq} \beta \smile \alpha$.
  - **Naturality**: $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.
- **Cap product** is a mapping
$$\frown: H^p(X; R) \otimes H_q(X; R) \to H_{q-p}(X; R)$$
defined at the chain-cochain level by: for $\alpha \in C^p(X; R)$, $\sigma \in C_q(X; R)$, with $q \geq p$,
$$\alpha \frown \sigma := \alpha(\sigma|[v_0, \dots, v_p]) \cdot \sigma|[v_p, \dots, v_q] \in C_{q-p}(X; R),$$
where $\sigma|[v_0, \dots, v_p]$ and $\sigma|[v_p, \dots, v_q]$ are the front and back faces of the singular simplex σ, respectively. It satisfies the following properties:
  - **Boundary rule**: $\partial(\alpha \frown \sigma) = (-1)^p(\delta\alpha \frown \sigma - \alpha \frown \partial\sigma)$.
  - **Naturality**: $f_*(f^*(\alpha) \frown x) = \alpha \frown f_*(x)$.
  - **Compatibility with cup product**: $(\alpha \smile \beta) \frown \sigma = \alpha \frown (\beta \frown \sigma)$.
- **Dualities**:
  - **Poincaré Duality Theorem** [22, Theorem 6.18]: For a compact connected orientable n-manifold M endowed with orientation $s : M \to \mathcal{F}$ and associated fundamental class z, the homomorphism
$$D : H^k(M; G) \to H_{n-k}(M; G)$$
via the cap product $D(x) = x \frown z$ is an isomorphism for all k.
  - **Poincaré–Lefschetz Duality** [22, Theorem 6.25]: For a compact orientable n-manifold M with boundary ∂M and fundamental class $z \in H_n(M, \partial M)$, the duality maps
$$D : H^k(M, \partial M) \cong H_{n-k}(M), \quad D' : H^k(M) \cong H_{n-k}(M, \partial M)$$
given by the cap products with z are both isomorphisms for all k.

1.8.5 Homotopy and Homology

We give a short comparison between homotopy and homology groups.

Concept	Homotopy Theory	Homology Theory
Fundamental groups	$\pi_n(X)$, nonabelian for $n=1$	$H_n(X)$, always abelian
Product formula	$\pi_n(X\times Y)\cong\pi_n(X)\oplus\pi_n(Y)$	Künneth formula + Eilenberg-Zilber theorem
Subdivision formula	van Kampen theorem	Mayer–Vietoris sequence

Theorem 1.8.8 (Hurewicz Theorem). *For any path-connected space X and strictly positive integer n there exists a group homomorphism*

$$h_*:\pi_n(X)\to H_n(X),$$

called the ***Hurewicz homomorphism****, from the n-th homotopy group to the n-th homology group (with integer coefficients). It is given in the following way: choose a canonical generator $u_n\in H_n(S^n)$, then a homotopy class of maps $f\in\pi_n(X)$ is taken to $f_*(u_n)\in H_n(X)$.*

1. *For $n\geq 2$, if X is $(n-1)$-connected (that is: $\pi_i(X)=0$ for all $i<n$), then $\tilde{H}_i(X)=0$ for all $i<n$, and the Hurewicz map $h_*:\pi_n(X)\to H_n(X)$ is an isomorphism (see [4, Theorem 4.32]). This implies, in particular, that the homological connectivity equals the homotopical connectivity when the latter is at least 1. In addition, the Hurewicz map $h_*:\pi_{n+1}(X)\to H_{n+1}(X)$ is an epimorphism in this case (see [4, p.390 Exercise 23]).*
2. *For $n=1$, the Hurewicz homomorphism induces an isomorphism $\tilde{h}_*:\pi_1(X)/[\pi_1(X),\pi_1(X)]\to H_1(X)$, between the abelianization of the first homotopy group (the fundamental group) and the first homology group (see [22, Proposition 4.21]).*

1.9 De Rham Cohomology

Imagine a point-particle of mass m moves within an open set $U\subseteq\mathbb{R}^3$ under a force field $\vec{F}(x,y,z)$. Let $\vec{x}(t)$ =position of the particle through time. Define the **work of a path** $\gamma:[a,b]\to\mathbb{R}^3$ with $\gamma(a)=p$, $\gamma(b)=q$ as the **line integral**

$$W_\gamma:=\int_\gamma\vec{F}\cdot\,\mathrm{d}\vec{x}:=\int_a^b\vec{F}(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t.$$

A **conservative vector field** F is a vector field on $U\subseteq\mathbb{R}^n$ whose line integerals around all piecewise-C^∞ closed paths γ are zero.
$\iff$ its line integrals are **path-independent**, i.e., its line integerals around piecewise-C^∞ paths γ with $\gamma(a)=p$, $\gamma(b)=q$ are all equal.
$\overset{\text{[11, Problem 11-15]}}{\iff}$ $\exists$ a function $V\in C^\infty(U)$, called **potential for** F, such that $F=-\nabla V$ ($\iff$ $\exists V\in C^\infty(U)$ s.t. $F=\nabla V$.)

Exercise 1.9.1. *Let F be a conservative vector field on an open set $U\subseteq\mathbb{R}^n$. Prove that for a piecewise-C^∞ path γ starting from p ending at q, it has work $W_\gamma=V(p)-V(q)$.*

Solution. Now, if F is conservative, and if we consider 1-form $\omega_x(v)=F(x)\cdot v$, then

$$\begin{aligned}&\omega_x(v)=-\nabla V(x)\cdot v=-\sum_i\frac{\partial V}{\partial x_i}v_i=-\,\mathrm{d}V(v)\\ \Longrightarrow\ & W_\gamma=\int_a^b\omega_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t\ \overset{\text{[11, Prop.11.38]}}{=\!=\!=\!=}\ \int_\gamma\omega=-\int_\gamma\,\mathrm{d}V\\ \Longrightarrow\ & -W_\gamma=V(q)-V(p)\text{ by [11, Thm.11.39].}\end{aligned}$$

♦

Completely analogously, we have the following concepts and results for the covector fields on smooth manifolds:

A **conservative covector field** ω is a C^∞ covector field on smooth manifold M such that the line integrals around all piecewise-C^∞ closed paths γ are zero. An **exact covector field** ω is a C^∞ covector field on smooth manifold M such that there exists a function $f \in C^\infty(M)$ such that $\omega = \mathrm{d}f$. Note that
a smooth covector field ω is conservative
$\overset{[11,\text{ Prop.11.40}]}{\iff}$ the line integrals of ω are path-independent
$\overset{[11,\text{ Thm.11.42}]}{\iff}$ ω is exact.

A differential form ω is exact if $\omega = \mathrm{d}\alpha$ for some α; it is closed if $\mathrm{d}\omega = 0$. Since $\mathrm{d} \circ \mathrm{d} = 0$ we see exact forms are closed, but not vice versa as the following example shows.

Example 1.9.2. Let $M = \mathbb{R}^2 \backslash \{0\}$. Define a covector field $\omega \in \mathfrak{X}^*(M)$ by

$$\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}.$$

1. The covector field ω is not conservative and thus not exact: it suffices to show that the line integral along one closed path γ is not zero: Consider curve segment $\gamma : [0, 2\pi] \to M$ defined by $\gamma(t) = (\cos t, \sin t)$ (if we identify $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{C}^*$ this $\gamma(t)$ is just $t \mapsto e^{it}$). We compute that

$$\begin{aligned}
\int_\gamma \omega &\overset{\text{line integral defn.}}{=\!=\!=} \int_{[0,2\pi]} \gamma^*\omega \\
&\overset{\text{Lemma 1.5.1}}{=\!=\!=} \int_{[0,2\pi]} \left[\left(\frac{x}{x^2+y^2} \right) \circ \gamma \,\mathrm{d}(y \circ \gamma) - \left(\frac{y}{x^2+y^2} \right) \circ \gamma \,\mathrm{d}(x \circ \gamma) \right] \\
&= \int_{[0,2\pi]} \left[\frac{\cos t}{1} \,\mathrm{d}(\sin t) - \frac{\sin t}{1} \,\mathrm{d}(\cos t) \right] \\
&= \int_{[0,2\pi]} \left[\cos^2 t \,\mathrm{d}t + \sin^2 \,\mathrm{d}t \right] \\
&= \int_{[0,2\pi]} \mathrm{d}t = 2\pi.
\end{aligned}$$

2. The covector field ω is closed: we compute that

$$\begin{aligned}
\mathrm{d}\omega &= \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\overset{(1.10)}{=\!=} \left[\frac{\partial}{\partial x} \left(-\frac{y}{x^2+y^2} \right) - \frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2} \right) \right] \mathrm{d}x \wedge \mathrm{d}y \text{ if we let } x^1 = x, x^2 = y \\
&= \left[-\frac{0 - y(2x)}{(x^2+y^2)^2} - \frac{x(2y)}{(x^2+y^2)^2} \right] \mathrm{d}x \wedge \mathrm{d}y \\
&= \frac{2xy - 2xy}{(x^2+y^2)^2} \mathrm{d}x \wedge \mathrm{d}y = 0.
\end{aligned}$$

3. However, ω is locally exact: if we restrict $M = \mathbb{R}^2 \setminus \{0\}$ to $U = \{(x, y) | x > 0\}$, then x is never 0 and ω is now exact on U. Consider the angle function $\theta(x, y) = \tan^{-1} \frac{y}{x}$ on U. We show $\mathrm{d}\theta = \omega$: consider

another local coordinate system $x = r\cos\theta$ and $y = r\sin\theta$ and compute that

$$\begin{aligned}
\omega &= \frac{x}{x^2+y^2}\,\mathrm{d}y - \frac{y}{x^2+y^2}\,\mathrm{d}x \\
&= \frac{r\cos\theta}{r^2}\,\mathrm{d}(r\sin\theta) - \frac{r\sin\theta}{r^2}\,\mathrm{d}(r\cos\theta) \\
&= \frac{\cos\theta}{r}(\sin\theta\,\mathrm{d}r + \cos\theta r\,\mathrm{d}\theta) - \frac{\sin\theta}{r}(\cos\theta\,\mathrm{d}r - \sin\theta r\,\mathrm{d}\theta) \\
&= \frac{1}{r}(\sin\theta\cos\theta\,\mathrm{d}r - \sin\theta\cos\theta\,\mathrm{d}r) + \frac{r\cos^2\theta + r\sin^2\theta}{r}\,\mathrm{d}\theta \\
&= \mathrm{d}\theta.
\end{aligned}$$

In contrast, there is no continuous angle (argument) function θ (called a branch of argument function) on $\mathbb{C}^* \cong \mathbb{R}\setminus\{0\}$. There is the principal branch of argument $\mathrm{Arg}:\mathbb{C}\backslash(-\infty,0]\to\mathbb{C}$.

♣

1.9.1 De Rham Cohomology

We now define de Rham cohomology groups to measure the failure of closed forms to be exact.

Definitions

We reserve n for the dimension of the manifold M and use p for the degree of the differential form.

Definition 1.9.3 (The de Rham Cohomology).
Codifferential *of the cochain complex: the linear map* $\mathrm{d}:\Omega^p(M)\to\Omega^{p+1}(M)$.
Cochain complex: $0\to\Omega^0(M)\to\cdots\to\Omega^{p-1}(M)\xrightarrow{d}\Omega^p(M)\xrightarrow{d}\Omega^{p+1}(M)\to\cdots\to\Omega^n(M)\to 0$.
Cocycle group: $Z^p(M)=\mathrm{Ker}(\mathrm{d}:\Omega^p(M)\to\Omega^{p+1}(M))=\{\omega\in\Omega^p(M)|\,\mathrm{d}\omega=0\}=\{$*closed* p*-forms on* $M\}$.
Coboundary group: $B^p(M)=\mathrm{Im}(\mathrm{d}:\Omega^{p-1}(M)\to\Omega^p(M))=\{\mathrm{d}\omega|\omega\in\Omega^{p-1}(M)\}=\{$*exact* p*-forms on* $M\}$.
De Rham cohomology group: $H^p_{\mathrm{dR}}(M)=H^p_{\mathrm{deRham}}(M)=Z^p(M)/B^p(M)$.
Pullback operator *and* ***induced homomorphism***: *from* $F:M\xrightarrow{C^\infty}N$ *we have pullback* $F^*:\Omega^p(N)\to\Omega^p(M)$*. It is a cochain map, i.e.,* $d\circ F^*=F^*\circ d$*, so it induces a homomorphism, still denoted by* F^**,* $H^p_{\mathrm{dR}}(N)\to H^p_{\mathrm{dR}}(N)$ *(exactly because cocycles are sent to cocycles and coboundaries are sent to coboundaries.)* $H^p_{\mathrm{dR}}(-)$ *is then a contravariant functor:* $(G\circ F)^*=F^*\circ G^*$*;* $(\mathrm{id}_M)^*=\mathrm{id}_{H^p_{\mathrm{dR}}(M)}$.

Remark 1.9.4.

(a) $d\circ d=0 \iff$ every exact form is closed $\iff B^p(M)\subseteq Z^p(M)$.

(b) Note that $\Omega^p(M)$ and thus $H^p_{\mathrm{dR}}(M)$ are zero when $p<0$ and $p>n$
$\implies B^0(M)=0$ and $Z^n(M)=\Omega^n(M)$
$\implies$ when $p=0$: $H^0_{\mathrm{dR}}(M)=Z^0(M)=\{f\in\Omega^0(M)|\,\mathrm{d}f=0\}\xlongequal{\text{if } M \text{ conn.}}\{$constant functions$\}=\mathbb{R}\{f\equiv 1\}\cong\mathbb{R}$; when $p=n$: $H^n_{\mathrm{dR}}(M)=\Omega^n(M)/B^n(M)$.

(c) For $0\le p\le n$, $H^p_{\mathrm{dR}}(M)=0 \iff Z^p(M)=B^p(M) \iff$ all closed p-forms are exact.

♠

Some other basic results analogous to those in section 1.8 are in order.

Proposition 1.9.5.

(a) If $F:M\to N$ *is a diffeomorphism between manifolds with or without boundaries then* $F^*:H^p_{\mathrm{dR}}(N)\to H^p_{\mathrm{dR}}(N)$ *is an isomorphism.*

(b) *If $M = \coprod_j M_j$ splits into components, then $\coprod_j \iota_j^* : H^p_{\mathrm{dR}}(M) \to \bigoplus_j H^p_{\mathrm{dR}}(M_j)$ collected from homomorphisms ι_j^* induced by inclusions $\iota_j : M_j \to M$ is an isomorphism.*

(c) *If M is 0-dimensional but not necessarily connected, then $H^0_{\mathrm{dR}}(M)$ is a direct sum of 1-dimensional vector spaces by Proposition 1.9.5 (b) and Remark 1.9.4 (b). All other $H^p_{\mathrm{dR}}(M)$'s are zero as $p \geq 1 > 0 = \dim M$.*

(d) *Analogous to Proposition 1.8.1, we have:*

  (i) *If $F, G : M \to N$ are smooth maps between manifolds with or without boundaries and there exists a **chain homotopy** $h : \Omega^*(N) \to \Omega^*(M)$ between cochain maps $F^*, G^* : \Omega^*(N) \to \Omega^*(M)$, i.e., a linear map of degree -1 such that $\mathrm{d} \circ h + h \circ \mathrm{d} = G^* - F^*$, then the induced homomorphisms $F^*, G^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ are equal.*

  (ii) *If $F, G : M \to N$ are smooth maps between manifolds with or without boundaries and F is homotopic to G (thus smoothly homotopic by [11, Theorem 9.28]), then there is a chain homotopy Φ between cochain maps F^* and G^*.*

  (iii) *As a corollary, one can use Whitney approximation theorem ([11, Theorem 9.27]) to show that two smooth manifolds of the same homotopy type have isomorphic de Rham cohomology. Here, for the homotopy equivalence $F : M \xrightarrow{C^0} N$ we cannot use F^* as an isomorphism because F is not even smooth to define F^*. We use Whitney approximation theorem to get some smooth approximation $\widetilde{F} : M \xrightarrow{C^\infty} N$ so that $\widetilde{F}^*$ serves as the isomorphism.*

Computations

Since we know the cohomology groups of a single point (a zero-manifold) and that contractible spaces are of the same homotopy type as a single point, we have the following corollary.

Corollary 1.9.6. *If M is a contractible smooth manifold with or without boundary, then $H^p_{\mathrm{dR}}(M) = 0$ for $p \geq 1$.*

Using the straight-line homotopy one can show a star-shaped set is contractible, so we have proved the Poincaré lemma:

Corollary 1.9.7 (Poincaré Lemma)**.** *If U is a star-shaped open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, then $H^p_{\mathrm{dR}}(U) = 0$ for $p \geq 1$.*

Corollary 1.9.8 (Local Exactness of Closed Forms)**.** *Let M be a smooth manifold with or without boundary. Each point of M has a neighborhood on which every closed form is exact.*

Proof. Every point of M has a neighborhood diffeomorphic to an open ball in $\mathbb{R}^n$ or an open half-ball in $\mathbb{H}^n$, each of which is star-shaped. The result follows from the Poincaré lemma and the diffeomorphism invariance of de Rham cohomology. ■

Corollary 1.9.9 (Cohomology of Euclidean Spaces and Half-Spaces)**.** *For any integers $n \geq 0$ and $p \geq 1$, $H^p_{\mathrm{dR}}(\mathbb{R}^n) = 0$ and $H^p_{\mathrm{dR}}(\mathbb{H}^n) = 0$.*

Proof. Both $\mathbb{R}^n$ and $\mathbb{H}^n$ are star-shaped. ■

Another analogous result is the Mayer-Vietoris sequence. Since inclusions will be involved, some subtleties regarding them should be adressed first.

Suppose S is an immersed submanifold of smooth manifold M. Let ι_S be the inclusion map. There are two types of vanishing we need to distinguish from each other. Let $\omega \in \Gamma(T^k T^* M)$. Then

(1) ω **vanishes along** S: ω vanishes at points of S, i.e., $\omega_p = 0$, $\forall p \in S$.

(2) The **restriction of** ω **on** S **vanishes**, or the **pullback of** ω **to** S **vanishes**: $\omega_p|_{T_pS} = 0$, $\forall p \in S$. Equivalently, $(\iota^*\omega)_p = 0$, $\forall p \in S$.

Clearly, (1) is a stronger condition, as the example below illustrates. However, [11, Proposition 3.9] states that when $S = U$ is an open submanifold of M, then $\mathrm{d}\iota_p : T_pU \to T_pM$ is an isomorphism. Thus, in this case types (1) and (2) are the same.

Example 1.9.10 ([11] Example 11.29)**.** Let $\omega = \mathrm{d}y$ on $\mathbb{R}^2$, and let S be the x-axis, considered as an embedded submanifold of $\mathbb{R}^2$. As a covector field on $\mathbb{R}^2, \omega$ is nonzero everywhere, because one of its component functions is always 1. However, the restriction $\iota^*\omega$ is identically zero, because y vanishes identically on S:

$$\iota^*\omega = \iota^*\,\mathrm{d}y = \mathrm{d}(y \circ \iota) = 0.$$

♣

Theorem 1.9.11 (Mayer-Vietoris Sequence for de Rham Cohomology)**.** *Let M be a smooth manifold with or without boundary, and let U, V be open subsets of M whose union is M. For each p, there is a linear map $\Delta : H^p_{\mathrm{dR}}(U \cap V) \to H^{p+1}_{\mathrm{dR}}(M)$ making the following a LES:*

$$\cdots \to H^p_{\mathrm{dR}}(M) \xrightarrow{k^* \oplus l^*} H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \xrightarrow{i^* - j^*} H^p_{\mathrm{dR}}(U \cap V) \xrightarrow{\Delta} H^{p+1}_{\mathrm{dR}}(M) \to \cdots$$

Example 1.9.12 (de Rham Cohomology Groups of Spheres)**.** Using Mayer-Vietoris sequence one can show that for $n \geq 1$,

$$H^p_{\mathrm{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R}, & p = 0 \text{ or } n \\ 0, & 0 < p < n. \end{cases}$$

♣

Exercise 1.9.13. *Show that $\eta \in \Omega^n(\mathbb{S}^n)$ is exact if and only if $\int_{\mathbb{S}^n} \eta = 0$.*

Solution. $\eta \in \Omega^n(\mathbb{S}^n)$ and η exact, so $\eta = \mathrm{d}\alpha$ and $\int_{\mathbb{S}^n} \eta = \int_{\mathbb{S}^n} \mathrm{d}\alpha \overset{\text{Stokes}}{=\!=} \int_{\varnothing} \alpha = 0$ as spheres are closed manifolds.

Conversely, $H^n_{\mathrm{dR}}(\mathbb{S}^n) = \mathbb{R}\{[\omega]\}$ for an orientation form $\omega \in \Omega^n(\mathbb{S}^n)$. Thus, $[\eta] = c[\omega]$ for some $c \in \mathbb{R}$, i.e., $\eta = c\omega + \mathrm{d}\alpha$. Then $0 \overset{\text{given}}{=\!=} \int_{\mathbb{S}^n} \eta \overset{\text{Stokes}}{=\!=} c \int_{\mathbb{S}^n} \omega \implies c = 0 \implies [\eta] = 0$ in $H^n_{\mathrm{dR}}(\mathbb{S}^n)$, i.e., η is exact. ♦

Corollary 1.9.14 (Cohomology of Punctured Euclidean Space)**.** *Suppose $n \geq 2$ and $x \in \mathbb{R}^n$, and let $M = \mathbb{R}^n \backslash \{x\}$. The only nontrivial de Rham groups of M are $H^0_{\mathrm{dR}}(M)$ and $H^{n-1}_{\mathrm{dR}}(M)$, both of which are 1-dimensional. A closed $(n-1)$-form η on M is exact if and only if $\int_S \eta = 0$ for some (and hence every) $(n-1)$-dimensional sphere $S \subseteq M$ centered at x.*

Proof. Let $S \subseteq M$ be any $(n-1)$-dimensional sphere centered at x. Because inclusion $\iota : S \hookrightarrow M$ is a homotopy equivalence, $\iota^* : H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{dR}}(S)$ is an isomorphism for each p, so the assertion about the dimension of $H^p_{\mathrm{dR}}(M)$ follows from Example 1.9.12. If η is a closed $(n-1)$-form on M, it follows that η is exact if and only if $\iota^*\eta$ is exact on S, which in turn is true if and only if $\int_S \eta = \int_S \iota^*\eta = 0$ by Exercise 1.9.13. ∎

Exercise 1.9.15. *Check that the statement and proof of Corollary 1.9.14 remain true if $\mathbb{R}^n \backslash \{x\}$ is replaced by $\mathbb{R}^n \backslash \bar{B}$ for some closed ball $\bar{B} \subseteq \mathbb{R}^n$.*

Two Features of de Rham Cohomology

1. A distinct feature of de Rham cohomology is the use of integration as a linear map.

Theorem 1.9.16 (A Hurewicz-type Result)**.** *Let M be a connected smooth manifold. Consider the mapping*

$$\begin{aligned} \Phi : H^1_{\mathrm{dR}}(M) &\longrightarrow \mathrm{Hom}(\pi_1(M,q),\mathbb{R}) \\ [\omega] &\longmapsto \begin{pmatrix} \pi_1(M,q) \to \mathbb{R} \\ [\gamma] \mapsto \int_{\widetilde{\gamma}} \omega \end{pmatrix} \end{aligned}$$

where $\widetilde{\gamma}$ is any piecewise-C^∞ curve in $[\gamma]$. Then [11, Theorem 17.17 + Problem 18-2] claim that the linear mapping Φ is an isomorphism. A corollary is that if M is a simply connected smooth manifold, then by [11, Corollary 16.27], $H^1_{\mathrm{dR}}(M) = 0$ and thus $\mathrm{Hom}(\pi_1(M,q),\mathbb{R}) = 0$. [11, Corollary 17.18 + Exercise 17.19] generalize this fact: if $\pi_1(M,q)$ is finite or torsion, then $H^1_{\mathrm{dR}}(M) = 0$ and thus $\mathrm{Hom}(\pi_1(M,q),\mathbb{R}) = 0$.

2. Another distinct feature of the de Rham cohomology (and in general all cohomology theories) is the products of cohomology groups.

For forms $\omega \in Z^k(M)$ and $\eta \in Z^l(M)$, one has

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta = 0$$

i.e. $\omega \wedge \eta \in Z^{k+l}(M)$. Moreover, for any $\xi_1 \in \Omega^{k-1}(M)$ and $\xi_2 \in \Omega^{l-1}(M)$,

$$(\omega + \mathrm{d}\xi_1) \wedge (\eta + \mathrm{d}\xi_2) = \omega \wedge \eta + \mathrm{d}\left[(-1)^k \omega \wedge \xi_2 + (-1)^{k-1} \xi_1 \wedge \eta + (-1)^{k-1} \xi_1 \wedge \mathrm{d}\xi_2\right]$$

In other words, $[\omega \wedge \eta]$ is independent of the choice of ω and η in $[\omega]$ and $[\eta]$. So we can define the **cup product** between $[\omega] \in H^k_{\mathrm{dR}}(M)$ and $[\eta] \in H^l_{\mathrm{dR}}(M)$ as

$$[\omega] \smile [\eta] := [\omega \wedge \eta] \in H^{k+l}_{\mathrm{dR}}(M).$$

De Rham cohomology is built from the complex $\Omega^*(M)$ of differential forms, but there is no corresponding de Rham homology built from a chain complex that "pairs naturally" with $\Omega^*(M)$. Thus there is no intrinsic **cap product** for de Rham cohomology. To get around this one may use currents to think about the cap product.

1.9.2 Compactly Supported de Rham Cohomology

We follow this lecture note.

From Exercise 1.14.24 (or [11, Problem 18-8]), we see that if a smooth manifold is compact (or homotopic equivalent to a compact manifold), then the de Rham cohomology groups are "simpler": They are finite-dimensional, and nice formulae like Künneth formula hold. When M is orientable, a very useful tool to study cohomology classes, especially the top-degree classes, is "integration on manifolds." Unfortunately, if M is non-compact, the integration of a top-degree form is not a nicely defined unless the differential form is compactly supported. Recall that for $\omega \in M$, the support of ω is defined to be

$$\mathrm{supp}(\omega) = \overline{\{p \in M \mid \omega_p \neq 0\}}$$

As usual, we say ω is compactly supported if $\mathrm{supp}(\omega)$ is compact in M. It's natural to let

$$\Omega^k_c(M) = \left\{\omega \in \Omega^k(M) \mid \omega \text{ is compactly supported } \right\}$$

be the set of all compactly supported smooth p-forms. Obviously,

(1) if ω_1, ω_2 are compactly supported p-forms, so is $c_1\omega_1 + c_2\omega_2$;

(2) if ω is compactly supported, so is $\mathrm{d}\omega$.

So $\Omega_c^k(M)$'s are vector spaces, and the exterior derivative makes these vector spaces a cochain complex. As in the ordinary de Rham theory, we denote **p-th compactly supported cocycle group**, **p-th compactly supported coboundary group**, and **p-th compactly supported de Rham cohomology group** as

$$\begin{aligned} Z_c^p(M) &= \{\omega \in \Omega_c^p(M) \mid \mathrm{d}\omega = 0\} \\ B_c^p(M) &= \left\{\omega \in \Omega_c^p(M) \mid \omega = \mathrm{d}\eta \text{ for some } \eta \in \Omega_c^{p-1}(M)\right\} \\ H_c^p(M) &= Z_c^p(M)/B_c^p(M). \end{aligned}$$

Remark 1.9.17. If M is compact, then for all p, $\Omega_c^p(M) = \Omega^p(M)$ and $H_c^p(M) = H_{\mathrm{dR}}^p(M)$. ♠

Lemma 1.9.18 (Poincaré Lemma with Compact Support). *Let $n \geq p \geq 1$, and suppose ω is a compactly supported closed p-form on $\mathbb{R}^n$. If $p = n$, suppose in addition that $\int_{\mathbb{R}^n} \omega = 0$. Then there exists a compactly supported smooth $(p-1)$-form η on $\mathbb{R}^n$ such that $\mathrm{d}\eta = \omega$.*

Remark 1.9.19. Of course, we know that ω is exact by the Poincaré lemma, so the novelty here is the claim that it is the exterior derivative of a compactly supported form. ♠

Proof. When $n = p = 1$, we can write $\omega = f\ \mathrm{d}x$ for some smooth, compactly supported function $f \in C^\infty(\mathbb{R})$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t$$

By the fundamental theorem of calculus, $\mathrm{d}F = F'\ \mathrm{d}x = f\ \mathrm{d}x = \omega$. Choose $R > 0$ such that $\operatorname{supp} f \subseteq [-R, R]$. When $x < -R$, $F(x) = 0$ by our choice of R. When $x > R$, the fact that $\int_{\mathbb{R}} \omega = 0$ translates to

$$F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t = \int_{-\infty}^{\infty} f(t)\ \mathrm{d}t = 0$$

so, in fact, $\operatorname{supp} F \subseteq [-R, R]$. This completes the proof for the case $n = p = 1$. Now assume $n \geq 2$, and let $B, B' \subseteq \mathbb{R}^n$ be open balls centered at the origin such that $\operatorname{supp} \omega \subseteq B \subseteq \bar{B} \subseteq B'$. By the ordinary Poincaré lemma, there exists a smooth (but not necessarily compactly supported) $(p-1)$-form η_0 on $\mathbb{R}^n$ such that $\mathrm{d}\eta_0 = \omega$. This implies, in particular, that $\mathrm{d}\eta_0 = 0$ on $\mathbb{R}^n \backslash \bar{B}$. To complete the proof, we consider three cases.

Case 1: $p = 1$. In this case η_0 is a smooth function. Because $\mathbb{R}^n \backslash \bar{B}$ is connected when $n \geq 2$, it follows that η_0 is equal to a constant c there. Letting $\eta = \eta_0 - c$, we find that η is compactly supported and satisfies $\mathrm{d}\eta = \omega$ as claimed.

Case 2: $1 < p < n$. Now the restriction of η_0 to $\mathbb{R}^n \backslash \bar{B}$ is a closed $(p-1)$-form. Because $H_{\mathrm{dR}}^{p-1}\left(\mathbb{R}^n \backslash \bar{B}\right) = 0$ by Exercise 1.9.15, there is a smooth $(p-2)$-form γ on $\mathbb{R}^n \backslash \bar{B}$ such that $\mathrm{d}\gamma = \eta_0$ there. If we let ψ be a smooth bump function that is supported in $\mathbb{R}^n \backslash \bar{B}$ and equal to 1 on $\mathbb{R}^n \backslash B'$, then $\eta = \eta_0 - \mathrm{d}(\psi\gamma)$ is smooth on all of $\mathbb{R}^n$ and satisfies $\mathrm{d}\eta = \mathrm{d}\eta_0 = \omega$. Because $\mathrm{d}(\psi\gamma) = \mathrm{d}\gamma = \eta_0$ on $\mathbb{R}^n \backslash B'$, η is compactly supported.

Case 3: $p = n$. In this case, we cannot use the same argument as in Case 2 because $H_{\mathrm{dR}}^{n-1}\left(\mathbb{R}^n \backslash \bar{B}\right) \neq 0$. However, it follows from Corollary 1.9.14 and Exercise 1.9.15 that the restriction of η_0 to $\mathbb{R}^n \backslash \bar{B}$ is exact provided its integral is zero over some sphere centered at the origin and contained in $\mathbb{R}^n \backslash \bar{B}$. Stokes's theorem implies that

$$0 = \int_{\mathbb{R}^n} \omega = \int_{\bar{B}'} \omega = \int_{\bar{B}'} \mathrm{d}\eta_0 = \int_{\partial B'} \eta_0$$

Thus η_0 is exact on $\mathbb{R}^n \backslash \bar{B}$, and the proof proceeds exactly as in Case 2. ■

$H_c^*(M)$ **v.s.** $H_{\mathrm{dR}}^*(M)$

In what follows we indicate the main differences between the compact supported de Rham cohomology groups and the ordinary de Rham cohomology groups.

(1) By definition we have $Z_c^k(M) = Z^k(M) \cap \Omega_c^k(M)$. However, in general,

$$B_c^k(M) \neq B^k(M) \cap \Omega_c(M). \quad \text{(Can you find an example?)}$$

(2) For $k = 0$, by definition

$$H_c^0(M) = Z_c^0(M) = \left\{ f \in C^\infty(M) \mid \mathrm{d}f = 0 \text{ and } \operatorname{supp}(f) \text{ is compact} \right\}.$$

But $\mathrm{d}f = 0$ if and only if f is locally constant, i.e. f is constant on each connected component. On the other hand, a locally constant compactly supported function has to be zero on any non-compact connected component. So we conclude

$$H_c^0(M) \cong \mathbb{R}^{m_c}$$

where m_c is the number of compact connected components of M. In particular,

$$H_c^0(\mathrm{pt}) = \mathbb{R} \quad \text{and} \quad H_c^0(\mathbb{R}^n) = 0, \quad \forall n \geq 1$$

where pt is a singleton. Since $\mathbb{R}^n$ is homotopy equivalent to $\{\mathrm{pt}\}$, we conclude

$$H_c^k(M)\text{'s are no longer homotopy invariants.}$$

(3) Now let $\varphi : M \to N$ be a smooth map. Then by definition,

$$\operatorname{supp}(\varphi^*\omega) \subset \varphi^{-1}(\operatorname{supp}(\omega)).$$

So if $\omega \in \Omega_c^k(N)$, in general we may have $\varphi^*\omega \notin \Omega_c^k(M)$. In particular, In general we cannot pullback compactly-supported cohomology classes on N to compactly-supported cohomology classes on M! In the ordinary theory, we used the pullback to prove the homotopy invariance and to construct the M-V sequence. It turns out that in the "compactly-supported theory," we can use proper maps to cover the first and pushforward the inclusions to cover the second purpose. For the latter, see Exercise 1.14.25.

Pullback of **proper maps**: If $\varphi : M \to N$ is proper, i.e., preimages of compact sets are compact, then the pullback $\varphi^*\omega$ of a compactly supported differential form $\omega \in \Omega_c^k(N)$ is still compactly supported. So the map

$$\varphi^* : H_c^k(N) \to H_c^k(M)$$

is still well-defined. In this case one can prove the following result.

Theorem 1.9.20. *If $\varphi_0, \varphi_1 : M \to N$ are proper smooth maps that are properly homotopic* [†]*, then the induced maps*

$$\varphi_1^* = \varphi_2^* : H_c^k(N) \to H_c^k(M)$$

Note that any homeomorphism is proper. So in particular the compactly supported de Rham cohomology groups are still topological invariants:

Corollary 1.9.21. *If M is homeomorphic to N, then $H_c^k(M) = H_c^k(N)$.*

[†] i.e. there exists a homotopy $\Phi : M \times [0,1] \to N$ connecting φ_0 and φ_1 that is a proper map.

Example 1.9.22 ($H_c^k(\mathbb{R}^m)$ for $k < m$)**.** Suppose $m \geq 1$. We have seen $H_c^0(\mathbb{R}^m) = 0$. Now we prove $H_c^k(\mathbb{R}^m) = 0$ for $1 \leq k < m$. We identify $\mathbb{R}^n$ with $S^m - \{N\}$, where N is the north pole. Then we get an "inclusion" map $\iota : \mathbb{R}^m \to \mathbb{S}^m$, and the words "compactly supported in $\mathbb{R}^m$" is equivalent to "supported in a subset of S^m that is away from N."

Case 1: $1 = k < m$. Take any $\omega \in Z_c^1(\mathbb{R}^m)$. Then $\iota_*\omega \in Z^1(\mathbb{S}^m)$ which is supported in $\mathbb{S}^m - U$ for some neighborhood U of p. Since $H^1(\mathbb{S}^m) = 0$, the closed 1-form $\iota_*\omega$ is exact, i.e. there exists $\eta \in \Omega^0(\mathbb{S}^m) = C^\infty(\mathbb{S}^m)$ so that $\iota_*\omega = \mathrm{d}\eta$. Moreover, the fact $\mathrm{d}\eta = \iota_*\omega = 0$ on U implies that η equals some constant c on U. It follows that if we take $\tilde{\eta} = \eta - c$, then $\tilde{\eta} \in \Omega_c^0(\mathbb{S}^m - \{N\}) = \Omega_c^0(\mathbb{R}^m)$ and $d\tilde{\eta} = \omega$.

Case 2: $1 < k < m$ Again we take $\omega \in Z_c^k(\mathbb{R}^m)$ and consider $\iota_*\omega \in Z^k(\mathbb{S}^m)$, which is supported in some $S^m - U$. Since $H_{\mathrm{dR}}^k(\mathbb{S}^m) = 0$, one can find $\eta \in \Omega^{k-1}(\mathbb{S}^m)$ such that $\iota_*\omega = \mathrm{d}\eta$. By shrinking the neighborhood U of p, we can assume that U is contractible. Then the fact $\mathrm{d}\eta = \iota_*\omega = 0$ in U implies that η is exact in U, i.e. one can find a $\mu \in \Omega^{k-2}(U)$ such that $\eta = d\mu$. Now one pick a bump function ρ on $\mathbb{S}^m$ which vanishes on $\mathbb{S}^m - U$ and equals 1 near p. Then $\tilde{\eta} = \eta - \mathrm{d}(\rho\mu) \in \Omega^{k-1}(\mathbb{S}^m)$ and $\tilde{\eta} = 0$ near p, i.e. it defines a compactly supported $(k-1)$-form on $\mathbb{R}^m$. By construction, $\mathrm{d}\tilde{\eta} = \mathrm{d}\eta = \omega$. ♣

Top-degree Cohomology of Manifolds

In this part we would like to calculate $H_c^n(M)$ and $H_{\mathrm{dR}}^n(M)$ for smooth n-manifold M.

We start with an example.

Example 1.9.23 ($H_c^1(\mathbb{R})$)**.** Let's try to compute $H_c^1(\mathbb{R})$. To do so we consider the integration map

$$\int_{\mathbb{R}} : Z_c^1(\mathbb{R}) = \Omega_c^1(\mathbb{R}) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}} \omega$$

This map is clearly linear and surjective. Moreover, it vanishes on $B_c^1(\mathbb{R})$ by the fundamental theorem of calculus, so it induces a surjective linear map

$$\int_{\mathbb{R}} : H_c^1(\mathbb{R}) \to \mathbb{R}$$

Moreover, if $\int_{\mathbb{R}} f(t)dt = 0$, where $f \in C_c^\infty(\mathbb{R})$, then the function $g(t) = \int_{-\infty}^t f(\tau)d\tau$ is smooth and compactly supported and $dg = f(t)dt$. In other words, $f(t)dt \in B_c^1(\mathbb{R})$, i.e. $[f(t)dt] = 0$ in $H_c^1(\mathbb{R})$. So $\int_{\mathbb{R}}$ is an isomorphism between $H_c^1(\mathbb{R})$ and $\mathbb{R}$, i.e.

$$H_c^1(\mathbb{R}) \cong \mathbb{R}.$$

Compare this with the fact that $H^1(\mathbb{R}) = 0$. ♣

Essentially the same method works in higher dimension. Let M be any n-dimensional connected oriented manifold, and $\omega \in \Omega_c^n(M)$ a compactly supported top-degree form. Then ω is closed, and we have defined the integral $\int_M \omega$. So we get a map

$$\int_M : \Omega_c^n(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega.$$

Now suppose $\omega \in B_c^n(M)$, i.e. $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega_c^{m-1}(M)$. Since the manifold M is locally compact Hausdorff, we can take a compact set K in M such that $\mathrm{supp}(\eta) \subseteq \mathrm{Int}(K)$. Thus, η is zero on ∂K. By the Stokes's formula,

$$\int_M \omega = \int_M \mathrm{d}\eta = \int_K \mathrm{d}\eta = \int_{\partial K} \eta = 0 \tag{1.13}$$

$\int_M$ then induces a linear map, called the **integration map**:

$$I : \int_M : H_c^n(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

Proposition 1.9.24. *Suppose M is a connected oriented smooth n-manifold. Then $\int_M : \Omega_c^n(M) \to \mathbb{R}$ is surjective and thus the integration map $I : H_c^n(M) \to \mathbb{R}$ is also surjective.*

Proof. Fix a volume form ω on M. We can choose a closed set A within a smooth chart U and get a smooth bump function ψ using [11, Proposition 2.25] such that $\psi \equiv 1$ on A and $\operatorname{supp}(\psi) \subseteq U$. Let c be any real number. If $\int \psi\omega = a$, then define $f = \frac{c}{a}\psi$ so that $\int f\omega = \frac{c}{a}\int \psi\omega = c$. ∎

Corollary 1.9.25. *Suppose M is a connected oriented smooth n-manifold. Then there is an exact sequence:*

$$\Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c^n(M) \xrightarrow{\int_M} \mathbb{R} \to 0$$

The integration map induces an isomorphism:

$$I : H_c^n(M) \xrightarrow{\cong} \mathbb{R}.$$

Proof. We have shown $\int_M$ is surjective. Now one inclusion of $\operatorname{Im}(d) = \operatorname{Ker}\left(\int_M\right)$ is covered by equation (1.13) and the other is due to the compactly-supported Poincaré lemma (see [11, Theorem 17.30]). The exactness is thus proved. The isomorphism follows from the first isomorphism theorem. ∎

We state without (complete) proof the following theorem that computes the top-degree (compactly-supported) de Rham cohomology groups of manifolds.

Theorem 1.9.26 (Top-degree $H_{\mathrm{dR}}(M)$ and $H_c(M)$)**.**

(a) [11, Theorem 17.30] (Orientable Compact Support Case): If M is a connected oriented smooth n-manifold, then the integration map $I : H_c^n(M) \to \mathbb{R}$ is an isomorphism, so $H_c^n(M)$ is 1-dimensional.

(b) [11, Theorem 17.31] (Orientable Compact Case): If M is a compact connected orientable smooth n-manifold, then $H_{\mathrm{dR}}^n(M)$ is 1-dimensional, and is spanned by the cohomology class of any smooth orientation form.

(c) [11, Theorem 17.32] (Orientable Noncompact Case): If M is a noncompact connected orientable smooth n-manifold, then $H_{\mathrm{dR}}^n(M) = 0$.

(d) [11, Theorem 17.34] (Nonorientable Case). If M is a connected nonorientable smooth n-manifold, then $H_c^n(M) = 0$ and $H_{\mathrm{dR}}^n(M) = 0$.

Proof. (a) is the same as Corollary 1.9.25. (b) follows from (a) as M compact implies $H_c^n(M) = H_{\mathrm{dR}}^n(M)$ and the integral of any orientation form is nonzero. (c) has an incorrect proof in Lee's original version; see his errata for a correct proof. For (d), see [11, Theorem 17.34]. ∎

1.9.3 Degree Theory

In homology theory, the degree of a continuous map between two spheres of the same dimension is defined as the multiple of the image of the generator under the induced homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$.

In the cohomology theory of de Rham, we can define a similar notion for smooth maps between compact, connected, oriented manifolds of the same dimension $F : M \to N$.

Compactness $\implies$ $H_{\mathrm{dR}}^n = H_c^n$. Corollary 1.9.25 $\implies$ commutative diagram

$$\begin{tikzcd} H_{\mathrm{dR}}^n(N) \arrow[r, "F^*"] \arrow[d, "I", "\cong"'] & H_{\mathrm{dR}}^n(M) \arrow[d, "I", "\cong"'] \\ \mathbb{R} \arrow[r, "\deg(F)"'] & \mathbb{R} \end{tikzcd}$$

via the mutiplication $x \mapsto \deg(F)x$. This multiplication is determined by its behavior on the basis 1 of $\mathbb{R}$. Let θ be a smooth n-form on N such that $\int_N \theta = 1$. Then define the **degree of a smooth map** F as

$$\deg(F) := I([F^*\theta]) = \int_M F^*\theta. \tag{1.14}$$

The multiplication via this number makes the diagram commutative and thus for a smooth n-form ω on N,

$$\int_M F^*\omega = k \int_N \omega.$$

[11, Theorem 17.35] shows that the degree of a smooth map has an alternative characterization. It is the unique integer k satisfying the following condition: If $q \in N$ is a regular value of F, then

$$k = \sum_{x \in F^{-1}(q)} \operatorname{sgn}(\mathrm{d}F_x) \in \mathbb{Z}, \tag{1.15}$$

where

$$\operatorname{sgn}(\mathrm{d}F_x) = \begin{cases} 1, & \text{if } \mathrm{d}F_x \text{ is orientation-preserving,} \\ -1, & \text{if } \mathrm{d}F_x \text{ is orientation-reversing,} \end{cases} \tag{1.16}$$

Much of the power of degree theory arises from the fact that the two different characterizations of the degree can be played off against each other. For example, it is often easy to compute the degree of a particular map simply by counting the points in the preimage of a regular value, with appropriate signs. On the other hand, the characterization in terms of differential forms makes it easy to prove many important properties, such as the ones given in the next proposition.

Proposition 1.9.27 (Properties of the Degree). *Suppose M, N, and P are compact, connected, oriented, smooth n-manifolds.*

(a) If $F : M \to N$ and $G : N \to P$ are both smooth maps, then $\deg(G \circ F) = (\deg G)(\deg F)$.

(b) If $F : M \to N$ is a diffeomorphism, then $\deg F = +1$ if F is orientation-preserving and -1 if it is orientation-reversing.

(c) If two smooth maps $F_0, F_1 : M \to N$ are homotopic, then they have the same degree.

This proposition allows us to define the **degree of a continuous map** $F : M \to N$ between compact, connected, oriented, smooth n-manifolds, by letting $\deg F$ be the degree of any smooth map that is homotopic to F. The Whitney approximation theorem guarantees that there is such a map, and the preceding proposition guarantees that the degree is the same for every map homotopic to F.

Theorem 1.9.28. *Suppose N is a compact, connected, oriented, smooth n-manifold, and X is a compact, oriented, smooth $(n+1)$-manifold with connected boundary. If $f : \partial X \to N$ is a continuous map that has a continuous extension $F : X \to N$ to X, then $\deg f = 0$.*

Proof. By the Whitney approximation theorem, there is a smooth map $\widetilde{F} : X \to N$ that is homotopic to F. Replacing F by $\tilde{F}$ and f by $\tilde{F}|_{\partial X}$, we may assume that both f and F are smooth (because the statement $\deg f = 0$ we want to show is a homotopic invariant and $F \sim \widetilde{F}$, $f = F|_{\partial X} \sim \widetilde{F}|_{\partial X}$.)

Let ω be any smooth n-form on N. Then $\mathrm{d}\omega = 0$ because it is an $(n+1)$-form on an n-manifold. From Stokes's theorem, we obtain

$$\int_{\partial X} f^*\omega = \int_{\partial X} F^*\omega = \int_X \mathrm{d}(F^*\omega) = \int_X F^*\,\mathrm{d}\omega = 0$$

It follows from the definition of smooth map (1.14) that f has degree zero. ∎

For noncompact manifolds, the degree theory can be extended to proper maps, which are maps that pull back compact sets to compact sets. The following exercises illustrate this extension and its properties.

Exercise 1.9.29 ([11] Problem 17-11)**.** *This problem shows that some parts of degree theory can be extended to proper maps between noncompact manifolds. Suppose M and N are noncompact, connected, oriented, smooth n-manifolds.*

*(a) Suppose $F : M \to N$ is a proper smooth map. Prove that there is a unique integer k called the **degree of** F such that for each smooth, compactly supported n-form ω on N,*

$$\int_M F^* \omega = k \int_N \omega$$

and for each regular value q of F,

$$k = \sum_{x \in F^{-1}(q)} \operatorname{sgn}\left(\mathrm{d} F_x\right)$$

where $\operatorname{sgn}(dF_x)$ is defined in (1.16).

(b) By considering the maps $F, G : \mathbb{C} \to \mathbb{C}$ given by $F(z) = z$ and $G(z) = z^2$, show that the degree of a proper map is not a homotopy invariant.

Exercise 1.9.30 ([11] Problem 17-12)**.** *Suppose M and N are compact, connected, oriented, smooth n-manifolds, and $F : M \to N$ is a smooth map. Prove that if $\int_M F^* \eta \neq 0$ for some $\eta \in \Omega^n(N)$, then F is surjective. Give an example to show that F can be surjective even if $\int_M F^* \eta = 0$ for every $\eta \in \Omega^n(N)$.*

Exercise 1.9.31 ([11] Problem 17-13)**.** *Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus. Consider the two maps $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(w, z) = (w, z)$ and $g(w, z) = (z, \bar{w})$. Show that f and g have the same degree, but are not homotopic. [Suggestion: consider the induced homomorphisms on the first cohomology group or the fundamental group.]*

Application of Degree Theory

Application of the degree theory includes Brouwer fixed-point theorem, Hairy-ball theorem, and separation theorems, which can also be obtained using ordinary homology theory (see [22] for example). An application distinct from the homology theory is the **linking number**: given any two non-intersecting smooth curve $\gamma_i : S^1 \to \mathbb{R}^3 (i = 1, 2)$, we can define the linking number $\operatorname{Link}(\gamma_1, \gamma_2)$ to be

$$\operatorname{Link}(\gamma_1, \gamma_2) := \deg\left(\Gamma_{\gamma_1, \gamma_2}\right),$$

where $\Gamma_{\gamma_1, \gamma_2}$ is the Gauss map

$$\Gamma_{\gamma_1, \gamma_2} : \mathbb{T}^2 \to S^2, \quad \left(e^{is}, e^{it}\right) \mapsto \frac{\gamma_1\left(e^{is}\right) - \gamma_2\left(e^{it}\right)}{\left|\gamma_1\left(e^{is}\right) - \gamma_2\left(e^{it}\right)\right|}$$

Geometrically, the linking number represents the number of times that each curve winds around the other, which may be positive or negative since we count the orientation of the two curves. For higher generalizations, see [16] *From Calculus to Cohomology* Definition 11.12 (**linking number** of two disjoint compact oriented connected smooth submanifolds of $\mathbb{R}^{n+1}$). It has the property that the linking number is zero if the two submanifolds can be separated by a hyperplane.

1.9.4 Index Theory

In this subsection we will state without proof the famous Poincaré-Hopf theorem. We first define the local index of a smooth vector field on an open subset $V \subseteq \mathbb{R}^n$. Then on a manifold we use the coordinate map (which is a diffeomorphism) to push the vector field to $\mathbb{R}^n$ to define the local index of a smooth vector field on a manifold.

Consider a smooth vector field X on an open subset $U \subseteq \mathbb{R}^n$. Tangent spaces on U are identified with $\mathbb{R}^n$. X can thus be viewed as a smooth function in $C^\infty(U, \mathbb{R}^n)$. Assume 0 is an isolated zero (or **singularity**) of X, i.e. there is some closed disk $D_r = \overline{B_r}$ of radius r centered at the origin within U such that $X(x) \neq 0$ for all $x \in D_r \setminus \{0\}$. Then we can consider the map

$$F_r : \partial D_r = \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$$
$$x \mapsto \frac{X(x)}{\|X(x)\|}$$

The homotopy class of F_r is independent of the choice of r. [†] Thus, Proposition 1.9.27 (c) and second characterization of degree (1.15) imply that $\deg F_r$ is an integer independent of the choice of r.

Definition 1.9.32. *The **local index of** X **at** 0 is defined as*

$$\iota(X; 0) = \deg F_r.$$

Given a diffeomorphism $\varphi : U \to V$ between open subsets $U, V \subseteq \mathbb{R}^n$, we can push the vector field X on V to U by $\varphi_* X(y) = \mathrm{d}\varphi_{\varphi^{-1}(y)}(X(\varphi^{-1}(y)))$. We have the following lemma.

Lemma 1.9.33 (Diffeomorphic Invariance of Local Index). *If $X \in C^\infty(U, \mathbb{R}^n)$ has an isolated zero at $0 \in U$ and $\varphi : U \to V$ is a diffeomorphism to an open subset $V \subseteq \mathbb{R}^n$ with $\varphi(0) = 0$, then*

$$\iota(\varphi_* X; 0) = \iota(X; 0).$$

Proof. See [16, Lemma 11.18]. ■

Definition 1.9.34. *Let X be a smooth vector field on a smooth manifold M. Let p be an isolated singularity of X on M. That is, there is a chart (U, φ) around p such that $\varphi(p) = 0$ and there are no other singularities of X in U. Consider the pushforward $F = \varphi_* X|_U$ of X to the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The **local index of** X **at** p is defined as*

$$\iota(X; p) = \iota(F; 0).$$

*The point p is called a **non-degenerate singularity** if $\mathrm{d}F_0 : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism where F is viewed as a smooth mapping $F \in C^\infty(\widehat{U}, \mathbb{R}^n)$.*

Remark 1.9.35. The local index is independent of the choice of chart $(U, \varphi : U \to \widehat{U})$ around p. Indeed, if $(V, \phi : V \to \widehat{V})$ is another chart around p such that $\phi(p) = 0$, then the transition map $\phi \circ \varphi^{-1} : \widehat{U} \cap \widehat{V} \to \widehat{U} \cap \widehat{V}$ is a diffeomorphism fixing the origin, and thus Lemma 1.9.33 implies that

$$\iota(\varphi_* X|_U; 0) = \iota((\phi \circ \varphi^{-1})^*(\varphi_* X|_U); 0) = \iota(\phi_* X|_V; 0)$$

♠

Lemma 1.9.36. *If p is a non-degenerate singularity of X, then*

$$\iota(X; p) = \operatorname{sgn}(\mathrm{d}F_0).$$

Proof. See [16, Lemma 11.20]. ■

Definition 1.9.37. *Let X be a smooth vector field on a smooth manifold M with only finitely many isolated singularities. The **(total) index of** X is defined as*

$$\operatorname{Ind}(X) = \sum_{p : X_p = 0} \iota(X; p).$$

[†]Simply let $H(x,t) = \frac{X(r(t)x)}{\|X(r(t)x)\|}$ where $r(t) = (tr_1 + (1-t)r_0)$ for radii r_0, r_1.

Since we won't prove the Poincaré-Hopf theorem, which however involves the Euler characteristic, we will state the theorem and define Euler characteristic later (see subsection 1.9.4).

Theorem 1.9.38 (Poincaré-Hopf theorem)**.** *Let M be a compact oriented smooth manifold, and X a smooth vector field on M, with only finitely many isolated zeroes. If M has boundary, X is required to point outward at all boundary points. Then*

$$\operatorname{Ind}(X) = \chi(M),$$

Proof. Above statement comes from Page 35 of [17], which gives a neat outline of the proof. For the full proof, we list three references here:
- Chapter 12 and Appendix C of [16], which uses dynamical systems and ODEs.
- Theorem 2.11 of lecture note using Lefschetz fixed-point theorem (one also need [16, Corollary 11.26] and this link).
- Page 134 of [3], which uses the intersection theory. ■

Poincaré Duality

Let M be an oriented manifold of dimension n. We have the following maps:

- $\smile: H^k_{\mathrm{dR}}(M) \times H^l_c(M) \to H^{k+l}_c(M), \quad ([\omega],[\eta]) \mapsto [\omega \wedge \eta]$.
- $I : H^n_c(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$.

For any $0 \le k \le n$, consider the natural pairing

$$P^k_M = I \circ \smile: H^k_{\mathrm{dR}}(M) \times H^{n-k}_c(M) \to \mathbb{R}, \quad P^k_M([\omega],[\eta]) = \int_M \omega \wedge \eta$$

The map P^k_M induces the following **Poincaré duality operator**

$$\mathcal{P}^k_M : H^k_{\mathrm{dR}}(M) \to \left(H^{n-k}_c(M)\right)^*, \quad \mathcal{P}^k([\omega]) = \left\{[\eta] \mapsto \int_M \omega \wedge \eta\right\}$$

For example, if M is connected, from Remark 1.9.4 (b) we pick the generator $f \equiv 1$ from $H^0_{\mathrm{dR}}(M) = \{\text{constant functions}\}$ and see that $\mathcal{P}^0_M$ maps it to the linear map

$$\int_M : H^n_c(M) \to \mathbb{R}, \quad [\eta] \mapsto \int_M \eta$$

so that one can think of $\int_M$ as an element in $(H^n_c(M))^*$. The major theorem for the Poincaré duality operator is, unsurprisingly, that it is an isomorphism.

Theorem 1.9.39 (Poincaré Duality)**.** *For any oriented manifold M and any k, the Poincaré duality map $\mathcal{P}^k_M$ is a linear isomorphism:*

$$\mathcal{P}^k_M : H^k_{\mathrm{dR}}(M) \xrightarrow{\cong} \left(H^{n-k}_c(M)\right)^*$$

Exercise 1.9.40. *Show the Poincaré duality theorem 1.9.39.*

Hint: imitate the proof of the de Rham theorem 1.9.48, with "de Rham manifold" replaced by "PD manifold." You will need Lemma 1.9.18 and Problem 1.14.25. In order to use Lemma 1.9.18, you'll need to prove the following fact: Every bounded convex open subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. To prove this, let U be such a subset, and without loss of generality assume $0 \in U$. First show that there exists a smooth nonnegative function $f \in C^\infty(U)$ such that $f(0) = 0$ and $f(x) \ge 1/d(x)$ away from a small neighborhood of 0, where $d(x)$ is the distance from x to ∂U. Next, show that $g(x) = 1 + \int_0^1 t^{-1} f(tx)\,\mathrm{d}t$ is a smooth positive exhaustion function on U that is nondecreasing along each ray starting at 0. Finally, show that the map $F : U \to \mathbb{R}^n$ given by $F(x) = g(x)x$ is a bijective local diffeomorphism. Also, you may use the fact that the conclusion of the five lemma is still true even if the appropriate diagram commutes only up to sign.

Corollary 1.9.41. *If $\dim H_c^{n-k}(M)<\infty$, then $\left(H_c^{n-k}(M)\right)^*$ is isomorphic to $H_c^{n-k}(M)$. So we get*

$$H_{\mathrm{dR}}^k(M) \cong H_c^{n-k}(M)$$

Example 1.9.42. For $M=\mathbb{R}^n$, we have

$$H_{\mathrm{dR}}^k\left(\mathbb{R}^n\right) \cong \begin{cases}\mathbb{R}, & k=n, \\ 0, & k \neq n .\end{cases} \quad \text{and} \quad H_c^k\left(\mathbb{R}^n\right) \cong \begin{cases}\mathbb{R}, & k=0 \\ 0, & k \neq 0\end{cases}$$

For $M=\mathbb{S}^n$, we have

$$H_c^k\left(\mathbb{S}^n\right)=H_{\mathrm{dR}}^k\left(\mathbb{S}^n\right) \cong \begin{cases}\mathbb{R}, & k=0, n, \\ 0, & k \neq 0, n .\end{cases}$$

♣

Betti Numbers and Euler Characteristic

Let M be a smooth n-manifold all of whose de Rham groups are finite-dimensional (this is always the case if M is compact). We define the **k-th Betti number** of M as

$$\beta_k=\dim H_{\mathrm{dR}}^k(M),$$

and the **Euler characteristic** of M as

$$\chi(M)=\sum_{k=0}^n(-1)^k \beta_k$$

Note that above definition via de Rham cohomology is equivalent to the one via singular homology due to de Rham theorem 1.9.48.

Proposition 1.9.43. *Let M be a compact orientable manifold of dimension n, then*

(1) For any $k, \beta_k=\beta_{n-k}$.

(2) If $n=4m+2$, then β_{2m+1} is even.

Proof. (1) follows from Corollary 1.9.41 and the fact that $H_c^{n-k}(M)$ is just $H_{\mathrm{dR}}^{n-k}(M)$ for compact manifolds:

$$H_{\mathrm{dR}}^k(M) \cong H_c^{n-k}(M)=H_{\mathrm{dR}}^{n-k}(M)$$

(2) follows from the non-degeneracy of the pairing

$$P_M^{2m+1}: H_{\mathrm{dR}}^{2m+1}(M) \times H_{\mathrm{dR}}^{2m+1}(M) \rightarrow \mathbb{R}$$

If fact, for any $[\omega],[\eta] \in H_{\mathrm{dR}}^{2m+1}(M)$, we have

$$P_M^{2m+1}([\omega],[\eta])=\int_M \omega \wedge \eta=\int_M(-1)^{(2m+1)(2m+1)} \eta \wedge \omega=-P_M^{2m+1}([\eta],[\omega])$$

It follows that the matrix for the pairing $P_M^{2m+1}: H_{dR}^{2m+1}(M) \times H_{\mathrm{dR}}^{2m+1}(M) \rightarrow \mathbb{R}$ is an antisymmetric $\beta_{2m+1} \times \beta_{2m+1}$ matrix. So

$$\det\left(P_M^{2m+1}\right)=\det\left(\left(P_M^{2m+1}\right)^T\right)=(-1)^{\beta_{2m+1}} \det\left(P_M^{2m+1}\right) .$$

So β_{2m+1} must be an even number, otherwise $\det\left(P_M^{2m+1}\right)=0$ and thus P_M^{2m+1} is not a nondegenerate pairing. ■

Exercise 1.9.44. *Show that $\chi(M)$ is a homotopy invariant of M, and $\chi(M)=0$ when M is compact, orientable, and odd-dimensional.*

1.9.5 Topology: Differential vs. Algebraic

Suppose M is a smooth manifold, ω is a closed p-form on M, and σ is a smooth p-simplex in M. We define the **integral of** ω **over** σ to be

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^* \omega$$

This makes sense because Δ_p is a smooth p-submanifold with corners embedded in $\mathbb{R}^p$ (see [11, p.415]), and it inherits the orientation of $\mathbb{R}^p$. (Or we could just consider Δ_p as a domain of integration in $\mathbb{R}^p$.) Observe that when $p = 1$, this is the same as the line integral of ω over the smooth curve segment $\sigma : [0,1] \to M$.

If $c = \sum_{i=1}^k c_i\sigma_i$ is a smooth p-chain, the **integral of** ω **over** c is defined as

$$\int_c \omega = \sum_{i=1}^k c_i \int_{\sigma_i} \omega$$

We need an analogue of Proposition 1.7.10 for manifolds with corners.

Lemma 1.9.45. *The statement of Proposition 1.7.10 is true if M is replaced by the boundary of an oriented smooth $(n+1)$-manifold with corners.*

Theorem 1.9.46 (Stokes's Theorem for Chains)**.** *If c is a smooth p-chain in a smooth manifold M, and ω is a smooth $(p-1)$-form on M, then*

$$\int_{\partial c} \omega = \int_c \mathrm{d}\omega$$

Proof. It suffices to prove the theorem when c is just a smooth simplex σ. Since Δ_p is a manifold with corners, Stokes's theorem says that

$$\int_\sigma d\omega = \int_{\Delta_p} \sigma^*\, \mathrm{d}\omega = \int_{\Delta_p} \mathrm{d}\sigma^*\omega = \int_{\partial\Delta_p} \sigma^*\omega$$

The maps $\{F_{i,p} : 0 = 1, \ldots, p\}$ are parametrizations of the boundary faces of Δ_p satisfying the conditions of Lemma 1.9.45, except possibly that they might not be orientation-preserving. To check the orientations, note that $F_{i,p}$ is the restriction to $\Delta_p \cap \partial\mathbb{H}^p$ of the affine diffeomorphism sending the simplex $[e_0, \ldots, e_p]$ to $[e_0, \ldots, \widehat{e_i}, \ldots, e_p, e_i]$. This is easily seen to be orientation-preserving if and only if $(e_0, \ldots, \widehat{e_i}, \ldots, e_p, e_i)$ is an even permutation of $(e_0, \ldots, e_p)$, which is the case if and only if $p - i$ is even. Since the standard coordinates on $\partial\mathbb{H}^p$ are positively oriented if and only if p is even, the upshot is that $F_{i,p}$ is orientation-preserving for $\partial\Delta_p$ if and only if i is even. Thus, by Lemma 1.9.45,

$$\int_{\partial\Delta_p} \sigma^*\omega = \sum_{i=0}^p (-1)^i \int_{\Delta_{p-1}} F_{i,p}^*\sigma^*\omega = \sum_{i=0}^p (-1)^i \int_{\Delta_{p-1}} (\sigma \circ F_{i,p})^*\,\omega = \sum_{i=0}^p (-1)^i \int_{\sigma\circ F_{i,p}} \omega.$$

By definition of the singular boundary operator, this is equal to $\int_{\partial\sigma} \omega$. ■

Using this theorem, we define a natural linear map $\mathcal{I} : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$, called the **de Rham homomorphism**, as follows. For any $[\omega] \in H^p_{\mathrm{dR}}(M)$ and $[c] \in H_p(M) \cong H_p^\infty(M)$, we define

$$\mathcal{I}[\omega][c] = \int_{\tilde{c}} \omega$$

where $\tilde{c}$ is any smooth p-cycle representing the homology class $[c]$. This is well defined, because if $\tilde{c}, \tilde{c}'$ are smooth cycles representing the same homology class, then Theorem 1.8.6 guarantees that $\tilde{c} - \tilde{c}' = \partial\tilde{b}$ for some smooth $(p+1)$-chain $\tilde{b}$, which implies

$$\int_{\tilde{c}} \omega - \int_{\tilde{c}'} \omega = \int_{\partial\tilde{b}} \omega = \int_{\tilde{b}} \mathrm{d}\omega = 0$$

while if $\omega = \mathrm{d}\eta$ is exact, then
$$\int_{\widetilde{c}} \omega = \int_{\widetilde{c}} \mathrm{d}\eta = \int_{\partial\widetilde{c}} \eta = 0$$
(Note that $\partial\widetilde{c} = 0$ because $\widetilde{c}$ represents a homology class, and $\mathrm{d}\omega = 0$ because ω represents a cohomology class.) Clearly, $\mathcal{I}[\omega]\,[c+c'] = \mathcal{I}[\omega][c] + \mathcal{I}[\omega]\,[c']$, and the resulting homomorphism $\mathcal{I}[\omega] : H_p(M) \to \mathbb{R}$ depends linearly on ω. Thus, $\mathcal{I}[\omega]$ is a well-defined element of $\operatorname{Hom}(H_p(M), \mathbb{R}) \cong H^p(M;\mathbb{R})$.

Proposition 1.9.47 (Naturality of the de Rham Homomorphism)**.** *Let M be a smooth manifold and $p \geq 0$ an integer. Let*
$$\mathcal{I} : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$$
denote the de Rham homomorphism.

(a) If $F : M \to N$ is a smooth map, then the following diagram commutes:

$$\begin{tikzcd} H^p_{\mathrm{dR}}(N) \arrow[r, "F^*"] \arrow[d, "\mathcal{I}"'] & H^p_{\mathrm{dR}}(M) \arrow[d, "\mathcal{I}"] \\ H^p(N;\mathbb{R}) \arrow[r, "F^*"'] & H^p(M;\mathbb{R}) \end{tikzcd}$$

(b) If M is a smooth manifold and $U, V \subset M$ are open subsets such that $U \cup V = M$, then the following diagram commutes:

$$\begin{tikzcd} H^{p-1}_{\mathrm{dR}}(U \cap V) \arrow[r, "\delta"] \arrow[d, "\mathcal{I}"'] & H^p_{\mathrm{dR}}(M) \arrow[d, "\mathcal{I}"] \\ H^{p-1}(U \cap V;\mathbb{R}) \arrow[r, "\partial^*"'] & H^p(M;\mathbb{R}) \end{tikzcd}$$

where δ and ∂^* are the connecting homomorphisms of the Mayer–Vietoris long exact sequences for de Rham and singular cohomology, respectively.

Theorem 1.9.48. *For every smooth manifold M and integer $p \geq 0$, the de Rham homomorphism*
$$\mathcal{I} : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$$
is an isomorphism.

Sketch of Proof; see [11] Theorem 18.14 for details. We say that a smooth manifold M is a **de Rham manifold** if the de Rham homomorphism
$$\mathcal{I} : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$$
is an isomorphism for all p. It suffices to show that every smooth manifold is de Rham. Since $\mathcal{I}$ commutes with pullbacks under smooth maps (Proposition 1.9.47), any manifold diffeomorphic to a de Rham manifold is also de Rham.

If M is any smooth manifold, let us call an open cover $\{U_i\}$ of M a **de Rham cover** if each subset U_i is a de Rham manifold, and every finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is de Rham. A de Rham cover that is also a basis for the topology of M is called a **de Rham basis** for M.

Step 1: If $\{M_j\}$ is any countable collection of de Rham manifolds, then their disjoint union is de Rham. Both de Rham and singular cohomology commute with disjoint unions. That is,
$$H^p(\coprod_j M_j) \cong \prod_j H^p(M_j), \quad H^p_{\mathrm{dR}}(\coprod_j M_j) \cong \prod_j H^p_{\mathrm{dR}}(M_j),$$
and the de Rham homomorphisms commute with these identifications, so $\mathcal{I}$ is an isomorphism on the disjoint union.

Step 2: Every convex open subset of $\mathbb{R}^n$ is de Rham. For such U, $H^p_{\mathrm{dR}}(U) = 0$ for $p > 0$ by the Poincaré lemma. On the other hand, since U is contractible, $H^p(U;\mathbb{R}) = 0$ for $p > 0$ and both $H^0_{\mathrm{dR}}(U) \cong \mathbb{R} \cong H^0(U;\mathbb{R})$, so $\mathcal{I}$ is an isomorphism.

Step 3: If M has a finite de Rham cover, then M is de Rham. This is the key inductive step. Suppose $M = U_1 \cup \cdots \cup U_k$ with each U_i and all finite intersections de Rham. One inducts on k using the Mayer–Vietoris sequence and naturality of $\mathcal{I}$, and applies the five lemma to show that $\mathcal{I} : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$ is an isomorphism.

Step 4: If M has a de Rham basis, then M is de Rham. Let $\{U_\alpha\}$ be a de Rham basis and $f : M \to \mathbb{R}$ an exhaustion function. Define compact sets

$$A_m := \{x \in M : m \leq f(x) \leq m+1\}, \quad A'_m := \{x \in M : m - \tfrac{1}{2} < f(x) < m + \tfrac{3}{2}\}.$$

Cover each compact A_m by finitely many basis elements, and let B_m be the union of these. Then B_m is de Rham by Step 3. Define $U := \bigcup_{m \text{ odd}} B_m$, $V := \bigcup_{m \text{ even}} B_m$, and apply Step 3 again.

Step 5: Every open subset of $\mathbb{R}^n$ is de Rham. Any such open set has a basis of convex open subsets (e.g. Euclidean balls), each de Rham by Step 2. So the open set has a de Rham basis, hence is de Rham by Step 4.

Step 6: Every smooth manifold is de Rham. Any smooth manifold has an atlas of coordinate charts $U_\alpha \cong \mathbb{R}^n$. By Step 5 and the stability of intersections of basis sets under finite intersection, this atlas gives a de Rham basis. Apply Step 4 to conclude. ■

This theorem is a powerful result that connects differential topology with algebraic topology. Note that there is no such thing called de Rham homology and that singular cohomology can be defined without the smooth structure. Another similar connection between differential and algebraic topology is that smooth homology groups are isomorphic to singular homology groups (see Theorem 1.8.6).

1.10 Fiber Bundles and Vector Bundles

In addition to [11], we will use [6], [13], [21], and [18] for this section. The last one has a new latex version available online.

1.10.1 Fiber Bundles

The surface of the cylinder can be seen as a disjoint union of a family of line segments continuously parametrized by points of a circle. The Möbius band can be presented in similar way. The two dimensional torus embedded in the three dimensional space can presented as a union of a family of circles (meridians) parametrized by points of another circle (a parallel). The tangent bundle TM of a smooth manifold M is a union of vector spaces T_pM.

The examples considered above share two important properties: (a) any two fibers are homeomorphic; (b) despite the fact that the whole space cannot be presented as a Cartesian product of a fiber with the base (the parameter space), if we restrict our consideration to some small region of the base the part of the fiber space over this region is such a Cartesian product. The two properties above are the basis of the following definition.

Definition 1.10.1 (Fiber Bundle). *A **fiber bundle** $\pi : E \to M$ consists of*

(i) three topological spaces:
*E, called the **total space** of the bundle,*
*M, called the **base space** of the bundle, and*
*F, called the **standard fiber** or **model fiber** of the bundle;*

*(ii) a surjective continuous map $\pi : E \to M$, called the **projection**; and*

*(iii) a local trivialization condition: for each $x \in M$, there exist a neighborhood U of x in M and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$, called the **local trivialization of** E **over** U, such that $\pi = \mathrm{pr}_1 \circ \varphi$.*

*A C^∞ **fiber bundle** is a fiber bundle in the category of smooth manifold: E, M, F are smooth manifolds, continuous mappings above are replaced with smooth mappings, and homeomorphism is replaced with diffeomorphism.*

*The projection π is sometimes also called a **fibration of** E. $F \to E \xrightarrow{\pi} M$ is also used to denote a fiber bundle.*

*A **trivial fiber bundle** is a fiber bundle that admits a local trivialization over the entire space, i.e., a **global trivialization**. In this sense, one can also call **locally trivial fiber bundle** for a general fiber bundle.*

*A **section** of a bundle is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma(x) = x$ for each $x \in M$.*

Remark 1.10.2. Note that if $E \to M$ is a fiber bundle with model fiber F then $\pi^{-1}(U) = \coprod_{x \in U} E_x$ where $E_x = \pi^{-1}(x)$ is called a **fiber (over** x**)**. The condition $\pi = \mathrm{pr}_1 \circ \Phi$ requires that for any $\xi \in E_x = \{\xi \in E | \pi(\xi) = x\}$, $\varphi|_{E_x}(\xi) = (x, y)$ for some $y \in F$. Thus, $\varphi|_{E_x} : E_x \to \{x\} \times F$. It is easy to see it has an inverse by restricting the inverse of φ to $\{x\} \times F$. Thus, each fiber E_x is homeomorphic to the model fiber F. ♠

The above definition of bundle is not sufficiently restrictive. A bundle will be required to carry additional structure involving a group G of homeomorphisms of F called the **group of the bundle**. Before imposing the additional requirements, consideration of a collection of examples, new and old, will show the need for these.

Example 1.10.3 (The Product Bundle)**.** The first example is the **product bundle** or product space $E = M \times F$. In this case, the projection is given by $\pi(x, y) = x$. Taking $U = X$ and $\varphi = \mathrm{id}$, the last condition is fulfilled. The sections of $E \to M$ are just the graphs of any continuous maps $M \to F$. The fibers are, of course, all homeomorphic, however there is a natural unique homeomorphism $E_x \to F$ given by $\mathrm{pr}_2 : (x, y) \to y$. As will be seen, this is equivalent to the statement that the group G of the bundle consists of the identity alone. In this case, the bundle is called the **trivial bundle**. ♣

Example 1.10.4 (The Möbius Band)**.** The second example is the Möbius band. The base space E is a circle $M = S^1$ obtained from a line segment L by identifying its ends. The fiber F is a line segment as well. The bundle M is obtained from the product $L \times Y$ by matching the two ends with a twist. The projection $L \times F \to L$ carries over under this matching into a projection $\pi : E \to M$. There are numerous crosssections; any curve as indicated with end points that match provides a section. It is clear that any two cross-sections must agree on at least one point. There is no natural unique homeomorphism of E_x with F. However there are two such which differ by the map g of F on itself obtained by reflecting in its midpoint. In this case the group G is the cyclic group of order 2 generated by g. ♣

Example 1.10.5 (Klein Bottle and the Twisted Torus)**.** Check [21] Example 1.4 & 1.5. Here, $G = \mathbb{Z}_2$ as well. ♣

Example 1.10.6 (Covering Space)**.** A covering space E of a space X is another example of a bundle. The projection $\pi : E \to X$ is the covering map. The usual definition of a covering space is the definition of bundle modified by requiring that each E_x is a discrete subspace of E so that $\pi^{-1}(U) = \coprod_{\xi \in E_x} V_\xi$, and that $\pi|_{V_\xi} : V_\xi \to U$ is a homeomorphism for every $\xi \in E_x$. If X is path-connected, motion of a point x along a curve γ in X from x_1 to x_2 can be covered by a continuous motion of E_x in E from E_{x_1} to E_{x_2}. Choosing a base point x_0 each E_x can be put in 1-1 correspondence with $F = E_{x_0}$ using a curve in X. This correspondence depends only on the homotopy class of the curve. Considering the action on F of closed curves from x_0 to x_0, the fundamental group $\pi_1(X)$ appears as a group of permutations on F. Any two correspondences of E_x with F differ by a permutation corresponding to an element of $\pi_1(X)$ due to unique lifting property. Thus, for covering spaces, the group of the bundle is a factor group of the fundamental group of the base space. ♣

Example 1.10.7 (Tangent Bundle)**.** This is the familiar one. Here, $G = \mathrm{GL}(V)$ where $V \cong T_pM$. ♣

These examples show that a bundle carries, as part of its structure, a group G of transformations of the fiber F. In the last exmaple, the group G has a topology. It is necessary to weave G and its topology into the definition of the bundle. This will be achieved through the intermediate notion of a fiber bundle with coordinate systems (briefly: "coordinate bundle"). The coordinate systems are eliminated by a notion of equivalence of coordinate bundles, and a passage to equivalence classes.

Definition 1.10.8 (Group Action)**.** *A topological group G is a set which has a group structure and a topology such that maps $g \mapsto g^{-1}$ and $(g_1, g_2) \mapsto g_1 g_2$ are continuous. If G is a topological group, and F is a topological space, a **group action of** G **on** F is a map $\eta : G \times F \to F$ such that*

(i) η is continuous,

(ii) $\eta(e, y) = y$ where e is the identity of G, and

(iii) $\eta(g_1 g_2, y) = \eta(g_1, \eta(g_2, y))$ for all g_1, g_2 in G and y in F.

*G is also called a **topological transformation group** of F and F is called a G**-space**. We shall abbreviate the group action $\eta(g, y)$ by $g \cdot y$. For any fixed g, the map $y \to g \cdot y$ is a homeomorphism of F onto itself; for it has the continuous inverse $y \to g^{-1} \cdot y$. In this way η provides a homomorphism of G into the group of homeomorphisms of F:*

$$\begin{aligned} \eta : G \longrightarrow & \operatorname{Homeo}(F) \\ g \longmapsto & \begin{pmatrix} F \to F \\ y \mapsto g \cdot y \end{pmatrix} \end{aligned}$$

Specifying such a homomorphism is equivalent to specifying a group action of G on F.

*We shall say that the group action is **effective** if $g \cdot y = y, \forall y \implies g = e$. Alternatively, the group action is effective if $\eta(g) = \mathrm{id}_F \implies g = e$, i.e., η is injective. In this case, G is isomorphic to a subgroup of the group of homeomorphisms of F. Then one can identify G with that group of homeomorphisms, however we shall frequently allow the same G to operate on several spaces.*

Before we talk about the coordinate bundle, we need to introduce the concept of transition map. Suppose there are two open sets U_α, U_β in which $x \in M$ lies. Let φ_α, φ_β be their associated homeomorphisms. We can regard them as

$$\begin{aligned} \varphi_\alpha : \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times F \\ \varphi_\beta : \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times F \end{aligned}$$

Now consider the map $f_{\alpha\beta,x} := \varphi_\alpha|_{E_x} \circ (\varphi_\beta|_{E_x})^{-1} : F \to F$ (in fact, the restrictions of φ_α and φ_β to E_x are maps from E_x to $\{x\} \times F \cong F$; see remark 1.10.2.) Note that $f_{\alpha\beta,x}$ is a homeomorphism of F. We have the **transition map** $f_{\alpha\beta}$ defined as

$$\begin{aligned} f_{\alpha\beta} : U_\alpha \cap U_\beta & \to \operatorname{Homeo}(F) \\ x & \mapsto f_{\alpha\beta,x} \end{aligned}$$

The map has two properties:

(1) $\forall x \in U_\alpha \cap U_\beta$, $(f_{\alpha\beta}(x))^{-1} = f_{\beta\alpha}(x)$ in the group $\operatorname{Homeo}(F)$;

(2) $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$, $f_{\alpha\gamma}(x) = f_{\alpha\beta}(x) f_{\beta\gamma}(x)$ in the group $\operatorname{Homeo}(F)$.

It is easy to see that satisfying (1) and (2) are equivalent to satisfying the following two conditions, called the **cocycle conditions**:

(1) $\forall x \in U_\alpha$, $f_{\alpha\alpha}(x) = \mathrm{id}_F(x)$ in the group $\operatorname{Homeo}(F)$

(2) $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$, $f_{\alpha\beta}(x) f_{\beta\gamma}(x) f_{\gamma\alpha}(x) = \mathrm{id}_F$ in the group $\operatorname{Homeo}(F)$.

Also note that if we denote $\cdot$ as the action of $\mathrm{Homeo}(F)$ on F, then $\forall \xi \in E_x$, $x \in U_\alpha \cap U_\beta$,

$$f_{\alpha\beta}(\pi(\xi)) \cdot \varphi_\beta(\xi) = f_{\alpha\beta}(x) \cdot \varphi_\beta(\xi) = \varphi_\alpha|_{E_x} \circ (\varphi_\beta|_{E_x})^{-1}(\varphi_\beta(\xi)) = \varphi_\alpha(\xi).$$

Definition 1.10.9 (Coordinate Bundle with Structure Group)**.** *A* ***coordinate bundle with structure group*** *G is a fiber bundle $F \to E \xrightarrow{\pi} M$ with an effective topological transformation group G acting on F by η and a family $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of* ***coordinate charts****, consisting of open sets covering M and their associated homeomorphisms, such that*

- *any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $\alpha, \beta \in A$ are G-**compatible**: either $U_\alpha \cap U_\beta = \varnothing$ or there exists a continuous map g from $U_\alpha \cap U_\beta$ to G such that $f_{\alpha\beta}(x) = \eta_{g(x)} \in \mathrm{Homeo}(F)$ for all $x \in U_\alpha \cap U_\beta$. In this case we shall identify $f_{\alpha\beta}$ with g, and consider $f_{\alpha\beta}$ as a continuous map from $U_\alpha \cap U_\beta$ to G; this yields the identity $f_{\alpha\beta}(x) \cdot y = \eta\left(f_{\alpha\beta}(x), y\right)$, $\forall x \in U_\alpha \cap U_\beta, y \in F$.*

A fiber bundle chart is ***admissible*** *as a bundle chart on the bundle E with structure group G if and only if it is G-compatible with every element of the given G-bundle atlas.*

We revisit the definition of fiber bundle now.

Definition 1.10.10. *Two coordinate bundles $(F \to E \xrightarrow{\pi} M, G)$ and $(F' \to E' \xrightarrow{\pi'} M', G')$ are said to be* ***equivalent in the strict sense*** *if they have the same total space, base space, projection, fiber, and group, and their coordinate functions $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, $\{(U_\beta, \varphi_\beta)\}_{\beta \in B}$ combine to give a coordinate bundle atlas, i.e., every pair $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $(\alpha, \beta) \in A \times B$ are G-compatible.* †

Definition 1.10.11 (Fiber Bundle as Equivalence Class)**.** *With above notion of equivalence, a* ***fiber bundle*** *is defined to be an equivalence class of coordinate bundles.*

One may regard a fiber bundle as a "maximal" coordinate bundle having all possible coordinate functions of an equivalence class. As our indexing sets are unrestricted, this involves the usual logical difficulty connected with the use of the word "all."

Remark 1.10.12 (Category of smooth manifold)**.** We will stick with the category of smooth manifolds now. The fiber bundle $F \to E \xrightarrow{\pi} M$ is required to be a C^∞ fiber bundle, the transformation group G is required to be a Lie group, and the group $\mathrm{Homeo}(F)$ becomes the group $\mathrm{Diff}(F)$. Continuous maps become smooth maps. Homeomorphisms become diffeomorphisms. We will use p, q to refer to typical elements in M and reserve x, y for elements in F. ♠

We have a smooth analogue of the remark 1.10.2; for its proof, see [6] Proposition 1.5.

Proposition 1.10.13. *The projection map $\pi : E \to M$ of a C^∞ fiber bundle is a submersion, that is, for each point ξ in E, the differential $\mathrm{d}\pi_\xi : E_\xi \to M_{\pi(\xi)}$ is surjective; furthermore, for each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ in E over p is an embedded submanifold diffeomorphic to the standard fiber F of the bundle.*

1.10.2 Vector Bundles

We give three equivalent way of defining a vector bundle, each of which has smooth analogue as remarked in Remark 1.10.12.

Definition 1.10.14 ($\mathbb{F}$-Vector Bundle: Definition I)**.** *Let $\mathbb{F}$ be one of the fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and let V be an k-dimensional vector space over $\mathbb{F}$. A fiber bundle E over a topological n-manifold M with standard fiber V is called an* ***$\mathbb{F}$-vector bundle of rank k*** *if:*

(i) Each fiber $E_p := \pi^{-1}(p)$ is endowed with the structure of a k-dimensional $\mathbb{F}$-vector space;

†This is because the pairs $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $\alpha, \beta \in A$ or $\alpha, \beta \in B$ are already G-compatible.

*(ii) There exists a bundle atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$, called a **vector bundle atlas**, such that each bundle chart*

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V \approx U_\alpha \times \mathbb{F}^k$$

*is a homeomorphism and is **fiberwise linear**, meaning that for each $p \in U_\alpha$, the restriction*

$$\varphi_\alpha|_{E_p} : E_p \to \{p\} \times V \cong V \cong \mathbb{F}^k$$

is an $\mathbb{F}$-linear isomorphism.

*If M and E are smooth manifolds with or without boundary, π is a smooth map, and the local trivialization can be chosen to be diffeomorphisms, then E is called a **smooth $\mathbb{F}$-vector bundle**. In this case, we call any local trivialization that is a diffeomorphism onto its image a **smooth local trivialization**.*

We will not talk about $\mathbb{H}$-vector bundles in this note, so we assume $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Remark 1.10.15. Note that every vector space is a topological space and a smooth manifold (see [11] Example 1.24). Each fiber E_p is homeomorphic (diffeomorphic) to V by Remark 1.10.2 (Proposition 1.10.13), and the homeomorphism (diffeomorphism) is obtained by the restriction of the bundle chart φ_α to E_p. Thus, condition (i) and (ii) are merely saying that each fiber E_p is not just homeomorphic (diffeomorphic) to the standard fiber V via $\varphi_\alpha|_{E_p}$ but also isomorphic to the standard fiber V via the same map $\varphi_\alpha|_{E_p}$. ♠

We show that elevating $\varphi_\alpha|_{E_p}$ to an isomorphism is equivalent to imposing a structure group to make the fiber bundle a coordinate bundle.

Proposition 1.10.16 ($\mathbb{F}$-vector bundle: Definition II)**.** *Let V be a vector space over $\mathbb{F}$. A fiber bundle E over M with standard fiber V is an $\mathbb{F}$-vector bundle if and only if E admits $\mathrm{GL}(V)$ as a structure group, where $\mathrm{GL}(V)$ is the group of all invertible linear transformations of the vector space V.*

Proof. Let V be a k-dimensional vector space over $\mathbb{F}$ and suppose that $V \to E \xrightarrow{\pi} M$ is a fiber bundle with standard fiber V.

(⇒) Suppose E is an $\mathbb{F}$-vector bundle with vector bundle atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$. For any overlapping pair $U_\alpha \cap U_\beta \neq \varnothing$, we have the transition function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(V)$ (or $\mathrm{Diff}(V)$) by

$$f_{\alpha\beta,p} := \varphi_\alpha|_{E_p} \circ \left(\varphi_\beta|_{E_p}\right)^{-1} : V \to V.$$

Since both $\varphi_\alpha|_{E_p}$ and $\varphi_\beta|_{E_p}$ are linear isomorphisms, their composition $f_{\alpha\beta,p}$ is an element of $\mathrm{GL}(V)$. Thus, simply let η be the inclusion $\mathrm{GL}(V) \to \mathrm{Homeo}(V)$ (or $\mathrm{Diff}(V)$) and let g be the map $U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ sending p to $f_{\alpha\beta,p}$. $V \to E \xrightarrow{\pi} M$ is now a coordinate bundle with structure group $\mathrm{GL}(V)$.

(⇐) Conversely, suppose $V \to E \xrightarrow{\pi} M$ is now a coordinate bundle with structure group $\mathrm{GL}(V)$. For p in a chart $(U_\alpha, \varphi_\alpha)$ define a vector space structure on E_p by

$$cu + v := (\varphi_\alpha|_{E_p})^{-1}(c\varphi_\alpha|_{E_p}(u) + \varphi_\alpha|_{E_p}(v)) \quad c \in \mathbb{F}, u, v \in E_p \tag{1.17}$$

By definition, φ_α maps E_p isomorphically to V. To show the vector space structure is well-defined, we need to show that it does not depend on the choice of chart. That is, if $p \in U_\alpha \cap U_\beta$, then we want to show

$$(\varphi_\alpha|_{E_p})^{-1}(c\varphi_\alpha|_{E_p}(u) + \varphi_\alpha|_{E_p}(v)) = (\varphi_\beta|_{E_p})^{-1}(c\varphi_\beta|_{E_p}(u) + \varphi_\beta|_{E_p}(v))$$

But notice that $f_{\alpha\beta,p} = \varphi_\alpha|_{E_p} \circ (\varphi_\beta|_{E_p})^{-1} \in \mathrm{GL}(V)$, so

$$f_{\alpha\beta,p}(c\varphi_\beta|_{E_p}(u) + \varphi_\beta|_{E_p}(v)) = cf_{\alpha\beta,p}(\varphi_\beta|_{E_p}(u)) + f_{\alpha\beta,p}(\varphi_\beta|_{E_p}(v)) = c\varphi_\alpha|_{E_p}(u) + \varphi_\alpha|_{E_p}(v)$$

and

$$\varphi_\alpha^{-1}(c\varphi_\alpha(u) + \varphi_\alpha(v)) = \varphi_\alpha^{-1} \circ f_{\alpha\beta,p}(c\varphi_\beta|_{E_p}(u) + \varphi_\beta|_{E_p}(v)) = (\varphi_\beta|_{E_p})^{-1}(c\varphi_\beta|_{E_p}(u) + \varphi_\beta|_{E_p}(v)).$$

■

Proposition 1.10.17 ($\mathbb{F}$-Vector Bundle: Definition III)**.** *Let V be a k-dimensional vector space over field $\mathbb{F}$. Given an open cover (U_α) of M, and a collection of continuous maps*

$$f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$$

satisfying the cocycle condition (see Subsection 1.10.1), we can reconstruct a vector bundle by gluing the product bundles $E_\alpha := U_\alpha \times V$ on the overlaps $U_\alpha \cap U_\beta$ according to the gluing rules

- *The point $(p, x) \in E_\alpha$ is identified with the point $(f_{\beta\alpha}(p)x, p) \in E_\beta \ \forall p \in U_\alpha \cap U_\beta$.*

The resulting space E with quotient topology has natural local trivializations $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ to make it a vector bundle over M with standard fiber V. We will say that the map $f_{\beta\alpha}$ is the transition from the α-trivialization to the β trivialization, and we will refer to the collection of maps $(f_{\beta\alpha})$ satisfying the cocycle condition as a ***gluing cocycle****.*

Proof. Exercise. ∎

Some concepts from tangent bundles are shared with vector bundles: rough, continuous, and smooth sections, local and global frames. Smooth local sections that are linearly indepdent in each T_pM can be completed to be a smooth local frame; and a smooth local frame over a closed set (in particular a singleton) can be extended to a smooth local frame over a larger open neighborhood (see [11, Proposition 10.15]). Besides, it turns out that smooth local (global) frames and local (global) trivializations have close relationship:

Proposition 1.10.18 (Local Frames and Local Trivializations)**.** *Let $V \to E \xrightarrow{\pi} M$ be a smooth vector bundle.*

(i) Given a smooth local trivialization $\varphi : \pi^{-1}(U) \to U \times \mathbb{F}^k$ of E, we can use the method as in equation (1.17) to construct a local frame for E over U: define maps $\sigma_1, \ldots, \sigma_k : U \to E$ by $\sigma_i(p) = \varphi^{-1}(p, e_i) = (\varphi|_{E_p})^{-1}(e_i)$ for a set of basis vectors $e_1, \ldots, e_k$ of $\mathbb{F}^k$. (σ_i) is called the ***smooth local frame associated with*** *φ.*

(ii) Conversely, given a smooth local frame $\sigma_1, \ldots, \sigma_k : U \to E$ over U, we can define a smooth local trivialization $\varphi : \pi^{-1}(U) \to U \times \mathbb{F}^k$ as the inverse of a diffeomorphism $\phi : U \times \mathbb{F}^k \to \pi^{-1}(U)$; $(p, (v^1, \cdots, v^k)) \mapsto v^i\sigma_i(p)$. φ is called the ***smooth local trivialization associated with*** *(σ_i).*

(iii) (i) and (ii) say that there is a one-to-one correspondence between smooth local trivializations and smooth local frames over an open subset $U \subseteq M$. Furthermore, the constructions in (i) and (ii) are inverse to each other: the smooth local frame associated with a smooth local trivialization φ constructed in (ii) is given by $\varphi^{-1}(p, e_i) = \phi(p, (0, \cdots, 1, \cdots, 0)) = \sigma_i(p)$; the smooth local trivialization associated with a smooth local frame (σ_i) constructed in (i) is given by the inverse of $\phi_p(v) = v^i\sigma_i(p) = v^i(\varphi|_{E_p})^{-1}(e_i) = (\varphi|_{E_p})^{-1}(v^ie_i) = (\varphi|_{E_p})^{-1}(v)$, which is just $\phi_p^{-1} = ((\varphi|_{E_p})^{-1})^{-1} = \varphi|_{E_p}$.

(iv) As a corollary, a smooth vector bundle is smoothly trivial (admits a smooth global trivialization) $\iff$ it admits a global smooth frame. When the vector bundle is TM and it admits a global smooth frame, we say M is ***parallelizable****.*

Proposition 1.10.19. *Suppose $E \to M$ is a smooth vector bundle with standard fiber V, defined by an open cover $(U_\alpha)_{\alpha\in\mathcal{A}}$, and gluing cocycles*

$$f_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(V).$$

Then there exists a natural bijection between the vector space $\Gamma(E)$ of smooth sections of E, and the set of families of smooth maps $\{s_\alpha : U_\alpha \to V; \alpha \in \mathcal{A}\}$, satisfying the following gluing condition on the overlaps

$$s_\alpha(x) = f_{\alpha\beta}(x)s_\beta(x), \quad \forall x \in U_\alpha \cap U_\beta.$$

Proof. Fix a trivializing atlas $\{\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V\}_{\alpha\in\mathcal{A}}$ whose transition maps are

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, f_{\alpha\beta}(x)v), \qquad x \in U_{\alpha\beta},\ v \in V.$$

Define the map $\Gamma(E) \to \{(s_\alpha)\}$ *and check compatibility.* Given $s \in \Gamma(E)$, set

$$s_\alpha \;:=\; \mathrm{pr}_2 \circ \varphi_\alpha \circ s|_{U_\alpha} \;:\; U_\alpha \to V.$$

On $U_{\alpha\beta}$ we have

$$s_\alpha(x) \;=\; \mathrm{pr}_2\big(\varphi_\alpha(s(x))\big) = \mathrm{pr}_2\big(\varphi_\alpha \circ \varphi_\beta^{-1}\big(\varphi_\beta(s(x))\big)\big) = \mathrm{pr}_2\big(x, f_{\alpha\beta}(x)\, s_\beta(x)\big) = f_{\alpha\beta}(x)\, s_\beta(x),$$

so (s_α) satisfies the gluing condition.

Construct the inverse map. Conversely, suppose we are given smooth maps $s_\alpha : U_\alpha \to V$ with $s_\alpha = f_{\alpha\beta} s_\beta$ on $U_{\alpha\beta}$. Define $s : M \to E$ by

$$s(x) \;:=\; \varphi_\alpha^{-1}\big(x,\, s_\alpha(x)\big) \qquad \text{for } x \in U_\alpha.$$

This is well-defined: if $x \in U_{\alpha\beta}$, then

$$\varphi_\alpha^{-1}\big(x, s_\alpha(x)\big) = \varphi_\alpha^{-1}\big(x, f_{\alpha\beta}(x) s_\beta(x)\big) = \varphi_\beta^{-1}\big(x, s_\beta(x)\big)$$

because $\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, f_{\alpha\beta}(x)v)$. The map s is smooth since in each U_α it is the composition of smooth maps. Clearly $\pi \circ s = \mathrm{id}_M$, so $s \in \Gamma(E)$.

Bijectivity and linearity. The two constructions are inverse to each other by definition, hence yield a bijection. Both directions are linear (pointwise on V), so the bijection is a vector space isomorphism.

Therefore the assignment $s \leftrightarrow (s_\alpha)$ defines a natural bijection between $\Gamma(E)$ and the families (s_α) satisfying $s_\alpha = f_{\alpha\beta} s_\beta$ on overlaps. ∎

If $\pi : E \to M$ and $\pi' : E' \to M'$ are vector bundles, a continuous map $F : E \to E'$ is called a **bundle homomorphism** if there exists a map $f : M \to M'$ satisfying $\pi' \circ F = f \circ \pi$, with the property that for each $p \in M$, the restricted map $F|_{E_p} : E_p \to E'_{f(p)}$ is linear. The relationship between F and f is expressed by saying that F **covers** f.

Proposition 1.10.20. *Suppose $\pi : E \to M$ and $\pi' : E' \to M'$ are vector bundles and $F : E \to E'$ is a bundle homomorphism covering $f : M \to M'$. Then f is continuous and is uniquely determined by F. If the bundles and F are all smooth, then f is smooth as well.*

Proof. All of the conclusions follow from the easily verified fact that $f = \pi' \circ F \circ \zeta$, where $\zeta : M \to E$ is the zero section. ζ is a continuous section and when the bundle is smooth ζ is smooth as well. ∎

A bijective bundle homomorphism $F : E \to E'$ whose inverse is also a bundle homomorphism is called a **bundle isomorphism**; if F is also a diffeomorphism, it is called a **smooth bundle isomorphism**.

In the special case in which both E and E' are vector bundles over the same base space M, a slightly more restrictive notion of bundle homomorphism is usually more useful. A **bundle homomorphism over** M is a bundle homomorphism covering the identity map of M, or in other words, a continuous map $F : E \to E'$ such that $\pi' \circ F = \pi$, and whose restriction to each fiber is linear. If there exists a bundle homomorphism $F : E \to E'$ over M that is also a (smooth) bundle isomorphism, then we say that E and E' are **(smoothly) isomorphic over** M. The next proposition shows that it is not necessary to check smoothness of the inverse.

Proposition 1.10.21. *Suppose E and E' are smooth vector bundles over a smooth manifold M with or without boundary, and $F : E \to E'$ is a bijective smooth bundle homomorphism over M. Then F is a smooth bundle isomorphism.*

Exercise 1.10.22. *Show that a smooth rank-k vector bundle over M is smoothly trivial if and only if it is smoothly isomorphic over M to the product bundle $M \times \mathbb{F}^k$.*

Example 1.10.23. If $F: M \rightarrow N$ is a smooth map, the global differential $\mathrm{d}F: TM \rightarrow TN$ is a smooth bundle homomorphism covering F.

If $E \rightarrow M$ is a smooth vector bundle and $S \subseteq M$ is an immersed submanifold with or without boundary, then the inclusion map $\left.E\right|_S \hookrightarrow E$ is a smooth bundle homomorphism covering the inclusion of S into M. ♣

Lemma 1.10.24 (Bundle Homomorphism Characterization Lemma). *Let $\pi: E \rightarrow M$ and $\pi^{\prime}: E^{\prime} \rightarrow M$ be smooth vector bundles over a smooth manifold M with or without boundary, and let $\Gamma(E), \Gamma\left(E^{\prime}\right)$ denote their spaces of smooth sections. A map $\mathcal{F}: \Gamma(E) \rightarrow \Gamma\left(E^{\prime}\right)$ is linear over $C^{\infty}(M)$ if and only if there is a smooth bundle homomorphism $F: E \rightarrow E^{\prime}$ over M such that $\mathcal{F}(\sigma)=F \circ \sigma$ for all $\sigma \in \Gamma(E)$.*

Proof. See [11] Lemma 10.29. ■

Given a vector bundle $\pi_E: E \rightarrow M$, a **subbundle of** E is a vector bundle $\pi_D: D \rightarrow M$, in which D is a topological subspace of E and π_D is the restriction of π_E to D, such that for each $p \in M$, the subset $D_p=D \cap E_p$ is a linear subspace of E_p, and the vector space structure on D_p is the one inherited from E_p. Note that the condition that D be a vector bundle over M implies that all of the fibers D_p must be nonempty and have the same dimension. If $E \rightarrow M$ is a smooth bundle, then a subbundle of E is called a **smooth subbundle** if it is a smooth vector bundle and an embedded submanifold with or without boundary in E. In this case, the inclusion map $\iota: D \hookrightarrow E$ is a smooth bundle homomorphism over M.

The following lemma gives a convenient condition for checking that a union of subspaces $\left\{D_p \subseteq E_p: p \in M\right\}$ is a smooth subbundle.

Lemma 1.10.25 (Local Frame Criterion for Subbundles). *Let $\pi: E \rightarrow M$ be a smooth vector bundle, and suppose that for each $p \in M$ we are given an m dimensional linear subspace $D_p \subseteq E_p$. Then $D=\bigcup_{p \in M} D_p \subseteq E$ is a smooth subbundle of E if and only if the following condition is satisfied:*

- *Each point of M has a neighborhood U on which there exist smooth local sections $\sigma_1, \ldots, \sigma_m: U \rightarrow E$ with the property that $\sigma_1(q), \ldots, \sigma_m(q)$ form a basis for D_q at each $q \in U$.*

Proof. See [11] Lemma 10.32. ■

Example 1.10.26. Suppose $E \rightarrow M$ is any trivial bundle, and let $\left(E_1, \ldots, E_k\right)$ be a smooth global frame for E. If $0 \leq m \leq k$, the subset $D \subseteq E$ defined by $D_p=\operatorname{span}\left(\left.E_1\right|_p, \ldots,\left.E_m\right|_p\right)$ for each $p \in M$ is a smooth subbundle of E.

Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an immersed k-submanifold with or without boundary. TS is a smooth rank-k subbundle of the ambient tangent bundle $\left.TM\right|_S$. ♣

The next theorem shows how to obtain many more subbundles. Suppose $E \rightarrow M$ and $E^{\prime} \rightarrow M$ are vector bundles and $F: E \rightarrow E^{\prime}$ is a bundle homomorphism over M. For each $p \in M$, the rank of the linear map $\left.F\right|_{E_p}$ is called the **rank of** F **at** p. We say that F has **constant rank** if its rank is the same for all $p \in M$.

Theorem 1.10.27 ([11] Theorem 10.34). *Let E and $E^{\prime}$ be smooth vector bundles over a smooth manifold M, and let $F: E \rightarrow E^{\prime}$ be a smooth bundle homomorphism over M. Define subsets $\operatorname{Ker} F \subseteq E$ and $\operatorname{Im} F \subseteq E^{\prime}$ by*

$$
\operatorname{Ker} F=\bigcup_{p \in M} \operatorname{Ker}\left(\left.F\right|_{E_p}\right), \quad \operatorname{Im} F=\bigcup_{p \in M} \operatorname{Im}\left(\left.F\right|_{E_p}\right)
$$

Then $\operatorname{Ker} F$ and $\operatorname{Im} F$ are smooth subbundles of E and $E^{\prime}$, respectively, if and only if F has constant rank.

Example 1.10.28 (The tautological line bundle over $\mathbb{RP}^n$ and $\mathbb{CP}^n$). A rank one vector bundle is usually called a **line bundle**. We consider only the complex case. The total space of the tautological or universal line bundle over $\mathbb{CP}^n$ is the space

$$
\mathcal{U}_n=\mathcal{U}_n^{\mathbb{C}}=\left\{(z, L) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n ; \quad z \text { belongs to the line } L \subset \mathbb{C}^{n+1}\right\}
$$

Let $\pi : \mathcal{U}_n^{\mathbb{C}} \to \mathbb{CP}^n$ denote the projection onto the second component. Note that for every $L \in \mathbb{CP}^n$, the fiber through $\pi^{-1}(L) = \mathcal{U}_{n,L}^{\mathbb{C}}$ coincides with the one-dimensional subspace in $\mathbb{C}^{n+1}$ defined by L. ♣

Example 1.10.29 (The tautological vector bundle over a Grassmannian)**.** We consider here for brevity only complex Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$. The real case is completely similar. The total space of this bundle is

$$\mathcal{U}_{k,n} = \mathcal{U}_{k,n}^{\mathbb{C}} = \{(z, L) \in \mathbb{C}^n \times \mathrm{G}_k(\mathbb{C}^n) \,; z \text{ belongs to the subspace } L \subset \mathbb{C}^n\}$$

If π denotes the natural projection $\pi : \mathcal{U}_{k,n} \to \mathrm{G}_k(\mathbb{C}^n)$, then for each $L \in \mathrm{G}_k(\mathbb{C}^n)$ the fiber over L coincides with the subspace in $\mathbb{C}^n$ defined by L. Note that $\mathcal{U}_{n-1}^{\mathbb{C}} = \mathcal{U}_{1,n}^{\mathbb{C}}$. ♣

Exercise 1.10.30. *Prove that $\mathcal{U}_n^{\mathbb{C}}$ and $\mathcal{U}_{k,n}^{\mathbb{C}}$ are indeed smooth vector bundles. Describe a gluing cocycle for $\mathcal{U}_n^{\mathbb{C}}$.*

Example 1.10.31. A complex line bundle over a smooth manifold M is described by an open cover $(U_\alpha)_{\alpha \in \mathcal{A}}$, and smooth maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$$

satisfying the cocycle condition

$$g_{\gamma\alpha}(x) = g_{\gamma\beta}(x) \cdot g_{\beta\alpha}(x), \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma$$

Consider for example the manifold $M = S^2 \subset \mathbb{R}^3$. Denote as usual by N and S the North and respectively South pole. We have an open cover

$$S^2 = U_0 \cup U_\infty, \quad U_0 = S^2 \backslash \{S\}, \quad U_1 = S^2 \backslash \{N\}$$

In this case, we have only a single nontrivial overlap, $U_N \cap U_S$. Identify U_0 with the complex line $\mathbb{C}$, so that the North pole becomes the origin $z = 0$.

For every $n \in \mathbb{Z}$ we obtain a complex line bundle $L_n \to S^2$, defined by the open cover $\{U_0, U_1\}$ and gluing cocycle

$$g_{10}(z) = z^{-n}, \quad \forall z \in \mathbb{C}^* = U_0 \backslash \{0\}$$

A smooth section of this line bundle is described by a pair of smooth functions

$$u_0 : U_0 \to \mathbb{C}, \quad u_1 : U_1 \to \mathbb{C},$$

which along the overlap $U_0 \cap U_1$ satisfy the equality $u_1(z) = z^{-n} u_0(z)$. For example, if $n \geq 0$, the pair of functions

$$u_0(z) = z^n, \quad u_1(p) = 1, \quad \forall p \in U_1$$

defines a smooth section of L_n. ♣

Exercise 1.10.32. *We know that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$. Prove that the universal line bundle $\mathcal{U}_n \to \mathbb{CP}^1$ is isomorphic with the line bundle L_{-1} constructed in the above example.*

Methods for Constructing Vector Bundles

One can extend operations on vector spaces to operations on vector bundles. We give an outline here. Details are in [18] Chapter 3. Constructing New Vector Bundles Out of Old.

1. Given any vector bundle $V \to E \xrightarrow{\pi} M$, one can define the **dual vector bundle** by replacing each E_p with its dual E_p^*. (How to define local trivializations? What are the transition maps $f_{\beta\alpha}$'s?) For example, T^*M is the dual bundle of TM.
2. Let $V_1 \to E_1 \xrightarrow{\pi_1} M$ and $V_2 \to E_2 \xrightarrow{\pi_2} M$ be two vector bundles over M of rank r_1 and r_2 respectively. Then the **direct sum bundle** $V_1 \oplus V_2 \to E_1 \oplus E_2 \xrightarrow{\pi_1 \oplus \pi_2} M$ is the rank $r_1 + r_2$ vector bundle over M with fiber $(\pi_1 \oplus \pi_2)^{-1}(p) = (E_1)_p \oplus (E_2)_p$. (local trivializations? transition maps $f_{\beta\alpha}$'s?)

3. The **tensor product bundle** $V_1 \otimes V_2 \to E_1 \otimes E_2 \xrightarrow{\pi_1 \otimes \pi_2} M$ is the rank $r_1 r_2$ vector bundle over M with fiber $(\pi_1 \otimes \pi_2)^{-1}(p) = (E_1)_p \otimes (E_2)_p$. (local trivializations? transition maps $f_{\beta\alpha}$'s?) For example, we have the (k,l)-tensor bundle $\otimes^{k,l} TM := (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$ over M.

4. The vector space $\mathrm{Hom}(V_1, V_2)$ of linear transformations from V_1 to V_2 builds a vector bundle $\mathrm{Hom}(E_1, E_2)$ called the **homomorphism vector bundle** for two vector bundles $V_1 \to E_1 \xrightarrow{\pi_1} M_1$ and $V_2 \to E_2 \xrightarrow{\pi_2} M_2$. Notice that a continuous (smooth) section of $\mathrm{Hom}(E_1, E_2)$ is a (smooth) bundle homomorphism between the two vector bundles. We denote $\underline{\mathrm{Hom}}(E_1, E_2) := \Gamma(\mathrm{Hom}(E_1, E_2))$ as the space of all smooth bundle homomorphisms between E_1 and E_2.

5. One can define the **exterior power bundle** $\Lambda^k E$, whose fiber at point $p \in M$ is the linear space $\Lambda^k E_p$. See section 3 of this note for the definition of exterior power of a module (and thus a vector space). (local trivializations? transition maps $f_{\beta\alpha}$'s?)

Apart from using vector space operations, there is another method for generating a large class of exmaples:

6. Let $f : N \to M$ be a smooth map, and $V \to E \xrightarrow{\pi} M$ a vector bundle over M with open cover (U_α) and gluing cocycles $(f_{\alpha\beta})$. Then one can define a **pullback bundle** f^*E over N by setting the fiber over $x \in N$ to be the fiber of $E_{f(x)}$ It is equipped with open cover $f^{-1}(U_\alpha)$ and gluing cocycles $(g_{\alpha\beta} \circ f)$.

   In particular, the **restriction of a vector bundle** $V \to E \xrightarrow{\pi} M$ to a submanifold N of the base manifold M is a vector bundle over the submanifold N. (Note: this is not a vector subbundle of $V \to E \xrightarrow{\pi} M$!)

   The pullback operation defines a linear map from $\Gamma(E)$ to $\Gamma(f^*E)$, still denoted as f^*. More precisely, in view of Proposition 1.10.19, if $s \in \Gamma(E)$ is defined by the smooth maps (s_α) on the open cover (U_α), then the pullback f^*s is defined by the smooth maps $(s_\alpha \circ f)$ on the open cover $f^{-1}(U_\alpha)$.

1.10.3 Principal G-Bundles

Use [7] Section 6.8. Principal and Associated Bundles

1.11 (Almost) Complex Manifolds

Let M be a smooth manifold of (real) dimension $2n$ ($\dim_{\mathbb{R}} M = 2n$).

An **almost complex structure** on a manifold M is a smooth field of complex structures on the tangent spaces:

$$p \longmapsto J_p : T_pM \to T_pM \quad \text{linear,} \quad \text{and} \quad J_p^2 = -\mathrm{id}.$$

The pair (M, J) is then called an almost complex manifold. Smoothness is understood as the requirement that $J_p \in \mathrm{End}(T_pM) \cong T^{(1,1)}T_pM$ varies smoothly with $p \in M$ (see Proposition 1.1.5), or that J is a smooth section of the bundle $\mathrm{End}(TM)$.

A **complex structure** on M is an atlas $\mathcal{A}$ of charts taking values in open subsets of $\mathbb{C}^n$ with biholomorphic chart transitions. The pair $(M, \mathcal{A})$ is then called a **complex manifold**.

Proposition 1.11.1. *Any complex manifold has a canonical almost complex structure.*

Proof. See Ana Cannas da Silva's *Lectures on Symplectic Geometry* Proposition 15.2. ∎

Coordinates

Let M be a complex manifold. Each (U, φ) in the complex structure can be written as

$$\varphi_\alpha = (x^1, y^1, \cdots, x^n, y^n) = (z^1, \cdots, z^n)$$

with identifications $z^j = x^j + \sqrt{-1}y^j$ and $\overline{z}^j = x^j - \sqrt{-1}y^j$.

The Complexification of Tangent and Cotangent Bundles of Almost Complex Manifolds

Let M be a complex manifold. At p in the chart (U, φ), we have the real vector space T_pM and the **complexification of tangent space** $T_pM \otimes \mathbb{C} = T_pM \otimes_{\mathbb{R}} \mathbb{C}$, a complex vector space with $\dim_{\mathbb{C}} T_pM \otimes_{\mathbb{R}} \mathbb{C} = 2n$:

$$T_pM = \mathbb{R} - \operatorname{span}\left\{ \left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p \right\} = \left\{ \sum_{j=1}^n r_x^j \left.\frac{\partial}{\partial x_j}\right|_p + \sum_{j=1}^n r_y^j \left.\frac{\partial}{\partial y_j}\right|_p \middle| r_x^j, r_y^j \in \mathbb{R} \right\}$$

$$T_pM \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} - \operatorname{span}\left\{ \left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p \right\} = \left\{ \sum_{j=1}^n c_x^j \left.\frac{\partial}{\partial x_j}\right|_p + \sum_{j=1}^n c_y^j \left.\frac{\partial}{\partial y_j}\right|_p \middle| c_x^j, c_y^j \in \mathbb{C} \right\}$$

as in [24] Example 2.66 + Proposition 2.58: $T_pM \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{R}^{2n}) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus^{2n}(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}) \cong \bigoplus^{2n} \mathbb{C}$ with identification

$$\left(\sum_{j=1}^n r_x^j \left.\frac{\partial}{\partial x_j}\right|_p + \sum_{j=1}^n r_y^j \left.\frac{\partial}{\partial y_j}\right|_p \right) \otimes c \longleftrightarrow \sum_{j=1}^n (r_x^j c) \left.\frac{\partial}{\partial x_j}\right|_p + \sum_{j=1}^n (r_y^j c) \left.\frac{\partial}{\partial y_j}\right|_p \tag{1.18}$$

We then have the **complexified tangent bundle** $TM \otimes \mathbb{C}$, which is a complex vector bundle over M with fiber $(TM \otimes \mathbb{C}) = T_pM \otimes \mathbb{C}$ at each point $p \in M$.

We may extend J linearly to $TM \otimes \mathbb{C}$:

$$J(v \otimes c) = Jv \otimes c, \quad v \in TM, \quad c \in \mathbb{C}.$$

Since $J^2 = -\mathrm{Id}$, on the complex vector space $(TM \otimes \mathbb{C})_p$, the linear map J_p has eigenvalues $\pm i$:

$$\begin{aligned}
T_{1,0} &= \{v \in TM \otimes \mathbb{C} \mid Jv = +iv\} = (+i)\text{-eigenspace of } J \\
&= \{v \otimes 1 - Jv \otimes i \mid v \in TM\} \\
&\overset{(1.18)}{=\!=} \{r_x^j \partial_{x^j} + r_y^j \partial_{y^j} - (r_x^j \sqrt{-1} \partial_{y^j} - r_y^j \sqrt{-1} \partial_{x^j}) | r_x^j, r_y^j \in \mathbb{R}\} \\
&= \left\{ \sum_{j=1}^n (r_x^j + r_y^j \sqrt{-1})(\partial_{x^j} - \sqrt{-1} \partial_{y^j}) \middle| r_x^j, r_y^j \in \mathbb{R} \right\} \\
&= \left\{ \sum_{j=1}^n (2c_j) \left(\frac{1}{2} \left(\partial_{x^j} - \sqrt{-1} \partial_{y^j} \right) \right) \middle| c^j \in \mathbb{C} \right\} \\
&= \mathbb{C} - \operatorname{span}\left\{ \frac{\partial}{\partial z^j} : j = 1, \ldots, n \right\} \\
&= \textbf{(}J\textbf{-)holomorphic tangent vectors}; \\
T_{0,1} &= \{v \in TM \otimes \mathbb{C} \mid Jv = -iv\} = (-i)\text{-eigenspace of } J \\
&= \{v \otimes 1 + Jv \otimes i \mid v \in TM\} \\
&= \cdots = \left\{ \sum_{j=1}^n (2c_j) \left(\frac{1}{2} \left(\partial_{x^j} + \sqrt{-1} \partial_{y^j} \right) \right) \middle| c^j \in \mathbb{C} \right\} \\
&= \mathbb{C} - \operatorname{span}\left\{ \frac{\partial}{\partial \bar{z}^j} : j = 1, \ldots, n \right\} \\
&= \textbf{(}J\textbf{-)anti-holomorphic tangent vectors}.
\end{aligned}$$

$\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j}$ are called **Wirtinger derivatives**. For each $p \in M$, they form a basis of the complex vector space $(TM \otimes \mathbb{C})_p$ with complex dimension $2n$:

$$T_pM \otimes_{\mathbb{R}} \mathbb{C} = (T_{1,0})_p \oplus (T_{0,1})_p$$

Since

$$\begin{aligned}\pi_{1,0}: TM &\longrightarrow T_{1,0}\\ v &\longmapsto \frac{1}{2}(v\otimes 1 - Jv\otimes i)\end{aligned}$$

is a (real) bundle isomorphism such that $\pi_{1,0}\circ J = i\pi_{1,0}$, and

$$\begin{aligned}\pi_{0,1}: TM &\longrightarrow T_{0,1}\\ v &\longmapsto \frac{1}{2}(v\otimes 1 + Jv\otimes i)\end{aligned}$$

is also a (real) bundle isomorphism such that $\pi_{0,1}\circ J = -i\pi_{0,1}$, we conclude that we have isomorphisms of complex vector bundles

$$(TM, J) \cong T_{1,0} \cong \overline{T_{0,1}},$$

where $\overline{T_{0,1}}$ denotes the complex conjugate bundle of $T_{0,1}$. Extending $\pi_{1,0}$ and $\pi_{0,1}$ to projections of $TM\otimes\mathbb{C}$, we obtain an isomorphism

$$(\pi_{1,0}, \pi_{0,1}): TM\otimes\mathbb{C} \xrightarrow{\cong} T_{1,0}\oplus T_{0,1}.$$

Similarly, we have the **complexification of the cotangent space** T_p^*M:

$$\begin{aligned}T_p^*M\otimes_{\mathbb{R}}\mathbb{C} &= \mathbb{C}-\operatorname{span}\left\{\left.\mathrm{d}x^j\right|_p,\ \left.\mathrm{d}y^j\right|_p : j=1,\ldots,n\right\}\\ &= \mathbb{C}-\operatorname{span}\left\{\left.\mathrm{d}x^j\right|_p + \sqrt{-1}\left.\mathrm{d}y^j\right|_p : j=1,\ldots,n\right\} \oplus \mathbb{C}-\operatorname{span}\left\{\left.\mathrm{d}x^j\right|_p - \sqrt{-1}\left.\mathrm{d}y^j\right|_p : j=1,\ldots,n\right\}\\ &= \mathbb{C}-\operatorname{span}\left\{\left.\mathrm{d}z^j\right|_p : j=1,\ldots,n\right\} \oplus \mathbb{C}-\operatorname{span}\left\{\left.\mathrm{d}\bar{z}^j\right|_p : j=1,\ldots,n\right\}\\ &= \left(T^{1,0}\right)_p \oplus \left(T^{0,1}\right)_p\end{aligned}$$

and **complexified cotangent bundle**:

$$\left(\pi^{1,0}, \pi^{0,1}\right): T^*M\otimes\mathbb{C} \xrightarrow{\cong} T^{1,0}\oplus T^{0,1}$$

where

$$\begin{aligned}T^{1,0} &= (T_{1,0})^* = \{\eta\in T^*M\otimes\mathbb{C} \mid \eta(J\omega) = i\eta(\omega), \forall\omega\in TM\otimes\mathbb{C}\}\\ &= \{\xi\otimes 1 - (\xi\circ J)\otimes i \mid \xi\in T^*M\}\\ &= \textbf{complex-linear cotangent vectors}\end{aligned}$$

$$\begin{aligned}T^{0,1} &= (T_{0,1})^* = \{\eta\in T^*M\otimes\mathbb{C} \mid \eta(J\omega) = -i\eta(\omega), \forall\omega\in TM\otimes\mathbb{C}\}\\ &= \{\xi\otimes 1 + (\xi\circ J)\otimes i \mid \xi\in T^*M\}\\ &= \textbf{complex-antilinear cotangent vectors}\end{aligned}$$

and $\pi^{1,0}, \pi^{0,1}$ are the two natural projections

$$\begin{aligned}\pi^{1,0}: T^*M\otimes\mathbb{C} &\longrightarrow T^{1,0}\\ \eta &\longmapsto \eta^{1,0} := \frac{1}{2}(\eta - i\eta\circ J)\\ \pi^{0,1}: T^*M\otimes\mathbb{C} &\longrightarrow T^{0,1}\\ \eta &\longmapsto \eta^{0,1} := \frac{1}{2}(\eta + i\eta\circ J)\end{aligned}$$

The Complexification of Differential Forms on Almost Complex Manifolds

We first record two facts:

Proposition 1.11.2. *There is a canonical isomorphism:*

$$\Lambda^k(V) \otimes_{\mathbb{R}} \mathbb{C} \cong \Lambda^k(V \otimes_{\mathbb{R}} \mathbb{C})$$

for a finite-dimensional real vector space V.

Proof. Define the map

$$((v_1 \wedge \cdots \wedge v_k) \otimes z) \mapsto (v_1 \otimes 1) \wedge \cdots \wedge (v_k \otimes 1) \cdot z$$

and its inverse

$$(v_1 \wedge \cdots \wedge v_k) \otimes (z_1 \cdots z_k) \leftarrow (v_1 \otimes z_1) \wedge \cdots \wedge (v_k \otimes z_k).$$

■

Theorem 1.11.3 (Künneth formula for exterior powers)**.** *Let M and N be R-modules. For $k \geq 0$, there is an R-module isomorphism*

$$\Lambda^k(M \oplus N) \cong \bigoplus_{i=0}^{k} \left(\Lambda^i(M) \otimes_R \Lambda^{k-i}(N)\right).$$

Proof. See Theorem 9.1 of this note. ■

For an almost complex manifold (M, J), consider k-th exterior power of the complex vector bundle $T^*M \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1}$ and its decomposition via above formula:

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \Lambda^k\left(T^{1,0} \oplus T^{0,1}\right) \xlongequal{(1.11.3)} \bigoplus_{l+m=k} \underbrace{\left(\Lambda^l T^{1,0}\right) \otimes_{\mathbb{R}} \left(\Lambda^m T^{0,1}\right)}_{=:\Lambda^{l,m}} = \bigoplus_{l+m=k} \Lambda^{l,m} \tag{1.19}$$

In particular, $\Lambda^{1,0} = T^{1,0}$ and $\Lambda^{0,1} = T^{0,1}$. Use Proposition 1.11.2 and pass the operation of vector space to vector bundle (see part 1.10.2) so that

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \Lambda^k(T^*M) \otimes \mathbb{C}. \tag{1.20}$$

We define

$$\Omega^{l,m} := \Gamma(\Lambda^{l,m}) = \text{smooth sections of vector bundle } \Lambda^{l,m}.$$

An element of $\Omega^{l,m}$ is called a (l,m)**-form**. We also define

$$\Omega^k(M;\mathbb{C}) := \Gamma(\Lambda^k(T^*M \otimes \mathbb{C})) = \text{smooth sections of vector bundle } \Lambda^k(T^*M \otimes \mathbb{C}).$$

An element of $\Omega^k(M;\mathbb{C})$ is called a **complex-valued k-form on M**, thanks to (1.20):

$$\begin{aligned}
&\Omega^k(M;\mathbb{C}) \\
=&\Omega^k(M) \otimes \mathbb{C} \text{ by } (1.20) \\
=&\{\text{complex-linear combinations of real } k\text{-forms}\} \\
=&\{\text{complex-valued } k\text{-forms}\} \\
=&\left\{\beta = \sum_{|J|+|K|=k} a_{J,K}\, \mathrm{d}x^J \wedge \mathrm{d}y^K, \text{ with } a_{J,K} \in C^\infty(U;\mathbb{C})\right\} \\
&\text{on a chart } (U, (x^j, y^j)),
\end{aligned}$$

where in the third equality we noticed that every complex-valued functions is a complex-linear combination of real functions and vice versa; all other equalities directly follow from definitions. In other words, $\Omega^k(M;\mathbb{C}) \cong \Omega^k(M) \otimes \mathbb{C} \cong \mathrm{Hom}(\Omega^k(M), \mathbb{C})$ via [24, Proposition 2.58].

Suppose further that M is a complex manifold. In a chart $(U, \varphi = (z^j))$, we can explicitly write down the elements in $\Omega^{l,m}$ and thus $\Omega^k(M;\mathbb{C})$. If we use multi-index notation:

$$\begin{aligned} J &= (j_1, \ldots, j_m) \quad 1 \le j_1 < \ldots < j_m \le n \\ |J| &= m \\ \mathrm{d}z^J &= \mathrm{d}z^{j_1} \wedge \mathrm{d}z^{j_2} \wedge \ldots \wedge \mathrm{d}z^{j_m} \end{aligned}$$

then

$$\Omega^{l,m} = \left\{ \sum_{|J|=l, |K|=m} b_{J,K} \,\mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K \,\middle|\, b_{J,K} \in C^\infty(U, \mathbb{C}) \right\}.$$

In particular,

$$\begin{aligned} (1,0)\text{-forms } &= \left\{ \sum_{j=1}^n b_j \,\mathrm{d}z^j \mid b_j \in C^\infty(U,\mathbb{C}) \right\} \\ (0,1)\text{-forms } &= \left\{ \sum_{j=1}^n b_j \,\mathrm{d}\bar{z}^j \mid b_j \in C^\infty(U,\mathbb{C}) \right\} \\ (2,0)\text{-forms } &= \left\{ \sum_{1 \le j_1 < j_2 \le n} b_{j_1,j_2} \,\mathrm{d}z^{j_1} \wedge \mathrm{d}z^{j_2} \,\middle|\, b_{j_1,j_2} \in C^\infty(U,\mathbb{C}) \right\} \\ (1,1)\text{-forms } &= \left\{ \sum_{1 \le j_1, j_2 \le n} b_{j_1,j_2} \,\mathrm{d}z^{j_1} \wedge \mathrm{d}\bar{z}^{j_2} \,\middle|\, b_{j_1,j_2} \in C^\infty(U,\mathbb{C}) \right\} \\ (0,2)\text{-forms } &= \left\{ \sum_{1 \le j_1 < j_2 \le n} b_{j_1,j_2} \,\mathrm{d}\bar{z}^{j_1} \wedge \mathrm{d}\bar{z}^{j_2} \,\middle|\, b_{j_1,j_2} \in C^\infty(U,\mathbb{C}) \right\} \end{aligned}$$

Let $\pi^{l,m}$ denote the natural projection operators

$$\pi^{l,m} : \Omega^k(M;\mathbb{C}) \longrightarrow \Omega^{l,m}, \quad l + m = k.$$

We can apply the tensor functor $- \otimes \mathbb{C}$ to the cochain complex

$$\cdots \to \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots$$

to get

$$\cdots \to \Omega^{k-1}(M) \otimes \mathbb{C} \xrightarrow{d \otimes \mathrm{id}} \Omega^k(M) \otimes \mathbb{C} \xrightarrow{d \otimes \mathrm{id}} \Omega^{k+1}(M) \otimes \mathbb{C} \to \cdots$$

If we still denote $\mathrm{d} \otimes \mathrm{id}$ by d, then this becomes

$$\cdots \to \Omega^{k-1}(M;\mathbb{C}) \xrightarrow{d} \Omega^k(M;\mathbb{C}) \xrightarrow{d} \Omega^{k+1}(M;\mathbb{C}) \to \cdots$$

On a complex chart U, a complex-valued k-form $\beta \in \Omega^k(M;\mathbb{C})$ can be written as

$$\beta = \sum_{|J|+|K|=k} a_{J,K} \,\mathrm{d}x^J \wedge \mathrm{d}y^K, \quad \text{with } a_{J,K} \in C^\infty(U;\mathbb{C})$$

As noted, it is also a complex-linear combination of real k-forms in $\Omega^k(M)$:

$$\beta = \sum_{|J|+|K|=k} \mathrm{Re}(a_{J,K}) \,\mathrm{d}x^J \wedge \mathrm{d}y^K + \sqrt{-1} \sum_{|J|+|K|=k} \mathrm{Im}(a_{J,K}) \,\mathrm{d}x^J \wedge \mathrm{d}y^K$$

Thus,

$$\begin{aligned}\overbrace{\mathrm{d}\beta}^{(\mathrm{d}\otimes\mathrm{id})\beta} &= \mathrm{d}\left(\sum_{|J|+|K|=k}\operatorname{Re}(a_{J,K})\,\mathrm{d}x^J\wedge\,\mathrm{d}y^K\right)+\sqrt{-1}\,\mathrm{d}\left(\sum_{|J|+|K|=k}\operatorname{Im}(a_{J,K})\,\mathrm{d}x^J\wedge\,\mathrm{d}y^K\right)\\ &= \left(\sum_{|J|+|K|=k}\mathrm{d}\left(\operatorname{Re}(a_{J,K})\right)\wedge\,\mathrm{d}x^J\wedge\,\mathrm{d}y^K\right)+\sqrt{-1}\left(\sum_{|J|+|K|=k}\mathrm{d}\left(\operatorname{Im}(a_{J,K})\right)\wedge\,\mathrm{d}x^J\wedge\,\mathrm{d}y^K\right)\end{aligned}$$

If we define

$$\mathrm{d}f=\,\mathrm{d}\operatorname{Re}(f)+\sqrt{-1}\,\mathrm{d}\operatorname{Im}(f)$$

for any smooth complex-valued function on $f: M\to\mathbb{C}$, then

$$\mathrm{d}\beta=\sum_{|J|+|K|=k}\mathrm{d}a_{J,K}\wedge\,\mathrm{d}x^J\wedge\,\mathrm{d}y^K$$

Definition 1.11.4. *A function f is J-**holomorphic at** $p\in M$ if $\mathrm{d}f_p$ is complex linear, i.e., $\mathrm{d}f_p\circ J=\sqrt{-1}\,\mathrm{d}f_p$. A function f is J-**holomorphic** if it is J-holomorphic at all $p\in M$. A function f is J-**anti-holomorphic at** $p\in M$ if $\mathrm{d}f_p$ is complex antilinear, i.e., $\mathrm{d}f_p\circ J=-\sqrt{-1}\,\mathrm{d}f_p$. A function f is J-**anti-holomorphic** if it is J-anti-holomorphic at all $p\in M$.*

Exercise 1.11.5. *Show that*

$$f \text{ is } J\text{-holomorphic at } p \iff \mathrm{d}f_p\circ J=\sqrt{-1}\,\mathrm{d}f_p \iff \mathrm{d}f_p\in T_p^{1,0} \iff \pi_p^{0,1}\,\mathrm{d}f_p=\bar\partial f=0.$$

and

$$\begin{aligned}f \text{ is } J\text{-anti-holomorphic at } p \iff \mathrm{d}f_p\circ J=-\sqrt{-1}\,\mathrm{d}f_p &\iff \mathrm{d}f_p\in T_p^{0,1} \iff \pi_p^{1,0}\,\mathrm{d}f_p=\partial f=0\\ &\iff \mathrm{d}\bar f_p\in T_p^{1,0} \iff \bar f \text{ is } J\text{-holomorphic at } p.\end{aligned}$$

Solution. The first claim follows directly from the definition:

$$T_p^{1,0}=\left\{\eta\in T_p^*\otimes\mathbb{C}\mid \eta(J\omega)=i\eta(\omega)\right\}=\left\{\eta\mid i\eta\circ J(\omega)=i\cdot i\eta(\omega)\right\}=\left\{\eta\mid \pi_p^{0,1}(\eta)=\eta\circ J+i\eta\circ J=0\right\}$$

The second claim can be similarly obtained. The last line is due to the facts $\mathrm{d}\bar f=\overline{\mathrm{d}f}$ and $\overline{T^{0,1}}\cong T^{1,0}$. ♦

We also define differential operators

$$\begin{aligned}\partial &:= \pi^{l+1,m}\circ\,\mathrm{d}|_{\Omega^{l,m}}:\Omega^{l,m}\longrightarrow\Omega^{k+1}(M;\mathbb{C})\longrightarrow\Omega^{l+1,m}\\ \bar\partial &:= \pi^{l,m+1}\circ\,\mathrm{d}|_{\Omega^{l,m}}:\Omega^{l,m}\longrightarrow\Omega^{k+1}(M;\mathbb{C})\longrightarrow\Omega^{l,m+1}.\end{aligned}$$

We then extend ∂ and $\bar\partial$ to $\Omega^k(M;\mathbb{C})=\bigoplus_{l+m=k}\Omega^{l,m}$ by complex linearity.

Exterior Differentiation

Let M be a complex manifold. We would like to know whether the following equality holds:

$$\mathrm{d}\beta\overset{?}{=}(\partial+\bar\partial)\beta.$$

If $\beta\in\Omega^{l,m}(M)$, with $k=l+m$, then $\mathrm{d}\beta\in\Omega^{k+1}(M;\mathbb{C})$:

$$\mathrm{d}\beta=\sum_{r+s=k+1}\pi^{r,s}\,\mathrm{d}\beta=\pi^{k+1,0}\,\mathrm{d}\beta+\cdots+\underbrace{\pi^{l+1,m}\,\mathrm{d}\beta}_{\partial\beta}+\underbrace{\pi^{l,m+1}\,\mathrm{d}\beta}_{\bar\partial\beta}+\cdots+\pi^{0,k+1}\,\mathrm{d}\beta \tag{1.21}$$

When $k=0$, $b\in\Omega^0(M;\mathbb{C})=$ complex-valued smooth functions, and $\mathrm{d}:\Omega^0(M;\mathbb{C})\to\Omega^1(M;\mathbb{C})=\Omega^{0,1}\oplus\Omega^{1,0}$, above equality becomes

$$\mathrm{d}b=\pi^{1,0}\,\mathrm{d}b+\pi^{0,1}\,\mathrm{d}b=\partial b+\bar{\partial}b \tag{1.22}$$

Thus, on functions, $\mathrm{d}=\partial+\bar{\partial}$. Locally, in terms of z and $\bar{z}$ and using the identities

$$\begin{cases}\mathrm{d}x^j+i\,\mathrm{d}y^j=\mathrm{d}z^j\\ \mathrm{d}x^j-i\,\mathrm{d}y^j=\mathrm{d}\bar{z}^j\end{cases}\Longleftrightarrow\begin{cases}\mathrm{d}x_j=\frac{1}{2}(\mathrm{d}z_j+\mathrm{d}\bar{z}_j)\\ \mathrm{d}y_j=\frac{1}{2i}(\mathrm{d}z_j-\mathrm{d}\bar{z}_j)\end{cases},$$

we obtain the following formulae:

$$\begin{aligned}
\mathrm{d}b&=\mathrm{d}\overbrace{\mathrm{Re}(b)}^{u}+i\,\mathrm{d}\overbrace{\mathrm{Im}(b)}^{v}\\
&=\left(\sum_j\frac{\partial u}{\partial x^j}\,\mathrm{d}x^j+\sum_j\frac{\partial u}{\partial y^j}\,\mathrm{d}y^j\right)+i\left(\sum_j\frac{\partial v}{\partial x^j}\,\mathrm{d}x^j+\sum_j\frac{\partial v}{\partial y^j}\,\mathrm{d}y^j\right)\\
&=\sum_j\left(\frac{\partial u}{\partial x^j}+i\frac{\partial v}{\partial x^j}\right)\mathrm{d}x^j+\sum_j\left(\frac{\partial u}{\partial y^j}+i\frac{\partial v}{\partial y^j}\right)\mathrm{d}y^j\\
&=\sum_j\left(\frac{\partial b}{\partial x^j}\,\mathrm{d}x^j+\frac{\partial b}{\partial y^j}\,\mathrm{d}y^j\right)\\
&=\sum_j\left[\frac{1}{2}\left(\frac{\partial b}{\partial x^j}-i\frac{\partial b}{\partial y^j}\right)(\mathrm{d}x^j+i\,\mathrm{d}y^j)+\frac{1}{2}\left(\frac{\partial b}{\partial x^j}+i\frac{\partial b}{\partial y^j}\right)(\mathrm{d}x^j-i\,\mathrm{d}y^j)\right]\\
&=\sum_j\left(\frac{\partial b}{\partial z^j}\,\mathrm{d}z^j+\frac{\partial b}{\partial\bar{z}^j}\,\mathrm{d}\bar{z}^j\right).
\end{aligned}$$

Hence,

$$\begin{cases}\partial b=\pi^{1,0}\,\mathrm{d}b=\sum_j\frac{\partial b}{\partial z_j}\,\mathrm{d}z_j\\ \bar{\partial}b=\pi^{0,1}\,\mathrm{d}b=\sum_j\frac{\partial b}{\partial\bar{z}_j}\,\mathrm{d}\bar{z}_j\end{cases} \tag{1.23}$$

When $k\geq 1$, we substitute and reshuffle the indices under the identities on $\mathrm{d}z^j$ and $\mathrm{d}\bar{z}^j$ to write $\beta\in\Omega^k(M;\mathbb{C})$ locally as

$$\beta=\underbrace{\sum_{|J|+|K|=k}b_{J,K}\,\mathrm{d}z^J\wedge\mathrm{d}\bar{z}^K}_{\in\Omega^k(M;\mathbb{C})=\bigoplus_{l+m=k}\Omega^{l,m}}=\sum_{l+m=k}\underbrace{\left(\sum_{|J|=l,|K|=m}b_{J,K}\,\mathrm{d}z^J\wedge\mathrm{d}\bar{z}^K\right)}_{\in\Omega^{l,m}}.$$

Because $\mathrm{d}(\sum_I' \omega_I \, \mathrm{d}x^I) = \sum_I' \, \mathrm{d}\omega_I \wedge \mathrm{d}x^I$ for real forms and we extended d linearly to complex forms, so we have

$$\begin{aligned}
\partial\beta &= \partial\left(\sum_{l+m=k}\left(\sum_{|J|=l,|K|=m} b_{J,K}\, \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K\right)\right) \\
&= \sum_{l+m=k}\left(\sum_{|J|=l,|K|=m} \pi^{l+1,m}\left(\mathrm{d}b_{J,K} \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K\right)\right) \\
&= \sum_{l+m=k}\left(\sum_{|J|=l,|K|=m} \pi^{l+1,m}\left(\left(\sum_{j=1}^n \frac{\partial b_{J,K}}{\partial z^j}\, \mathrm{d}z^j + \sum_{k=1}^n \frac{\partial b_{J,K}}{\partial \bar{z}^k}\, \mathrm{d}\bar{z}^k\right) \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K\right)\right) \\
&= \sum_{l+m=k}\left(\sum_{|J|=l,|K|=m}\left(\sum_{j=1}^n \frac{\partial b_{J,K}}{\partial z^j}\, \mathrm{d}z^j\right) \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K\right) \\
&\quad \text{(the } \mathrm{d}\bar{z}^\alpha \text{ terms go to } (l, m+1)\text{-component and are ignored in this projection)} \\
&= \sum_{l+m=k}\left(\sum_{|J|=l,|K|=m} \partial b_{J,K} \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K\right) \\
&\quad \text{(due to (1.23))}
\end{aligned}$$

and similarly,

$$\bar{\partial}\beta = \sum_{l+m=k}\left(\sum_{|J|=l,|K|=m} \bar{\partial} b_{J,K} \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K\right)$$

Lastly,

$$\begin{aligned}
\mathrm{d}\beta &= \sum_{l+m=k}\left(\sum_{|J|=l,|K|=m} \mathrm{d}b_{J,K} \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K\right) \\
&= \sum_{l+m=k}\sum_{|J|=l,|K|=m}\left(\partial b_{J,K} + \bar{\partial} b_{J,K}\right) \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K \\
&= \sum_{l+m=k}\left(\underbrace{\sum_{|J|=l,|K|=m} \partial b_{J,K} \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K}_{\in\Omega^{l+1,m}} + \underbrace{\sum_{|J|=l,|K|=m} \bar{\partial} b_{J,K} \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K}_{\in\Omega^{l,m+1}}\right) \\
&= \partial\beta + \bar{\partial}\beta.
\end{aligned}$$

Therefore,

$$\mathrm{d} = \partial + \bar{\partial}$$

on forms of any degree for a complex manifold.

In the case where $\beta \in \Omega^{\ell,m}$, we have

$$\mathrm{d}\beta = \partial\beta + \bar{\partial}\beta = (l+1, m)\text{-form} + (l, m+1)\text{-form}$$

That is, all other terms in (1.21) vanish. Since $\mathrm{d} \circ \mathrm{d} = 0$ for real forms, $\mathrm{d} \circ \mathrm{d} = 0$ for complex forms by linearity. We have

$$0 = \mathrm{d}^2\beta = \underbrace{\partial^2\beta}_{\in\Omega^{l+2,m}} + \underbrace{\partial\bar{\partial}\beta + \bar{\partial}\partial\beta}_{\in\Omega^{l+1,m+1}} + \underbrace{\bar{\partial}^2\beta}_{\in\Omega^{\ell,m+2}},$$

Since they are forms of different types, this forces

$$\begin{cases}\bar{\partial}^2=0\\ \partial\bar{\partial}+\bar{\partial}\partial=0\\ \partial^2=0\end{cases}$$

Since $\bar{\partial}^2=0$, the chain

$$0\longrightarrow\Omega^{l,0}\xrightarrow{\bar{\partial}}\Omega^{l,1}\xrightarrow{\bar{\partial}}\Omega^{l,2}\xrightarrow{\bar{\partial}}\cdots$$

is a differential complex, called **Dolbeault complex**; its cohomology groups

$$H^{l,m}_{\text{Dolbeault}}(M):=\frac{\ker\bar{\partial}:\Omega^{l,m}\to\Omega^{l,m+1}}{\operatorname{im}\bar{\partial}:\Omega^{\ell,m-1}\to\Omega^{\ell,m}}$$

are called the **Dolbeault cohomology groups**.

1.12 Symplectic Manifolds

1.13 Kähler Manifolds

1.14 Problems

Exercise 1.14.1 ([11] 1-7). *Let N denote the north pole $(0,\ldots,0,1)\in\mathbb{S}^n\subseteq\mathbb{R}^{n+1}$, and let S denote the south pole $(0,\ldots,0,-1)$. Define the* ***stereographic projection*** *$\sigma:\mathbb{S}^n\backslash\{N\}\to\mathbb{R}^n$ by*

$$\sigma\left(x^1,\ldots,x^{n+1}\right)=\frac{\left(x^1,\ldots,x^n\right)}{1-x^{n+1}}$$

Let $\widetilde{\sigma}(x)=-\sigma(-x)$ for $x\in\mathbb{S}^n\backslash\{S\}$.

(a) For any $x\in\mathbb{S}^n\backslash\{N\}$, show that $\sigma(x)=u$, where $(u,0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1}=0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called ***stereographic projection from the south pole****.)*

(b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1,\ldots,u^n\right)=\frac{\left(2u^1,\ldots,2u^n,|u|^2-1\right)}{|u|^2+1}$$

(c) Compute the transition map $\tilde{\sigma}\circ\sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n\backslash\{N\},\sigma)$ and $(\mathbb{S}^n\backslash\{S\},\tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\tilde{\sigma}$ are called stereographic coordinates.)

(d) Show that this smooth structure is the same as the one defined in [11, Example 1.31].

Solution. (a) Quick solution: Let O be the origin and $\pi(x)=\left(x^1,\ldots,x^n,0\right)$ the projection to the hyperplane $H=\left\{x\in\mathbb{R}^{n+1}\mid x^{n+1}=0\right\}$. The triangle $N,O,(u,0)$ is similar to the triangle $x,\pi(x),(u,0)$. Hence, $(u,0)=\pi(x)/\left(1-x^{n+1}\right)$. The line passing through N and $-x$ is the mirror image of the line passing through S and x, which yields $\tilde{\sigma}(x)=-\sigma(-x)$.

Slow solution: The line passing through $N=(0,\cdots,0,1)$ and $x=\left(x^1,\cdots,x^n,x^{n+1}\right)$ can be written as

$$N+t(x-N),\quad t\in\mathbb{R}$$

or

$$\begin{aligned} N+t(x-N) &= (0,\cdots,0,1)+t\left(\left(x^1,\cdots,x^n,x^{n+1}\right)-(0,\cdots,0,1)\right) \\ &= \left(tx^1,\cdots,tx^n,t\left(x^{n+1}-1\right)+1\right), \quad t\in\mathbb{R} \end{aligned}$$

let $t\left(x^{n+1}-1\right)+1=0$ to get its intersection with the hyperplane $x^{n+1}=0$ and we have

$$t\left(x^{n+1}-1\right)+1=0 \Rightarrow t=\frac{1}{1-x^{n+1}}, \quad x^{n+1}\neq 1 \Leftrightarrow t=\frac{1}{1-x^{n+1}}, \quad x\in\mathbb{S}^n\backslash\{N\}$$

Plugging t into the original line to get

$$(u,0)=\left.\left(tx^1,\cdots,tx^n,t\left(x^{n+1}-1\right)+1\right)\right|_{t=\frac{1}{1-x^{n+1}}}=\left(\frac{x^1}{1-x^{n+1}},\cdots,\frac{x^n}{1-x^{n+1}},0\right)=(\sigma(x),0)$$

which shows that $\sigma(x)=u$.

For the other part, let u' be similarly defined: $(u',0)$ is the point where the line through S and x interests the linear subspace $x^{n+1}=0$ The line passing through $S=(0,\cdots,0,-1)$ and $x=\left(x^1,\cdots,x^n,x^{n+1}\right)$ can be written as

$$S+t(x-S), \quad t\in\mathbb{R}$$

or

$$\begin{aligned} S+t(x-S) &= (0,\cdots,0,-1)+t\left(\left(x^1,\cdots,x^n,x^{n+1}\right)-(0,\cdots,0,-1)\right) \\ &= \left(tx^1,\cdots,tx^n,t\left(x^{n+1}+1\right)-1\right), \quad t\in\mathbb{R} \end{aligned}$$

Let $t\left(x^{n+1}+1\right)-1=0$ to get its intersection with the hyperplane $x^{n+1}=0$ and we have

$$t\left(x^{n+1}+1\right)-1=0 \Rightarrow t=\frac{1}{x^{n+1}+1}, \quad x^{n+1}\neq -1 \Leftrightarrow t=\frac{1}{x^{n+1}+1}, \quad x\in\mathbb{S}^n\backslash\{S\}$$

Notice that

$$\tilde{\sigma}(x)=-\sigma(-x)=-\frac{\left(-x^1,\cdots,-x^n\right)}{1+x^{n+1}}=\frac{\left(x^1,\cdots,x^n\right)}{1+x^{n+1}}.$$

Thus plugging t into the original line and get

$$(u',0)=\left.\left(tx^1,\cdots,tx^n,t\left(x^{n+1}+1\right)-1\right)\right|_{t=\frac{1}{x^{n+1}+1}}=\left(\frac{x^1}{x^{n+1}+1},\cdots,\frac{x^n}{x^{n+1}+1},0\right)=(\tilde{\sigma}(x),0)$$

which shows that $\tilde{\sigma}(x)=u'$.

(b) Quick solution: Note that

$$|\sigma(x)|^2=\frac{\left(x^1\right)^2+\cdots+\left(x^n\right)^2}{\left(1-x^{n+1}\right)^2}=\frac{1-\left(x^{n+1}\right)^2}{\left(1-x^{n+1}\right)^2}=\frac{1+x^{n+1}}{1-x^{n+1}} \Longrightarrow x^{n+1}=\frac{|\sigma(x)|^2-1}{|\sigma(x)|^2+1}=1-\frac{2}{|\sigma(x)|^2+1}.$$

This shows that the left inverse of σ is

$$\tau\left(u^1,\ldots,u^n\right):=\frac{\left(2u^1,\ldots,2u^n,|u|^2-1\right)}{|u|^2+1}$$

Since the tangent space to $\mathbb{S}^n$ at N is a translate of H, any line from H to N intersects $\mathbb{S}^n$ at a second point, which shows that σ is surjective; consequently, τ is its right inverse as well.

Slow solution: get the equation of the line passing through $(u,0),x,N$ and solve parameter t, get left and right inverse, and compute their compositions.

(c) We have

$$(\tilde{\sigma} \circ \tau)\left(u^1, \ldots, u^n\right) = -\sigma\left[\frac{\left(-2u^1, \ldots, -2u^n, 1-|u|^2\right)}{1+|u|^2}\right] = \frac{\left(2u^1, \ldots, 2u^n\right)}{1-\left(1-|u|^2\right)/\left(1+|u|^2\right)} = \frac{\left(u^1, \ldots, u^n\right)}{|u|^2}.$$

This transition map is smooth and involutive (so that its inverse is itself), so the atlas defines a smooth structure on $\mathbb{S}^n$.

(d) Let the atlas defined in [11, Example 1.31] be

$$\mathcal{A} = \left\{\left(U_{n-1}^{+}, \varphi_{n+1}^{+}\right), \left(U_{n+1}^{-}, \varphi_{n+1}^{-}\right), \left(U_i^{\pm}, \varphi_i^{\pm}\right)\right\}_{i \in \{1, \cdots, n\}}$$

and let the atlas defined in this exercise be

$$\mathcal{B} = \left\{\left(\left(\mathbb{S}^n \backslash \{N\}, \sigma\right), \left(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma}\right)\right)\right\}.$$

We want to use [11, Proposition 1.17 (b)] to show that the two atlases determine the same smooth structure by proving that each of the chart in the $\mathcal{A}$ is compatible with both charts in $\mathcal{B}$, because this implies that their union is an atlas ($\mathcal{A}$ and $\mathcal{B}$ are already smooth atlases). We check the smooth compatibilities:

We see that $U_i^{+} = \left\{\left(x^1, \cdots, x^{n+1}\right) \in \mathbb{R}^{n+1} : x^i > 0\right\}$ and $U_i^{-} = \left\{\left(x^1, \cdots, x^{n+1}\right) \in \mathbb{R}^{n+1} : x^i < 0\right\}$,

$$\varphi_i^{\pm} : U_i^{\pm} \cap \mathbb{S}^n \rightarrow \mathbb{B}^n; \quad \varphi_i^{\pm}\left(x^1, \cdots, x^{n+1}\right) = \left(x^1, \cdots, \widehat{x}^2, \cdots x^{n+1}\right)$$

Then, for $\left(U_{n-1}^{+}, \varphi_{n+1}^{+}\right), \left(U_{n+1}^{-}, \varphi_{n+1}^{-}\right)$,

$$\varphi_{n+1}^{\pm} \circ \sigma^{-1}\left(u^1, \cdots, u^n\right) = \varphi_{n+1}^{+}\left(\frac{\left(2u^1, \cdots, 2u^n, \|u\|_n^2 - 1\right)}{\|u\|_n^2 + 1}\right) = \frac{\left(2u^1, \cdots, 2u^n\right)}{\|u\|_n^2 + 1}.$$

Similarly,

$$\varphi_{n+1}^{\pm} \circ \tilde{\sigma}^{-1}\left(u^1, \cdots, u^n\right) = \varphi_{n+1}^{+}\left(\frac{\left(2u^1, \cdots, 2u^n, 1-\|u\|_n^2\right)}{\|u\|_n^2 + 1}\right) = \frac{\left(2u^1, \cdots, 2u^n\right)}{\|u\|_n^2 + 1}$$

The two maps are both smooth by the same reasoning we have shown in part (c). The inverses of them (noticing that it is the n+1-th coordinate that is truncated under $\varphi_{n+1}^{\pm}$) are

$$\sigma \circ \left(\varphi_{n+1}^{\pm}\right)^{-1}\left(x^1, \cdots, x^n, \widehat{x^{n+1}}\right) = \sigma\left(x^1, \cdots, x^n, \pm\sqrt{1-\|x\|_n^2}\right) = \frac{\left(x^1, \cdots, x^n\right)}{1 \mp \sqrt{1-\|x\|_n^2}}$$

and

$$\tilde{\sigma} \circ \left(\varphi_{n+1}^{\pm}\right)^{-1}\left(x^1, \cdots, x^n, \widehat{x^{n+1}}\right) = \tilde{\sigma}\left(x^1, \cdots, x^n, \pm\sqrt{1-\|x\|_n^2}\right) = \frac{\left(x^1, \cdots, x^n\right)}{1 \pm \sqrt{1-\|x\|_n^2}}.$$

$\sigma \circ \left(\varphi_{n+1}^{-}\right)^{-1}$ has a positive denominator and is thus smooth and $\sigma \circ \left(\varphi_{n+1}^{+}\right)^{-1}$ is smooth because the denominator is zero iff the $\left(x^1, \cdots, x^n, \widehat{x^{n+1}}\right) = \left(0, \cdots, 0, \widehat{x^{n+1}}\right)$ while $\sigma : \mathbb{S}^n \backslash \{N = (0, \cdots, 0, 1)\} \rightarrow \mathbb{R}^n$ excludes that possibility. Argument for $\tilde{\sigma}$ is similar, where exclusion of the south pole from domain of $\tilde{\sigma}$ is working. Lastly, we want to prove that $\left(U_i^{\pm}, \varphi_i^{\pm}\right)$for $i \in \{1, \cdots, n\}$ are all compatible with ($\left(\mathbb{S}^n \backslash \{N\}, \sigma\right)$ (argument for ($\mathbb{S}^n \backslash \{S\}, \tilde{\sigma}$)) is similar). Notice that these $2n$ charts do not contain S and N. We have the smooth transition maps

$$\varphi_i^{\pm} \circ \sigma^{-1}\left(u^1, \cdots, u^n\right) = \frac{\left(2u^1, \cdots, 2u^l, \cdots, 2u^n, \|u\|_n^2 - 1\right)}{\|u\|_n^2 + 1}$$

and its inverse

$$\sigma \circ \left(\varphi_i^{\pm}\right)^{-1}\left(x^1, \cdots, x^2, \cdots x^{n+1}\right) = \frac{\left(x^1, \cdots, \sqrt{1-\|x\|_n^2}, \cdots, x^{n+1}\right)}{1 - x^{n+1}}.$$

The inverse is smooth also by avoiding the singularity in the denominator ($2n$ charts do no contain S and N where the $(\mathrm{n}+1)$-th coordinate is 1). ♦

Exercise 1.14.2 ([11] 1-8). *By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An **angle function** on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.*

Solution. No Global Angle Function:
Poor Man's solution: If θ existed, then $\theta\left(\mathbb{S}^1\right) \subset \mathbb{R}$ would be compact and connected, i.e., $\theta\left(\mathbb{S}^1\right) = [a, b]$. The restriction $\theta : \mathbb{S}^1 \to [a, b]$ is continuous and surjective by definition, making its left inverse a two-sided inverse. The existence of a continuous inverse (alternatively, the fact that θ is continuous and bijective with a compact domain and Hausdorff codomain) makes θ a homeomorphism. But $\mathbb{S}^1 \neq [a, b]$ because only the latter has cut-points.

Fancy solution: Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be given by $\pi(t) = \exp(it)$. Recall that $H_1\left(\mathbb{S}^1, \mathbb{Z}\right) = \mathbb{Z}$ and $H_1(\mathbb{R}, \mathbb{Z}) = 0$. If a global angle function $\theta : \mathbb{S}^1 \to \mathbb{R}$ existed, then

$$\pi_* \circ \theta_* = \mathrm{id}_{\mathbb{Z}} \Longrightarrow \theta_* : \mathbb{Z} \to 0 \text{ is injective,}$$

a clear contradiction.

Yes Local Angle Function: Let $U \subsetneq \mathbb{S}^1$ be open and $\theta : U \to \mathbb{R}$ defined by $\theta(z) = \arg(z)$ with a branch cut along the ray $\{\lambda p \mid \lambda \geq 0\}$ for some point $p \in \mathbb{S}^1 \backslash U$.

Smooth Structure: We observe that arg is the only possible choice of angle function (up to an element of $2\pi\mathbb{Z}$ for each connected component of U, which is irrelevant for our purposes). It is an injective continuous open map (open arcs form a base for the topology of U). If we restrict its codomain to its image (which is necessarily an open subset of $\mathbb{R}$), then it is also surjective, making it a homeomorphism with an open subset of $\mathbb{R}$. It remains to check that the chart $(U, \arg)$ is compatible with any of the standard charts on $\mathbb{S}^1$, say, (U_x^+, φ_x^+). We have

$$\varphi_x^+\left(\arg^{-1}(t)\right) = \varphi_x^+\left(e^{it}\right) = \sin(t)$$

which is smooth with a smooth inverse on $(-\pi/2, \pi/2)$. ⧫

Exercise 1.14.3 ([11] 1-11). *Let $M = \overline{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ is the stereographic projection (Problem 1.14.1) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]*

Solution. Since $M \subset \mathbb{R}^n$, it is second countable and Hausdorff. In order to come up with the atlas, we observe that the inverse stereographic projection $\mathbb{R}^n \to \mathbb{S}^n \backslash \{N\}$ maps the ball $\mathbb{B}^n$ to the southern hemisphere and its boundary to the equator. Hence, we can restrict the codomain to the lower half-space. We compose with a chart dropping any but the last coordinate to get a map $\overline{\mathbb{B}^n} \to \mathbb{H}^n$. Concretely, let

$$\varphi_i^{\pm}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, \widehat{2u^i}, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1} \qquad \pm u^i > 0$$

where the hat denotes omission. Its inverse is given by

$$\left(\varphi_i^{\pm}\right)^{-1}\left(x^1, \ldots, x^n\right) = \frac{\left(x^1, \ldots, \pm\sqrt{1 - |x|^2}, x^i, \ldots, x^{n-1}\right)}{1 - x^n}$$

These are trivially compatible with the interior chart $(\mathbb{B}^n, \mathrm{id})$. The transition functions are

$$\varphi_i^{\mp} \circ \left(\varphi_j^{\pm}\right)^{-1} = \begin{cases} \left(x^1, \ldots, \widehat{x^i}, \ldots, \pm\sqrt{1-|x|^2}, x^j, \ldots, x^n\right) & i < j \\ \left(x^1, \ldots, x^n\right) & i = j \\ \left(x^1, \ldots, \pm\sqrt{1-|x|^2}, x^j, \ldots, \widehat{x^{i-1}}, \ldots, x^n\right) & i > j \end{cases}$$

which are smooth because their domains exclude $|x| = 1$. ♦

Exercise 1.14.4 ([11] 2-3)**.** *For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.*

(a) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ *is the nth power map for* $n \in \mathbb{Z}$*, given in complex notation by* $p_n(z) = z^n$*.*

(b) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ *is the antipodal map* $\alpha(x) = -x$*.*

(c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ *is given by* $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$*, where we think of* $\mathbb{S}^3$ *as the subset* $\{(w, z) : |w|^2 + |z|^2 = 1\}$ *of* $\mathbb{C}^2$*.*

Solution. (a) For sufficiently small angular charts around z and z^n, the map in coordinates is $\theta \mapsto n\theta + 2\pi m$, where $m \in \mathbb{Z}$ is a constant depending on the choice of charts.

(b) Using standard charts $(U_i^{\pm}, \varphi_i^{\pm})$, the map in coordinates is $u \mapsto -u$.

(c) The map F expressed with real numbers is $F(a, b, c, d) = \left(2(ac + bd), 2(bc - ad), a^2 + b^2 - \left(c^2 + d^2\right)\right)$. For example, using the standard charts $\left(U_4^-, \varphi_4^-\right)$on $\mathbb{S}^3$ and $\left(U_3^-, \varphi_3^-\right)$ containing the points $(0, 0, 0, -1)$ and $(0, 0, -1)$, respectively, we have

$$\left(\left(\varphi_3^-\right) \circ F \circ \left(\varphi_4^-\right)^{-1}\right)(x, y, z) = (2(xz + yw), 2(yz - xw)) \quad w = -\sqrt{1 - (x^2 + y^2 + z^2)}.$$

We should restrict the domain to

$$V = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \left(x^2 + y^2 + z^2 < 1\right) \wedge \left(x^2 + y^2 < \frac{1}{2}\right) \right\}$$

so that $F(V) \subset U_3^-$. † The calculations are similar for the remaining charts. In all three parts, we have smooth maps $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ which restrict to maps of spheres $\mathbb{S}^n \to \mathbb{S}^m$. These restrictions are always smooth because we can compose with inclusion to get a smooth map $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ and restrict the codomain to $\mathbb{S}^m$ via [11, Corollary 5.30]. ♦

Exercise 1.14.5 ([11] 3-1)**.** *Suppose* M *and* N *are smooth manifolds with or without boundary, and* $F : M \to N$ *is a smooth map. Show that* $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ *is the zero map for each* $p \in M$ *if and only if* F *is constant on each component of* M*.*

Solution. Because manifolds are locally connected, we know that F is constant on each connected component iff F is locally constant. It therefore suffices to work in coordinates and prove that a smooth map $\widehat{F} : \mathbb{R}^m \to \mathbb{R}^n$ is constant iff its Jacobian is identically zero. If $\widehat{F}$ is constant, then its Jacobian clearly vanishes. If $\widehat{F}$ is nonconstant, then pick points p and q such that $\widehat{F}(p) \neq \widehat{F}(q)$. Let $v := q - p, L(t) := p + tv$, and $f := \widehat{F} \circ L$. Since $f(0) \neq f(1)$, the mean value theorem guarantees us a time $t \in (0, 1)$ such that $f'(t) = (D\widehat{F} \circ L)(t)v \neq 0$, which implies $D\widehat{F} \not\equiv 0$. ♦

Exercise 1.14.6 ([11] 3-2)**.** *Prove the [11, Proposition 3.14] below.*

†The complement of set V in the unit 3-ball is a napkin ring; see this YouTube video.

Proposition 1.14.7 (The Tangent Space to a Product Manifold). *Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map*

$$\alpha : T_p (M_1 \times \cdots \times M_k) \to T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k$$

defined by

$$\alpha(v) = \left(d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v)\right)$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. We will take $k = 2$ to simplify notation. Let $\iota_M : M \hookrightarrow M \times N$ be the map sending x to (x, p_2) and $\iota_N : N \hookrightarrow M \times N$ be the map sending y to (p_1, y). Note that they are sections of bundles π_M and π_N, respectively. Note that $\pi_M \circ \iota_N$ and $\pi_N \circ \iota_M$ are constant (so their differentials are zero); $\pi_M \circ \iota_M$ and $\pi_N \circ \iota_N$ are identity maps (so their differentials are identities). We claim that $\beta(u, w) := d\iota_M(u) + d\iota_N(w)$ is the inverse of α (checking one side is enough since these are vector spaces of the same dimension):

$$\begin{aligned}
(\alpha \circ \beta)(u, w) &= \alpha\left(d\iota_M(u) + d\iota_N(w)\right) \\
&= \left(d(\pi_M \circ \iota_M)(u) + d(\pi_M \circ \iota_N)(w), d(\pi_N \circ \iota_M)(u) + d(\pi_N \circ \iota_N)(w)\right) \\
&= (u, w).
\end{aligned}$$

What we are saying in terms of coordinates is that, for a smooth curve $\gamma = (\gamma_M, \gamma_N) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^n$, the data of $\gamma'(0)$ and $(\gamma_M'(0), \gamma_N'(0))$ are equivalent. ∎

Exercise 1.14.8 ([11] 3-3). *Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.*

Solution. By [11, Proposition 3.14], we have an identification between $T_p M^n \oplus T_q N^m$ and $T_{(p,q)}(M^n \times N^m)$. The diffeomorphism between them is given by the map

$$\begin{gathered}
F : T(M \times N) \to TM \times TN \\
((p, q), u \oplus v) \mapsto ((p, u), (q, v))
\end{gathered}$$

F is bijective with inverse $F^{-1} : ((p, u), (q, v)) \mapsto ((p, q), u \oplus v)$. Given any point $(p, q) \in M \times N$, we have a smooth chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ for $M \times N$. Then by [11, Proposition 3.18], we have a smooth chart $(\pi^{-1}(U_1 \times U_2), \tilde{\varphi})$ where

$$\begin{gathered}
\tilde{\varphi} : \pi^{-1}(U_1 \times U_2) \to \mathbb{R}^{2(m+n)} \\
\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_{(p,q)} \oplus w^j \frac{\partial}{\partial y^j}\bigg|_{(p,q)} \right) = (\varphi_1(p), \varphi_2(q), v, w)
\end{gathered}$$

where $\pi : T(M \times N) = \coprod T_{(p,q)}(M \times N) \approx \coprod T_p M \oplus T_q N \to M \times N$ is the natural projection. The smooth chart of the image point $((p, u), (q, v))$ for the manifold $TM \times TN$ is given by $(\pi_M^{-1}(U_1) \times \pi_N^{-1}(U_2), \bar{\varphi})$ where

$$\begin{gathered}
\bar{\varphi} : \pi_M^{-1}(U_1) \times \pi_N^{-1}(U_2) \to \mathbb{R}^{2(m+n)} \\
\bar{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p, w^j \frac{\partial}{\partial y^j}\bigg|_q \right) = (\varphi_1(p), v, \varphi_2(q), w)
\end{gathered}$$

where $\pi_M : TM \to M$ and $\pi_N : TN \to N$ are natural projections and we again omit including the points as index. In fact in both $\tilde{\varphi}$ and $\bar{\varphi}$, the indexes are implied by the subscripts along with the coordinate vectors. In the above maps u and w are both coordinates of the derivations $(u^1, \cdots, u^n), (w^1, \cdots, w^m)$. Both of these charts are standard charts constructed form product and bundle in different order. The remaining is to show that the coordinate representations of F and F^{-1} are smooth:

$$\begin{aligned}
\hat{F}(x,y,v,w) &= \bar{\varphi}\circ F\circ\tilde{\varphi}^{-1}(x,y,v,w)\\
&= \bar{\varphi}\left(F\left(v^i\frac{\partial}{\partial x^i}\bigg|_{\left(\varphi_1^{-1}(x),\varphi_2^{-1}(y)\right)}\oplus w^j\frac{\partial}{\partial y^j}\bigg|_{\left(\varphi_1^{-1}(x),\varphi_2^{-1}(y)\right)}\right)\right)\\
&= \bar{\varphi}\left(v^i\frac{\partial}{\partial x^i}\bigg|_{\varphi_1^{-1}(x),w^j}\frac{\partial}{\partial y^j}\bigg|_{\varphi_2^{-1}(x)}\right)\\
&= \left(\varphi_1\left(\varphi_1^{-1}(x)\right),v,\varphi_2\left(\varphi_2^{-1}(x)\right),w\right)\\
&= (x,v,y,w)
\end{aligned}$$

and similarly

$$\widehat{F}^{-1}(x,v,y,w)=\tilde{\varphi}\circ F\circ\bar{\varphi}^{-1}(x,v,y,w)=(x,y,v,w)$$

Both of them are just switching the coordinates or a multiplication of the matrix

$$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

and are thus linear and smooth. Therefore F is a diffeomorphism. ♦

Exercise 1.14.9 ([11] 3-4)**.** *Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1\times\mathbb{R}$.*

Solution. Let (U,θ_U) and (V,θ_V) be two angle charts covering $\mathbb{S}^1$. The transition function $\theta_U\circ\theta_V^{-1}$ is of the form $\theta\mapsto\theta+C$, where C is a locally constant function. The derivative of this function is the identity, so $\partial/\partial\theta_U$ and $\partial/\partial\theta_V$ glue together into a smooth vector field $\partial/\partial\theta$ on $\mathbb{S}^1$ which is nowhere-vanishing. We claim that the map $F:\mathbb{S}^1\times\mathbb{R}\to T\mathbb{S}^1$ given by $F(p,r):=\left(p,r\frac{\partial}{\partial\theta}\right)$ is a diffeomorphism. Because $\partial/\partial\theta$ is nowhere-vanishing, F is bijective. We also need to check that F is a local diffeomorphism, but this is trivial. See [11, Corollary 10.20] for the generalization of this idea. ♦

Exercise 1.14.10. *Prove that $\mathbb{S}^2\times\mathbb{R}$ is parallelizable. Explain why this does not contradict the combined observations that $\mathbb{S}^2$ is not parallelizable and $T\left(\mathbb{S}^2\times\mathbb{R}\right)$ is diffeomorphic to $T\mathbb{S}^2\times T\mathbb{R}$.*

Solution. Let $\mathbb{S}^2\subset\mathbb{R}^3$ be the unit sphere. The map $(p,r)\mapsto e^rp$ is a diffeomorphism from $\mathbb{S}^2\times\mathbb{R}$ to $\mathbb{R}^3\backslash\{0\}$. The latter is an open subset of $\mathbb{R}^3$ and therefore has a trivial tangent bundle. There is no contradiction because a vector field on $\mathbb{S}^2\times\mathbb{R}$ can be nonzero in the $\mathbb{R}$ direction and zero in the $\mathbb{S}^2$ direction. In fact, the vector bundle $T\mathbb{S}^2$ is even worse than nontrivial: it has a nonzero Euler class, so it does not admit a single nowhere-vanishing global section, let alone a global frame (this is the hairy ball theorem, hehe). The vector fields $\partial/\partial x,\partial/\partial y,\partial/\partial z$, which parallelize $\mathbb{R}^3\backslash\{0\}$, each vanish at some point when restricted to $\mathbb{S}^2$. ♦

Exercise 1.14.11 ([11] 4-12)**.** *Using the covering map $\varepsilon^2:\mathbb{R}^2\to\mathbb{T}^2$ (see [11, Example 4.35]), show that the immersion $X:\mathbb{R}^2\to\mathbb{R}^3$ defined in [11, Example 4.2(d)] descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\tilde{X}:\mathbb{T}^2\to\mathbb{R}^3$, and then show that $\tilde{X}$ is a smooth embedding whose image is the given surface of revolution.*

Exercise 1.14.12 ([11] 5-1)**.** *Consider the map $\Phi:\mathbb{R}^4\to\mathbb{R}^2$ defined by*

$$\Phi(x,y,s,t)=\left(x^2+y,x^2+y^2+s^2+t^2+y\right).$$

Show that $(0,1)$ is a regular value of Φ, and that the level set $\Phi^{-1}(0,1)$ is diffeomorphic to $\mathbb{S}^2$.

Solution. The Jacobian of Φ is

$$D\Phi = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{pmatrix}$$

Suppose $x^2 + y = 0$ and $x^2 + y^2 + s^2 + t^2 + y = y^2 + s^2 + t^2 = 1$. If $s \neq 0$ or $t \neq 0$, then $D\Phi$ is surjective. If $s = t = 0$, then $y = -1$ and $x = \pm 1$, which makes the first two columns linearly independent. Now, let $M = \Phi^{-1}(0,1)$. By the regular level set theorem, $M \subset \mathbb{R}^4$ is an embedded submanifold. The smooth map $\varphi : M \to \mathbb{S}^2$ given by $\varphi(x,y,s,t) = \left(x, s/\sqrt{1+x^2}, t/\sqrt{1+x^2}\right)$ has smooth inverse $\varphi^{-1}(a,b,c) = \left(a, -a^2, b\sqrt{1+a^2}, c\sqrt{1+a^2}\right)$. Hence, it is a diffeomorphism. ♦

Exercise 1.14.13 ([11] 7-1)**.** *Show that for any Lie group G, the multiplication map $m : G \times G \to G$ is a smooth submersion. [Hint: use local sections.]*

Solution. 1. Let $(g,h) \in G \times G$. The smooth map $\sigma_h(k) := \left(m\left(k, h^{-1}\right), h\right) = \left(kh^{-1}, h\right)$ is a section of m, and $\sigma_h(gh) = (g,h)$. By the local section theorem, m is a submersion.
2. The map m has constant rank because it intertwines the G-actions $g \cdot (h,k) := (gh,k)$ and $g \cdot h := gh$. It is surjective and therefore a submersion.
3. Let $\iota : G \hookrightarrow G \times G$ be the inclusion $\iota(g) := (g,e)$. Since $m \circ \iota = \mathrm{id}_G$ is a submersion, m must be a submersion.
4. Using techniques from Problems 1.14.14 and 1.14.15, we can directly compute the derivative $dm_{(g,h)}(X,Y) = dR_h(X) + dL_g(Y)$. For any $Z \in T_{gh}G$, we have $dm_{(g,h)}\left(dR_{h^{-1}}(Z), 0\right) = Z$.

As my calculus professor would say, we have four proofs, so it must be true! ♦

Exercise 1.14.14 ([11] 7-2)**.** *Let G be a Lie group.*

(a) Let $m : G \times G \to G$ denote the multiplication map. Using Proposition 1.14.7 to identify $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$, show that the differential $\mathrm{d}m_{(e,e)} : T_eG \oplus T_eG \to T_eG$ is given by

$$\mathrm{d}m_{(e,e)}(X,Y) = X + Y$$

[Hint: compute $\mathrm{d}m_{(e,e)}(X,0)$ and $\mathrm{d}m_{(e,e)}(0,Y)$ separately.]

(b) Let $i : G \to G$ denote the inversion map. Show that $\mathrm{d}i_e : T_eG \to T_eG$ is given by $\mathrm{d}i_e(X) = -X$.

Solution. (a) Let $X, Y \in T_eG$. Let $\ell : G \to G \times G$ be the section of m given by $\ell(g) := (g,e)$. Since $\mathrm{d}\ell(X) = (X,0)$, we have

$$\mathrm{d}m(X,0) = (\mathrm{d}m \circ \mathrm{d}\ell)(X) = \mathrm{d}(m \circ \ell)(X) = \mathrm{d}\left(\mathrm{id}_G\right)(X) = X$$

A symmetric argument shows that $\mathrm{d}m(0,Y) = Y$.

(b) Let $\iota : G \to G \times G$ be given by $\iota(g) := (g, i(g)) = \left(g, g^{-1}\right)$. Since $\mathrm{d}\iota(X) = (X, \mathrm{d}i(X))$, and $m \circ \iota$ is constant, we have

$$0 = \mathrm{d}(m \circ \iota)(X) = \mathrm{d}m(d\iota(X)) = \mathrm{d}m(X, \mathrm{d}i(X)) = X + \mathrm{d}i(X).$$

♦

Exercise 1.14.15 ([11] 7-3)**.** *Our definition of Lie groups includes the requirement that both the multiplication map and the inversion map are smooth. Show that smoothness of the inversion map is redundant: if G is a smooth manifold with a group structure such that the multiplication map $m : G \times G \to G$ is smooth, then G is a Lie group. [Hint: show that the map $F : G \times G \to G \times G$ defined by $F(g,h) = (g,gh)$ is a bijective local diffeomorphism.]*

Solution. We observe that F has a set-theoretic two-sided inverse given by $F^{-1}(g,h) = \left(g, g^{-1}h\right)$. It suffices to show that F^{-1} is smooth, since $i(g) = \pi_2\left(F^{-1}(g,e)\right)$, where π_2 is projection onto the second factor. Since $\mathrm{d}F_{e,e}(X,Y) = (X, X+Y)$ is invertible, F is a diffeomorphism on a neighborhood $U \ni (e,e)$. Let L_g and R_g denote left and right multiplication by g, respectively. They are diffeomorphisms for all $g \in G$ with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$. Note that $V := (L_g \times R_h)(U)$ is a neighborhood of (g,h), and

$$F|_V = (L_g \times (L_g \circ R_h)) \circ F|_U \circ \left(L_{g^{-1}} \times R_{h^{-1}}\right),$$

which is a diffeomorphism. Since F is a local diffeomorphism, its inverse is smooth. ♦

Exercise 1.14.16 ([11] 7-13). *For each $n \geq 1$, prove that $\mathrm{U}(n)$ is a properly embedded n^2-dimensional Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$.*

Exercise 1.14.17 ([11] 10-2). *Let E be a vector bundle over a topological space M. Show that the projection map $\pi : E \to M$ is a homotopy equivalence.*

Exercise 1.14.18 ([11] 10-10). *Suppose M is a compact smooth manifold and $E \to M$ is a smooth vector bundle of rank k. Use transversality to prove that E admits a smooth section σ with the following property: if $k > \dim M$, then σ is nowhere vanishing; while if $k \leq \dim M$, then the set of points where σ vanishes is a smooth compact codimension- k submanifold of M. Use this to show that M admits a smooth vector field with only finitely many singular points.*

Exercise 1.14.19 ([11] 8-16). *For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$, compute the Lie bracket $[X,Y]$.*

(a) $X = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}; \quad Y = \frac{\partial}{\partial y}$.

(b) $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}; \quad Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$.

(c) $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}; \quad Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$.

Solution. We can compute Lie brackets using the definition, [11, Proposition 8.26], or [11, Proposition 8.28]. We will illustrate one approach for each of the parts:

(a) Given an arbitrary function f,

$$\begin{aligned}
X(Yf) - Y(Xf) &= \left(y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}\right)\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\left(y\frac{\partial f}{\partial z} - 2xy^2\frac{\partial f}{\partial y}\right) \\
&= y\frac{\partial^2 f}{\partial z\partial y} - 2xy^2\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial z} - y\frac{\partial^2 f}{\partial y\partial z} + 4xy\frac{\partial f}{\partial y} + 2xy^2\frac{\partial^2 f}{\partial y^2} \\
&= \left(4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right) f \\
&\Longrightarrow [X,Y] = 4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}
\end{aligned}$$

(b)

$$[X,Y] = \left(XY^1 - YX^1\right)\frac{\partial}{\partial x} + \left(XY^2 - YX^2\right)\frac{\partial}{\partial y} + \left(XY^3 - YX^3\right)\frac{\partial}{\partial z} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

(c)

$$[X,Y] = 2\left[x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] = 2\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right).$$

♦

Exercise 1.14.20 ([11] 9-3). *Compute the flow of each of the following vector fields on $\mathbb{R}^2$:*

(a) $V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

(b) $W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$.

(c) $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

(d) $Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$.

Solution. (a) $F_t(x,y) = \left(x + yt + t^2/2, y + t\right)$.
(b) $F_t(x,y) = \left(xe^t, ye^{2t}\right)$.
(c) $F_t(x,y) = (xe^t, ye^{-t})$.
(d) $F_t(x,y) = \left((x+y)e^t + (x-y)e^{-t}, (y+x)e^t + (y-x)e^{-t}\right)/2$. ♦

Exercise 1.14.21 ([11] 9-5). *Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. Show that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.*

Solution. Let X be a nowhere-vanishing vector field on M. It is complete because M is compact. Thus, we have a smooth family of diffeomorphisms $\{G_t : M \to M\}_{t\in\mathbb{R}}$ with $G_0 = \mathrm{id}$. Every member is homotopic to the identity by sending $t \to 0$. Every point $p \in M$ is a regular point, so there are neighborhoods $p \in V_p \subset U_p$ and times $\epsilon_p > 0$ such that X has the canonical form $\partial/\partial x^1$ on U_p, and no point of V_p is fixed by G_t for all $0 < t \le \epsilon_p$. Reduce $\{V_p\}_{p\in M}$ to a finite subcover, and let $\epsilon = \min_p \epsilon_p > 0$. Then G_ϵ has no fixed points and is the required map. ♦

Exercise 1.14.22 ([11] 11-5). *For any smooth manifold M, show that T^*M is a trivial vector bundle if and only if TM is trivial.*

Exercise 1.14.23 ([11] 11-7). *In the following problems, M and N are smooth manifolds, $F : M \to N$ is a smooth map, and $\omega \in \mathfrak{X}^*(N)$. Compute $F^*\omega$ in each case.*
(a) $M = N = \mathbb{R}^2$, $F(s,t) = (st, e^t)$, $\omega = xdy - ydx$
(b) $M = \mathbb{R}^2$ *and* $N = \mathbb{R}^3$, $F(\theta,\varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$, $\omega = z^2dx$
(c) $M = \left\{(s,t) \in \mathbb{R}^2 : s^2 + t^2 < 1\right\}$ *and* $N = \mathbb{R}^3\backslash\{0\}$, $F(s,t) = \left(s, t, \sqrt{1 - s^2 - t^2}\right)$, $\omega = \left(1 - x^2 - y^2\right)dz$

Solution. (a)

$$\begin{aligned}
F^*\omega &= ((-y)\circ F)d(x\circ F) + (x\circ F)d(y\circ F) \\
&= -e^t d(st) + std\left(e^t\right) \\
&= -e^t\left(\frac{\partial(st)}{\partial s}ds + \frac{\partial(st)}{\partial t}dt\right) + st\left(\frac{\partial\left(e^t\right)}{\partial s}ds + \frac{\partial\left(e^t\right)}{\partial t}dt\right) \\
&= -e^t(tds + sdt) + st\left(e^t dt\right) \\
&= se^t(t-1)dt - e^t tds
\end{aligned}$$

(b)

$$\begin{aligned}
F^*\omega &= \left(\left(z^2\right)\circ F\right)d(x\circ F) + (0\circ F)d(y\circ F) + (0\circ F)d(z\circ F) \\
&= \left(\sin^2\varphi\right)d((\cos\varphi + 2)\cos\theta) \\
&= \sin^2\varphi\left(\frac{\partial((\cos\varphi + 2)\cos\theta)}{\partial\varphi}d\varphi + \frac{\partial((\cos\varphi + 2)\cos\theta)}{\partial\theta}d\theta\right) \\
&= \sin^2\varphi(-\sin\varphi\cos\theta d\varphi - (\cos\varphi + 2)\sin\theta d\theta) \\
&= -\sin^3\varphi\cos\theta d\varphi - (\cos\varphi + 2)\sin\theta\sin^2\varphi d\theta
\end{aligned}$$

(c)

$$\begin{aligned}
F^*\omega &= (0\circ F)d(x\circ F) + (0\circ F)d(y\circ F) + \left(\left(1-x^2-y^2\right)\circ F\right)d(z\circ F)\\
&= \left(1-s^2-t^2\right)d\left(\sqrt{1-s^2-t^2}\right)\\
&= \left(1-s^2-t^2\right)\left(\frac{\partial\left(\left(1-s^2-t^2\right)^{\frac{1}{2}}\right)}{\partial s}ds + \frac{\partial\left(\left(1-s^2-t^2\right)^{\frac{1}{2}}\right)}{\partial t}dt\right)\\
&= \left(1-s^2-t^2\right)\left(\frac{1}{2}\left(1-s^2-t^2\right)^{-\frac{1}{2}}(-2s)ds + \frac{1}{2}\left(1-s^2-t^2\right)^{-\frac{1}{2}}(-2t)dt\right)\\
&= -s\sqrt{1-s^2-t^2}ds - t\sqrt{1-s^2-t^2}dt
\end{aligned}$$

♦

Exercise 1.14.24. *Reading: see lecture note Definition 2.3, Theorem 2.4, and Theorem 2.6. It can be shown that there are many smooth manifolds (at least those with "good covers") with finite-dimensional de Rham cohomology groups. In particular, the compact ones all have finite-dimensional de Rham cohomology groups. Furthermore, we have a Künneth-type formula: if M and N are smooth manifolds with finite good covers, then for any $0 \le k \le \dim M + \dim N$, one has*

$$H^k_{\mathrm{dR}}(M\times N) \cong \bigoplus_{i=0}^{k} H^i_{\mathrm{dR}}(M)\otimes H^{k-i}_{\mathrm{dR}}(N)$$

Exercise 1.14.25. *For any smooth manifold M, let $H^p_c(M)$ denote the p th compactly supported de Rham cohomology group of M.*

(a) Given an open subset $U \subseteq M$, let $\iota : U \hookrightarrow M$ denote the inclusion map, and define a linear map $\iota_\sharp : \Omega^p_c(U) \to \Omega^p_c(M)$ by extending each compactly supported form to be zero on $M\backslash U$. Show that $d\circ\iota_\sharp = \iota_\sharp\circ d$, and so $\iota_\sharp$ induces a linear map on compactly supported cohomology, denoted by $\iota_* : H^p_c(U) \to H^p_c(M)$.

(b) Mayer-Vietoris with Compact Supports: Suppose M is a smooth manifold and $U, V \subseteq M$ are open subsets whose union is M. Prove that for each nonnegative integer p, there is a linear map $\delta_* : H^p_c(M) \to H^{p+1}_c(U\cap V)$ such that the following sequence is exact:

$$\cdots \xrightarrow{\delta_*} H^p_c(U\cap V) \xrightarrow{i_*\oplus(-j_*)} H^p_c(U)\oplus H^p_c(V) \xrightarrow{k_*+l_*} H^p_c(M) \xrightarrow{\delta_*} H^{p+1}_c(U\cap V) \xrightarrow{i_*\oplus(-j_*)} \cdots,$$

where i, j, k, l are the inclusion maps.

(c) Let $H^p_c(M)^*$ denote the algebraic dual space to $H^p_c(M)$, that is, the vector space of all linear maps from $H^p_c(M)$ to $\mathbb{R}$. Show that the following sequence is also exact:

$$\cdots \xrightarrow{(\delta_*)^*} H^p_c(M)^* \xrightarrow{(k_*)^*\oplus(l_*)^*} H^p_c(U)^*\oplus H^p_c(V)^* \xrightarrow{(i_*)^*-(j_*)^*} H^p_c(U\cap V)^* \xrightarrow{(\delta_*)^*} H^{p-1}_c(M)^* \xrightarrow{(k_*)^*\oplus(l_*)^*} \cdots.$$

Chapter 2

Riemannian Manifolds

Given a vector space V (which we always assume to be real), an **inner product** on V is a map $V \times V \to \mathbb{R}$, typically written $(v, w) \mapsto \langle v, w\rangle$, that satisfies the following properties for all $v, w, x \in V$ and $a, b \in \mathbb{R}$:

(i) SYMMETRY: $\langle v, w\rangle = \langle w, v\rangle$.

(ii) BILINEARITY: $\langle av + bw, x\rangle = a\langle v, x\rangle + b\langle w, x\rangle = \langle x, av + bw\rangle$.

(iii) POSITIVE DEFINITENESS: $\langle v, v\rangle \geq 0$, with equality if and only if $v = 0$.

A vector space endowed with a specific inner product is called an **inner product space**.

An inner product on V allows us to make sense of geometric quantities such as lengths of vectors and angles between vectors. First, we define the **length** or **norm** of a vector $v \in V$ as

$$|v| = \langle v, v\rangle^{1/2}.$$

Polarization identity

$$\langle v, w\rangle = \frac{1}{4}(\langle v + w, v + w\rangle - \langle v - w, v - w\rangle).$$

shows that an inner product is completely determined by knowledge of the lengths of all vectors. The **angle** between two nonzero vectors $v, w \in V$ is defined as the unique $\theta \in [0, \pi]$ satisfying

$$\cos\theta = \frac{\langle v, w\rangle}{|v||w|}$$

Two vectors $v, w \in V$ are said to be **orthogonal** if $\langle v, w\rangle = 0$, which means that either their angle is $\pi/2$ or one of the vectors is zero. If $S \subseteq V$ is a linear subspace, the set $S^{\perp} \subseteq V$, consisting of all vectors in V that are orthogonal to every vector in S, is also a linear subspace, called the **orthogonal complement** of S.

Vectors $v_1, \ldots, v_k$ are called **orthonormal** if they are of length 1 and pairwise orthogonal, or equivalently if $\langle v_i, v_j\rangle = \delta_{ij}$ (where δ_{ij} is the Kronecker delta symbol). The following well-known proposition shows that every finite-dimensional inner product space has an orthonormal basis.

Proposition 2.0.1 (Gram-Schmidt Algorithm)**.** *Let V be an n-dimensional inner product space, and suppose $(v_1, \ldots, v_n)$ is any ordered basis for V. Then there is an orthonormal ordered basis $(b_1, \ldots, b_n)$ satisfying the following conditions:*

$$\operatorname{span}(b_1, \ldots, b_k) = \operatorname{span}(v_1, \ldots, v_k) \quad \textit{for each } k = 1, \ldots, n$$

Proof. See [12, Proposition 2.3]. ∎

Let M be a smooth manifold with or without boundary. A **Riemannian metric** on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$ whose value g_p at each $p \in M$ is an inner product on T_pM; thus g is a symmetric 2-tensor field that is positive definite in the sense that $g_p(v, v) \geq 0$ for each $p \in M$ and each $v \in T_pM$, with equality if and only if $v = 0$. A **Riemannian manifold** is a pair (M, g), where M is a smooth manifold and g is a specific choice of Riemannian metric on M. If M is understood to be endowed with a specific Riemannian metric, we sometimes say "M is a Riemannian manifold." The next proposition shows that Riemannian metrics exist in great abundance.

Proposition 2.0.2. *Every smooth manifold with or without boundary admits a Riemannian metric.*

Proof. See [11, Proposition 13.3]. ■

Let g be a Riemannian metric on a smooth manifold M with or without boundary. Because g_p is an inner product on T_pM for each $p \in M$, we often use the following angle-bracket notation for $v, w \in T_pM$:

$$\langle v, w\rangle_g = g_p(v, w).$$

Using this inner product, we can define lengths of tangent vectors, angles between nonzero tangent vectors, and orthogonality of tangent vectors as described above. The length of a vector $v \in T_pM$ is denoted by $|v|_g = \langle v, v\rangle_g^{1/2}$. If the metric is understood, we sometimes omit it from the notation, and write $\langle v, w\rangle$ and $|v|$ in place of $\langle v, w\rangle_g$ and $|v|_g$, respectively.

The starting point for Riemannian geometry is the following fundamental example.

Example 2.0.3 (The Euclidean Metric)**.** The **Euclidean metric** is the Riemannian metric $\bar{g}$ on $\mathbb{R}^n$ whose value at each $x \in \mathbb{R}^n$ is just the usual dot product on $T_x\mathbb{R}^n$ under the natural identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$. This means that for $v, w \in T_x\mathbb{R}^n$ written in standard coordinates $\left(x^1, \ldots, x^n\right)$ as $v = \sum_i v^i \partial_i\big|_x, w = \sum_j w^j \partial_j\Big|_x$, we have

$$\langle v, w\rangle_{\bar{g}} = \sum_{i=1}^n v^i w^i.$$

When working with $\mathbb{R}^n$ as a Riemannian manifold, we always assume we are using the Euclidean metric unless otherwise specified. ♣

Suppose (M, g) is a Riemannian manifold with or without boundary. If $\left(x^1, \ldots, x^n\right)$ are any smooth local coordinates on an open subset $U \subseteq M$, then g can be written locally in U as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$$

for some collection of n^2 smooth functions g_{ij} for $i, j = 1, \ldots, n$. The component functions of this tensor field constitute a matrix-valued function (g_{ij}), characterized by $g_{ij}(p) = \left\langle \partial_i|_p, \partial_j|_p \right\rangle$, where $\partial_i = \partial/\partial x^i$ is the i th coordinate vector field; this matrix is symmetric in i and j and depends smoothly on $p \in U$. If $v = v^i \partial_i\big|_p$ is a vector in T_pM such that $g_{ij}(p)v^j = 0$, it follows that $\langle v, v\rangle = g_{ij}(p)v^i v^j = 0$, which implies $v = 0$; thus the matrix $(g_{ij}(p))$ is always nonsingular. The notation for g can be shortened by expressing it in terms of the symmetric product: using the symmetry of g_{ij}, we compute

$$\begin{aligned}
g &= g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \\
&= \frac{1}{2}\left(g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ji}\, \mathrm{d}x^i \otimes \mathrm{d}x^j\right) \qquad (g_{ij} = g_{ji}) \\
&= \frac{1}{2}\left(g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}\, \mathrm{d}x^j \otimes \mathrm{d}x^i\right) \qquad \Big(\sum_i \sum_j = \sum_j \sum_i\Big) \\
&= g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \qquad \text{(due to eq. (1.9))}
\end{aligned}$$

For example, the Euclidean metric on $\mathbb{R}^n$ (Example 2.0.3) can be expressed in standard coordinates in several ways:

$$\bar{g}=\sum_i \mathrm{d}x^i \mathrm{d}x^i=\sum_i\left(\mathrm{d}x^i\right)^2=\delta_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

The matrix of $\bar{g}$ in these coordinates is thus $\bar{g}_{ij}=\delta_{ij}$. More generally, if $\left(E_1, \ldots, E_n\right)$ is any smooth local frame for TM on an open subset $U \subseteq M$ and $\left(\varepsilon^1, \ldots, \varepsilon^n\right)$ is its dual coframe, we can write g locally in U as

$$g=g_{ij} \varepsilon^i \varepsilon^j, \tag{2.1}$$

where $g_{ij}(p)=\left\langle\left.E_i\right|_p,\left.E_j\right|_p\right\rangle$, and the matrix-valued function $\left(g_{ij}\right)$ is symmetric and smooth as before.

A Riemannian metric g acts on smooth vector fields $X, Y \in \mathfrak{X}(M)$ to yield a real-valued function $\langle X, Y\rangle$. In terms of any smooth local frame, this function is expressed locally by $\langle X, Y\rangle=g_{ij} X^i Y^j$ and therefore is smooth. Similarly, we obtain a nonnegative real-valued function $|X|=\langle X, X\rangle^{1/2}$, which is continuous everywhere and smooth on the open subset where $X \neq 0$.

A local frame $\left(E_i\right)$ for M on an open set U is said to be an orthonormal frame if the vectors $\left.E_1\right|_p, \ldots,\left.E_n\right|_p$ are an orthonormal basis for $T_p M$ at each $p \in U$. Equivalently, $\left(E_i\right)$ is an orthonormal frame if and only if

$$\left\langle E_i, E_j\right\rangle=\delta_{ij}$$

in which case g has the local expression

$$g=\left(\varepsilon^1\right)^2+\cdots+\left(\varepsilon^n\right)^2$$

where $\left(\varepsilon^i\right)^2$ denotes the symmetric product $\varepsilon^i \varepsilon^i=\varepsilon^i \otimes \varepsilon^i$.

Proposition 2.0.4 (Existence of Orthonormal Frames). *Let (M, g) be a Riemannian n-manifold with or without boundary. If $\left(X_j\right)$ is any smooth local frame for TM over an open subset $U \subseteq M$, then there is a smooth orthonormal frame $\left(E_j\right)$ over U such that* $\operatorname{span}\left(\left.E_1\right|_p, \ldots,\left.E_k\right|_p\right)=\operatorname{span}\left(\left.X_1\right|_p, \ldots,\left.X_k\right|_p\right)$ *for each $k=1, \ldots, n$ and each $p \in U$. In particular, for every $p \in M$, there is a smooth orthonormal frame $\left(E_j\right)$ defined on some neighborhood of p.*

Proof. See [12] Proposition 2.8. ■

Warning: A common mistake made by beginners is to assume that one can find coordinates near p such that the coordinate frame $\left(\partial_i\right)$ is orthonormal. Above proposition does not show this. In fact, as we will see in Chapter **??**, this is possible only when the metric is flat, that is, locally isometric to the Euclidean metric.

For a Riemannian manifold (M, g) with or without boundary, we define the unit tangent bundle to be the subset $UTM \subseteq TM$ consisting of unit vectors:

$$UTM=\left\{(p, v) \in TM:|v|_g=1\right\}.$$

Proposition 2.0.5 (Properties of the Unit Tangent Bundle). *If (M, g) is a Riemannian manifold with or without boundary, its unit tangent bundle UTM is a smooth, properly embedded codimension-1 submanifold with boundary in TM, with $\partial(UTM)=\pi^{-1}(\partial M)$(where $\pi: UTM \rightarrow M$ is the canonical projection). The unit tangent bundle is connected if and only if M is connected (when* $\dim M>1$*), and compact if and only if M is compact.*

Exercise 2.0.6. *Use local orthonormal frames to prove the preceding proposition.*

2.1 Pullback Metrics and Isometries

If two vector spaces V and W are both equipped with inner products, denoted by $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, respectively, then a map $F: V \rightarrow W$ is called a **linear isometry** if it is a vector space isomorphism that preserves inner products: $\langle F(v), F(v')\rangle_W = \langle v, v'\rangle_V$. If V and W are inner product spaces of dimension n, then given any choices of orthonormal bases $(v_1, \ldots, v_n)$ for V and $(w_1, \ldots, w_n)$ for W, the linear map $F: V \rightarrow W$ determined by $F(v_i) = w_i$ is easily seen to be a linear isometry. Thus all inner product spaces of the same finite dimension are linearly isometric to each other.

Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary. An **isometry from** (M, g) **to** $(\widetilde{M}, \widetilde{g})$ is a diffeomorphism $\varphi: M \rightarrow \widetilde{M}$ such that $\varphi^* \widetilde{g} = g$. We say (M, g) and $(\widetilde{M}, \widetilde{g})$ are isometric if there exists an isometry between them.

Proposition 2.1.1. *When $\partial M = \varnothing$, $\varphi: (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$ is an isometry if and only if φ is a smooth bijection and each differential $d\varphi_p: T_p M \rightarrow T_{\varphi(p)} \widetilde{M}$ is a linear isometry.*

Proof. "$\Rightarrow$": Notice that

$$(\varphi^* \widetilde{g})_p (v, v') = \widetilde{g}_{\varphi(p)} (d\varphi_p(v), d\varphi_p(v')) = \langle d\varphi_p(v), d\varphi_p(v')\rangle_{\widetilde{g}} \tag{2.2}$$

and

$$g_p(v, v') = \langle v, v'\rangle_g$$

Since φ is an isometry, the RHS of above two equations are equal. So do their LHS. This shows that $d\varphi_p: (T_p M, \langle\cdot,\cdot\rangle_g) \rightarrow \left(T_{\varphi(p)} \widetilde{M}, \langle\cdot,\cdot\rangle_{\widetilde{g}}\right)$ is a linear isometry. φ as a diffeomorphism is smooth and bijective.

"$\Leftarrow$":

Suppose φ is smooth (this condition first ensures $d\varphi_p$ can be defined) and bijective and $d\varphi_p: (T_p M, \langle\cdot,\cdot\rangle_g) \rightarrow \left(T_{\varphi(p)} \widetilde{M}, \langle\cdot,\cdot\rangle_{\widetilde{g}}\right)$ is a linear isometry. We first show that φ is a diffeomorphism: by [11] Theorem 4.14 (c), it suffices to show it has constant rank. But this is resulted from φ being a smooth immersion. That's because isometry implies injectivity by the positivedefiniteness of the norm: for linear map $A: V \rightarrow W$, let $v \in V$ s.t. $Av = \mathbf{0}$; then $0 = \|\mathbf{0}\|_W = \|Av\|_W = \|v\|_V \Rightarrow v = \mathbf{0}$; thus $A^{-1}(\mathbf{0}) = \{\mathbf{0}\} \Rightarrow d\varphi_p$ is injective. The remaining is to pass $\langle v, w\rangle_g = \langle d\varphi_p(v), d\varphi_p(w)\rangle_{\widetilde{g}}$ to $\varphi^* \widetilde{g} = g$, but this argument is the same as the "$\Rightarrow$" direction because the metrics are pointwise defined. ■

A composition of isometries and the inverse of an isometry are again isometries, so being isometric is an equivalence relation on the class of Riemannian manifolds with or without boundary. Our subject, Riemannian geometry, is concerned primarily with properties of Riemannian manifolds that are preserved by isometries.

If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds, a map $\varphi: M \rightarrow \widetilde{M}$ is a **local isometry** if each point $p \in M$ has a neighborhood U such that $\left.\varphi\right|_U$ is an isometry onto an open subset of $\widetilde{M}$. That is, φ is said to be a local isometry if $\forall p \in M$, there is a neighborhood U of p such that $\phi: U \rightarrow \varphi(U)$, defined as the restriction of $\left.\varphi\right|_U: U \rightarrow \widetilde{M}$ onto codomain $\varphi(U)$, is diffeomorphism from (open) Riemannian submanifold $(U, \iota_U^* g)$ to (open) Riemannian submanifold $\left(\varphi(U), \iota_{\varphi(U)}^* \widetilde{g}\right)$ with $\phi^*\left(\iota_{\varphi(U)}^* \widetilde{g}\right) = \iota_U^* g$. We need to first explain how $\iota^* \widetilde{g}$ gives a Riemannian metric on M (called **pullback metric**). In fact,

Lemma 2.1.2. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold with or without boundary, M is a smooth manifold with or without boundary, and $F: M \rightarrow \widetilde{M}$ is a smooth map. The smooth 2-tensor field $g = F^* \widetilde{g}$ is a Riemannian metric on M if and only if F is an immersion.*

Proof. We have $g_p(v, w) = \widetilde{g}_{F(p)}(dF_p(v), dF_p(w))$. Thus, symmetry and bilinearity of g_p follows from that of $\widetilde{g}_{F(p)}$, and positive definiteness is true iff $\mathrm{d}F_p(v) = 0$ implies $v = 0$, i.e., $\mathrm{d}F_p$ is injective. ■

A **Riemannian submanifold** (M, g) is then a manifold $M \subseteq \widetilde{M}$ equipped with the metric $g = \iota^* \widetilde{g}$ induced by the pullback of the inclusion $\iota : M \to \widetilde{M}$:

$$g_p(v, w) = \widetilde{g}_p\left(\mathrm{d}\iota_p(v),\ \mathrm{d}\iota_p(w)\right).$$

Because we usually identify T_pM with its image in $T_p\widetilde{M}$ under $d\iota_p$, and think of $\mathrm{d}\iota_p$ as an inclusion map, what this really amounts to is $g_p(v, w) = \widetilde{g}_p(v, w)$ for $v, w \in T_pM$. In other words, the induced metric g is just the restriction of $\widetilde{g}$ to vectors tangent to M.

We go back to prove an exercise ([12] Exercise 2.7) on local isometry.

Exercise 2.1.3. *Prove that if $(\widetilde{M}, \widetilde{g})$ and (M, g) with $\partial M = \varnothing$ are Riemannian manifolds of the same dimension, a smooth map $\varphi : M \to \widetilde{M}$ is a local isometry if and only if $\varphi^* \widetilde{g} = g$.*

Proof. "$\Leftarrow$":

$\varphi^*\widetilde{g} = g \xRightarrow{\text{proof of (2.2)}}$ each $d\varphi_p$ is a linear isometry $\implies$ $d\varphi_p$ is injective $\implies$ φ is a smooth immersion $\xRightarrow{\text{[11]4.8(b)},\partial M=\varnothing,\dim M=\dim \widetilde{M}=n}$ φ is a local diffeomorphism $\xRightarrow{\text{defn.}}$ $\forall p \in M, \exists$ nbd U of p s.t. $\phi : U \to \varphi(U)$ is a diffeomorphism. The left is to check $\phi^*\left(\iota^*_{\varphi(U)}\widetilde{g}\right) = \iota^*_U g$. The following commutative diagram is helpful. We see that $\iota_{\varphi(U)} \circ \phi = \varphi|_U = \varphi \circ \iota_U$.

$$\begin{tikzcd}
U \arrow[r, "\phi"] \arrow[d, hook, "\iota_U"'] \arrow[dr, "\varphi|_U"] & \varphi(U) \arrow[d, hook, "\iota_{\varphi(U)}"] \\
M \arrow[r, "\phi"'] & \widetilde{M}
\end{tikzcd}$$

Note that $\phi^*\left(\iota^*_{\varphi(U)}\widetilde{g}\right) = \left(\iota_{\varphi(U)} \circ \phi\right)^* \widetilde{g} = \left(\varphi|_U\right)^* \widetilde{g}$. Noticing that $\varphi|_U = \varphi \circ \iota_U$ and that [11] Proposition 3.9 tells us $d\iota_p : T_pU \to T_pM$ is an isomorphism, we for $v, w \in T_pU$ have

$$\begin{aligned}
&\left[\left(\varphi|_U\right)^* \widetilde{g}\right]_p (v, w) \\
&= \widetilde{g}_{\varphi\circ\iota_U(p)}\left(\mathrm{d}\left(\varphi \circ (l_U)\right)_p(v),\ \mathrm{d}\left(\varphi \circ (l_U)\right)_p(w)\right) \\
&= \widetilde{g}_{\varphi(p)}\left(\mathrm{d}\varphi_{\iota_U(p)} \circ\ \mathrm{d}\left(\iota_U\right)_p(v),\ \mathrm{d}\varphi_{\iota_U(p)} \circ\ \mathrm{d}\left(\iota_U\right)_p(w)\right) \\
&= \widetilde{g}_{\varphi(p)}\left(\mathrm{d}\varphi_p\left(d\left(l_U\right)_p(v)\right),\ \mathrm{d}\varphi_p\left(d\left(\iota_U\right)_p(w)\right)\right) \\
&= \left(\varphi^*\widetilde{g}\right)_p\left(\mathrm{d}\left(\iota_U\right)_p(v),\ \mathrm{d}\left(\iota_U\right)_p(w)\right) \\
&\overset{\text{given}}{=}\ g_p\left(d\left(l_U\right)_p(v),\ \mathrm{d}\left(\iota_U\right)_p(w)\right) \\
&= \left(\iota_U^* g\right)_p(v, w)
\end{aligned} \tag{2.3}$$

Thus $\left(\varphi|_U\right)^* \widetilde{g} = \iota^*_U g$ on T_pU.

"$\Rightarrow$":

Now φ is a local isometry. $\forall p \in M$, there exists a neighborhood U of p s.t. $\phi : U \to \varphi(U)$ is a diffeomorphism from (open) Riemannian submanifold $(U, \iota^*_U g)$ to (open) Riemannian submanifold $\left(\varphi(U), \iota^*_{\varphi(U)}\widetilde{g}\right)$

with $(\varphi|_U)^*\left(l^*_{\varphi(U)}\widetilde{g}\right) = (\varphi|_U)^*\widetilde{g} = \iota_U^* g$. Since $\mathrm{d}(\iota_U)_p : T_pU \to T_pM$ is an isomorphism (see [11] Proposition 3.9), we for $\widehat{v}, \widehat{w} \in T_pM$ have $v = \left[\mathrm{d}(\iota_U)_p\right]^{-1}(\widehat{v}), w = \left[\mathrm{d}(\iota_U)_p\right]^{-1}(\widehat{w}) \in T_pU$ and

$$\begin{aligned} g_p(\widehat{v}, \widehat{w}) &= g_p\Big(\mathrm{d}(\iota_U)_p(v),\ \mathrm{d}(\iota_U)_p(w)\Big) \\ &= (\iota_U^* g)_p(v, w) \\ &\overset{\text{given}}{=} \left[(\varphi|_U)^*\widetilde{g}\right]_p(v, w) \\ &\overset{(2.2)}{=\!=} (\varphi^*\widetilde{g})_p\Big(\mathrm{d}(\iota_U)_p(v),\ \mathrm{d}(\iota_U)_p(w)\Big) \\ &= (\varphi^*\widetilde{g})_p(\widehat{v}, \widehat{w}) \end{aligned}$$

This shows $g = \varphi^*\widetilde{g}$. ■

Remark 2.1.4. We enforced $\partial M = \varnothing$ to use [11] Proposition 4.8 (b), which is used in the proof of [11] Theorem 4.14 (c). Also note that we don't need $\partial\widetilde{M} = \varnothing$ due to [11] 4.9. ♠

2.2 Methods for Constructing Riemannian Metrics

2.2.1 Riemannian Submanifold

As we have seen, every submanifold M of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ inherits a Riemannian metric $g = \iota^*\widetilde{g}$.

Example 2.2.1 (Spheres). For each positive integer n, the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is an embedded n-dimensional submanifold. The Riemannian metric induced on $\mathbb{S}^n$ by the Euclidean metric is denoted by $\mathring{g}$ and known as the **round metric** or **standard metric on** $\mathbb{S}^n$. ♣

The next proposition describes one of the most important tools for studying Riemannian submanifolds. If $(\widetilde{M}, \widetilde{g})$ is an m-dimensional smooth Riemannian manifold and $M \subseteq \widetilde{M}$ is an n-dimensional submanifold (both with or without boundary), a local frame $(E_1, \ldots, E_m)$ for $\widetilde{M}$ on an open subset $\widetilde{U} \subseteq \widetilde{M}$ is said to be **adapted to** M if the first n vector fields $(E_1, \ldots, E_n)$ are tangent to M. (see remark below.)

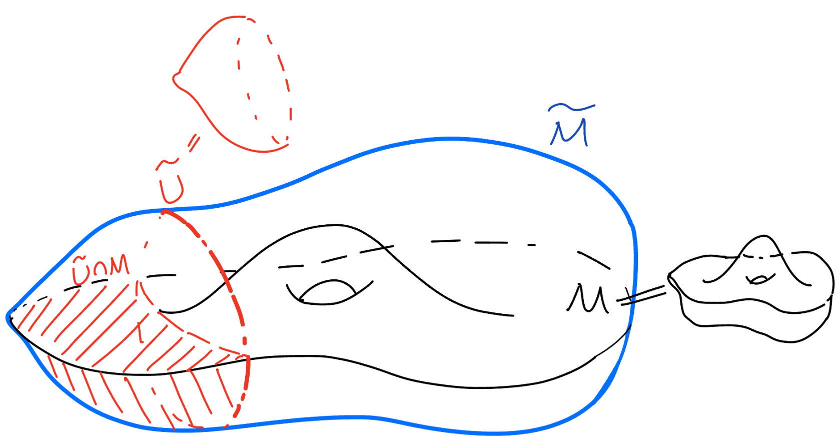

Figure 2.1: Adapted local frame

Remark 2.2.2. From [11] p.116, we can see that T_pM can be seen as a subspace of $T_p\widetilde{M}$. Thus, $n = \dim T_pM \leq \dim T_p\widetilde{M} = m$. When we say $(E_1, \ldots, E_n)$ are tangent to M we mean for each $p \in \widetilde{U} \cap M$, we have $(E_i)_p \in T_pM$ (notice that $(E_i)_p$ is defined in $T_p\widetilde{M}$ but not necessarily in $T_pM \subseteq T_p\widetilde{M}$.) ♠

Proposition 2.2.3 (Existence of Adapted Orthonormal Frames)**.** *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold (without boundary), and let $M \subseteq \widetilde{M}$ be an embedded smooth submanifold with or without boundary. Given $p \in M$, there exist a neighborhood $\widetilde{U}$ of p in $\widetilde{M}$ and a smooth orthonormal frame for $\widetilde{M}$ on $\widetilde{U}$ that is adapted to M.*

Exercise 2.2.4. *Prove the preceding proposition. [Hint: Apply the Gram-Schmidt algorithm to a coordinate frame in slice coordinates (see [12] Proposition A.22).]*

Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold (without boundary) and $M \subseteq \widetilde{M}$ is a smooth submanifold with or without boundary in $\widetilde{M}$. Given $p \in M$, a vector $\nu \in T_p\widetilde{M}$ is said to be **normal to** M if $\langle \nu, w \rangle = 0$ for every $w \in T_pM$. The space of all vectors normal to M at p is a subspace of $T_p\widetilde{M}$, called the **normal space at** p and denoted by $N_pM = (T_pM)^\perp$. At each $p \in M$, the ambient tangent space $T_p\widetilde{M}$ splits as an orthogonal direct sum $T_p\widetilde{M} = T_pM \oplus N_pM$. A section N of the ambient tangent bundle $T\widetilde{M}\big|_M$ is called a **normal vector field along** M if $N_p \in N_pM$ for each $p \in M$. The set

$$NM = \coprod_{p \in M} N_pM$$

is called the **normal bundle of** M. Fig. 2.2 illustrates an example where vector $v \in T_p\widetilde{M}$ is normal to T_pM for $\widetilde{M} \subseteq \mathbb{R}^3$.

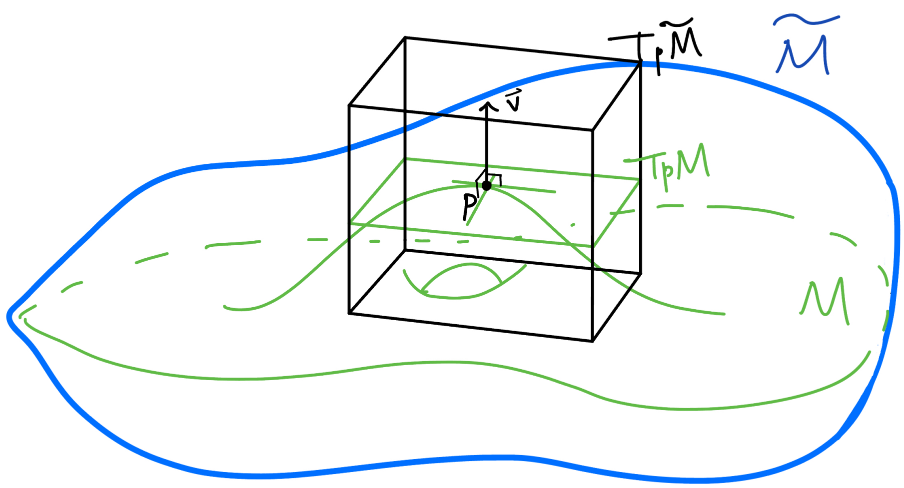

Figure 2.2: Tangent space of Riemannian submanifold

Proposition 2.2.5 (The Normal Bundle)**.** *If $\widetilde{M}$ is a Riemannian m-manifold (without boundary) and $M \subseteq \widetilde{M}$ is an immersed or embedded n-dimensional submanifold with or without boundary, then NM is a smooth rank-$(m-n)$ vector subbundle of the ambient tangent bundle $T\widetilde{M}\big|_M$. There are smooth bundle homomorphisms*

$$\pi^\top : T\widetilde{M}\big|_M \to TM, \quad \pi^\perp : T\widetilde{M}\big|_M \to NM$$

*called the **tangential** and **normal projections**, that for each $p \in M$ restrict to orthogonal projections from $T_p\widetilde{M}$ to T_pM and N_pM, respectively.*

Proof. See [12] Proposition 2.16. ∎

In case $\widetilde{M}$ is a manifold with boundary, the preceding constructions do not always work, because there is not a fully general construction of slice coordinates in that case. However, there is a satisfactory result in case the submanifold is the boundary itself, using boundary coordinates in place of slice coordinates.

Suppose (M, g) is a Riemannian manifold with boundary. We will always consider ∂M to be a Riemannian submanifold with the induced metric.

Proposition 2.2.6 (Existence of Outward-Pointing Normal). *If (M, g) is a smooth Riemannian manifold with boundary, the normal bundle to ∂M is a smooth rank-1 vector bundle over ∂M, and there is a unique smooth outward-pointing unit normal vector field along all of ∂M.*

Exercise 2.2.7. *See [11] Proposition 15.33.*

Corollary 2.2.8. *If (M, g) is an oriented Riemannian manifold with boundary and $\tilde{g}$ is the induced Riemannian metric on ∂M, then the volume form of $\tilde{g}$ is*

$$\omega_{\tilde{g}} = \iota_{\partial M}^{*}\left(N \lrcorner \omega_{g}\right),$$

where N is the outward unit normal vector field along ∂M.

Computations on a submanifold $M \subseteq \widetilde{M}$ are usually carried out most conveniently in terms of a **smooth local parametrization**: this is a smooth map $X : U \to \widetilde{M}$, where U is an open subset of $\mathbb{R}^n$ (or $\mathbb{R}^n_+$ in case M has a boundary), such that $X(U)$ is an open subset of M, and such that X, regarded as a map from U into M, is a diffeomorphism onto its image. Note that we can think of X either as a map into M or as a map into $\widetilde{M}$; both maps are typically denoted by the same symbol X. If we put $V = X(U) \subseteq M$ and $\varphi = X^{-1} : V \to U$, then (V, φ) is a smooth coordinate chart on M. Note that coordinate chart need not to be smooth to define an atlas for a smooth manifold, but there X, φ are smooth (diffeomorphisms).

Suppose (M, g) is a Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$ and $X : U \to \widetilde{M}$ is a smooth local parametrization of M. The coordinate representation of g in these coordinates is given by the following 2-tensor field on U:

$$\left(\varphi^{-1}\right)^{*} g = X^{*} g = X^{*} \iota^{*} \widetilde{g} = (\iota \circ X)^{*} \widetilde{g}.$$

Since $\iota \circ X$ is just the map X itself, regarded as a map into $\widetilde{M}$, this is really just $X^{*}\widetilde{g}$. The simplicity of the formula for the pullback of a tensor field makes this expression exceedingly easy to compute, once a coordinate expression for $\widetilde{g}$ is known. For example, if M is an immersed n-dimensional Riemannian submanifold of $\mathbb{R}^m$ and $X : U \to \mathbb{R}^m$ is a smooth local parametrization of M, the induced metric on U is just

$$g = X^{*}\bar{g} \xlongequal{\text{Prop.1.1.20}} \sum_{i=1}^{m}\left(\mathrm{d}X^{i}\right)^{2} = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} \frac{\partial X^{i}}{\partial u^{j}} \mathrm{d}u^{j}\right)^{2} = \sum_{i=1}^{m} \sum_{j,k=1}^{n} \frac{\partial X^{i}}{\partial u^{j}} \frac{\partial X^{i}}{\partial u^{k}} \mathrm{d}u^{j} \mathrm{d}u^{k}. \tag{2.4}$$

where (u^i) stands for the coordinates of $\mathbb{R}^n \supseteq U$.

Example 2.2.9 (Metrics in Graph Coordinates). If $U \subseteq \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}$ is a smooth function, then the **graph of** f is the subset $\Gamma(f) = \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^{n+1}$, which is an embedded submanifold of dimension n. It has a global parametrization $X : U \to \mathbb{R}^{n+1}$ called a **graph parametrization**, given by $X(u) = (u, f(u))$; the corresponding coordinates $\left(u^1, \ldots, u^n\right)$ on M are called **graph coordinates**. In graph coordinates, by (2.4), the induced metric of $\Gamma(f)$ is

$$X^{*}\bar{g} = \sum_{i=1}^{n}\left(\overbrace{\sum_{j=1}^{n} \underbrace{\frac{\partial u^{i}}{\partial u^{j}}}_{=\delta_{ij}} \mathrm{d}u^{j}}^{=\mathrm{d}u^{i}}\right)^{2} + \left(\overbrace{\sum_{j=1}^{n} \frac{\partial f(u)}{\partial u^{j}} \mathrm{d}u^{j}}^{=\mathrm{d}f}\right)^{2} = \left(\mathrm{d}u^{1}\right)^{2} + \cdots + \left(\mathrm{d}u^{n}\right)^{2} + \mathrm{d}f^{2}.$$

Applying this to the upper hemisphere of $\mathbb{S}^2$ with the parametrization $X : \mathbb{B}^2 \to \mathbb{R}^3$ given by

$$X(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right),$$

we see that the round metric on $\mathbb{S}^2$ can be written locally as

$$\begin{aligned}\mathring{g} = X^*\bar{g} = &\ \mathrm{d}u^2 + dv^2 + \left(\frac{u\,\mathrm{d}u + v\,\mathrm{d}v}{\sqrt{1-u^2-v^2}}\right)^2 \\ = &\ \frac{\left(1-v^2\right)\,\mathrm{d}u^2 + \left(1-u^2\right)\,\mathrm{d}v^2 + 2uv\,\mathrm{d}u\,\mathrm{d}v}{1-u^2-v^2}.\end{aligned}$$

♣

Example 2.2.10 (Surfaces of Revolution)**.** Let H be the half-plane $\{(r,z) : r > 0\}$, and suppose $C \subseteq H$ is an embedded 1-dimensional submanifold. The **surface of revolution** determined by C is the subset $S_C \subseteq \mathbb{R}^3$ given by

$$S_C = \left\{(x,y,z) : \left(\sqrt{x^2+y^2}, z\right) \in C\right\}.$$

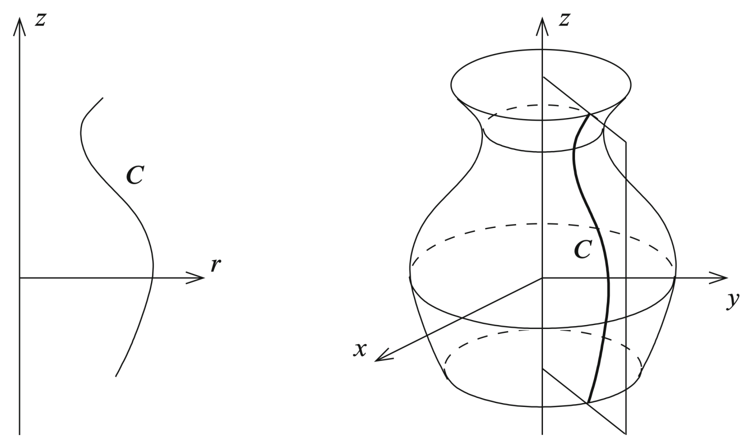

Figure 2.3: A surface of revolution

The set C is called its **generating curve** (see Fig. 2.3). Every smooth local parametrization $\gamma(t) = (a(t), b(t))$ for C yields a smooth local parametrization for S_C of the form

$$X(t,\theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t)),$$

provided that (t,θ) is restricted to a sufficiently small open set in the plane. The t-coordinate curves $t \mapsto X(t,\theta_0)$ are called **meridians**, and the θ-coordinate curves $\theta \mapsto X(t_0,\theta)$ are called **latitude circles**. The induced metric on S_C is

$$\begin{aligned}X^*\bar{g} =&\ d(a(t)\cos\theta)^2 + \mathrm{d}(a(t)\sin\theta)^2 + \mathrm{d}(b(t))^2 \\ =&\ (a'(t)\cos\theta\,\mathrm{d}t - a(t)\sin\theta\,\mathrm{d}\theta)^2 \\ &+ (a'(t)\sin\theta\,\mathrm{d}t + a(t)\cos\theta\,\mathrm{d}\theta)^2 + (b'(t)\,\mathrm{d}t)^2 \\ =&\ \left(a'(t)^2 + b'(t)^2\right)\,\mathrm{d}t^2 + a(t)^2\,\mathrm{d}\theta^2.\end{aligned}$$

In particular, if γ is a **unit-speed curve** (meaning that $|\gamma'(t)|^2 = a'(t)^2 + b'(t)^2 \equiv 1$), this reduces to $\mathrm{d}t^2 + a(t)^2\,\mathrm{d}\theta^2$. Here are some examples of surfaces of revolution and their induced metrics.

- If C is the semicircle $r^2 + z^2 = 1$, parametrized by $\gamma(\varphi) = (\sin\varphi, \cos\varphi)$ for $0 < \varphi < \pi$, then S_C is the unit sphere (minus the north and south poles). The map $X(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ constructed above is called the **spherical coordinate parametrization**, and the induced metric is $\mathrm{d}\varphi^2 + \sin^2\varphi\, \mathrm{d}\theta^2$. (This example is the source of the terminology for meridians and latitude circles.)
- If C is the circle $(r-2)^2 + z^2 = 1$, parametrized by $\gamma(t) = (2 + \cos t, \sin t)$, we obtain a torus of revolution, whose induced metric is $\mathrm{d}t^2 + (2+\cos t)^2\, \mathrm{d}\theta^2$.
- If C is a vertical line parametrized by $\gamma(t) = (1, t)$, then S_C is the unit cylinder $x^2 + y^2 = 1$, and the induced metric is $\mathrm{d}t^2 + \mathrm{d}\theta^2$. Note that this means that the parametrization $X : \mathbb{R}^2 \to \mathbb{R}^3$ is an isometric immersion.

♣

Example 2.2.11 (The $\boldsymbol{n}$-Torus as a Riemannian Submanifold)**.** The n**-torus** is the manifold $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, regarded as the subset of $\mathbb{R}^{2n}$ defined by $(x^1)^2 + (x^2)^2 + \cdots + (x^{2n-1})^2 + (x^{2n})^2 = 1$. The smooth covering map $X : \mathbb{R}^n \to \mathbb{T}^n$, defined by $X(u^1, \cdots, u^n) = (\cos u^1, \sin u^1, \cdots, \cos u^n, \sin u^n)$, restricts to a smooth local parametrization on any sufficiently small open subset of $\mathbb{R}^n$, and the induced metric is equal to the Euclidean metric in $\left(u^i\right)$ coordinates, and therefore the induced metric on $\mathbb{T}^n$ is flat. ♣

2.2.2 Riemannian Products

Next we consider products. If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, the product manifold $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$, called the product metric, defined by

$$g_{(p_1,p_2)}\left((v_1, v_2), (w_1, w_2)\right) = g_1|_{p_1}(v_1, w_1) + g_2|_{p_2}(v_2, w_2),$$

where (v_1, v_2) and (w_1, w_2) are elements of $T_{p_1}M_1 \oplus T_{p_2}M_2$, which is naturally identified with $T_{(p_1,p_2)}(M_1 \times M_2)$. Smooth local coordinates $\left(x^1, \ldots, x^n\right)$ for M_1 and $\left(x^{n+1}, \ldots, x^{n+m}\right)$ for M_2 give coordinates $\left(x^1, \ldots, x^{n+m}\right)$ for $M_1 \times M_2$. In terms of these coordinates, the product metric has the local expression $g = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$, where (g_{ij}) is the block diagonal matrix

$$(g_{ij}) = \begin{pmatrix} (g_1)_{ab} & 0 \\ 0 & (g_2)_{cd} \end{pmatrix}$$

here the indices a, b run from 1 to n, and c, d run from $n+1$ to $n+m$. Product metrics on products of three or more Riemannian manifolds are defined similarly.

Exercise 2.2.12. *Show that the induced metric on $\mathbb{T}^n$ described in Example 2.2.11 is equal to the product metric obtained from the usual induced metric on $\mathbb{S}^1 \subseteq \mathbb{R}^2$.*

Here is an important generalization of product metrics. Suppose (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds, and $f : M_1 \to \mathbb{R}^+$is a strictly positive smooth function. The **warped product** $M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ endowed with the Riemannian metric $g = g_1 \oplus f^2 g_2$, defined by

$$g_{(p_1,p_2)}\left((v_1, v_2), (w_1, w_2)\right) = g_1|_{p_1}(v_1, w_1) + f(p_1)^2\, g_2\Big|_{p_2}(v_2, w_2),$$

where $(v_1, v_2), (w_1, w_2) \in T_{p_1}M_1 \oplus T_{p_2}M_2$ as before. (Despite the similarity with the notation for product metrics, $g_1 \oplus f^2 g_2$ is generally not a product metric unless f is constant.) A wide variety of metrics can be constructed in this way; here are just a few examples.

Example 2.2.13 (Warped Products)**.**

(a) With $f \equiv 1$, the warped product $M_1 \times fM_2$ is just the space $M_1 \times M_2$ with the product metric.

(b) Every surface of revolution can be expressed as a warped product, as follows. Let H be the half-plane $\{(r, z) : r > 0\}$, let $C \subseteq H$ be an embedded smooth 1-dimensional submanifold, and let $S_C \subseteq \mathbb{R}^3$

denote the corresponding surface of revolution as in Example 2.2.10. Endow C with the Riemannian metric induced from the Euclidean metric on H, and let $\mathbb{S}^1$ be endowed with its standard metric. Let $f : C \to \mathbb{R}$ be the distance to the z-axis: $f(r, z) = r$. Then [12] Problem 2-3 shows that S_C is isometric to the warped product $C \times_f \mathbb{S}^1$.

(c) If we let ρ denote the standard coordinate function on $\mathbb{R}^+ \subseteq \mathbb{R}$, then the map $\Phi(\rho, \omega) = \rho\omega$ gives an isometry from the warped product $\mathbb{R}^+ \times_\rho \mathbb{S}^{n-1}$ to $\mathbb{R}^n \backslash \{0\}$ with its Euclidean metric (see [12] Problem 2-4).

♣

2.2.3 Riemannian Submersions

Unlike submanifolds and products, the quotient of Riemannian manifolds only inherit Riemannian metrics under very special circumstances. Suppose $\widetilde{M}$ and M are smooth manifolds, $\pi : \widetilde{M} \to M$ is a smooth submersion, and $\widetilde{g}$ is a Riemannian metric on $\widetilde{M}$. By the submersion level set theorem (see [11] Cor.5.13), each level set $\widetilde{M_y} = \pi^{-1}(y)$ is regular (as π is a smooth submersion) and a properly embedded smooth submanifold of $\widetilde{M}$, and π is a defining map for $\widetilde{M_y}$ (see [11] p.107). Then by [11] Proposition 5.38, $T_x\widetilde{M_y} = \operatorname{Ker}\left(\mathrm{d}\pi_x : T_x\widetilde{M} \to T_{\pi(x)}M\right)$ for any $x \in \widetilde{M_y}$. Therefore, at each point $x \in \widetilde{M}$, we define two subspaces of the tangent space $T_x\widetilde{M}$ as follows: the **vertical tangent space at** x is

$$V_x = \operatorname{Ker}\ \mathrm{d}\pi_x = T_x\left(\widetilde{M}_{\pi(x)}\right)$$

(that is, the tangent space to the fiber containing x), and the **horizontal tangent space at** x is its orthogonal complement:

$$H_x = (V_x)^\perp := \left\{ v \in T_x\widetilde{M} \mid \forall w \in T_x\left(\widetilde{M}_{\pi(x)}\right), \langle v, w\rangle_g = 0 \right\}$$

Then the tangent space $T_x\widetilde{M}$ decomposes as an orthogonal direct sum $T_x\widetilde{M} = H_x \oplus V_x$. Note that the vertical space is well defined for every submersion, because it does not refer to the metric; but the horizontal space depends on the metric.

A vector field on $\widetilde{M}$ is said to be a **horizontal vector field** if its value at each point lies in the horizontal space at that point; a **vertical vector field** is defined similarly. Given a vector field X on M, a vector field $\widetilde{X}$ on $\widetilde{M}$ is called a **horizontal lift of** X if $\widetilde{X}$ is horizontal and π-related to X. (The latter property means that $\mathrm{d}\pi_x\left(\widetilde{X}_x\right) = X_{\pi(x)}$ for each $x \in \widetilde{M}$.) In other words, the following diagram is commutative ($\widetilde{X}$ is so-called "lift"):

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\widetilde{X}"] \arrow[d, "\pi"'] & T\widetilde{M} = \coprod_{x\in\widetilde{M}} T_x\widetilde{M} \arrow[d, "d\pi"] \\
M \arrow[r, "X"'] & TM = \coprod_{x\in M} T_xM
\end{tikzcd}$$

The next proposition is the principal tool for doing computations on Riemannian submersions.

Proposition 2.2.14 (Properties of Horizontal Vector Fields). *Let $\widetilde{M}$ and M be smooth manifolds, let $\pi : \widetilde{M} \to M$ be a smooth submersion, and let $\widetilde{g}$ be a Riemannian metric on $\widetilde{M}$.*

(a) Every smooth vector field W on $\widetilde{M}$ can be expressed uniquely in the form $W = W^H + W^V$, where W^H is horizontal, W^V is vertical, and both W^H and W^V are smooth.

(b) Every smooth vector field on M has a unique smooth horizontal lift to $\widetilde{M}$.

(c) For every $x \in \widetilde{M}$ and $v \in H_x$, there is a vector field $X \in \mathfrak{X}(M)$ whose horizontal lift $\widetilde{X}$ satisfies $\widetilde{X}_x = v$.

Proof. [12] Proposition 2.25. ■

The fact that every horizontal vector at a point of $\widetilde{M}$ can be extended to a horizontal lift on all of $\widetilde{M}$ (part (c) of the preceding proposition) is highly useful for computations. It is important to be aware, though, that not every horizontal vector field on $\widetilde{M}$ is a horizontal lift, as the next exercise shows.

Exercise 2.2.15. *Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection map $\pi(x, y) = x$, and let W be the smooth vector field $y\partial_x$ on $\mathbb{R}^2$. Show that W is horizontal, but there is no vector field on $\mathbb{R}$ whose horizontal lift is equal to W.*

Now we can identify some quotients of Riemannian manifolds that inherit metrics of their own. Let us begin by describing what such a metric should look like.

Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are Riemannian manifolds, and $\pi : \widetilde{M} \to M$ is a smooth submersion. Then π is said to be a **Riemannian submersion** if for each $x \in \widetilde{M}$, the differential $\mathrm{d}\pi_x$ restricts to a linear isometry from H_x onto $T_{\pi(x)}M$. In other words, $\widetilde{g}_x(v, w) = g_{\pi(x)}\left(\mathrm{d}\pi_x(v), \mathrm{d}\pi_x(w)\right)$ whenever $v, w \in H_x$.

Remark 2.2.16. Note that $\mathrm{d}\pi_x : T_x\widetilde{M} = V_x \oplus H_x = \operatorname{Ker}(\mathrm{d}\pi_x) \oplus (V_x)^{\perp} \to T_{\pi(x)}M$ is a C^∞ submersion and is thus onto. Thus, $\forall v' \in T_{\pi(x)}M, \exists v = v_{V_x} + v_{H_x}$ s.t $v' = \mathrm{d}\pi_x(v) = \mathrm{d}\pi_x(v_{V_x}) + \mathrm{d}\pi_x(v_{H_x}) = 0 + \mathrm{d}\pi_x(v_{H_x}) = \mathrm{d}\pi_x(v_{H_x})$. This shows that $\left.\mathrm{d}\pi_x\right|_{H_x} : H_x \to T_{\pi(x)}M$ is also onto. Therefore, in the above definition, the only requirement is linear isometry. ♠

Example 2.2.17 (Riemannian Submersions)**.**

(a) The projection $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a Riemannian submersion if $\mathbb{R}^{n+k}$ and $\mathbb{R}^n$ are both endowed with their Euclidean metrics.

(b) If M and N are Riemannian manifolds and $M \times N$ is endowed with the product metric, then both projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are Riemannian submersions.

(c) If $M \times_f N$ is a warped product manifold, then the projection $\pi_M : M \times_f N \to M$ is a Riemannian submersion, but π_N typically is not.

♣

Exercise 2.2.18. *Verify above example.*

Solution. We do (a) and (b), leaving (c) as a fact.

(a): For

$$\begin{aligned} \pi : \mathbb{R}^{n+k} &\to \mathbb{R}^n \\ (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+k}) &\mapsto (x_1, \cdots, x_n) \end{aligned}$$

we have components $\pi_i(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+k}) = x_i$ for $1 \le i \le n$ and Jacobian

$$J_\pi(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times (n+k)}$$

Then

$$\mathrm{d}\pi_x\left(\sum_i^{n+k} v^i \frac{\partial}{\partial x^i}\right) = \left[J_\pi(x)\begin{pmatrix} v^1 \\ \vdots \\ v^{n+k} \end{pmatrix}\right] \cdot \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{pmatrix} = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{pmatrix} = \sum_i^n v^i \frac{\partial}{\partial x^i}$$

Since

$$V_x = \operatorname{Ker}(\mathrm{d}\pi_x) = \left\{ \sum_i^{n+k} v^i \frac{\partial}{\partial x^i} \,\middle|\, v^1 = \cdots = v^n = 0 \right\}$$

and

$$H_x = (V_x)^{\perp} = \left\{ \sum_i^{n+k} v^i \frac{\partial}{\partial x^i} \,\middle|\, v^{n+1} = \cdots = v^{n+k} = 0 \right\},$$

we see $\forall v, w \in H_x$,

$$\begin{aligned} \widetilde{g}(v,w) = \bar{g}(v,w) &= \sum\nolimits^{n+k} |v_i - w_i|^2 = \sum\nolimits^{n} |v_i - w_i|^2 + \sum\nolimits^{k} |v_{n+i} - w_{n+i}|^2 \\ &= \sum\nolimits^{n} |v_i - w_i|^2 = \bar{g}(\mathrm{d}\pi_x(v),\ \mathrm{d}\pi_x(w)) = g(\mathrm{d}\pi_x(v),\ \mathrm{d}\pi_x(w)) \end{aligned}$$

The map π is thus a Riemannian submersion.

(b): Let (M, g_M) and (N, g_N) be Riemannian manifolds. Equip the product manifold $M \times N$ with the product metric $g_{M\times N} := g_M \oplus g_N$, defined at each point $(p,q) \in M \times N$ by:

$$g_{M\times N}((X_1,Y_1),(X_2,Y_2)) = g_M(X_1,X_2) + g_N(Y_1,Y_2),$$

for all $(X_1,Y_1),(X_2,Y_2) \in T_{(p,q)}(M \times N) \cong T_pM \oplus T_qN$.

We consider the projection:

$$\pi_M : M \times N \to M, \quad (p,q) \mapsto p.$$

We claim that π_M is a Riemannian submersion. First, note that π_M is a smooth submersion since its differential at each point (p,q) is:

$$\mathrm{d}\pi_M : T_pM \oplus T_qN \to T_pM, \quad (X,Y) \mapsto X,$$

which is clearly surjective. The vertical space at (p,q) is:

$$V_{(p,q)} := \ker(\mathrm{d}\pi_M) = \{(0,Y) \mid Y \in T_qN\}.$$

The horizontal space is the orthogonal complement of $V_{(p,q)}$ with respect to the product metric:

$$H_{(p,q)} := V_{(p,q)}^{\perp} = \{(X,0) \mid X \in T_pM\}.$$

We now verify that the differential $\mathrm{d}\pi_M$ restricts to a linear isometry from $H_{(p,q)}$ to T_pM: For any $v = (X,0)$ and $w = (Y,0)$ in $H_{(p,q)}$, we have:

$$g_{M\times N}(v,w) = g_{M\times N}((X,0),(Y,0)) = g_M(X,Y),$$

and

$$g_M(\mathrm{d}\pi_M(v),\ \mathrm{d}\pi_M(w)) = g_M(X,Y).$$

Therefore, the restriction $\mathrm{d}\pi_M|_{H_{(p,q)}} : H_{(p,q)} \to T_pM$ is a linear isometry. Hence, π_M is a Riemannian submersion. ♦

Given a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and a surjective submersion $\pi : \widetilde{M} \to M$, it is almost never the case that there is a metric on M that makes π into a Riemannian submersion. It is not hard to see why: for this to be the case, whenever $p_1, p_2 \in \widetilde{M}$ are two points in the same fiber $\pi^{-1}(y)$, the linear maps $\left(\mathrm{d}\pi_{p_i}|_{H_{p_i}}\right)^{-1} : T_yM \to H_{p_i}$ both have to pull $\widetilde{g}$ back to the same inner product on T_yM.

There is, however, an important special case in which there is such a metric. Suppose $\pi : \widetilde{M} \to M$ is a smooth surjective submersion, and G is a group acting on $\widetilde{M}$. (See [12] Appendix C for a review of the basic definitions and terminology regarding group actions on manifolds.) We say that the action is **vertical** if every element $\varphi \in G$ takes each fiber to itself, meaning that $\pi(\varphi \cdot p) = \pi(p)$ for all $p \in \widetilde{M}$. The action is **transitive on fibers** if for each $p, q \in \widetilde{M}$ such that $\pi(p) = \pi(q)$, there exists $\varphi \in G$ such that $\varphi \cdot p = q$.

If in addition $\widetilde{M}$ is endowed with a Riemannian metric, the action is said to be an **isometric action** or an **action by isometries**, and the metric is said to be **invariant under** G, if the map $x \mapsto \varphi \cdot x$ is an isometry for each $\varphi \in G$. In that case, provided the action is effective (so that different elements of G define different isometries of $\widetilde{M}$), we can identify G with a subgroup of $\operatorname{Iso}(\widetilde{M}, g)$. Since an isometry is, in particular, a diffeomorphism, every isometric action is an action by diffeomorphisms.

Theorem 2.2.19. *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, let $\pi : \widetilde{M} \to M$ be a surjective smooth submersion, and let G be a group acting on $\widetilde{M}$. If the action is isometric, vertical, and transitive on fibers, then there is a unique Riemannian metric on M such that π is a Riemannian submersion.*

Proof. Problem 2.6.6. ■

The next corollary describes one important situation to which the preceding theorem applies.

Corollary 2.2.20. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold, and G is a Lie group acting smoothly, freely, properly, and isometrically on $\widetilde{M}$. Then the orbit space $M = \widetilde{M}/G$ has a unique smooth manifold structure and Riemannian metric such that π is a Riemannian submersion.*

Proof. Under the given hypotheses, the quotient manifold theorem (see [11] Theorem 21.10) shows that M has a unique smooth manifold structure such that the quotient map $\pi : \widetilde{M} \to M$ is a smooth submersion. It follows easily from the definitions in that case that the given action of G on $\widetilde{M}$ is vertical and transitive on fibers. Since the action is also isometric, Theorem 2.2.19 shows that M inherits a unique Riemannian metric making π into a Riemannian submersion. ■

Here is an important example of a Riemannian metric defined in this way. A larger class of such metrics is described in Problem 2.6.7.

Example 2.2.21 (The Fubini-Study Metric)**.** Let n be a positive integer, and consider the complex projective space $\mathbb{CP}^n$ defined in [12] Example C.19. That example shows that the map $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ sending each point in $\mathbb{C}^{n+1} \backslash \{0\}$ to its span is a surjective smooth submersion. Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ endowed with its Euclidean metric, we can view the unit sphere $\mathbb{S}^{2n+1}$ with its round metric $\mathring{g}$ as an embedded Riemannian submanifold of $\mathbb{C}^{n+1} \backslash \{0\}$. Let $p : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ denote the restriction of the map π. Then p is smooth, and it is surjective, because every 1-dimensional complex subspace contains elements of unit norm. We need to show that it is a submersion. Let $z_0 \in \mathbb{S}^{2n+1}$ and set $\zeta_0 = p(z_0) \in \mathbb{CP}^n$. Since π is a smooth submersion, it has a smooth local section $\sigma : U \to \mathbb{C}^{n+1}$ defined on a neighborhood U of ζ_0 and satisfying $\sigma(\zeta_0) = z_0$ (Theorem A.17). Let $v : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{S}^{2n+1}$ be the radial projection onto the sphere:

$$\nu(z) = \frac{z}{|z|}.$$

Since dividing an element of $\mathbb{C}^{n+1}$ by a nonzero scalar does not change its span, it follows that $p \circ \nu = \pi$. Therefore, if we set $\widetilde{\sigma} = \nu \circ \sigma$, we have $p \circ \widetilde{\sigma} = p \circ \nu \circ \sigma = \pi \circ \sigma = \mathrm{Id}_U$, so $\widetilde{\sigma}$ is a local section of p. By the local secrtion theorem (see [11] Theorem 4.26), this shows that p is a submersion. Define an action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ by complex multiplication:

$$\lambda \cdot \left(z^1, \ldots, z^{n+1}\right) = \left(\lambda z^1, \ldots, \lambda z^{n+1}\right)$$

for $\lambda \in \mathbb{S}^1$ (viewed as a complex number of norm 1) and $z = \left(z^1, \ldots, z^{n+1}\right) \in \mathbb{S}^{2n+1}$. This is easily seen to be isometric, vertical, and transitive on fibers of p. By Theorem 2.2.19, therefore, there is a unique metric on $\mathbb{CP}^n$ such that the map $p : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a Riemannian submersion. This metric is called the Fubini-Study metric. ♣

2.2.4 Riemannian Coverings

Another important special case of Riemannian submersions occurs in the context of covering maps. Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are Riemannian manifolds. A smooth covering map $\pi : M \to M$ is called a **Riemannian covering** if it is a local isometry.

Proposition 2.2.22. *Suppose $\pi : \widetilde{M} \to M$ is a smooth normal covering map, and $\widetilde{g}$ is any metric on $\widetilde{M}$ that is invariant under all covering automorphisms. Then there is a unique metric g on M such that π is a Riemannian covering.*

Proof. "invariant under $\Gamma = \operatorname{Aut}_\pi(\widetilde{M})$," is defined in last subsection; for normal covering map and beyond, see [10] p.293, 309-314 and [11] Chapter 21; for proof of the proposition, see [12] Proposition 2.31. ■

Proposition 2.2.23. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold, and Γ is a discrete Lie group acting smoothly, freely, properly, and isometrically on $\widetilde{M}$. Then $\widetilde{M}/\Gamma$ has a unique Riemannian metric such that the quotient map $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$ is a normal Riemannian covering.*

Proof. This is an immediate consequence of [11] Theorem 21.13 and above proposition. ■

Corollary 2.2.24. *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds, $\pi : \widetilde{M} \to M$ is a normal Riemannian covering map, and $\Gamma = \operatorname{Aut}_\pi(\widetilde{M})$. Then M is isometric to $\widetilde{M}/\Gamma$.*

Proof. Proof by [11] Proposition 21.12 & [11] Theorem 21.13 & [11] Theorem 4.31. ■

Example 2.2.25. The two-element group $\Gamma = \{\pm 1\}$ acts smoothly, freely, properly, and isometrically on $\mathbb{S}^n$ by multiplication. [12] Example C.24 shows that the quotient space is diffeomorphic to the real projective space $\mathbb{RP}^n$ and the quotient map $q : \mathbb{S}^n \to \mathbb{RP}^n$ is a smooth normal covering map. Because the action is isometric, Proposition 2.2.23 shows that there is a unique metric on $\mathbb{RP}^n$ such that q is a Riemannian covering. ♣

Example 2.2.26 (The Open Möbius Band)**.** The **open Möbius band** is the quotient space $M = \mathbb{R}^2/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}^2$ by $n \cdot (x, y) = (x + n, (-1)^n y)$. This action is smooth, free, proper, and isometric, and therefore M inherits a flat Riemannian metric such that the quotient map is a Riemannian covering. (See Problem 2.6.8) ♣

Exercise 2.2.27. *Let $\mathbb{T}^n \subseteq \mathbb{R}^{2n}$ be the n-torus with its induced metric. Show that the map $X : \mathbb{R}^n \to \mathbb{T}^n$ of Example 2.2.11 is a Riemannian covering.*

2.3 Basic Constructions on Riemannian Manifolds

2.3.1 Raising and Lowering Indices

Given a Riemannian metric g on a smooth manifold M with or without boundary, we define a bundle homomorphism $\widehat{g} : TM \to T^*M$ as follows. For each $p \in M$ and each $v \in T_pM$, we let $\widehat{g}(v) \in T_p^*M$ be the covector defined by

$$\widehat{g}(v)(w) = g_p(v, w) \quad \text{for all } w \in T_pM.$$

To see that this is a smooth bundle homomorphism, it is easiest to consider its action on smooth vector fields:

$$\widehat{g}(X)(Y) = g(X, Y) \quad \text{for } X, Y \in \mathfrak{X}(M).$$

Because $\widehat{g}(X)(Y)$ is linear over $C^\infty(M)$ as a function of Y, it follows from the tensor characterization lemma 1.1.19 that $\widehat{g}(X)$ is a smooth covector field; and because $\widehat{g}(X)$ is linear over $C^\infty(M)$ as a function of X, this defines $\widehat{g}$ as a smooth bundle homomorphism by the bundle homomorphism characterization lemma ([11] Lemma 10.29). As usual, we use the same symbol for both the pointwise bundle homomorphism $\widehat{g}: TM \to T^*M$ and the linear map on sections $\widehat{g}: \mathfrak{X}(M) \to \mathfrak{X}^*(M)$.

Note that $\widehat{g}$ is injective at each point, because $\widehat{g}(v) = 0$ for some $v \in T_pM$ implies

$$0 = \widehat{g}(v)(v) = \langle v, v \rangle_g,$$

which in turn implies $v = 0$. For dimensional reasons, therefore, $\widehat{g}$ is bijective, so it is a bundle isomorphism (see [11] Proposition 10.26).

Given a smooth local frame (E_i) and its dual coframe $\left(\varepsilon^i\right)$, let $g = g_{ij}\varepsilon^i\varepsilon^j$ be the local expression for g (see (2.1)). If $X = X^i E_i$ is a smooth vector field, the covector field $\widehat{g}(X)$ has the coordinate expression

$$\widehat{g}(X) = \left(g_{ij}X^i\right)\varepsilon^j, \tag{2.5}$$

as $\widehat{g}(X)(Y) = g_{ij}\varepsilon^i \otimes \varepsilon^j(X, Y) = g_{ij}X^i\varepsilon^j(Y) \implies \widehat{g}(X) = \left(g_{ij}X^i\right)\varepsilon^j$.

Exercise 2.3.1. *Write down the matrix of $\widehat{g}$ and conclude that the matrix of $\widehat{g}$ in any local frame is the same as the matrix of g itself.*

Solution. For each p, $\widehat{g}$ as a linear transformation from vector space T_pM to vector space T_p^*M. Its matrix is computed using Definition 1.1.9. Since

$$\widehat{g}\left(E_k\right) = (g_{ij}(E_k)^i)\varepsilon^j = \sum_j g_{kj}\varepsilon^j.$$

and the basis of T_pM is (E_k) and the basis of T_p^*M is (ε^j), the matrix of $\widehat{g}$ is

$$A = \left[\begin{array}{ccc} | & & | \\ \widehat{g}\left(E_1\right) & \cdots & \widehat{g}\left(E_n\right) \\ | & & | \end{array}\right] = \begin{bmatrix} g_{11} & \cdots & g_{n1} \\ \vdots & \ddots & \vdots \\ g_{1n} & \cdots & g_{nn} \end{bmatrix}$$

We note that this is the transpose of the matrix of g. However, since the matrix of g is symmetric, we proved the statement. ♦

Given a vector field X, it is standard practice to denote the components of the covector field $\widehat{g}(X)$ by

$$X_j = g_{ij}X^i,$$

so that

$$\widehat{g}(X) = X_j\varepsilon^j,$$

and we say that $\widehat{g}(X)$ is obtained from X by **lowering an index**. With this in mind, the covector field $\widehat{g}(X)$ is denoted by $X^\flat$ and called X **flat**, borrowing from the musical notation for lowering a tone. That is, we also use $\flat$ to denote $\widehat{g}$, which is a smooth bundle isomorphism, as we remarked above.

The matrix of the inverse map $\widehat{g}^{-1}: T_p^*M \to T_pM$ is the inverse of (g_{ij}). (Because (g_{ij}) is the matrix of the isomorphism $\widehat{g}$, it is invertible at each point.) We let $\left(g^{ij}\right)$ denote the matrix-valued function whose value at $p \in M$ is the inverse of the matrix $(g_{ij}(p))$, so that

$$g^{ij}g_{jk} = g_{kj}g^{ji} = \delta^i_k. \tag{2.6}$$

Because g_{ij} is a symmetric matrix, so is g^{ij}. Thus for a covector field $\omega \in \mathfrak{X}^*(M)$, the vector field $\widehat{g}^{-1}(\omega)$ has the coordinate representation

$$\widehat{g}^{-1}(\omega) = \omega^i E_i \tag{2.7}$$

where
$$\omega^i = g^{ij}\omega_j. \tag{2.8}$$
If ω is a covector field, the vector field $\widehat{g}^{-1}(\omega)$ is called (what else?) ω **sharp** and denoted by $\omega^\sharp$, and we say that it is obtained from ω by **raising an index**. Likewise, as the inverse of $\widehat{g}$, the map $\widehat{g}^{-1} = \sharp : \Gamma(T^*M) \to \Gamma(TM); T^*M \to TM$ is a smooth bundle isomorphism.

The two inverse isomorphisms $\flat$ and $\sharp$ are known as the **musical isomorphisms**.

Definition 2.3.2. *The flat and sharp operators can be applied to tensors of any rank, in any index position, to convert tensors from covariant to contravariant or vice versa. Formally, this operation is defined as follows: if F is any (k,l)-tensor and $i \in \{1,\ldots,k+l\}$ is any covariant index position for F (meaning that the i th argument is a vector, not a covector), we can form a new tensor $F^\sharp$ of type $(k+1,l-1)$ by setting*
$$F^\sharp\left(\alpha_1,\ldots,\alpha_{k+l}\right) = F\left(\alpha_1,\ldots,\alpha_{i-1},\alpha_i^\sharp,\alpha_{i+1},\ldots,\alpha_{k+l}\right)$$
whenever $\alpha_1,\ldots,\alpha_{k+l}$ are vectors or covectors as appropriate. In any local frame, the components of $F^\sharp$ are obtained by multiplying the components of F by g^{pq} and contracting one of the indices of g^{pq} with the i th index of F. Similarly, if i is a contravariant index position, we can define a $(k-1,l+1)$-tensor $F^\flat$ by
$$F^\flat\left(\alpha_1,\ldots,\alpha_{k+l}\right) = F\left(\alpha_1,\ldots,\alpha_{i-1},\alpha_i^\flat,\alpha_{i+1},\ldots,\alpha_{k+l}\right).$$

In components, it is computed by multiplying by g_{pq} and contracting.

Example 2.3.3. For example, if A is a mixed 3-tensor given in terms of a local frame by
$$A = A_i{}^j{}_k\varepsilon^i \otimes E_j \otimes \varepsilon^k,$$
we can lower its middle index to obtain a covariant 3-tensor $A^\flat$ with components
$$A_{ijk} = g_{jl}A_i{}^l{}_k.$$

♣

To avoid overly cumbersome notation, we use the symbols $F^\sharp$ and $F^\flat$ without explicitly specifying which index position the sharp or flat operator is to be applied to; when there is more than one choice, we will always stipulate in words what is meant.

Another important application of the flat and sharp operators is to extend the trace operator introduced to covariant tensors. If F is any covariant k-tensor field on a Riemannian manifold with $k \geq 2$, we can raise one of its indices (say the last one for definiteness) and obtain a $(1,k-1)$-tensor $h^\sharp$. The trace of $F^\sharp$ is thus a well-defined covariant $(k-2)$-tensor field. We define the **trace of F with respect to g** as
$$\operatorname{tr}_g F = \operatorname{tr}\left(F^\sharp\right).$$

Sometimes we may wish to raise an index other than the last, or to take the trace on a pair of indices other than the last covariant and contravariant ones. In each such case, we will say in words what is meant.

The most important case is that of a covariant 2-tensor field. We will see that the additional information g allows us to do some identifications beyond $T^{(1,1)}(V) = V \otimes V^* \cong \mathcal{L}(V^*,V;\mathbb{R}) \cong \operatorname{End}(V)$. If $L : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a smooth tensor field of type $(1,1)$, then the map $\widehat{L} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ given by
$$\widehat{L}(X,Y) = g(L(X),Y), \tag{2.9}$$
for each $X,Y \in \mathfrak{X}(M)$ is a smooth tensor field of type $(0,2)$ on M. Conversely, if $F : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ is a smooth tensor field of type $(0,2)$ on M, then there exists a unique smooth tensor field $\check{F}$:

$(M) \to (M)$ of type $(1,1)$ such that $F(X,Y) = g(\breve{F}(X),Y)$ for each $X,Y \in \mathfrak{X}(M)$. Indeed, for each $X \in \mathfrak{X}(M)$ we define $\omega_X : \mathfrak{X}(M) \to C^\infty(M)$ by $\omega_X(Y) = F(X,Y)$. Then let $\breve{F} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be the map

$$\breve{F}(X) = \omega_X^\sharp := \sharp(\omega_X) = \widehat{g}^{-1}(\omega_X). \tag{2.10}$$

Notice that

$$g(\breve{F}(X),Y) = g(\omega_X^\sharp, Y) = \widehat{g}(\omega_X^\sharp)(Y) = \omega_X(Y) = F(X,Y)$$

Therefore the mapping

$$\begin{gathered} \Gamma(T^{(1,1)}(M)) \to \Gamma(T^{(0,2)}(M)) \\ F \mapsto \widehat{F} \\ \breve{F} \leftarrow\!\shortmid F \end{gathered} \tag{2.11}$$

defines an isomorphism between the tensor fields of type $(0,2)$ on M and the tensor fields of type $(1,1)$ on M. In particular we shall use the trace of the endomorphism field $\breve{F}$ to define that of the $(0,2)$-tensor field F:

$$\operatorname{tr}_g(F) := \operatorname{tr}(\breve{F}) \in C^\infty(M) \tag{2.12}$$

The trace of a linear endomorphism $A : V \to V$ on a finite-dimensional vector space is invariant under the choice of change of basis, so for an orthonormal basis of V, any vector is written as $v = \sum_i \langle v, e_i \rangle e_i$ and thus $\operatorname{tr}(A) =$ sum of diagonal elements of $\begin{bmatrix} | & & | \\ A(e_1) & \cdots & A(e_n) \\ | & & | \end{bmatrix} = \sum_i^n \langle A(e_i), e_i \rangle$. Thus, for a local orthonormal frame $(E_1, \cdots, E_n)$ on (M,g), we can write

$$\operatorname{tr}_g(F) := \operatorname{tr}(\breve{F}) = \sum_{i=1}^n g(\breve{F}(E_i), E_i) = \sum_{i=1}^n F(E_i, E_i) = F_{ii} \tag{2.13}$$

In general, for any local frame $(E_1, \cdots, E_n)$ on (M,g),

$$\operatorname{tr}_g(F) := \operatorname{tr}(\breve{F}) = \sum_{i=1}^n \underbrace{(\breve{F}(E_i))_i}_{i\text{-th coeff of vector}} = \sum_{i=1}^n (\omega_{E_i}^\sharp)_i \overset{(2.7),(2.8)}{=\!=\!=} \sum_{i=1}^n g^{ij} \underbrace{(\omega_{E_i})_j}_{=\omega_{E_i}(E_j) = F(E_i,E_j) = F_{ij}} = g^{ij} F_{ij}. \tag{2.14}$$

When $g = (\delta_{ij})$, (2.14) recovers (2.13). Note that coefficients like F_{ij} and F_{ii} are always understood under a certain choice of basis.

2.3.2 Inner Products of Tensors

A Riemannian metric yields, by definition, an inner product on tangent vectors at each point. Because of the musical isomorphisms between vectors and covectors, it is easy to carry the inner product over to covectors as well.

Suppose g is a Riemannian metric on M, and $x \in M$. We can define an inner product on the cotangent space $T_x^* M$ by

$$\langle \omega, \eta \rangle_g = \left\langle \omega^\sharp, \eta^\sharp \right\rangle_g. \tag{2.15}$$

(Just as with inner products of vectors, we might sometimes omit g from the notation when the metric is understood.) To see how to compute this, we just use the basis formula (2.8) for the sharp operator, together with the relation $g_{kl} g^{ki} = g_{lk} g^{ki} = \delta_l^i$, to obtain

$$\begin{aligned} \langle \omega, \eta \rangle &= \left\langle g^{ki} \omega_i E_k, g^{lj} \eta_j E_l \right\rangle \\ &= \langle E_k, E_l \rangle \left(g^{ki} \omega_i\right) \left(g^{lj} \eta_j\right) \\ &= g_{kl} \left(g^{ki} \omega_i\right) \left(g^{lj} \eta_j\right) \\ &= \delta_l^i g^{lj} \omega_i \eta_j \\ &= g^{ij} \omega_i \eta_j. \end{aligned} \tag{2.16}$$

In other words, the inner product on covectors is represented by the inverse matrix $\left(g^{ij}\right)$. Using our convention (2.8), this can also be written

$$\langle\omega,\eta\rangle=\omega_i\eta^i=\omega^j\eta_j.$$

Exercise 2.3.4. *Let (M,g) be a Riemannian manifold with or without boundary, let (E_i) be a local frame for M, and let $\left(\varepsilon^i\right)$ be its dual coframe. Show that the following are equivalent:*

(a) (E_i) *is orthonormal.*

(b) $\left(\varepsilon^i\right)$ *is orthonormal.*

(c) $\left(\varepsilon^i\right)^\sharp=E_i$ *for each i.*

This construction can be extended to tensor bundles of any rank, as the following proposition shows. First a bit of terminology: if $E\to M$ is a smooth vector bundle, a **smooth fiber metric on** E is an inner product on each fiber E_p that varies smoothly, in the sense that for any (local) smooth sections σ,τ of E, the inner product $\langle\sigma,\tau\rangle$ is a smooth function.

Proposition 2.3.5 (Inner Products of Tensors)**.** *Let (M,g) be an n-dimensional Riemannian manifold with or without boundary. There is a unique smooth fiber metric on each tensor bundle $T^{(k,l)}TM$ with the property that if $\alpha_1,\ldots,\alpha_{k+l}$, $\beta_1,\ldots,\beta_{k+l}$ are vector or covector fields as appropriate, then*

$$\left\langle\alpha_1\otimes\cdots\otimes\alpha_{k+l},\beta_1\otimes\cdots\otimes\beta_{k+l}\right\rangle=\left\langle\alpha_1,\beta_1\right\rangle\cdot\cdots\cdot\left\langle\alpha_{k+l},\beta_{k+l}\right\rangle. \tag{2.17}$$

With this inner product, if $(E_1,\ldots,E_n)$ is a local orthonormal frame for TM and $\left(\varepsilon^1,\ldots,\varepsilon^n\right)$ is the corresponding dual coframe, then the collection of tensor fields $E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes\varepsilon^{j_1}\otimes\cdots\otimes\varepsilon^{j_l}$ as all the indices range from 1 to n forms a local orthonormal frame for $T^{(k,l)}TM$. In terms of any (not necessarily orthonormal) frame, this fiber metric satisfies

$$\langle F,G\rangle=g_{i_1r_1}\cdots g_{i_kr_k}g^{j_1s_1}\cdots g^{j_ls_l}F^{i_1\ldots i_k}_{j_1\ldots j_l}G^{r_1\ldots r_k}_{s_1\ldots s_l}. \tag{2.18}$$

If F and G are both covariant, this can be written

$$\langle F,G\rangle=F_{j_1\ldots j_l}G^{j_1\cdots j_l},\text{ where }G^{j_1\cdots j_l}=g^{j_1s_1}\ldots g^{j_ls_l}G_{s_1\ldots s_l} \tag{2.19}$$

represents the components of G with all of its indices raised (see Definition 2.3.2).

Proof. Problem 2.6.11. ∎

2.3.3 Integration of Riemannian Volume Form

Let (M,g) be a Riemannian manifold and for $p\in M$, let $(U,\varphi=(x_i))$ be a coordinate chart. $\{\partial x_i\}$ is then a smooth local frame defined on U. Apply Gram-Schmidt orthogonalization process we can obtain a smooth orthonormal frame $\{E_i\}$ defined on U. If (M,g) in endowed with an orientation, by replacing E_i with $-E_i$ if ncessary, we can find an oriented orthonormal frame in U. Thus, every point has a neighborhood on which we can find an oriented orthonormal frame.

Proposition 2.3.6 (Riemannian Volume Form)**.** *Suppose (M,g) is an oriented Riemannian n-manifold with or without boundary, and $n\geq 1$. There is a unique smooth orientation form $\omega_g\in\Omega^n(M)$, called the* ***Riemannian volume form****, that satisfies*

$$\omega_g\left(E_1,\ldots,E_n\right)=1 \tag{2.20}$$

for every local oriented orthonormal frame (E_i) for M.

Proof. Suppose first that such a form ω_g exists. If $(E_1, \ldots, E_n)$ is any local oriented orthonormal frame on an open subset $U \subseteq M$ and $\left(\varepsilon^1, \ldots, \varepsilon^n\right)$ is the dual coframe, we can write $\omega_g = f\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ on U. The condition (2.20) then reduces to $f = 1$, so

$$\omega_g = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n \tag{2.21}$$

This proves that such a form is uniquely determined.

To prove existence, we would like to define ω_g in a neighborhood of each point by (2.21), so we need to check that this definition is independent of the choice of oriented orthonormal frame. If $\left(\tilde{E}_1, \ldots, \tilde{E}_n\right)$ is another oriented orthonormal frame, with dual coframe $\left(\widetilde{\varepsilon}^1, \ldots, \widetilde{\varepsilon}^n\right)$, let

$$\widetilde{\omega}_g = \widetilde{\varepsilon}^1 \wedge \cdots \wedge \widetilde{\varepsilon}^n.$$

We can write

$$\widetilde{E}_i = A_i^j E_j$$

for some matrix $\left(A_i^j\right)$ of smooth functions. The fact that both frames are orthonormal means that $\left(A_i^j(p)\right) \in \mathrm{O}(n)$ for each p, so $\det\left(A_i^j\right) = \pm 1$, and the fact that the two frames are consistently oriented forces the positive sign. We compute

$$\omega_g\left(\tilde{E}_1, \ldots, \tilde{E}_n\right) = \det\left(\varepsilon^j\left(\tilde{E}_i\right)\right) = \det\left(A_i^j\right) = 1 = \tilde{\omega}_g\left(\tilde{E}_1, \ldots, \tilde{E}_n\right).$$

Thus $\omega_g = \tilde{\omega}_g$, so defining ω_g in a neighborhood of each point by (2.21) with respect to some smooth oriented orthonormal frame yields a global n-form. The resulting form is clearly smooth and satisfies (2.20) for every oriented orthonormal frame. ■

Lemma 2.3.7. *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are positive-dimensional oriented Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is an orientation-preserving local isometry. Show that $F^*\omega_{\tilde{g}} = \omega_g$.*

Proof. Let (E_i) be a local oriented orthonormal frame on M. Then $(\tilde{E}_i) = (F_* E_i)$ gives a basis for each $T_q\tilde{M}$, $q \in F(U)$ (dF_p is an isomorphism for each $p \in U$) and local isometry condition keeps the orthonormality of $(\tilde{E}_i)$ on $F(U)$. It is also oriented with respect to the given orientation of $\tilde{M}$, so $(\tilde{E}_i)$ on $F(U)$ is a local oriented orthonormal frame on $\tilde{M}$. Now,

$$1 = \omega_{\tilde{g}}(\tilde{E}_1, \cdots, \tilde{E}_n) = \omega_{\tilde{g}}(F_* E_1, \cdots, F_* E_n) = F^*\omega_{\tilde{g}}(E_1, \cdots, E_n),$$

which by uniquness of the Riemannian volume form implies that $F^*\omega_{\tilde{g}} = \omega_g$. ■

Although the expression for the Riemannian volume form with respect to an oriented orthonormal frame is particularly simple, it is also useful to have an expression for it in coordinates.

Proposition 2.3.8. *Let (M, g) be an oriented Riemannian n-manifold with or without boundary, $n \geq 1$. In any oriented smooth coordinates (x^i), the Riemannian volume form has the local coordinate expression*

$$\omega_g = \sqrt{\det\left(g_{ij}\right)}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

where g_{ij} are the components of g in these coordinates.

Proof. Let $\left(U, \left(x^i\right)\right)$ be an oriented smooth chart, and let $p \in M$. In these coordinates, $\omega_g = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ for some positive coefficient function f. To compute f, let (E_i) be any smooth oriented orthonormal frame defined on a neighborhood of p, and let $\left(\varepsilon^i\right)$ be the dual coframe. If we write the coordinate frame in terms of the orthonormal frame as

$$\frac{\partial}{\partial x^i}=A_i^j E_j$$

then we can compute

$$f=\omega_g\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)=\varepsilon^1 \wedge \cdots \wedge \varepsilon^n\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)=\det\left(\varepsilon^j\left(\frac{\partial}{\partial x^i}\right)\right)=\det\left(A_i^j\right).$$

On the other hand, observe that

$$g_{ij}=\left\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle_g=\left\langle A_i^k E_k, A_j^l E_l\right\rangle_g=A_i^k A_j^l\left\langle E_k, E_l\right\rangle_g=\sum_k A_i^k A_j^k$$

This last expression is the (i, j)-entry of the matrix product $A^T A$, where $A=\left(A_i^j\right)$. Thus,

$$\det\left(g_{ij}\right)=\det\left(A^T A\right)=\det A^T \det A=(\det A)^2$$

from which it follows that $f=\det A=\pm\sqrt{\det\left(g_{ij}\right)}$, where we also note that $\left(g_{ij}\right)$ is positive-definite and has positive determinant. Since both frames $\left(\partial / \partial x^i\right)$ and $\left(E_j\right)$ are oriented, the sign must be positive. ■

Suppose (M, g) is an oriented Riemannian manifold with or without boundary, and let ω_g denote its Riemannian volume form. If f is a compactly supported continuous real-valued function on M, then $f \omega_g$ is a compactly supported n-form, so we can define the **integral of** f **over** M to be $\int_M f \omega_g$. If M itself is compact, we define the **volume of** M by $\operatorname{Vol}(\boldsymbol{M})=\int_{\boldsymbol{M}} \omega_g$.

Because of these definitions, the Riemannian volume form is often denoted by $\mathrm{d} V_g$ (or $\mathrm{d} A_g$ or $\mathrm{d} s_g$ in the 2-dimensional or 1-dimensional case, respectively). Then the integral of f over M is written $\int_M f \mathrm{~d} V_g$, and the volume of M as $\int_M \mathrm{~d} V_g$. Be warned, however, that this notation is not meant to imply that the volume form is the exterior derivative of an $(n-1)$-form; in fact, as we will see when we study de Rham cohomology, this is never the case on a compact manifold.

Proposition 2.3.9. *Let (M, g) be a nonempty oriented Riemannian manifold with or without boundary, and suppose f is a compactly supported continuous real-valued function on M. Then*

(a) If $f \geq 0$, then $\int_M f \mathrm{~d} V_g \geq 0$, with equality if and only if $f \equiv 0$.

(b) $\left|\int_M f \mathrm{~d} V_g\right| \leq \int_M|f| \mathrm{~d} V_g$.

Proof. See [12, Proposition 16.8-9]. For part (a), evoking Proposition 1.7.9 (c) should suffice. ■

2.3.4 Integration of Riemannian Density

Proposition 2.3.10 (The Riemannian Density). *If (M, g) is any Riemannian manifold, then there is a unique smooth positive density μ on M, called the **Riemannian density**, with the property that*

$$\mu\left(E_1, \ldots, E_n\right)=1 \tag{2.22}$$

for every local orthonormal frame $\left(E_i\right)$.

Proof. Uniqueness is immediate, because any two densities that agree on a basis must be equal. Given any point $p \in M$, let U be a connected smooth coordinate neighborhood of p. Since U is diffeomorphic to an open subset of Euclidean space, it is orientable. Any choice of orientation of U uniquely determines a Riemannian volume form ω_g on U, with the property that $\omega_g\left(E_1, \ldots, E_n\right)=1$ for any oriented orthonormal frame. If we put $\mu_g=\left|\omega_g\right|$, it follows easily that μ_g is a smooth positive density on U satisfying (2.22). If U and V are two overlapping smooth coordinate neighborhoods, the two definitions of μ_g agree where they overlap by uniqueness, so this defines μ_g globally. ■

As analogues of Lemma 2.3.7 and Proposition 2.3.8, we have the following results.

Exercise 2.3.11. *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is a local isometry. Show that $F^*\mu_{\widetilde{g}} = \mu_g$.*

Solution. The proof is similar to that of Lemma 2.3.7: use the local isometry to transfer a local orthonormal frame (E_i) on M to a local orthonormal frame $(\tilde{E}_i)$ on $\widetilde{M}$, and then compute to verify the defining property of the Riemannian density (2.22) to evoke uniqueness of the Riemannian density. ♦

Exercise 2.3.12. *Let (M, g) be a Riemannian manifold with or without boundary, and let (x^i) be any smooth coordinates on M. Show that the Riemannian density μ_g has the local coordinate expression*

$$\mu_g = \sqrt{\det(g_{ij})}|\ \mathrm{d}x^1 \wedge \cdots \wedge\ \mathrm{d}x^n|.$$

Solution. Let $\mu_g = u|\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|$. Pick a local orthonormal frame (E_i) on M and write $\partial_{x_i} = A_i^j E_j$. Then $u = \mu_g(\partial_{x_1}, \cdots, \partial_{x_n}) = |\det(A)|$. Since $g_{ij} = (A^T A)_{ij}$, we have $\det(g_{ij}) = (\det(A))^2$, $\det(A) = \pm\sqrt{\det(g_{ij})}$, and $u = |\det(A)| = \sqrt{\det(g_{ij})}$. Thus, $\mu_g = \sqrt{\det(g_{ij})}|\ \mathrm{d}x^1 \wedge \cdots \wedge\ \mathrm{d}x^n|$. ♦

Exercise 2.3.13. *Let (M, g) be an oriented Riemannian manifold with or without boundary and let ω_g be its Riemannian volume form.*

(a) Show that the Riemannian density of M is given by $\mu_g = |\omega_g|$.

(b) For any compactly supported continuous function $f : M \to \mathbb{R}$, show that

$$\int_M f\mu_g = \int_M f\omega_g$$

Solution. Use Exercise 2.3.12 and notice that the definition of integration of density traces back to equation (1.12). ♦

Similar to Proposition 2.3.9, by evoking Proposition 1.7.17 (b), we obtain

Proposition 2.3.14. *Let (M, g) be a Riemannian manifold with or without boundary with its Riemannian density μ_g. Let f be a compactly supported continuous real-valued function on M. If $f \geq 0$, then $\int_M f\,\mu_g \geq 0$, with equality if and only if $f \equiv 0$.*

Because of Exercise 2.3.13 (b), it is customary to denote the Riemannian density simply by $\mathrm{d}V_g$, and to specify when necessary whether the notation refers to a density or a form. If $f : M \to \mathbb{R}$ is a compactly supported continuous function, the **integral of f over M** is defined to be $\int_M f\ \mathrm{d}V_g$. Exercise 2.3.13 shows that when M is oriented, it does not matter whether we interpret dV_g as the Riemannian volume form or the Riemannian density. (If the orientation of M is changed, then both the integral and $\mathrm{d}V_g$ change signs, so the result is the same.) When M is not orientable, however, we have no choice but to interpret it as a density.

2.3.5 Riemannian Measure

Let (M, g) be a Riemannian manifold. If M is oriented, $\mathrm{d}V_g$ is interpreted as the Riemannian volume form; otherwise it is just the Riemannian density. Now, Propositions 2.3.9 and 2.3.14 reveal that the mapping

$$\begin{aligned} \Lambda_g : C_c(M) &\longrightarrow \mathbb{R} \\ f &\mapsto \int_M f\ \mathrm{d}V_g \end{aligned}$$

is a **positive linear functional**. It turns out that this "integration" we defined indeed matches up with an integral with respect to a measure. This is the content of the Riesz-Markov-Kakutani (RMK) representation theorem. We recall some concepts in measure theory following Folland's [2].

A topological space X is called **locally compact** if every point in X has a compact neighborhood. X is called a **Hausdorff** if every pair of distinct points in X have disjoint neighborhoods. A useful consequence is that compact sets are closed in Hausdorff spaces. A topological space X is called **σ-compact** if it can be written as a countable union of compact subsets. A measure μ is called **σ-finite** if space X can be written as a countable union of finite-measure sets.

Let X be a **locally compact Hausdorff (LCH)** space. A **Borel measure** is any measure μ defined on the σ-algebra $\mathcal{B}$ generated by the topology of X, called **Borel σ-algebra**. A Borel-measurable set E, i.e., $E \in \mathcal{B}$, is called a **Borel set**.

Let μ be a Borel measure on the LCH X and E a Borel subset of X. The measure μ is called **outer regular on E** if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open}\}$$

and **inner regular on E** if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$$

If μ is outer and inner regular on all Borel sets, μ is called **regular**. It turns out that regularity is a bit too much to ask for when X is not σ-compact, so we adopt the following definition. A **Radon measure on X** is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Proposition 2.3.15. *A Radon measure is inner regular on all of its σ-finite sets.*

Corollary 2.3.16.

1. *If a Radon measure is σ-finite then it is regular.*
2. *If X is σ-compact, every Radon measure on X is regular.*

The proof of the proposition (see section 7.2 of [2]) involves using the inner and outer regularity properties of the Radon measure to approximate the measure of σ-finite sets of finite measure, and for sets of infinite measure to apply that technique to a sequence of sets, each of which has finite measure. From this, the corollary follows directly from the definitions of σ-finiteness and σ-compactness.

Note that some authors define a Radon measure μ on the Borel σ-algebra of any Hausdorff space to be any Borel measure that is inner regular on open sets and **locally finite**, meaning that for every point $x \in X$ there is an open neighborhood of x with finite measure. For Hausdorff spaces, this implies that $\mu(C) < \infty$ for every compact set C; and for locally compact Hausdorff spaces, the two conditions are equivalent.

One further bit of notation: If U is open in X and $f \in C_c(X)$, we shall write

$$f \prec U$$

to mean that $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subset U$. (This is slightly stronger than the condition $0 \leq f \leq \chi_U$, which implies only that $\operatorname{supp}(f) \subset \bar{U}$.)

Theorem 2.3.17 (Riesz-Markov-Kakutani Representation Theorem)**.** *Let X be a **locally compact Hausdorff (LCH)** space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a unique Radon measure μ on σ-algebra $\mathcal{E}$ of X that represents Λ in following sense:*

$$\Lambda f = \int_X f \, \mathrm{d}\mu \quad \text{for every } f \in C_c(X).$$

Moreover, μ satisfies the following properties:

(a) $\mu(U) = \sup\{I(f) : f \in C_c(X), f \prec U\}$ *for all open* $U \subset X$.

(b) $\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\}$ *for all compact* $K \subset X$.

(d) μ *is such that the measure space* $(X, \mathcal{E}, \mu)$ *is* ***complete****: if* $E \in \mathcal{E}, A \subset E$*, and* $\mu(E) = 0$*, then* $A \in \mathcal{E}$.

Proof. The proof is very involved. See section 7.2 of [2], Theorem 2.14 of Rudin's *Real and Complex Analysis*, or Martikainen's class note. There is also a nice REU paper blending with elements in functional analysis to prove it. ■

For any topological manifold M, it is by definition Hausdorff and locally compact by [11, Proposition 1.12]. Furthermore, it is also σ-compact.

Proposition 2.3.18. *Every topological manifold* M *is* σ*-compact.*

Proof. By [11, Lemma 1.10], it has a countable basis of precompact coordinate balls. Therefore, there is a countable set of precompact coordinate balls which cover X, and the set of closure of these balls is the set of countably many compact subspaces which cover X. Thus, X is σ-compact. ■

Now, for Riemannian manifold (M, g), we can use RMK representation theorem to induce a Radon measure that we will denote by μ_g (recall $\mathrm{d}V_g$ denotes the Riemannian density/volume form). We call μ_g the **Riemannian measure**. Above proposition combines with Corollary 2.3.16 (b) to make this μ_g a regular Borel measure on M. One use standard machinery developed in real analysis to define integrable functions.

A function $F : M \to \dot{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called a **lower semicontinuous function** if $p_n \to p$ implies $\liminf F(p_n) \geq F(p)$. For a lower semicontinuous function $F \geq 0$ on M, let us define

$$\bar{\Lambda}_g(F) = \sup\{\Lambda_g(f) \mid f \in C_0(M) \text{ satisfies } f \leq F\}$$

and for any function $f \geq 0$ on M, let us define

$$\bar{\Lambda}_g(f) = \inf\left\{\bar{\Lambda}_g(F) \mid F \geq f \text{ is lower semicontinuous}\right\}.$$

A function f on M is said to be **integrable** if there exists a sequence $\{f_n\} \subset C_0(M)$ such that $\bar{\Lambda}_g(|f - f_n|) \to 0$. Then $\{\Lambda_g(f_n)\}$ is a convergent sequence, and its limit does not depend on the choice of $\{f_n\}$. We denote the limit by $\int_M f \,\mathrm{d}\mu_g$ and we call it the **integral of** f. In particular, any $f \in C_0(M)$ is integrable and its integral is just the above defined $\Lambda_g(f)$. A subset $K \subset M$ is said to be **integrable** if its characteristic function $\chi_K : M \to \mathbb{R}$ is integrable. In this case, $\int_M \chi_K \,\mathrm{d}\mu_g$ [also denoted by $\int_K \mathrm{d}\mu_g$ or $\mathrm{Vol}(K, g)$] is called the n**-dimensional volume of** K. Any compact (or more generally, precompact) subset K of M is integrable with $\mathrm{Vol}(K, g) < +\infty$. The following approximation theorem for Radon measure is useful.

Theorem 2.3.19. *Let* X *be a LCH space. If* μ *is a Radon measure on* $X, C_c(X)$ *is dense in* $L^p(\mu)$ *for* $1 \leq p < \infty$

Proof. See [2, Proposition 7.9]. ■

For any $1 \leq p < \infty$ we define the L^p norm on $C_c(M)$ via

$$\|f\|_{L^p} := \left(\int_M |f|^p \,\mathrm{d}\mu_g\right)^{1/p}$$

and define $L^p(M, g)$ to be the completion of $C_c(M)$ under the L^p norm. Similarly one can define $L^\infty(M, g)$. It is not hard to extend to the theory to complex-valued functions. In the special case $p = 2$, one can define an inner product structure on $L^2(M, g)$ by

$$\langle f_1, f_2 \rangle_{L^2} := \int_M f_1 \bar{f}_2 \,\mathrm{d}\mu_g$$

which make $L^2(M,g)$ into a Hilbert space.

We will come back to the analysis on Riemannian manifolds in Chapter 4. Before that we need to introduce several basic aspects of Riemannian manifolds, namely connection, geodesic, and curvature. Nevertheless, there are some differential operators we can study.

2.3.6 Some Differential Operators on Riemannian Manifolds

Let (M,g) be an n-dimensional Riemannian manifold. We define some some canonical linear differential operators in this subsection. The references we use here are [1], [7], and this note.

I. The Gradient Operator

We use the sharp operator to extend the classical gradient operator to Riemannian manifolds. Let $f: M \to \mathbb{R}$ be a smooth function, the **gradient of** f is the vector field $\operatorname{grad} f = (\mathrm{d}f)^\sharp = \hat{g}^{-1}(\mathrm{d}f)$ obtained from $\mathrm{d}f$ by raising an index. Unwinding the definitions, we see that $\operatorname{grad} f$ is characterized by the following equation

$$\mathrm{d}f(X) = \langle \operatorname{grad} f, X\rangle \qquad \text{for all } X \in \mathfrak{X}(M). \tag{2.23}$$

That's because $g(\operatorname{grad} f, X) = \widehat{g}(\operatorname{grad} f)(X) = \widehat{g}\left(\widehat{g}^{-1}(\mathrm{d}f)\right)(X) = \mathrm{d}f(X)$. The gradient has the local expression

$$\operatorname{grad} f \xlongequal{(2.7)} \left(g^{ij}(\mathrm{d}f)_i\right) E_j \xlongequal{(\mathrm{d}f)_i = (\mathrm{d}f)(E_i) \overset{[11]14.24}{=} E_i f} \left(g^{ij} E_i f\right) E_j. \tag{2.24}$$

Thus if (E_i) is an orthonormal frame (then $(g_{ij}) = I \implies (g^{ij}) = I^{-1} = I$), then $\operatorname{grad} f$ is the vector field whose components are the same as the components of $\mathrm{d}f$; but in other frames, this will not be the case.

If h is another smooth function on M, then

$$(\operatorname{grad} f)(h) = \mathrm{d}h(\operatorname{grad} f) = \langle \operatorname{grad} g, \operatorname{grad} f\rangle \xlongequal{(2.15)} \langle \mathrm{d}h, \mathrm{d}f\rangle.$$

which under a local frame writes as

$$(\operatorname{grad} f)(h) = g^{ij}(\mathrm{d}h)_i(\mathrm{d}f)_j$$

by (2.16). If we choose the frame to be the coordinate frame, then

$$(\operatorname{grad} f)(h) = g^{ij}\frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_j} \implies (\operatorname{grad} f) \xlongequal{g \text{ symmetric}} g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}.$$

The gradient operator $\operatorname{grad}: C^\infty(M) \to \mathfrak{X}(M)$ is linear and satisfies product rule: $\forall f_1, f_2 \in C^\infty(M)$,

$$\begin{aligned} \operatorname{grad}(f_1 + f_2) &= \operatorname{grad}(f_1) + \operatorname{grad}(f_2) \\ \operatorname{grad}(f_1 f_2) &= f_2(\operatorname{grad}(f_1)) + f_1(\operatorname{grad}(f_2)) \end{aligned} \tag{2.25}$$

If $\Phi: (M,g) \to (\widetilde{M}, \tilde{g})$ is an isometry, then

$$\Phi_*\left(\operatorname{grad}_g\left(\Phi^*(\tilde{f})\right)\right) = \operatorname{grad}_{\tilde{g}}(\tilde{f}) \tag{2.26}$$

for any $\tilde{f} \in C^\infty(\widetilde{M})$. Indeed, Φ is a diffeomorphism, so Definition 1.2.11 of isomorphisms Φ_*, Φ^* ensure that any $Y \in \mathfrak{X}(\widetilde{M})$ is some $\Phi_*(X)$ and give the following computation:

$$\begin{aligned} &\tilde{g}\left(\Phi_*\left(\operatorname{grad}_g\left(\Phi^*(\tilde{f})\right)\right), \Phi_*(X)\right) \circ \Phi^{-1} = \Phi^*\tilde{g}\left(\operatorname{grad}_g\left(\Phi^*(\tilde{f})\right), X\right) = g\left(\operatorname{grad}_g\left(\Phi^*(\tilde{f})\right), X\right) \circ \Phi^{-1} \\ &= \underbrace{X\ \underbrace{\left(\Phi^*(\tilde{f})\right)}_{=\tilde{f}\circ\Phi: M \to \mathbb{R}}}_{M \to \mathbb{R}} \circ \Phi^{-1} = \underbrace{(\underbrace{[\Phi_*(X)]}_{=Y \in \mathfrak{X}(\widetilde{M})}(\tilde{f}) \circ \Phi)}_{\widetilde{M} \to \mathbb{R}} \circ \Phi^{-1} = [\Phi_*(X)](\tilde{f}) = \tilde{g}\left(\operatorname{grad}_{\tilde{g}}(\tilde{f}), \Phi_*(X)\right), \end{aligned}$$

Moreover, note that (2.26) implies

$$g\left(\operatorname{grad}_g\left(\Phi^*\left(\tilde{f}_1\right)\right), \operatorname{grad}_g\left(\Phi^*\left(\tilde{f}_2\right)\right)\right)=\tilde{g}\left(\operatorname{grad}_{\tilde{g}}\left(\tilde{f}_1\right), \operatorname{grad}_{\tilde{g}}\left(\tilde{f}_2\right)\right) \circ \Phi \tag{2.27}$$

for any $\tilde{f}_1, \tilde{f}_2 \in C^{\infty}(\widetilde{M})$. Indeed, taking into account the definitions of the gradient of $\Phi^*\left(\tilde{f}_2\right)$ (resp. $\tilde{f}_2$) and of Φ^*, one gets

$$\begin{aligned} & g\left(\operatorname{grad}_g\left(\Phi^*\left(\tilde{f}_1\right)\right), \operatorname{grad}_g\left(\Phi^*\left(\tilde{f}_2\right)\right)\right)=\left[\operatorname{grad}_g\left(\Phi^*\left(\tilde{f}_1\right)\right)\right]\left(\Phi^*\left(\tilde{f}_2\right)\right) \\ = & \left[\Phi_*\left(\operatorname{grad}_g\left(\Phi^*\left(\tilde{f}_1\right)\right)\right)\right]\left(\tilde{f}_2\right) \circ \Phi=\left[\operatorname{grad}_{\tilde{g}}\left(\tilde{f}_1\right)\right]\left(\tilde{f}_2\right) \circ \Phi=\tilde{g}\left(\operatorname{grad}_{\tilde{g}}\left(\tilde{f}_1\right), \operatorname{grad}_{\tilde{g}}\left(\tilde{f}_2\right)\right) \circ \Phi\end{aligned}$$

as claimed.

The next proposition shows that the gradient has the same geometric interpretation on a Riemannian manifold as it does in Euclidean space. If f is a smooth real-valued function on a smooth manifold M, recall that a point $p \in M$ is called a **regular point of** f if $\mathrm{d} f_p \neq 0$, and a **critical point of** f otherwise; and a level set $f^{-1}(c)$ is called a **regular level set** if every point of $f^{-1}(c)$ is a regular point of f. [12] Corollary A.26 shows that each regular level set is an embedded smooth hypersurface in M.

Proposition 2.3.20. *Suppose (M, g) is a Riemannian manifold, $f \in C^{\infty}(M)$, and $\mathcal{R} \subseteq M$ is the set of regular points of f. For each $c \in \mathbb{R}$, the set $M_c=f^{-1}(c) \cap \mathcal{R}$, if nonempty, is an embedded smooth hypersurface in M, and grad f is everywhere normal to M_c.*

Proof. Problem 2.6.9. ■

II. The Divergence Operator

Suppose (M, g) is an oriented Riemannian n-manifold with or without boundary, and $\mathrm{d} V_g$ is its volume form. If X is a smooth vector field on M, then $X \lrcorner \mathrm{d} V_g$ is an $(n-1)$-form. The exterior derivative of this $(n-1)$-form is a smooth n-form, so it can be expressed as a smooth function multiplied by $\mathrm{d} V_g$. That function is called the **divergence of** X, and denoted by $\operatorname{div} X$; thus it is characterized by the following formula:

$$\mathrm{d}\left(X \lrcorner \mathrm{d} V_g\right)=(\operatorname{div} X) \mathrm{d} V_g . \tag{2.28}$$

Even if M is nonorientable, in a neighborhood of each point we can choose an orientation and define the divergence by (2.28), and then note that reversing the orientation changes the sign of $\mathrm{d} V_g$ on both sides of the equation, so $\operatorname{div} X$ is well defined, independently of the choice of orientation. In this way, we can define the divergence operator on any Riemannian manifold with or without boundary, by requiring that it satisfy (2.28) for any choice of orientation in a neighborhood of each point.

The next theorem is a fundamental result about vector fields on Riemannian manifolds. In the special case of a compact regular domain in $\mathbb{R}^3$, it is often referred to as **Gauss's theorem**.

Theorem 2.3.21 (The Divergence Theorem). *Let (M, g) be an Riemannian manifold with boundary $S=\partial M$ (both M and S can be nonorientable; unlike the case in Theorem 1.6.14). [11, Theorem 5.11] is still valid in this case, so S with subspace topology has a smooth structure making it an embedded hypersurface in M inheriting a metric $\tilde{g}=\iota_S^* g$. Then, for any compactly supported smooth vector field X on M,*

$$\int_M(\operatorname{div} X) \mathrm{d} V_g=\int_{\partial M}\langle X, N\rangle_g \mathrm{d} V_{\tilde{g}}$$

where N is the outward-pointing unit normal vector field along ∂M given by Proposition 2.2.6 (which does not require the ambient Riemannian manifold M to be oriented).

Proof. One need to first prove the orientable case [11, Theorem 16.32] and then use the orientation covering $\hat{\pi}: \widehat{M} \rightarrow M$ ot show the nonorientable case [11, Theorem 16.48]. ■

The term "divergence" is used because of the following geometric interpretation. A smooth flow θ on M is said to be volume-preserving if for every compact regular domain D, we have $\mathrm{Vol}(\theta_t(D)) = \mathrm{Vol}(D)$ whenever the domain of θ_t contains D. It is called **volume-increasing**, **volume-decreasing**, **volume-nonincreasing**, or **volume-nondecreasing** if for every such D, $\mathrm{Vol}(\theta_t(D))$ is **strictly increasing**, **strictly decreasing**, **nonincreasing**, or **nondecreasing**, respectively, as a function of t. Note that the properties of flow domains ensure that if D is contained in the domain of θ_t for some t, then the same is true for all times between 0 and t.

The next proposition shows that the divergence of a vector field can be interpreted as a measure of the tendency of its flow to "spread out," or diverge (see Fig. 2.4).

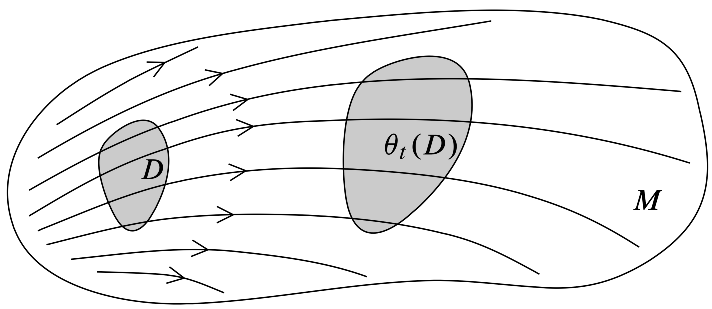

Figure 2.4: Geometric interpretation of divergence.

Proposition 2.3.22 (Geometric Interpretation of the Divergence). *Let M be an oriented Riemannian manifold, let $X \in \mathfrak{X}(M)$, and let θ be the flow of X. Then θ is*

(a) volume-preserving if and only if $\operatorname{div} X = 0$ everywhere on M.

(b) volume-nondecreasing if and only if $\operatorname{div} X \geq 0$ everywhere on M.

(c) volume-nonincreasing if and only if $\operatorname{div} X \leq 0$ everywhere on M.

(d) volume-increasing if and only if $\operatorname{div} X > 0$ on a dense subset of M.

(e) volume-decreasing if and only if $\operatorname{div} X < 0$ on a dense subset of M.

Proof. See [11, Proposition 16.33]. Note that this theorem cannot be easily generalized to the nonorientable case. We need $\mathrm{d}V_g$ to be a differential form to use Cartan's magic formula, even though some results employed in the proof have analogues in the nonorientable case. ■

In any smooth local coordinates (x^i), we can express the divergence operator as

$$\operatorname{div}\left(X^i \frac{\partial}{\partial x^i}\right) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det g}\right) \tag{2.29}$$

where $\det g = \det(g_{kl})$ is the determinant of the component matrix of g in these coordinates. Indeed,

$$\begin{aligned}
\mathrm{d}(X \lrcorner\, \mathrm{d}V_g) &= \mathrm{d}\left(X \lrcorner \left(\sqrt{\det(g)}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right)\right) \\
&\overset{\text{Prop.1.4.8}}{=\!=\!=} \mathrm{d}\left(\sqrt{\det(g)} \sum_{i=1}^{n} (-1)^{i-1} \underbrace{\mathrm{d}x^i(X)}_{=X^i}\, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n\right) \\
&\overset{(1.10)}{=\!=} \sum_{i=1}^{n} (-1)^{i-1} \underbrace{\mathrm{d}\left(X^i \sqrt{\det(g)}\right)}_{=\sum_j \frac{\partial}{\partial x^j}\left(X^j\sqrt{\det(g)}\right)\mathrm{d}x^j} \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\left(X^j \sqrt{\det(g)}\right) \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n.
\end{aligned}$$

For one-forms, $\alpha \wedge \beta = -\beta \wedge \alpha$ and $\alpha \wedge \alpha = 0$, so above can be written as

$$\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det(g)}\right) \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$$

By $\alpha \wedge \beta = -\beta \wedge \alpha$ again, we have

$$\begin{aligned}
\mathrm{d}(X \lrcorner\, \mathrm{d}V_g) &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det(g)}\right) (-1)^{i-1}\, \mathrm{d}x^1 \wedge \cdots\cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det(g)}\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det(g)}\right) \sqrt{\det(g)}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det g}\right) \mathrm{d}V_g
\end{aligned}$$

implying (2.29).

Like the gradient operator, the divergence operator $\operatorname{div} : \mathfrak{X}(M) \to C^\infty(M)$ also satisfies linearity and product rule: if $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$, then

$$\begin{aligned}
\operatorname{div}(X+Y) &= \operatorname{div}(X) + \operatorname{div}(Y) \\
\operatorname{div}(fX) &= f \operatorname{div} X + \langle \operatorname{grad} f, X \rangle_g
\end{aligned} \tag{2.30}$$

Exercise 2.3.23. *Use Equation (2.32) to show above two properties of the divergence operator.*

Assuming some knowledge about covariant deriavtives in later chapters, we can show

$$\operatorname{div}(X) = \operatorname{tr}(\nabla X) \tag{2.31}$$

and

$$\operatorname{div}(X) = \sum_{i=1}^{n} \left(\frac{\partial X^i}{\partial x^i} + \sum_{k=1}^{n} X^k \Gamma_{ik}^i\right) \tag{2.32}$$

Ideed, for any local chart $(U,(x^i))$, we first observe that (3.23) implies

$$
\begin{aligned}
(*):\quad \sum_{i=1}^{n}\Gamma_{ij}^{i} &= \sum_{i=1}^{n}\left(\frac{1}{2}\sum_{l=1}^{n} g^{il}(\partial_i g_{jl}+\partial_j g_{il}-\partial_l g_{ij})\right)\\
&= \frac{1}{2}\left[\sum_{i,l} g^{il}\partial_j g_{il}+\frac{1}{2}\sum_{i,l} g^{il}\partial_i g_{jl}-\frac{1}{2}\sum_{i,l} g^{il}\partial_l g_{ij}\right]\\
&= \frac{1}{2}\left[\sum_{i,l} g^{il}\partial_j g_{il}+\frac{1}{2}\sum_{i,l} g^{il}\partial_i g_{jl}-\frac{1}{2}\sum_{i,l} g^{li}\partial_i g_{lj}\right]\\
&\xlongequal{g_{ij},g^{ij}\text{ symmetric}} \frac{1}{2}\sum_{i,l} g^{il}\partial_j g_{il}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\operatorname{div}(X) &= \frac{1}{\sqrt{\det(g_{ij})}}\sum_{k=1}^{n}\frac{\partial}{\partial x^k}\left(\sqrt{\det(g_{ij})}X^k\right)\\
&= \frac{1}{\sqrt{\det(g_{ij})}}\sum_{k=1}^{n}\left(\frac{\partial\sqrt{\det(g_{ij})}}{\partial x^k}X^k+\frac{\partial X^k}{\partial x^k}\sqrt{\det(g_{ij})}\right)\\
&= \frac{1}{\sqrt{\det(g_{ij})}}\sum_{k=1}^{n}\left(\frac{1}{2\sqrt{\det(g_{ij})}}\frac{\partial\det(g_{ij})}{\partial x^k}X^k+\frac{\partial X^k}{\partial x^k}\sqrt{\det(g_{ij})}\right)\\
&= \sum_{k=1}^{n}\left(\frac{1}{2\det(g_{ij})}\frac{\partial\det(g_{ij})}{\partial x^k}X^k+\frac{\partial X^k}{\partial x^k}\right)\\
&= \sum_{k=1}^{n}\left(\frac{1}{2}X^k\sum_{m,l=1}^{n} g^{ml}\frac{\partial g_{ml}}{\partial x^k}+\frac{\partial X^k}{\partial x^k}\right)\\
&= \sum_{k=1}^{n}\left(\frac{\partial X^k}{\partial x^k}+\sum_{i=1}^{n}X^k\Gamma_{ik}^{i}\right)\quad\text{by } (*)
\end{aligned}
$$

where we used Jacobi's formula for derivative of determinant:

$$
\begin{aligned}
\frac{\partial}{\partial x^k}\det(g_{ij}) &= \det(g_{ij})\cdot\operatorname{tr}\left((g_{ij})^{-1}\frac{\partial(g_{ij})}{\partial x^k}\right)=\det(g_{ij})\cdot\operatorname{tr}\left(\underbrace{(g^{ij})\frac{\partial(g_{ij})}{\partial x^k}}_{\text{matrix multiplication}}\right)\\
&= \det(g_{ij})\sum_{l,m=1}(g^{ij})_{ml}\left(\frac{\partial g_{ij}}{\partial x^k}\right)_{lm}\xlongequal{g\text{ symmetric}}\det(g_{ij})\sum_{l,m=1} g^{ml}\frac{\partial g_{ml}}{\partial x^k}.
\end{aligned}
$$

Now,

$$
\operatorname{tr}(\nabla X)\xlongequal{\text{Prop.1.1.13}}\sum_{i=1}^{n}X^i{}_{;i}\xlongequal{(3.9)}\sum_{i=1}^{n}\left(\frac{\partial X^i}{\partial x^i}+\sum_{k=1}^{n}X^k\Gamma_{ik}^{i}\right)=\operatorname{div}(X).
$$

On a local orthonormal frame of a Riemannian manifold (M,g), the trace of the linear endomorphism $\nabla X:\mathfrak{X}(M)\to\mathfrak{X}(M);\ Y\mapsto\nabla_Y X$ can be computed in the same way as (2.14). Thus

$$
\operatorname{div}(X)=\operatorname{tr}(\nabla X)=\sum_{i=1}^{n}\langle\nabla_{E_i}X,E_i\rangle_g. \tag{2.33}
$$

If $\Phi:(M,g)\to(\widetilde{M},\widetilde{g})$ is an isometry, then

$$\Phi^*\left(\operatorname{div}_{\widetilde{g}}(\Phi_*(X))\right)=\operatorname{div}_g(X) \tag{2.34}$$

for any $X\in\mathfrak{X}(M)$. Indeed, let $(E_1,\ldots,E_n)$ be a local orthonormal frame on (M,g). Then $(\Phi_*(E_1),\ldots,\Phi_*(E_n))$ is a local orthonormal frame on $(\widetilde{M},\widetilde{g})$. By Proposition 3.2.8,

$$\begin{aligned}
&\Phi^*(\operatorname{div}_{\widetilde{g}}(\Phi_*(X)))=\operatorname{div}_{\widetilde{g}}(\Phi_*(X))\circ\Phi\\
=&\sum_{i=1}^n\widetilde{g}\left(\widetilde{\nabla}_{\Phi_*E_i}\Phi_*(X),\Phi_*E_i\right)\circ\Phi\\
=&\sum_{i=1}^n\widetilde{g}\left(\Phi_*(\Phi^{-1})_*\widetilde{\nabla}_{\Phi_*E_i}\Phi_*(X),\Phi_*E_i\right)\circ\Phi\\
=&\sum_{n=1}^n\Phi^*\widetilde{g}\left((\Phi^{-1})_*\widetilde{\nabla}_{\Phi_*E_i}\Phi_*(X),E_i\right)\\
=&\sum_{n=1}^n\Phi^*\widetilde{g}\left((\Phi^*\widetilde{\nabla})_{E_i}X,E_i\right)\text{ by (3.13)}\\
=&\sum_{n=1}^n g\left(\nabla_{E_i}X,E_i\right)=\operatorname{div}_g(X).
\end{aligned}$$

III. The Hessian Operators

Let f be a smooth function on M. Then $\nabla f\in\Gamma\left(T^{(0,1)}TM\right)=\Omega^1(M)$ is just the 1-form $\mathrm{d}f$, because both tensors have the same action on vectors: $\nabla f(X)=\nabla_X f=Xf=\mathrm{d}f(X)$. The 2-tensor $H_f:=\nabla^2 f=\nabla(\mathrm{d}f)$ is called the **covariant Hessian of** f. Proposition 3.1.19 shows that its action on smooth vector fields X,Y can be computed by the following formula:

$$H_f(Y,X)=\nabla^2 f(Y,X)=\nabla^2_{X,Y}f=\nabla_X\left(\nabla_Y f\right)-\nabla_{(\nabla_X Y)}f=X(Yf)-\left(\nabla_X Y\right)f. \tag{2.35}$$

In any local coordinates, it is

$$H_f=\nabla^2 f=f_{;ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j,\quad\text{ with } f_{;ij}=\partial_j\partial_i f-\Gamma^k_{ji}\partial_k f. \tag{2.36}$$

If M is equipped with a Riemannian metric g and ∇ is the Levi-Civita connection, then $(\nabla_X Y)f-(\nabla_Y X)f=[X,Y]f=X(Yf)-Y(Xf)$. Equation (2.35) then implies the symmetry of covariant Hessian operator:

$$H_f(Y,X)=H_f(X,Y). \tag{2.37}$$

With the additional information g, we also define the **Hesse tensor field** $h_f:\mathfrak{X}(M)\to\mathfrak{X}(M)$ by $h_f=\nabla(\operatorname{grad}f)$:

$$h_f(X)=\nabla_X(\operatorname{grad}f) \tag{2.38}$$

It defines a smooth $(1,1)$-tensor field on M. Note that qquation (2.9) shows $H_f=\widehat{h_f}$. Indeed,

$$\begin{aligned}
\widehat{h_f}(X,Y)&=g(h_f(X),Y)=\langle\nabla_X(\operatorname{grad}f),Y\rangle\\
&\xlongequal{g\text{'s metric property}}\nabla_X\langle\operatorname{grad}f,Y\rangle-\langle\operatorname{grad}f,\nabla_X Y\rangle\\
&=\nabla_X(Yf)-(\nabla_X Y)(f)\\
&=X(Yf)-(\nabla_X Y)(f)\\
&=H_f(X,Y)
\end{aligned} \tag{2.39}$$

By the isomorphism (2.11), we can also write $H_f=\widehat{h_f}$ as $h_f=\check{H}_f$.

IV. The Laplace-Beltrami Operator

One of the most fundamental objects associated to a Riemannian manifold (M, g) is its **Laplace-Beltrami operator**, a second-order elliptic self-adjoint partial differential operator which acts on $C^\infty(M)$. It is defined as

$$\begin{aligned}\Delta : C^\infty(M) &\to C^\infty(M)\\ f &\mapsto -\operatorname{div}(\operatorname{grad}(f))\end{aligned}$$

From equations (2.33), (2.39), and (2.14), one gets

$$\begin{aligned}\Delta f &= -\operatorname{div}(\operatorname{grad}(f)) = -\operatorname{tr}(\nabla(\operatorname{grad} f))\\ &= -\operatorname{tr}(h_f) = -\operatorname{tr}(\check{H}_f) = -\operatorname{tr}_g(H_f)\\ &= -\sum_{i,j=1}^{n} g^{ij} H_f(E_i, E_j)\\ &= -\sum_{i,j=1}^{n} g^{ij}\left(E_i(E_j f) - (\nabla_{E_i} E_j)(f)\right)\end{aligned} \tag{2.40}$$

under a local frame $(E_1, \cdots, E_n)$ on (M, g). If $(E_i) = (\partial_i)$ for a local chart, then one can go through the same process as in the proof of (2.31) to continue writing above equation to get

$$\Delta f = -\frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det(g)}\frac{\partial f}{\partial x^j}\right). \tag{2.41}$$

A quicker way is to use (2.29) and (2.24):

$$-\operatorname{div}(\operatorname{grad} f) = -\operatorname{div}\left(g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\right) = -\frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^j}\left(g^{ij}\frac{\partial f}{\partial x^i}\sqrt{\det(g)}\right)$$

For $f_1, f_2 \in C^\infty(M)$, one has

$$\begin{aligned}&\operatorname{div}\left(f_2\left(\operatorname{grad}\left(f_1\right)\right)\right) = -f_2\left(\Delta f_1\right) + g\left(\operatorname{grad}\left(f_2\right), \operatorname{grad}\left(f_1\right)\right)\\ &\Delta\left(f_1 f_2\right) = f_2\left(\Delta f_1\right) - 2g\left(\operatorname{grad}\left(f_1\right), \operatorname{grad}\left(f_2\right)\right) + f_1\left(\Delta f_2\right)\end{aligned} \tag{2.42}$$

To derive (2.42) (1), one simply substitutes $f = f_2$ and $X = \operatorname{grad}(f_1)$ in (2.30) (2) and one takes into account the definition of Δ. On the other hand, the definition of Δ, (2.25), (2.30) (1), and (2.42) (1) imply that

$$\begin{aligned}\Delta\left(f_1 f_2\right) &= -\operatorname{div}\left(\operatorname{grad}\left(f_1 f_2\right)\right)\\ &= -\operatorname{div}\left(f_2\left(\operatorname{grad} f_1\right)\right) - \operatorname{div}\left(f_1\left(\operatorname{grad}\left(f_2\right)\right)\right)\\ &= f_2\left(\Delta f_1\right) - g\left(\operatorname{grad}\left(f_2\right), \operatorname{grad}\left(f_1\right)\right) + f_1\left(\Delta f_2\right) - g\left(\operatorname{grad}\left(f_1\right), \operatorname{grad}\left(f_2\right)\right)\\ &= f_2\left(\Delta f_1\right) - 2g\left(\operatorname{grad}\left(f_1\right), \operatorname{grad}\left(f_2\right)\right) + f_1\left(\Delta f_2\right)\end{aligned}$$

i.e. (2.42) (2) is valid.

Let (M, g) be a Riemannian manifold with or without boundary. A function $u \in C^\infty(M)$ is said to be **harmonic** if $\Delta u = 0$.

Theorem 2.3.24 (Green's identities). *Suppose (M, g) is compact. Let N, $\widetilde{g}$ be as in Theorem 2.3.21. Then,*

$$\begin{aligned}\int_M u\Delta v \, \mathrm{d}V_g &= \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, \mathrm{d}V_g - \int_{\partial M} uNv \, \mathrm{d}V_{\tilde{g}}\\ \int_M (u\Delta v - v\Delta u) \, \mathrm{d}V_g &= \int_{\partial M} (vNu - uNv) \, \mathrm{d}V_{\tilde{g}}\end{aligned}$$

Proof. Exercise. ■

Exercise 2.3.25.

(a) *Show that if M is compact and connected and $\partial M = \varnothing$, the only harmonic functions on M are the constants.*

(b) *Show that if M is compact and connected, $\partial M \neq \varnothing$, and u, v are harmonic functions on M whose restrictions to ∂M agree, then $u \equiv v$.*

If $\Phi : (M, g) \to (\widetilde{M}, \widetilde{g})$ is an isometry, then

$$\Delta_g \circ \Phi^* = \Phi^* \circ \Delta_{\widetilde{g}} \tag{2.43}$$

Indeed, using the definition of Δ, (2.34) and (2.26), one gets

$$\begin{aligned} \Delta_g \left(\Phi^*(\tilde{f})\right) &= -\operatorname{div}_g \left(\operatorname{grad}_g \Phi^*(\tilde{f})\right) = -\Phi^* \left(\operatorname{div}_{\widetilde{g}} \left(\Phi_* \left(\operatorname{grad}_g \Phi^*(\tilde{f})\right)\right)\right) \\ &= -\Phi^* \left(\operatorname{div}_{\widetilde{g}} \left(\operatorname{grad}_{\widetilde{g}}(\tilde{f})\right)\right) = \Phi^* \left(\Delta_{\widetilde{g}} \tilde{f}\right) \end{aligned}$$

for any $\tilde{f} \in C^\infty(\widetilde{M})$.

In fact, one can show for a Riemannian manifold (M, g) that the only diffeomorphisms $\Phi : M \to M$ which commute with the Laplace-Beltrami operator Δ are the isometries (see Helgason [1962], p. 387).

V. The Hodge-de Rham Operators

Let (M, g) be an oriented n-dimensional Riemannian manifold. Since $\mathrm{d}V_g$ is a top-degree form serving as the basis of $\Omega^n(M)$, every n-form in $\Omega^n(M)$ can be written as $f\ \mathrm{d}V_g$ for some smooth function f on M. Thus, multiplication by the Riemannian volume form defines a smooth bundle isomorphism $* : C^\infty(M) \to \Omega^n(M)$:

$$*f := f\ \mathrm{d}V_g.$$

For example, the divergence operator can be written as $\operatorname{div}(X) = *^{-1}(\mathrm{d}(X \lrcorner\, \mathrm{d}V_g))$. We will generalize this to a smooth bundle homomorphism over all forms, called Hodge star operator. First, the Riemannian metric g induces a fiber metric $\langle \cdot | \cdot \rangle$ on $\Lambda^k(T^*M)$. We give two equivalent definitions:

Definition 2.3.26 (Fiber metric on n-forms)**.**

Definition 1: If

$$\begin{aligned} \omega &= \alpha^1 \wedge \cdots \wedge \alpha^k \\ \eta &= \beta^1 \wedge \cdots \wedge \beta^k, \end{aligned}$$

then

$$\langle \omega | \eta \rangle = \det\left(\langle \alpha^i, \beta^j \rangle\right) = \det\left(\langle \sharp\alpha^i, \sharp\beta^j \rangle\right)$$

and extend linearly (via universal property). This is well-defined and gives a symmetric bilinear map that is positive-definite if we specify an ordering. See [7] p.404-408.

Definition 2: If $\{e_i\}$ is an orthonormal basis of T_pM, let $\{\varepsilon^i\}$ denote the dual basis, defined by $\varepsilon^i(e_j) = \delta^i_j$. Then declare that

$$\{\varepsilon^I := \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

is an orthonormal basis of $\Lambda^k(T_p^*M)$ and show that it satisfies $\langle \omega | \eta \rangle = \det\left(\langle \alpha^i, \beta^j \rangle\right)$ and is independent of the choice of frame. See [12] Problem 2-16.

We already have a fiber metric $\langle\cdot,\cdot\rangle$ for all tensor fields. We wish to compare it with the fiber metric $\langle\cdot|\cdot\rangle$ just defined on forms. If $\alpha = \sum \alpha_I e^I$ and $\beta = \sum \beta_I e^I$ are k-forms considered as covariant tensor fields, then by (2.19) we have

$$\langle\alpha,\beta\rangle = \sum_I \alpha_{i_1\ldots i_k}\beta^{i_1\ldots i_k} = \sum_I \alpha_I\beta^I \text{ where } \beta^I \text{ have raised indices.}$$

Now let $e^1,\ldots,e^n$ be an orthonormal basis for V^*. Then in this basis,

$$\langle\alpha,\beta\rangle = \sum_I \alpha_I\beta^I = \sum_I \alpha_I\beta_I$$

and we have

$$\begin{aligned}\langle\alpha|\beta\rangle &= \sum_{I,J}{}' \left\langle \alpha_I e^I | \beta_J e^J\right\rangle = \sum_{I,J}{}'\alpha_I\beta_J\left\langle e^I|e^J\right\rangle \\ &= \sum_I{}'\alpha_I\beta_I = \frac{1}{k!}\sum_I \alpha_I\beta_I = \frac{1}{k!}\langle\alpha,\beta\rangle.\end{aligned}$$

We conclude that

$$\langle\alpha|\beta\rangle = \frac{1}{k!}\langle\alpha,\beta\rangle,$$

so the two inner products differ by a factor of $k!$.

Exercise 2.3.27. *Let (M,g) be an oriented Riemannian n-manifold. Show that the Riemannian volume form $\mathrm{d}V_g$ is the unique positively oriented n-form that has unit norm with respect to the fiber metric $\langle\cdot|\cdot\rangle$.*

We are now ready to define the Hodge star operator.

Proposition 2.3.28 (Hodge Star Operator)**.** *Let (M,g) be an oriented Riemannian manifold and $\mathrm{d}V_g$ be its Riemannian volume form. For each $k=0,\ldots,n$, there is a unique smooth bundle homomorphism $*:\Lambda^kT^*M\to\Lambda^{n-k}T^*M$, called the* ***Hodge star operator****, satisfying*

$$\omega\wedge *\eta = \langle\omega|\eta\rangle\,\mathrm{d}V_g$$

for all smooth k-forms ω,η. For $k=0$, we interpret the inner product as ordinary multiplication, which recovers the original case that $*f = f\,\mathrm{d}V_g$.

Proof. We prove uniqueness first:

suppose there is some $*:\Lambda^kT^*M\to\Lambda^{n-k}T^*M$ such that $\omega\wedge *\eta = \langle\omega|\eta\rangle\,\mathrm{d}V_g$. In a local oriented orthonormal frame $(E_1,\cdots,E_n)$, we consider in particular $\omega=\eta=\varepsilon^I=\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k}$ and suppose $*\omega = \sum_J{}' a_J\varepsilon^J$. If we denote the complementary indices of $I=(i_1,\cdots,i_k)$ in $(1,\cdots,n)$ as $I^-=(i_1^-,\cdots,i_{n-k}^-)$, which is ordered as increasing indices and is thus unique for each I, then the rule that $\alpha\wedge\beta=-\beta\wedge\alpha$ for one-forms implies

$$\begin{aligned}\varepsilon^I\wedge\left(\sum_J{}'a_J\varepsilon^M\right) &= \langle\varepsilon^I|\varepsilon^I\rangle\varepsilon^1\wedge\cdots\wedge\varepsilon^n \\ \varepsilon^I\wedge a_{I^-}\varepsilon^{I^-} &= \varepsilon^1\wedge\cdots\wedge\varepsilon^n \\ a_{I^-}\cdot(-1)^{\tau(i_1,\cdots,i_k,i_1^-,\cdots,i_{n-k}^-)}\varepsilon^1\wedge\cdots\wedge\varepsilon^n &= \varepsilon^1\wedge\cdots\wedge\varepsilon^n \\ a_{I^-}\cdot(-1)^{\tau(i_1,\cdots,i_k,i_1^-,\cdots,i_{n-k}^-)} &= 1\end{aligned}$$

where τ is the number of inversions for the permutation $(i_1,\cdots,i_k,i_1^-,\cdots,i_{n-k}^-)$. $(-1)^\tau$ is ±1, so a_{I^-} must have the same sign as it, i.e., $(-1)^\tau$. Thus,

$$*(\varepsilon^I) = (-1)^{\tau(i_1,\cdots,i_k,i_1^-,\cdots,i_{n-k}^-)}\varepsilon^{I^-}.$$

Specification of the assignment of $*$ on the whole orthonormal basis uniquely determines $*$.

The next is the existence of $*$:

We simply define $*$ locally as

$$*(\varepsilon^I) = (-1)^{\tau(i_1,\cdots,i_k,i_1^-,\cdots,i_{n-k}^-)}\varepsilon^{I^-}$$

for any local oriented orthonormal coframe (ε^i) and extend it linearly over the whole space $\Lambda^k T^*M$ via the universal property of the tensor product. To show it is independent of the choice of coframe, proceed similarly as in Proposition 2.3.6.

Notice that the basis for $\Lambda^0 T^*M$ is the constant function 1 on M, and its complementary indices are simply $1,\cdots,n$. Thus, $*1 = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n = \mathrm{d}V_g$. By linearity, $*f = f \cdot *1 = f\,\mathrm{d}V_g$, recovering the base case. ■

Proposition 2.3.29 (General Formula of Hodge Star Operator)**.** *Let (M,g) be an oriented Riemannian n-manifold. Let (E_i) be a local orthonormal frame and (ε^i) be its coframe. Then*

$$*\left(\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k}\right) = \sqrt{\det(g)}\left(\sum_{(j_1,\cdots,j_k)\in S\{i_1,\cdots,i_k\}} (-1)^{\tau(j_1\ldots j_k, i_{k+1}\ldots i_n)} g^{i_1 j_k}\cdots g^{i_k j_k}\right)\varepsilon^{i_{k+1}}\wedge\cdots\wedge\varepsilon^{i_n} \quad (2.44)$$

where $(i_{k+1},\cdots,i_n)$ are the complementary indices of $(i_1,\cdots,i_k)$ and τ is the number of inversions of the permutation $(j_1,\cdots,j_k,i_{k+1},\cdots,i_n)$. Here, $S\{i_1,\cdots,i_k\}$ means the set of all self-bijections of the set $\{i_1,\cdots,i_k\}$.

Proof. See [7, Theorem 9.26]. ■

For example, when $n=5$, for the basis (∂_i), $(\mathrm{d}x^i)$, we have

$$\begin{aligned}
*(\mathrm{d}x^1\wedge\mathrm{d}x^3) &= \sqrt{\det(g)}\sum_{(j_1,j_2)\in S\{1,3\}}(-1)^{\tau(j_1\,2\,4\,5)}g^{1j_1}g^{3j_2}\,\mathrm{d}x^2\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5\\
&= \sqrt{\det(g)}\left((-1)^{\tau(1\,3\,2\,4\,5)}g^{11}g^{33} + (-1)^{\tau(3\,1\,2\,4\,5)}g^{13}g^{31}\right)\mathrm{d}x^2\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5\\
&= \sqrt{\det(g)}\left(-g^{11}g^{33}+g^{13}g^{31}\right)\mathrm{d}x^2\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5.
\end{aligned}$$

For $\mathbb{R}^3$ with Euclidean metric, we have

$$\begin{aligned}
*\,\mathrm{d}x &= \mathrm{d}y\wedge\mathrm{d}z\\
*\,\mathrm{d}y &= \mathrm{d}z\wedge\mathrm{d}x\\
*\,\mathrm{d}z &= \mathrm{d}x\wedge\mathrm{d}y
\end{aligned}$$

Proposition 2.3.30. *The Hodge star operator satisfies the following properties:*

(a) The Hodge star is an isometry from $\Lambda^k T^*M$ to $\Lambda^{n-k}T^*M$ with respect to the fiber metric $\langle\cdot|\cdot\rangle$.

(b) $*1 = \mathrm{d}V_g$, and $*\,\mathrm{d}V_g = 1$.

(c) For any $\omega\in\Lambda^k T^*M$, $*^2\omega = (-1)^{k(n-k)}\omega$.

(d) For $\alpha,\beta\in\Lambda^k T^*M$, $\langle\alpha|\beta\rangle = \langle *\alpha|*\beta\rangle = *(\alpha\wedge *\beta) = *(\beta\wedge *\alpha)$.

Proof. We showed (b) in Proposition 2.3.28 because $*$ is an isomorphism for $k=0$ case. For the proof of (c), see [7] Proposition 9.25 (3). We show (a): since $\omega\wedge\eta = (-1)^{kl}\eta\wedge\omega$ for k-form ω and l-form η, we see

$$\begin{aligned}
\langle *\alpha|*\beta\rangle\,\mathrm{d}V_g &= (*\alpha)\wedge(*(*\beta))\\
&= (*\alpha)\wedge\left((-1)^{k(n-k)}\beta\right)\\
&= (-1)^{k(n-k)} *\alpha\wedge\beta\\
&= (-1)^{k(n-k)}(-1)^{k(n-k)}\beta\wedge *\alpha\\
&= \langle\beta|\alpha\rangle\,\mathrm{d}V_g = \langle\alpha|\beta\rangle\,\mathrm{d}V_g.
\end{aligned}$$

(d): the first equality is just (a). For the second notice that $*(\alpha \wedge *\beta) = *(\langle \alpha|\beta\rangle \, \mathrm{d}V_g) = \langle \alpha|\beta\rangle * \, \mathrm{d}V_g = \langle \alpha|\beta\rangle$ by (b). The third equality is similar. ∎

Let (M, g) be an oriented Riemannian n-manifold. For $1 \leq k \leq n$, we define the **codifferential operater**

$$\begin{aligned}\delta = \mathrm{d}^* : \Omega^k(M) &\to \Omega^{k-1}(M)\\ \omega &\mapsto (-1)^{n(k+1)+1} * \, \mathrm{d} * \omega\end{aligned}$$

and extend this definition to 0-forms by defining $\delta\omega = 0$ for $\omega \in \Omega^0(M)$. Here, from right to left, $* : \Omega^k(M) \to \Omega^{n-k}(M)$, $\mathrm{d} : \Omega^{n-k}(M) \to \Omega^{n-k+1}(M)$, and $* : \Omega^{n-k+1}(M) \to \Omega^{k-1}(M)$. Recall that d is called the **exterior differential operator**. By Proposition 2.3.30 (c) and the fact that $\mathrm{d} \circ \mathrm{d} = 0$, the square of the codifferential operator also vanishes:

$$\delta \circ \delta = 0. \tag{2.45}$$

Also, $\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M)$ and $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ are adjoint with respect to the fiber metric $\langle\cdot|\cdot\rangle$. That is, for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k-1}(M)$,

$$\langle \delta\omega|\eta\rangle = \langle \omega| \, \mathrm{d}\eta\rangle \tag{2.46}$$

Indeed, Proposition 2.3.30 (c) implies $*\delta = (-1)^{n(k+1)+1}(-1)^{(n-k+1)(k-1)} \, \mathrm{d}* = (-1)^{k(2n-k+2)} \, \mathrm{d}* \implies \mathrm{d}* = (-1)^{k(k-2n-2)} * \delta$. Then Theorem 1.5.4 (ii) gives

$$\begin{aligned}0 = \mathrm{d} \overbrace{(\omega \wedge *\eta)}^{k+(n-(k-1))=n+1 \text{ form}} &= \mathrm{d}\omega \wedge *\eta + (-1)^k \omega \wedge \, \mathrm{d}(*\eta)\\ &= \mathrm{d}\omega \wedge *\eta + (-1)^k(-1)^{(k-1)(k-1-2n-2)}\omega \wedge *\delta\eta\\ &= \mathrm{d}\omega \wedge *\eta - \omega \wedge *\delta\eta \quad \text{by analyzing the parity}\\ \omega \wedge *\delta\eta &= \mathrm{d}\omega \wedge *\eta\\ \langle \omega|\delta\eta\rangle \, \mathrm{d}V_g &= \langle \, \mathrm{d}\omega|\eta\rangle \, \mathrm{d}V_g \quad \text{by definition of } *\end{aligned}$$

Exercise 2.3.31 ([11] 16-22). *Let (M, g) be an oriented compact Riemannian n-manifold. Show that the formula*

$$(\omega, \eta) = \int_M \langle \omega|\eta\rangle_g \, \mathrm{d}V_g$$

defines an inner product on $\Omega^k(M)$ for each k.

We also have a generalization of the zero-th order Laplace-Beltrami operator Δ.

Definition 2.3.32 (Hodge-de Rham operator). *Let (M, g) be an oriented compact Riemannian n-manifold. For each $0 \leq k \leq n$, the **Hodge-de Rham operator** or **Laplace-Beltrami operator on k-forms** $\Delta : \Omega^k(M) \to \Omega^k(M)$ is the linear map defined by*

$$\Delta\omega = \mathrm{d}\delta\omega + \delta \, \mathrm{d}\omega$$

Exercise 2.3.33 ([11] 17-3). *When $k = 0$, by definition, $\delta f = 0$, so $\Delta f = \delta \, \mathrm{d}f = (-1)^{n+1} * \, \mathrm{d} * (\, \mathrm{d}f)$. Show that this coincides with Δf defined by (2.40).*

Solution. There is in fact a generalization of the divergence operator over $\Omega^k(M)$ by

$$\operatorname{div} \omega = \sum_{i=1} \iota_{E_i}(\nabla_{E_i}\omega), \tag{2.47}$$

and one can show that $\delta = -\operatorname{div}$ (see Proposition 5.4 of this note for example). For the one-form $\mathrm{d}f$ we have

$$\begin{aligned}\operatorname{div}(\mathrm{d}f) &= \sum_{i=1} \iota_{E_i}(\nabla_{E_i} \mathrm{d}f) = \sum_{i=1} \nabla_{E_i} \mathrm{d}f(E_i) \\ &\overset{(3.6)}{=\!=} \sum_{i=1} \left(E_i\left(E_i f\right) - \left(\nabla_{E_i} E_i\right)(f)\right)\end{aligned}$$

On the other hands,

$$\begin{aligned}\operatorname{div}(\operatorname{grad} f) &= \sum_{i=1} g\left(\nabla_{E_i} \operatorname{grad} f, E_i\right) \\ &= \sum_{i=1} E_i \underbrace{\left(g\left(\operatorname{grad} f, E_i\right)\right)}_{=\mathrm{d}f(E_i)=E_i(f)} - \underbrace{g\left(\operatorname{grad} f, \nabla_{E_i} E_i\right)}_{=\mathrm{d}f(\nabla_{E_i}E_i)=(\nabla_{E_i}E_i)(f)} \\ &= \sum_{i=1} \left(E_i\left(E_i f\right) - \left(\nabla_{E_i} E_i\right)(f)\right).\end{aligned}$$

Thus, $\operatorname{div}(\mathrm{d}f) = \operatorname{div}(\operatorname{grad} f)$ and $\Delta f = \delta\,\mathrm{d}f = -\operatorname{div}(\operatorname{grad} f)$, recovering the original definition of the zero-th order Laplace-Beltrami operator. ♦

Exercise 2.3.34. *Read Section 4.1 of the note for the definition of divergence operator over all (k,l)-tensor fields. Proposition 4.1 of the note shows that for a vector field X,*

$$\operatorname{div} X = \operatorname{div}(X^\flat).$$

This fact generalizes the equality $\operatorname{div}(\mathrm{d}f) = \operatorname{div}(\operatorname{grad} f)$ we showed above.

Proposition 2.3.35 ([1] Proposition 2.10)**.** *The operator $\Delta : \Omega^k(M) \to \Omega^k(M)$, for each $0 \le k \le n$, has the following properties:*

(i) Δ is self-adjoint, i.e., $\langle \Delta\omega | \eta \rangle = \langle \omega | \Delta\eta \rangle$;

(ii) Δ is positive, i.e. $\langle \Delta\omega | \omega \rangle \ge 0$ for all $\omega \in \Omega^k(M)$;

(iii) $\Delta\omega = 0$ if and only if $\mathrm{d}\omega = 0$ and $\delta\omega = 0$;

(iv) $\mathrm{d}\Delta = \Delta\,\mathrm{d}$, $\delta\Delta = \Delta\delta$, $\Delta * = *\Delta$.

2.4 Generalizations of Riemannian Metrics

There are other common ways of measuring "lengths" of tangent vectors on smooth manifolds. Let's digress briefly to mention three that play important roles in other branches of mathematics: pseudo-Riemannian metrics, sub-Riemannian metrics, and Finsler metrics. Each is defined by relaxing one of the requirements in the definition of Riemannian metric: a pseudo-Riemannian metric is obtained by relaxing the requirement that the metric be positive; a sub-Riemannian metric by relaxing the requirement that it be defined on the whole tangent space; and a Finsler metric by relaxing the requirement that it be quadratic on each tangent space.

Pseudo-Riemannian Metrics

A **pseudo-Riemannian metric** (occasionally also called a **semi-Riemannian metric**) on a smooth manifold M is a symmetric 2-tensor field g that is nondegenerate at each point $p \in M$. This means that the only vector orthogonal to everything is the zero vector. More formally, $g(X,Y) = 0$ for all $Y \in T_pM$ if and only if $X = 0$. If $g = g_{ij}\varphi^i\varphi^j$ in terms of a local coframe, nondegeneracy just means that the matrix g_{ij} is invertible. If g is

Riemannian, nondegeneracy follows immediately from positive-definiteness, so every Riemannian metric is also a pseudo-Riemannian metric; but in general pseudo-Riemannian metrics need not be positive.

Given a pseudo-Riemannian metric g and a point $p \in M$, by a simple extension of the Gram-Schmidt algorithm one can construct a basis $(E_1, \ldots, E_n)$ for T_pM in which g has the expression

$$g = -\left(\varphi^1\right)^2 - \cdots - \left(\varphi^r\right)^2 + \left(\varphi^{r+1}\right)^2 + \cdots + \left(\varphi^n\right)^2 \tag{2.48}$$

for some integer $0 \leq r \leq n$. This integer r, called the index of g, is equal to the maximum dimension of any subspace of T_pM on which g is negative definite. Therefore the index is independent of the choice of basis, a fact known classically as *Sylvester's law of inertia*.

By far the most important pseudo-Riemannian metrics (other than the Riemannian ones) are the **Lorentz metrics**, which are pseudo-Riemannian metrics of index 1. The most important example of a Lorentz metric is the **Minkowski metric**; this is the Lorentz metric m on $\mathbb{R}^{n+1}$ that is written in terms of coordinates $\left(\xi^1, \ldots, \xi^n, \tau\right)$ as

$$m = \left(\mathrm{d}\xi^1\right)^2 + \cdots + \left(\mathrm{d}\xi^n\right)^2 - (\mathrm{d}\tau)^2. \tag{2.49}$$

In the special case of $\mathbb{R}^4$, the Minkowski metric is the fundamental invariant of Einstein's special theory of relativity, which can be expressed succinctly by saying that in the absence of gravity, the laws of physics have the same form in any coordinate system in which the Minkowski metric has the expression (2.49). The differing physical characteristics of "space" (the ξ directions) and "time" (the τ direction) arise from the fact that they are subspaces on which g is positive definite and negative definite, respectively. The general theory of relativity includes gravitational effects by allowing the Lorentz metric to vary from point to point.

Many aspects of the theory of Riemannian metrics apply equally well to pseudo-Riemannian metrics. Although we do not treat pseudo-Riemannian geometry directly in this book, we will attempt to point out as we go along which aspects of the theory apply to pseudo-Riemannian metrics. As a rule of thumb, proofs that depend only on the invertibility of the metric tensor, such as existence and uniqueness of the Riemannian connection and geodesics, work fine in the pseudo-Riemannian setting, while proofs that use positivity in an essential way, such as those involving distance-minimizing properties of geodesics, do not.

For an introduction to the mathematical aspects of pseudo-Riemannian metrics, see the excellent book [O'N83] (Barrett O'Neill, *Semi-Riemannian Geometry with Applications to General Relativity*); a more physical treatment can be found in [HE73] (Stephen W. Hawking and George F. R. Ellis, *The Large-Scale Structure of Space-Time*.)

Sub-Riemannian Metrics

A **sub-Riemannian metric** (aka. **singular Riemannian metric** or **Carnot-Carathéodory metric**) on a manifold M is a fiber metric on a smooth distribution $S \subset TM$ (i.e., a k-plane field or sub-bundle of TM). Since lengths make sense only for vectors in S, the only curves whose lengths can be measured are those whose tangent vectors lie everywhere in S. Therefore one usually imposes some condition on S that guarantees that any two nearby points can be connected by such a curve. This is, in a sense, the opposite of the Frobenius integrability condition, which would restrict every such curve to lie in a single leaf of a foliation.

Sub-Riemannian metrics arise naturally in the study of the abstract models of real submanifolds of complex space $\mathbb{C}^n$, called *CR manifolds*. (Here CR stands for "Cauchy-Riemann.") CR manifolds are real manifolds endowed with a distribution $S \subset TM$ whose fibers carry the structure of complex vector spaces (with an additional integrability condition that need not concern us here). In the model case of a submanifold $M \subset \mathbb{C}^n$, S is the set of vectors tangent to M that remain tangent after multiplication by $i = \sqrt{-1}$ in the ambient complex coordinates. If S is sufficiently far from being integrable, choosing a fiber metric on S results in a sub-Riemannian metric whose geometric properties closely reflect the complex-analytic properties of M as a subset of $\mathbb{C}^n$.

Another motivation for studying sub-Riemannian metrics arises from *control theory*. In this subject, one is given a manifold with a vector field depending on parameters called controls, with the goal being to vary the controls so as to obtain a solution curve with desired properties, often one that minimizes some function such as arc length. If the vector field is everywhere tangent to a distribution S on the manifold (for example, in the case of a robot arm whose motion is restricted by the orientations of its hinges), then the function can often be modeled as a sub-Riemannian metric and optimal solutions modeled as sub-Riemannian geodesics.

A useful introduction to the geometry of sub-Riemannian metrics is provided in the article [Str86] (Robert s. Strichartz, *Sub-Riemannian Geometry*.)

Finsler Metrics

A **Finsler metric** on a manifold M is a continuous function $F : TM \to \mathbb{R}$, smooth on the complement of the zero section, that defines a norm on each tangent space T_pM. This means that $F(X) > 0$ for $X \neq 0, F(cX) = |c|F(X)$ for $c \in \mathbb{R}$, and $F(X+Y) \leq F(X)+F(Y)$. Again, the norm function associated with any Riemannian metric is a special case.

The inventor of Riemannian geometry himself, G. F. B. Riemann, clearly envisaged an important role in n-dimensional geometry for what we now call Finsler metrics; he restricted his investigations to the "Riemannian" case purely for simplicity (see Spivak, volume 2). However, only very recently have Finsler metrics begun to be studied seriously from a geometric point of view.

The recent upsurge of interest in Finsler metrics has been motivated largely by the fact that two different Finsler metrics appear very naturally in the theory of several complex variables: at least for bounded strictly convex domains in $\mathbb{C}^n$, the Kobayashi metric and the Carathéodory metric are intrinsically defined, biholomorphically invariant Finsler metrics. Combining differential-geometric and complex-analytic methods has led to striking new insights into both the function theory and the geometry of such domains.

2.5 Model Riemannian Manifolds

We will study the model Riemannian manifolds, Euclidean spaces, spheres, hyperbolic spaces, which are highly-symmetric spaces, i.e., spaces with a large isometry group. After that, we explore some more general classes of Riemannian manifolds with symmetry–the invariant metrics on Lie groups, homogeneous spaces, and symmetric spaces.

Let (M, g) be a Riemannian manifold. Recall that $\operatorname{Iso}(M, g)$ denotes the set of all isometries from M to itself, which is a group under composition. We say that (M, g) is a homogeneous Riemannian manifold if $\operatorname{Iso}(M, g)$ acts transitively on M, which is to say that for each pair of points $p, q \in M$, there is an isometry $\varphi : M \to M$ such that $\varphi(p) = q$.

The isometry group does more than just act on M itself. For every $\varphi \in \operatorname{Iso}(M, g)$, the global differential $d\varphi$ maps TM to itself and restricts to a linear isometry $d\varphi_p : T_pM \to T_{\varphi(p)}M$ for each $p \in M$.

Given a point $p \in M$, let $\operatorname{Iso}_p(M, g)$ denote the isotropy subgroup at p, that is, the subgroup of Iso (M, g) consisting of isometries that fix p. For each $\varphi \in$ Iso $_p(M, g)$, the linear map $d\varphi_p$ takes T_pM to itself, and the map $I_p : \operatorname{Iso}_p(M, g) \to \mathrm{GL}(T_pM)$ given by $I_p(\varphi) = d\varphi_p$ is a representation of $\operatorname{Iso}_p(M, g)$, called the **isotropy representation**. We say that M is **isotropic at** p if the isotropy representation of $\operatorname{Iso}_p(M, g)$ acts transitively on the set of unit vectors in T_pM. If M is isotropic at every point, we say simply that M is **isotropic**.

There is an even stronger kind of symmetry than isotropy. Let $\mathrm{O}(M)$ denote the set of all orthonormal bases for all tangent spaces of M (and recall as in [11] we assume a basis is ordered):

$$\mathrm{O}(M) = \coprod_{p\in M} \{\text{ orthonormal bases for } T_pM\}$$

There is an induced action of $\operatorname{Iso}(M, g)$ on $\mathrm{O}(M)$, defined by using the differential of an isometry φ to push an orthonormal basis at p forward to an orthonormal basis at $\varphi(p)$:

$$\varphi \cdot (b_1, \ldots, b_n) = (\, d\varphi_p\,(b_1)\,, \ldots,\, d\varphi_p\,(b_n)).$$

We say that (M, g) is **frame-homogeneous** if this induced action is transitive on $\mathrm{O}(M)$, or in other words, if for all $p, q \in M$ and choices of orthonormal bases at p and q, there is an isometry taking p to q and the chosen basis at p to the one at q. (Warning: Some authors use the term isotropic to refer to the property we have called frame-homogeneous.)

A homogeneous Riemannian manifold looks geometrically the same at every point, while an isotropic one looks the same in every direction. We have the following proposition/examples.

Proposition 2.5.1. *Let (M, g) be a Riemannian manifold.*

(a) If M is isotropic at a point $p \in M$ and homogeneous, then it is isotropic.

(b) If M is frame-homogeneous, then it is homogeneous and isotropic.

(c) If M is isotropic, then it is homogeneous.

(d) M can be isotropic at one point without being isotropic.

(e) M can be homogeneous without being isotropic anywhere.

(f) M can be isotropic and homogeneous without being frame-homogeneous.

Proof. (a): Suppose it is isotropic at p and homogeneous. Then for any $q \in M$, there is an isometry $\varphi : M \to M$ such that $\varphi(p) = q$. Let $u_1, u_2 \in T_q M$ be unit vectors. We want to show that there is an isometry $\psi : M \to M$ such that $d\psi_q(u_1) = u_2$. We find ϕ such that $d\phi_p$ sends $v_1 := d\varphi_p^{-1}(u_1)$ to $v_2 := d\varphi_p^{-1}(u_2)$, which are unit vectors as ϕ is an isometry. Then let $\psi = \varphi \circ \phi \circ \varphi^{-1}$.

(b): Homogeneity is automatic. To show M is isotropic, let $p \in M$ and $u_1, u_2 \in T_p M$ be unit vectors. Complete u_i to orthonormal bases of $T_p M$ and then use frame-homogeneity.

(c): Problem 3.6.15.

(d): Problem 2.6.34: A counterexample is the paraboloid $z = x^2 + y^2$ in $\mathbb{R}^3$ with the induced metric.

(e): Problem 2.6.22, Problem 2.6.23: A counterexample is the Berger metrics on $\mathbb{S}^3$.

(f): Problem 3.6.6: A counterexample is the Fubini-Study metrics on complex projective spaces $\mathbb{CP}^n$. ■

A deep theorem of Sumner B. Myers and Norman E. Steenrod [15] shows that if M has finitely many components, then $\operatorname{Iso}(M, g)$ has a topology and smooth structure making it into a finite-dimensional Lie group acting smoothly on M. We will neither prove nor use the Myers-Steenrod theorem, but if you are interested, a good source for the proof is [8].

2.5.1 Euclidean Spaces

The simplest and most important model Riemannian manifold is of course n-dimensional Euclidean space, which is just $\mathbb{R}^n$ with the Euclidean metric $\bar{g} = \delta_{ij} dx^i dx^j = (dx^i)^2$.

Somewhat more generally, if V is any n-dimensional real vector space endowed with an inner product, we can set $g(v, w) = \langle v, w\rangle$ for any $p \in V$ and any $v, w \in T_p V \cong V$. Choosing an orthonormal basis $(b_1, \ldots, b_n)$ for V defines a basis isomorphism from $\mathbb{R}^n$ to V that sends $\left(x^1, \ldots, x^n\right)$ to $x^i b_i$; this is easily seen to be an isometry of (V, g) with $(\mathbb{R}^n, \bar{g})$, so all n-dimensional inner product spaces are isometric to each other as Riemannian manifolds.

It is easy to construct isometries of the Riemannian manifold $(\mathbb{R}^n, \bar{g})$: for example, every orthogonal linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ preserves the Euclidean metric, as does every translation $x \mapsto b + x$. It follows that every affine transformation $x \mapsto b + Ax$ with A orthogonal is an isometry.

It turns out that the set of all such isometries can be realized as a Lie group acting smoothly on $\mathbb{R}^n$. Regard $\mathbb{R}^n$ as a Lie group under addition, and let $\theta : \mathrm{O}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ be the natural action of $\mathrm{O}(n)$ on $\mathbb{R}^n$. Define the Euclidean group $\mathrm{E}(n)$ to be the semidirect product $\mathbb{R}^n \rtimes_\theta \mathrm{O}(n)$ determined by this action: this is the Lie group whose underlying manifold is the product space $\mathbb{R}^n \times \mathrm{O}(n)$, with multiplication given by $(b, A)(b', A') = (b + Ab', AA')$. It has a faithful representation given by the map $\rho : \mathrm{E}(n) \to \mathrm{GL}(n+1, \mathbb{R})$ defined in block form by

$$\rho(b, A) = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

where b is considered an $n \times 1$ column matrix. The Euclidean group acts on $\mathbb{R}^n$ via

$$(b, A) \cdot x = b + Ax \tag{2.50}$$

Note that $\mathrm{E}(n) = \mathbb{R}^n \rtimes_\theta \mathrm{O}(n)$ as a semidirect product of Lie groups is a Lie group (see [11, p.168]) and that (2.50) indeed defines a group action: $(b', A') \cdot ((b, A) \cdot x) = b + A'b + A'Ax = ((b', A')(b, A)) \cdot x$, with $\mathrm{id} = (0, I)$.

Each Euclidean space is frame-homogeneous:

Exercise 2.5.2. *Show that (2.50) defines a smooth isometric action of* $\mathrm{E}(n)$ *on* $(\mathbb{R}^n, \bar{g})$*, and the induced action on the orthogonal frame bundle* $\mathrm{O}(\mathbb{R}^n)$ *is transitive.*

2.5.2 Spheres

Our second class of model Riemannian manifolds comes in a family, with one for each positive real number. Given $R > 0$, let $\mathbb{S}^n(R)$ denote the sphere of radius R centered at the origin in $\mathbb{R}^{n+1}$, endowed with the metric $\mathring{g}_R$ (called the **round metric of radius** R) induced from the Euclidean metric on $\mathbb{R}^{n+1}$, i.e., $\mathring{g}_R = \iota^*_{\mathbb{S}^n(R)} \bar{g}$. When $R = 1$, it is the round metric on $\mathbb{S}^n(1) = \mathbb{S}^n$, and we use the notation $\mathring{g} = \mathring{g}_1$.

One of the first things one notices about the spheres is that like Euclidean spaces, they are highly symmetric. We can immediately write down a large group of isometries of $\mathbb{S}^n(R)$ by observing that the linear action of the orthogonal group $\mathrm{O}(n+1)$ on $\mathbb{R}^{n+1}$ preserves $\mathbb{S}^n(R)$ and the Euclidean metric, so its restriction to $\mathbb{S}^n(R)$ acts isometrically on the sphere. Problem 2.6.31 will show that this is the full isometry group.

Proposition 2.5.3. *The group* $\mathrm{O}(n+1)$ *acts transitively on* $\mathrm{O}(\mathbb{S}^n(R))$*, and thus each round sphere is frame-homogeneous.*

Proof. See [12, Proposition 3.2]. ■

Another important feature of the round metrics–one that is much less evident than their symmetry–is that they bear a certain close relationship to the Euclidean metrics, which we now describe. Two metrics g_1 and g_2 on a manifold M are said to be **conformally related** (or **pointwise conformal** or just **conformal**) to each other if there is a positive function $f \in C^\infty(M)$ such that $g_2 = f g_1$. Given two Riemannian manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$, a diffeomorphism $\varphi : M \to \widetilde{M}$ is called a **conformal diffeomorphism** (or a **conformal transformation**) if it pulls $\tilde{g}$ back to a metric that is conformal to g:

$$\varphi^* \widetilde{g} = f g \text{ for some positive } f \in C^\infty(M)$$

Problem 2.6.17 shows that conformal diffeomorphisms are the same as angle-preserving diffeomorphisms. Two Riemannian manifolds are said to be **conformally equivalent** if there is a conformal diffeomorphism between them.

A Riemannian manifold (M, g) is said to be **locally conformally flat** if every point of M has a neighborhood that is conformally equivalent to an open set in $(\mathbb{R}^n, \bar{g})$.

Exercise 2.5.4. *(a) Show that for every smooth manifold $\boldsymbol{M}$, conformality is an equivalence relation on the set of all Riemannian metrics on M.*

(b) Show that conformal equivalence is an equivalence relation on the class of all Riemannian manifolds.

(c) Suppose g_1 and $g_2 = f g_1$ are conformally related metrics on an oriented n-manifold. Show that their volume forms are related by $\mathrm{d}V_{g_2} = f^{n/2}\,\mathrm{d}V_{g_1}$.

A conformal equivalence between $\mathbb{R}^n$ and $\mathbb{S}^n(R)$ minus a point is provided by stereographic projection from the north pole. This is the map $\sigma : \mathbb{S}^n(R)\backslash\{N\} \to \mathbb{R}^n$ that sends a point $P \in \mathbb{S}^n(R)\backslash\{N\}$, written $P = \left(\xi^1, \ldots, \xi^n, \tau\right)$, to $u = \left(u^1, \ldots, u^n\right) \in \mathbb{R}^n$, where $U = \left(u^1, \ldots, u^n, 0\right)$ is the point where the line through N and P intersects the hyperplane $\{(\xi, \tau) : \tau = 0\}$ in $\mathbb{R}^{n+1}$. Thus U is characterized by the fact that $(U - N) = \lambda(P - N)$ for some nonzero scalar λ. Writing $N = (0, R), U = (u, 0)$, and $P = (\xi, \tau) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, we obtain the system of equations

$$\begin{aligned} u^i &= \lambda \xi^i \\ -R &= \lambda(\tau - R) \end{aligned} \tag{2.51}$$

Solving the second equation for λ and plugging it into the first equation, we get the following formula for stereographic projection from the north pole of the sphere of radius R:

$$\sigma(\xi, \tau) = u = \frac{R\xi}{R - \tau} \tag{2.52}$$

It follows from this formula that σ is defined and smooth on all of $\mathbb{S}^n(R)\backslash\{N\}$. The easiest way to see that it is a diffeomorphism is to compute its inverse. Solving the two equations of (2.51) for τ and ξ^i gives

$$\xi^i = \frac{u^i}{\lambda}, \quad \tau = R\frac{\lambda - 1}{\lambda} \tag{2.53}$$

The point $P = \sigma^{-1}(u)$ is characterized by these equations and the fact that P is on the sphere. Thus, substituting (2.53) into $|\xi|^2 + \tau^2 = R^2$ gives

$$\frac{|u|^2}{\lambda^2} + R^2\frac{(\lambda - 1)^2}{\lambda^2} = R^2$$

from which we conclude

$$\lambda = \frac{|u|^2 + R^2}{2R^2}$$

Inserting this back into (2.53) gives the formula

$$\sigma^{-1}(u) = (\xi, \tau) = \left(\frac{2R^2 u}{|u|^2 + R^2}, R\frac{|u|^2 - R^2}{|u|^2 + R^2}\right) \tag{2.54}$$

which by construction maps $\mathbb{R}^n$ back to $\mathbb{S}^n(R)\backslash\{N\}$ and shows that σ is a diffeomorphism.

Proposition 2.5.5. *Stereographic projection is a conformal diffeomorphism between $\mathbb{S}^n(R)\backslash\{N\}$ and $\mathbb{R}^n$.*

Proof. Rigorously we have

$$\sigma^{-1} : (\mathbb{R}^n, \bar{g}) \longrightarrow (M = \mathbb{S}^n(R) \setminus N, \overbrace{\iota_M^* \iota_{\mathbb{S}^n}^* \bar{g}}^{=\iota^* \bar{g}})$$

where $\iota : M \to \mathbb{R}^{n+1}$ is the composition $\iota_{\mathbb{S}^n} \circ \iota_M$. In light of the context of equation (2.4), σ^{-1} is a smooth parametrization of M. In this case, the manifold we parametrize is an immersed Riemannian submanifold of the Euclidean space $\mathbb{R}^{n+1}$, so the equation (2.4) works perfectly. Due to (2.54), we compute:

$$\left(\sigma^{-1}\right)^* \mathring{g}_R = \left(\sigma^{-1}\right)^* \bar{g} = \sum_{j=1}^{n+1} \left(\mathrm{d}(u^j \circ \sigma^{-1})\right)^2 = \sum_{j=1}^{n} \left(\mathrm{d}\left(\frac{2R^2 u^j}{|u|^2+R^2}\right)\right)^2 + \left(\mathrm{d}\left(R\frac{|u|^2-R^2}{|u|^2+R^2}\right)\right)^2.$$

If we expand each of these terms individually, we get

$$\mathrm{d}\left(\frac{2R^2 u^j}{|u|^2+R^2}\right) = -\sum_{i\neq j} \frac{2R^2 \cdot 2u_i u_j}{(|u|^2+R^2)^2}\,\mathrm{d}u_j + \frac{(|u|^2+R^2)2R^2 - (2R^2u_j)(2u_j)}{(|u|^2+R^2)^2}\,\mathrm{d}u_j = \frac{2R^2\,\mathrm{d}u^j}{|u|^2+R^2} - \frac{4R^2u^j\sum_i u^i\,\mathrm{d}u^i}{\left(|u|^2+R^2\right)^2};$$

$$\mathrm{d}\left(R\frac{|u|^2-R^2}{|u|^2+R^2}\right) = \frac{2R\sum_i u^i\,\mathrm{d}u^i}{|u|^2+R^2} - \frac{2R\left(|u|^2-R^2\right)\sum_i u^i\,\mathrm{d}u^i}{\left(|u|^2+R^2\right)^2} = \frac{4R^3\sum_i u^i\,\mathrm{d}u^i}{\left(|u|^2+R^2\right)^2}.$$

Therefore,

$$\begin{aligned}\left(\sigma^{-1}\right)^* \mathring{g}_R =& \frac{4R^4\sum_j\left(\mathrm{d}u^j\right)^2}{\left(|u|^2+R^2\right)^2} - \frac{16R^4\left(\sum_i u^i\,\mathrm{d}u^i\right)^2}{\left(|u|^2+R^2\right)^3} + \frac{16R^4|u|^2\left(\sum_i u^i\,\mathrm{d}u^i\right)^2}{\left(|u|^2+R^2\right)^4} + \frac{16R^6\left(\sum_i u^i\,\mathrm{d}u^i\right)^2}{\left(|u|^2+R^2\right)^4}\\ =& \frac{4R^4\sum_j(\mathrm{d}u^j)^2}{\left(|u|^2+R^2\right)^2}\end{aligned}$$

In other words,

$$\left(\sigma^{-1}\right)^* \mathring{g}_R = f\bar{g}, \text{ where } f(u) = \frac{4R^4}{\left(|u|^2+R^2\right)^2} \tag{2.55}$$

Here, $\bar{g}$ now represents the Euclidean metric on $\mathbb{R}^n$, and so σ is a conformal diffeomorphism. ∎

Corollary 2.5.6. *Each sphere with a round metric is locally conformally flat.*

Proof. Stereographic projection gives a conformal equivalence between a neighborhood of any point except the north pole and Euclidean space; applying a suitable rotation and then stereographic projection (or stereographic projection from the south pole), we get such an equivalence for a neighborhood of the north pole as well. ∎

2.5.3 Hyperbolic Spaces

Our third class of model Riemannian manifolds is perhaps less familiar than the other two. For each $n \geq 1$ and each $R > 0$ we will define a frame-homogeneous Riemannian manifold $\mathbb{H}^n(R)$, called hyperbolic space of radius R. There are four equivalent models of the hyperbolic spaces, each of which is useful in certain contexts. In the next theorem, we introduce all of them and show that they are isometric.

Theorem 2.5.7 (Models of Hyperbolic Space)**.** *Let n be an integer greater than 1. For each fixed $R > 0$, the following Riemannian manifolds are all mutually isometric.*

(a) *(HYPERBLOID MODEL) $\mathbb{H}^n(R)$ is the submanifold of Minkowski space $\mathbb{R}^{n,1}$ defined in standard coordinates $\left(\xi^1, \ldots, \xi^n, \tau\right)$ as the "upper sheet" $\{\tau > 0\}$ of the two-sheeted hyperboloid $\left(\xi^1\right)^2 + \cdots + (\xi^n)^2 - \tau^2 = -R^2$, with the induced metric*

$$\breve{g}_R^1 = \iota^* \bar{q},$$

where $\iota : \mathbb{H}^n(R) \to \mathbb{R}^{n,1}$ is inclusion, and $\bar{q} = \bar{q}^{(n,1)}$ is the Minkowski metric:

$$\bar{q} = \left(\mathrm{d}\xi^1\right)^2 + \cdots + (\mathrm{d}\xi^n)^2 - (\mathrm{d}\tau)^2$$

(b) *(BELTRAMI-KLEIN MODEL) $\mathbb{K}^n(R)$ is the ball of radius R centered at the origin in $\mathbb{R}^n$, with the metric given in coordinates $(w^1,\ldots,w^n)$ by*

$$\breve{g}_R^2 = R^2\frac{(\mathrm{d}w^1)^2+\cdots+(\mathrm{d}w^n)^2}{R^2-|w|^2} + R^2\frac{(w^1\,\mathrm{d}w^1+\cdots+w^n\,\mathrm{d}w^n)^2}{(R^2-|w|^2)^2}$$

(c) *(POINCARÉ BALL MODEL) $\mathbb{B}^n(R)$ is the ball of radius R centered at the origin in $\mathbb{R}^n$, with the metric given in coordinates $(u^1,\ldots,u^n)$ by*

$$\breve{g}_R^3 = 4R^4\frac{(\mathrm{d}u^1)^2+\cdots+(\mathrm{d}u^n)^2}{(R^2-|u|^2)^2}$$

(d) *(POINCARÉ HALF-SPACE MODEL) $\mathbb{U}^n(R)$ is the upper half-space in $\mathbb{R}^n$ defined in coordinates $(x^1,\ldots,x^{n-1},y)$ by $\mathbb{U}^n(R)=\{(x,y): y>0\}$, endowed with the metric*

$$\breve{g}_R^4 = R^2\frac{(\mathrm{d}x^1)^2+\cdots+(\mathrm{d}x^{n-1})^2+\mathrm{d}y^2}{y^2}$$

Proof. Figure 2.5 below illustrates the four models of hyperbolic space.

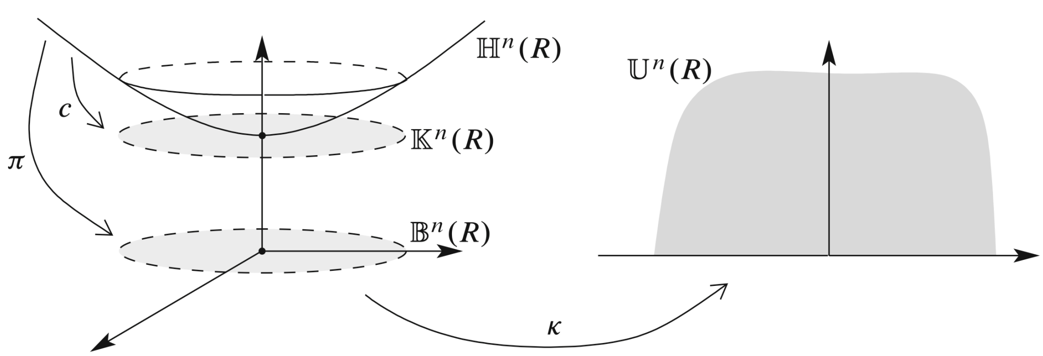

Figure 2.5: The four hyperbolic models.

The three isometries are **central projection** $c:\mathbb{H}^n(R)\to\mathbb{K}^n(R)$, **hyperbolic stereographic projection** $\pi:\mathbb{H}^n(R)\to\mathbb{B}^n(R)$, and **generalized Cayley transform** $\kappa:\mathbb{U}^n(R)\to\mathbb{B}^n(R)$. For the explicit formulas of these isometries, see [12] Theorem 3.7. ■

We often use the generic notation $\mathbb{H}^n(R)$ to refer to any one of the Riemannian manifolds of Theorem 2.5.7, and $\breve{g}_R$ to refer to the corresponding metric; the special case $R=1$ is denoted by $(\mathbb{H}^n,\breve{g})$ and is called simply **hyperbolic space**, or in the 2-dimensional case, the **hyperbolic plane**.

Because all of the models for a given value of R are isometric to each other, when analyzing them geometrically we can use whichever model is most convenient for the application we have in mind. The next corollary is an example in which the Poincaré ball and half-space models serve best.

Corollary 2.5.8. *Each hyperbolic space is locally conformally flat.*

Proof. In either the Poincaré ball model or the half-space model, the identity map gives a global conformal equivalence with an open subset of Euclidean space. ■

The examples presented so far might give the impression that most Riemannian manifolds are locally conformally flat. This is far from the truth, but we do not yet have the tools to prove it. See [12, Problem 8-25] for some explicit examples of Riemannian manifolds that are not locally conformally flat.

The symmetries of $\mathbb{H}^n(R)$ are most easily seen in the hyperboloid model. Let $\mathrm{O}(n,1)$ denote the group of linear maps from $\mathbb{R}^{n,1}$ to itself that preserve the Minkowski metric, called the $(n+1)$**-dimensional Lorentz group**. Note that each element of $\mathrm{O}(n,1)$ preserves the hyperboloid $\left\{\tau^2-|\xi|^2=R^2\right\}$, which has two components determined by $\tau>0$ and $\tau<0$. We let $\mathrm{O}^+(n,1)$ denote the subgroup of $\mathrm{O}(n,1)$ consisting of maps that take the $\tau>0$ component of the hyperboloid to itself. (This is called the **orthochronous Lorentz group**, because physically it represents coordinate changes that preserve the forward time direction.) Then $\mathrm{O}^+(n,1)$ preserves $\mathbb{H}^n(R)$, and because it preserves $\bar{q}$ it acts isometrically on $\mathbb{H}^n(R)$. (Problem 3.6.1 will show that this is the full isometry group.) Recall that $\mathrm{O}\left(\mathbb{H}^n(R)\right)$ denotes the set of all orthonormal bases for all tangent spaces of $\mathbb{H}^n(R)$.

Proposition 2.5.9 ([12] Proposition 3.9)**.** *The group $\mathrm{O}^+(n,1)$ acts transitively on $\mathrm{O}\left(\mathbb{H}^n(R)\right)$, and therefore $\mathbb{H}^n(R)$ is frame-homogeneous.*

2.5.4 Invariant Metrics on Lie Groups

Let G be a Lie group. A Riemannian metric g on G is said to be **left-invariant** if it is left-invariant as a tensor field, i.e., invariant under all left translations:

$$\forall \varphi \in G, \quad L_\varphi^* g=g$$

Similarly, g is **right-invariant** if it is invariant under all right translations, and **bi-invariant** if it is both left- and right-invariant. The next lemma shows that left-invariant metrics are easy to come by.

Lemma 2.5.10. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra of left-invariant vector fields.*

(a) A Riemannian metric g on G is left-invariant if and only if for all $X, Y\in\mathfrak{g}$, the function $g(X,Y)$ is constant on G.

(b) The restriction map $g\mapsto g_e\in\Sigma^2\left(T_e^*G\right)$ together with the natural identification $T_eG\cong\mathfrak{g}$ gives a bijection between left-invariant Riemannian metrics on G and inner products on $\mathfrak{g}$.

Proof. (a): For a fixed $p\in G$, every $q\in G$ can be written as $q=\varphi p$ for some $\varphi\in G$ (just letting $\varphi=qp^{-1}$). Thus, $g(X,Y)$ is constant on G iff $\forall\varphi\in G$, $g_p(X_p,Y_p)=g_{\varphi p}(X_{\varphi p},Y_{\varphi p})$, which, by left-invariance of X and Y, is equal to $g_{L_\varphi(p)}(\mathrm{d}(L_\varphi)_p(X_p),\ \mathrm{d}(L_\varphi)_p(Y_p))=(L_\varphi^*g)_p(X_p,Y_p)$.

(b): We give the inverse of the map $g\mapsto g_e\in\Sigma^2\left(T_e^*G\right)$. Given $g_e\in\Sigma^2\left(T_e^*G\right)$ we define g by letting

$$\forall u,v\in T_pG,\quad g_p(u,v)=g_e(\mathrm{d}(L_{p^{-1}})_p(u),\ \mathrm{d}(L_{p^{-1}})_p(v)). \tag{2.56}$$

For each $p\in G$, g_p defined in this way is certainly an inner product. It is not hard to show g is smooth. In light of part (a), to show g is left-invariant, it suffices to show $p\mapsto g_p(X_p,Y_p)$ is a constant for any pair of left-invariant vector fields on G: $\mathrm{d}\left(L_{p^{-1}}\right)_p X_p=X_{L_{p^{-1}}(p)}=X_e$ and similarly for Y_e imply that $g_p(X_p,Y_p)=g_e(X_e,Y_e)$ for any $p\in G$. That is, $g(X,Y)$ is constant on G. ■

Thus all we need to do to construct a left-invariant metric is choose any inner product on $\mathfrak{g}$, and define a metric on G by applying that inner product to leftinvariant vector fields. Right-invariant metrics can be constructed in a similar way using right-invariant vector fields. Since a Lie group acts transitively on itself by either left or right translation, every left-invariant or right-invariant metric is homogeneous.

Much more interesting are the bi-invariant metrics, because, as you will be able to prove later (Problems 3.6.3 and 3.6.4), their curvatures are intimately related to the structure of the Lie algebra of the group. But bi-invariant metrics are generally much rarer than left-invariant or right-invariant ones; in fact, some

Lie groups have no bi-invariant metrics at all (see [12, Problems 3-12 and 3-13]). Fortunately, there is a complete answer to the question of which Lie groups admit bi-invariant metrics, which we present in this section.

We begin with a proposition that shows how to determine whether a given left-invariant metric is bi-invariant, based on properties of the adjoint representation of the group. Recall that this is the representation $\mathrm{Ad}: G \rightarrow \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}(\varphi)=\left(C_{\varphi}\right)_{*}: \mathfrak{g} \rightarrow \mathfrak{g}$, † where $C_{\varphi}: G \rightarrow G$ is the automorphism defined by conjugation: $C_{\varphi}(\psi)=\varphi \psi \varphi^{-1}$.

Proposition 2.5.11. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Suppose g is a left-invariant Riemannian metric on G, and let $\langle\cdot, \cdot\rangle$ denote the corresponding inner product on $\mathfrak{g}$ as in Lemma 2.5.10. Then g is bi-invariant if and only if $\langle\cdot, \cdot\rangle$ is invariant under the action of $\mathrm{Ad}(G) \subseteq \mathrm{GL}(\mathfrak{g})$, in the sense that $\langle\mathrm{Ad}(\varphi) X, \mathrm{Ad}(\varphi) Y\rangle=\langle X, Y\rangle$ for all $X, Y \in \mathfrak{g}$ and $\varphi \in G$.*

Proof. We begin the proof with some preliminary computations. Suppose g is left-invariant and $\langle\cdot, \cdot\rangle$ is the associated inner product on $\mathfrak{g}$. Let $\varphi \in G$ be arbitrary, and note that C_{φ} is the composition of left multiplication by φ followed by right multiplication by φ^{-1}. Thus for every $X \in \mathfrak{g}$, left-invariance implies $\left(R_{\varphi^{-1}}\right)_{*} X=\left(R_{\varphi^{-1}}\right)_{*}\left(L_{\varphi}\right)_{*} X=\left(C_{\varphi}\right)_{*} X=\mathrm{Ad}(\varphi) X$. Therefore, for all $\psi \in G$ and $X, Y \in \mathfrak{g}$, we have

$$
\begin{aligned}
\left(\left(R_{\varphi^{-1}}\right)^{*} g\right)_{\psi}\left(X_{\psi}, Y_{\psi}\right) & =g_{\psi \varphi^{-1}}\left(\left(\left(R_{\varphi^{-1}}\right)_{*} X\right)_{\psi \varphi^{-1}},\left(\left(R_{\varphi^{-1}}\right)_{*} Y\right)_{\psi \varphi^{-1}}\right) \\
& =g_{\psi \varphi^{-1}}\left((\mathrm{Ad}(\varphi) X)_{\psi \varphi^{-1}},(\mathrm{Ad}(\varphi) Y)_{\psi \varphi^{-1}}\right) \\
& =\langle\mathrm{Ad}(\varphi) X, \mathrm{Ad}(\varphi) Y\rangle
\end{aligned}
$$

Now assume that $\langle\cdot, \cdot\rangle$ is invariant under $\mathrm{Ad}(G)$. Then the expression on the last line above is equal to $\langle X, Y\rangle=g_{\psi}\left(X_{\psi}, Y_{\psi}\right)$, which shows that $\left(R_{\varphi^{-1}}\right)^{*} g=g$. Since this is true for all $\varphi \in G$, it follows that g is bi-invariant.

Conversely, assuming that g is bi-invariant, we have $\left(R_{\varphi^{-1}}\right)^{*} g=g$ for each $\varphi \in G$, so the above computation yields

$$
\langle X, Y\rangle=g_{\psi}\left(X_{\psi}, Y_{\psi}\right)=\left(\left(R_{\varphi^{-1}}\right)^{*} g\right)_{\psi}\left(X_{\psi}, Y_{\psi}\right)=\langle\mathrm{Ad}(\varphi) X, \mathrm{Ad}(\varphi) Y\rangle
$$

which shows that $\langle\cdot, \cdot\rangle$ is $\mathrm{Ad}(G)$-invariant. ■

In order to apply the preceding proposition, we need a lemma about finding invariant inner products on vector spaces. Recall that for every finite-dimensional real vector space $V, \mathrm{GL}(V)$ denotes the Lie group of all invertible linear maps from V to itself. If H is a subgroup of $\mathrm{GL}(V)$, an inner product $\langle\cdot, \cdot\rangle$ on V is said to be H**-invariant** if $\langle h x, h y\rangle=\langle x, y\rangle$ for all $x, y \in V$ and $h \in H$.

Lemma 2.5.12. *Suppose V is a finite-dimensional real vector space and H is a subgroup of $\mathrm{GL}(V)$. There exists an H-invariant inner product on V if and only if H has compact closure in $\mathrm{GL}(V)$.*

Proof. Assume first that there exists an H-invariant inner product $\langle\cdot, \cdot\rangle$ on V. This implies that H is contained in the orthogonal subgroup $\mathrm{O}(V) \subseteq \mathrm{GL}(V)$ defined as

$$
\mathrm{O}(V)=\{A \in \mathrm{GL}(V) \mid\langle A x, A y\rangle=\langle x, y\rangle\} .
$$

Choosing an orthonormal basis of V yields a Lie group isomorphism between $\mathrm{O}(V)$ and $\mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ (where $n=\operatorname{dim} V$), so $\mathrm{O}(V)$ is compact; and the closure of H is a closed subset of this compact group, and thus is itself compact.

Conversely, suppose H has compact closure in $\mathrm{GL}(V)$, and let K denote the closure. A simple limiting argument shows that K is itself a subgroup, and thus it is a Lie group by the closed subgroup theorem.

†Here, F_* is called the **induced Lie algebra homomorphism** from the Lie group homomorphism $F: G \rightarrow H$ as in [11] Theorem 8.44, but it is in fact the same as dF_e if we identify $\mathfrak{g} \cong T_e G$ and $\mathfrak{h} \cong TeH$. If dF_e sends u to v, then F_* sends u^L to v^L.

Let $\langle\cdot,\cdot\rangle_0$ be an arbitrary inner product on V, and let μ be a right-invariant density on K (for example, the Riemannian density of some right-invariant metric on K; see subsection 2.3.4). * For fixed $x, y \in V$, define a smooth function $f_{x,y} : K \to \mathbb{R}$ by $f_{x,y}(k) = \langle kx, ky\rangle_0$. Then define a new inner product $\langle\cdot,\cdot\rangle$ on V by

$$\langle x, y\rangle = \int_K f_{x,y}\mu$$

Here, compactness of K ensures that the integral is well-defined. It follows directly from the definition that $\langle\cdot,\cdot\rangle$ is symmetric and bilinear over $\mathbb{R}$. For each nonzero $x \in V$, we have $f_{x,x} > 0$ everywhere on K, so $\langle x, x\rangle > 0$, showing that $\langle\cdot,\cdot\rangle$ is indeed an inner product.

To see that it is invariant under K, let $k_0 \in K$ be arbitrary. Then for all $x, y \in V$ and $k \in K$, we have

$$f_{k_0x,k_0y}(k) = \langle kk_0x, kk_0y\rangle_0 = f_{x,y} \circ R_{k_0}(k),$$

where $R_{k_0} : K \to K$ is right translation by k_0. Because μ is right-invariant, it follows from diffeomorphism invariance of the integral that

$$\begin{aligned}\langle k_0x, k_0y\rangle = \int_K f_{k_0x,k_0y}\mu &= \int_K \left(f_{x,y} \circ R_{k_0}\right)\mu \\ &\overset{\text{right-invariance}}{=\!=\!=} \int_K \left(f_{x,y} \circ R_{k_0}\right) R_{k_0}^*\mu \\ &\overset{\text{Prop.1.1.20(a)}}{=\!=\!=} \int_K R_{k_0}^*\left(f_{x,y}\mu\right) \\ &\overset{\text{Prop.1.7.9(d)}}{=\!=\!=} \int_K f_{x,y}\mu = \langle x, y\rangle\end{aligned}$$

Thus $\langle\cdot,\cdot\rangle$ is K-invariant, and it is also H-invariant because $H \subseteq K$. ■

Theorem 2.5.13 (Existence of Bi-invariant Metrics). *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then G admits a bi-invariant metric if and only if $\operatorname{Ad}(G)$ has compact closure in $\operatorname{GL}(\mathfrak{g})$.*

Proof. Proposition 2.5.11 shows that there is a bi-invariant metric on G if and only if there is an $\operatorname{Ad}(G)$-invariant inner product on $\mathfrak{g}$, and Lemma 2.5.12 in turn shows that the latter is true if and only if $\operatorname{Ad}(G)$ has compact closure in $\operatorname{GL}(\mathfrak{g})$. ■

The most important application of the preceding theorem is to compact groups.

Corollary 2.5.14 (Compact Lie Groups). *Every compact Lie group admits a bi-invariant Riemannian metric.*

Proof. If G is compact, then $\operatorname{Ad}(G)$ is a compact subgroup of $\operatorname{GL}(\mathfrak{g})$ because $\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g})$ is continuous. Now, a continuous mapping from a compact space to a Hausdorff space is a closed map, so the closure of $\operatorname{Ad}(G)$ just itself. The previous theorem thus applies. ■

Another important application is to prove that certain Lie groups do not admit bi-invariant metrics. One way to do this is to note that if $\operatorname{Ad}(G)$ has compact closure in $\operatorname{GL}(\mathfrak{g})$, then every orbit of $\operatorname{Ad}(G)$ must be a bounded subset of $\mathfrak{g}$ with respect to any choice of norm, because it is contained in the image of the compact set $\overline{\operatorname{Ad}(G)}$ under a continuous map of the form $\varphi \mapsto \varphi\left(X_0\right)$ from $\operatorname{GL}(\mathfrak{g})$ to $\mathfrak{g}$. Thus if one can find an element

*I don't see why we cannot just use Riemannian volume form but Riemannian density here. Whichever we use, we should note that left (right)-invariant metrics produce left (right)-invariant Riemannian volume forms/Riemannian densities; see Lemma 2.6.21. To use Riemannian volume form, we need to first give K a right-invariant orientation in the same way as Proposition 1.7.18 and then define a right-invariant metric in the same way as (2.56). Perhaps Jack Lee doesn't bother putting too much in his errata to explain the right other-half.

$X_0 \in \mathfrak{g}$ and a subset $S \subseteq G$ such that the elements of the form $\operatorname{Ad}(\varphi)X$ are unbounded in $\mathfrak{g}$ for $\varphi \in S$, then there is no bi-invariant metric.

Here are some examples.

Example 2.5.15 (Invariant Metrics on Lie Groups).

(a) Every left-invariant metric on an abelian Lie group is bi-invariant, because the adjoint representation is trivial. Thus the Euclidean metric on $\mathbb{R}^n$ and the flat metric on $\mathbb{T}^n$ of Example 2.2.11 are both bi-invariant.

(b) If a metric g on a Lie group G is left-invariant, then the induced metric on every Lie subgroup $H \subseteq G$ is easily seen to be left-invariant. Similarly, if g is bi-invariant, then the induced metric on H is bi-invariant.

(c) The Lie group $\mathrm{SL}(2,\mathbb{R})$ (the group of 2×2 real matrices of determinant 1) admits many left-invariant metrics (as does every positive-dimensional Lie group), but no bi-invariant ones. To see this, recall that the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to the algebra $\mathfrak{sl}(2,\mathbb{R})$ of trace-free 2×2 matrices, and the adjoint representation is given by $\operatorname{Ad}(A)X = AXA^{-1}$. If we let $X_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$ and $A_c = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ for $c>0$, then $\operatorname{Ad}(A_c)X_0 = \begin{pmatrix} 0 & c^2 \\ 0 & 0 \end{pmatrix}$, which is unbounded as $c \to \infty$. Thus the orbit of X_0 is not contained in any compact subset, which implies that there is no bi-invariant metric on $\mathrm{SL}(2,\mathbb{R})$. A similar argument shows that $\mathrm{SL}(n,\mathbb{R})$ admits no bi-invariant metric for any $n \geq 2$. In view of (b) above, this shows also that $\mathrm{GL}(n,\mathbb{R})$ admits no bi-invariant metric for $n\geq 2$. (Of course, $\mathrm{GL}(1,\mathbb{R})$ does admit bi-invariant metrics because it is abelian.)

(d) With $\mathbb{S}^3$ regarded as a submanifold of $\mathbb{C}^2$, the map

$$(w,z) \mapsto \begin{pmatrix} w & z \\ -\bar{z} & \bar{w} \end{pmatrix} \tag{2.57}$$

gives a diffeomorphism from $\mathbb{S}^3$ to $\mathrm{SU}(2)$. Under the inverse of this map, the round metric on $\mathbb{S}^3$ pulls back to a bi-invariant metric on $\mathrm{SU}(2)$, as Problem 2.6.22 shows.

(e) Let $\mathfrak{o}(n)$ denote the Lie algebra of $\mathrm{O}(n)$, identified with the algebra of skewsymmetric $n\times n$ matrices, and define a bilinear form on $\mathfrak{o}(n)$ by

$$\langle A, B\rangle = \operatorname{tr}\left(A^T B\right)$$

This is an Ad-invariant inner product, and thus determines a bi-invariant Riemannian metric on $\mathrm{O}(n)$ (see Problem 2.6.24).

(f) Let $\mathbb{U}^n$ be the upper half-space as defined in Theorem 2.5.7. We can regard $\mathbb{U}^n$ as a Lie group by identifying each point $(x,y) = \left(x^1,\ldots,x^{n-1},y\right) \in \mathbb{U}^n$ with an invertible $n\times n$ matrix as follows:

$$(x,y) \quad \longleftrightarrow \begin{pmatrix} I_{n-1} & 0 \\ x^T & y \end{pmatrix}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. Then the hyperbolic metric $\breve{g}_R^4$ is left-invariant on $\mathbb{U}^n$ but not right-invariant (see Problem 2.6.25).

(g) For $n \geq 1$, the $(2n+1)$-dimensional **Heisenberg group** is the Lie subgroup $H_n \subseteq \mathrm{GL}(n+2,\mathbb{R})$ defined by

$$H_n = \left\{ \begin{pmatrix} 1 & x^T & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\}$$

where x and y are treated as column matrices. These are the simplest examples of **nilpotent Lie groups**, meaning that the series of subgroups $G \supseteq [G,G] \supseteq [G,[G,G]] \supseteq \cdots$ eventually reaches the trivial subgroup (where for any subgroups $G_1, G_2 \subseteq G$, the notation $[G_1, G_2]$ means the subgroup of G generated by all elements of the form $x_1 x_2 x_1^{-1} x_2^{-1}$ for $x_i \in G_i$). There are many leftinvariant metrics on H_n, but no bi-invariant ones, as Problem 2.6.26 shows.

(h) Our last example is a group that plays an important role in the classification of 3-manifolds. Let Sol denote the following 3-dimensional Lie subgroup of $\mathrm{GL}(3,\mathbb{R})$:

$$\mathrm{Sol} = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$$

This group is the simplest non-nilpotent example of a **solvable Lie group**, meaning that the series of subgroups $G \supseteq [G,G] \supseteq [[G,G],[G,G]] \supseteq \cdots$ eventually reaches the trivial subgroup. Like the Heisenberg groups, Sol admits left-invariant metrics but not bi-invariant ones (Problem 2.6.27).

♣

2.5.5 Other Homogeneous Riemannian Manifolds

There are many homogeneous Riemannian manifolds besides the frame-homogeneous ones and the Lie groups with invariant metrics. To identify other examples, it is natural to ask the following question: If M is a smooth manifold endowed with a smooth, transitive action by a Lie group G (called a **homogeneous G-space** or just a **homogeneous space**), is there a Riemannian metric on M that is invariant under the group action?

The next theorem gives a necessary and sufficient condition for existence of an invariant Riemannian metric that is usually easy to check.

Theorem 2.5.16 (Existence of Invariant Metrics on Homogeneous Spaces)**.** *Suppose G is a Lie group and M is a homogeneous G-space. Let p_0 be a point in M, and let $I_{p_0} : G_{p_0} \to \mathrm{GL}\left(T_{p_0} M\right)$ denote the isotropy representation at p_0. There exists a G-invariant Riemannian metric on M if and only if $I_{p_0}\left(G_{p_0}\right)$ has compact closure in $\mathrm{GL}(T_{p_0} M)$.*

Proof. See [12, Theorem 3.17]. ■

The next corollary, which follows immediately from Theorem 2.5.16, addresses the most commonly encountered case. (Other necessary and sufficient conditions for the existence of invariant metrics are given in Walter Poor's *Differential Geometric Structures* 6.58-6.59.)

Corollary 2.5.17. *If a Lie group G acts smoothly and transitively on a smooth manifold M with compact isotropy groups, then there exists a G-invariant Riemannian metric on M.*

Locally Homogeneous Riemannian Manifolds

A Riemannian manifold (M,g) is said to be **locally homogeneous** if for every pair of points $p,q \in M$ there is a Riemannian isometry from a neighborhood of p to a neighborhood of q that takes p to q. Similarly, we say that (M,g) is **locally frame-homogeneous** if for every $p,q \in M$ and every pair of orthonormal bases (v_i) for $T_p M$ and (w_i) for $T_q M$, there is an isometry from a neighborhood of p to a neighborhood of q that takes p to q, and whose differential takes v_i to w_i for each i.

Every homogeneous Riemannian manifold is locally homogeneous, and every frame-homogeneous one is locally frame-homogeneous. Every proper open subset of a homogeneous or frame-homogeneous Riemannian manifold is locally homogeneous or locally frame-homogeneous, respectively. More interesting examples arise in the following way.

Proposition 2.5.18 ([12] Proposition 3.20)**.** *Suppose $(\widetilde{M}, \widetilde{g})$ is a homogeneous Riemannian manifold, (M, g) is a Riemannian manifold, and $\pi : \widetilde{M} \to M$ is a Riemannian covering. Then (M, g) is locally homogeneous. If $(\widetilde{M}, \widetilde{g})$ is frame-homogeneous, then (M, g) is locally frame-homogeneous.*

Locally homogeneous Riemannian metrics play an important role in classification theorems for manifolds, especially in low dimensions. The most fundamental case is that of compact 2-manifolds, for which we have the following important theorem.

Theorem 2.5.19 (Uniformization of Compact Surfaces)**.** *Every compact, connected, smooth 2-manifold admits a locally frame-homogeneous Riemannian metric, and a Riemannian covering by the Euclidean plane, hyperbolic plane, or round unit sphere.*

Proof. See [12, Theorem 3.22]. The proof relies on the topological classification of compact surfaces (see, for example, [10, Thms. 6.15 and 10.22]). ∎

Locally homogeneous metrics also play a key role in the classification of compact 3-manifolds. In 1982, William Thurston made a conjecture about the classification of such manifolds, now known as the Thurston geometrization conjecture. The conjecture says that every compact, orientable 3-manifold can be expressed as a connected sum of compact manifolds, each of which either admits a Riemannian covering by a homogeneous Riemannian manifold or can be cut along embedded tori so that each piece admits a finite-volume locally homogeneous Riemannian metric. An important ingredient in the analysis leading up to the conjecture was his classification of all simply connected homogeneous Riemannian 3-manifolds that admit finite-volume Riemannian quotients. Thurston showed that there are exactly eight such manifolds (see [Thu97] or [Sco83] for a proof):

- $\mathbb{R}^3$ with the Euclidean metric
- $\mathbb{S}^3$ with a round metric
- $\mathbb{H}^3$ with a hyperbolic metric
- $\mathbb{S}^2 \times \mathbb{R}$ with a product of a round metric and the Euclidean metric
- $\mathbb{H}^2 \times \mathbb{R}$ with a product of a hyperbolic metric and the Euclidean metric
- The Heisenberg group H_1 of Example 2.5.15 (g) with a left-invariant metric
- The group Sol of Example 2.5.15 (h) with a left-invariant metric
- The universal covering group of $\mathrm{SL}(2, \mathbb{R})$ with a left-invariant metric

The Thurston geometrization conjecture was proved in 2003 by Grigori Perelman. The proof is described in several books [BBBMP, KL08, MF10, MT14].

Symmetric Spaces

We end this section with a brief introduction to another class of Riemannian manifolds with abundant symmetry, called symmetric spaces. They turn out to be intermediate between frame-homogeneous and homogeneous Riemannian manifolds (see Problem 3.6.16).

Here is the definition. If (M, g) is a Riemannian manifold and $p \in M$, a **point reflection at** p is an isometry $\varphi : M \to M$ that fixes p and satisfies $d\varphi_p = -\operatorname{Id} : T_pM \to T_pM$. A Riemannian manifold (M, g) is called a **(Riemannian) symmetric space** if it is connected and for each $p \in M$ there exists a point reflection at p. More generally, (M, g) is called a **(Riemannian) locally symmetric space** if each $p \in M$ has a neighborhood U on which there exists an isometry $\varphi : U \to U$ that is a point reflection at p. Clearly every Riemannian symmetric space is locally symmetric.

The next lemma can be used to facilitate the verification that a given Riemannian manifold is symmetric.

Lemma 2.5.20. *If (M,g) is a connected homogeneous Riemannian manifold that possesses a point reflection at one point, then it is symmetric.*

Proof. Suppose (M,g) satisfies the hypothesis, and let $\varphi: M \to M$ be a point reflection at $p \in M$. Given any other point $q \in M$, by homogeneity there is an isometry $\psi: M \to M$ satisfying $\psi(p)=q$. Then $\widetilde{\varphi}=\psi\circ\varphi\circ\psi^{-1}$ is an isometry that fixes q. Because $d\psi_p$ is linear, it commutes with multiplication by -1, so

$$d\tilde{\varphi}_q = d\psi_p \circ \left(-\mathrm{Id}_{T_pM}\right)\circ d\left(\psi^{-1}\right)_q = \left(-\mathrm{Id}_{T_qM}\right)\circ d\psi_p \circ d\left(\psi^{-1}\right)_q = -\mathrm{Id}_{T_qM}$$

Thus $\widetilde{\varphi}$ is a point reflection at q. ■

Example 2.5.21 (Riemannian Symmetric Spaces)**.**

(a) Suppose (M,g) is any connected frame-homogeneous Riemannian manifold. Then for each $p \in M$, we can choose an orthonormal basis (b_i) for T_pM, and frame homogeneity guarantees that there is an isometry $\varphi: M \to M$ that fixes p and sends (b_i) to $(-b_i)$, which implies that $d\varphi_p = -\mathrm{Id}$. Thus every frame-homogeneous Riemannian manifold is a symmetric space. In particular, all Euclidean spaces, spheres, and hyperbolic spaces are symmetric.

(b) Suppose G is a connected Lie group with a bi-invariant Riemannian metric g. If we define $\Phi: G \to G$ by $\Phi(x)=x^{-1}$, then it is straightforward to check that $d\Phi_e(v)=-v$ for every $v \in T_eG$, from which it follows that $d\Phi_e^*\left(g_e\right)=g_e$. To see that Φ is an isometry, let $p \in G$ be arbitrary. The identity $q^{-1}=\left(p^{-1}q\right)^{-1}p^{-1}$ for all $q \in G$ implies that $\Phi = R_{p^{-1}}\circ\Phi\circ L_{p^{-1}}$, and therefore it follows from bi-invariance of g that

$$\left(\Phi^* g\right)_p = d\Phi_p^* g_{p^{-1}} = d\left(L_{p^{-1}}\right)_p^* \circ d\Phi_e^* \circ d\left(R_{p^{-1}}\right)_e^* g_{p^{-1}} = g_p$$

Therefore Φ is an isometry of g and hence a point reflection at e. Lemma 2.5.20 then implies that (G,g) is a symmetric space.

(c) The complex projective spaces (Example 2.2.21) and the Grassmann manifolds (Problem 2.6.7) are all Riemannian symmetric spaces (see Problems 2.6.29 and 2.6.30).

(d) Every product of Riemannian symmetric spaces is easily seen to be a symmetric space when endowed with the product metric. A symmetric space is said to be irreducible if it is not isometric to a product of positive-dimensional symmetric spaces.

♣

2.6 Problems

Exercise 2.6.1 ([12] 1-4)**.** *A topological space is said to be σ-compact if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.*

Exercise 2.6.2 ([12] 2-1)**.** *Show that every Riemannian* 1*-manifold is flat.*

Exercise 2.6.3 ([11] 13-1)**.** *If (M,g) is a Riemannian n-manifold with or without boundary, let $UM \subseteq TM$ be the subset $UM=\{(x,v)\in TM : |v|_g = 1\}$, called the unit tangent bundle of M. Show that UM is a smooth fiber bundle over M with model fiber $\mathbb{S}^{n-1}$.*

Solution. Since 1 is a regular value of the map $\varphi(p,v):=|v|_g^2$, we know that $UM=\varphi^{-1}(1)$ is an embedded submanifold, and we can restrict $\pi: TM \to M$ to obtain $\pi|_{UM}: UM \to M$. It is impossible in general to choose coordinates around each point which make g look like the Euclidean metric $dx^i \otimes dx^i$. However, we do have an orthonormal local frame $\sigma_1,\ldots,\sigma_n : U \to \pi^{-1}(U)$, which induces a local trivialization $U\times\mathbb{R}^n \cong \pi^{-1}(U)$ of TM via $\left(p,c^1,\ldots,c^n\right)\mapsto\left(p,c^i\sigma_i(p)\right)$. But then $\pi^{-1}(U)\cap UM = \pi^{-1}\big|_{UM}(U)\cong U\times S^{n-1}$, proving that UM is a fiber bundle with model fiber S^{n-1}. ♦

Exercise 2.6.4 ([12] 2-2)**.** *Suppose V and W are finite-dimensional real inner product spaces of the same dimension, and $F : V \to W$ is any map (not assumed to be linear or even continuous) that preserves the origin and all distances: $F(0) = 0$ and $|F(x) - F(y)| = |x - y|$ for all $x, y \in V$. Prove that F is a linear isometry. [Hint:First show that F preserves inner products, and then show that it is linear.]*

Exercise 2.6.5 ([12] 2-5)**.** *Prove parts (b) and (c) of Proposition 2.2.14 (properties of horizontal vector fields).*

Exercise 2.6.6 ([12] 2-6)**.** *Prove Theorem 2.2.19 (if $\pi : \widetilde{M} \to M$ is a surjective smooth submersion, and a group acts on $\widetilde{M}$ isometrically, vertically, and transitively on fibers, then M inherits a unique Riemannian metric such that π is a Riemannian submersion).*

Exercise 2.6.7 ([12] 2-7)**.** *For $0 < k < n$, the set $\mathrm{G}_k(\mathbb{R}^n)$ of k-dimensional linear subspaces of $\mathbb{R}^n$ is called a **Grassmann manifold** or **Grassmannian**. The group $\mathrm{GL}(n, \mathbb{R})$ acts transitively on $\mathrm{G}_k(\mathbb{R}^n)$ in an obvious way, and $\mathrm{G}_k(\mathbb{R}^n)$ has a unique smooth manifold structure making this action smooth (see [11] Example 21.21).*

*(a) Let $V_k(\mathbb{R}^n)$ denote the set of orthonormal ordered k-tuples of vectors in $\mathbb{R}^n$. By arranging the vectors in k columns, we can view $V_k(\mathbb{R}^n)$ as a subset of the vector space $\mathrm{M}(n \times k, \mathbb{R})$ of all $n \times k$ real matrices. Prove that $V_k(\mathbb{R}^n)$ is a smooth submanifold of $\mathrm{M}(n \times k, \mathbb{R})$ of dimension $k(2n - k - 1)/2$, called a **Stiefel manifold**. [Hint: Consider the map $\Phi : \mathrm{M}(n \times k, \mathbb{R}) \to \mathrm{M}(k \times k, \mathbb{R})$ given by $\Phi(A) = A^T A$.]*

(b) Show that the map $\pi : V_k(\mathbb{R}^n) \to \mathrm{G}_k(\mathbb{R}^n)$ that sends a k-tuple to its span is a surjective smooth submersion.

(c) Give $V_k(\mathbb{R}^n)$ the Riemannian metric induced from the Euclidean metric on $\mathrm{M}(n \times k, \mathbb{R})$. Show that the right action of $\mathrm{O}(k)$ on $V_k(\mathbb{R}^n)$ by matrix multiplication on the right is isometric, vertical, and transitive on fibers of π, and thus there is a unique metric on $\mathrm{G}_k(\mathbb{R}^n)$ such that π is a Riemannian submersion. [Hint: It might help to note that the Euclidean inner product on $\mathrm{M}(n \times k, \mathbb{R})$ can be written in the form $\langle A, B\rangle = \operatorname{tr}\left(A^T B\right)$.]

Exercise 2.6.8 ([12] 2-8)**.** *Prove that the action of $\mathbb{Z}$ on $\mathbb{R}^2$ defined in Example 2.2.26 is smooth, free, proper, and isometric, and therefore the open Möbius band inherits a flat Riemannian metric such that the quotient map is a Riemannian covering.*

Exercise 2.6.9 ([12] 2-9)**.** *Prove Proposition 2.3.20 (the gradient is orthogonal to regular level sets).*

Exercise 2.6.10 ([12] 2-10)**.** *Suppose (M, g) is a Riemannian manifold, $f \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$ is a nowhere-vanishing vector field. Prove that $X = \operatorname{grad} f$ if and only if $Xf \equiv \|X\|_g^2$ and X is orthogonal to the level sets of f at all regular points of f.*

Exercise 2.6.11 ([12] 2-11)**.** *Prove Proposition 2.17 (inner products on tensor bundles).*

Exercise 2.6.12 ([11] 16-19)**.** *Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric and the standard orientation.*

(a) Calculate $* \, \mathrm{d}x^i$ for $i = 1, \ldots, n$.

(b) Calculate $*(\mathrm{d}x^i \wedge \mathrm{d}x^j)$ in the case $n = 4$.

Exercise 2.6.13 ([11] 16-20)**.** *Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be **self-dual** if $*\omega = \omega$, and **anti-self-dual** if $*\omega = -\omega$.*

(a) Show that every 2-form ω on M can be written uniquely as a sum of a self-dual form and an anti-self-dual form.

(b) On $M = \mathbb{R}^4$ with the Euclidean metric, determine the self-dual and anti-self-dual forms in standard coordinates.

Exercise 2.6.14 ([11] 16-21). *Let (M, g) be an oriented Riemannian manifold and $X \in \mathfrak{X}(M)$. Show that*

$$X \lrcorner \, \mathrm{d}V_g = *X^\flat$$
$$\operatorname{div} X = * \, \mathrm{d} * X^\flat$$

and, when $\dim M = 3$, we can define $\operatorname{curl} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $\operatorname{curl}(X) = \beta^{-1} \, \mathrm{d}(X^\flat)$ where $\beta : TM \to \Lambda^2 T^* M$ is such that by $\beta(X) = X \lrcorner \, \mathrm{d}V_g$. See [11] p.426 for more details. Show that

$$\operatorname{curl} X = \left(* \, \mathrm{d}X^\flat \right)^\sharp$$

Exercise 2.6.15 ([12] 3-2). *Prove that the metric on $\mathbb{RP}^n$ described in Example 2.2.25 is frame-homogeneous.*

Exercise 2.6.16 ([12] 3-5).

(a) Prove that $\left(\mathbb{S}^n(R), \mathring{g}_R\right)$ is isometric to $\left(\mathbb{S}^n, R^2 \mathring{g}\right)$ for each $R > 0$.

(b) Prove that $(\mathbb{H}^n(R), \breve{g}_R)$ is isometric to $\left(\mathbb{H}^n, R^2 \breve{g}\right)$ for each $R > 0$.

(c) We could also have defined a family of metrics on $\mathbb{R}^n$ by $\bar{g}_R = R^2 \bar{g}$. Why did we not bother?

Exercise 2.6.17 ([12] 3-6). *Show that two Riemannian metrics g_1 and g_2 are conformal if and only if they define the same angles but not necessarily the same lengths, and that a diffeomorphism is a conformal equivalence if and only if it preserves angles. [Hint: Let (E_i) be a local orthonormal frame for g_1, and consider the g_2-angle between E_i and $(\cos\theta) E_i + (\sin\theta) E_j$]*

Exercise 2.6.18 ([12] 3-7). *Let $\mathbb{U}^2$ denote the upper half-plane model of the hyperbolic plane (of radius 1), with the metric $\breve{g} = \left(dx^2 + dy^2\right)/y^2$. Let $\mathrm{SL}(2, \mathbb{R})$ denote the group of 2×2 real matrices of determinant 1. Regard $\mathbb{U}^2$ as a subset of the complex plane with coordinate $z = x + iy$, and let*

$$A \cdot z = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Show that this defines a smooth, transitive, orientation-preserving, and isometric action of $\mathrm{SL}(2, \mathbb{R})$ on $(\mathbb{U}^2, \breve{g})$. Is the induced action transitive on $\mathrm{O}(\mathbb{U}^2)$?

Exercise 2.6.19 ([12] 3-8). *Let $\mathbb{B}^2$ denote the Poincaré disk model of the hyperbolic plane (of radius 1), with the metric $\breve{g} = \left(du^2 + dv^2\right) / \left(1 - u^2 - v^2\right)^2$, and let $G \subseteq \mathrm{GL}(2, \mathbb{C})$ be the subgroup defined by*

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 > 0 \right\}.$$

Regarding $\mathbb{B}^2$ as a subset of the complex plane with coordinate $w = u + iv$, let G act on $\mathbb{B}^2$ by

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot w = \frac{\alpha w + \beta}{\bar{\beta} w + \bar{\alpha}}$$

Show that this defines a smooth, transitive, orientation-preserving, and isometric action of G on $(\mathbb{B}^2, \breve{g})$. [Hint: One way to proceed is to define an action of G on the upper half-plane by $A \cdot z = \kappa^{-1} \circ A \circ \kappa(z)$, where κ is the ***Cayley transform*** *defined by*

$$\kappa(z) = w = iR \frac{z - iR}{z + iR}$$

in the case $R = 1$, and use the result of Problem 2.6.18.]

Exercise 2.6.20 ([12] 3-9). *Suppose G is a compact connected Lie group with a left-invariant metric g and a left-invariant orientation. Show that the Riemannian volume form $\mathrm{d}V_g$ is bi-invariant. [Hint: Show that dV_g is equal to the Riemannian volume form for a bi-invariant metric.]*

Lemma 2.6.21. *Let G be a Lie group and let g be a left (right)-invariant Riemannian metric on G. If it is oriented, let $\mathrm{d}V_g$ be the Riemannian volume form on G; otherwise, let μ_g be the Riemannian density on G. Then $\mathrm{d}V_g$ or μ_g is left (right)-invariant.*

Proof. Left (right)-invariance of the metric means exactly that left (right) translations are isometries. Now use Lemma 2.3.7 (for $\mathrm{d}V_g$) or Exercise 2.3.11 (for μ_g). ∎

Exercise 2.6.22 ([12] 3-10)**.** *Consider the basis*

$$X=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \quad Y=\left(\begin{array}{cc} 0 & i \\ i & 0 \end{array}\right), \quad Z=\left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right)$$

for the Lie algebra $\mathfrak{su}(2)$. For each positive real number a, define a leftinvariant metric g_a on the group $\mathrm{SU}(2)$ by declaring X, Y, aZ to be an orthonormal frame.

(a) Show that g_a is bi-invariant if and only if $a=1$.

(b) Show that the map defined by (2.57) is an isometry between $\left(\mathbb{S}^3, \mathring{g}\right)$ and $(\mathrm{SU}(2), g_1)$.

[Remark: $\mathrm{SU}(2)$ with any of these metrics is called a ***Berger sphere****, named after Marcel Berger.]*

Exercise 2.6.23 ([12] 8-16)**.** *For each $a>0$, let g_a be the Berger metric on $\mathrm{SU}(2)$. Compute the sectional curvatures with respect to g_a of the planes spanned by $(X, Y),(Y, Z)$, and (Z, X). Prove that if $a \neq 1$, then $(\mathrm{SU}(2), g_a)$ is homogeneous but not isotropic anywhere.*

Exercise 2.6.24 ([12] 3-11)**.** *Prove that the formula $\langle A, B\rangle=\operatorname{tr}\left(A^T B\right)$ defines a bi-invariant Riemannian metric on $\mathrm{O}(n)$.*

Exercise 2.6.25 ([12] 3-12)**.** *Regard the upper half-space $\mathbb{U}^n$ as a Lie group as described in Example 2.5.7 (f).*

(a) Show that for each $R>0$, the hyperbolic metric $\breve{g}_R^4$ on $\mathbb{U}^n$ is leftinvariant.

(b) Show that $\mathbb{U}^n$ does not admit any bi-invariant metrics.

Exercise 2.6.26 ([12] 3-13)**.** *Write down an explicit formula for an arbitrary left-invariant metric on the Heisenberg group H_n of Example 2.5.15 (g) in terms of global coordinates $\left(x^1, \ldots, x^n, y^1, \ldots, y^n, z\right)$, and show that the group has no bi-invariant metrics.*

Exercise 2.6.27 ([12] 3-14)**.** *Repeat Problem 2.6.26 for the group Sol of Example 2.5.15 (h).*

Exercise 2.6.28 ([12] 3-18)**.** *Let $\Gamma \subseteq \mathrm{E}(2)$ be the subgroup defined by (3.20). Prove that Γ acts freely and properly on $\mathbb{R}^2$ and the orbit space is homeomorphic to the Klein bottle, and conclude that the Klein bottle has a flat metric and a Riemannian covering by the Euclidean plane.*

Exercise 2.6.29 ([12] 3-19)**.** *Show that the Fubini-Study metric on $\mathbb{CP}^n$ is homogeneous, isotropic, and symmetric.*

Exercise 2.6.30 ([12] 3-20)**.** *Show that the metric on the Grassmannian $\mathrm{G}_k\left(\mathbb{R}^n\right)$ defined in [12, Problem 2-7] is homogeneous, isotropic, and symmetric.*

Exercise 2.6.31 ([12] 5-11)**.** *Recall the groups $\mathrm{E}(n), \mathrm{O}(n+1)$, and $\mathrm{O}^{+}(n, 1)$ defined previously, which act isometrically on the model Riemannian manifolds $\left(\mathbb{R}^n, \bar{g}\right),\left(\mathbb{S}^n(R), \mathring{g}_R\right)$, and $\left(\mathbb{H}^n(R), \breve{g}_R\right)$, respectively.*

(a) Show that

$$\begin{aligned} \operatorname{Iso}\left(\mathbb{R}^n, \bar{g}\right) & =\mathrm{E}(n), \\ \operatorname{Iso}\left(\mathbb{S}^n(R), \mathring{g}_R\right) & =\mathrm{O}(n+1), \\ \operatorname{Iso}\left(\mathbb{H}^n(R), \overset{y}{g}_R\right) & =\mathrm{O}^{+}(n, 1). \end{aligned}$$

(b) *Show that in each case, for each point p in $\mathbb{R}^n, \mathbb{S}^n(R)$, or $\mathbb{H}^n(R)$, the isotropy group at p is a subgroup isomorphic to $\mathrm{O}(n)$.*

(c) *Strengthen the result above by showing that if (M, g) is one of the Riemannian manifolds $(\mathbb{R}^n, \bar{g})$, $\left(\mathbb{S}^n(R), \mathring{g}_R\right)$, or $(\mathbb{H}^n(R), \breve{g}_R)$, U is a connected open subset of M, and $\varphi : U \to M$ is a local isometry, then φ is the restriction to U of an element of $\mathrm{Iso}(M, g)$.*

Exercise 2.6.32 ([12] 6-2)**.** *Let n be a positive integer and R a positive real number.*

(a) *Prove that the Riemannian distance between any two points p, q in $\mathbb{S}^n(R)$ with the round metric is given by*

$$d_{g_R}(p, q) = R \arccos \frac{\langle p, q\rangle}{R^2},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^{n+1}$.

(b) *Prove that the metric space $(\mathbb{S}^n(R), d_{g_R})$ has diameter πR.*

Exercise 2.6.33 ([12] 6-3)**.** *Let n be a positive integer and R a positive real number. Prove that the Riemannian distance between any two points in the Poincaré ball model $(\mathbb{B}^n(R), \breve{g}_R)$ of hyperbolic space of radius R is given by*

$$d_{\breve{g}_R}(p, q) = R \operatorname{arccosh}\left(1 + \frac{2R^2|p-q|^2}{(R^2 - |p|^2)(R^2 - |q|^2)}\right),$$

where $|\cdot|$ represents the Euclidean norm in $\mathbb{R}^n$. [Hint: First use the result of Problem 2.6.16 to show that it suffices to consider the case $R = 1$. Then use a rotation to reduce to the case $n = 2$, and use the group action of Problem 2.6.19 to show that it suffices to consider the case in which p is the origin.]

Exercise 2.6.34 ([12] 8-5)**.** *Let $S \subseteq \mathbb{R}^3$ be the paraboloid given by $z = x^2 + y^2$, with the induced metric. Prove that S is isotropic at only one point.*

Chapter 3

Connections, Curvatures, and Geodesics

In essense, for an abstract manifold, we can define the velocity vector of a curve $\gamma : I \to M$ by $\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\big|_{t_0}\right)$ but we cannot define the acceleration of the curve. To define $\gamma''(t)$ by differentiating $\gamma'(t)$ with respect to t, we have to take a limit of a difference quotient involving the vectors $\gamma'(t+h)$ and $\gamma'(t)$ which, however, live in different vector spaces $T_{\gamma(t+h)}M$ and $T_{\gamma(t)}M$. For submanifold $M \subseteq \mathbb{R}^n$ these two vector spaces can be identified with $\mathbb{R}^n$, but for an abstract manifold we need a new notion of differentiation to "connect" the two vector spaces.

Two remedies:

- Lie derivative $\mathcal{L}_X$ of tensor fields (including vector fields) along a vector field; see equation (1.8).
- Covariant derivative ∇_X of tensor fields (including vector fields) along a vector field; which we shall soon define and see that this derivative is equivalent to the concept of parallel transport.

Together with another differentiation,

- Exterior derivative d of differential forms (including functions),

we have three differentiations associated to a smooth manifold. d and $\mathcal{L}_X$ are related via the Cartan's magic formula $\mathcal{L}_X = \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}$. When ∇ is torsion-free, the algebraic relation between d and ∇_X and that between ∇X and $\mathcal{L}_X$ are detailed in [7, Proposition 12.54] and [7, Theorem 12.56] respectively.

3.1 Connections of Vector Bundles

Let M be a smooth manifold. Let $E \to M$ be a $\mathbb{F}$-vector bundle. The space $C^\infty(M, \mathbb{F})$ of all $\mathbb{F}$-valued smooth funcitons over M is a ring with pointewise addition and multiplication. The space $\Gamma(E) = \Gamma(M; E)$ of all smooth sections can be seen as a $C^\infty(M, \mathbb{F})$-module with scalar multiplication

$$\begin{aligned} C^\infty(M, \mathbb{F}) \times \Gamma(M; E) &\longrightarrow \Gamma(M; E) \\ (f, s) &\longmapsto \begin{pmatrix} f \cdot s : M \to E \\ p \mapsto f(p) \cdot s(p) \end{pmatrix}. \end{aligned}$$

Definition 3.1.1 (Connection of a Vector Bundle I)**.** *Let $\pi : E \to M$ be a smooth $\mathbb{F}$-vector bundle over a smooth manifold M with or without boundary, and let $\Gamma(E)$ denote the space of smooth sections of E. A **connection in** E is a map*

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, Y) &\longmapsto \nabla_X Y, \end{aligned}$$

satisfying the following properties:

(i) $\nabla_X(-)$ *is* $\mathbb{F}$*-linear: for* $a_1, a_2 \in \mathbb{F}$ *and* $Y_1, Y_2 \in \Gamma(E)$*,*

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2$$

(ii) $\nabla_{(-)} Y$ *is linear over* $C^\infty(M,\mathbb{R})$*: for* $f \in C^\infty(M) := C^\infty(M,\mathbb{R})$ *and* $X_1, X_2 \in \mathfrak{X}(M)$*,*

$$\nabla_{X_1 + f X_2} Y = \nabla_{X_1} Y + f \nabla_{X_2} Y$$

(iii) Leibniz rule: for $f \in C^\infty(M,\mathbb{F})$ *and* $Y \in \Gamma(E)$*,*

$$\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y.$$

The symbols ∇ *reads as "del" or "nabla" and is called* ***Koszul connection****,* ***affine connection****, or* ***linear connection****.* $\nabla_X Y$ *is called the* ***covariant derivative of*** Y ***along*** X*.*

The conditions (i) and (ii) can be rephrased as follows: for any $s \in \Gamma(E)$, the map

$$\begin{aligned} \nabla Y : \mathfrak{X}(M) &\longrightarrow \Gamma(E) \\ X &\longmapsto \nabla_X Y \end{aligned}$$

is $C^\infty(M)$-linear so that it defines a smooth bundle homorphism between vector bundles $TM \to M$ and $E \to M$:

$$\nabla Y \in \underline{\mathrm{Hom}}(TM, E) = \Gamma(\mathrm{Hom}(TM,E)) \cong \Gamma\left(T^*M \otimes E\right).$$

(see part 1.10.2 for the notation).

Summarizing, we can formulate the following equivalent definition.

Definition 3.1.2 (Connection of a Vector Bundle II)**.** *An affine connection on* E *is a* $\mathbb{F}$*-linear map*

$$\nabla : \Gamma(E) \to \Gamma\left(T^*M \otimes E\right),$$

such that for all $f \in C^\infty(M,\mathbb{F})$ *and* $Y \in \Gamma(E)$*, we have*

$$\nabla(fY) = \mathrm{d}f \otimes Y + f\nabla Y,$$

where we used the identification in Proposition 1.1.8.

Definition 3.1.3. *We set*

$$\Omega^k(E) := \Gamma\left(\Lambda^k T^*M \otimes E\right).$$

We will refer to these sections as differential k*-forms with coefficients in the vector bundle* E*, or* E***-valued*** k***-forms****. Note that*

$$\Omega^0(E) = \Gamma(T^0T^*M \otimes E) = \Gamma(\coprod_{p\in M} \underbrace{T^0T_p^*M}_{=\mathbb{R}} \otimes_{\mathbb{R}} E_p) \cong \Gamma(\coprod_{p\in M} \underbrace{\mathbb{R} \otimes_{\mathbb{R}} E_p}_{=E_p}) \cong \Gamma(E)$$

and

$$\Omega^1(E) = \Gamma(T^*M \otimes E) \cong \Gamma(\mathrm{Hom}(TM,E)).$$

Example 3.1.4. Let $\underline{\mathbb{F}}^r_M \cong M \times \mathbb{F}^r$ be the rank-r trivial vector bundle over M. The space $\Gamma\left(\underline{\mathbb{F}}^r_M\right)$ of smooth sections coincides with the space $C^\infty\left(M,\mathbb{F}^r\right)$ of $\mathbb{F}^r$-valued smooth functions on M. We can define

$$\begin{gathered} \nabla^0 : C^\infty\left(M,\mathbb{F}^r\right) \to \Gamma\left(T^*M \otimes \mathbb{F}^r\right) \\ \nabla^0\left(f_1,\ldots,f_r\right) = \left(\mathrm{d}f_1,\ldots,\mathrm{d}f_r\right) \end{gathered}$$

One checks easily that ∇ is a connection. This is called the **trivial connection**. ♣

Example 3.1.5 (The Euclidean Connection)**.** In $T\mathbb{R}^n$, define the Euclidean connection $\overline{\nabla}$ by the following formula (see [12] (4.3)):

$$\overline{\nabla}_X Y = X\left(Y^1\right)\frac{\partial}{\partial x^1} + \cdots + X\left(Y^n\right)\frac{\partial}{\partial x^n}$$

Note that

$$\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \sum_{k=1}^n \frac{\partial}{\partial x^i}\left(\delta_{jk}\right)\frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^i}(1)\frac{\partial}{\partial x^j} = 0\frac{\partial}{\partial x^j} = 0.$$

This shows that the connection coefficients (3.2) of the Euclidean connection are zero. ♣

Example 3.1.6 (The Tangential Connection on a Submanifold of $\mathbb{R}^n$)**.** Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. Define a connection $\nabla^\top$ on TM, called the **tangential connection**, by setting

$$\nabla^\top_X Y = \pi^\top\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}\Big|_M\right),$$

where $\pi^\top$ is the orthogonal projection onto TM, $\overline{\nabla}$ is the Euclidean connection on $\mathbb{R}^n$ (Example 3.1.5), and $\widetilde{X}$ and $\widetilde{Y}$ are smooth extensions of X and Y to an open set in $\mathbb{R}^n$. (Such extensions exist by the result of [12] Exercise A.23.) Since the value of $\overline{\nabla}_{\widetilde{X}}\widetilde{Y}$ at a point $p \in M$ depends only on $\widetilde{X}_p = X_p$, this just boils down to defining $\left(\nabla^\top_X Y\right)_p$ to be equal to the tangential directional derivative $\nabla^\top_{X_p} Y$ we intuitively defined in [12] (4.4). To show it is indeed a connection, see [12] Example 4.9. ♣

We have defined and shown several examples of connection of vector bundles. Before we go to the properties and other constructions, we first address the existence and abundance of connections.

Proposition 3.1.7. *Let E be a vector bundle. The space $\mathcal{A}(E)$ of affine connections on E is an affine space modeled on $\Omega^1(\operatorname{End}(E))$.*

Proof. We first prove that $\mathcal{A}(E)$ is not empty. To see this, choose an open cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha}$ is trivial $\forall\alpha$. Next, pick a smooth partition of unity (μ_β) subordinated to this cover.

Since $E|_{U_\alpha}$ is trivial, it admits at least one connection, the trivial one, as in the Example 3.1.4. Denote such a connection by ∇^α. Now define

$$\nabla := \sum_\alpha \mu_\alpha \nabla^\alpha$$

One checks easily that ∇ is a connection so that $\mathcal{A}(E)$ is nonempty. To check that $\mathcal{A}(E)$ is an affine space, consider two connections ∇^0 and ∇^1. Their difference $A = \nabla^1 - \nabla^0$ is an operator

$$A : \Gamma(E) \to \Gamma\left(T^*M \otimes E\right) = \Omega^1(E)$$

satisfying $A(fu) = fA(u), \forall u \in \Gamma(E)$. Thus,

$$A \in \Gamma\left(\operatorname{Hom}\left(E, T^*M \otimes E\right)\right) \cong \Gamma\left(T^*M \otimes E^* \otimes E\right) \cong \Omega^1\left(E^* \otimes E\right) \cong \Omega^1(\operatorname{End}(E))$$

Conversely, given $\nabla^0 \in \mathcal{A}(E)$ and $A \in \Omega^1(\operatorname{End}(E))$ one can verify that the operator

$$\nabla^A = \nabla^0 + A : \Gamma(E) \to \Omega^1(E)$$

is a linear connection. This concludes the proof of the proposition. ■

Although a connection is defined by its action on global sections, it follows from the definitions that it is actually a local operator, as the next lemma shows.

Lemma 3.1.8 (Locality)**.** *Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every $X \in \mathfrak{X}(M), Y \in \Gamma(E)$, and $p \in M$, the covariant derivative $\nabla_X Y\big|_p$ depends only on the values of X and Y in an arbitrarily small neighborhood of p. More precisely, if $X = \widetilde{X}$ on a neighborhood of p, then $\nabla_X Y\big|_p = \nabla_{\widetilde{X}} Y\big|_p$; if $Y = \widetilde{Y}$ on a neighborhood of p, then $\nabla_X Y\big|_p = \nabla_X \widetilde{Y}\big|_p$. (The proof is similar to that of [11] Proposition 3.8)*

Proof. First consider Y. Replacing Y by $Y-\widetilde{Y}$ shows that it suffices to prove $\nabla_X Y|_p = 0$ if Y vanishes on a neighborhood of p.

Thus suppose Y is a smooth section of E that is identically zero on a neighborhood U of p. Choose a bump function $\varphi \in C^\infty(M,\mathbb{R}) \subseteq C^\infty(M,\mathbb{C})$ (see [11] p.42) with support in U such that $\varphi(p)=1$. The hypothesis that Y vanishes on U implies that $\varphi Y \equiv 0$ on all of M, so for every $X \in \mathfrak{X}(M)$, we have $\nabla_X(\varphi Y) = \nabla_X(0\cdot\varphi Y) = 0\nabla_X(\varphi Y) = 0$. Thus the Leibniz rule gives

$$0 = \nabla_X(\varphi Y) = \overbrace{(X\varphi)Y}^{=0} + \varphi\left(\nabla_X Y\right) \Rightarrow 0 = \varphi(\nabla_X Y)$$

Now $Y \equiv 0$ on the support of φ, so the first term on the right is identically zero. Evaluating above equation at p shows that $\nabla_X Y|_p = 0$.

The argument for X is similar. It suffices to show $\nabla_X Y|_p = 0$ if X vanishes on a neighborhood of p. Again, $\varphi X \equiv 0$. Thus, $0 = \nabla_{\varphi X} Y = \varphi \nabla_X Y$. Then evaluate both sides at p. ■

Proposition 3.1.9 (Restriction of a Connection)**.** *Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every open subset $U \subseteq M$, there is a unique connection ∇^U on the restricted bundle $E|_U$ that satisfies the following relation for every $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$:*

$$\nabla^U_{\left(X|_U\right)}\left(Y|_U\right) = \left(\nabla_X Y\right)|_U . \tag{3.1}$$

where $X|_U$ and $Y|_U$ are smooth sections of the vector bundles $TM|_U$ and $E|_U$ respectively (see [11, Example 10.8]) and $(\nabla_X Y)|_U$ is both a local section of E via restriction and a global section of $E|_U$ as output of ∇^U.

Proof. [12] Proposition 4.3. ■

In the situation of this proposition, we typically just refer to the restricted connection as ∇ instead of ∇^U; the proposition guarantees that there is no ambiguity in doing so.

Lemma 3.1.8 tells us that we can compute the value of $\nabla_X Y$ at p knowing only the values of X and Y in a neighborhood of p. In fact, as the next proposition shows, we need only know the value of X at p itself.

Proposition 3.1.10. *Under the hypotheses of Lemma 3.1.8, $\nabla_X Y|_p$ depends only on the values of Y in a neighborhood of p and the value of X at p. (Since the claim about Y was proved in Lemma 3.1.8, this is to prove $X_p = \widetilde{X}_p \Rightarrow \nabla_X Y|_p = \nabla_{\widetilde{X}} Y\big|_p$. Equivalently, $(X-\widetilde{X})_p = 0_p \in T_pM \Rightarrow \nabla_{X-\widetilde{X}} Y\Big|_p =$ zero section ζ.)*

Proof. The claim about Y was proved in Lemma 3.1.8. To prove the claim about X, it suffices by linearity to assume that $X_p = 0$ and show that $\nabla_X Y|_p = 0$. Choose a coordinate neighborhood U of p, and write $X = X^i\partial_i$ in coordinates on U, with $X^i(p) = 0$. Thanks to Proposition 3.1.9, it suffices to work with the restricted connection on U, which we also denote by ∇. For every $Y \in \Gamma\left(E|_U\right)$, we have

$$\nabla_X Y|_p = \nabla_{X^i\partial_i} Y|_p = X^i(p)\nabla_{\partial_i} Y\big|_p = 0.$$

■

Remark 3.1.11. Thanks to Propositions 3.1.9 and 3.1.10, we can make sense of the expression $\nabla_v Y$ when v is some element of T_pM and Y is a smooth local section of E defined only on some neighborhood of p. To evaluate it, let X be a vector field on a neighborhood of p whose value at p is v, and set $\nabla_v Y = \nabla_X Y|_p$. Proposition 3.1.10 shows that the result does not depend on the extension chosen. Henceforth, we will interpret covariant derivatives of local sections of bundles in this way without further comment. ♠

3.1.1 Connections on Tangent and Tensor Bundles

We consider the case $E = TM$ and connection

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M).$$

Let (E_i) be a smooth local frame for TM on an open subset $U \subseteq M$. For every choice of the indices i and j, we can express the vector field $\nabla_{E_i} E_j$ in terms of this same frame:

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} E_k \tag{3.2}$$

As i, j, and k range from 1 to n, this defines n^3 smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$, called the **connection coefficients** of ∇ with respect to the given frame. We also call them **Christoffel symbols**, which [12] reserves for the connection coefficients of Levi-Civita connection on Riemannian manifold. In fact, the set of connections on TM is 1-1 corresponded to the set of n^3 smooth funcitons $\{\Gamma_{ij}^k\}$: fix a smooth local frame, given ∇ we have funcitons $\{\Gamma_{ij}^k\}$ and the following formula:

$$\nabla_X Y = \left(X\left(Y^k\right) + X^i Y^j \Gamma_{ij}^k\right) E_k. \tag{3.3}$$

Conversely, specifying funcitons $\{\Gamma_{ij}^k\}$ determines ∇ via (3.3). In fact, it is easy to see that the vector field $\nabla_X Y$ defined via (3.3) is smooth and that ∇ is linear over $\mathbb{R}$ in Y and linear over $C^\infty(M)$ in X. The Leibniz rule is also satisfied:

$$\begin{aligned}
\nabla_X(fY) &= \left(X\left(fY^k\right) + X^i f Y^j \Gamma_{ij}^k\right) E_k \\
&\xlongequal{[11]\ (8.5)} \left(fX\left(Y^k\right) + Y^k Xf + fX^i Y^j \Gamma_{ij}^k\right) E_k \\
&= f\left(X\left(Y^k\right) + X^i Y^j \Gamma_{ij}^k\right) E_k + \left(Y^k Xf\right) E_k \\
&= f\nabla_X Y + (Xf) Y^k E_k \\
&= f\nabla_X Y + (Xf) Y.
\end{aligned}$$

Once the connection coefficients (and thus the connection) have been determined in some local frame, they can be determined in any other local frame on the same open set by the result of the following proposition.

Proposition 3.1.12 (Transformation Law for Connection Coefficients)**.** *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose we are given two smooth local frames (E_i) and $\left(\widetilde{E}_j\right)$ for TM on an open subset $U \subseteq M$, related by $\widetilde{E}_i = A_i^j E_j$ for some matrix of functions $\left(A_i^j\right)$. Let Γ_{ij}^k and $\widetilde{\Gamma}_{ij}^k$ denote the connection coefficients of ∇ with respect to these two frames. Then*

$$\widetilde{\Gamma}_{ij}^k = \left(A^{-1}\right)_p^k A_i^q A_j^r \Gamma_{qr}^p + \left(A^{-1}\right)_p^k A_i^q E_q\left(A_j^p\right).$$

Proof. We note that

$$\begin{pmatrix} \widetilde{E}_1 \\ \vdots \\ \widetilde{E}_n \end{pmatrix} = \begin{pmatrix} A_1^1 & \cdots & A_1^n \\ \vdots & \ddots & \vdots \\ A_n^1 & \cdots & A_n^n \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix}$$

Hence, $E_p = (A^{-1})_p^k \widetilde{E}_k$. By (3.3), we see that

$$\begin{aligned}
\nabla_{\widetilde{E}_i} \widetilde{E}_j &= \left[\widetilde{E}_i(\widetilde{E}_j^p) + \widetilde{E}_i^q \widetilde{E}_j^r \Gamma_{qr}^p\right] E_p \\
&= \left[\left(A_i^q E_q\right)\left(A_j^p\right) + A_i^q A_j^r \Gamma_{qr}^p\right]\left((A^{-1})_p^k \widetilde{E}_k\right) \\
&= \left(A^{-1}\right)_p^k A_i^q A_j^r \Gamma_{qr}^p + \left(A^{-1}\right)_p^k A_i^q E_q\left(A_j^p\right)
\end{aligned}$$

■

We now show that every connection in TM automatically induces connections in all tensor bundles over M,

$$\nabla : \mathfrak{X}(M) \times \Gamma\left(T^{(k,l)}TM\right) \to \Gamma\left(T^{(k,l)}TM\right)$$

and thus gives a way to compute covariant derivatives of tensor fields of any type.

Proposition 3.1.13. *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Then ∇ uniquely determines a connection in each tensor bundle $T^{(k,l)}TM$, also denoted by ∇, such that the following four conditions are satisfied.*

(i) *In $T^{(1,0)}TM = TM, \nabla$ agrees with the given connection.*

(ii) *In $T^{(0,0)}TM = M \times \mathbb{R}, \nabla$ is given by ordinary differentiation of functions:*

$$\nabla_X f = Xf$$

(For the identification $T^{(0,0)}TM = M \times \mathbb{R}$, see [11] p.317: for any vector space V, [11] p. 312 notes that $T^0V = \mathbb{R}$ by convention. Now

$$T^0T^*M = \coprod_{p\in M} T^0\left(T_p^*M\right) = \coprod_{p\in M} \mathbb{R} = M \times \mathbb{R}$$

Similarly, $T^0TM = M \times \mathbb{R}$. Thus, $T^{(0,0)}TM$, either interpreted as T^0T^*M or T^0TM, equals to $M \times \mathbb{R}$. And the space of smooth sections $\Gamma\left(T^{(0,0)}TM\right) = \Gamma(M \times \mathbb{R}) = C^\infty(M)$ is just the space of smooth functions.)

(iii) *∇ obeys the following Leibniz rule with respect to tensor products:*

$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G).$$

(iv) *∇ commutes with all contractions: if "tr" denotes a trace on any pair of indices, one covariant and one contravariant, then*

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F)$$

This connection also satisfies the following additional properties:

(a) *∇ obeys the following Leibniz rule with respect to the natural pairing between a covector field ω and a vector field Y :*

$$\nabla_X\langle\omega, Y\rangle = \langle\nabla_X\omega, Y\rangle + \langle\omega, \nabla_X Y\rangle.$$

(Note: $\langle\omega, Y\rangle_p := \langle\omega_p, Y_p\rangle = \omega_p(Y_p)$. So $\langle\omega, Y\rangle \in C^\infty(M)$.)

(b) *(b) For all $F \in \Gamma\left(T^{(k,l)}TM\right)$, smooth 1-forms $\omega^1, \ldots, \omega^k$, and smooth vector fields $Y_1, \ldots, Y_l$,*

$$\begin{aligned}\left(\nabla_X F\right)\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right) &= X\left(F\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right)\right) \\ &- \sum_{i=1}^k F\left(\omega^1, \ldots, \nabla_X\omega^i, \ldots, \omega^k, Y_1, \ldots, Y_l\right) \\ &- \sum_{j=1}^l F\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_l\right).\end{aligned} \tag{3.4}$$

Proof. First we show that every family of connections on all tensor bundles satisfying (i)-(iv) also satisfies (a) and (b). Suppose we are given such a family of connections, all denoted by ∇. Recall for $\omega \in \mathfrak{X}^*(M), Y \in \mathfrak{X}(M), \omega \otimes Y$ denotes the tensor fields defined by $(\omega \otimes Y)_p := \omega_p \otimes Y_p$ (see [11] p.317), and $\langle\omega, Y\rangle$ is also pointwise defined: $\langle\omega, Y\rangle_p := \langle\omega_p, Y_p\rangle = \omega_p(Y_p)$. Also note that

$$\omega_p \otimes Y_p \in T_p^*M \otimes T_pM = T^{(1,1)}T_p^*M \cong \operatorname{End}\left(T_p^*M\right)$$

so that $\omega \otimes Y \in \Gamma\left(T^{(1,1)} T^{*} M\right)$. Then the trace of $\omega_p \otimes Y_p \in T^{(1,1)} T_p^{*} M$ is the sum of the diagonal elements of the matrix representation of $\omega_p \otimes Y_p$ identified as a linear endomorphism. Plugging $k=l=0$ into formula (1.3) gives

$$\operatorname{tr}\left(\omega_p \otimes Y_p\right)=\sum_{1 \leq m \leq n}\left(\omega_p \otimes Y_p\right)_m^m \xlongequal{[11]\ 12.22}\left(\omega_p\right)_m\left(Y_p\right)^m .$$

On the other hand, if $Y_p=\left(Y_p\right)^i E_i, \omega_p=\left(\omega_p\right)_j \varepsilon^j$ then $\varepsilon^j\left(E_i\right)=\delta_i^j$ gives that

$$\omega_p\left(Y_p\right)=\left(\omega_p\right)_j \varepsilon^j\left[\left(Y_p\right)^i E_i\right]=\left(\omega_p\right)_j\left(Y_p\right)^j=\left(\omega_p\right)_m\left(Y_p\right)^m$$

Thus $\operatorname{tr}\left(\omega_p \otimes Y_p\right)=\omega_p\left(Y_p\right)$ and $\langle\omega, Y\rangle=\operatorname{tr}(\omega \otimes Y)$. Therefore, (i)-(iv) imply

$$\begin{aligned}
\nabla_X(\omega(Y)) & =\nabla_X\langle\omega, Y\rangle=\nabla_X(\operatorname{tr}(\omega \otimes Y))=\operatorname{tr}\left(\nabla_X(\omega \otimes Y)\right) \\
& =\operatorname{tr}\left(\left(\nabla_X \omega\right) \otimes Y+\omega \otimes\left(\nabla_X Y\right)\right) \quad \text{(by (iv))} \\
& =\operatorname{tr}\left(\left(\nabla_X \omega\right) \otimes Y\right)+\operatorname{tr}\left(\omega \otimes\left(\nabla_X Y\right)\right) \quad (\text{linearity of } \operatorname{tr}) \\
& =\left\langle\nabla_X \omega, Y\right\rangle+\left\langle\omega, \nabla_X Y\right\rangle \quad \left(\nabla_X \omega \text{ is a 1-form, } \in \Gamma(T^{(0,1)} T M) \text{ while } \nabla_X Y \text{ is a vector field, } \in \Gamma(T^{(1,0)} T M)\right)
\end{aligned} \tag{3.5}$$

Then (b) is proved by induction using a similar computation applied to

$$F\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right)=\underbrace{\operatorname{tr} \circ \cdots \circ \operatorname{tr}}_{k+l}\left(F \otimes \omega^1 \otimes \cdots \otimes \omega^k \otimes Y_1 \otimes \cdots \otimes Y_l\right),$$

where each trace operator acts on an upper index of F and the lower index of the corresponding 1-form, or a lower index of F and the upper index of the corresponding vector field. In fact, (3.4) can be easily generalized from the case $k=l=1$:

$$\begin{aligned}
(\nabla_X F)(\omega, Y) & =\nabla_X(\operatorname{tr} \circ \operatorname{tr}(F \otimes \omega \otimes Y)) \\
& =\operatorname{tr} \circ \operatorname{tr}\left(\nabla_X(F \otimes(\omega \otimes Y))\right) \\
& =\operatorname{tr} \circ \operatorname{tr}((\nabla_X F) \otimes(\omega \otimes Y)+F \otimes(\nabla_X(\omega \otimes Y))) \\
& =\operatorname{tr} \circ \operatorname{tr}((\nabla_X F) \otimes(\omega \otimes Y)+F \otimes((\nabla_X \omega \otimes Y+\omega \otimes \nabla_X Y))) \\
& =\operatorname{tr} \circ \operatorname{tr}((\nabla_X F) \otimes \omega \otimes Y+F \otimes \nabla_X \omega \otimes Y+F \otimes \omega \otimes \nabla_X Y) \\
& =(\nabla_X F)(\omega, Y)+F(\nabla_X \omega, Y)+F(\omega, \nabla_X Y) \\
\Longrightarrow & (\nabla_X F)(\omega, Y)=\nabla_X F(\omega, Y)-F(\nabla_X \omega, Y)-F(\omega, \nabla_X Y)
\end{aligned}$$

Next we address uniqueness. Assume again that ∇ represents a family of connections satisfying (i)-(iv), and hence also (a) and (b). Observe that (ii) and (a) imply that the covariant derivative of every 1-form ω can be computed by

$$\left(\nabla_X \omega\right)(Y)=X(\omega(Y))-\omega\left(\nabla_X Y\right). \tag{3.6}$$

It follows that the connection on 1-forms is uniquely determined by the original connection in TM, which is $\nabla_X Y$. Similarly, (b) gives a formula determining the covariant derivative of every tensor field F in terms of covariant derivatives of vector fields and 1-forms, so the connection in every tensor bundle is uniquely determined.

Now to prove existence, we first define covariant derivatives of 1-forms by (3.6), and then we use (3.4) to define ∇ on all other tensor bundles. The first thing that needs to be checked is that the resulting expression is multilinear over $C^{\infty}(M)$ in each ω^i and Y_j, and therefore defines a smooth tensor field. This is done by inserting $f \omega^i$ in place of ω^i, or $f Y_j$ in place of Y_j, and expanding the right-hand side, noting that the two terms in which f is differentiated cancel each other out. Once we know that $\nabla_X F$ is a smooth tensor field, we need to check that it satisfies the defining properties of a connection. Linearity over $C^{\infty}(M)$ in X and linearity over $\mathbb{R}$ in F are both evident from (3.4) and (3.6), and the Leibniz rule in F follows easily from the fact that differentiation of functions by X satisfies the Leibniz rule. It is then a straightforward computational exercise to show that the resulting connection satisfies conditions (i)-(iii). To prove (iv), first observe that every (k,l)-tensor field can be written locally as a sum of tensor fields of the form $Z_1 \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \zeta^l$, and for such a tensor field the trace on the i th contravariant index and the j th covariant one satisfies

$$\operatorname{tr}\left(Z_1 \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \zeta^l\right) = \zeta^j\left(Z_i\right) Z_1 \otimes \cdots \otimes \widehat{Z_i} \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \widehat{\zeta^j} \otimes \cdots \otimes \zeta^l.$$

Then (iv) follows by applying (3.4) and (3.6) to this formula. ■

While (3.4) and (3.6) are useful for proving the existence and uniqueness of the connections in tensor bundles, they are not very practical for computation, because computing the value of $\nabla_X F$ at a point requires extending all of its arguments to vector fields and covector fields in an open set, and computing a great number of derivatives. For computing the components of a covariant derivative in terms of a local frame, the formulas in the following proposition are far more useful.

Proposition 3.1.14. *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose (E_i) is a local frame for M, $\left(\varepsilon^j\right)$ is its dual coframe, and $\left\{\Gamma_{ij}^k\right\}$ are the connection coefficients of ∇ with respect to this frame. Let X be a smooth vector field, and let $X^i E_i$ be its local expression in terms of this frame.*

(a) The covariant derivative of a 1-form $\omega = \omega_i \varepsilon^i$ is given locally by

$$\nabla_X \omega = \left(X\left(\omega_k\right) - X^j \omega_i \Gamma_{jk}^i\right) \varepsilon^k.$$

(b) If $F \in \Gamma\left(T^{(k,l)} TM\right)$ is a smooth mixed tensor field of any rank, expressed locally as

$$F = F_{j_1 \ldots j_l}^{i_1 \ldots i_k} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l},$$

then the covariant derivative of F is given locally by

$$\nabla_X F = \left(X\left(F_{j_1 \ldots j_l}^{i_1 \ldots i_k}\right) + \sum_{s=1}^k X^m F_{j_1 \ldots j_l}^{i_1 \ldots p \ldots i_k} \Gamma_{mp}^{i_s} - \sum_{s=1}^l X^m F_{j_1 \ldots p \ldots j_l}^{i_1 \ldots i_k} \Gamma_{mj_s}^p\right) \times \\ E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}.$$

Proof. To show (a), we only need to show

$$(\nabla_X \omega)(E_k) = X(\omega_k) - X^j \omega_i \Gamma_{jk}^i$$

By (3.6), we see

$$\begin{aligned}(\nabla_X \omega)(E_k) &= X(\omega(E_k)) - \omega(\nabla_X E_k) \\ &= X(\omega_k) - \omega\left[(X(\underbrace{\delta_k^i}_{\text{constant}}) + X^j \delta_k^r \Gamma_{jr}^i) E_i\right] \\ &= X(\omega_k) - \omega\left[(0 + X^j \Gamma_{jk}^i) E_i\right] \\ &= X(\omega_k) - \omega_i X^j \Gamma_{jk}^i\end{aligned}$$

To show (b), we only need to show

$$(\nabla_X F)(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}) = X\left(F_{j_1 \ldots j_l}^{i_1 \ldots i_k}\right) + \sum_{s=1}^k X^m F_{j_1 \ldots j_l}^{i_1 \ldots p \ldots i_k} \Gamma_{mp}^{i_s} - \sum_{s=1}^l X^m F_{j_1 \ldots p \ldots j_l}^{i_1 \ldots i_k} \Gamma_{mj_s}^p$$

By (3.4), we see

$$
\begin{aligned}
&(\nabla_X F)\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}\right) \\
=&X\left(F(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l})\right) - \sum_{s=1}^{k} F\left(\varepsilon^{i_1}, \ldots, \nabla_X \varepsilon^{i_s}, \ldots, \varepsilon^{i_k}, E_{j_1}, \ldots, E_{j_l}\right) \\
&- \sum_{s=1}^{l} F\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \ldots, \nabla_X E_{j_s}, \ldots, E_{j_l}\right) \\
=&F^{i_1 \ldots i_k}_{j_1 \ldots j_l} - \sum_{s=1}^{k} F\left(\varepsilon^{i_1}, \ldots, -X^m \Gamma^{i_s}_{mp} \varepsilon^p, \ldots, \varepsilon^{i_k}, E_{j_1}, \ldots, E_{j_l}\right) \\
&- \sum_{s=1}^{l} F\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \ldots, X^m \Gamma^{p}_{mj_s} E_p, \ldots, E_{j_l}\right) \quad \text{(by (a) and (3.3))} \\
=&F^{i_1 \ldots i_k}_{j_1 \ldots j_l} + \sum_{s=1}^{k} X^m F^{i_1 \ldots p \ldots i_k}_{j_1 \ldots j_l} \Gamma^{i_s}_{mp} - \sum_{s=1}^{l} X^m F^{i_1 \ldots i_k}_{j_1 \ldots p \ldots j_l} \Gamma^{p}_{mj_s}
\end{aligned}
$$

■

Because the covariant derivative $\nabla_X F$ of a tensor field (or, as a special case, a vector field) is linear over $C^\infty(M)$ in X, the covariant derivatives of F in all directions can be handily encoded in a single tensor field whose rank is one more than the rank of F, as follows.

Proposition 3.1.15 (The Total Covariant Derivative). *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every $F \in \Gamma\left(T^{(k,l)}TM\right)$, the map*

$$
\nabla F : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ copies}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l+1 \text{ copies}} \to C^\infty(M)
$$

given by

$$
(\nabla F)\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X\right) = (\nabla_X F)\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right) \tag{3.7}
$$

*defines a smooth $(k, l+1)$-tensor field on M called the **total covariant derivative of** F.*

Proof. This follows immediately from the tensor characterization lemma (Lemma 1.1.19): $\nabla_X F$ is a tensor field, so it is multilinear over $C^\infty(M)$ in its $k+l$ arguments; and it is linear over $C^\infty(M)$ in X by definition of a connection. ■

Remark 3.1.16. Note that the smooth $(k, l+1)$-tensor field induced by ∇F is called the total covariant derivative of F and is denoted by ∇F as well. One can think of the covariant derivative of a tensor field as directional derivatives while the total covariant derivative is the total derivative of the tensor field. ♠

When we write the components of a total covariant derivative in terms of a local frame, it is standard practice to use a semicolon to separate indices resulting from differentiation from the indices resulting from the "+1" insertion. For example, let Y be a vector field. That is, $Y \in \mathfrak{X}(M) = \Gamma\left(T^{(1.0)}TM\right)$ where $k=1, l=0$ in the above proposition. We write it in coordinates as $Y = Y^i E_i$. Then the components of the $(1,1)$-tensor field ∇Y are written as $Y^i{}_{;j}$, i.e.,

$$
\nabla Y = Y^i{}_{;j} E_i \otimes \varepsilon^j \tag{3.8}
$$

where $Y^i{}_{;j}$ is obtained by the following:

$$\begin{aligned}
Y^i{}_{;j} &= \nabla Y\left(\varepsilon^i, E_j\right) = \left(\nabla_{E_j} Y\right)\left(\varepsilon^i\right) \\
&\overset{(3.3)}{=\!=} \left(E_j\left(Y^l\right) + \left(E^j\right)^m Y^k \Gamma_{mk}^l\right) E_l\left(\varepsilon^i\right) \\
&= E_j\left(Y^i\right) + \left(E^j\right)^m Y^k \Gamma_{mk}^i \\
&= E_j\left(Y^i\right) + Y^k \Gamma_{jk}^i
\end{aligned} \tag{3.9}$$

For a one-form $\omega \in \Gamma(T^{(0,1)}TM)$ where $k=0, l=1$ in above proposition, we have a $(0,2)$-tensor field $\nabla\omega$. If we write $\omega = \omega_m \varepsilon^m$, then the components of the $\nabla\omega$ are written as $\omega_{i;j}$, i.e.,

$$\nabla \omega = \omega_{i;j} \varepsilon^i \otimes \varepsilon^j$$

where $\omega_{i;j}$ is obtained by the following:

$$\begin{aligned}
\omega_{i;j} &= \nabla \omega\left(E_i, E_j\right) = \left(\nabla_{E_j} \omega\right)\left(E_i\right) \\
&\overset{(3.6)}{=\!=} E_j\left(\omega\left(E_i\right)\right) - \omega\left(\nabla_{E_j} E_i\right) \\
&\overset{(3.2)}{=\!=} E_j\left(\omega_m \varepsilon^m\left(E_i\right)\right) - \omega_m \varepsilon^m\left(\Gamma_{ji}^k E_k\right) \\
&= E_j \omega_i - \omega_k \Gamma_{ji}^k
\end{aligned}$$

More generally, replacing (3.3) and (3.6) with (3.4) and using the definition of coefficient Γ_{ij}^k we get a formula for the components of total covariant derivatives of arbitrary tensor fields as shown in the next lemma.

Proposition 3.1.17. *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM; and let (E_i) be a smooth local frame for TM and $\left\{\Gamma_{ij}^k\right\}$ the corresponding connection coefficients. The components of the total covariant derivative of a (k,l)-tensor field F with respect to this frame are given by*

$$F_{j_1 \ldots j_l; m}^{i_1 \ldots i_k} = E_m\left(F_{j_1 \ldots j_l}^{i_1 \ldots i_k}\right) + \sum_{s=1}^k F_{j_1 \ldots j_l}^{i_1 \ldots p \ldots i_k} \Gamma_{mp}^{i_s} - \sum_{s=1}^l F_{j_1 \ldots p \ldots j_l}^{i_1 \ldots i_k} \Gamma_{m j_s}^p .$$

Proof.

$$\Gamma\left(T^{(k,l+1)} TM\right) \ni \nabla F = F_{j_1, \cdots, j_l; m}^{i_1, \cdots, i_k} E_{i_1} \otimes \cdots \otimes F_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} \otimes \varepsilon^m$$

$$\begin{aligned}
F_{j \cdots\cdots j; m}^{i_1 \cdots i_j} &= \nabla F\left(\varepsilon^i, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}, E_m\right) \\
&\overset{(3.7)}{=\!=} \nabla_{E_m} F\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}\right) \\
&\overset{\text{prop. 3.1.14(b)}}{=\!=\!=\!=} E_m\left(F_{j_1 \cdots j_l}^{i_1 \cdots i_k}\right) + \sum_{s=1}^k \left(E_m\right)^q F_{j, \cdots, j_l}^{i_1 \cdots p \cdots i_k} \Gamma_{qp}^{i_s} - \sum_{s=1}^l \left(E_m\right)^q F_{j_1 \cdots q \cdots j_l}^{i_1 \cdots i_k} \Gamma_{q j_s}^p \\
&= E_m\left(F_{j_1 \cdots j_l}^{i_1 \cdots i_k}\right) + \sum_{s=1}^k F_{j_1 \cdots j_l}^{i_1 \cdots p \cdots i_k} \Gamma_{mp}^{i_s} - \sum_{s=1}^l F_{j, \cdots, j_l}^{i_1 \cdots i_k} \Gamma_{m j_s}^p
\end{aligned}$$

■

Exercise 3.1.18. *Suppose F is a smooth (k,l)-tensor field and G is a smooth (r,s) tensor field. Show that the components of the total covariant derivative of $F \otimes G$ are given by*

$$(\nabla(F \otimes G))_{j_1 \ldots j_l q_1 \ldots q_s; m}^{i_1 \ldots i_k p_1 \ldots p_r} = F_{j_1 \ldots j_l; m}^{i_1 \ldots i_k} G_{q_1 \ldots q_s}^{p_1 \ldots p_r} + F_{j_1 \ldots j_l}^{i_1 \ldots i_k} G_{q_1 \ldots q_s : m}^{p_1 \ldots p_r} .$$

[Remark: This formula is often written in the following way, more suggestive of the Leibniz rule for ordinary derivatives:

$$\left(F_{j_1\ldots j_l}^{i_1\ldots i_k} G_{q_1\ldots q_s}^{p_1\ldots p_r}\right)_{;m} = F_{j_1\ldots j_l;m}^{i_1\ldots i_k} G_{q_1\ldots q_s}^{p_1\ldots p_r} + F_{j_1\ldots j_l}^{i_1\ldots i_k} G_{q_1\ldots q_s;m}^{p_1\ldots p_r}.$$

Notice that this does not say that $\nabla(F\otimes G)=(\nabla F)\otimes G+F\otimes(\nabla G)$, because in the first term on the right-hand side of this latter formula, the index resulting from differentiation is not the last lower index.]

Having defined the tensor field ∇F for a (k,l)-tensor field F, we can in turn take its total covariant derivative and obtain a $(k,l+2)$-tensor field $\nabla^2 F=\nabla(\nabla F)$. Given vector fields $X,Y\in\mathfrak{X}(M)$, let us introduce the notation $\nabla^2_{X,Y}F$ for the (k,l)-tensor field obtained by inserting X,Y in the last two slots of $\nabla^2 F$:

$$\nabla^2_{X,Y}F(\ldots)=\nabla^2 F(\ldots,Y,X).$$

Note the reversal of order of X and Y : this is necessitated by our convention that the last index position in ∇F is the one resulting from differentiation, while it is conventional to let $\nabla^2_{X,Y}$ stand for differentiating first in the Y direction, then in the X direction. (For this reason, some authors adopt the convention that the new index position introduced by differentiation is the first instead of the last. As usual, be sure to check each author's conventions when you read.)

It is important to be aware that $\nabla^2_{X,Y}F$ is not the same as $\nabla_X\left(\nabla_Y F\right)$. The main reason is that the former is linear over $C^\infty(M)$ in Y, while the latter is not. The relationship between the two expressions is given in the following proposition.

Proposition 3.1.19. *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth vector field or tensor field F,*

$$\nabla^2_{X,Y}F=\nabla_X\left(\nabla_Y F\right)-\nabla_{\left(\nabla_X Y\right)}F.$$

Proof. For $Y\in\mathfrak{X}(M)=\Gamma\left(T^{(1,0)}TM\right), \nabla F\in\Gamma\left(T^{(k,l+1)}TM\right)$, we have $\nabla F\otimes Y\in\Gamma\left(T^{(k+1,l+1)}TM\right)$. The covariant derivative $\left(\nabla_Y F\right)(\cdots)\xlongequal{(3.7)}\nabla F(\cdots,Y)$ can be expressed as the trace of $\nabla F\otimes Y$ on its last two indices. We have

$$\nabla_Y F=\operatorname{tr}(\nabla F\otimes Y)=C_{l+1}^{k+1}(\nabla F\otimes Y) \tag{3.10}$$

as we can verify by computing their components: proposition 3.1.14 shows that

$$\left(\nabla_Y F\right)_{j_1\cdots j_l}^{i_1\cdots i_k}=Y\left(F_{j_1\cdots j_l}^{i_1\cdots i_k}\right)+\sum_{s=1}^k Y^m F_{j_1\cdots j_l}^{i_1\cdots p\cdots i_k}\Gamma_{mp}^{i_s}-\sum_{s=1}^l Y^m F_{j_1\cdots p\cdots j_l}^{i_1\cdots i_k}\Gamma_{mj_s}^p \xlongequal{\text{prop. 3.1.17}} F_{j_1\cdots j_l;m}^{i_1\cdots\cdots j_k}Y^m \tag{3.11}$$

On the other hand,

$$\begin{aligned}
\left[\operatorname{tr}(\nabla F\otimes Y)\right]_{j_1\cdots j_l}^{i_1\cdots i_k} &\xlongequal{(1.3)} (\nabla F\otimes Y)_{j_1\cdots j_l m}^{i_1\cdots i_k m}\\
&=(\nabla F\otimes Y)\left(\varepsilon^{i_1},\cdots,\varepsilon^{i_k},\varepsilon^m,E_{j_1},\cdots,E_{j_l},E_m\right)\\
&=\nabla F\left(\varepsilon^{i_1},\cdots,\varepsilon^{i_k},E_{j_1},\cdots,E_{j_l},E_m\right)Y\left(\varepsilon^m\right)\\
&=F_{j_1\cdots j_l;m}^{i_1\cdots i_k}Y^m
\end{aligned} \tag{3.12}$$

Similarly, $\nabla^2_{X,Y}F$ can be expressed as an iterated trace:

$$\nabla^2_{X,Y}F=\operatorname{tr}\left(\operatorname{tr}\left(\nabla^2 F\otimes X\right)\otimes Y\right).$$

(First trace the last index of $\nabla^2 F$ with that of X, and then trace the last remaining free index-originally the second-to-last in $\nabla^2 F$-with that of Y.)

We notice that for $X \in \mathfrak{X}(M)=\Gamma\left(T^{(1,0)} T M\right), \nabla F \in \Gamma\left(T^{(k, l+1)} T M\right)$, we have $\nabla_X(\nabla F) \in \Gamma\left(T^{(k, l+1)} T M\right)$, $\nabla(\nabla F) \in \Gamma\left(T^{(k, l+2)} T M\right)$, $\nabla(\nabla F) \otimes X \in \Gamma\left(T^{(k+1, l+2)} T M\right)$, and $\nabla_X(\nabla F) \otimes Y \in \Gamma\left(T^{(k+1, l+1)} T M\right)$. We write the iterated expression as

$$C_{l+1}^{k+1}\left(C_{l+2}^{k+1}(\nabla(\nabla F) \otimes X) \otimes Y\right) \xlongequal{(3.10)} C_{l+1}^{k+1}\left(\nabla_X(\nabla F) \otimes Y\right)=\nabla(\nabla F)(\cdots, Y, X):=\nabla_{X, Y}^2 F,$$

where the second equality comes from the following reasoning:

$$[C_{l+1}^{k+1}(\nabla_X(\nabla F) \otimes Y)]_{j_1 \cdots j_l}^{i_1 \cdots i_k} \xlongequal{(3.12)}[\nabla_X(\nabla F)]_{j_1 \cdots j_l \underbrace{j_{l+1}}_{=q}}^{i_1 \cdots i_k} Y^q \xlongequal{(3.11)}(\nabla F)_{j_1 \cdots j_l \underbrace{j_{l+1}}_{=q} \underbrace{j_{l+2}}_{=m}}^{i_1 \cdots i_k} X^m Y^q,$$

where $j_{l+1}=q$ and $j_{l+2}=m$ are just renaming of indices. On the other hand, $F \in T^{(k, l)}(V) \Rightarrow \nabla F \in T^{(k, l+1)}(V) \Rightarrow[\nabla(\nabla F)] \in T^{(k, l+2)}(V) \Rightarrow[\nabla(\nabla F)(\cdots, Y, X)] \in T^{(k, l)}(V)$ where

$$\nabla(\nabla F)(\cdots, Y, X):\left(\omega^1, \cdots, \omega^k, Y_1, \cdots, Y_l\right) \mapsto \nabla(\nabla F)\left(\omega^1, \cdots, \omega^k, Y_1, \cdots, Y_l, Y, X\right)$$

Now,

$$\begin{aligned}
& [\nabla(\nabla F)(\cdots, Y, X)]_{j_1 \cdots j_l}^{i_1 \cdots i_k}=\nabla(\nabla F)\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}, Y, X\right) \\
& =\nabla_X(\nabla F)\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}, Y\right) \\
& \xlongequal{\text{prop. 3.1.14}}\left[X\left((\nabla F)_{j_1 \cdots j_{l+1}}^{i_1 \cdots j_k}\right)+\sum_{s=1}^k X^m(\nabla F)_{j_1 \cdots j_{l+1}}^{i_1 \cdots p \cdots i_k} \Gamma_{m p}^{i_s}-\sum_{s=1}^l X^m(\nabla F)_{j_1 \cdots p \cdots j_{l+1}}^{i_1 \cdots i_k} \Gamma_{m j_s}^p\right] \\
& \times \underbrace{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_{l+1}}\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}, Y\right)}_{=Y^{j_{l+1}}} \\
& \xlongequal{\text{prop. 3.1.17}}(\nabla F)_{j_1 \cdots j_l \underbrace{j_{l+1}}_{=q} \underbrace{j_{l+2}}_{=m}}^{i_1 \cdots i_k} X^m Y^q
\end{aligned}$$

This shows $\left[C_{l+1}^{k+1}\left(\nabla_X(\nabla F) \otimes Y\right)\right]_{j_1 \cdots j_l}^{i_1 \cdots, i_k}=[\nabla(\nabla F)(\cdots, Y, X)]_{j_1 \cdots j_l}^{i_1 \cdots, i_k}$. So

$$\nabla_{X, Y}^2 F=C_{l+1}^{k+1}\left(C_{l+2}^{k+1}(\nabla(\nabla F) \otimes X) \otimes Y\right)$$

Therefore, since ∇_X commutes with contraction (see prop. 3.1.13 (iv)) and satisfies the Leibniz rule with respect to tensor products (see prop. 3.1.13 (iii)), we have

$$\begin{aligned}
\nabla_X\left(\nabla_Y F\right) & =\nabla_X(\operatorname{tr}(\nabla F \otimes Y)) \\
& =\operatorname{tr}\left(\nabla_X(\nabla F \otimes Y)\right) \\
& =\operatorname{tr}\left(\nabla_X(\nabla F) \otimes Y+\nabla F \otimes \nabla_X Y\right) \\
& =\operatorname{tr}\left(\operatorname{tr}\left(\nabla^2 F \otimes X\right) \otimes Y\right)+\operatorname{tr}\left(\nabla F \otimes \nabla_X Y\right) \\
& =\nabla_{X, Y}^2 F+\nabla_{\left(\nabla_X Y\right)} F
\end{aligned}$$

■

3.1.2 Constructions of Connections

Apart from the restriction ∇^U of connection on a vector bundle, there are many other ways to induce connections on vector bundles. For example, the tensorial operations on vector bundles extend naturally to vector bundles with connections. The guiding principle behind this fact is the Leibniz rule.

- **Tensor bundle connection** $\nabla^{E_1 \otimes E_2}$: If E_i $(i=1,2)$ are two bundles with connections ∇^i, then $E_1 \otimes E_2$ has a naturally induced connection $\nabla^{E_1 \otimes E_2}$ uniquely determined by the Leibniz rule,

$$\nabla_X^{E_1 \otimes E_2}\left(u_1 \otimes u_2\right)=\left(\nabla_X^1 u_1\right) \otimes u_2+u_1 \otimes \nabla_X^2 u_2 .$$

- **Dual bundle connection** ∇^*: A connection ∇ on a bundle E induced a connection ∇^* on the dual bundle E^* determined by the identity

$$X\langle u^*, u\rangle = \langle \nabla^*_X u^*, u\rangle + \langle u^*, \nabla_X u\rangle, \quad \forall u^* \in \Gamma(E^*), u \in \Gamma(E), X \in \mathfrak{X}(M),$$

where

$$\langle \cdot, \cdot \rangle : \Gamma(E^*) \times \Gamma(E) \to C^\infty(M)$$

is the natural pairing between the fibers of E^* and E. Note that when $E = TM$ and $E^* = T^*M$, above formula of connection is just (3.5). Moreover, from ∇^* and $\nabla^{E_1 \otimes E_2}$ we can essentially build the connections on all the tensor bundle $T^{(k,l)}TM$.

- **Endomorphism bundle connection** $\nabla^{\mathrm{End}(E)}$: We can use the connection $\nabla^{E^* \otimes E}$ of the vector bundle $E^* \otimes E$ to define a connection on the vector bundle $\mathrm{End}(E)$. Recall the identification $V^* \otimes V \cong \mathrm{End}(V)$ via extending the mapping $\phi \otimes u \mapsto (v \mapsto \phi(v)u)$ linearly. Given $T = u^* \otimes u \in \Gamma(E^* \otimes E)$ and a section $v \in \Gamma(E)$, we have

$$\begin{aligned}
\left[\nabla_X^{E^* \otimes E}(u^* \otimes u)\right](v) &= \left[(\nabla^*_X u^*) \otimes u\right](v) + \left[u^* \otimes \nabla^E_X u\right](v) \\
&= (\nabla^*_X u^*)(v) \cdot u + u^*(v) \cdot \nabla^E_X u \quad \text{via identification} \\
&= \nabla^E_X(u^*(v)) \cdot u - u^*(\nabla^E_X v) \cdot u + u^*(v) \cdot \nabla^E_X u \quad \text{defn. of } \nabla^* \\
&= \nabla^E_X(u^*(v) \cdot u) - u^*(\nabla^E_X v) \cdot u \quad \text{Leibniz rule of } \nabla^E \\
&= \nabla^E_X(T(v)) - T(\nabla^E_X v) \quad \text{via identification}
\end{aligned}$$

Thus, we can define $\nabla^{\mathrm{End}(E)}$ as

$$\left(\nabla_X^{\mathrm{End}(E)} T\right)(v) = \nabla^E_X(Tv) - T\left(\nabla^E_X v\right) = \left[\nabla^E_X, T\right]u, \quad \forall T \in \mathrm{End}(E), v \in \Gamma(E).$$

- **Pullback connection**: If ∇ is a connection on smooth $\mathbb{F}$-vector bundle $E \to M$, and $f : N \to M$ is a smooth map, then $f^*E \to N$ is a smooth $\mathbb{F}$-vector bundle over N (see part 1.10.2). We define the pullback connection $f^*\nabla$ on bundle f^*E via the following map

$$\begin{aligned}
f^*\nabla : \mathfrak{X}(N) \times \Gamma(f^*E) &\to \Gamma(f^*E) \\
(Y, f^*s) &\mapsto f^*\left(\nabla_{f_* Y} s\right).
\end{aligned} \tag{3.13}$$

where we note that

$$\Gamma(f^*E) = \left\{ \sum_i \varphi_i f^* s_i \mid \varphi_i \in C^\infty(N), s_i \in \Gamma(E) \right\}.$$

3.1.3 Constructions on Connections

Let $E \to M$ be a $\mathbb{F}$-vector bundle over M. A **metric on** E is a section h of $E^* \otimes_{\mathbb{F}} \overline{E^*} (\overline{E} = E$ if $\mathbb{F} = \mathbb{R})$ such that, for any $p \in M$, h_p defines a metric on E_p (Euclidean if $\mathbb{F} = \mathbb{R}$ or Hermitian if $\mathbb{F} = \mathbb{C}$). Let h be a real metric on $E^* \otimes E^*$. A connection ∇ on E is said to be **compatible with a metric** h **on** E if

$$\nabla^{E^* \otimes E^*} h = 0$$

If ∇ is compatible with a metric h then, for any $s_1, s_2 \in \Gamma(E)$ and any $X \in \mathfrak{X}(M)$ we have

$$X(h(s_1, s_2)) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2)$$

It is often useful to have a local description of a covariant derivative.

Gauge Transformations

Let $E \to M$ be a $\mathbb{F}$-vector bundle of rank r with a connection ∇ over the smooth manifold M. A **moving frame**, or **local gauge**, is a bundle isomorphism $\phi : \underline{\mathbb{F}}_U^r = U \times \mathbb{F}^r \to E|_U$. Recall that a choice of a local frame field over an open set $U \subset M$ is equivalent to a local trivialization, which is a mapping from $E|_U$ to $\underline{\mathbb{F}}_U^r$. We will make a distinction between the two at the end.

Consider the sections $E_j = \phi(b_j), j = 1, \ldots, r$, where b_j's are the basis of the trivial bundle $\mathbb{F}_U^r$. As x moves in U, the collection $(E_1(x), \ldots, E_r(x))$ describes a basis of the moving fiber E_x, whence the terminology moving frame. Pick local coordinates (x^k) over U. Then

$$\nabla E_j \in \Omega^1(E|_U) = \Gamma(T^*M|_U \otimes E|_U)$$

and

$$\nabla E_j(X) = \nabla E_j(X^k \partial_k) = X^k \nabla_{\partial_k} E_j = X^k \Gamma_{kj}^i E_i = \Gamma_{kj}^i \, \mathrm{d}x^k(X) E_i = (\Gamma_{kj}^i \, \mathrm{d}x^k \otimes E_i)(X), \quad \Gamma_{kj}^i \in C^\infty(U, \mathbb{F}).$$

Thus,

$$\nabla E_j = \Gamma_{kj}^i \, \mathrm{d}x^k \otimes E_i$$

For any section $u = u^j E_j$ of $E|_U$ we have

$$\nabla u = \mathrm{d}u^j \otimes E_j + u^j \nabla E_j = \mathrm{d}u^i \otimes E_i + u^j \Gamma_{kj}^i \, \mathrm{d}x^k \otimes E_i$$

We define a one-form $\Gamma^i{}_j$ on $E|_U$ by

$$\Gamma^i{}_j(u) := u^k \Gamma_{kj}^i, \text{ or } \Gamma^i{}_j := \Gamma_{kj}^i \, \mathrm{d}x^k = \mathrm{d}x^k \otimes \Gamma_{kj}^i$$

called the **connection one-form** with respect to the local coordinates (x^k) of U and the frame (E_j) of $E|_U$. Then,

$$\nabla E_j = \Gamma^i{}_j \otimes E_i.$$

Furthermore, we can define Γ_k to be the matrix of smooth functions (Γ_{kj}^i) whose (i,j)-entry is Γ_{kj}^i so that the matrix-valued 1-form $\Gamma = (\Gamma^i{}_j)$ can be expressed as

$$\Gamma = \mathrm{d}x^k \otimes \Gamma_k,$$

where we adopt a matrix-wise notation: for a matrix $A \in M_{r\times r}(\mathbb{F})$ we define a matrix-valued $(\varepsilon \otimes A)_{ij} := \varepsilon \otimes A_{ij}$. With this notation the expression for ∇u becomes

$$\nabla u = \left(\mathrm{d}u^i + u^j \Gamma^i{}_j\right) \otimes E_i.$$

Identifying $u = (u^1, \ldots, u^r)^T$ and $\mathrm{d}u = (\mathrm{d}u^1, \ldots, \mathrm{d}u^r)^T$ as $\mathbb{F}^r$-valued functions and 1-forms respectively, the above can be written compactly as

$$\nabla u = (\mathrm{d}u + \Gamma u) \otimes E.$$

(The i-th entry of the column Γu is $(\Gamma u)^i = \sum_j \Gamma^i{}_j u^j$.)

A natural question arises: how does the connection 1-form change with the change of the local gauge?

Let $\mathcal{F} = (F_i)$ be another moving frame of $E|_U$. For each point $p \in U$, the change-of-basis matrix $\mathcal{M}_{\mathcal{F}_p \to \mathcal{E}_p}$ defines an automorphism of E_p. We then get a smooth map $g : U \to \mathrm{GL}(r; \mathbb{F})$ sending p to $\mathcal{M}_{\mathcal{F}_p \to \mathcal{E}_p}$, whose (i,j)-entry is denoted $g^i{}_j$. The map g is called the **local gauge transformation relating $\mathcal{E}$ to $\mathcal{F}$** and it has the property that $[v]_{\mathcal{E}} = g[v]_{\mathcal{F}}$ and $F_j = \sum_i g^i{}_j E_i$.

Let $\widehat{\Gamma}$ denote the connection 1-form corresponding to the new moving frame, i.e.

$$\nabla F_i = \widehat{\Gamma}^j{}_i \otimes F_j.$$

Consider a section σ of $E|_U$. With respect to the local frame (E_i) the section σ has a decomposition

$$\sigma = u^i E_i,$$

while with respect to (F_i) it has a decomposition

$$\sigma = \widehat{u}^i F_i.$$

The two decompositions are related by

$$u = [\sigma]_{\mathcal{E}} = g[\sigma]_{\mathcal{F}} = g\,\widehat{u}.$$

Similarly, we can identify the E-valued 1-form $\nabla\sigma$ in two ways: using the frame $\mathcal{E}$, we have

$$\nabla\sigma = \left(\mathrm{d}u^i + \Gamma^i{}_j u^j \right) \otimes E_i = (\mathrm{d}u + \Gamma u) \otimes E.$$

Using $\mathcal{F}$, we have

$$\begin{aligned}\nabla\sigma &= \left(\mathrm{d}\hat{u}^i + \hat{\Gamma}^i{}_j \hat{u}^j \right) \otimes F_i = \left(\mathrm{d}\hat{u}^i + \hat{\Gamma}^i{}_j \hat{u}^j \right) \otimes \left(g^k{}_i E_k \right) \\ &= (g^k{}_i \,\mathrm{d}\hat{u}^i + g^k{}_i (\hat{\Gamma}\hat{u})^i) \otimes E_k = (g(\mathrm{d}\hat{u} + \hat{\Gamma}\hat{u})) \otimes E.\end{aligned}$$

Thus,

$$\mathrm{d}u + \Gamma u = g(\mathrm{d}\hat{u} + \hat{\Gamma}\hat{u}).$$

On the other hand, from $u = g\widehat{u}$ we have

$$\mathrm{d}(g\hat{u}) + \Gamma(g\hat{u}) = (\mathrm{d}g)\hat{u} + g\,\mathrm{d}\hat{u} + \Gamma(g\hat{u}).$$

Then

$$\begin{aligned}&g(\mathrm{d}\hat{u} + \hat{\Gamma}\hat{u}) = g\,\mathrm{d}\hat{u} + g\hat{\Gamma}\hat{u} = (\mathrm{d}g)\hat{u} + g\,\mathrm{d}\hat{u} + \Gamma(g\hat{u}) \\ \Longrightarrow\ & g\hat{\Gamma}\hat{u} = (\mathrm{d}g)\hat{u} + \Gamma g\hat{u} \\ \Longrightarrow\ & \hat{\Gamma}\hat{u} = g^{-1}(\mathrm{d}g)\hat{u} + g^{-1}\Gamma g\hat{u}\end{aligned}$$

Since this holds for all $\widehat{u}$, we conclude the following **transformation law for connection forms**:

$$\widehat{\Gamma} = g^{-1}dg + g^{-1}\Gamma g,$$

Remark 3.1.20. The identification

$$\{\text{moving frames}\} \;\cong\; \{\text{local trivializations}\}$$

must be used with care. Fix two trivializations

$$f_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{F}^r, \qquad f_\beta : E|_{U_\beta} \xrightarrow{\cong} U_\beta \times \mathbb{F}^r,$$

and write $U_{\alpha\beta} = U_\alpha \cap U_\beta$. The transition map from α to β is

$$t_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(r, \mathbb{F})$$

with

$$(f_\beta \circ f_\alpha^{-1})(p, v) = (p, t_{\beta\alpha}(p)\,v), \qquad v \in \mathbb{F}^r.$$

The standard basis (b_i) of $\mathbb{F}^r$ induces moving frames

$$E_{\alpha,i} := f_\alpha^{-1}(b_i), \qquad E_{\beta,i} := f_\beta^{-1}(b_i).$$

On $U_{\alpha\beta}$ there is a unique map

$$g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(r, \mathbb{F})$$

characterized by the requirement that, for every $p \in U_{\alpha\beta}$ and every $v = (v^1, \ldots, v^r)^T \in \mathbb{F}^r$,

$$\sum_{i=1}^{r} v^i E_{\beta,i}(p) = \sum_{j=1}^{r} \left(g_{\alpha\beta}(p)\, v\right)^j E_{\alpha,j}(p).$$

Equivalently, for the standard basis $v = e_i$ of $\mathbb{F}^r$,

$$E_{\beta,i}(p) = \sum_{j=1}^{r} \left(g_{\alpha\beta}(p)\right)^j{}_i\, E_{\alpha,j}(p).$$

For any $w \in E_p$ we write $f_\alpha(w) = (p, u)$ and $f_\beta(w) = (p, \hat{u})$ with $u, \hat{u} \in \mathbb{F}^r$. By definition of the transition function,

$$\hat{u} = t_{\beta\alpha}(p)\, u,$$

while by the defining property of $g_{\alpha\beta}$,

$$u = g_{\alpha\beta}(p)\, \hat{u}.$$

Hence, for every $p \in U_{\alpha\beta}$,

$$t_{\beta\alpha}(p) = g_{\alpha\beta}(p)^{-1}.$$

But $g_{\alpha\beta}(p)^{-1} = g_{\beta\alpha}$, so

$$t_{\beta\alpha} = g_{\beta\alpha}.$$

Thus, local gauge transformation is precisely the opposite way the two trivializations are identified via their transition map. ♠

The above arguments can be reversed to produce the following global result.

Proposition 3.1.21. *Let $E \to M$ be a rank-r smooth vector bundle, and (U_α) a trivializing cover with transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r; \mathbb{F})$. Then any collection of matrix valued 1-forms $\Gamma_\alpha \in \Omega^1\left(\mathrm{End}\,\underline{\mathbb{F}}^r_{U_\alpha}\right)$ satisfying*

$$\Gamma_\beta = g_{\alpha\beta}^{-1}\left(\,\mathrm{d}g_{\alpha\beta}\right) + g_{\alpha\beta}^{-1}\Gamma_\alpha g_{\alpha\beta} = -\left(\,\mathrm{d}g_{\beta\alpha}\right) g_{\beta\alpha}^{-1} + g_{\beta\alpha}\Gamma_\alpha g_{\beta\alpha}^{-1} \text{ over } U_\alpha \cap U_\beta \tag{3.14}$$

uniquely defines a covariant derivative on E. Here $g_{\alpha\beta} = (g_{\beta\alpha})^{-1}$, so differentiation of $g_{\alpha\beta}g_{\beta\alpha} = I_r$ gives $\mathrm{d}g_{\alpha\beta} = -g_{\alpha\beta}\,\mathrm{d}g_{\beta\alpha}g_{\beta\alpha}^{-1}$.

We can use the local description in Proposition 3.1.21 to give an equivalent notion of pullback of a connection. Suppose we are given the following data.

- A smooth map $f : N \to M$.
- A rank-r $\mathbb{F}$-vector bundle $E \to M$ defined by the open cover (U_α), and transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}\left(\mathbb{F}^r\right)$.
- A connection ∇ on E defined by the 1 -forms $\Gamma_\alpha \in \Omega^1\left(\mathrm{End}\left(\underline{\mathbb{F}}^r_{U_\alpha}\right)\right)$ satisfying the gluing conditions (3.14).

Then, these data define a connection $f^*\nabla$ on f^*E described by the open cover $f^{-1}\left(U_\alpha\right)$, transition maps $g_{\beta\alpha} \circ f$ and 1-forms $f^*\Gamma_\alpha$. Connection $f^*\nabla$ is independent of the various choices and it is called the **pullback connection**.

Exercise 3.1.22. *Show that the above definition of pullback connection is the same as the one defined by equation (3.13).*

Parallel Transport

Let $\gamma : I \to M$ be a smooth curve on a smooth manifold M, and let $E \to M$ be a smooth $\mathbb{F}$-vector bundle over M endowed with a connection ∇. By Definition 3.13, the pullback bundle γ^*E is a vector bundle over the interval I, and the pullback connection $\gamma^*\nabla$ is a connection on γ^*E.

Since TI is canonically trivialized by the coordinate vector field $\frac{d}{dt}$, the pullback connection yields an operator

$$D_t = D_t^{\nabla} := (\gamma^*\nabla)_{\frac{d}{dt}} \ : \ \Gamma(\gamma^*E) \longrightarrow \Gamma(\gamma^*E),$$

called the **covariant derivative along** γ. Explicitly, for a section $\gamma^*s \in \Gamma(\gamma^*E)$,

$$(D_t\gamma^*s)(t_0) \ = \ \left((\gamma^*\nabla)_{\frac{d}{dt}}\gamma^*s\right)(t_0) = \gamma^*\left(\nabla_{d\gamma(\frac{d}{dt})}s\right)(t_0) = \left(\nabla_{\gamma'(t)}s\right)(\gamma(t_0))$$

Definition 3.1.23 (Parallel Transport). *A section $\sigma(t) : I \to \gamma^*E$ in $\Gamma(\gamma^*E)$ is called **parallel** (with respect to ∇) if it satisfies*

$$\left(D_t\sigma(t)\right)(t) = 0 \quad \text{for all } t \in I.$$

Proposition 3.1.24. *Given an initial value $v_0 \in E_{\gamma(t_0)}$, there exists a unique parallel section $\sigma(t)$ such that $\sigma(t_0) = v_0$. The map*

$$P^{\gamma}_{t_0 t} : E_{\gamma(t_0)} \longrightarrow E_{\gamma(t)}, \quad v_0 \mapsto \sigma(t)$$

*is called the **parallel transport** along γ from t_0 to t.*

Remaining questions:

1. Does above definition have the property that $D_t\sigma(t) = \nabla_{\gamma'(t)}\widetilde{\sigma}$ as in [LeeRM] Theorem 4.24.
2. Notice how connection and parallel transport are identified with each other as in p.165 of Jost's.
3. Latex for Lee's on optimal transport is reserved in RG_C4.tex.

3.2 The Levi-Civita Connection

3.2.1 The Tangential Connection Revisited

We are eventually going to show that on each Riemannian manifold there is a natural connection that is particularly well suited to computations in Riemannian geometry. Since we get most of our intuition about Riemannian manifolds from studying submanifolds of $\mathbb{R}^n$ with the induced metric, let us start by examining that case.

Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. A geodesic in M should be "as straight as possible." A reasonable way to make this rigorous is to require that the geodesic have no acceleration in directions tangent to the manifold, or in other words that its acceleration vector have zero orthogonal projection onto TM.

The tangential connection $\nabla^{\top}_X(Y) = \pi^{\top}\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}|_M\right)$ defined in Example 3.1.6 is perfectly suited to this task, because it computes covariant derivatives on M by taking ordinary derivatives in $\mathbb{R}^n$ and projecting them orthogonally to TM.

It is easy to compute covariant derivatives along curves in M with respect to the tangential connection. Suppose $\gamma : I \to M$ is a smooth curve. Then γ can be regarded as either a smooth curve in M or a smooth curve in $\mathbb{R}^n$, and a smooth vector field V along γ that takes its values in TM can be regarded as either a vector field along γ in M or a vector field along γ in $\mathbb{R}^n$. Let $\overline{D}_tV$ denote the covariant derivative of V along γ (as a curve in $\mathbb{R}^n$) with respect to the Euclidean connection $\overline{\nabla}$, and let $D_t^{\top}V$ denote its covariant derivative along γ (as a curve in M) with respect to the tangential connection $\nabla^{\top}$. [12] Proposition 5.1 shows a simple relationship between them: $\forall t \in I$, $D_t^{\top}V(t) = \pi^{\top}(\overline{D}_tV(t))$. Via plugging the zero connection coefficients of the Euclidean connection on $\mathbb{R}^n$ into (4.2), we see that $\overline{D}_t\gamma'(t) = \gamma''(t)$. Thus, the smooth curve $\gamma : I \to M$ is a geodesic with respect to the tangential connection on M if and only if its ordinary acceleration $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for all $t \in I$.

Analogs for embedded Riemannian or pseudo-Riemannian manifolds in pseudo-Euclidean space $\mathbb{R}^{r,s}$ are provided in [12] p.117 as well.

3.2.2 Connections on Abstract Riemannian Manifolds

There is a celebrated (and hard) theorem of John Nash that says that every Riemannian metric on a smooth manifold can be realized as the induced metric of some embedding in a Euclidean space. That theorem was later generalized independently by Robert Greene and Chris J. S. Clarke to pseudo-Riemannian metrics. Thus, in a certain sense, we would lose no generality by studying only submanifolds of Euclidean and pseudo-Euclidean spaces with their induced metrics, for which the tangential connection would suffice. However, when we are trying to understand *intrinsic* properties of a Riemannian manifold, an embedding introduces a great deal of extraneous information, and in some cases actually makes it harder to discern which geometric properties depend only on the metric. Our task in this chapter is to distinguish some important properties of the tangential connection that make sense for connections on an abstract Riemannian or pseudo-Riemannian manifold, and to use them to single out a unique connection in the abstract case.

Metric Connections

The Euclidean connection on $\mathbb{R}^n$ has one very nice property with respect to the Euclidean metric: it satisfies the Leibniz rule

$$\overline{\nabla}_X\langle Y, Z\rangle = \left\langle\overline{\nabla}_X Y, Z\right\rangle + \left\langle Y, \overline{\nabla}_X Z\right\rangle, \tag{3.15}$$

as you can verify easily by computing in terms of the standard basis. (In this formula, the left-hand side represents the covariant derivative of the real-valued function $\langle Y, Z\rangle$ regarded as a $(0,0)$-tensor field, which is really just $X\langle Y, Z\rangle$ by virtue of property (ii) of Prop. 3.1.13.) The Euclidean connection has the same property with respect to the pseudo-Euclidean metric on $\mathbb{R}^{r,s}$. It is almost immediate that the tangential connection on a Riemannian or pseudo-Riemannian submanifold satisfies the same Leibniz rule, if we now interpret all the vector fields as being tangent to M and interpret the inner products as being taken with respect to the induced metric on M (see Prop. 3.2.3 below).

This property makes sense on an abstract Riemannian or pseudo-Riemannian manifold. Let g be a Riemannian or pseudo-Riemannian metric on a smooth manifold M (with or without boundary). A connection ∇ on TM is said to be **compatible with** g, or to be a **metric connection**, if it satisfies the following Leibniz rule for all $X, Y, Z \in \mathfrak{X}(M)$:

$$\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle .$$

The next proposition gives several alternative characterizations of compatibility with a metric, any one of which could be used as the definition.

Proposition 3.2.1 (Characterizations of Metric Connections). *Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be a connection on TM. The following conditions are equivalent:*

(a) ∇ is compatible with $g : \nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

(b) g is parallel with respect to $\nabla : \nabla g \equiv 0$.

(c) In terms of any smooth local frame (E_i), the connection coefficients of ∇ satisfy

$$\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il} = E_k\left(g_{ij}\right). \tag{3.16}$$

(d) If V, W are smooth vector fields along any smooth curve γ, then

$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle . \tag{3.17}$$

(e) If V, W are parallel vector fields along a smooth curve γ in M, then $\langle V, W\rangle$ is constant along γ.

(f) Given any smooth curve γ in M, every parallel transport map along γ is a linear isometry.

(g) Given any smooth curve γ in M, every orthonormal basis at a point of γ can be extended to a parallel orthonormal frame along γ.

Proof. First we prove (a) $\Leftrightarrow$ (b). By (3.7) and (3.4), the total covariant derivative of the symmetric 2-tensor g is given by

$$(\nabla g)(Y, Z, X) = (\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) .$$

This is zero for all X, Y, Z if and only if (3.15) is satisfied for all X, Y, Z. To prove (b) $\Leftrightarrow$ (c), note that Proposition 3.1.17 shows that the components of ∇g in terms of a smooth local frame (E_i) are

$$g_{ij;k} = E_k(g_{ij}) - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il} .$$

These are all zero if and only if (3.16) is satisfied. Next we prove (a) $\Leftrightarrow$ (d). Assume (a), and let V, W be smooth vector fields along a smooth curve $\gamma : I \to M$. Given $t_0 \in I$, in a neighborhood of $\gamma(t_0)$ we may choose coordinates (x^i) and write $V = V^i \partial_i$ and $W = W^j \partial_j$ for some smooth functions $V^i, W^j (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$. Applying (3.15) to the extendible vector fields ∂_i, ∂_j, we obtain

$$\begin{aligned}
\frac{d}{dt}\langle V, W\rangle &= \frac{d}{dt}\left(V^i W^j \langle \partial_i, \partial_j\rangle\right) \\
&= \left(\dot{V}^i W^j + V^i \dot{W}^j\right)\langle \partial_i, \partial_j\rangle + V^i W^j \left(\left\langle \nabla_{\gamma'(t)}\partial_i, \partial_j\right\rangle + \left\langle \partial_i, \nabla_{\gamma'(t)}\partial_j\right\rangle\right) \\
&= \langle D_t V, W\rangle + \langle V, D_t W\rangle ,
\end{aligned}$$

which proves (d). Conversely, if (d) holds, then in particular it holds for extendible vector fields along γ, and then (a) follows from part (iii) of Theorem 4.4.2.

Now we will prove (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (d). Assume first that (d) holds. If V and W are parallel along γ, then (3.17) shows that $\langle V, W\rangle$ has zero derivative with respect to t, so it is constant along γ.

Now assume (e). Let v_0, w_0 be arbitrary vectors in $T_{\gamma(t_0)}M$, and let V, W be their parallel transports along γ, so that $V(t_0) = v_0, W(t_0) = w_0, P_{t_0 t_1}^\gamma v_0 = V(t_1)$, and $P_{t_0 t_1}^\gamma w_0 = W(t_1)$. Because $\langle V, W\rangle$ is constant along γ, it follows that $\left\langle P_{t_0 t_1}^\gamma v_0, P_{t_0 t_1}^\gamma w_0\right\rangle = \langle V(t_1), W(t_1)\rangle = \langle V(t_0), W(t_0)\rangle = \langle v_0, w_0\rangle$, so $P_{t_0 t_1}^\gamma$ is a linear isometry.

Next, assuming (f), we suppose $\gamma : I \to M$ is a smooth curve and (b_i) is an orthonormal basis for $T_{\gamma(t_0)}M$, for some $t_0 \in I$. We can extend each b_i by parallel transport to obtain a smooth parallel vector field E_i along γ, and the assumption that parallel transport is a linear isometry guarantees that the resulting n-tuple (E_i) is an orthonormal frame at all points of γ.

Finally, assume that (g) holds, and let (E_i) be a parallel orthonormal frame along γ. Given smooth vector fields V and W along γ, we can express them in terms of this frame as $V = V^i E_i$ and $W = W^j E_j$. The fact that the frame is orthonormal means that the metric coefficients $g_{ij} = \langle E_i, E_j\rangle$ are constants along $\gamma(\pm 1$ or 0), and the fact that it is parallel means that $D_t V = \dot{V}^i E_i$ and $D_t W = \dot{W}^i E_i$. Thus both sides of (3.17) reduce to the following expression:

$$g_{ij}\left(\dot{V}^i W^j + V^i \dot{W}^j\right) .$$

This proves (d). ∎

Corollary 3.2.2. *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold with or without boundary, ∇ is a metric connection on M, and $\gamma : I \to M$ is a smooth curve.*

(a) $|\gamma'(t)|$ is constant if and only if $D_t\gamma'(t)$ is orthogonal to $\gamma'(t)$ for all $t \in I$.

(b) If γ is a geodesic, then $|\gamma'(t)|$ is a constant.

Proof. Let $V(t) = W(t) = \gamma'(t)$ in proposition 3.2.1(d). ■

Proposition 3.2.3. *If M is an embedded Riemannian or pseudo-Riemannian submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$, the tangential connection on M is compatible with the induced Riemannian or pseudo-Riemannian metric.*

Proof. We will show that $\nabla^\top$ satisfies (3.15). Suppose $X, Y, Z \in \mathfrak{X}(M)$, and let $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ be smooth extensions of them to an open subset of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$. At points of M, we have

$$\begin{aligned}
\nabla_X^\top \langle Y, Z\rangle = X\langle Y, Z\rangle &= \widetilde{X}\langle \widetilde{Y}, \widetilde{Z}\rangle \\
&= \overline{\nabla}_{\widetilde{X}}\langle \widetilde{Y}, \widetilde{Z}\rangle \\
&= \left\langle \overline{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}\right\rangle + \left\langle \widetilde{Y}, \overline{\nabla}_{\widetilde{X}}\widetilde{Z}\right\rangle \\
&= \left\langle \pi^\top\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}\right), \widetilde{Z}\right\rangle + \left\langle \widetilde{Y}, \pi^\top\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Z}\right)\right\rangle \\
&= \left\langle \nabla_X^\top Y, Z\right\rangle + \left\langle Y, \nabla_X^\top Z\right\rangle,
\end{aligned}$$

where the next-to-last equality follows from the fact that $\widetilde{Z}$ and $\widetilde{Y}$ are tangent to M. ■

Symmetric Connections

It turns out that every abstract Riemannian or pseudo-Riemannian manifold admits many different metric connections (see [12] Problem 5-1), so requiring compatibility with the metric is not sufficient to pin down a unique connection on such a manifold. To do so, we turn to another key property of the tangential connection. Recall the definition of the Euclidean connection. The expression on the right-hand side of that definition is reminiscent of part of the coordinate expression for the Lie bracket:

$$[X, Y] = X\left(Y^i\right)\frac{\partial}{\partial x^i} - Y\left(X^i\right)\frac{\partial}{\partial x^i}.$$

In fact, the two terms in the Lie bracket formula are exactly the coordinate expressions for $\overline{\nabla}_X Y$ and $\overline{\nabla}_Y X$. Therefore, the Euclidean connection satisfies the following identity for all smooth vector fields X, Y :

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y].$$

This expression has the virtue that it is coordinate-independent and makes sense for every connection on the tangent bundle. We say that a connection ∇ on the tangent bundle of a smooth manifold M is **symmetric** if

$$\nabla_X Y - \nabla_Y X \equiv [X, Y] \text{ for all } X, Y \in \mathfrak{X}(M).$$

The symmetry condition can also be expressed in terms of the **torsion tensor** of the connection, which was introduced in Problem 3.6.7; this is the smooth $(1,2)$-tensor field $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Thus a connection ∇ is symmetric if and only if its torsion vanishes identically. It follows from the result of Problem 3.6.7 that a connection is symmetric if and only if its connection coefficients in every coordinate frame satisfy $\Gamma_{ij}^k = \Gamma_{ji}^k$; this is the origin of the term "symmetric."

Proposition 3.2.4. *If M is an embedded (pseudo-)Riemannian submanifold of a (pseudo-)Euclidean space, then the tangential connection on M is symmetric.*

Proof. Let M be an embedded Riemannian or pseudo-Riemannian submanifold of $\mathbb{R}^n$, where $\mathbb{R}^n$ is endowed either with the Euclidean metric or with a pseudoEuclidean metric $\overline{q}^{(r,s)}, r+s=n$. Let $X, Y \in \mathfrak{X}(M)$, and let $\widetilde{X}, \widetilde{Y}$ be smooth extensions of them to an open subset of the ambient space. If $\iota : M \hookrightarrow \mathbb{R}^n$ represents the inclusion map, it follows that X and Y are ι-related to $\widetilde{X}$ and $\widetilde{Y}$, respectively, and thus by the naturality of the Lie bracket ([12] Prop. A.39), $[X,Y]$ is ι-related to $[\widetilde{X},\widetilde{Y}]$. In particular, $[\widetilde{X},\widetilde{Y}]$ is tangent to M, and its restriction to M is equal to $[X,Y]$. Therefore,

$$\begin{aligned}\nabla^{\top}_X Y - \nabla^{\top}_Y X = \pi^{\top}\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}\Big|_M - \overline{\nabla}_{\widetilde{Y}}\widetilde{X}\Big|_M\right)\\ =\pi^{\top}\left([\widetilde{X},\widetilde{Y}]\Big|_M\right) = [\widetilde{X},\widetilde{Y}]\Big|_M = [X,Y].\end{aligned}$$

■

The last two propositions show that if we wish to single out a connection on each Riemannian or pseudo-Riemannian manifold in such a way that it matches the tangential connection when the manifold is presented as an embedded submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$ with the induced metric, then we must require at least that the connection be compatible with the metric and symmetric. It is a pleasant fact that these two conditions are enough to determine a unique connection.

Theorem 3.2.5 (Fundamental Theorem of Riemannian Geometry)**.** *Let (M,g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary). There exists a unique connection ∇ on TM that is compatible with g and symmetric. It is called the* ***Levi-Civita connection*** *of g (or also, when g is positive definite, the* ***Riemannian connection****).*

Proof. We prove uniqueness first, by deriving a formula for ∇. Suppose, therefore, that ∇ is such a connection, and let $X, Y, Z \in \mathfrak{X}(M)$. Writing the compatibility equation three times with X, Y, Z cyclically permuted, we obtain

$$\begin{aligned}X\langle Y,Z\rangle &= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\\ Y\langle Z,X\rangle &= \langle\nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle\\ Z\langle X,Y\rangle &= \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle\end{aligned}$$

Using the symmetry condition on the last term in each line, this can be rewritten as

$$\begin{aligned}X\langle Y,Z\rangle &= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y,[X,Z]\rangle\\ Y\langle Z,X\rangle &= \langle\nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z,[Y,X]\rangle\\ Z\langle X,Y\rangle &= \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X,[Z,Y]\rangle\end{aligned}$$

Adding the first two of these equations and subtracting the third, we obtain

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = 2\left\langle\nabla_X Y, Z\right\rangle + \langle Y,[X,Z]\rangle + \langle Z,[Y,X]\rangle - \langle X,[Z,Y]\rangle.$$

Finally, solving for $\langle\nabla_X Y, Z\rangle$, we get

$$\langle\nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle). \tag{3.18}$$

Now suppose ∇^1 and ∇^2 are two connections on TM that are symmetric and compatible with g. Since the right-hand side of (3.18) does not depend on the connection, it follows that $\left\langle\nabla^1_X Y - \nabla^2_X Y, Z\right\rangle = 0$ for all X, Y, Z. This can happen only if $\nabla^1_X Y = \nabla^2_X Y$ for all X and Y, so $\nabla^1 = \nabla^2$.

To prove existence, we use (3.18), or rather a coordinate version of it. It suffices to prove that such a connection exists in each coordinate chart, for then uniqueness ensures that the connections in different charts agree where they overlap.

Let $\left(U,\left(x^i\right)\right)$ be any smooth local coordinate chart. Applying (3.18) to the coordinate vector fields, whose Lie brackets are zero, we obtain

$$\langle \nabla_{\partial_i}\partial_j, \partial_l\rangle = \frac{1}{2}\left(\partial_i \langle \partial_j, \partial_l\rangle + \partial_j \langle \partial_l, \partial_i\rangle - \partial_l \langle \partial_i, \partial_j\rangle\right). \tag{3.19}$$

Recall the definitions of the metric coefficients and the connection coefficients:

$$g_{ij} = \langle \partial_i, \partial_j\rangle, \quad \nabla_{\partial_i}\partial_j = \Gamma_{ij}^m \partial_m.$$

Inserting these into (3.19) yields

$$\Gamma_{ij}^m g_{ml} = \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{3.20}$$

Finally, multiplying both sides by the inverse matrix g^{kl} and noting that $g_{ml}g^{kl} = \delta_m^k$, we get

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{3.21}$$

This formula certainly defines a connection in each chart, and it is evident from the formula that $\Gamma_{ij}^k = \Gamma_{ji}^k$, so the connection is symmetric by Problem 3.6.7(b). Thus only compatibility with the metric needs to be checked. Using (3.20) twice, we get

$$\begin{aligned}\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il} &= \frac{1}{2}\left(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ki}\right) + \frac{1}{2}\left(\partial_k g_{ji} + \partial_j g_{ki} - \partial_i g_{kj}\right)\\ &= \partial_k g_{ij}\end{aligned}$$

By Proposition 3.2.1 (c), this shows that ∇ is compatible with g. ■

A bonus of this proof is that it gives us explicit formulas that can be used for computing the Levi-Civita connection in various circumstances.

Corollary 3.2.6 (Formulas for the Levi-Civita Connection). *Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be its Levi-Civita connection.*

(a) IN TERMS OF VECTOR FIELDS: If X, Y, Z are smooth vector fields on M, then

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}(&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle\\ &- \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle)\end{aligned} \tag{3.22}$$

(This is known as ***Koszul's formula****.)*

(b) IN COORDINATES: In any smooth coordinate chart for M, the coefficients of the Levi-Civita connection are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{3.23}$$

(c) IN A LOCAL FRAME: Let (E_i) be a smooth local frame on an open subset $U \subseteq M$, and let $c_{ij}^k : U \to \mathbb{R}$ be the n^3 smooth functions defined by

$$[E_i, E_j] = c_{ij}^k E_k. \tag{3.24}$$

Then the coefficients of the Levi-Civita connection in this frame are

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm} c_{il}^m - g_{lm} c_{ji}^m + g_{im} c_{lj}^m\right). \tag{3.25}$$

(d) *IN A LOCAL ORTHONORMAL FRAME: If g is Riemannian, (E_i) is a smooth local orthonormal frame, and the functions c_{ij}^k are defined by (5.11), then*

$$\Gamma_{ij}^k = \frac{1}{2}\left(c_{ij}^k - c_{ik}^j - c_{jk}^i\right). \tag{3.26}$$

Proof. We derived (3.22) and (3.23) in the proof of Theorem 3.2.5. To prove (3.25), apply formula (3.22) with $X = E_i, Y = E_j$, and $Z = E_l$, to obtain

$$\begin{aligned}\Gamma_{ij}^q g_{ql} &= \langle \nabla_{E_i} E_j, E_l \rangle \\ &= \frac{1}{2}\left(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm} c_{il}^m - g_{lm} c_{ji}^m + g_{im} c_{lj}^m\right).\end{aligned}$$

Multiplying both sides by g^{kl} and simplifying yields (3.25). Finally, under the hypotheses of (d), we have $g_{ij} = \delta_{ij}$, so (3.25) reduces to (3.26) after rearranging and using the fact that c_{ij}^k is antisymmetric in i, j. ■

On every Riemannian or pseudo-Riemannian manifold, we will always use the Levi-Civita connection from now on without further comment. Geodesics with respect to this connection are called **Riemannian** (or **pseudo-Riemannian**) **geodesics**, or simply "geodesics" as long as there is no risk of confusion. The connection coefficients Γ_{ij}^k of the Levi-Civita connection in coordinates, given by (3.23), are called the **Christoffel symbols** of g.

The next proposition shows that these connections are familiar ones in the case of embedded submanifolds of Euclidean or pseudo-Euclidean spaces.

Proposition 3.2.7.

(a) *The Levi-Civita connection on a (pseudo-)Euclidean space is equal to the Euclidean connection.*

(b) *Suppose M is an embedded (pseudo-)Riemannian submanifold of a (pseudo-)Euclidean space. Then the Levi-Civita connection on M is equal to the tangential connection $\nabla^\top$.*

Proof. We observed earlier in this chapter that the Euclidean connection is symmetric and compatible with both the Euclidean metric $\bar{g}$ and the pseudo-Euclidean metrics $\bar{q}^{(r,s)}$, which implies (a). Part (b) then follows from Propositions 3.2.3 and 3.2.4. ■

An important consequence of the definition is that because Levi-Civita connections are defined in coordinate-independent terms, they behave well with respect to isometries.

Proposition 3.2.8 (Naturality of the Levi-Civita Connection)**.** *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and let ∇ denote the Levi-Civita connection of g and $\widetilde{\nabla}$ that of $\widetilde{g}$. If $\varphi : M \to \widetilde{M}$ is an isometry, then $\varphi^* \widetilde{\nabla} = \nabla$.*

Proof. By uniqueness of the Levi-Civita connection, it suffices to show that the pullback connection $\varphi^* \widetilde{\nabla}$ is symmetric and compatible with g. The fact that φ is an isometry means that for any $X, Y \in \mathfrak{X}(M)$ and $p \in M$,

$$\langle Y_p, Z_p \rangle = \langle d\varphi_p(Y_p), d\varphi_p(Z_p) \rangle = \left\langle (\varphi_* Y)_{\varphi(p)}, (\varphi_* Z)_{\varphi(p)} \right\rangle,$$

where $\varphi^* Y$ is a vector field, called the pushforward of Y by φ; see [11] p.183. In other words, $\langle Y, Z \rangle =$

$\langle \varphi_* Y, \varphi_* Z\rangle \circ \varphi$. Therefore,

$$\begin{aligned}
X\langle Y, Z\rangle &= X\left(\langle \varphi_* Y, \varphi_* Z\rangle \circ \varphi\right) \\
&\xlongequal{\text{[11]Cor.8.21}} \left(\left(\varphi_* X\right)\langle \varphi_* Y, \varphi_* Z\rangle\right) \circ \varphi \\
&\xlongequal{\widetilde{\nabla}\text{ a metric conn.}} \left(\left\langle \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right), \varphi_* Z\right\rangle + \left\langle \varphi_* Y, \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Z\right)\right\rangle\right) \circ \varphi \\
&\xlongequal{\text{see below}} \left\langle \left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right), Z\right\rangle + \left\langle Y, \left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Z\right)\right\rangle \\
&\xlongequal{(3.13)} \left\langle \left(\varphi^* \widetilde{\nabla}\right)_X Y, Z\right\rangle + \left\langle Y, \left(\varphi^* \widetilde{\nabla}\right)_X Z\right\rangle,
\end{aligned}$$

which shows that the pullback connection is compatible with g. The fourth equality is true in the same manner as $\langle Y, Z\rangle = \langle \varphi_* Y, \varphi_* Z\rangle \circ \varphi$. Specifically, $\left\langle \varphi_*\left(\left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right)\right), \varphi_* Z\right\rangle \circ \varphi = \left\langle \left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right), Z\right\rangle$. Symmetry of the pullback connection is proved as follows:

$$\begin{aligned}
\left(\varphi^* \widetilde{\nabla}\right)_X Y - \left(\varphi^* \widetilde{\nabla}\right)_Y X &\xlongequal{(3.13)} \left(\varphi^{-1}\right)_* \left(\widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right) - \widetilde{\nabla}_{\varphi_* Y}\left(\varphi_* X\right)\right) \\
&\xlongequal{\widetilde{\nabla}\text{ a sym. conn.}} \left(\varphi^{-1}\right)_* \left[\varphi_* X, \varphi_* Y\right] \\
&= [X, Y]
\end{aligned}$$

■

Corollary 3.2.9 (Naturality of Geodesics). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is a local isometry. If γ is a geodesic in M, then $\varphi \circ \gamma$ is a geodesic in $\widetilde{M}$.*

Proof. To check a curve is a geodesic is to check the geodesic equation at a local neighborhood. Thus, we can restrict our attention to open submanifolds $\widetilde{U} \subseteq \widetilde{M}$ and $U := \varphi^{-1}(\widetilde{U}) \subseteq M$. φ is a diffeomorphism with respect to them, pulling connection $\widetilde{\nabla}^{\widetilde{U}}$ to connection ∇^U. Proposition 4.4.13 then concludes. ■

Like every connection on the tangent bundle, the Levi-Civita connection induces connections on all tensor bundles.

Proposition 3.2.10. *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold. The connection induced on each tensor bundle by the Levi-Civita connection is compatible with the induced inner product on tensors, in the sense that $X\langle F, G\rangle = \langle \nabla_X F, G\rangle + \langle F, \nabla_X G\rangle$ for every vector field X and every pair of smooth tensor fields $F, G \in \Gamma\left(T^{(k,l)} TM\right)$.*

Proof. Since every tensor field can be written as a sum of tensor products of vector and/or covector fields, it suffices to consider the case in which $F = \alpha_1 \otimes \cdots \otimes \alpha_{k+l}$ and $G = \beta_1 \otimes \cdots \otimes \beta_{k+l}$, where α_i and β_i are covariant or contravariant 1-tensor fields, as appropriate. In this case, the formula follows from (2.17) by a routine computation. ■

Proposition 3.2.11. *Let (M, g) be an oriented Riemannian manifold. The Riemannian volume form of g is parallel with respect to the Levi-Civita connection.*

Proof. Let $p \in M$ and $v \in T_p M$ be arbitrary, and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. Let $(E_1, \ldots, E_n)$ be a parallel oriented orthonormal frame along γ. Since $dV_g\left(E_1, \ldots, E_n\right) \equiv 1$ and $D_t E_i \equiv 0$ along γ, formula (3.4) shows that $\nabla_v\left(dV_g\right) = D_t\left(dV_g\right)\big|_{t=0} = 0$. ■

Proposition 3.2.12. *The musical isomorphisms commute with the total covariant derivative operator: if F is any smooth tensor field with a contravariant ith index position, and $\flat$ represents the operation of lowering the i th index, then*

$$\nabla\left(F^{\flat}\right)=(\nabla F)^{\flat}. \tag{3.27}$$

Similarly, if G has a covariant ith position and $\sharp$ denotes raising the ith index, then

$$\nabla\left(G^{\sharp}\right)=(\nabla G)^{\sharp}. \tag{3.28}$$

Proof. The discussion on subsection 2.3.1 shows that $F^{\flat}=\operatorname{tr}(F\otimes g)$, where the trace is taken on the i th and last indices of $F\otimes g$. Because g is parallel, for every vector field X we have $\nabla_X(F\otimes g)=(\nabla_X F)\otimes g$. Because ∇_X commutes with traces, therefore,

$$\nabla_X\left(F^{\flat}\right)=\nabla_X(\operatorname{tr}(F\otimes g))=\operatorname{tr}\left((\nabla_X F)\otimes g\right)=(\nabla_X F)^{\flat}.$$

This shows that when X is inserted into the last index position on both sides of (3.27), the results are equal. Since X is arbitrary, this proves (3.27). Because the sharp and flat operators are inverses of each other when applied to the same index position, (3.28) follows by substituting $F=G^{\sharp}$ into (3.27) and applying $\sharp$ to both sides. ■

3.2.3 Exponential Map

Note that the results in this section are generally true for all connection in TM, not just for the Levi-Civita connection. For simplicity, we restrict attention here to the latter case. We also restrict to manifolds without boundary, in order to avoid complications with geodesics running into a boundary.

The next lemma shows that geodesics with proportional initial velocities are related in a simple way.

Lemma 3.2.13 (Rescaling Lemma)**.** *For every $p\in M, v\in T_pM$, and $c,t\in\mathbb{R}$,*

$$\gamma_{cv}(t)=\gamma_v(ct),$$

whenever either side is defined.

Proof. See [12] Lemma 5.18. ■

The assignment $v\mapsto\gamma_v$ defines a map from TM to the set of geodesics in M. More importantly, by virtue of the rescaling lemma, it allows us to define a map from (a subset of) the tangent bundle to M itself, which sends each line $\{cv\}$ through the origin in T_pM to a geodesic. Define a subset $\mathcal{E}\subseteq TM$, the **domain of the exponential map**, by

$$\mathcal{E}=\{v\in TM:\gamma_v \text{ is defined on an interval containing } [0,1]\},$$

and then define the **exponential map** $\exp:\mathcal{E}\to M$ by

$$\exp(v)=\gamma_v(1)$$

For each $p\in M$, the **restricted exponential map at** p, denoted by $\exp_p$, is the restriction of exp to the set $\varepsilon_p=\mathcal{E}\cap T_pM$.

The exponential map of a Riemannian manifold should not be confused with the exponential map of a Lie group. To avoid confusion, we always designate the exponential map of a Lie group G by $\exp^G$, and reserve the undecorated notation $\exp$ for the Riemannian exponential map.

The next proposition describes some essential features of the exponential map. Recall that a subset of a vector space V is said to be star-shaped with respect to a point $x\in S$ if for every $y\in S$, the line segment from x to y is contained in S.

Proposition 3.2.14 (Properties of the Exponential Map). *Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and let $\exp : \mathcal{E} \to M$ be its exponential map.*

(a) $\mathcal{E}$ is an open subset of TM containing the image of the zero section, and each set $\varepsilon_p \subseteq T_pM$ is star-shaped with respect to 0 .

(b) For each $v \in TM$, the geodesic γ_v is given by

$$\gamma_v(t) = \exp(tv)$$

for all t such that either side is defined.

(c) The exponential map is smooth.

(d) For each point $p \in M$, the differential $d\left(\exp_p\right)_0 : T_0\left(T_pM\right) \cong T_pM \to T_pM$ is the identity map of T_pM, under the usual identification of $T_0\left(T_pM\right)$ with T_pM.

Proof. Write $n = \dim M$. The rescaling lemma with $t = 1$ says precisely that $\exp(cv) = \gamma_{cv}(1) = \gamma_v(c)$ whenever either side is defined; this is (b). Moreover, if $v \in \mathcal{E}_p$, then by definition γ_v is defined at least on $[0, 1]$. Thus for $0 \leq t \leq 1$, the rescaling lemma says that

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

is defined. Thus, $\{tv : t \in [0, 1]\} \subseteq \mathcal{E}_p \implies$ the segment $[0, v]$ is in $\mathcal{E}_p$. This shows that ε_p is star-shaped with respect to 0.

Next we will show that $\mathcal{E}$ is open and exp is smooth. To do so, we revisit the proof of the theorem of existence and uniqueness theorem for geodesics 4.4.5 and reformulate it in a more invariant way. Let $\left(x^i\right)$ be any smooth local coordinates on an open set $U \subseteq M$, let $\pi : TM \to M$ be the projection, and let $\left(x^i, v^i\right)$ denote the associated natural coordinates for $\pi^{-1}(U) \subseteq TM$. In terms of these coordinates, formula (4.5) defines a smooth vector field G on $\pi^{-1}(U)$. The integral curves of G are the curves $\eta(t) = \left(x^1(t), \ldots, x^n(t), v^1(t), \ldots, v^n(t)\right)$ that satisfy the system of ODEs given by (4.4), which is equivalent to the geodesic equation under the substitution $v^k = \dot{x}^k$, as we observed in the proof of Theorem 4.4.5. Stated somewhat more invariantly, every integral curve of G on $\pi^{-1}(U)$ projects to a geodesic under $\pi : TM \to M$ (which in these coordinates is just $\pi(x, v) = x$); conversely, every geodesic $\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$ in U lifts to an integral curve of G in $\pi^{-1}(U)$ by setting $v^i(t) = \dot{x}^i(t)$.

The importance of G stems from the fact that it actually defines a global vector field on the total space of TM, called the **geodesic vector field**. Then the unique C^∞ maximal flow θ obtained from fundamental theorem on flows 1.2.8 is called **geodesic flow**. We could verify that G defines a global vector field by computing the transformation law for the components of G under a change of coordinates and showing that they take the same form in every coordinate chart; but fortunately there is a way to avoid that messy computation. The key observation, to be proved below, is that G acts on a function $f \in C^\infty(TM)$ by

$$Gf(p, v) = G_{(p,v)}f = \left.\frac{d}{dt}\right|_{t=0} f\left(\gamma_v(t), \gamma_v'(t)\right). \tag{3.29}$$

(Here and whenever convenient, we use the notations (p, v) and v interchangeably for an element $v \in T_pM$, depending on whether we wish to emphasize the point at which v is tangent.) Since this formula is independent of coordinates, it shows that the various definitions of G given by (4.5) in different coordinate systems agree.

To prove that G satisfies (3.29), we write the components of the geodesic $\gamma_v(t)$ as $x^i(t)$ and those of its velocity as $v^i(t) = \dot{x}^i(t)$. Using the chain rule and the geodesic equation in the form (4.4), we can write the right-hand side of (3.29) as

$$\left.\left(\frac{\partial f}{\partial x^k}(\overbrace{x(t)}^{=\gamma_v(t)}, \overbrace{v(t)}^{=\gamma_v'(t)})\dot{x}^k(t) + \frac{\partial f}{\partial v^k}(x(t), v(t))\dot{v}^k(t)\right)\right|_{t=0}$$
$$\xlongequal{\gamma_v = x(t) \text{ geodesic} \Leftrightarrow (4.4)} \frac{\partial f}{\partial x^k}(p,v)v^k - \frac{\partial f}{\partial v^k}(p,v)v^i v^j \Gamma_{ij}^k(p)$$
$$\xlongequal{(4.5)} Gf(p,v).$$

The fundamental theorem on flows shows that there exist an open set $\mathcal{D} \subseteq \mathbb{R} \times TM$ containing $\{0\} \times TM$ and a smooth map $\theta : \mathcal{D} \to TM$, such that each curve $\theta^{(p,v)}(t) = \theta(t, (p, v))$ is the unique maximal integral curve of G starting at (p, v), defined on an open interval containing 0.

Now suppose $(p, v) \in \mathcal{E}$. This means that the geodesic γ_v is defined at least on the interval $[0, 1]$, and therefore so is the integral curve of G starting at $(p, v) \in TM$. Since $(1, (p, v)) \in \mathcal{D}$, there is a neighborhood of $(1, (p, v))$ in $\mathbb{R} \times TM$ on which the flow of G is defined (Fig. 3.1). In particular, this means that there is a neighborhood of (p, v) on which the flow exists for $t \in [0, 1]$, and therefore on which the exponential map is defined. This shows that $\mathcal{E}$ is open.

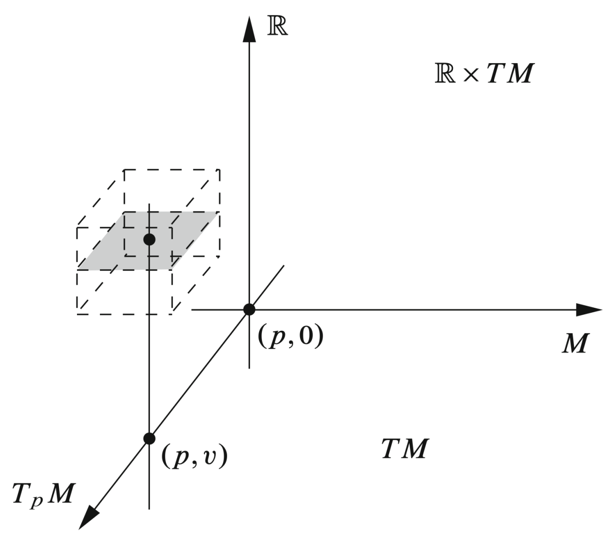

Figure 3.1: $\mathcal{E}$ is open.

Since geodesics are projections of integral curves of G, it follows that the exponential map can be expressed as

$$\exp_p(v) = \gamma_v(1) = \pi \circ \theta(1, (p, v))$$

wherever it is defined, and therefore $\exp_p(v)$ is a smooth function of (p, v). To compute $d\left(\exp_p\right)_0(v)$ for an arbitrary vector $v \in T_pM$, we just need to choose a curve τ in T_pM starting at 0 whose initial velocity is v, and compute the initial velocity of $\exp_p \circ \tau$. A convenient curve is $\tau(t) = tv$, which yields

$$d\left(\exp_p\right)_0(v) = \left.\frac{d}{dt}\right|_{t=0} \left(\exp_p \circ \tau\right)(t) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t) = v.$$

Thus $d\left(\exp_p\right)_0$ is the identity map. ∎

Corollary 3.2.9 on the naturality of geodesics translates into the following important property of the exponential map.

Proposition 3.2.15 (Naturality of the Exponential Map). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi : M \to \widetilde{M}$ is a local isometry. Then for every $p \in M$, the following diagram commutes:*

$$\begin{tikzcd} \mathcal{E}_p \arrow[r, "d\varphi_p"] \arrow[d, "\exp_p"'] & \widetilde{\mathcal{E}}_{\varphi(p)} \arrow[d, "\exp_{\varphi(p)}"] \\ M \arrow[r, "\varphi"'] & \widetilde{M} \end{tikzcd}$$

where $\mathcal{E}_p \subseteq T_pM$ and $\widetilde{\mathcal{E}}_{\varphi(p)} \subseteq T_{\varphi(p)}\widetilde{M}$ are the domains of the restricted exponential maps $\exp_p$ (with respect to g) and $\exp_{\varphi(p)}$ (with respect to $\widetilde{g}$), respectively.

Exercise 3.2.16. *Prove above proposition*

An important consequence of the naturality of the exponential map is the following proposition, which says that local isometries of connected manifolds are completely determined by their values and differentials at a single point.

Proposition 3.2.17. *Proposition 5.22. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian or pseudo-Riemannian manifolds, with M connected. Suppose $\varphi, \psi : M \to \widetilde{M}$ are local isometries such that for some point $p \in M$, we have $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi \equiv \psi$.*

Proof. [12] Problem 5-10. ■

A Riemannian or pseudo-Riemannian manifold (M, g) is said to be **geodesically complete** if every maximal geodesic is defined for all $t \in \mathbb{R}$, or equivalently if the domain of the exponential map is all of TM. It is easy to construct examples of manifolds that are not geodesically complete; for example, n every proper open subset of $\mathbb{R}^n$ with its Euclidean metric or with a pseudo-Euclidean metric, there are geodesics that reach the boundary in finite time. Similarly, on $\mathbb{R}^n$ with the metric $\left(\sigma^{-1}\right)^* g$ obtained from the sphere by stereographic projection, there are geodesics that escape to infinity in finite time.

3.2.4 Normal Neighborhood and Normal Coordinates

We continue to let (M, g) be a Riemannian or pseudo-Riemannian manifold of dimension n (without boundary). Recall that for every $p \in M$, the restricted exponential map $\exp_p$ maps the open subset $\mathcal{E}_p \subseteq T_pM$ smoothly into M. Because $d\left(\exp_p\right)_0$ is invertible, the inverse function theorem guarantees that there exist a neighborhood V of the origin in T_pM and a neighborhood U of p in M such that $\exp_p : V \to U$ is a diffeomorphism. A neighborhood U of $p \in M$ that is the diffeomorphic image under $\exp_p$ of a star-shaped neighborhood of $0 \in T_pM$ is called a **normal neighborhood of** p.

Every orthonormal basis (b_i) for T_pM determines a basis isomorphism $B : \mathbb{R}^n \to T_pM$ by $B\left(x^1, \ldots, x^n\right) = x^i b_i$. If $U = \exp_p(V)$ is a normal neighborhood of p, we can combine this isomorphism with the exponential map to get a smooth coordinate map $\varphi = B^{-1} \circ \left(\left.\exp_p\right|_V\right)^{-1} : U \to \mathbb{R}^n$:

$$\begin{tikzcd} T_pM \arrow[r, "B^{-1}"] & \mathbb{R}^n \\ U \arrow[u, "(\exp_p|_V)^{-1}"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Such coordinates are called **(Riemannian or pseudo-Riemannian) normal coordinates centered at** p.

Proposition 3.2.18 (Uniqueness of Normal Coordinates). *Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, p a point of M, and U a normal neighborhood of p. For every normal coordinate chart on U centered at p, the coordinate basis is orthonormal at p; and for every orthonormal basis (b_i) for T_pM, there is a unique normal coordinate chart $\left(x^i\right)$ on U such that $\partial_i|_p = b_i$ for $i = 1, \ldots, n$. In the Riemannian case, any two normal coordinate charts $\left(x^i\right)$ and $\left(\widetilde{x}^j\right)$ are related by*

$$\widetilde{x}^j = A_i^j x^i \tag{3.30}$$

for some (constant) matrix $\left(A_i^j\right) \in \mathrm{O}(n)$.

Proof. Let φ be a normal coordinate chart on U centered at p, with coordinate functions $\left(x^i\right)$. By definition, this means that $\varphi = B^{-1} \circ \exp_p^{-1}$, where $B : \mathbb{R}^n \to T_pM$ is the basis isomorphism determined by some orthonormal basis (b_i) for T_pM. Note that $d\varphi_p^{-1} = d\left(\exp_p\right)_0 \circ dB_0 = B$ because $d\left(\exp_p\right)_0$ is the identity and B is linear. Thus $\partial_i|_p = d\varphi_p^{-1}\left(\partial_i|_0\right) = B\left(\partial_i|_0\right) = b_i$, which shows that the coordinate basis is orthonormal at p. Conversely, every orthonormal basis (b_i) for T_pM yields a basis isomorphism B and thus a normal coordinate chart $\varphi = B^{-1} \circ \exp_p^{-1}$, which satisfies $\partial_i|_p = b_i$ by the computation above. If $\widetilde{\varphi} = \widetilde{B}^{-1} \circ \exp_p^{-1}$ is another such chart, then

$$\widetilde{\varphi} \circ \varphi^{-1} = \widetilde{B}^{-1} \circ \exp_p^{-1} \circ \exp_p \circ B = \widetilde{B}^{-1} \circ B,$$

which is a linear isometry of $\mathbb{R}^n$ and therefore has the form (3.30) in terms of standard coordinates on $\mathbb{R}^n$. Since $\left(\widetilde{x}^j\right)$ and $\left(x^i\right)$ are the same coordinates if and only if $\left(A_i^j\right)$ is the identity matrix, this shows that the normal coordinate chart associated with a given orthonormal basis is unique. ■

Proposition 3.2.19 (Properties of Normal Coordinates). *Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, and let $\left(U, \left(x^i\right)\right)$ be any normal coordinate chart centered at $p \in M$.*

(a) The coordinates of p are $(0, \ldots, 0)$.

(b) The components of the metric at p are $g_{ij} = \delta_{ij}$ if g is Riemannian, and $g_{ij} = \pm\delta_{ij}$ otherwise.

(c) For every $v = v^i \partial_i\big|_p \in T_pM$, the geodesic γ_v starting at p with initial velocity v is represented in normal coordinates by the line

$$\gamma_v(t) = \left(tv^1, \ldots, tv^n\right), \tag{3.31}$$

as long as t is in some interval I containing 0 such that $\gamma_v(I) \subseteq U$.

(d) The Christoffel symbols in these coordinates vanish at p.

(e) All of the first partial derivatives of g_{ij} in these coordinates vanish at p.

Proof. Part (a) follows directly from the definition of normal coordinates, and parts (b) and (c) follow from Propositions 3.2.18 and 3.2.14(b), respectively.

To prove (d), let $v = v^i \partial_i\big|_p \in T_pM$ be arbitrary. The geodesic equation (4.3) for $\gamma_v(t) = \left(tv^1, \ldots, tv^n\right)$ simplifies to

$$\Gamma_{ij}^k(tv) v^i v^j = 0.$$

Evaluating this expression at $t = 0$ shows that $\Gamma_{ij}^k(0) v^i v^j = 0$ for every index k and every vector v. In particular, with $v = \partial_a$ for some fixed a, this shows that $\Gamma_{aa}^k = 0$ for each a and k (no summation). Substituting $v = \partial_a + \partial_b$ and $v = \partial_b - \partial_a$ for any fixed pair of indices a and b and subtracting, we conclude also that $\Gamma_{ab}^k = 0$ at p for all a, b, k. Finally, (e) follows from (d) together with (3.16) in the case $E_k = \partial_k$. ■

Because they are given by the simple formula (3.31), the geodesics starting at p and lying in a normal neighborhood of p are called **radial geodesics**. (But be warned that geodesics that do not pass through p do not in general have a simple form in normal coordinates.)

3.3 Curvatures

Recall that a Riemannian manifold is said to be **flat** if it is locally isometric to a Euclidean space, that is, if every point has a neighborhood that is isometric to an open set in $\mathbb{R}^n$ with its Euclidean metric. Similarly, a pseudo-Riemannian manifold is flat if it is locally isometric to a pseudo-Euclidean space. For Euclidean connection on $\mathbb{R}^n$, we see that $\overline{\nabla}_X \overline{\nabla}_Y Z = XY\left(Z^k\right) \partial_k$, $\overline{\nabla}_Y \overline{\nabla}_X Z = YX\left(Z^k\right) \partial_k$, and $\left(XY\left(Z^k\right) - YX\left(Z^k\right)\right) \partial_k = \overline{\nabla}_{[X,Y]} Z$ due to Example 4.8. Thus, the following relation holds for all vector fields X, Y, Z defined on an open subset of $\mathbb{R}^n$:

$$\overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z = \overline{\nabla}_{[X,Y]} Z.$$

We say that a connection ∇ on a smooth manifold M satisfies the **flatness criterion** if whenever X, Y, Z are smooth vector fields defined on an open subset of M, the following identity holds:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z. \tag{3.32}$$

Example 3.3.1. The metric on the n-torus induced by the embedding in $\mathbb{R}^{2n}$ given in Example 2.2.11 is flat, because each point has a coordinate neighborhood in which the metric is Euclidean. ♣

Proposition 3.3.2. *If (M, g) is a flat Riemannian or pseudo-Riemannian manifold, then its Levi-Civita connection satisfies the flatness criterion.*

Proof. We just showed that the Euclidean connection on $\mathbb{R}^n$ satisfies (3.32). By naturality (see Proposition 3.2.8), the Levi-Civita connection on every manifold that is locally isometric to a Euclidean or pseudo-Euclidean space must also satisfy the same identity. ■

3.3.1 Curvature Tensor

Motivated by the computation in the preceding section, we make the following definition. Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and define a map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3.33}$$

Proposition 3.3.3. *The map R defined above is multilinear over $C^\infty(M)$, and thus defines a (1,3)-tensor field on M.*

Proof. The map R is obviously multilinear over $\mathbb{R}$. For $f \in C^\infty(M)$,

$$\begin{aligned}
R(X, fY)Z &= \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z \\
&\xlongequal{\text{prop.1.2.4}} \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{f[X,Y]+(Xf)Y} Z \\
&\xlongequal{\text{defn.3.1.1}} (Xf)\nabla_Y Z + f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z - (Xf)\nabla_Y Z \\
&= fR(X, Y)Z.
\end{aligned}$$

The same proof shows that R is linear over $C^\infty(M)$ in X, because $R(X, Y)Z = -R(Y, X)Z$ from the definition. The remaining case to be checked is linearity over $C^\infty(M)$ in Z: using definition of connection and Lie Bracket, we see

$$\begin{aligned}
R(X, Y)fZ &= \nabla_X \nabla_Y fZ - \nabla_Y \nabla_X fZ - \nabla_{[X,Y]} fZ \\
&= \nabla_X (f\nabla_Y Z + YfZ) - \nabla_Y (f\nabla_X Z + XfZ) - f\nabla_{[X,Y]} Z - [X, Y]fZ \\
&= f\nabla_X \nabla_Y Z + Xf\nabla_Y Z + Yf\nabla_X Z + X(Yf)Z \\
&\quad - f\nabla_Y \nabla_X Z - Yf\nabla_X Z - Xf\nabla_Y Z - Y(Xf)Z \\
&\quad - f\nabla_{[X,Y]} Z - [X, Y]fZ \\
&= fR(X, Y)Z
\end{aligned}$$

By the tensor characterization lemma 1.1.19, the fact that R is multilinear over $C^{\infty}(M)$ implies that it is a $(1,3)$-tensor field (R takes in three vectors and output one vector, so $R \in \mathcal{L}(V,V,V;V) \cong V \otimes V^* \otimes V^* \otimes V^* = T^{(1,3)}(V)$.) ■

Thanks to this proposition, for each pair of vector fields $X, Y \in \mathfrak{X}(M)$, the map $R(X,Y): \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $Z \mapsto R(X,Y)Z$ is a smooth bundle endomorphism of TM (see [11] 10.29), called the **curvature endomorphism determined by X and Y**. The tensor field R itself is called the **(Riemann) curvature endomorphism** or the **$(1,3)$-curvature tensor** or the **Riemann curvature tensor of the second kind** (Some authors call it simply the curvature tensor, but we reserve that name instead for another closely related tensor field, defined below.)

As a $(1,3)$-tensor field, the curvature endomorphism can be written in terms of any local frame with one upper and three lower indices. We adopt the convention that the last index is the contravariant (upper) one. (This is contrary to our default assumption that covector arguments come first.) Thus, for example, the curvature endomorphism can be written in terms of local coordinates $\left(x^i\right)$ as

$$R = R_{ijk}{}^{l} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

where the coefficients $R_{ijk}{}^{l}$ are defined by

$$R\left(\partial_i, \partial_j\right) \partial_k = R_{ijk}{}^{l} \partial_l. \tag{3.34}$$

We explain this a bit: looking back at proposition 1.1.5, we write above equation really to mean that for the $R \in T^{(1,3)}TM$,

$$\Psi(R)\left(\partial_i, \partial_j, \partial_k\right) = R_{ijk}{}^{l} \partial_l,$$

or

$$\tau(R(\,\cdot\,, \partial_i, \partial_j, \partial_k)) = R_{ijk}{}^{l} \partial_l.$$

Since τ is an isomorphism whose inverse sends a vector to its evaluation map $\bar{v}$, showing this equality is exactly showing that

$$\overline{R_{ijk}{}^{l} \partial_l} = R(\,\cdot\,, \partial_i, \partial_j, \partial_k),$$

but $\overline{R_{ijk}{}^{l} \partial_l}(dx^m) = \sum_l R_{ijk}{}^{l} dx^m(\partial_l) = R_{ijk}{}^{m}$ and $R(\,\cdot\,, \partial_i, \partial_j, \partial_k)(dx^m) = R(dx^m, \partial_i, \partial_j, \partial_k) = R_{ijk}{}^{m}$.

The next proposition shows how to compute the components of R in coordinates.

Proposition 3.3.4. *Let (M,g) be a Riemannian or pseudo-Riemannian manifold. In terms of any smooth local coordinates, the components of the $(1,3)$-curvature tensor are given by*

$$R_{ijk}{}^{l} = \partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{m} \Gamma_{im}^{l} - \Gamma_{ik}^{m} \Gamma_{jm}^{l}. \tag{3.35}$$

Proof.

$$\begin{aligned} R(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k - \nabla_{[\partial_i, \partial_j]} \partial_k \\ &\overset{[11](8.10)}{=\!=\!=} \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k \\ &\overset{(3.2)}{=\!=} \nabla_{\partial_i}(\Gamma_{jk}^{m} \partial_m) - \nabla_{\partial_j}(\Gamma_{ik}^{m} \partial_m) \\ &\overset{\Gamma\text{'s are functions}}{=\!=\!=\!=\!=} \Gamma_{jk}^{m} \nabla_{\partial_i} \partial_m + \partial_i \Gamma_{jk}^{m} \partial_m - \Gamma_{ik}^{m} \nabla_{\partial_j} \partial_m - \partial_j \Gamma_{ik}^{m} \partial_m \\ &= \Gamma_{jk}^{m} \Gamma_{im}^{l} \partial_l + \partial_i \Gamma_{jk}^{l} \partial_l - \Gamma_{ik}^{m} \Gamma_{jm}^{l} \partial_l - \partial_j \Gamma_{ik}^{l} \partial_l \\ &= [\partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{m} \Gamma_{im}^{l} - \Gamma_{ik}^{m} \Gamma_{jm}^{l} \partial_l \end{aligned}$$

The characterization (3.34) then concludes. ■

Importantly for our purposes, the curvature endomorphism also measures the failure of second covariant derivatives along families of curves to commute. Given a smooth one-parameter family of curves $\Gamma : J \times I \to M$, recall from previous chapter that the velocity fields $T(s,t) = \partial_t \Gamma(s,t) = (\Gamma_s)'(t)$ and $S(s,t) = \partial_s \Gamma(s,t) = \Gamma^{(t)\prime}(s)$ are smooth vector fields along Γ.

Proposition 3.3.5. *Suppose (M,g) is a smooth Riemannian or pseudo-Riemannian manifold and $\Gamma : J \times I \to M$ is a smooth one-parameter family of curves in M. Then for every smooth vector field V along Γ,*

$$D_s D_t V - D_t D_s V = R(\partial_s \Gamma, \partial_t \Gamma) V. \tag{3.36}$$

Proof. This is a local question, so for each $(s,t) \in J \times I$, we can choose smooth coordinates (x^i) defined on a neighborhood of $\Gamma(s,t)$ and write

$$\Gamma(s,t) = \left(\gamma^1(s,t), \ldots, \gamma^n(s,t)\right), \quad V(s,t) = V^j(s,t) \partial_j\big|_{\Gamma(s,t)}.$$

The Leibniz rule for covariant derivatives along curves yields

$$D_t V = \frac{\partial V^i}{\partial t} \partial_i + V^i D_t \partial_i.$$

Therefore, applying Leibniz rule again, we get

$$D_s D_t V = \frac{\partial^2 V^i}{\partial s \partial t} \partial_i + \frac{\partial V^i}{\partial t} D_s \partial_i + \frac{\partial V^i}{\partial s} D_t \partial_i + V^i D_s D_t \partial_i.$$

Interchanging s and t and subtracting, we see that all the terms except the last cancel:

$$D_s D_t V - D_t D_s V = V^i \left(D_s D_t \partial_i - D_t D_s \partial_i\right). \tag{3.37}$$

Now we need to compute the commutator in parentheses. For brevity, let us write

$$S = \partial_s \Gamma = \frac{\partial \gamma^k}{\partial s} \partial_k; \quad T = \partial_t \Gamma = \frac{\partial \gamma^j}{\partial t} \partial_j.$$

Because ∂_i is extendible,

$$D_t \partial_i = \nabla_T \partial_i = \frac{\partial \gamma^j}{\partial t} \nabla_{\partial_j} \partial_i,$$

and therefore, because $\nabla_{\partial_j} \partial_i$ is also extendible,

$$\begin{aligned} D_s D_t \partial_i &= D_s \left(\frac{\partial \gamma^j}{\partial t} \nabla_{\partial_j} \partial_i \right) \\ &= \frac{\partial^2 \gamma^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial \gamma^j}{\partial t} \nabla_S \left(\nabla_{\partial_j} \partial_i\right) \\ &= \frac{\partial^2 \gamma^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial \gamma^j}{\partial t} \frac{\partial \gamma^k}{\partial s} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i. \end{aligned}$$

Interchanging $s \leftrightarrow t$ and $j \leftrightarrow k$ and subtracting, we find that the first terms cancel, and we get

$$\begin{aligned} D_s D_t \partial_i - D_t D_s \partial_i &= \frac{\partial \gamma^j}{\partial t} \frac{\partial \gamma^k}{\partial s} \left(\nabla_{\partial_k} \nabla_{\partial_j} \partial_i - \nabla_{\partial_j} \nabla_{\partial_k} \partial_i\right) \\ &= \frac{\partial \gamma^j}{\partial t} \frac{\partial \gamma^k}{\partial s} R\left(\partial_k, \partial_j\right) \partial_i = R(S,T) \partial_i \end{aligned}$$

Finally, inserting this into (3.37) yields the result. ■

For many purposes, the information contained in the curvature endomorphism is much more conveniently encoded in the form of a covariant 4-tensor. We define the **(Riemann) curvature tensor (of the first kind)** to be the $(0,4)$-tensor field $Rm = R^\flat$ (also denoted by $Riem$ by some authors) obtained from the $(1,3)$-curvature tensor R by lowering its last index. Its action on vector fields is given by

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle_g \tag{3.38}$$

(Thr LHS is $R^\flat(X,Y,Z,W) = R(X,Y,Z,W^\flat)$; what this really means is that for $R \in L(V,V,V;V)$ given by (3.33), $\Phi(R)(X,Y,Z,W^\flat)$ =RHS. This is true as $\Phi(R)(X,Y,Z,W^\flat) = W^\flat(R(X,Y,Z)) = \widehat{g}(W)(R(X,Y)Z) = g(W,R(X,Y)Z) = g(R(X,Y)Z,W)$.) In terms of any smooth local coordinates it is written

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where $R_{ijkl} = g_{lm} R_{ijk}{}^m$ (see Example 2.3.3). Thus (3.35) yields

$$R_{ijkl} = g_{lm}\left(\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^p \Gamma_{ip}^m - \Gamma_{ik}^p \Gamma_{jp}^m\right). \tag{3.39}$$

It is appropriate to note here that there is much variation in the literature with respect to index positions in the definitions of the curvature endomorphism and curvature tensor. While almost all authors define the $(1,3)$-curvature tensor as we have, there are a few whose definition is the negative of ours. There is much less agreement on the definition of the $(0,4)$-curvature tensor: whichever definition is chosen for the curvature endomorphism, you will see the curvature tensor defined as in (3.38) but with various permutations of (X,Y,Z,W) on the right-hand side. After applying the symmetries of the curvature tensor that we will prove later in this chapter, however, all of the definitions agree up to sign. There are various arguments to support one choice or another; we have made a choice that makes equation (3.38) easy to remember. You just have to be careful when you begin reading any book or article to determine the author's sign convention.

The next proposition gives one reason for our interest in the curvature tensor.

Proposition 3.3.6. *The curvature tensor is a local isometry invariant: if (M,g) and $(\widetilde{M},\widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi : M \to \widetilde{M}$ is a local isometry, then $\varphi^*\widetilde{Rm} = Rm$.*

Exercise 3.3.7. *Prove above proposition.*

3.3.2 Flat Manifolds

To give a qualitative geometric interpretation to the curvature tensor, we will show that it is precisely the obstruction to being locally isometric to Euclidean (or pseudo-Euclidean) space. (In next chapter, after we have developed more machinery, we will be able to give a far more detailed quantitative interpretation.) The crux of the proof is the following lemma.

Lemma 3.3.8. *Suppose M is a smooth manifold, and ∇ is any connection on M satisfying the flatness criterion. Given $p \in M$ and any vector $v \in T_pM$, there exists a parallel vector field V on a neighborhood of p such that $V_p = v$.*

Proof. Let $p \in M$ and $v \in T_pM$ be arbitrary, and let $\left(x^1,\ldots,x^n\right)$ be any smooth coordinates for M centered at p. By shrinking the coordinate neighborhood if necessary, we may assume that the image of the coordinate map is an open cube $C_\varepsilon = \left\{x : \left|x^i\right| < \varepsilon, i = 1,\ldots,n\right\}$. We use the coordinate map to identify the coordinate domain with C_ε.

Begin by parallel transporting v along the x^1-axis; then from each point on the x^1-axis, parallel transport along the coordinate line parallel to the x^2-axis; then successively parallel transport along coordinate lines parallel to the x^3 through x^n axes (Fig. 7.2). The result is a vector field V defined in C_ε. The fact that V is smooth follows from by an inductive application of Theorem 1.2.8 to vector fields of the form $W_k|_{(x,v)} = \partial/\partial x^k - v^i \Gamma_{ki}^j(x) \partial/\partial v^j$ on $C_\varepsilon \times \mathbb{R}^n$; the details are left as an exercise.

Since $\nabla_X V$ is linear over $C^\infty(M)$ in X, to show that V is parallel, it suffices to show that $\nabla_{\partial_i} V = 0$ for each $i = 1, \ldots, n$. By construction, $\nabla_{\partial_1} V = 0$ on the x^1-axis, $\nabla_{\partial_2} V = 0$ on the $\left(x^1, x^2\right)$-plane, and in general $\nabla_{\partial_k} V = 0$ on the slice $M_k \subseteq C_\varepsilon$ defined by $x^{k+1} = \cdots = x^n = 0$. We will prove the following fact by induction on k :

$$\nabla_{\partial_1} V = \cdots = \nabla_{\partial_k} V = 0 \quad \text{on } M_k.$$

For $k = 1$, this is true by construction, and for $k = n$, it means that V is parallel on the whole cube C_ε. So assume that (7.9) holds for some k. By construction, $\nabla_{\partial_{k+1}} V = 0$ on all of M_{k+1}, and for $i \leq k$, the inductive hypothesis shows that $\nabla_{\partial_i} V = 0$ on the hyperplane $M_k \subseteq M_{k+1}$. Since $[\partial_{k+1}, \partial_i] = 0$, the flatness criterion gives

$$\nabla_{\partial_{k+1}} \left(\nabla_{\partial_i} V\right) = \nabla_{\partial_i} \left(\nabla_{\partial_{k+1}} V\right) = 0 \quad \text{on } M_{k+1}.$$

This shows that $\nabla_{\partial_i} V$ is parallel along the x^{k+1}-curves starting on M_k. Since $\nabla_{\partial_i} V$ vanishes on M_k and the zero vector field is the unique parallel transport of zero, we conclude that $\nabla_{\partial_i} V$ is zero on each x^{k+1}-curve. Since every point of M_{k+1} is on one of these curves, it follows that $\nabla_{\partial_i} V = 0$ on all of M_{k+1}. This completes the inductive step to show that V is parallel. ■

Exercise 3.3.9. *Prove that the vector field V constructed in the preceding proof is smooth.*

Theorem 3.3.10. *A Riemannian or pseudo-Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Proof. One direction is immediate: Proposition 3.3.2 showed that the Levi-Civita connection of a flat metric satisfies the flatness criterion, so its curvature endomorphism is identically zero, which implies that the curvature tensor is also zero.

Now suppose (M, g) has vanishing curvature tensor. This means that the curvature endomorphism vanishes as well, so the Levi-Civita connection satisfies the flatness criterion. We begin by showing that g shares one important property with Euclidean and pseudo-Euclidean metrics: it admits a parallel orthonormal frame in a neighborhood of each point.

Let $p \in M$, and choose an orthonormal basis $(b_1, \ldots, b_n)$ for $T_p M$. In the pseudo-Riemannian case, we may assume that the basis is in standard order (with positive entries before negative ones in the matrix $g_{ij} = g_p\left(b_i, b_j\right)$). Lemma 3.3.8 shows that there exist parallel vector fields $E_1, \ldots, E_n$ on a neighborhood U of p such that $\left.E_i\right|_p = b_i$ for each $i = 1, \ldots, n$. Because parallel transport preserves inner products, the vector fields (E_j) are orthonormal (and hence linearly independent) in all of U. Because the Levi-Civita connection is symmetric, we have

$$[E_i, E_j] \xlongequal{\text{symmetric conn.}} \nabla_{E_i} E_j - \nabla_{E_j} E_i = \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) E_k \xlongequal{\text{Pb.3.6.7}} 0.$$

Thus the vector fields $(E_1, \ldots, E_n)$ form a commuting orthonormal frame on U. The canonical form theorem for commuting vector fields ([11] proposition **??**) shows that there are coordinates $\left(y^1, \ldots, y^n\right)$ on a (possibly smaller) neighborhood of p such that $E_i = \partial / \partial y^i$ for $i = 1, \ldots, n$. In any such coordinates, $g_{ij} = g\left(\partial_i, \partial_j\right) = g\left(E_i, E_j\right) = \pm \delta_{ij}$, so the map $y = \left(y^1, \ldots, y^n\right)$ is an isometry from a neighborhood of p to an open subset of the appropriate Euclidean or pseudo-Euclidean space. ■

Using similar ideas, we can give a more precise interpretation of the meaning of the curvature tensor: it is a measure of the extent to which parallel transport around a small rectangle fails to be the identity map.

Theorem 3.3.11. *Let (M, g) be a Riemannian or pseudo-Riemannian manifold; let I be an open interval containing 0; let $\Gamma : I \times I \to M$ be a smooth one-parameter family of curves; and let $p = \Gamma(0,0), x = \partial_s \Gamma(0,0)$, and $y = \partial_t \Gamma(0,0)$ (see Fig.3.5). For any $s_1, s_2, t_1, t_2 \in I$, let $P_{s_1, t_1}^{s_1, t_2} : T_{\Gamma(s_1, t_1)} M \to T_{\Gamma(s_1, t_2)} M$ denote parallel transport along the curve $\left.\Gamma_{s_1}\right|_{[t_1, t_2]} : t \mapsto \Gamma\left(s_1, t\right)$ from time t_1 to time t_2, and let $P_{s_1, t_1}^{s_2, t_1} : T_{\Gamma(s_1, t_1)} M \to$*

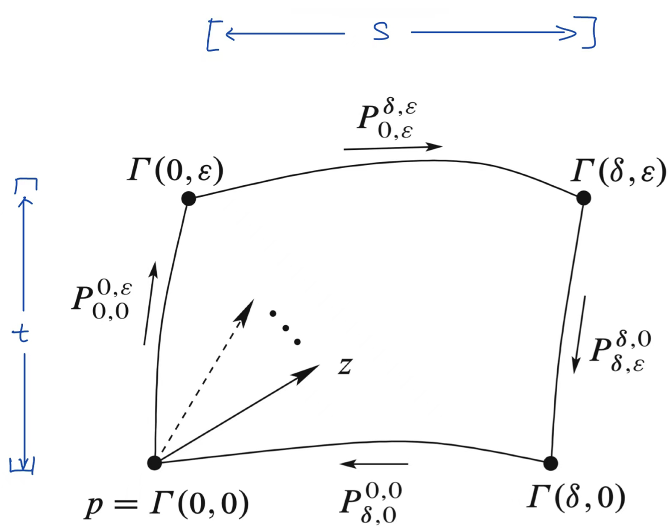

Figure 3.2: The curvature endomorphism and parallel transport around a closed loop.

$T_{\Gamma(s_2,t_1)}M$ denote parallel transport along the curve $\Gamma^{(t_1)}\big|_{[s_1,s_2]} : s \mapsto \Gamma(s,t_1)$ from time s_1 to time s_2. (See Fig.3.2) Then for every $z \in T_pM$,

$$R(x,y)z = \lim_{\delta,\varepsilon\to 0} \frac{P_{\delta,0}^{0,0} \circ P_{\delta,\varepsilon}^{\delta,0} \circ P_{0,\varepsilon}^{\delta,\varepsilon} \circ P_{0,0}^{0,\varepsilon}(z) - z}{\delta\varepsilon}. \tag{3.40}$$

Proof. Define a vector field Z along Γ by first parallel transporting z along the curve $t \mapsto \Gamma(0,t)$, and then for each t, parallel transporting $Z(0,t)$ along the curve $s \mapsto \Gamma(s,t)$. The resulting vector field along Γ is smooth by another application of Theorem 1.2.8 as in the proof of lemma 3.3.8; and by construction, it satisfies $D_tZ(0,t) = 0$ for all $t \in I$, and $D_sZ(s,t) = 0$ for all $(s,t) \in I \times I$. Proposition 3.3.5 shows that

$$R(x,y)z = D_sD_tZ(0,0) - D_tD_sZ(0,0) = D_sD_tZ(0,0).$$

Thus we need only show that $D_sD_tZ(0,0)$ is equal to the limit on the right-hand side of (7.10). From Theorem 4.4.10, we have

$$(D_tZ)(s,0) = \lim_{\varepsilon\to 0} \frac{P_{s,\varepsilon}^{s,0}(Z(s,\varepsilon)) - Z(s,0)}{\varepsilon}, \tag{3.41}$$

$$(D_s(D_tZ))(0,0) = \lim_{\delta\to 0} \frac{P_{\delta,0}^{0,0}\left(D_tZ(\delta,0)\right) - D_tZ(0,0)}{\delta}. \tag{3.42}$$

Evaluating (3.41) first at $s = \delta$ and then at $s = 0$, and inserting the resulting expressions into (3.42), we obtain

$$(D_s(D_tZ))(0,0) = \lim_{\delta,\varepsilon\to 0} \frac{P_{\delta,0}^{0,0} \circ P_{\delta,\varepsilon}^{\delta,0}(Z(\delta,\varepsilon)) - P_{\delta,0}^{0,0}(Z(\delta,0)) - P_{0,\varepsilon}^{0,0}(Z(0,\varepsilon)) + Z(0,0)}{\delta\varepsilon}. \tag{3.43}$$

Here we have used the fact that parallel transport is linear, so the ε-limit can be pulled past $P_{\delta,0}^{0,0}$.

Now, the fact that Z is parallel along $t \mapsto \Gamma(0,t)$ and along all of the curves $s \mapsto \Gamma(s,t)$ implies

$$P_{\delta,0}^{0,0}(Z(\delta,0)) = P_{0,\varepsilon}^{0,0}(Z(0,\varepsilon)) = Z(0,0) = z$$
$$Z(\delta,\varepsilon) = P_{0,\varepsilon}^{\delta,\varepsilon}(Z(0,\varepsilon)) = P_{0,\varepsilon}^{\delta,\varepsilon} \circ P_{0,0}^{0,\varepsilon}(z).$$

Inserting these relations into (3.43) yields (3.40). ■

3.3.3 Symmetries of the Curvature Tensor

The curvature tensor on a Riemannian or pseudo-Riemannian manifold has a number of symmetries besides the obvious skew-symmetry in its first two arguments.

Proposition 3.3.12 (Symmetries of the Curvature Tensor). *Let (M, g) be a Riemannian or pseudo-Riemannian manifold. The $(0, 4)$-curvature tensor of g has the following symmetries for all vector fields W, X, Y, Z:*

(a) $Rm(W, X, Y, Z) = -Rm(X, W, Y, Z)$.

(b) $Rm(W, X, Y, Z) = -Rm(W, X, Z, Y)$.

(c) $Rm(W, X, Y, Z) = Rm(Y, Z, W, X)$.

(d) $Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0$.

Remark 3.3.13. Before we begin the proof, a few remarks are in order. First, as the proof will show, (a) is a trivial consequence of the definition of the curvature endomorphism; (b) follows from the compatibility of the Levi-Civita connection with the metric; (d) follows from the symmetry of the connection; and (c) follows from (a), (b), and (d). The identity in (d) is called the **algebraic Bianchi identity** (or more traditionally but less informatively, the **first Bianchi identity**). It is easy to show using (a)-(d) that a three-term sum obtained by cyclically permuting any three arguments of Rm is also zero. Finally, it is useful to record the form of these symmetries in terms of components with respect to any basis:

(a') $R_{ijkl} = -R_{jikl}$.

(b') $R_{ijkl} = -R_{ijlk}$.

(c') $R_{ijkl} = R_{klij}$

(d') $R_{ijkl} + R_{jkil} + R_{kijl} = 0$.

♠

Proof. Identity (a) is immediate from the definition of the curvature tensor, because $R(W, X)Y = -R(X, W)Y$. To prove (b), it suffices to show that $Rm(W, X, Y, Y) = 0$ for all Y, denoted as identity $(\star)$ for then (b) follows from the expansion of $Rm(W, X, Y + Z, Y + Z) = 0$:

$$\begin{aligned}
0 &\overset{(\star)}{=\!=} Rm(W, X, Y + Z, Y + Z) \\
&= \langle R(W, X)(Y + Z), Y + Z\rangle_g \\
&= \langle R(W, X)Y + R(W, X)Z, Y + Z\rangle_g \\
&= \langle R(W, X)\underline{Y, Y}\rangle_g + \langle R(W, X)Y, Z\rangle_g + \langle R(W, X)Z, Y\rangle_g + \langle R(W, X)\underline{Z, Z}\rangle_g \\
&\overset{(\star)}{=\!=} \langle R(W, X)Y, Z\rangle_g + \langle R(W, X)Z, Y\rangle_g \\
\implies &\langle R(W, X)Y, Z\rangle_g = -\langle R(W, X)Z, Y\rangle_g, \text{ or } Rm(W, X, Y, Z) = -Rm(W, X, Z, Y)
\end{aligned}$$

we now show $(\star)$: the compatibility with the metric gives

$$\begin{aligned}
&\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
\implies &\underbrace{\nabla_X\langle Y, Y\rangle}_{X|Y|^2} = \langle\nabla_X Y, Y\rangle + \langle Y, \nabla_X Y\rangle = 2\langle\nabla_X Y, Y\rangle \qquad (*)
\end{aligned}$$

and

$$\begin{aligned}
&\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
\implies &\underbrace{\nabla_W\langle\nabla_X Y, Z\rangle}_{W\langle\nabla_X Y, Z\rangle} = \langle\nabla_W\nabla_X Y, Z\rangle + \langle\nabla_X Y, \nabla_W Z\rangle \qquad (**)
\end{aligned}$$

Thus,

$$
\begin{aligned}
WX|Y|^2 &\overset{(*)(**)}{=\!=\!=} W\left(2\left\langle\nabla_X Y, Y\right\rangle\right)=2\left\langle\nabla_W \nabla_X Y, Y\right\rangle+2\left\langle\nabla_X Y, \nabla_W Y\right\rangle ; \\
XW|Y|^2 &\overset{(*)(**)}{=\!=\!=} X\left(2\left\langle\nabla_W Y, Y\right\rangle\right)=2\left\langle\nabla_X \nabla_W Y, Y\right\rangle+2\left\langle\nabla_W Y, \nabla_X Y\right\rangle ; \\
[W, X]|Y|^2 &\overset{(*)}{=\!=} 2\left\langle\nabla_{[W, X]} Y, Y\right\rangle .
\end{aligned}
$$

When we subtract the second and third equations from the first, the left-hand side is zero. The terms $2\left\langle\nabla_X Y, \nabla_W Y\right\rangle$ and $2\left\langle\nabla_W Y, \nabla_X Y\right\rangle$ cancel on the right-hand side, giving

$$
\begin{aligned}
0 & =2\left\langle\nabla_W \nabla_X Y, Y\right\rangle-2\left\langle\nabla_X \nabla_W Y, Y\right\rangle-2\left\langle\nabla_{[W, X]} Y, Y\right\rangle \\
& =2\langle R(W, X) Y, Y\rangle \\
& =2 Rm(W, X, Y, Y) .
\end{aligned}
$$

Next we prove (d). From the definition of Rm, this will follow immediately from

$$
R(W, X) Y+R(X, Y) W+R(Y, W) X=0 .
$$

Using the definition of R and the symmetry of the connection, the left-hand side expands to

$$
\begin{aligned}
& \left(\nabla_W \nabla_X Y-\nabla_X \nabla_W Y-\nabla_{[W, X]} Y\right) \\
& +\left(\nabla_X \nabla_Y W-\nabla_Y \nabla_X W-\nabla_{[X, Y]} W\right) \\
& +\left(\nabla_Y \nabla_W X-\nabla_W \nabla_Y X-\nabla_{[Y, W]} X\right) \\
& =\nabla_W\left(\nabla_X Y-\nabla_Y X\right)+\nabla_X\left(\nabla_Y W-\nabla_W Y\right)+\nabla_Y\left(\nabla_W X-\nabla_X W\right) \\
& -\nabla_{[W, X]} Y-\nabla_{[X, Y]} W-\nabla_{[Y, W]} X \\
& =\nabla_W[X, Y]+\nabla_X[Y, W]+\nabla_Y[W, X] \\
& -\nabla_{[W, X]} Y-\nabla_{[X, Y]} W-\nabla_{[Y, W]} X \\
& =[W,[X, Y]]+[X,[Y, W]]+[Y,[W, X]] .
\end{aligned}
$$

This is zero by the Jacobi identity (see property 1.2.4).

Finally, we show that identity (c) follows from the other three. Writing the algebraic Bianchi identity four times with indices cyclically permuted gives

$$
\begin{aligned}
Rm(W, X, Y, Z)+Rm(X, Y, W, Z)+Rm(Y, W, X, Z) & =0 \\
Rm(X, Y, Z, W)+Rm(Y, Z, X, W)+Rm(Z, X, Y, W) & =0 \\
Rm(Y, Z, W, X)+Rm(Z, W, Y, X)+Rm(W, Y, Z, X) & =0 \\
Rm(Z, W, X, Y)+Rm(W, X, Z, Y)+Rm(X, Z, W, Y) & =0 .
\end{aligned}
$$

Now add up all four equations. Applying (b) four times makes all the terms in the first two columns cancel. Then applying (a) and (b) in the last column yields $2 Rm(Y, W, X, Z)-2 Rm(X, Z, Y, W)=0$, which is equivalent to (c). ■

There is one more identity that is satisfied by the covariant derivatives of the curvature tensor on every Riemannian manifold. Classically, it was called the second Bianchi identity, but modern authors tend to use the more informative name differential Bianchi identity.

Proposition 3.3.14 (Differential Bianchi Identity)**.** *The total covariant derivative of the curvature tensor satisfies the following identity:*

$$
\nabla Rm(X, Y, Z, V, W)+\nabla Rm(X, Y, V, W, Z)+\nabla Rm(X, Y, W, Z, V)=0 . \tag{3.44}
$$

In components, this is

$$
R_{ijkl;m}+R_{ijlm;k}+R_{ijmk;l}=0 . \tag{3.45}
$$

Proof. First of all, we show that by the symmetries of Rm, (3.44) is equivalent to

$$\nabla Rm(Z, V, X, Y, W) + \nabla Rm(V, W, X, Y, Z) + \nabla Rm(W, Z, X, Y, V) = 0. \tag{3.46}$$

For example,

$$\begin{aligned}
&\nabla Rm(X, Y, Z, V, W) = (\nabla_W Rm)(X, Y, Z, V) \\
\overset{(3.4)}{=\!=}&W(Rm(X, Y, Z, V)) - Rm(\nabla_W X, Y, Z, V) - Rm(X, \nabla_W Y, Z, V) - Rm(X, Y, \nabla_W Z, V) - Rm(X, Y, Z, \nabla_W V) \\
\overset{\text{prop.3.3.12}}{=\!=}&\underbrace{-W(\underbrace{Rm(Z, V, X, Y)}_{\text{a function}})}_{=-\nabla_W(Rm(Z,V,X,Y))} - \underbrace{Rm(\nabla_W X, Y, Z, V)}_{(1)} - \underbrace{Rm(X, \nabla_W Y, Z, V)}_{(2)} - \underbrace{Rm(X, Y, \nabla_W Z, V)}_{(3)} - \underbrace{Rm(X, Y, Z, \nabla_W V)}_{(4)}
\end{aligned}$$

and

$$\begin{aligned}
&\nabla Rm(Z, V, X, Y, W) \\
=&W(Rm(Z, V, X, Y)) - \overbrace{Rm(\nabla_W Z, V, X, Y)}^{(3)} - \overbrace{Rm(Z, \nabla_W V, X, Y)}^{(4)} - \overbrace{Rm(Z, V, \nabla_W X, Y)}^{(1)} - \overbrace{Rm(Z, V, X, \nabla_W Y)}^{(2)}
\end{aligned}$$

Equation (3.46) be proved by a long and tedious computation, but there is a standard shortcut for such calculations in Riemannian geometry that makes our task immeasurably easier. To prove that (3.46) holds at a particular point p, it suffices by multilinearity to prove the formula when X, Y, Z, V, W are basis vectors with respect to some frame. The shortcut consists in choosing a special frame for each point p to simplify the computations there.

Let p be an arbitrary point, let $\left(x^i\right)$ be normal coordinates centered at p, and let X, Y, Z, V, W be arbitrary <u>coordinate basis vector fields</u>. These vectors satisfy two properties that simplify our computations enormously: (1) their commutators vanish identically, since $[\partial_i, \partial_j] \equiv 0$; and (2) their covariant derivatives vanish at p, since $\Gamma_{ij}^k(p) = 0$ (Prop.3.2.19(d)).

Using these facts and the compatibility of the connection with the metric, the first term in (3.46) evaluated at p becomes

$$\begin{aligned}
(\nabla_W Rm)(Z, V, X, Y) &= \nabla_W(Rm(Z, V, X, Y)) \\
&= \nabla_W \langle R(Z, V)X, Y\rangle \\
&= \nabla_W \left\langle \nabla_Z \nabla_V X - \nabla_V \nabla_Z X - \nabla_{[Z,V]} X, Y\right\rangle \\
&= \langle \nabla_W \nabla_Z \nabla_V X - \nabla_W \nabla_V \nabla_Z X, Y\rangle .
\end{aligned}$$

Write this equation three times, with the vector fields W, Z, V cyclically permuted. Summing all three gives

$$\begin{aligned}
&\nabla Rm(Z, V, X, Y, W) + \nabla Rm(V, W, X, Y, Z) + \nabla Rm(W, Z, X, Y, V) \\
&= \langle \nabla_W \nabla_Z \nabla_V X - \nabla_W \nabla_V \nabla_Z X \\
&\quad + \nabla_Z \nabla_V \nabla_W X - \nabla_Z \nabla_W \nabla_V X \\
&\quad + \nabla_V \nabla_W \nabla_Z X - \nabla_V \nabla_Z \nabla_W X, Y\rangle \\
&= \langle R(W, Z)(\nabla_V X) + R(Z, V)(\nabla_W X) + R(V, W)(\nabla_Z X), Y\rangle \\
&= 0,
\end{aligned}$$

where the last line follows because $\nabla_V X = \nabla_W X = \nabla_Z X = 0$ at p. ■

3.3.4 The Ricci Identities

The curvature endomorphism also appears as the obstruction to commutation of total covariant derivatives. Recall that if F is any smooth tensor field of type (k, l), then its second covariant derivative $\nabla^2 F = \nabla(\nabla F)$ is a smooth $(k, l+2)$-tensor field, and for vector fields X and Y, the notation $\nabla^2_{X,Y}F$ denotes $\nabla^2 F(\ldots, Y, X)$. Given vector fields X and Y, let $R(X,Y)^* : T^*M \to T^*M$ denote the **dual map to** $R(X,Y)$, defined by

$$(R(X,Y)^*\eta)(Z) = \eta(R(X,Y)Z).$$

Theorem 3.3.15 (Ricci Identities)**.** *On a Riemannian or pseudo-Riemannian manifold M, the second total covariant derivatives of vector and tensor fields satisfy the following identities. If Z is a smooth vector field,*

$$\nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z = R(X,Y)Z. \tag{3.47}$$

If β is a smooth 1-form,

$$\nabla^2_{X,Y}\beta - \nabla^2_{Y,X}\beta = -R(X,Y)^*\beta. \tag{3.48}$$

And if B is a smooth (k,l)-tensor field,

$$\begin{aligned}
&\left(\nabla^2_{X,Y}B - \nabla^2_{Y,X}B\right)\left(\omega^1, \ldots, \omega^k, V_1, \ldots, V_l\right) \\
&= B\left(R(X,Y)^*\omega^1, \omega^2, \ldots, \omega^k, V_1, \ldots, V_l\right) + \cdots \\
&+ B\left(\omega^1, \ldots, \omega^{k-1}, R(X,Y)^*\omega^k, V_1, \ldots, V_l\right) \\
&- B\left(\omega^1, \ldots, \omega^k, R(X,Y)V_1, V_2, \ldots, V_l\right) - \cdots \\
&- B\left(\omega^1, \ldots, \omega^k, V_1, \ldots, V_{l-1}, R(X,Y)V_l\right)
\end{aligned} \tag{3.49}$$

for all covector fields ω^i and vector fields V_j. In terms of any local frame, the component versions of these formulas read

$$Z^i;pq - Z^i;qp = -R_{pqm}{}^i Z^m, \tag{3.50}$$

$$\beta_{j;pq} - \beta_{j;qp} = R_{pqj}{}^m\beta_m, \tag{3.51}$$

$$\begin{aligned}
B^{i_1\ldots i_k}_{j_1\ldots j_l;pq} - B^{i_1\ldots i_k}_{j_1\ldots j_l;qp} = &- R_{pqm}{}^{i_1} B^{mi_2\ldots i_k}_{j_1\ldots j_l} - \cdots - R_{pqm}{}^{i_k} B^{i_1\ldots i_k m}_{j_1\ldots j_l} \\
&+ R_{pqj_1}{}^m B^{i_1\ldots i_k}_{mj_2\ldots j_l} + \cdots + R_{pqj_l}{}^m B^{i_1\ldots i_k}_{j_1\ldots j_{l-1}m}.
\end{aligned} \tag{3.52}$$

Proof. For any tensor field B and vector fields X, Y, Proposition 3.1.19 implies

$$\begin{aligned}
\nabla^2_{X,Y}B - \nabla^2_{Y,X}B &= \nabla_X\nabla_Y B - \nabla_{(\nabla_X Y)}B - \nabla_Y\nabla_X B + \nabla_{(\nabla_Y X)}B \\
&= \nabla_X\nabla_Y B - \nabla_Y\nabla_X B - \nabla_{[X,Y]}B,
\end{aligned} \tag{3.53}$$

where the last equality follows from the symmetry of the connection. In particular, this holds when $B = Z$ is a vector field, so (3.47) follows directly from the definition of the curvature endomorphism. Next we prove (3.48). Using (3.6) repeatedly, we compute

$$\begin{aligned}
(\nabla_X\nabla_Y\beta)(Z) &= X((\nabla_Y\beta)(Z)) - (\nabla_Y\beta)(\nabla_X Z) \\
&= X(Y(\beta(Z)) - \beta(\nabla_Y Z)) - (\nabla_Y\beta)(\nabla_X Z) \\
&= XY(\beta(Z)) - (\nabla_X\beta)(\nabla_Y Z) - \beta(\nabla_X\nabla_Y Z) - (\nabla_Y\beta)(\nabla_X Z).
\end{aligned} \tag{3.54}$$

Reversing the roles of X and Y, we get

$$(\nabla_Y\nabla_X\beta)(Z) = YX(\beta(Z)) - (\nabla_Y\beta)(\nabla_X Z) - \beta(\nabla_Y\nabla_X Z) - (\nabla_X\beta)(\nabla_Y Z), \tag{3.55}$$

and applying (3.6) one more time yields

$$\left(\nabla_{[X,Y]}\beta\right)(Z) = [X,Y](\beta(Z)) - \beta\left(\nabla_{[X,Y]}Z\right). \tag{3.56}$$

Now subtract (3.55) and (3.56) from (3.54): all but three of the terms cancel, yielding

$$\begin{aligned}
\left(\nabla_X\nabla_Y\beta - \nabla_Y\nabla_X\beta - \nabla_{[X,Y]}\beta\right)(Z) &= -\beta\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\right) \\
&= -\beta(R(X,Y)Z),
\end{aligned}$$

which is equivalent to (3.48). Next consider the action of $\nabla^2_{X,Y} - \nabla^2_{Y,X}$ on an arbitrary tensor product:

$$\begin{aligned}
&\left(\nabla^2_{X,Y} - \nabla^2_{Y,X}\right)(F \otimes G)\\
&\quad= \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right)(F \otimes G)\\
&\quad= \nabla_X \nabla_Y F \otimes G + \nabla_Y F \otimes \nabla_X G + \nabla_X F \otimes \nabla_Y G + F \otimes \nabla_X \nabla_Y G\\
&\qquad - \nabla_Y \nabla_X F \otimes G - \nabla_X F \otimes \nabla_Y G - \nabla_Y F \otimes \nabla_X G - F \otimes \nabla_Y \nabla_X G\\
&\qquad - \nabla_{[X,Y]} F \otimes G - F \otimes \nabla_{[X,Y]} G\\
&\quad= \left(\nabla^2_{X,Y} F - \nabla^2_{Y,X} F\right) \otimes G + F \otimes \left(\nabla^2_{X,Y} G - \nabla^2_{Y,X} G\right).
\end{aligned}$$

A simple induction using this relation together with (3.47) and (3.48) shows that for all smooth vector fields $W_1, \ldots, W_k$ and 1-forms $\eta^1, \ldots, \eta^l$,

$$\begin{aligned}
&\left(\nabla^2_{X,Y} - \nabla^2_{Y,X}\right)\left(W_1 \otimes \cdots \otimes W_k \otimes \eta^1 \otimes \cdots \otimes \eta^l\right)\\
&= (R(X,Y)W_1) \otimes W_2 \otimes \cdots \otimes W_k \otimes \eta^1 \otimes \cdots \otimes \eta^l + \cdots\\
&+ W_1 \otimes \cdots \otimes W_{k-1} \otimes (R(X,Y)W_k) \otimes \eta^1 \otimes \cdots \otimes \eta^l\\
&+ W_1 \otimes \cdots \otimes W_k \otimes \left(-R(X,Y)^* \eta^1\right) \otimes \eta^2 \otimes \cdots \otimes \eta^l + \cdots\\
&+ W_1 \otimes \cdots \otimes W_k \otimes \eta^1 \otimes \cdots \otimes \eta^{l-1} \otimes \left(-R(X,Y)^* \eta^l\right).
\end{aligned}$$

Since every tensor field can be written as a sum of tensor products of vector fields and 1-forms, this implies (3.49). Finally, the component formula (3.52) follows by applying (3.49) to

$$\left(\nabla^2_{E_q,E_p} B - \nabla^2_{E_p,E_q} B\right)\left(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}, E_{j_1}, \ldots, E_{j_l}\right),$$

where (E_j) and $\left(\varepsilon^i\right)$ represent a local frame and its dual coframe, respectively, and using

$$\begin{aligned}
R\left(E_q, E_p\right) E_j &= R_{qpj}{}^m E_m = -R_{pqj}{}^m E_m,\\
R\left(E_q, E_p\right)^* \varepsilon^i &= R_{qpm}{}^i \varepsilon^m = -R_{pqm}{}^i \varepsilon^m.
\end{aligned}$$

The other two component formulas are special cases of (3.52). ■

3.3.5 Ricci and Scalar Curvature

Suppose (M, g) is an n-dimensional Riemannian or pseudo-Riemannian manifold. Because 4-tensors are so complicated, it is often useful to construct simpler tensors that summarize some of the information contained in the curvature tensor. The most important such tensor is the **Ricci curvature** or **Ricci tensor**, denoted by Rc (or often Ric in the literature), which is the covariant 2-tensor field defined as the trace of the curvature endomorphism on its first and last indices. That is, $Rc = C^1_3(R)$ where C^1_3 is the unique linear mapping from $T^{(1,3)}(TM)$ to $T^{(0,2)}(TM)$ such that

$$\omega_1 \otimes \omega_2 \otimes \omega_3 \otimes v_1 \mapsto \langle \omega_1, v_1 \rangle \omega_2 \otimes \omega_3$$

(note that we didn't write $v_1 \otimes \omega_1 \otimes \omega_2 \otimes \omega_3$ because we want to be aligned with the convention that the contravariant index is placed at last for Riemannian endomorphism; as in definition 2.3.2 the order of covariant and contravariant is assumed to be dropped.) Now, since

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

we see that C^1_3 sends R to

$$R_{ijk}{}^l \langle dx^i, \partial_l \rangle dx^j \otimes dx^k = R_{ijk}{}^l \delta_{il} dx^j \otimes dx^k = R_{pjk}{}^p dx^j \otimes dx^k = \overbrace{R_{kij}{}^k dx^i \otimes dx^j}^{Rc}.$$

The components of Rc are usually denoted as R_{ij}, so above equation implies

$$R_{ij} = R_{kij}{}^{k}$$

Proposition 3.3.16.

(1) For vector fields X, Y,

$$Rc(X, Y) = \operatorname{tr}(Z \mapsto R(Z, X)Y).$$

(2) For orthonormal basis (E_i), we have

$$\operatorname{tr}(Z \mapsto R(Z, X)Y) = \sum_i \langle R(E_i, X)Y, E_i \rangle_g$$

(3) $R_{ij} = g^{km} R_{kijm}$.

Proof. (1): We denote $Z \mapsto R(Z, X)Y$ as the operator $A \in \operatorname{End}(TM)$. Then $f = \Phi(A) \in T^{(1,1)}(TM)$ is defined by

$$f(W, Z) = \Phi(A)(W, Z) = W(R(Z, X)Y)$$

To get the trace of $f = \Phi(A)$, we compute $f(dx^i, \partial_j)$:

$$f_j^i = f(dx^i, \partial_j) = dx^i(R(\partial_j, X^k \partial_k)(Y^m \partial_m)) \xlongequal{(3.34)} dx^i \left(R_{jkm}{}^{l} X^k Y^m \partial_l\right) = R_{jkm}{}^{i} X^k Y^m.$$

Thus, the trace of f is

$$\sum_i f_i^i = R_{ikm}{}^{i} X^k Y^m = R_{kij}{}^{k} X^i Y^j$$

which is the same as $R_{kij}{}^{k} dx^i \otimes dx^j(X, Y) = Rc(X, Y)$.

(2) In general, for $f \in \operatorname{End}(V)$,

$$\operatorname{tr}(f) = \sum_i \langle E_i, f(E_i) \rangle.$$

That's because $f(E_i) = \sum_j f_{ji} E_j$ where (f_{ij}) is the matrix of f, and $\langle E_i, f(E_i) \rangle = \sum_j f_{ji} \langle E_i, E_j \rangle = \sum_j f_{ji} \delta_{ij} = f_{\text{II}}$.

(3) It is known that the components of Riemann curvature tensor satisfies $R_{ijkl} = g_{lm} R_{ijk}{}^{m}$. Thus

$$\begin{aligned} g^{km} R_{kijm} &= g^{km} g_{mp} R_{kij}{}^{p} \\ &\xlongequal{(2.6)} \delta_p^k R_{kij}{}^{p} \\ &= R_{kij}{}^{k} = R_{ij} \end{aligned}$$

∎

The **scalar curvature** is the function S pointewise defined as the trace of the Ricci tensor:

$$S = \operatorname{tr}_g Rc = R_i{}^{i} = g^{ij} R_{ij}.$$

where we used equation (2.14). Note that $(Rc_p)^\sharp(v, \omega) = Rc_p(v, \omega^\sharp)$ and $S_p = \operatorname{tr}((Rc_p)^\sharp)$ (note that it is the last index, or the second covariant, that is raised, so we write (v, w) instead of (ω, v); just as in definition 2.3.2, the order of covariant and contravariant is assumed to be dropped).

Lemma 3.3.17. *The Ricci curvature is a symmetric 2-tensor field. It can be expressed in any of the following ways:*

$$R_{ij} = R_{kij}{}^{k} = R_{ik}{}^{k}{}_{j} = -R_{ki}{}^{k}{}_{j} = -R_{ikj}{}^{k}.$$

Proof. To show $R_{ij} = R_{ik}{}^{k}{}_{j}$, we use Example 2.3.3. By the symmetry of Riemann curvature tensor we obtain

$$R_{ik}{}^{k}{}_{j} = g^{km}R_{ikmj} = g^{km}(-R_{kimj}) = g^{km}(-(-R_{kijm})) = g^{km}R_{kijm} \xlongequal{prop.3.3.5(3)} R_{ij}$$

Similarly,

$$-R_{ki}{}^{k}{}_{j} = -g^{km}R_{kimj} = g^{km}R_{kijm} = R_{ij},$$

and

$$-R_{ikj}{}^{k} = -g^{km}R_{ikjm} = g^{km}R_{kijm} = R_{ij}.$$

■

It is sometimes useful to decompose the Ricci tensor into a multiple of the metric and a complementary piece with zero trace. Define the **traceless Ricci tensor of** g as the following symmetric 2-tensor:

$$\overset{\circ}{Rc} = Rc - \frac{1}{n}Sg.$$

Proposition 3.3.18. *Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold. Then $\operatorname{tr}_g Rc \equiv 0$, and the Ricci tensor decomposes orthogonally as*

$$Rc = \overset{\circ}{Rc} + \frac{1}{n}Sg. \tag{3.57}$$

Therefore, in the Riemannian case,

$$|Rc|_g^2 = |Rc|_g^2 + \frac{1}{n}S^2 \tag{3.58}$$

Remark 3.3.19. The statement about norms, and others like it that we will prove below, works only in the Riemannian case because of the additional absolute value signs required to compute norms in the pseudo-Riemannian case. The pseudo-Riemannian analogue would be $\langle Rc, Rc\rangle_g = \langle Rc, Rc\rangle_g + \frac{1}{n}S^2$, but this is not as useful. ♠

Proof. Note that in every local frame, we have

$$\operatorname{tr}_g g = g_{ij}g^{ji} = \delta_i^i = n.$$

It then follows directly from the definition of $\overset{\circ}{Rc}$ that $\operatorname{tr}_g \overset{\circ}{Rc} \equiv 0$ and (3.57) holds:

$$\operatorname{tr}_g \overset{\circ}{Rc} = \operatorname{tr}_g(Rc - \frac{1}{n}Sg) \xlongequal{\text{linearity}} \operatorname{tr}_g Rc - \frac{1}{n}S\operatorname{tr}_g g = S - \frac{1}{n}Sn = 0$$

where we again note that S is a function and S_p is thus only a scalar. The fact that the decomposition is orthogonal follows easily from the fact that for every symmetric 2-tensor h, we have

$$\langle h, g\rangle = g^{ik}g^{jl}h_{ij}g_{kl} = g^{ij}h_{ij} = \operatorname{tr}_g h,$$

and therefore $\langle \overset{\circ}{Rc}, g\rangle = \operatorname{tr}_g \overset{\circ}{Rc} = 0$. Finally, (3.58) follows from (3.57) and the fact that $\langle g, g\rangle = \operatorname{tr}_g g = n$. ■

The next proposition, which follows directly from the differential Bianchi identity, expresses some important relationships among the covariant derivatives of the various curvature tensors. To express it concisely, it is useful to introduce another operator on tensor fields. If T is a smooth 2-tensor field on a Riemannian or pseudo-Riemannian manifold, we define the **exterior covariant derivative of** T to be the 3-tensor field DT defined by

$$(DT)(X, Y, Z) = -(\nabla T)(X, Y, Z) + (\nabla T)(X, Z, Y).$$

In terms of components, this is

$$(DT)_{ijk} = -T_{ij;k} + T_{ik;j}$$

(This operator is a generalization of the ordinary exterior derivative of a 1-form, which can be expressed in terms of the total covariant derivative by $(d\eta)(Y,Z) = -(\nabla\eta)(Y,Z) + (\nabla\eta)(Z,Y)$ by the result of [12] Problem 5-13. The exterior covariant derivative can be generalized to other types of tensors as well, but this is the only case we need.)

Proposition 3.3.20 (Contracted Bianchi Identities). *Let (M,g) be a Riemannian or pseudo-Riemannian manifold. The covariant derivatives of the Riemann, Ricci, and scalar curvatures of g satisfy the following identities:*

$$\operatorname{tr}_g(\nabla Rm) = -D(Rc), \tag{3.59}$$

$$\operatorname{tr}_g(\nabla Rc) = \frac{1}{2}dS, \tag{3.60}$$

where the trace in each case is on the first and last indices. In components, this is

$$R_{ijkl;}{}^{i} = R_{jk;l} - R_{jl;k}, \tag{3.61}$$

$$R_{il;}{}^{i} = \frac{1}{2}S_{;l}. \tag{3.62}$$

Proof. Start with the component form (3.45) of the differential Bianchi identity, raise the index m, and then contract on the indices i,m to obtain (3.61). (Note that covariant differentiation commutes with contraction by Proposition 3.1.13 and with the musical isomorphisms by Proposition 3.2.12, so it does not matter whether the indices that are raised and contracted come before or after the semicolon.) Then do the same with the indices j,k and simplify to obtain (3.62). The coordinate-free formulas (3.59) and (3.60) follow by expanding everything out in components. ■

It is important to note that if the sign convention chosen for the curvature tensor is the opposite of ours, then the Ricci tensor must be defined as the trace of Rm on the first and third (or second and fourth) indices. (The trace on the first two or last two indices is always zero by antisymmetry.) The definition is chosen so that the Ricci and scalar curvatures have the same meaning for everyone, regardless of the conventions chosen for the full curvature tensor. So, for example, if a manifold is said to have positive scalar curvature, there is no ambiguity as to what is meant.

A Riemannian or pseudo-Riemannian metric is said to be an **Einstein metric** if its Ricci tensor is a constant multiple of the metric-that is,

$$Rc = \lambda g \quad \text{for some constant } \lambda. \tag{3.63}$$

This equation is known as the **Einstein equation**. As the next proposition shows, for connected manifolds of dimension greater than 2, it is not necessary to assume that λ is constant; just assuming that the Ricci tensor is a function times the metric is sufficient.

Proposition 3.3.21 (Schur's Lemma). *Suppose (M,g) is a connected Riemannian or pseudo-Riemannian manifold of dimension $n \geq 3$ whose Ricci tensor satisfies $Rc = fg$ for some smooth real-valued function f. Then f is constant and g is an Einstein metric.*

Proof. Proof. Taking traces of both sides of $Rc = fg$ shows that $f = \frac{1}{n}S$, so the traceless Ricci tensor is identically zero. It follows that $\mathring{Rc} \equiv 0$. Because the covariant derivative of the metric is zero, this implies the following equation in any coordinate chart:

$$0 = R_{ij;k} - \frac{1}{n}S_{;k}g_{ij}$$

Tracing this equation on i and k, and comparing with the contracted Bianchi identity (3.62), we conclude that

$$0 = \frac{1}{2}S_{;j} - \frac{1}{n}S_{;j}$$

Because $n \geq 3$, this implies $S_{;j} = 0$. But $S_{;j}$ is the component of $\nabla S = dS$, so connectedness of M implies that S is constant and thus so is f. ■

Corollary 3.3.22. *If (M, g) is a connected Riemannian or pseudo-Riemannian manifold of dimension $n \geq 3$, then g is Einstein if and only if $\mathring{Rc} = 0$.*

Proof. Suppose first that g is an Einstein metric with $Rc = \lambda g$. Taking traces of both sides, we find that $\lambda = \frac{1}{n}S$, and therefore $\mathring{Rc} = Rc - \lambda g = 0$. Conversely, if $\mathring{Rc} = 0$, Schur's lemma implies that g is Einstein. ■

3.4 Geometry of Submanifolds

3.4.1 The Second Fundamental Form

Suppose (M, g) is a Riemannian submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Recall that this means that M is a submanifold of $\widetilde{M}$ endowed with the induced metric $g = \iota_M^* \widetilde{g}$ (where $\iota_M : M \hookrightarrow \widetilde{M}$ is the inclusion map). We will study the relationship between the geometry of M and that of $\widetilde{M}$. We assume that $(\widetilde{M}, \widetilde{g})$ is a Riemannian or pseudo-Riemannian manifold of dimension m, and (M, g) is an embedded n dimensional Riemannian submanifold of $\widetilde{M}$. For other cases, see [12] p.226 for more explanation.

Our first main task is to compare the Levi-Civita connection of M with that of $\widetilde{M}$. The starting point for doing so is the orthogonal decomposition of sections of the ambient tangent bundle $T\widetilde{M}\big|_M$ into tangential and orthogonal components. Just as we did for submanifolds of $\mathbb{R}^n$, we define orthogonal projection maps called **tangential** and **normal projections**:

$$\begin{aligned} \pi^\top &: T\widetilde{M}\Big|_M \to TM, \\ \pi^\perp &: T\widetilde{M}\Big|_M \to NM. \end{aligned}$$

In terms of an adapted orthonormal frame $(E_1, \ldots, E_m)$ for M in $\widetilde{M}$, these are just the usual projections onto $\operatorname{span}(E_1, \ldots, E_n)$ and $\operatorname{span}(E_{n+1}, \ldots, E_m)$ respectively, so both projections are smooth bundle homomorphisms (i.e., they are linear on fibers and map smooth sections to smooth sections). If X is a section of $T\widetilde{M}\Big|_M$, we often use the shorthand notations $X^\top = \pi^\top X$ and $X^\perp = \pi^\perp X$ for its tangential and normal projections.

If X, Y are vector fields in $\mathfrak{X}(M)$, we can extend them to vector fields on an open subset of $\widetilde{M}$ (still denoted by X and Y), apply the ambient covariant derivative operator $\widetilde{\nabla}$, and then decompose at points of M to get

$$\widetilde{\nabla}_X Y = \left(\widetilde{\nabla}_X Y\right)^\top + \left(\widetilde{\nabla}_X Y\right)^\perp. \tag{3.64}$$

We wish to interpret the two terms on the right-hand side of this decomposition. Let us focus first on the normal component. We define the **second fundamental form** of M to be the map $\mathrm{II} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM)$ (read "two") given by

$$\mathrm{II}(X, Y) = \left(\widetilde{\nabla}_X Y\right)^\perp,$$

where X and Y are extended arbitrarily to an open subset of $\widetilde{M}$. Since $\pi^\perp$ maps smooth sections to smooth sections, $\mathrm{II}(X, Y)$ is a smooth section of NM.

The term **first fundamental form**, by the way, was originally used to refer to the induced metric g on M. Although that usage has mostly been replaced by more descriptive terminology, we seem unfortunately to be stuck with the name "second fundamental form." The word "form" in both cases refers to bilinear form, not differential form.

Proposition 3.4.1 (Properties of the Second Fundamental Form)**.** *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and let $X, Y \in \mathfrak{X}(M)$.*

(a) $\mathrm{II}(X, Y)$ *is independent of the extensions of X and Y to an open subset of $\widetilde{M}$.*

(b) $\mathrm{II}(X, Y)$ *is bilinear over $C^{\infty}(M)$ in X and Y.*

(c) $\mathrm{II}(X, Y)$ *is symmetric in X and Y.*

(d) The value of $\mathrm{II}(X, Y)$ *at a point $p \in M$ depends only on X_p and Y_p.*

Proof. Proof. Choose particular extensions of X and Y to a neighborhood of M in $\widetilde{M}$, and for simplicity denote the extended vector fields also by X and Y. We begin by proving that $\mathrm{II}(X, Y)$ is symmetric in X and Y when defined in terms of these extensions. The symmetry of the connection $\widetilde{\nabla}$ implies

$$\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = \left(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X\right)^{\perp} = [X, Y]^{\perp}.$$

Since X and Y are tangent to M at all points of M, so is their Lie bracket (Cor.1.2.6). Therefore $[X, Y]^{\perp} = 0$, so II is symmetric.

Because $\widetilde{\nabla}_X Y\Big|_p$ depends only on X_p, it follows that the value of $\mathrm{II}(X, Y)$ at p depends only on X_p, and in particular is independent of the extension chosen for X. Because $\widetilde{\nabla}_X Y$ is linear over $C^{\infty}(\widetilde{M})$ in X, and every $f \in C^{\infty}(M)$ can be extended to a smooth function on a neighborhood of M in $\widetilde{M}$, it follows that $\mathrm{II}(X, Y)$ is linear over $C^{\infty}(M)$ in X. By symmetry, the same claims hold for Y. ■

As a consequence of the preceding proposition, for every $p \in M$ and all vectors $v, w \in T_p M$, it makes sense to interpret $\mathrm{II}(v, w)$ as the value of $\mathrm{II}(V, W)$ at p, where V and W are any vector fields on M such that $V_p = v$ and $W_p = W$, and we will do so from now on without further comment.

The following theorem shows that for the normal part of the decomposition, we have a relationship similar to the Euclidean case: $(\widetilde{\nabla}_X Y)^{\top} = \nabla_X Y$.

Theorem 3.4.2 (The Gauss Formula)**.** *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. If $X, Y \in \mathfrak{X}(M)$ are extended arbitrarily to smooth vector fields on a neighborhood of M in $\widetilde{M}$, the following formula holds along M :*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y)$$

The Gauss formula can also be used to compare intrinsic and extrinsic covariant derivatives along curves. If $\gamma : I \to M$ is a smooth curve and X is a vector field along γ that is everywhere tangent to M, then we can regard X as either a vector field along γ in $\widetilde{M}$ or a vector field along γ in M. We let $\widetilde{D}_t X$ and $D_t X$ denote its covariant derivatives along γ as a curve in $\widetilde{M}$ and as a curve in M, respectively. The next corollary shows how the two covariant derivatives are related.

Corollary 3.4.3 (The Gauss Formula Along a Curve)**.** *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and $\gamma : I \to M$ is a smooth curve. If X is a smooth vector field along γ that is everywhere tangent to M, then*

$$\widetilde{D}_t X = D_t X + \mathrm{II}\left(\gamma', X\right).$$

Although the second fundamental form is defined in terms of covariant derivatives of vector fields tangent to M, it can also be used to evaluate extrinsic covariant derivatives of normal vector fields, as the following proposition shows. To express it concisely, we introduce one more notation. For each normal vector field $N \in \Gamma(NM)$, we obtain a scalar-valued symmetric bilinear form $\mathrm{II}_N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ by

$$\mathrm{II}_N(X,Y) = \langle N, \mathrm{II}(X,Y)\rangle. \tag{3.65}$$

Let $W_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denote the self-adjoint linear map associated with this bilinear form, characterized by

$$\langle W_N(X), Y\rangle = \mathrm{II}_N(X,Y) = \langle N, \mathrm{II}(X,Y)\rangle. \tag{3.66}$$

The map W_N is called the **Weingarten map in the direction of** N. Because the second fundamental form is bilinear over $C^\infty(M)$, it follows that W_N is linear over $C^\infty(M)$ and thus defines a smooth bundle homomorphism from TM to itself.

Proposition 3.4.4 (The Weingarten Equation)**.** *Suppose (M,g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M},\widetilde{g})$. For every $X \in \mathfrak{X}(M)$ and $N \in \Gamma(NM)$, the following equation holds:*

$$\left(\widetilde{\nabla}_X N\right)^\top = -W_N(X) \tag{3.67}$$

when N is extended arbitrarily to an open subset of $\widetilde{M}$.

In addition to describing the difference between the intrinsic and extrinsic connections, the second fundamental form plays an even more important role in describing the difference between the curvature tensors of $\widetilde{M}$ and M. The explicit formula, also due to Gauss, is given in the following theorem.

Theorem 3.4.5 (The Gauss Equation)**.** *Suppose (M,g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M},\widetilde{g})$. For all $W,X,Y,Z \in \mathfrak{X}(M)$, the following equation holds:*

$$\widetilde{Rm}(W,X,Y,Z) = Rm(W,X,Y,Z) - \langle \mathrm{II}(W,Z), \mathrm{II}(X,Y)\rangle + \langle \mathrm{II}(W,Y), \mathrm{II}(X,Z)\rangle.$$

There is one other fundamental submanifold equation, which relates the normal part of the ambient curvature endomorphism to derivatives of the second fundamental form. We will not have need for it, but we include it here for completeness. To state it, we need to introduce a connection on the normal bundle of a Riemannian submanifold.

If (M,g) is a Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M},\widetilde{g})$, the normal connection $\nabla^\perp : \mathfrak{X}(M) \times \Gamma(NM) \to \Gamma(NM)$ is defined by

$$\nabla \overset{\perp}{X} N = \left(\widetilde{\nabla}_X N\right)^\perp,$$

where N is extended to a smooth vector field on a neighborhood of M in $\widetilde{M}$.

Proposition 3.4.6. *If (M,g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M},\widetilde{g})$, then $\nabla^\perp$ is a well-defined connection in NM, which is compatible with $\widetilde{g}$ in the sense that for any two sections N_1, N_2 of NM and every $X \in \mathfrak{X}(M)$, we have*

$$X\langle N_1, N_2\rangle = \left\langle \nabla\frac{\perp}{X} N_1, N_2 \right\rangle + \left\langle N_1, \nabla_X^\perp N_2\right\rangle.$$

Exercise 3.4.7. *Prove the preceding proposition.*

We need the normal connection primarily to make sense of tangential covariant derivatives of the second fundamental form. To do so, we make the following definitions. Let $F \to M$ denote the bundle whose fiber at each point $p \in M$ is the set of bilinear maps $T_pM \times T_pM \to N_pM$. It is easy to check that F is a smooth vector bundle over M, and that smooth sections of F correspond to smooth maps $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM)$ that are bilinear over $C^\infty(M)$, such as the second fundamental form. Define a connection ∇^F in F as follows: if B is any smooth section of F, let $\nabla^F_X B$ be the smooth section of F defined by

$$\left(\nabla_X^F B\right)(Y, Z) = \nabla_X^\perp(B(Y, Z)) - B\left(\nabla_X Y, Z\right) - B\left(Y, \nabla_X Z\right).$$

Exercise 3.4.8. *Prove that ∇^F is a connection in F.*

Now we are ready to state the last of the fundamental equations for submanifolds.

Theorem 3.4.9 (The Codazzi Equation). *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For all $W, X, Y \in \mathfrak{X}(M)$, the following equation holds:*

$$(\widetilde{R}(W, X)Y)^\perp = \left(\nabla_W^F \mathrm{II}\right)(X, Y) - \left(\nabla_X^F \mathrm{II}\right)(W, Y). \tag{3.68}$$

Curvature of Curve

By studying the curvatures of curves, we can give a more geometric interpretation of the second fundamental form. Suppose (M, g) is a Riemannian or pseudoRiemannian manifold, and $\gamma : I \to M$ is a smooth unit-speed curve in M. We define the **(geodesic) curvature of** γ as the length of the acceleration vector field, which is the function $\kappa : I \to \mathbb{R}$ given by

$$\kappa(t) = \left|D_t \gamma'(t)\right|.$$

If γ is an arbitrary regular curve in a Riemannian manifold (not necessarily of unit speed), we first find a unit-speed reparametrization $\widetilde{\gamma} = \gamma \circ \varphi$, and then define the curvature of γ at t to be the curvature of $\widetilde{\gamma}$ at $\varphi^{-1}(t)$. In a pseudo-Riemannian manifold, the same approach works, but we have to restrict the definition to curves γ such that $|\gamma'(t)|$ is everywhere nonzero. [12] Problem 8-6 gives a formula that can be used in the Riemannian case to compute the geodesic curvature directly without explicitly finding a unit-speed reparametrization.

From the definition, it follows that a smooth unit-speed curve has vanishing geodesic curvature if and only if it is a geodesic, so the geodesic curvature of a curve can be regarded as a quantitative measure of how far it deviates from being a geodesic. If $M = \mathbb{R}^n$ with the Euclidean metric, the geodesic curvature agrees with the notion of curvature introduced in advanced calculus courses.

Now suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian or pseudo-Riemannian manifold and (M, g) is a Riemannian submanifold. Every regular curve $\gamma : I \to M$ has two distinct geodesic curvatures: its **intrinsic curvature** κ as a curve in M, and its **extrinsic curvature** $\widetilde{\kappa}$ as a curve in $\widetilde{M}$. The second fundamental form can be used to compute the relationship between the two.

Proposition 3.4.10 (Geometric Interpretation of II). *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, $p \in M$, and $v \in T_pM$.*

(a) $\mathrm{II}(v, v)$ *is the $\widetilde{g}$-acceleration at p of the g-geodesic γ_v.*

(b) If v is a unit vector, then $|\mathrm{II}(v, v)|$ is the $\widetilde{g}$-curvature of γ_v at p.

Note that the second fundamental form is completely determined by its values of the form $\mathrm{II}(v, v)$ for unit vectors v, by the following lemma.

Lemma 3.4.11. *Suppose V is an inner product space, W is a vector space, and $B, B' : V \times V \to W$ are symmetric and bilinear. If $B(v, v) = B'(v, v)$ for every unit vector $v \in V$, then $B = B'$.*

Because the intrinsic and extrinsic accelerations of a curve are usually different, it is generally not the case that a $\widetilde{g}$-geodesic that starts tangent to M stays in M; just think of a sphere in Euclidean space, for example. A Riemannian submanifold (M, g) of $(\widetilde{M}, \widetilde{g})$ is said to be **totally geodesic** if every $\widetilde{g}$-geodesic that is tangent to M at some time t_0 stays in M for all t in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Proposition 3.4.12. *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, The following are equivalent:*

(a) M is totally geodesic in $\widetilde{M}$.

(b) Every g-geodesic in M is also a $\widetilde{g}$-geodesic in $\widetilde{M}$.

(c) The second fundamental form of M vanishes identically.

3.4.2 Hypersurfaces

Now we specialize the preceding considerations to the case in which M is a hypersurface (i.e., a submanifold of codimension 1) in $\widetilde{M}$. Throughout this section, our default assumption is that (M, g) is an embedded n-dimensional Riemannian submanifold of an $(n+1)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. (The analogous formulas in the pseudo-Riemannian case are a little different; see [12] Problem 8-19.)

In this situation, at each point of M there are exactly two unit normal vectors. In terms of any local adapted orthonormal frame $(E_1, \ldots, E_{n+1})$, the two choices are $\pm E_{n+1}$. In a small enough neighborhood of each point of M, therefore, we can always choose a smooth unit normal vector field along M.

If both M and $\widetilde{M}$ are orientable, we can use an orientation to pick out a global smooth unit normal vector field along all of M. In general, though, this might or might not be possible. Since all of our computations in this chapter are local, we will always assume that we are working in a small enough neighborhood that a smooth unit normal field exists. We will address as we go along the question of how various quantities depend on the choice of normal vector field.

The Scalar Second Fundamental Form and the Shape Operator

Having chosen a distinguished smooth unit normal vector field N on the hypersurface $M \subseteq \widetilde{M}$, we can replace the vector-valued second fundamental form II by a simpler scalar-valued form. The **scalar second fundamental form of** M is the symmetric covariant 2-tensor field $h \in \Gamma\left(\Sigma^2 T^* M\right)$ defined by $h = \mathrm{II}_N$ (see (3.65)); in other words,

$$h(X, Y) = \langle N, \mathrm{II}(X, Y)\rangle. \tag{3.69}$$

Using the Gauss formula $\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y)$ and noting that $\nabla_X Y$ is orthogonal to N, we can rewrite the definition as

$$h(X, Y) = \left\langle N, \widetilde{\nabla}_X Y\right\rangle. \tag{3.70}$$

Also, since N is a unit vector spanning NM at each point, the definition of h is equivalent to

$$\mathrm{II}(X, Y) = h(X, Y)N. \tag{3.71}$$

Note that replacing N by $-N$ multiplies h by -1, so the sign of h depends on which unit normal is chosen; but h is otherwise independent of the choices.

The choice of unit normal field also determines a Weingarten map $W_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by (**??**); in the case of a hypersurface, we use the notation $s = W_N$ and call it the **shape operator of** M. Alternatively, we can think of s as the $(1,1)$-tensor field on M obtained from h by raising an index. It is characterized by

$$\langle sX, Y\rangle = h(X, Y) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

Because h is symmetric, s is a self-adjoint endomorphism of TM, that is,

$$\langle sX, Y\rangle = \langle X, sY\rangle \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

As with h, the sign of s depends on the choice of N.

Remark 3.4.13. We can think of s as the $(1,1)$-tensor field on M obtained from h by raising an index. In fact, by mimicing Example 2.3.3 and using (2.6), we see that

$$\begin{aligned}
&h_i{}^j = (h^\sharp)_i{}^j = g^{jl}h_{il} \\
\Longrightarrow\ &h^\flat = h_i{}^j dx^i \otimes \partial_j = g^{il}h_{il}dx^i \otimes \partial_j \\
\Longrightarrow\ &s(X) = \Psi(h^\sharp)(X) = \Psi(g^{il}h_{il}dx^i \otimes \partial_j)(X) \\
&\quad = g^{il}h_{il}[\Psi(dx^i \otimes \partial_j)](X) = g^{il}h_{il}dx^i(X)\partial_j \\
&\quad = g^{il}h_{il}X^i\partial_j \\
\Longrightarrow\ &\langle sX, Y\rangle = \langle g^{jl}h_{il}X^i\partial_j, Y^k\partial_k\rangle = g^{il}g_{jk}h_{il}X^iY^k \\
&\quad = \delta^l_k h_{il}X^iY^k = h_{ik}X^iY^k = h(X,Y)
\end{aligned}$$

where Ψ is the isomorphism from $T^{(1,1)}(TM)$ to $\mathrm{End}(TM)$. ♠

In terms of the tensor fields h and s, the formulas of the last section can be rewritten somewhat more simply. For this purpose, we will use the **Kulkarni-Nomizu product** of symmetric 2-tensors h, k:

$$\begin{aligned}
h \owedge k(w,x,y,z) = {}&h(w,z)k(x,y) + h(x,y)k(w,z) \\
&-h(w,y)k(x,z) - h(x,z)k(w,y),
\end{aligned}$$

and the **exterior covariant derivative** of a smooth symmetric 2-tensor field T is

$$(DT)(x,y,z) = -(\nabla T)(x,y,z) + (\nabla T)(x,z,y).$$

Theorem 3.4.14 (Fundamental Equations for a Hypersurface)**.** *Suppose (M,g) is a Riemannian hypersurface in a Riemannian manifold $(\widetilde{M},\widetilde{g})$, and N is a smooth unit normal vector field along M.*

(a) THE GAUSS FORMULA FOR A HYPERSURFACE: If $X, Y \in \mathfrak{X}(M)$ are extended to an open subset of $\widetilde{M}$, then

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)N.$$

(b) THE GAUSS FORMULA FOR A CURVE IN A HYPERSURFACE: If $\gamma : I \to M$ is a smooth curve and $X : I \to TM$ is a smooth vector field along γ, then

$$\widetilde{D}_t X = D_t X + h(\gamma', X)\, N.$$

(c) The WEINGARTEN EQUATION FOR A HYPERSURFACE: For every $X \in \mathfrak{X}(M)$,

$$\widetilde{\nabla}_X N = -sX$$

(d) The GAUSS EQUATION FOR A HYPERSURFACE: For all $W, X, Y, Z \in \mathfrak{X}(M)$,

$$\widetilde{Rm}(W,X,Y,Z) = Rm(W,X,Y,Z) - \frac{1}{2}(h \owedge h)(W,X,Y,Z).$$

(e) THE CODAZZI EQUATION FOR A HYPERSURFACE: For all $W, X, Y \in \mathfrak{X}(M)$,

$$\widetilde{Rm}(W,X,Y,N) = (Dh)(Y,W,X).$$

Principal Curvatures

At every point $p \in M$, we have seen that the shape operator s is a self-adjoint linear endomorphism of the tangent space T_pM. To analyze such an operator, we recall some linear-algebraic facts about self-adjoint endomorphisms.

Lemma 3.4.15. *Suppose V is a finite-dimensional inner product space and $s : V \to V$ is a self-adjoint linear endomorphism. Let C denote the set of unit vectors in V. There is a vector $v_0 \in C$ where the function $v \mapsto \langle sv, v\rangle$ achieves its maximum among elements of C, and every such vector is an eigenvector of s with eigenvalue $\lambda_0 = \langle sv_0, v_0\rangle$.*

Proposition 3.4.16 (Finite-Dimensional Spectral Theorem). *Suppose V is a finitedimensional inner product space and $s : V \to V$ is a self-adjoint linear endomorphism. Then V has an orthonormal basis of s-eigenvectors, and all of the eigenvalues are real.*

Proof. The proof is by induction on $n = \dim V$. The $n = 1$ result is easy, so assume that the theorem holds for some $n \geq 1$ and suppose $\dim V = n + 1$. Above lemma shows that s has a unit eigenvector b_0 with a real eigenvalue λ_0. Let $B \subseteq V$ be the span of b_0. Since $s(B) \subseteq B$, self-adjointness of s implies $s\left(B^\perp\right) \subseteq B^\perp$. The inductive hypothesis applied to $s|_B \perp$ implies that $B^\perp$ has an orthonormal basis $(b_1, \ldots, b_n)$ of s-eigenvectors with real eigenvalues, and then $(b_0, b_1, \ldots, b_n)$ is the desired basis of V. ■

Applying this proposition to the shape operator $s : T_pM \to T_pM$, we see that s has real eigenvalues $\kappa_1, \ldots, \kappa_n$, and there is an orthonormal basis $(b_1, \ldots, b_n)$ for T_pM consisting of s-eigenvectors, with $sb_i = \kappa_i b_i$ for each i (no summation). In this basis, both h and s are represented by diagonal matrices, and h has the expression

$$h(v, w) = \kappa_1 v^1 w^1 + \cdots + \kappa_n v^n w^n.$$

The eigenvalues of s at a point $p \in M$ are called the **principal curvatures of** M **at** p, and the corresponding eigenspaces are called the principal directions. The principal curvatures all change sign if we reverse the normal vector, but the principal directions and principal curvatures are otherwise independent of the choice of coordinates or bases.

There are two combinations of the principal curvatures that play particularly important roles for hypersurfaces. The **Gaussian curvature** is defined as $K = \det(s)$, and the **mean curvature** as $H = (1/n)\operatorname{tr}(s) = (1/n)\operatorname{tr}_g(h)$. Since the determinant and trace of a linear endomorphism are basis-independent, these are well defined once a unit normal is chosen. In terms of the principal curvatures, they are

$$K = \kappa_1\kappa_2\cdots\kappa_n, \quad H = \frac{1}{n}\left(\kappa_1 + \cdots + \kappa_n\right),$$

as can be seen by expressing s in terms of an orthonormal basis of eigenvectors. If N is replaced by $-N$, then H changes sign, while K is multiplied by $(-1)^n$.

Hypersurfaces in Euclidean Space

Now we specialize even further, to hypersurfaces in Euclidean space. In this section, we assume that $M \subseteq \mathbb{R}^{n+1}$ is an embedded n-dimensional submanifold with the induced Riemannian metric. The Euclidean metric will be denoted as usual by $\bar{g}$, and covariant derivatives and curvatures associated with $\bar{g}$ will be indicated by a bar. The induced metric on M will be denoted by g.

In this setting, because $\overline{Rm} \equiv 0$, the Gauss and Codazzi equations take even simpler forms:

$$\frac{1}{2} h \owedge h = Rm, \tag{3.72}$$

$$Dh = 0, \tag{3.73}$$

or in terms of a local frame for M,

$$h_{il}h_{jk} - h_{ik}h_{jl} = R_{ijkl}, \tag{3.74}$$
$$h_{ij;k} - h_{ik;j} = 0. \tag{3.75}$$

In particular, this means that the Riemann curvature tensor of a hypersurface in $\mathbb{R}^{n+1}$ is completely determined by the second fundamental form. A symmetric 2-tensor field that satisfies $Dh = 0$ is called a Codazzi tensor, so $Dh = 0$ can be expressed succinctly by saying that h is a Codazzi tensor.

Exercise 3.4.17. *Show that a smooth 2-tensor field h on a Riemannian manifold is a Codazzi tensor if and only if both h and ∇h are symmetric.*

The equations $\frac{1}{2}h \owedge h = Rm$ and $Dh = 0$ can be viewed as compatibility conditions for the existence of an embedding or immersion into Euclidean space with prescribed first and second fundamental forms. If (M, g) is a Riemannian n-manifold and h is a given smooth symmetric 2-tensor field on M, then Theorem 8.13 shows that these two equations are necessary conditions for the existence of an isometric immersion $M \to \mathbb{R}^{n+1}$ for which h is the scalar second fundamental form. (Note that an immersion is locally an embedding, so the theorem applies in a neighborhood of each point.) It is a remarkable fact that the Gauss and Codazzi equations are actually sufficient, at least locally. A sketch of a proof of this fact, called the fundamental theorem of hypersurface theory, can be found in [Pet16, pp. 108-109].

In the setting of a hypersurface $M \subseteq \mathbb{R}^{n+1}$, we can give some very concrete geometric interpretations of the quantities we have defined so far. We begin with curves. For every unit vector $v \in T_pM$, let $\gamma = \gamma_v : I \to M$ be the g-geodesic in M with initial velocity v. Then the Gauss formula shows that the ordinary Euclidean acceleration of γ at 0 is $\gamma''(0) = \bar{D}_t\gamma'(0) = h(v, v)N_p$. Thus $|h(v, v)|$ is the Euclidean curvature of γ at 0, and $h(v, v) = \langle \gamma''(0), N_p \rangle > 0$ if and only if $\gamma''(0)$ points in the same direction as N_p. In other words, $h(v, v)$ is positive if γ is curving in the direction of N_p, and negative if it is curving away from N_p.

Proposition 3.4.18. *Suppose $\gamma : I \to \mathbb{R}^m$ is a unit-speed curve, $t_0 \in I$, and $\kappa(t_0) \neq 0$.*

(a) There is a unique unit-speed parametrized circle $c : \mathbb{R} \to \mathbb{R}^m$, called the ***osculating circle*** *at $\gamma(\boldsymbol{t_0})$, with the property that c and γ have the same position, velocity, and acceleration at $t = t_0$.*

(b) The Euclidean curvature of γ at t_0 is $\kappa(t_0) = 1/R$, where R is the radius of the osculating circle.

Proof. An easy geometric argument shows that every circle in $\mathbb{R}^m$ with center q and radius R has a unit-speed parametrization of the form

$$c(t) = q + R\cos\left(\frac{t - t_0}{R}\right)v + R\sin\left(\frac{t - t_0}{R}\right)w,$$

where (v, w) is a pair of orthonormal vectors in $\mathbb{R}^m$. By direct computation, such a parametrization satisfies

$$c(t_0) = q + Rv, \quad c'(t_0) = w, \quad c''(t_0) = -\frac{1}{R}v.$$

Thus if we put

$$R = \frac{1}{|\gamma''(t_0)|} = \frac{1}{\kappa(t_0)}, \quad v = -R\gamma''(t_0), \quad w = \gamma'(t_0), \quad q = \gamma(t_0) - Rv$$

we obtain a circle satisfying the required conditions, and its radius is equal to $1/\kappa(t_0)$ by construction. Uniqueness is left as an exercise. ∎

Exercise 3.4.19. *Complete the proof of the preceding proposition by proving uniqueness of the osculating circle.*

Computations in Euclidean Space

When we wish to compute the invariants of a Euclidean hypersurface $M \subseteq \mathbb{R}^{n+1}$, it is usually unnecessary to go to all the trouble of computing Christoffel symbols. Instead, it is usually more effective to use either a defining function or a parametrization to compute the scalar second fundamental form, and then use (**??**) to compute the curvature. Here we describe several contexts in which this computation is not too hard.

Usually the computations are simplest if the hypersurface is presented in terms of a local parametrization. Suppose $M \subseteq \mathbb{R}^{n+1}$ is a smooth embedded hypersurface, and let $X : U \rightarrow \mathbb{R}^{n+1}$ be a smooth local parametrization of M. The coordinates $\left(u^1, \ldots, u^n\right)$ on $U \subseteq \mathbb{R}^n$ thus give local coordinates for M. The coordinate vector fields $\partial_i = \partial / \partial u^i$ push forward to vector fields $dX\left(\partial_i\right)$ on M, which we can view as sections of the restricted tangent bundle $T\mathbb{R}^{n+1}\big|_M$, or equivalently as $\mathbb{R}^{n+1}$-valued functions. If we think of $X(u) = \left(X^1(u), \ldots, X^{n+1}(u)\right)$ as a vector-valued function of u, these vectors can be written as

$$dX_u\left(\partial_i\right) = \partial_i X(u) = \left(\partial_i X^1(u), \ldots, \partial_i X^{n+1}(u)\right).$$

For simplicity, write $X_i = \partial_i X$. Once these vector fields are computed, a unit normal field can be computed as follows: Choose any coordinate vector field $\partial / \partial x^{j_0}$ that is not contained in span $\left(X_1, \ldots, X_n\right)$ (there will always be one, at least in a neighborhood of each point). Then apply the Gram-Schmidt algorithm to the local frame $\left(X_1, \ldots, X_n, \partial / \partial x^{j_0}\right)$ along M to obtain an adapted orthonormal frame $\left(E_1, \ldots, E_{n+1}\right)$. The two choices of unit normal are $N = \pm E_{n+1}$.

The next proposition gives a formula for the second fundamental form that is often easy to use for computation.

Proposition 3.4.20. *Suppose $M \subseteq \mathbb{R}^{n+1}$ is an embedded hypersurface, $X : U \rightarrow M$ is a smooth local parametrization of $M, \left(X_1, \ldots, X_n\right)$ is the local frame for TM determined by X, and N is a unit normal field on M. Then the scalar second fundamental form is given by*

$$h\left(X_i, X_j\right) = \left\langle \frac{\partial^2 X}{\partial u^i \partial u^j}, N \right\rangle.$$

Here is another approach. When it is practical to write down a smooth vector field $N = N^i \partial_i$ on an open subset of $\mathbb{R}^{n+1}$ that restricts to a unit normal vector field along M, then the shape operator can be computed straightforwardly using the Weingarten equation and observing that the Euclidean covariant derivatives of N are just ordinary directional derivatives in Euclidean space. Thus for every vector $X = X^j \partial_j$ tangent to M, we have

$$sX = -\bar{\nabla}_X N = -\sum_{i,j=1}^{n+1} X^j \left(\partial_j N^i\right) \partial_i$$

One common way to produce such a smooth vector field is to work with a local defining function for M : Recall that this is a smooth real-valued function defined on some open subset $U \subseteq \mathbb{R}^{n+1}$ such that $U \cap M$ is a regular level set of F (see [12] Prop. A.27). The definition ensures that grad F (the gradient of F with respect to $\bar{g}$) is nonzero on some neighborhood of $M \cap U$, so a convenient choice for a unit normal vector field along M is

$$N = \frac{\operatorname{grad} F}{|\operatorname{grad} F|}$$

Here is an application.

Example 3.4.21 (Shape Operators of Spheres). The function $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ defined by $F(x) = |x|^2$ is a smooth defining function for each sphere $\mathbb{S}^n(R)$. The gradient of this function is grad $F = 2\sum_i x^i \partial_i$, which has length $2R$ along $\mathbb{S}^n(R)$. The smooth vector field

$$N = \frac{1}{R} \sum_{i=1}^{n+1} x^i \partial_i$$

thus restricts to a unit normal along $\mathbb{S}^n(R)$. (It is the outward pointing normal.) The shape operator is now easy to compute:

$$sX = -\frac{1}{R}\sum_{i,j=1}^{n+1} X^j \left(\partial_j x^i\right)\partial_i = -\frac{1}{R}X.$$

Therefore $s = (-1/R)$ Id. The principal curvatures, therefore, are all equal to $-1/R$, and it follows that the mean curvature is $H = -1/R$ and the Gaussian curvature is $(-1/R)^n$. ♣

For surfaces in $\mathbb{R}^3$, either of the above methods can be used. When a parametrization X is given, the normal vector field is particularly easy to compute: because X_1 and X_2 span the tangent space to M at each point, their cross product is a nonzero normal vector, so one choice of unit normal is

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|}$$

Gauss's Theorema Egregium

Because the Gaussian and mean curvatures are defined in terms of a particular embedding of M into $\mathbb{R}^{n+1}$, there is little reason to suspect that they have much to do with the intrinsic Riemannian geometry of (M, g). The next exercise illustrates the fact that the mean curvature has no intrinsic meaning.

Exercise 3.4.22. *Let $M_1 \subseteq \mathbb{R}^3$ be the plane $\{z = 0\}$, and let $M_2 \subseteq \mathbb{R}^3$ be the cylinder $\left\{x^2 + y^2 = 1\right\}$. Show that M_1 and M_2 are locally isometric, but the former has mean curvature zero, while the latter has mean curvature $\pm\frac{1}{2}$, depending on which normal is chosen.*

The amazing discovery made by Gauss was that the Gaussian curvature of a surface in $\mathbb{R}^3$ is actually an intrinsic invariant of the Riemannian manifold (M, g). He was so impressed with this discovery that he called it Theorema Egregium, Latin for "excellent theorem."

Theorem 3.4.23 (Gauss's Theorema Egregium)**.** *Suppose (M,g) is an embedded 2-dimensional Riemannian submanifold of $\mathbb{R}^3$. For every $p \in M$, the Gaussian curvature of M at p is equal to one-half the scalar curvature of g at p, and thus the Gaussian curvature is a local isometry invariant of (M, g).*

Motivated by the Theorema Egregium, for an abstract Riemannian 2-manifold (M, g), not necessarily embedded in $\mathbb{R}^3$, we define the Gaussian curvature to be $K = \frac{1}{2}S$, where S is the scalar curvature. If M is a Riemannian submanifold of $\mathbb{R}^3$, then the Theorema Egregium shows that this new definition agrees with the original definition of K as the determinant of the shape operator.

Corollary 3.4.24. *If (M, g) is a Riemannian 2-manifold, the following relationships hold:*

$$Rm = \frac{1}{2}Kg \owedge g, \quad Rc = Kg, \quad S = 2K.$$

3.4.3 Sectional Curvature

Now, finally, we can give a quantitative geometric interpretation to the curvature tensor in dimensions higher than 2 . Suppose M is a Riemannian n-manifold (with $n \geq 2$), p is a point of M, and $V \subseteq T_pM$ is a star-shaped neighborhood of zero on which $\exp_p$ is a diffeomorphism onto an open set $U \subseteq M$. Let Π be any 2dimensional linear subspace of T_pM. Since $\Pi \cap V$ is an embedded 2-dimensional submanifold of V, it follows that $S_\Pi = \exp_p(\Pi \cap V)$ is an embedded 2-dimensional submanifold of $U \subseteq M$ containing p, called the **plane section determined by** Π. Note that S_Π is just the set swept out by geodesics whose initial velocities lie in Π, and T_pS_Π is exactly Π.

We define the sectional curvature of Π, denoted by $\sec(\Pi)$, to be the intrinsic Gaussian curvature at p of the surface S_Π with the metric induced from the embedding $S_\Pi \subseteq M$. If (v, w) is any basis for Π, we also use the notation $\sec(v, w)$ for $\sec(\Pi)$.

The next theorem shows how to compute the sectional curvatures in terms of the curvature of (M, g). To make the formula more concise, we introduce the following notation. Given vectors v, w in an inner product space V, we set

$$|v \wedge w| = \sqrt{|v|^2|w|^2 - \langle v, w\rangle^2}$$

It follows from the Cauchy-Schwarz inequality that $|v \wedge w| \geq 0$, with equality if and only if v and w are linearly dependent, and $|v \wedge w| = 1$ when v and w are orthonormal.

Proposition 3.4.25 (Formula for the Sectional Curvature)**.** *Let (M, g) be a Riemannian manifold and $p \in M$. If v, w are linearly independent vectors in T_pM, then the sectional curvature of the plane spanned by v and w is given by*

$$\sec(v, w) = \frac{Rm_p(v, w, w, v)}{|v \wedge w|^2} \tag{3.76}$$

Exercise 3.4.26. *Suppose (M, g) is a Riemannian manifold and $\widetilde{g} = \lambda g$ for some positive constant λ. Use Theorem 7.30 to prove that for every $p \in M$ and plane $\Pi \subseteq T_pM$, the sectional curvatures of Π with respect to $\widetilde{g}$ and g are related by $\widetilde{\sec}(\Pi) = \lambda^{-1}\sec(\Pi)$.*

The formula for the sectional curvature shows that one important piece of quantitative information provided by the curvature tensor is that it encodes the sectional curvatures of all plane sections. It turns out, in fact, that this is all of the information contained in the curvature tensor: as the following proposition shows, the sectional curvatures completely determine the curvature tensor.

Proposition 3.4.27. *Suppose R_1 and R_2 are algebraic curvature tensors on a finitedimensional inner product space V. Iffor every pair of linearly independent vectors $v, w \in V$,*

$$\frac{R_1(v, w, w, v)}{|v \wedge w|^2} = \frac{R_2(v, w, w, v)}{|v \wedge w|^2}$$

then $R_1 = R_2$.

Proposition 3.4.28 (Geometric Interpretation of Ricci and Scalar Curvatures)**.** *Let (M, g) be a Riemannian n-manifold and $p \in M$.*

(a) For every unit vector $v \in T_pM$, $Rc_p(v, v)$ is the sum of the sectional curvatures of the 2-planes spanned by $(v, b_2), \ldots, (v, b_n)$, where $(b_1, \ldots, b_n)$ is any orthonormal basis for T_pM with $b_1 = v$.

(b) The scalar curvature at p is the sum of all sectional curvatures of the 2-planes spanned by ordered pairs of distinct basis vectors in any orthonormal basis.

Proof. Given any unit vector $v \in T_pM$, let $(b_1, \ldots, b_n)$ be as in the hypothesis. Then $Rc_p(v, v)$ is given by

$$Rc_p(v, v) = R_{11}(p) = R_{k11}{}^{k}(p) = \sum_{k=1}^{n} Rm_p(b_k, b_1, b_1, b_k) = \sum_{k=2}^{n} \sec(b_1, b_k)$$

For the scalar curvature, we let $(b_1, \ldots, b_n)$ be any orthonormal basis for T_pM, and compute

$$S(p) = R_j{}^{j}(p) = \sum_{j=1}^{n} Rc_p(b_j, b_j) = \sum_{j,k=1}^{n} Rm_p(b_k, b_j, b_j, b_k) = \sum_{j \neq k} \sec(b_j, b_k).$$

■

One consequence of this proposition is that if (M, g) is a Riemannian manifold in which all sectional curvatures are positive, then the Ricci and scalar curvatures are both positive as well. The analogous statement holds if "positive" is replaced by "negative," "nonpositive," or "nonnegative."

If the opposite sign convention is chosen for the curvature tensor, then the righthand side of formula (3.76) has to be adjusted accordingly, with $Rm_p(v, w, v, w)$ taking the place of $Rm_p(v, w, w, v)$. This is so that whatever sign convention is chosen for the curvature tensor, the notion of positive or negative sectional, Ricci, or scalar curvature has the same meaning for everyone.

Sectional Curvatures of the Model Spaces

3.4.4 The Gauss-Bonnet Theorem

3.5 Geodesics and Jacobi Fields

3.5.1 Lengths

To say that $\gamma : I \to M$ is a **smooth curve** is to say that it is smooth as a map from the manifold (with boundary) I to M. If I has one or two endpoints and M has empty boundary, then γ is smooth if and only if it extends to a smooth curve defined on some open interval containing I. (If $\partial M \neq \varnothing$, then smoothness of γ has to be interpreted as meaning that each coordinate representation of γ has a smooth extension to an open interval.) A **curve segment** is a curve whose domain is a compact interval. A smooth curve $\gamma : I \to M$ has a well-defined velocity $\gamma'(t) \in T_{\gamma(t)}M$ for each $t \in I$. We say that γ is a **regular curve** if $\gamma'(t) \neq 0$ for $t \in I$. This implies that γ is an immersion, so its image has no "corners" or "kinks." If M is a smooth manifold with or without boundary, a (continuous) curve segment $\gamma : [a, b] \to M$ is said to be **piecewise regular** if there exists a partition $(a_0, \ldots, a_k)$ of $[a, b]$ such that $\gamma|_{[a_{i-1}, a_i]}$ is a regular curve segment (meaning it is smooth with nonvanishing velocity) for $i = 1, \ldots, k$. For brevity, we refer to a piecewise regular curve segment as an **admissible curve**, and any partition $(a_0, \ldots, a_k)$ such that $\gamma|_{[a_{i-1}, a_i]}$ is smooth for each i an **admissible partition for** γ. (There are many admissible partitions for a given admissible curve, because we can always add more points to the partition.) It is also convenient to consider any map $\gamma : \{a\} \to M$ whose domain is a single real number to be an admissible curve.

Suppose γ is an admissible curve and $(a_0, \ldots, a_k)$ is an admissible partition for it. At each of the intermediate partition points $a_1, \ldots, a_{k-1}$, there are two one-sided velocity vectors, which we denote by $\gamma'\left(a_i^-\right) = \lim_{t \nearrow a_i} \gamma'(t)$, $\gamma'\left(a_i^+\right) = \lim_{t \searrow a_i} \gamma'(t)$. They are both nonzero, but they need not be equal.

If $\gamma : I \to M$ is a smooth curve, we define a **reparametrization of** γ to be a curve of the form $\widetilde{\gamma} = \gamma \circ \varphi : I' \to M$, where $I' \subseteq \mathbb{R}$ is another interval and $\varphi : I' \to I$ is a diffeomorphism. Because intervals are connected, φ is either strictly increasing or strictly decreasing on I'. We say that $\widetilde{\gamma}$ is a **forward reparametrization** if φ is increasing, and a **backward reparametrization** if it is decreasing.

For an admissible curve $\gamma : [a, b] \to M$, we define a **reparametrization of** γ a little more broadly, as a curve of the form $\widetilde{\gamma} = \gamma \circ \varphi$, where $\varphi : [c, d] \to [a, b]$ is a homeomorphism for which there is a partition $(c_0, \ldots, c_k)$ of $[c, d]$ such that the restriction of φ to each subinterval $[c_{i-1}, c_i]$ is a diffeomorphism onto its image.

If $\gamma : [a, b] \to M$ is an admissible curve, we define the length of γ to be

$$L_g(\gamma) = \int_a^b \left|\gamma'(t)\right|_g dt$$

The integrand is bounded and continuous everywhere on $[a, b]$ except possibly at the finitely many points where γ is not smooth, so this integral is well defined.

Proposition 3.5.1 (Properties of Lengths)**.** *Suppose (M, g) is a Riemannian manifold with or without boundary, and $\gamma : [a, b] \to M$ is an admissible curve.*

(a) ADDITIVITY OF LENGTH: If $a < c < b$, then $L_g(\gamma) = L_g\left(\gamma|_{[a,c]}\right) + L_g\left(\gamma|_{[c,b]}\right)$.

(b) PARAMETER INDEPENDENCE: If $\widetilde{\gamma}$ is a reparametrization of γ, then $L_g(\gamma) = L_g(\widetilde{\gamma})$.

(c) *ISOMETRY INVARIANCE: If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds (with or without boundary) and $\varphi : M \to \widetilde{M}$ is a local isometry, then $L_g(\gamma) = L_{\widetilde{g}}(\varphi \circ \gamma)$.*

Exercise 3.5.2. *Prove above proposition.*

Suppose $\gamma : [a, b] \to M$ is an admissible curve. The **arc-length function of** γ is the function $s : [a, b] \to \mathbb{R}$ defined by

$$s(t) = L_g\left(\gamma|_{[a,t]}\right) = \int_a^t |\gamma'(u)|_g \, \mathrm{d}u. \tag{3.77}$$

It is continuous everywhere, and it follows from the fundamental theorem of calculus that it is smooth wherever γ is, with derivative $s'(t) = |\gamma'(t)|$. For this reason, if $\gamma : I \to M$ is any smooth curve (not necessarily a curve segment), we define the **speed of** γ at any time $t \in I$ to be the scalar $|\gamma'(t)|$. We say that γ is a **unit-speed curve** if $|\gamma'(t)| = 1$ for all t, and a **constant-speed curve** if $|\gamma'(t)|$ is constant. If γ is a piecewise smooth curve, we say that γ has unit speed if $|\gamma'(t)| = 1$ wherever γ is smooth. If $\gamma : [a, b] \to M$ is a unit-speed admissible curve, then its arc-length function has the simple form $s(t) = t - a$. If, in addition, its parameter interval is of the form $[0, b]$ for some $b > 0$, then the arc-length function is $s(t) = t$. For this reason, a unit-speed admissible curve whose parameter interval is of the form $[0, b]$ is said to be **parametrized by arc length**.

Proposition 3.5.3. *Suppose (M, g) is a Riemannian manifold with or without boundary.*

(a) *Every regular curve in M has a unit-speed forward reparametrization.*

(b) *Every admissible curve in M has a unique forward reparametrization by arc length.*

Proof. (a): For a regular curve $\gamma : I = [a, b] \to M$ we define $s(t)$ as in (3.77) and notice that $s'(t) = |\gamma'(t)| > 0$ due to regularity of the curve (γ' never vanishes). Thus, $s'(t) \neq 0$, making $s(t)$ a local diffeomorphism due to the inverse function theorem; $s'(t) > 0$ means $s(t)$ is strictly increasing and is thus injective. If we write $s : I \to s(I) =: I'$ then s is bijective. The (toy version of) global rank theorem tells us that s must be a diffeomorphism from I to I'. Then let $\varphi = s^{-1}$ be the reparametrization $\widetilde{\gamma} = \gamma \circ \varphi : I' \to M$ of $\gamma : I \to M$. Since inverse function theorem also gives the formula for the derivative of the inverse function, we see

$$|\widetilde{\gamma}'(s)| = |\gamma'(\varphi(s))\varphi'(s)| = |\gamma'(\varphi(s))| \left| \frac{1}{s'(\varphi(s))} \right| = 1,$$

showing that $\widetilde{\gamma}$ is a unit-speed reparametrization of γ.

(b): notice that in (a), $s(a) = 0$. If we denote $s(b) = c$, then the curve $\widetilde{\gamma}$ is a curve $[0, c] \to M$ parametrized by arc length. This works for the case where there is only one smooth segment in the admissible curve γ. If there are more than one, do induction. The details are shown in [12, Proposition 2.49]. ■

3.5.2 Riemannian Distance

Suppose (M, g) is a connected Riemannian manifold with or without boundary. For each pair of points $p, q \in M$, we define the **Riemannian distance from** p **to** q, denoted by $d_g(p, q)$, to be the infimum of the lengths of all admissible curves from p to q. The following proposition guarantees that $d_g(p, q)$ is a well-defined nonnegative real number for each $p, q \in M$.

Proposition 3.5.4. *If M is a connected smooth manifold (with or without boundary), then any two points of M can be joined by an admissible curve.*

For convenience, if (M, g) is a disconnected Riemannian manifold, we also let $d_g(p, q)$ denote the Riemannian distance from p to q, provided that p and q lie in the same connected component of M.

Proposition 3.5.5 (Isometry Invariance of the Riemannian Distance Function)**.** *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is an isometry. Then $d_{\widetilde{g}}(\varphi(x), \varphi(y)) = d_g(x, y)$ for all $x, y \in M$.*

Remark 3.5.6. Note that unlike lengths of curves, Riemannian distances are not necessarily preserved by local isometries. ♠

Theorem 3.5.7 ([12] Theorem 2.55). *Let (M, g) be a connected Riemannian manifold with or without boundary. With the distance function d_g, M is a metric space whose metric topology is the same as the given manifold topology.*

Thanks to the preceding theorem, it makes sense to apply all the concepts of the theory of metric spaces to a connected Riemannian manifold (M, g). For example, we say that M is **(metrically) complete** if every Cauchy sequence in M converges. A subset $A \subseteq M$ is **bounded** if there is a constant C such that $d_g(p, q) \leq C$ for all $p, q \in A$; if this is the case, the **diameter of** A is the smallest such constant, i.e., $\operatorname{diam}(A) = \sup \{d_g(p, q) : p, q \in A\}$. Since every compact metric space is bounded, every compact connected Riemannian manifold with or without boundary has finite diameter.

3.5.3 Geodesics and Minimizing Curves

Let (M, g) be a Riemannian manifold. An admissible curve γ in M is said to be a **minimizing curve** if $L_g(\gamma) \leq L_g(\widetilde{\gamma})$ for every admissible curve $\widetilde{\gamma}$ with the same endpoints. When M is connected, it follows from the definition of the Riemannian distance that γ is minimizing if and only if $L_g(\gamma)$ is equal to the distance between its endpoints.

Our first goal in this section is to show that all minimizing curves are geodesics. To do so, we will think of the length function L_g as a functional on the set of all admissible curves in M with fixed starting and ending points. (Real-valued functions whose domains are themselves sets of functions are typically called **functionals**.) Our project is to search for minima of this functional.

Families of Curves

Given intervals $I, J \subseteq \mathbb{R}$, a continuous map $\Gamma : J \times I \to M$ is called a **one-parameter family of curves**. Such a family defines two collections of curves in M: the **main curves** $\Gamma_s(t) = \Gamma(s, t)$ defined for $t \in I$ by holding s constant, and the **transverse curves** $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined for $s \in J$ by holding t constant.

If such a family Γ is smooth (or at least continuously differentiable), we denote the velocity vectors of the main and transverse curves by

$$\partial_t \Gamma(s, t) = (\Gamma_s)'(t) \in T_{\Gamma(s,t)} M; \quad \partial_s \Gamma(s, t) = \Gamma^{(t)\prime}(s) \in T_{\Gamma(s,t)} M.$$

Each of these is an example of a **vector field along** Γ, which is a continuous map $V : J \times I \to TM$ such that $V(s, t) \in T_{\Gamma(s,t)} M$ for each (s, t).

The families of curves that will interest us most in this chapter are of the following type. A one-parameter family Γ is called an **admissible family of curves** if (i) its domain is of the form $J \times [a, b]$ for some open interval J; (ii) there is a partition $(a_0, \ldots, a_k)$ of $[a, b]$ such that Γ is smooth on each rectangle of the form $J \times [a_{i-1}, a_i]$; and (iii) $\Gamma_s(t) = \Gamma(s, t)$ is an admissible curve for each $s \in J$ (Fig. 6.1). Every such partition is called an admissible partition for the family.

If $\gamma : [a, b] \to M$ is a given admissible curve, a **variation of** γ is an admissible family of curves $\Gamma : J \times [a, b] \to M$ such that J is an open interval containing 0 and $\Gamma_0 = \gamma$. It is called a **proper variation** if in addition, all of the main curves have the same starting and ending points: $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all $s \in J$.

In the case of an admissible family, the transverse curves are smooth on J for each t, but the main curves are in general only piecewise regular. Thus the velocity vector fields $\partial_s \Gamma$ and $\partial_t \Gamma$ are smooth on each rectangle $J \times [a_{i-1}, a_i]$, but not generally on the whole domain.

We can say a bit more about $\partial_s \Gamma$, though. If Γ is an admissible family, a **piecewise smooth vector field along** Γ is a (continuous) vector field along Γ whose restriction to each rectangle $J \times [a_{i-1}, a_i]$ is smooth

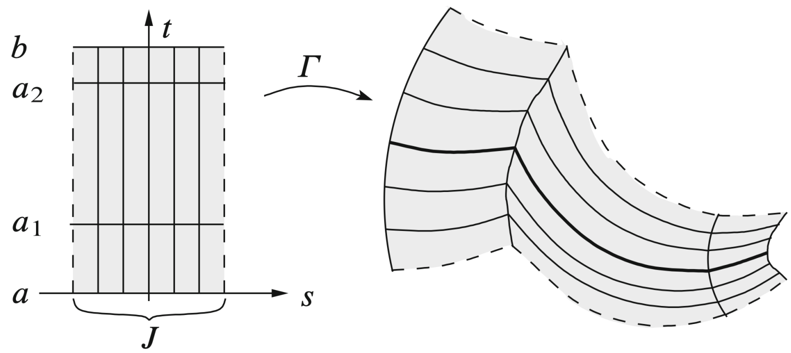

Figure 3.3: Admissble family of curves.

for some admissible partition $(a_0, \ldots, a_k)$ for Γ. In fact, $\partial_s\Gamma$ is always such a vector field. To see that it is continuous on the whole domain $J \times [a, b]$, note on the one hand that for each $i = 1, \ldots, k-1$, the values of $\partial_s\Gamma$ along the set $J \times \{a_i\}$ depend only on the values of Γ on that set, since the derivative is taken only with respect to the s variable; on the other hand, $\partial_s\Gamma$ is continuous (in fact smooth) on each subrectangle $J \times [a_{i-1}, a_i]$ and $J \times [a_i, a_{i+1}]$, so the right-hand and left-hand limits at $t = a_i$ must be equal. Therefore $\partial_s\Gamma$ is always a piecewise smooth vector field along Γ. (However, $\partial_t\Gamma$ is typically not continuous at $t = a_i$.)

If Γ is a variation of γ, the **variation field of** Γ is the piecewise smooth vector field $V(t) = \partial_s\Gamma(0, t)$ along γ. We say that a vector field V along γ is **proper** if $V(a) = 0$ and $V(b) = 0$; it follows easily from the definitions that the variation field of every proper variation is itself proper.

Lemma 3.5.8. *If γ is an admissible curve and V is a piecewise smooth vector field along γ, then V is the variation field of some variation of γ. If V is proper, the variation can be taken to be proper as well.*

Proof. Suppose γ and V satisfy the hypotheses, and set $\Gamma(s, t) = \exp_{\gamma(t)}(sV(t))$. By compactness of $[a, b]$, there is some positive ε such that Γ is defined on $(-\varepsilon, \varepsilon) \times [a, b]$. By composition, Γ is smooth on $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ for each subinterval $[a_{i-1}, a_i]$ on which V is smooth, and it is continuous on its whole domain. By the properties of the exponential map, the variation field of Γ is V. Moreover, if $V(a) = 0$ and $V(b) = 0$, the definition gives $\Gamma(s, a) \equiv \gamma(a)$ and $\Gamma(s, b) \equiv \gamma(b)$, so Γ is proper. ■

If V is a piecewise smooth vector field along Γ, we can compute the covariant derivative of V either along the main curves (at points where V is smooth) or along the transverse curves; the resulting vector fields along Γ are denoted by $\underline{D_tV \text{ and } D_sV}$ respectively.

A key ingredient in the proof that minimizing curves are geodesics is the symmetry of the Levi-Civita connection. It enters into our proofs in the form of the following lemma. (Although we state and use this lemma only for the Levi-Civita connection, the proof shows that it is actually true for every symmetric connection in TM.)

Lemma 3.5.9 (Symmetry Lemma)**.** *Let $\Gamma : J \times [a, b] \to M$ be an admissible family of curves in a Riemannian manifold. On every rectangle $J \times [a_{i-1}, a_i]$ where Γ is smooth, $D_s(\partial_t\Gamma) = D_t(\partial_s\Gamma)$.*

Proof. This is a local question, so we may compute in local coordinates (x^i) around a point $\Gamma(s_0, t_0)$. Writing the components of Γ as $\Gamma(s, t) = (x^1(s, t), \ldots, x^n(s, t))$, we have

$$\partial_t\Gamma = \frac{\partial x^k}{\partial t}\partial_k; \quad \partial_s\Gamma = \frac{\partial x^k}{\partial s}\partial_k.$$

Then, using the coordinate formula (4.2) for covariant derivatives along curves, we obtain

$$D_s\partial_t\Gamma = \left(\frac{\partial^2 x^k}{\partial s\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma_{ji}^k\right)\partial_k;\ D_t\partial_s\Gamma = \left(\frac{\partial^2 x^k}{\partial t\partial s} + \frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma_{ji}^k\right)\partial_k.$$

Now, the lemma follows from the following

$$\frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma_{ji}^k = \frac{\partial x^j}{\partial s}\frac{\partial x^i}{\partial t}\Gamma_{ji}^k \xlongequal{i\leftrightarrow j} \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma_{ij}^k \xlongequal{\text{Problem 3.6.7}} \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma_{ji}^k$$

■

Minimizing Curves are Geodesics

We can now compute an expression for the derivative of the length functional along a variation of a curve. Traditionally, the derivative of a functional on a space of maps is called its **first variation**.

Theorem 3.5.10 (First Variation Formula)**.** *Let (M, g) be a Riemannian manifold. Suppose $\gamma : [a, b] \to M$ is a unit-speed admissible curve, $\Gamma : J \times [a, b] \to M$ is a variation of γ, and V is its variation field (Fig. 3.5). Then $L_g(\Gamma_s)$ is a smooth function of s, and*

$$\left.\frac{d}{ds}\right|_{s=0} L_g(\Gamma_s) = -\int_a^b \langle V, D_t\gamma'\rangle\, dt - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\gamma'\rangle + \langle V(b), \gamma'(b)\rangle - \langle V(a), \gamma'(a)\rangle, \tag{3.78}$$

where $(a_0, \ldots, a_k)$ is an admissible partition for V, and for each $i = 1, \ldots, k-1$, $\Delta_i\gamma' = \gamma'(a_i^+) - \gamma'(a_i^-)$ is the "jump" in the velocity vector field γ' at a_i (Fig.3.4). In particular, if Γ is a proper variation, then

$$\left.\frac{d}{ds}\right|_{s=0} L_g(\Gamma_s) = -\int_a^b \langle V, D_t\gamma'\rangle\, dt - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\gamma'\rangle. \tag{3.79}$$

Figure 3.4: $\Delta_i\gamma'$ is the "jump" in γ' at a_i

Proof. On every rectangle $J \times [a_{i-1}, a_i]$ where Γ is smooth, since the integrand in $L_g(\Gamma_s)$ is smooth and the domain of integration is compact, we can differentiate under the integral sign as many times as we wish. Because $L_g(\Gamma_s)$ is a finite sum of such integrals, it follows that it is a smooth function of s. For brevity, let us introduce the notations (see the blue and red v.f. in Fig. 3.5)

$$T(s,t) = \partial_t\Gamma(s,t), \quad S(s,t) = \partial_s\Gamma(s,t).$$

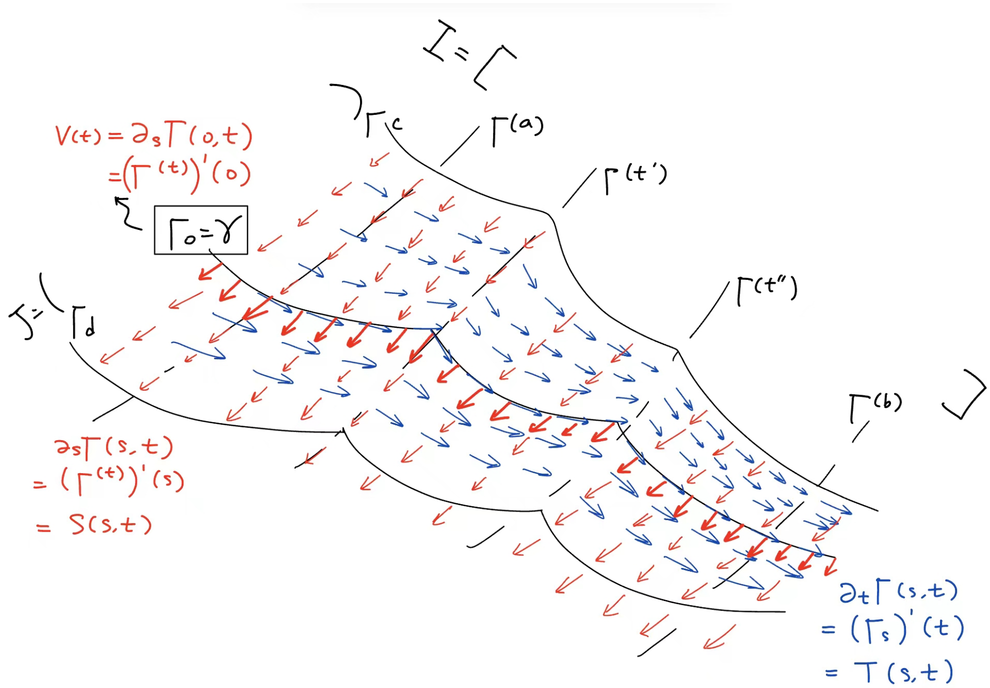

Figure 3.5: Vector fields V, S, and T

Differentiating on the interval $[a_{i-1}, a_i]$ yields

$$\begin{aligned}\frac{d}{ds} L_g \left(\Gamma_s|_{[a_{i-1},a_i]} \right) &= \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial s} \langle T, T \rangle^{1/2} dt \\ &\overset{(3.17)}{=\!=} \int_{a_{i-1}}^{a_i} \frac{1}{2} \langle T, T \rangle^{-1/2} 2 \langle D_s T, T \rangle \, dt \\ &\overset{\text{Lemma 3.5.9}}{=\!=} \int_{a_{i-1}}^{a_i} \frac{1}{|T|} \langle D_t S, T \rangle \, dt\end{aligned} \tag{3.80}$$

where we have used the symmetry lemma in the last line. Setting $s = 0$ and noting that $S(0,t) = V(t)$ and $T(0,t) = \gamma'(t)$ (which has length 1 given by assumption of the theorem), we get

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} L_g \left(\Gamma_s|_{[a_{i-1},a_i]} \right) &= \int_{a_{i-1}}^{a_i} \langle D_t V, \gamma' \rangle \, dt \\ &\overset{(3.17)}{=\!=} \int_{a_{i-1}}^{a_i} \left(\frac{d}{dt} \langle V, \gamma' \rangle - \langle V, D_t \gamma' \rangle \right) dt \\ &\overset{\text{FTC}}{=\!=} \left\langle V(a_i), \gamma'(a_i^-) \right\rangle - \left\langle V(a_{i-1}), \gamma'(a_{i-1}^+) \right\rangle - \int_{a_{i-1}}^{a_i} \langle V, D_t \gamma' \rangle \, dt.\end{aligned}$$

Finally, summing over i, we obtain (3.78). ■

Because every admissible curve has a unit-speed parametrization and length is independent of parametrization, the requirement in the above proposition that γ be of unit speed is not a real restriction, but rather just a computational convenience.

Theorem 3.5.11. *In a Riemannian manifold, every minimizing curve is a geodesic when it is given a unit-speed parametrization.*

Proof. Suppose $\gamma : [a, b] \to M$ is minimizing and of unit speed (so that we can use previous theorem), and $(a_0, \ldots, a_k)$ is an admissible partition for γ. If Γ is any proper variation of γ, then $L_g(\Gamma_s)$ is a smooth function of s that achieves its minimum at $s = 0$ (we are given that γ is minimizing), so it follows from elementary calculus that $d(L_g(\Gamma_s))/ds = 0$ when $s = 0$. Since every proper vector field along γ is the variation field of some proper variation (Lemma 3.5.8), the right-hand side of (3.79) must vanish for every such V.

First we show that $D_t\gamma' = 0$ on each subinterval $[a_{i-1}, a_i]$, so γ is a "broken geodesic." Choose one such interval, and let $\varphi \in C^\infty(\mathbb{R})$ be a bump function such that $\varphi > 0$ on (a_{i-1}, a_i) and $\varphi = 0$ elsewhere. Then (3.79) with $V = \varphi D_t\gamma'$ (which is proper and we can thus apply the last sentence of the first paragraph) becomes

$$0 = -\int_{a_{i-1}}^{a_i} \varphi \left|D_t\gamma'\right|^2 dt \qquad (*)$$

Since the integrand is nonnegative and $\varphi > 0$ on (a_{i-1}, a_i), this shows that $D_t\gamma' = 0$ on each such subinterval.

Next we need to show that $\Delta_i\gamma' = 0$ for each i between 0 and k, which is to say that γ has no corners. For each such i, we can use a bump function in a coordinate chart to construct a piecewise smooth vector field V along γ such that $V(a_i) = \Delta_i\gamma'$ and $V(a_j) = 0$ for $j \neq i$. Then (6.2) reduces to $-\left|\Delta_i\gamma'\right|^2 = 0$, so $\Delta_i\gamma' = 0$ for each i.

Finally, since the two one-sided velocity vectors of γ match up at each a_i, it follows from uniqueness of geodesics that $\gamma|_{[a_i, a_{i+1}]}$ is the continuation of the geodesic $\gamma|_{[a_{i-1}, a_i]}$, and therefore γ is smooth. ∎

The preceding proof has an enlightening geometric interpretation. Under the assumption that $D_t\gamma' \neq 0$, the first variation with $V = \varphi D_t\gamma'$ is negative (RHS of $(*)$), which shows that deforming γ in the direction of its acceleration vector field (since $\varphi > 0$) decreases its length (Fig. 3.6). Similarly, the length of a broken geodesic γ is decreased by deforming it in the direction of a vector field V such that $V(a_i) = \Delta_i\gamma'$ (Fig. 3.7). Geometrically, this corresponds to "rounding the corner."

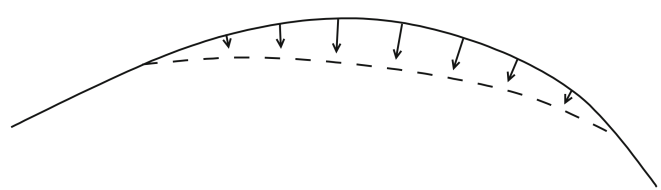

Figure 3.6: Deforming in the direction of the acceleration.

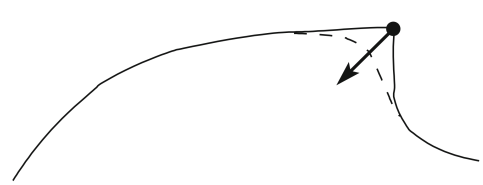

Figure 3.7: Rounding the corner

The first variation formula actually tells us a bit more than is claimed in Theorem 3.5.11. In proving that γ is a geodesic, we did not use the full strength of the assumption that the length of Γ_s takes a minimum when $s=0$; we only used the fact that its derivative is zero. We say that an admissible curve γ is a **critical point of** L_g if for every proper variation Γ_s of γ, the derivative of $L_g\left(\Gamma_s\right)$ with respect to s is zero at $s=0$. Therefore we can strengthen Theorem 3.5.11 in the following way.

Corollary 3.5.12. *A unit-speed admissible curve γ is a critical point for L_g if and only if it is a geodesic.*

Proof. If γ is a critical point, the proof of Theorem 3.5.11 goes through without modification to show that γ is a geodesic. Conversely, if γ is a geodesic, then the first term on the right-hand side of (3.79) vanishes by the geodesic equation, and the second term vanishes because γ' has no jumps. ■

The geodesic equation $D_t \gamma'=0$ thus characterizes the critical points of the length functional. In general, the equation that characterizes critical points of a functional on a space of maps is called the **variational equation** or the **Euler-Lagrange equation** of the functional. Many interesting equations in differential geometry arise as variational equations.

3.5.4 Geodesics Are Locally Minimizing

Next we turn to the converse of Theorem 3.5.11. It is easy to see that the literal converse is not true, because not every geodesic segment is minimizing. For example, every geodesic segment on $\mathbb{S}^n$ that goes more than halfway around the sphere is not minimizing, because the other portion of the same great circle is a shorter curve segment between the same two points. For that reason, we concentrate initially on local minimization properties of geodesics.

As usual, let (M, g) be a Riemannian manifold. A regular (or piecewise regular) curve $\gamma: I \rightarrow M$ is said to be **locally minimizing** if every $t_0 \in I$ has a neighborhood $I_0 \subseteq I$ such that whenever $a, b \in I_0$ with $a<b$, the restriction of γ to $[a, b]$ is minimizing.

Lemma 3.5.13. *Every minimizing admissible curve segment is locally minimizing.*

Exercise 3.5.14. *Prove the preceding lemma.*

Our goal in this section is to show that geodesics are locally minimizing. The proof will be based on a careful analysis of the geodesic equation in Riemannian normal coordinates.

If ε is a positive number such that $\exp_p$ is a diffeomorphism from the ball $B_{\varepsilon}(0) \subseteq T_p M$ to its image (where the radius of the ball is measured with respect to the norm defined by g_p), then the image set $\exp_p\left(B_{\varepsilon}(0)\right)$ is a normal neighborhood of p, called a **geodesic ball in** M, or sometimes an **open geodesic ball** for clarity.

Also, if the closed ball $\bar{B}_{\varepsilon}(0)$ is contained in an open set $V \subseteq T_p M$ on which $\exp_p$ is a diffeomorphism onto its image, then $\exp_p\left(\bar{B}_{\varepsilon}(0)\right)$ is called a **closed geodesic ball**, and $\exp_p\left(\partial B_{\varepsilon}(0)\right)$ is called a **geodesic sphere**. Given such a V, by compactness there exists $\varepsilon'>\varepsilon$ such that $B_{\varepsilon'}(0) \subseteq V$, so every closed geodesic ball is contained in an open geodesic ball of larger radius. In Riemannian normal coordinates centered at p, the open and closed geodesic balls and geodesic spheres centered at p are just the coordinate balls and spheres.

Suppose U is a normal neighborhood of $p \in M$. Given any normal coordinates $\left(x^i\right)$ on U centered at p, define the **radial distance function** $r: U \rightarrow \mathbb{R}$ by

$$r(x)=\sqrt{\left(x^1\right)^2+\cdots+\left(x^n\right)^2}, \tag{3.81}$$

and the **radial vector field** on $U \backslash\{p\}$, denoted by ∂_r, by

$$\partial_r=\frac{x^i}{r(x)} \frac{\partial}{\partial x^i} \tag{3.82}$$

In Euclidean space, $r(x)$ is the distance to the origin, and ∂_r is the unit vector field pointing radially outward from the origin. (The notation is suggested by the fact that ∂_r is a coordinate derivative in polar or spherical coordinates.)

Lemma 3.5.15. *In every normal neighborhood U of $p \in M$, the radial distance function and the radial vector field are well defined, independently of the choice of normal coordinates. Both r and ∂_r are smooth on $U \backslash \{p\}$, and r^2 is smooth on all of U.*

Proof. Proposition 3.2.18 shows that any two normal coordinate charts on U are related by $\widetilde{x}^i = A^i_j x^j$ for some orthogonal matrix $\left(A^i_j\right)$, and a straightforward computation shows that both r and ∂_r are invariant under such coordinate changes. The smoothness statements follow directly from the coordinate formulas. ■

The crux of the proof that geodesics are locally minimizing is the following deceptively simple geometric lemma.

Theorem 3.5.16 (The Gauss Lemma)**.** *Let (M, g) be a Riemannian manifold, let U be a geodesic ball centered at $p \in M$, and let ∂_r denote the radial vector field on $U \backslash \{p\}$. Then ∂_r is a unit vector field orthogonal to the geodesic spheres in $U \backslash \{p\}$.*

Proof. We will work entirely in normal coordinates $\left(x^i\right)$ on U centered at p, using the properties of normal coordinates described in Proposition 3.2.19.

Let $q \in U \backslash \{p\}$ be arbitrary, with coordinate representation $\left(q^1, \ldots, q^n\right)$, and let $b = r(q) = \sqrt{\left(q^1\right)^2 + \cdots + \left(q^n\right)^2}$, where r is the radial distance function defined by (3.82). It follows that $\left.\partial_r\right|_q$ has the coordinate representation

$$\left.\partial_r\right|_q = \frac{q^i}{b} \left.\frac{\partial}{\partial x^i}\right|_q .$$

Let $v = v^i \left.\partial_i\right|_p \in T_p M$ be the tangent vector at p with components $v^i = q^i / b$. By Proposition 3.2.19(c), the radial geodesic with initial velocity v is given in these coordinates by

$$\gamma_v(t) = \left(t v^1, \ldots, t v^n\right).$$

It satisfies $\gamma_v(0) = p, \gamma_v(b) = q$, and $\gamma_v'(b) = v^i \left.\partial_i\right|_q = \left.\partial_r\right|_q$. Because g_p is equal to the Euclidean metric in these coordinates, we have

$$\left|\gamma_v'(0)\right|_g = |v|_g = \sqrt{\left(v^1\right)^2 + \cdots + \left(v^n\right)^2} = \frac{1}{b} \sqrt{\left(q^1\right)^2 + \cdots + \left(q^n\right)^2} = 1,$$

so v is a unit vector, and thus γ_v is a unit-speed geodesic. It follows that $\left.\partial_r\right|_q = \gamma_v'(b)$ is also a unit vector.

To prove that ∂_r is orthogonal to the geodesic spheres let q, b, and v be as above, and let $\Sigma_b = \exp_p\left(\partial B_b(0)\right)$ be the geodesic sphere containing q. In these coordinates, Σ_b is the set of points satisfying the equation $\left(x^1\right)^2 + \cdots + \left(x^n\right)^2 = b^2$. Let $w \in T_q M$ be any vector tangent to Σ_b at q. We need to show that $\left\langle w, \left.\partial_r\right|_q \right\rangle_g = 0$.

Choose a smooth curve $\sigma : (-\varepsilon, \varepsilon) \rightarrow \Sigma_b$ satisfying $\sigma(0) = q$ and $\sigma'(0) = w$, and write its coordinate representation in $\left(x^i\right)$-coordinates as $\sigma(s) = \left(\sigma^1(s), \ldots, \sigma^n(s)\right)$. The fact that $\sigma(s)$ lies in Σ_b for all s means that

$$\left(\sigma^1(s)\right)^2 + \cdots + \left(\sigma^n(s)\right)^2 = b^2.$$

Define a smooth map $\Gamma : (-\varepsilon, \varepsilon) \times [0, b] \rightarrow U$ (Fig.3.8) by

$$\Gamma(s, t) = \left(\frac{t}{b} \sigma^1(s), \ldots, \frac{t}{b} \sigma^n(s)\right).$$

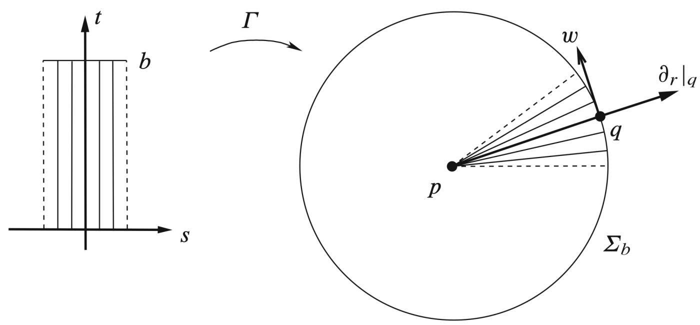

Figure 3.8: Proof of Gauss lemma.

For each $s \in (-\varepsilon, \varepsilon), \Gamma_s$ is a geodesic by Proposition 3.2.19(c). Its initial velocity is $\Gamma_s'(0) = (1/b)\sigma^i(s)\partial_i \mid p$, which is a unit vector by (6.6) and the fact that g_p is the Euclidean metric in coordinates; thus each Γ_s is a unit-speed geodesic. As before, let $S = \partial_s \Gamma$ and $T = \partial_t \Gamma$. It follows from the definitions that

$$\begin{aligned}
S(0,0) &= \left.\frac{d}{ds}\right|_{s=0} \Gamma_s(0) = 0; \\
T(0,0) &= \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t) = v; \\
S(0,b) &= \left.\frac{d}{ds}\right|_{s=0} \sigma(s) = w; \\
T(0,b) &= \left.\frac{d}{dt}\right|_{t=b} \gamma_v(t) = \gamma_v'(b) = \left.\partial_r\right|_q .
\end{aligned}$$

Therefore $\langle S, T\rangle$ is zero when $(s,t) = (0,0)$ and equal to $\left\langle w, \left.\partial_r\right|_q \right\rangle$ when $(s,t) = (0,b)$, so to prove the theorem it suffices to show that $\langle S, T\rangle$ is independent of t. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\langle S, T\rangle &= \langle D_t S, T\rangle + \langle S, D_t T\rangle && \text{(compatibility with the metric)} \\
&= \langle D_s T, T\rangle + \langle S, D_t T\rangle && \text{(symmetry lemma)} \\
&= \langle D_s T, T\rangle + 0 && \text{(each } \Gamma_s \text{ is a geodesic)} \\
&= \frac{1}{2}\frac{\partial}{\partial s}|T|^2 = 0 && (|T| = |\Gamma_s'| \equiv 1 \text{ for all } (s,t)) .
\end{aligned} \tag{3.83}$$

This proves the theorem. ■

We will use the Gauss lemma primarily in the form of the next corollary.

Corollary 3.5.17. *Let U be a geodesic ball centered at $p \in M$, and let r and ∂_r be the radial distance and radial vector field as defined by (3.81) and (3.82). Then $\operatorname{grad} r = \partial_r$ on $U \backslash \{p\}$.*

Proof. By the result of Problem 2.6.10, it suffices to show that ∂_r is orthogonal to the level sets of r and $\partial_r(r) \equiv |\partial_r|_g^2$. The first claim follows directly from the Gauss lemma, and the second from the fact that $\partial_r(r) \equiv 1$ by direct computation in normal coordinates, which in turn is equal to $|\partial_r|_g^2$ by the Gauss lemma. ■

Here is the payoff: our first step toward proving that geodesics are locally minimizing. Note that this is not yet the full strength of the theorem we are aiming for, because it shows only that for each point on a geodesic, sufficiently small segments of the geodesic starting at that point are minimizing. We will remove this restriction after a little more work below.

Proposition 3.5.18. *Let (M, g) be a Riemannian manifold. Suppose $p \in M$ and q is contained in a geodesic ball around p. Then (up to reparametrization) the radial geodesic from p to q is the unique minimizing curve in M from p to q.*

Proof. Choose $\varepsilon > 0$ such that $\exp_p(B_\varepsilon(0))$ is a geodesic ball containing q. Let $\gamma : [0, c] \to M$ be the radial geodesic from p to q parametrized by arc length, and write $\gamma(t) = \exp_p(tv)$ for some unit vector $v \in T_pM$. Then $L_g(\gamma) = c$, since γ has unit speed.

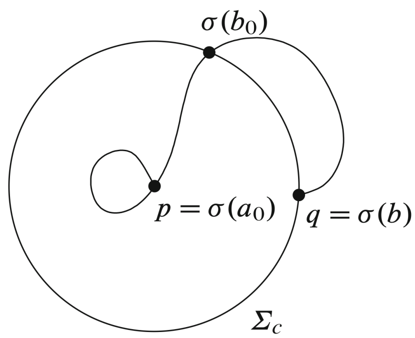

Figure 3.9: Radial geodesics are minimizing.

To show that γ is minimizing, we need to show that every other admissible curve from p to q has length at least c. Let $\sigma : [0, b] \to M$ be an arbitrary admissible curve from p to q, which we may assume to be parametrized by arc length as well. Let $a_0 \in [0, b]$ denote the last time that $\sigma(t) = p$, and $b_0 \in [0, b]$ the first time after a_0 that $\sigma(t)$ meets the geodesic sphere Σ_c of radius c around p (Fig.3.9). Then the composite function $r \circ \sigma$ is continuous on $[a_0, b_0]$ and piecewise smooth in (a_0, b_0), so we can apply the fundamental theorem of calculus to conclude that

$$\begin{aligned} r(\sigma(b_0)) - r(\sigma(a_0)) &= \int_{a_0}^{b_0} \frac{d}{dt} r(\sigma(t)) dt = \int_{a_0}^{b_0} dr(\sigma'(t))\, dt \\ &= \int_{a_0}^{b_0} \left\langle \operatorname{grad} r|_{\sigma(t)}, \sigma'(t) \right\rangle dt \leq \int_{a_0}^{b_0} |\operatorname{grad} r|_{\sigma(t)}||\sigma'(t)|\, dt \\ &= \int_{a_0}^{b_0} |\sigma'(t)|\, dt = L_g\left(\sigma|_{[a_0, b_0]}\right) \leq L_g(\sigma) \end{aligned} \tag{3.84}$$

Thus $L_g(\sigma) \geq r(\sigma(b_0)) - r(\sigma(a_0)) = c$, so γ is minimizing. Now suppose $L_g(\sigma) = c$. Then $b = c$, and both inequalities in (3.84) are equalities. Because we assume that σ is a unit-speed curve, the second of these equalities implies that $a_0 = 0$ and $b_0 = b = c$, since otherwise the segments of σ before $t = a_0$ and after $t = b_0$ would contribute positive lengths. The first equality then implies that the nonnegative expression $|\operatorname{grad} r|_{\sigma(t)}||\sigma'(t)| - \left\langle \operatorname{grad} r|_{\sigma(t)}, \sigma'(t) \right\rangle$ is identically zero on $[0, b]$, which is possible only if $\sigma'(t)$ is a positive multiple of grad $r|_{\sigma(t)}$ for each t. Since we assume that σ has unit speed, we must have $\sigma'(t) = \operatorname{grad} r|_{\sigma(t)} = \partial_r|_{\sigma(t)}$. Thus σ and γ are both integral curves of ∂_r passing through q at time $t = c$, so $\sigma = \gamma$. ∎

The next two corollaries show how radial distance functions, balls, and spheres in normal coordinates are related to their global metric counterparts.

Corollary 3.5.19. *Let (M, g) be a connected Riemannian manifold and $p \in M$. Within every open or closed geodesic ball around p, the radial distance function $r(x)$ defined by (3.81) is equal to the Riemannian distance from p to x in M.*

Proof. Since every closed geodesic ball is contained in an open geodesic ball of larger radius, we need only consider the open case. If x is in the open geodesic ball $\exp_p(B_c(0))$, the radial geodesic γ from p to x is minimizing by Proposition 3.5.18. Since its velocity is equal to ∂_r, which is a unit vector in both the g-norm and the Euclidean norm in normal coordinates, the g-length of γ is equal to its Euclidean length, which is $r(x)$. ∎

Corollary 3.5.20. *In a connected Riemannian manifold, every open or closed geodesic ball is also an open or closed metric ball of the same radius, and every geodesic sphere is a metric sphere of the same radius.*

Proof. Let (M, g) be a Riemannian manifold, and let $p \in M$ be arbitrary. First, let $V = \exp_p\left(\bar{B}_c(0)\right) \subseteq M$ be a closed geodesic ball of radius $c > 0$ around p. Suppose q is an arbitrary point of M. If $q \in V$, then Corollary 3.5.19 shows that q is also in the closed metric ball of radius c. Conversely, suppose $q \notin V$. Let S be the geodesic sphere $\exp_p(\partial B_c(0))$. The complement of S is the disjoint union of the open sets $\exp_p(B_c(0))$ and $M \backslash \exp_p\left(\bar{B}_c(0)\right)$, and hence disconnected. Thus if $\gamma : [a, b] \to M$ is any admissible curve from p to q, there must be a time $t_0 \in (a, b)$ when $\gamma(t_0) \in S$, and then Corollary 3.5.19 shows that the length of $\left.\gamma\right|_{[a,t_0]}$ must be at least c. Since $\left.\gamma\right|_{[t_0,b]}$ must have positive length, it follows that $d_g(p, q) > c$, so q is not in the closed metric ball of radius c around p.

Next, let $W = \exp_p(B_c(0))$ be an open geodesic ball of radius c. Since W is the union of all closed geodesic balls around p of radius less than c, and the open metric ball of radius c is similarly the union of all closed metric metric balls of smaller radii, the result of the preceding paragraph shows that W is equal to the open metric ball of radius c.

Finally, if $S = \exp_p(\partial B_c(0))$ is a geodesic sphere of radius c, the arguments above show that S is equal to the closed metric ball of radius c minus the open metric ball of radius c, which is exactly the metric sphere of radius c. ∎

The last corollary suggests a simplified notation for geodesic balls and spheres in M. From now on, we will use the notations $B_c(p) = \exp_p(B_c(0))$, $\bar{B}_c(p) = \exp_p\left(\bar{B}_c(0)\right)$, and $S_c(p) = \exp_p(\partial B_c(0))$ for open and closed geodesic balls and geodesic spheres, which we now know are also open and closed metric balls and spheres. (To avoid confusion, we refrain from using this notation for other metric balls and spheres unless explicitly stated.)

In order to prove that geodesics in (M, g) are locally minimizing, we need the following refinement of the concept of normal neighborhoods. A subset $W \subseteq M$ is called **uniformly normal** if there exists some $\delta > 0$ such that W is contained in a geodesic ball of radius δ around each of its points (Fig. 3.10). If δ is any such constant, we will also say that W is **uniformly δ-normal**. Clearly every subset of a uniformly δ-normal set is itself uniformly δ-normal.

Lemma 3.5.21 (Uniformly Normal Neighborhood Lemma)**.** *Given $p \in M$ and any neighborhood U of p, there exists a uniformly normal neighborhood of p contained in U.*

Proof. See [12, Lemma 6.14]. ∎

Theorem 3.5.22. *Every Riemannian geodesic is locally minimizing.*

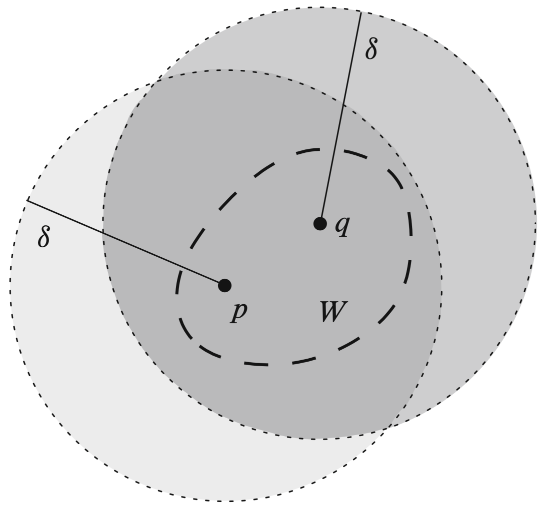

Figure 3.10: Uniformly normal Neighborhood.

Proof. Let (M, g) be a Riemannian manifold. Suppose $\gamma : I \to M$ is a geodesic, which we may assume to be defined on an open interval, and let $t_0 \in I$. Let W be a uniformly normal neighborhood of $\gamma(t_0)$, and let $I_0 \subseteq I$ be the connected component of $\gamma^{-1}(W)$ containing t_0. If $a, b \in I_0$ with $a < b$, then the definition of uniformly normal neighborhood implies that the image of $\gamma|_{[a,b]}$ is contained in a geodesic ball centered at $\gamma(a)$ (Fig. 3.11).

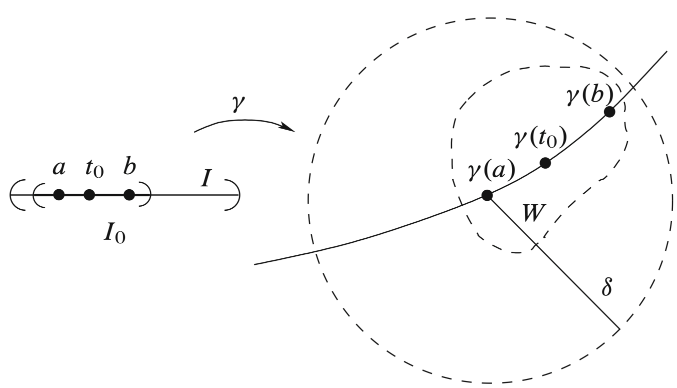

Figure 3.11: Geodesics are locally minimizing.

Proposition 3.2.19 shows that every geodesic segment lying in that ball and starting at $\gamma(a)$ is part of a radial geodesic, and Proposition 3.5.18 shows that each radial geodesic segment is minimizing. However, the restriction of γ to $[a, b]$ is also a geodesic segment from $\gamma(a)$ to $\gamma(b)$ lying in the same geodesic ball, and thus $\gamma|_{[a,b]}$ must coincide with this minimizing geodesic. ■

Given a Riemannian manifold (M, g) (without boundary), for each point $p \in M$ we define the **injectivity radius of** M **at** p, denoted by $\operatorname{inj}(p)$, to be the supremum of all $a > 0$ such that $\exp_p$ is a diffeomorphism from $B_a(0) \subseteq T_pM$ onto its image.

If there is no upper bound to the radii of such balls (as is the case, for example, on $\mathbb{R}^n$), then we set

$\operatorname{inj}(p)=\infty$. Then we define the **injectivity radius of** M, denoted by $\operatorname{inj}(M)$, to be the infimum of $\operatorname{inj}(p)$ as p ranges over points of M. It can be zero, positive, or infinite. (The terminology is explained by Problem 3.6.19.)

Lemma 3.5.23. *If (M,g) is a compact Riemannian manifold, then $\operatorname{inj}(M)$ is positive.*

Proof. For each $x\in M$, there is a positive number $\delta(x)$ such that x is contained in a uniformly $\delta(x)$-normal neighborhood W_x, and $\operatorname{inj}(x')\geq\delta(x)$ for each $x'\in W$. Since M is compact, it is covered by finitely many such neighborhoods $W_{x_1},\ldots,W_{x_k}$. Therefore, $\operatorname{inj}(M)$ is at least equal to the minimum of $\delta(x_1),\ldots,\delta(x_k)$. It cannot be infinite, because a compact metric space is bounded, and a geodesic ball of radius c contains points whose distances from the center are arbitrarily close to c. ■

In addition to uniformly normal neighborhoods, there is another, more specialized, kind of normal neighborhood that is frequently useful. Let (M,g) be a Riemannian manifold. A subset $U\subseteq M$ is said to be **geodesically convex** if for each $p,q\in U$, there is a unique minimizing geodesic segment from p to q in M, and the image of this geodesic segment lies entirely in U.

The next theorem says that every sufficiently small geodesic ball is geodesically convex.

Theorem 3.5.24. *Let (M,g) be a Riemannian manifold. For each $p\in M$, there exists $\varepsilon_0>0$ such that every geodesic ball centered at p of radius less than or equal to ε_0 is geodesically convex.*

Proof. Exercise. ■

3.5.5 Completeness

Recall that such a manifold is said to be **geodesically complete** if every maximal geodesic is defined for all $t\in\mathbb{R}$. For clarity, we will use the phrase **metrically complete** for a connected Riemannian manifold that is complete as a metric space with the Riemannian distance function, in the sense that every Cauchy sequence converges.

The Hopf-Rinow theorem, which we will state and prove below, shows that these two notions of completeness are equivalent for connected Riemannian manifolds. Before we prove it, let us establish a preliminary result, which will have other important consequences besides the Hopf-Rinow theorem itself.

Lemma 3.5.25. *Suppose (M,g) is a connected Riemannian manifold, and there is a point $p\in M$ such that $\exp_p$ is defined on the whole tangent space T_pM. Then*

(a) Given any other $q\in M$, there is a minimizing geodesic segment from p to q.

(b) M is metrically complete.

Remark 3.5.26. We can call the assumption in the lemma as "**locally geodesically complete**." ♠

Proof. Let $q\in M$ be arbitrary. If $\gamma:[a,b]\to M$ is a geodesic segment starting at p, let us say that γ **aims at** q if γ is minimizing and

$$d_g(p,q)=d_g(p,\gamma(b))+d_g(\gamma(b),q) \tag{3.85}$$

(This would be the case, for example, if γ were an initial segment of a minimizing geodesic from p to q; but we are not assuming that.) To prove (a), it suffices to show that there is a geodesic segment $\gamma:[a,b]\to M$ that begins at p, aims at q, and has length equal to $d_g(p,q)$, for then the fact that γ is minimizing means that $d_g(p,\gamma(b))=L_g(\gamma)=d_g(p,q)$, and (3.85) becomes

$$d_g(p,q)=d_g(p,q)+d_g(\gamma(b),q),$$

which implies $\gamma(b)=q$. Since γ is a segment from p to q of length $d_g(p,q)$, it is the desired minimizing geodesic segment.

Choose $\varepsilon > 0$ such that there is a closed geodesic ball $\bar{B}_\varepsilon(p)$ around p that does not contain q. Since the distance function on a metric space is continuous, there is a point x in the geodesic sphere $S_\varepsilon(p)$ where $d_g(x, q)$ attains its minimum on the compact set $S_\varepsilon(p)$. Let γ be the maximal unit-speed geodesic whose restriction to $[0, \varepsilon]$ is the radial geodesic segment from p to x (Fig. 3.12); by assumption, γ is defined for all $t \in \mathbb{R}$.

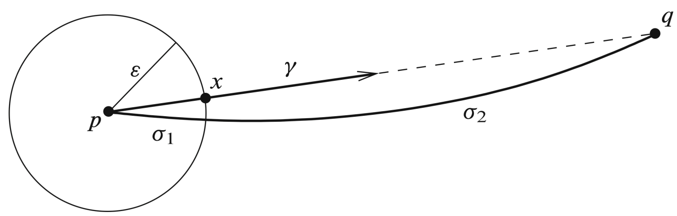

Figure 3.12: Proof that $\gamma|_{[0,\varepsilon]}$ aims at q.

We begin by showing that $\gamma|_{[0,\varepsilon]}$ aims at q. Since it is minimizing by Proposition 3.5.18 (noting that every closed geodesic ball is contained in a larger open one), we need only show that (3.85) holds with $b = \varepsilon$, or

$$d_g(p, q) = d_g(p, x) + d_g(x, q) \tag{3.86}$$

To this end, let $\sigma : [a_0, b_0] \to M$ be any admissible curve from p to q. Let t_0 be the first time σ hits $S_\varepsilon(p)$, and let σ_1 and σ_2 denote the restrictions of σ to $[a_0, t_0]$ and $[t_0, b_0]$, respectively (Fig. 3.12). Since every point in $S_\varepsilon(p)$ is at a distance ε from p, we have $L_g(\sigma_1) \geq d_g(p, \sigma(t_0)) = d_g(p, x)$; and by our choice of x we have $L_g(\sigma_2) \geq d_g(\sigma(t_0), q) \geq d_g(x, q)$. Putting these two inequalities together yields

$$L_g(\sigma) = L_g(\sigma_1) + L_g(\sigma_2) \geq d_g(p, x) + d_g(x, q)$$

Taking the infimum over all such σ, we find that $d_g(p, q) \geq d_g(p, x) + d_g(x, q)$. The opposite inequality is just the triangle inequality, so (3.86) holds.

Now let $T = d_g(p, q)$ and

$$\mathcal{A} = \left\{ b \in [0, T] : \gamma|_{[0,b]} \text{ aims at } q \right\}$$

We have just shown that $\varepsilon \in \mathcal{A}$. Let $A = \sup \mathcal{A} \geq \varepsilon$. By continuity of the distance function, it is easy to see that $\mathcal{A}$ is closed in $[0, T]$, and therefore $A \in \mathcal{A}$. If $A = T$, then $\gamma|_{[0,T]}$ is a geodesic of length $T = d_g(p, q)$ that aims at q, and by the remark above we are done. So we assume $A < T$ and derive a contradiction.

Let $y = \gamma(A)$, and choose $\delta > 0$ such that there is a closed geodesic ball $\bar{B}_\delta(y)$ around y, small enough that it does not contain q (Fig. 3.13).

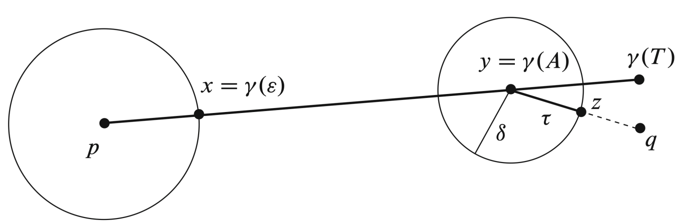

Figure 3.13: Proof that $A = T$.

The fact that $A \in \mathcal{A}$ means that

$$d_g(y, q) = d_g(p, q) - d_g(p, y) = T - A$$

Let $z \in S_\delta(y)$ be a point where $d_g(z, q)$ attains its minimum, and let $\tau : [0, \delta] \to M$ be the unit-speed radial geodesic from y to z. By exactly the same argument as before, τ aims at q, so

$$d_g(z, q) = d_g(y, q) - d_g(y, z) = (T - A) - \delta \tag{3.87}$$

By the triangle inequality and (3.87),

$$\begin{aligned} d_g(p, z) &\geq d_g(p, q) - d_g(z, q) \\ &= T - (T - A - \delta) = A + \delta \end{aligned}$$

Therefore, the admissible curve consisting of $\gamma|_{[0,A]}$ (of length A) followed by τ (of length δ) is a minimizing curve from p to z. This means that it has no corners, so z must lie on γ, and in fact, $z = \gamma(A + \delta)$. But then (3.87) says that

$$d_g(p, q) = T = (A + \delta) + d_g(z, q) = d_g(p, z) + d_g(z, q),$$

so $\gamma|_{[0,A+\delta]}$ aims at q and $A + \delta \in \mathcal{A}$, which is a contradiction. This completes the proof of (a).

To prove (b), we need to show that every Cauchy sequence in M converges. Let (q_i) be a Cauchy sequence in M. For each i, the assumption of local geodesical completeness allows us to get a unit-speed minimizing geodesic $\gamma_i(t) = \exp_p(tv_i)$ from p to q_i. Let $d_i = d_g(p, q_i)$, so that $q_i = \exp_p(d_i v_i)$. Since (q_i) is Cauchy and d_g is continuous, the sequence (d_i) is Cauchy and thus bounded in $\mathbb{R}$. Let (v_i) be a sequence consisting of unit vectors in T_pM. Then the sequence of vectors $(d_i v_i)$ in T_pM is bounded. Therefore a subsequence $(d_{i_k} v_{i_k})$ converges to some $v \in T_pM$ by the Balzano-Weierstrass theorem. By continuity of the exponential map, $q_{i_k} = \exp_p(d_{i_k} v_{i_k}) \to \exp_p v$, and since the original sequence (q_i) is Cauchy, it converges to the same limit.

■

The next theorem is the main result of this section.

Theorem 3.5.27 (Hopf-Rinow). *A connected Riemannian manifold is metrically complete if and only if it is geodesically complete.*

Proof. Let (M, g) be a connected Riemannian manifold. Suppose first that M is geodesically complete. Then in particular, it satisfies the hypothesis of Lemma 3.5.25, so it is metrically complete.

Conversely, suppose M is metrically complete, and assume for the sake of contradiction that it is not geodesically complete. Then there is some unit-speed geodesic $\gamma : [0, b) \to M$ that has no extension to a geodesic on any interval $[0, b')$ for $b' > b$. Let (t_i) be any increasing sequence in $[0, b)$ that approaches b, and set $q_i = \gamma(t_i)$. Since γ is parametrized by arc length, the length of $\gamma|_{[t_i,t_j]}$ is exactly $|t_j - t_i|$, so $d_g(q_i, q_j) \leq |t_j - t_i|$ and (q_i) is a Cauchy sequence in M. By completeness, (q_i) converges to some point $q \in M$. Let W be a uniformly δ-normal neighborhood of q for some $\delta > 0$. Choose j large enough that $t_j > b - \delta$ and $q_j \in W$ (Fig. 3.14).

The fact that $B_\delta(q_j)$ is a geodesic ball means that every unit-speed geodesic starting at q_j exists at least for $t \in [0, \delta)$. In particular, this is true of the geodesic σ with $\sigma(0) = q_j$ and $\sigma'(0) = \gamma'(t_j)$. Define $\widetilde{\gamma} : [0, t_j + \delta) \to M$ by

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(t), & t \in [0, b) \\ \sigma(t - t_j), & t \in (t_j - \delta, t_j + \delta) \end{cases}$$

Note that both expressions on the right-hand side are geodesics, and they have the same position and velocity when $t = t_j$. Therefore, by uniqueness of geodesics, the two definitions agree where they overlap. Since $t_j + \delta > b, \widetilde{\gamma}$ is an extension of γ past b, which is a contradiction. ■

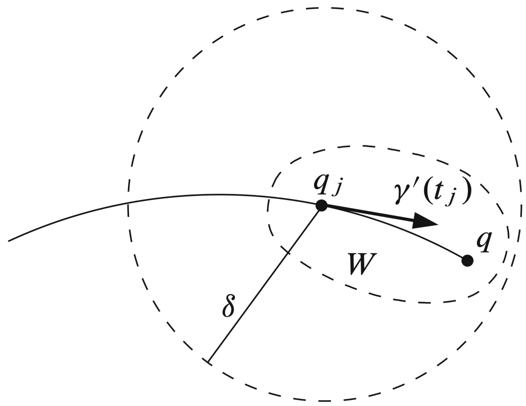

Figure 3.14: γ extends past q.

A connected Riemannian manifold is simply said to be **complete** if it is either geodesically complete or metrically complete; the Hopf-Rinow theorem then implies that it is both. For disconnected manifolds, we interpret "complete" to mean geodesically complete, which is equivalent to the requirement that each component be a complete metric space. As mentioned in the previous chapter, complete manifolds are the natural setting for global questions in Riemannian geometry.

We conclude this section by stating three important corollaries: whose proofs are easy applications of Lemma 3.5.25 and the Hopf-Rinow theorem.

Corollary 3.5.28. *If M is a connected Riemannian manifold and there exists a point $p \in M$ such that the restricted exponential map $\exp_p$ is defined on all of T_pM, then M is complete.*

Corollary 3.5.29. *If M is a complete, connected Riemannian manifold, then any two points in M can be joined by a minimizing geodesic segment.*

Since compact metric spaces are complete, we also have:

Corollary 3.5.30. *If M is a compact Riemannian manifold, then every maximal geodesic in M is defined for all time.*

The Hopf-Rinow theorem and Corollary 3.5.28 are key ingredients in the following theorem about Riemannian covering maps. This theorem will play a key role in the proofs of some of the local-to-global theorems.

Theorem 3.5.31. *Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are connected Riemannian manifolds with $\widetilde{M}$ complete, and $\pi : \widetilde{M} \to M$ is a local isometry. Then M is complete and π is a Riemannian covering map.*

Proof. A fundamental property of covering maps is the path-lifting property: if π is a covering map, then every continuous path $\gamma : I \to M$ lifts to a path $\widetilde{\gamma}$ in $\widetilde{M}$ such that $\pi \circ \widetilde{\gamma} = \gamma$. We begin by proving that π possesses the path-lifting property for geodesics (Fig. 3.15): if $p \in M$ is a point in the image of $\pi, \gamma : I \to M$ is any geodesic starting at p, and $\widetilde{p}$ is any point in $\pi^{-1}(p)$, then γ has a unique lift starting at $\widetilde{p}$. The lifted curve is necessarily also a geodesic because π is a local isometry.

To prove the path-lifting property for geodesics, suppose $p \in \pi(M)$ and $\widetilde{p} \in \pi^{-1}(p)$, and let $\gamma : I \to M$ be any geodesic with $p = \gamma(0)$. Let $v = \gamma'(0)$ and $\widetilde{v} = (d\pi_{\widetilde{p}})^{-1}(v) \in T_{\widetilde{p}}\widetilde{M}$ (which is well defined because $d\pi_{\widetilde{p}}$ is an isomorphism), and let $\widetilde{\gamma}$ be the geodesic in $\widetilde{M}$ with initial point $\widetilde{p}$ and initial velocity $\widetilde{v}$. Because $\widetilde{M}$ is complete, $\widetilde{\gamma}$ is defined on all of $\mathbb{R}$. Since π is a local isometry, it takes geodesics to geodesics; and since by construction $\pi(\widetilde{\gamma}(0)) = \gamma(0)$ and $d\pi_{\widetilde{p}}(\widetilde{\gamma}'(0)) = \gamma'(0)$, we must have $\pi \circ \widetilde{\gamma} = \gamma$ on I, so $\widetilde{\gamma}|_I$ is a lift of γ starting

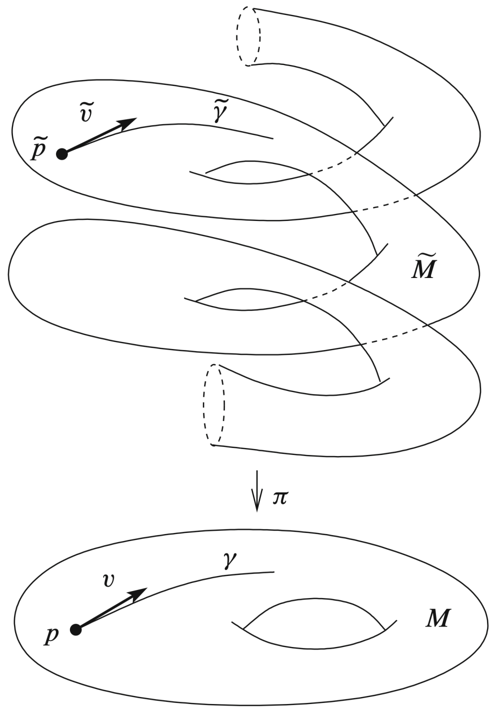

Figure 3.15: Lifting geodesics.

at $\widetilde{p}$. To show that M is complete, let p be any point in the image of π. If $\gamma : I \to M$ is any geodesic starting at p, then γ has a lift $\widetilde{\gamma} : I \to \widetilde{M}$. Because $\widetilde{M}$ is complete, $\pi \circ \widetilde{\gamma}$ is a geodesic defined for all t that coincides with γ on I, so γ extends to all of $\mathbb{R}$. Thus M is complete by Corollary 3.5.28.

Next we show that π is surjective. Choose some point $\widetilde{p} \in \widetilde{M}$, write $p = \pi(\widetilde{p})$, and let $q \in M$ be arbitrary. Because M is connected and complete, there is a minimizing unit-speed geodesic segment γ from p to q. Letting $\widetilde{\gamma}$ be the lift of γ starting at $\widetilde{p}$ and $r = d_g(p,q)$, we have $\pi(\widetilde{\gamma}(r)) = \gamma(r) = q$, so q is in the image of π.

To show that π is a smooth covering map, we need to show that every point of M has a neighborhood U that is evenly covered, which means that $\pi^{-1}(U)$ is a disjoint union of connected open sets $\widetilde{U}_\alpha$ such that $\pi \mid \widetilde{U}_\alpha : \widetilde{U}_\alpha \to U$ is a diffeomorphism. We will show, in fact, that every geodesic ball is evenly covered.

Let $p \in M$, and let $U = B_\varepsilon(p)$ be a geodesic ball centered at p. Write $\pi^{-1}(p) = \{\widetilde{p}_\alpha\}_{\alpha\in A}$, and for each α let $\widetilde{U}_\alpha$ denote the metric ball of radius ε around $\widetilde{p}_\alpha$ (we are not claiming that $\widetilde{U}_\alpha$ is a geodesic ball). The first step is to show that the various sets $\widetilde{U}_\alpha$ are disjoint. For every $\alpha \neq \beta$, there is a minimizing geodesic segment $\widetilde{\gamma}$ from $\widetilde{p}_\alpha$ to $\widetilde{p}_\beta$ because $\widetilde{M}$ is complete. The projected curve $\gamma = \pi \circ \widetilde{\gamma}$ is a geodesic segment that starts and ends at p (Fig. 3.16), whose length is the same as that of $\widetilde{\gamma}$. Such a geodesic must leave U and reenter it (since all geodesics passing through p and lying in U are radial line segments), and thus must have length at least 2ε. This means that $d_{\widetilde{g}}(\widetilde{p}_\alpha, \widetilde{p}_\beta) \geq 2\varepsilon$, and thus by the triangle inequality, $\widetilde{U}_\alpha \cap \widetilde{U}_\beta = \varnothing$.

The next step is to show that $\pi^{-1}(U) = \bigcup_\alpha \widetilde{U}_\alpha$. If $\widetilde{q}$ is any point in $\widetilde{U}_\alpha$, then there is a geodesic $\widetilde{\gamma}$ of length less than ε from $\widetilde{p}_\alpha$ to $\widetilde{q}$, and then $\pi \circ \widetilde{\gamma}$ is a geodesic of the same length from p to $\pi(\widetilde{q})$, showing that $\pi(\widetilde{q}) \in U$. It follows that $\bigcup_\alpha \widetilde{U}_\alpha \subseteq \pi^{-1}(U)$.

Conversely, suppose $\widetilde{q} \in \pi^{-1}(U)$, and set $q = \pi(\widetilde{q})$. This means that $q \in U$, so there is a minimizing radial geodesic γ in U from q to p, and $r = d_g(q,p) < \varepsilon$. Let $\widetilde{\gamma}$ be the lift of γ starting at $\widetilde{q}$ (Fig. 3.16). It follows that $\pi(\widetilde{\gamma}(r)) = \gamma(r) = p$. Therefore $\widetilde{\gamma}(r) = \widetilde{p}_\alpha$ for some α, and $d_{\widetilde{g}}(\widetilde{q}, \widetilde{p}_\alpha) \leq L_g(\widetilde{\gamma}) = r < \varepsilon$, so $\widetilde{q} \in \widetilde{U}_\alpha$.

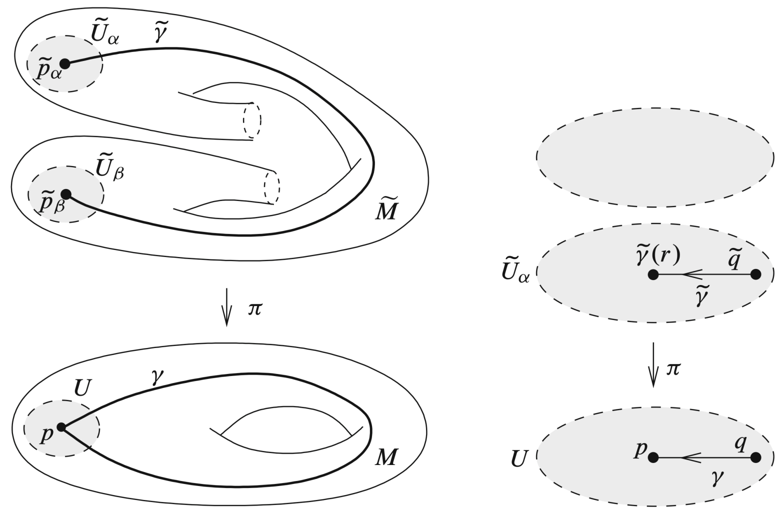

Figure 3.16: Left: proof that $\widetilde{U}_\alpha \cap \widetilde{U}_\beta = \varnothing$; right: proof that $\pi^{-1}(U) \subseteq \bigcup_\alpha \widetilde{U}_\alpha$.

It remains only to show that $\pi : \widetilde{U}_\alpha \to U$ is a diffeomorphism for each α. It is certainly a local diffeomorphism (because π is). It is bijective because its inverse can be constructed explicitly: it is the map sending each radial geodesic starting at p to its lift starting at $\widetilde{p}_\alpha$. This completes the proof. ∎

Corollary 3.5.32. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds, and $\pi : \widetilde{M} \to M$ is a Riemannian covering map. Then M is complete if and only if $\widetilde{M}$ is complete.*

Proof. A Riemannian covering map is, in particular, a local isometry. Thus if $\widetilde{M}$ is complete, π satisfies the hypotheses of Theorem 3.5.31, which implies that M is also complete. ∎

Conversely, suppose M is complete. Let $\widetilde{p} \in \widetilde{M}$ and $\widetilde{v} \in T_p\widetilde{M}$ be arbitrary, and let $p = \pi(\widetilde{p})$ and $v = d\pi_{\widetilde{p}}(\widetilde{v})$. Completeness of M implies that the geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$ is defined for all $t \in \mathbb{R}$, and then its lift $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ starting at $\widetilde{p}$ is a geodesic in $\widetilde{M}$ with initial velocity $\widetilde{v}$, also defined for all t.

Corollary 3.5.29 to the Hopf-Rinow theorem shows that any two points in a complete, connected Riemannian manifold can be joined by a minimizing geodesic segment. The next proposition gives a refinement of that statement.

Proposition 3.5.33. *Suppose (M, g) is a complete, connected Riemannian manifold, and $p, q \in M$. Every path-homotopy class of paths from p to q contains a geodesic segment γ that minimizes length among all admissible curves in the same path-homotopy class.*

Proof. Let $\pi : \widetilde{M} \to M$ be the universal covering manifold of M, endowed with the pullback metric $\widetilde{g} = \pi^* g$. Given $p, q \in M$ and a path $\sigma : [0,1] \to M$ from p to q, choose a point $\widetilde{p} \in \pi^{-1}(p)$, and let $\widetilde{\sigma} : [0,1] \to \widetilde{M}$ be the lift of σ starting at $\widetilde{p}$, and set $\widetilde{q} = \widetilde{\sigma}(1)$. By Corollary 3.5.29, there is a minimizing $\widetilde{g}$-geodesic segment $\widetilde{\gamma}$ from $\widetilde{p}$ to $\widetilde{q}$, and because π is a local isometry, $\gamma = \pi \circ \widetilde{\gamma}$ is a geodesic in M from p to q. If γ_1 is any other admissible curve from p to q in the same path-homotopy class, then by the monodromy theorem (see [10, Theorem 11.15]), its lift $\widetilde{\gamma}_1$ starting at $\widetilde{p}$ also ends at $\widetilde{q}$. Thus $\widetilde{\gamma}_1$ is no shorter than $\widetilde{\gamma}$, which implies γ_1 is no shorter than γ. ∎

3.5.6 Geodesics of the Model Spaces

Euclidean Space $(\mathbb{R}^n, \bar{g})$

The Levi-Civita connection of $(\mathbb{R}^n, \bar{g})$ is the Euclidean connection (by Proposition 3.2.7 (a)):

$$\overline{\nabla}_X Y = X\left(Y^1\right)\frac{\partial}{\partial x^1} + \cdots + X\left(Y^n\right)\frac{\partial}{\partial x^n}$$

Exercise 3.5.34. *Show that the maximal geodesics on $\mathbb{R}^n$ with respect to the Euclidean connection are exactly the constant curves (points) and the straight lines with constant-speed parametrizations.*

Exercise 3.5.35. *Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve, show that the smooh vector fields V along γ that are parallel along γ with respect to the Euclidean connection are exactly the constant-coefficient ones.*

Every Euclidean space is geodesically complete.

Spheres $(\mathbb{S}^n(R), \mathring{g})$

The round metric on the sphere $\mathbb{S}^n(R)$ is induced by the Euclidean metric on $\mathbb{R}^{n+1}$ (Example 2.2.1), so Proposition 3.2.7 (b) implies that the Levi-Civita connection of $(\mathbb{S}^n(R), \mathring{g})$ is the tangential connection (Example 3.1.6)

$$\nabla^\top_X Y := \pi^\top(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}|_M).$$

Define a **great circle** on $\mathbb{S}^n(R)$ to be any subset of the form $\mathbb{S}^n(R) \cap \Pi$, where $\Pi \subseteq \mathbb{R}^{n+1}$ is a 2-dimensional linear subspace.

Proposition 3.5.36. *A nonconstant curve on $\mathbb{S}^n(R)$ is a maximal geodesic if and only if it is a periodic constant-speed curve whose image is a great circle. Thus every sphere is geodesically complete.*

Proof. Let $p \in \mathbb{S}^n(R)$ be arbitrary. Because $f(x) = |x|^2$ is a defining function for $\mathbb{S}^n(R)$, [11, Proposition 5.38] shows that a vector $v \in T_p\mathbb{R}^{n+1}$ is tangent to $\mathbb{S}^n(R)$ if and only if $\mathrm{d}f_p(v) = 2\langle v, p\rangle = 0$, where we think of p as a vector by means of the usual identification of $\mathbb{R}^{n+1}$ with $T_p\mathbb{R}^{n+1}$. Thus $T_p\mathbb{S}^n(R)$ is exactly the set of vectors orthogonal to p.

Suppose v is an arbitrary nonzero vector in $T_p\mathbb{S}^n(R)$. Let $a = |v|/R$ and $\widehat{v} = v/a$ (so $|\widehat{v}| = R$), and consider the smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ given by

$$\gamma(t) = (\cos at)p + (\sin at)\hat{v}$$

By direct computation, $|\gamma(t)|^2 = R^2$, so $\gamma(t) \in \mathbb{S}^n(R)$ for all t. Moreover,

$$\gamma'(t) = -a(\sin at)p + a(\cos at)\hat{v}$$
$$\gamma''(t) = -a^2(\cos at)p - a^2(\sin at)\hat{v}$$

Recall from subsection 3.2.1 that a smooth curve $\gamma : I \to M$ on embedded submanifold M of $\mathbb{R}^n$ is a geodesic with respect to its tangential connection if and only if its ordinary acceleration $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for all $t \in I$. Now, $\gamma''(t)$ is proportional to $\gamma(t)$ (thinking of both as vectors in $\mathbb{R}^{n+1}$), it follows that $\gamma''(t)$ is $\bar{g}$-orthogonal to $T_{\gamma(t)}\mathbb{S}^n(R)$, so γ is a geodesic in $\mathbb{S}^n(R)$. Since $\gamma(0) = p$ and $\gamma'(0) = a\widehat{v} = v$, it follows that $\gamma = \gamma_v$. Each γ_v is periodic of period $2\pi/a$, and has constant speed by Corollary 3.2.2 (or by direct computation). The image of γ_v is the great circle formed by the intersection of $\mathbb{S}^n(R)$ with the linear subspace spanned by $\{p, \widehat{v}\}$, as one can check.

Conversely, suppose C is a great circle formed by intersecting $\mathbb{S}^n(R)$ with a 2-dimensional subspace Π, and let $\{v, w\}$ be an orthonormal basis for Π. Then C is the image of the geodesic with initial point $p = Rw$ and initial velocity v. ∎

Hyperbolic Spaces $(\mathbb{H}^n(R), \breve{g}_R)$

The geodesics of hyperbolic spaces can be determined by an analogous procedure using the hyperboloid model.

Proposition 3.5.37. *A nonconstant curve in a hyperbolic space is a maximal geodesic if and only if it is a constant-speed embedding of $\mathbb{R}$ whose image is one of the following:*

*(a) HYPERBOLOID MODEL: The intersection of $\mathbb{H}^n(R)$ with a 2-dimensional linear subspace of $\mathbb{R}^{n,1}$, called a **great hyperbola** (Fig. 3.17).*

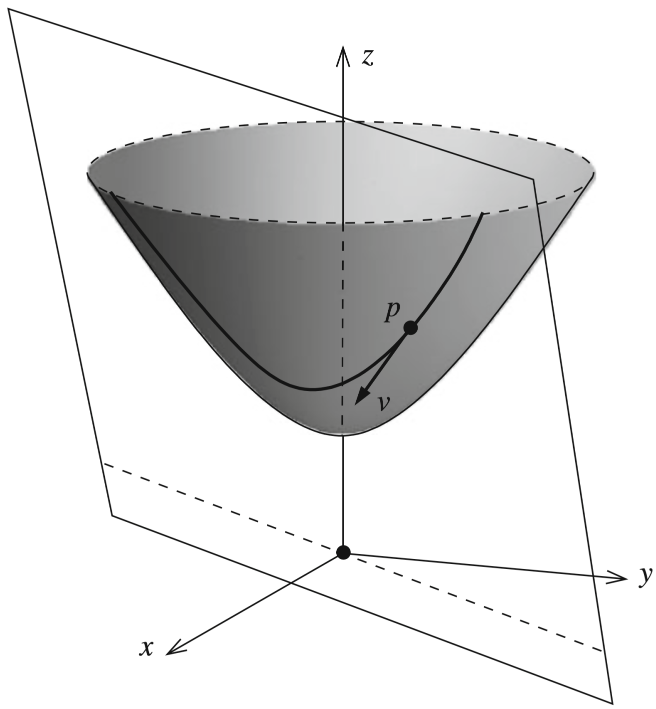

Figure 3.17: A great hyperbola.

(b) BELTRAMI-KLEIN MODEL: The interior of a line segment whose endpoints both lie on $\partial\mathbb{F}^n(R)$ (Fig. 3.18).

(c) BALL MODEL: The interior of a diameter of $\mathbb{B}^n(R)$, or the intersection of $\mathbb{B}^n(R)$ with a Euclidean circle that intersects $\partial\mathbb{B}^n(R)$ orthogonally (Fig. 3.19).

(d) HALF-SPACE MODEL: The intersection of $\mathbb{U}^n(R)$ with one of the following: a line parallel to the y-axis or a Euclidean circle with center on $\partial\mathbb{U}^n(R)$ (Fig. 3.20).

Every hyperbolic space is geodesically complete.

Proof. We begin with the hyperboloid model, for which the proof is formally quite similar to what we just did for the sphere. Since the Riemannian connection on $\mathbb{H}^n(R)$ is equal to the tangential connection by Proposition 5.12, it follows from Corollary 5.2 that a smooth curve $\gamma : I \to \mathbb{H}^n(R)$ is a geodesic if and only if its acceleration $\gamma''(t)$ is everywhere $\bar{q}$-orthogonal to $T_{\gamma(t)}\mathbb{H}^n(R)$ (where $\bar{q} = \bar{q}^{(n,1)}$ is the Minkowski metric).

Let $p \in \mathbb{H}^n(R)$ be arbitrary. Note that $f(x) = \bar{q}(x, x)$ is a defining function for $\mathbb{H}^n(R)$, and (3.10) shows that the gradient of f at p is equal to $2p$ (where we regard p as a vector in $T_p\mathbb{R}^{n,1}$ as before). It follows that a

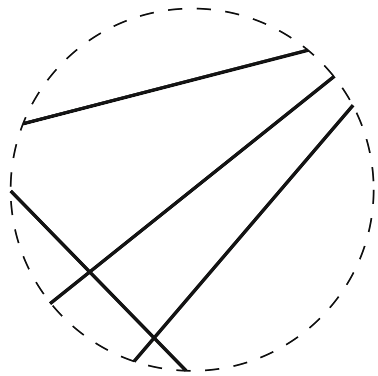

Figure 3.18: Geodesics of $\mathbb{F}^n(R)$.

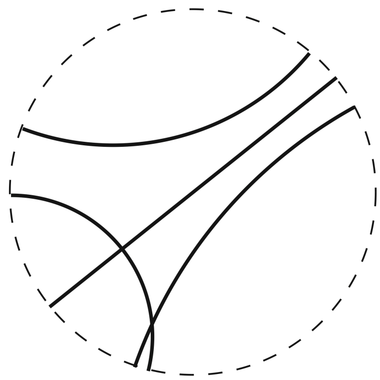

Figure 3.19: Geodesics of $\mathbb{B}^n(R)$.

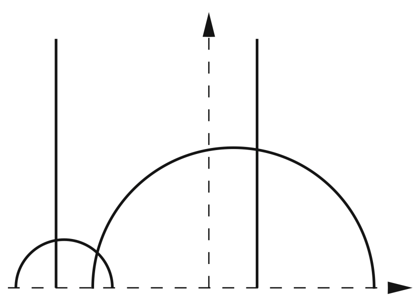

Figure 3.20: Geodesics of $\mathbb{U}^n(R)$.

vector $v \in T_p\mathbb{R}^{n,1}$ is tangent to $\mathbb{H}^n(R)$ if and only if $\bar{q}(p,v) = 0$. Let $v \in T_p\mathbb{H}^n(R)$ be an arbitrary nonzero vector. Put $a = |v|_{\bar{q}}/R = \bar{q}(v,v)^{1/2}/R$ and $\hat{v} = v/a$, and define $\gamma : \mathbb{R} \to \mathbb{R}^{n,1}$ by

$$\gamma(t) = (\cosh at)p + (\sinh at)\hat{v}$$

Direct computation shows that γ takes its values in $\mathbb{H}^n(R)$ and that its acceleration vector is everywhere proportional to $\gamma(t)$. Thus $\gamma''(t)$ is $\bar{q}$-orthogonal to $T_{\gamma(t)}\mathbb{H}^n(R)$, so γ is a geodesic in $\mathbb{H}^n(R)$ and therefore has constant speed. Because it satisfies the initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$, it is equal to γ_v. Note that γ_v is a smooth embedding of $\mathbb{R}$ into $\mathbb{H}^n(R)$ whose image is the great hyperbola formed by the intersection between $\mathbb{H}^n(R)$ and the plane spanned by $\{p, \widehat{v}\}$.

Conversely, suppose Π is any 2-dimensional linear subspace of $\mathbb{R}^{n,1}$ that has nontrivial intersection with $\mathbb{H}^n(R)$. Choose $p \in \Pi \cap \mathbb{H}^n(R)$, and let v be another nonzero vector in Π that is $\bar{q}$-orthogonal to p, which implies $v \in T_p\mathbb{H}^n(R)$. Using the computation above, we see that the image of the geodesic γ_v is the great hyperbola formed by the intersection of Π with $\mathbb{H}^n(R)$.

Before considering the other three models, note that since maximal geodesics in $\mathbb{H}^n(R)$ are constant-speed embeddings of $\mathbb{R}$, it follows from naturality that maximal geodesics in each of the other models are also constant-speed embeddings of $\mathbb{R}$. Thus each model is geodesically complete, and to determine the geodesics in the other models we need only determine their images.

Consider the Beltrami-Klein model. Recall the isometry $c : \mathbb{H}^n(R) \to \mathbb{F}^n(R)$ given by $c(\xi, \tau) = R\xi/\tau$ (see (3.11)). The image of a maximal geodesic in $\mathbb{H}^n(R)$ is a great hyperbola, which is the set of points $(\xi, \tau) \in \mathbb{H}^n(R)$ that solve a system of $n-1$ independent linear equations. Simple algebra shows that (ξ, τ) satisfies a linear equation $\alpha_i \xi^i + \beta\tau = 0$ if and only if $w = c(\xi, \tau) = R\xi/\tau$ satisfies the affine equation $\alpha_i w^i = -\beta R$. Thus c maps each great hyperbola onto the intersection of $\mathbb{F}^n(R)$ with an affine subspace of $\mathbb{R}^n$, and since it is the image of a smooth curve, it must be the intersection of $\mathbb{F}^n(R)$ with a straight line.

Next consider the Poincaré ball model. First consider the 2-dimensional case, and recall the inverse hyperbolic stereographic projection $\pi^{-1} : \mathbb{B}^2(R) \to \mathbb{H}^2(R)$ constructed in Chapter 3:

$$\pi^{-1}(u) = (\xi, \tau) = \left(\frac{2R^2 u}{R^2 - |u|^2}, R\frac{R^2 + |u|^2}{R^2 - |u|^2} \right).$$

In this case, a great hyperbola is the set of points on $\mathbb{H}^2(R)$ that satisfy a single linear equation $\alpha_i \xi^i + \beta\tau = 0$. In the special case $\beta = 0$, this hyperbola is mapped by π to a straight line segment through the origin, as can easily be seen from the geometric definition of π. If $\beta \neq 0$, we can assume (after multiplying through by a constant if necessary) that $\beta = -1$, and write the linear equation as $\tau = \alpha_i \xi^i = \alpha \cdot \xi$ (where the dot represents the Euclidean dot product between elements of $\mathbb{R}^2$). Under π^{-1}, this pulls back to the equation

$$R\frac{R^2 + |u|^2}{R^2 - |u|^2} = \frac{2R^2 \alpha \cdot u}{R^2 - |u|^2}$$

on the disk, which simplifies to

$$|u|^2 - 2R\alpha \cdot u + R^2 = 0$$

Completing the square, we can write this as

$$|u - R\alpha|^2 = R^2 \left(|\alpha|^2 - 1 \right) \tag{3.88}$$

If $|\alpha|^2 \leq 1$, this locus is either empty or a point on $\partial\mathbb{B}^2(R)$, so it contains no points in $\mathbb{B}^2(R)$. Since we are assuming that it is the image of a maximal geodesic, we must therefore have $|\alpha|^2 > 1$. In that case, (3.88) is the equation of a circle with center $R\alpha$ and radius $R\sqrt{|\alpha|^2 - 1}$. At a point u_0 where the circle intersects $\partial\mathbb{B}^2(R)$, the three points $0, u_0$, and $R\alpha$ form a triangle with sides $|u_0| = R, |R\alpha|$, and $|u_0 - R\alpha|$ (Fig. **??**), which satisfy the Pythagorean identity by (3.88); therefore the circle meets $\partial\mathbb{B}^2(R)$ in a right angle.

In the higher-dimensional case, a geodesic on $\mathbb{H}^n(R)$ is determined by a 2-plane. If the 2-plane contains the point $(0, \ldots, 0, R)$, then the corresponding geodesic on $\mathbb{B}^n(R)$ is a line through the origin as before. Otherwise, we can use an orthogonal transformation in the ($\xi^1, \ldots, \xi^n$) variables (which preserves $\breve{g}_R$) to move this 2-plane so that it lies in the $\left(\xi^1, \xi^2, \tau\right)$ subspace, and then we are in the same situation as in the 2-dimensional case.

Finally, consider the upper half-space model. The 2-dimensional case is easiest to analyze using complex notation. Recall the complex formula for the Cayley transform $\kappa : \mathbb{U}^2(R) \to \mathbb{B}^2(R)$ given in Chapter 3:

$$\kappa(z) = w = iR\frac{z - iR}{z + iR}$$

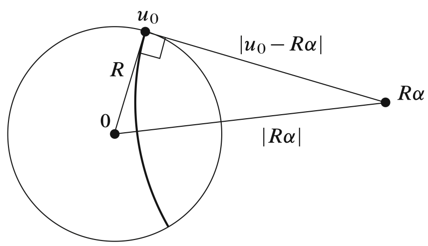

Figure 3.21: Geodesics are arcs of circles orthogonal to the boundary of $\mathbb{H}^2(R)$.

Substituting this into equation (3.88) and writing $w = u + iv$ and $\alpha = a + ib$ in place of $u = \left(u^1, u^2\right), \alpha = \left(\alpha^1, \alpha^2\right)$, we get

$$R^2 \frac{|z - iR|^2}{|z + iR|^2} - iR^2 \bar{\alpha} \frac{z - iR}{z + iR} + iR^2 \alpha \frac{\bar{z} + iR}{\bar{z} - iR} + R^2 |\alpha|^2 = R^2 \left(|\alpha|^2 - 1\right)$$

Multiplying through by $(z + iR)(\bar{z} - iR)/2R^2$ and simplifying yields

$$(1 - b)|z|^2 - 2aRx + (b + 1)R^2 = 0.$$

This is the equation of a circle with center on the x-axis, unless $b = 1$, in which case the condition $|\alpha|^2 > 1$ forces $a \neq 0$, and then it is a straight line $x =$ constant. The other class of geodesics on the ball, line segments through the origin, can be handled similarly.

In the higher-dimensional case, suppose first that $\gamma : \mathbb{R} \to \mathbb{U}^n(R)$ is a maximal geodesic such that $\gamma(0)$ lies on the y-axis and $\gamma'(0)$ is in the span of $\left\{\partial/\partial x^1, \partial/\partial y\right\}$. From the explicit formula (3.15) for κ, it follows that $\kappa \circ \gamma(0)$ lies on the v-axis in the ball, and $(\kappa \circ \gamma)'(0)$ is in the span of $\left\{\partial/\partial u^1, \partial/\partial v\right\}$. The image of the geodesic $\kappa \circ \gamma$ is either part of a line through the origin or an arc of a circle perpendicular to $\partial \mathbb{B}^n(R)$, both of which are contained in the $\left(u^1, v\right)$-plane. By the argument

in the preceding paragraph, it then follows that the image of γ is contained in the $\left(x^1, y\right)$-plane and is either a vertical half-line or a semicircle centered on the $y = 0$ hyperplane. For the general case, note that translations and orthogonal transformations in the x-variables preserve vertical half-lines and circles centered on the $y = 0$ hyperplane in $\mathbb{U}^n(R)$, and they also preserve the metric $\breve{g}_R^3$. Given an arbitrary maximal geodesic $\gamma : \mathbb{R} \to \mathbb{U}^n(R)$, after applying an x-translation we may assume that $\gamma(0)$ lies on the y-axis, and after an orthogonal transformation in the x variables, we may assume that $\gamma'(0)$ is in the span of $\left\{\partial/\partial x^1, \partial/\partial y\right\}$; then the argument above shows that the image of γ is either a vertical half-line or a semicircle centered on the $y = 0$ hyperplane. ■

3.6 Problems

Exercise 3.6.1 ([12] 5-1)**.** *Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and let ∇ be its Levi-Civita connection. Suppose $\widetilde{\nabla}$ is another connection on TM, and D is the difference tensor between ∇ and $\widetilde{\nabla}$ (Prop.**??**). Let D^b denote the covariant 3-tensor field defined by $D^b(X, Y, Z) = \langle D(X, Y), Z\rangle$. Show that $\widetilde{\nabla}$ is compatible with g if and only if D^b is antisymmetric in its last two arguments: $D^\flat(X, Y, Z) = -D^\flat(X, Z, Y)$ for all $X, Y, Z \in \mathfrak{X}(M)$. Conclude that on every Riemannian or pseudo-Riemannian manifold of dimension at least 2, the space of metric connections is an infinite-dimensional affine space.*

Exercise 3.6.2 ([12] 5-8)**.** *Let G be a Lie group and g its Lie algebra. Suppose $\mathfrak{g}$ is a bi-invariant Riemannian metric on G, and $\langle\cdot, \cdot\rangle$ is the corresponding inner product on $\mathfrak{g}$ (see [12] Proposition 3.12). Let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denote the adjoint representation of $\mathfrak{g}$ (see [12] Appendix C).*

(a) Show that $\operatorname{ad}(X)$ *is a skew-adjoint endomorphism of* $\mathfrak{g}$ *for every* $X \in \mathfrak{g}$*:*

$$\langle \operatorname{ad}(X)Y, Z\rangle = -\langle Y, \operatorname{ad}(X)Z\rangle.$$

[Hint: Take the derivative of $\left\langle \operatorname{Ad}\left(\exp^G tX\right) Y, \operatorname{Ad}\left(\exp^G tX\right) Z\right\rangle$ *with respect to* t *at* $t = 0$*, where* $\exp^G$ *is the Lie group exponential map of* G*, and use the fact that* $\operatorname{Ad}_* = \operatorname{ad}$*.]*

(b) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$ *whenever* X *and* Y *are left-invariant vector fields on* G*.*

(c) Show that the geodesics of g *starting at the identity are exactly the one-parameter subgroups. Conclude that under the canonical isomorphism of* $\mathfrak{g} \cong T_e G$ *described in [12] Proposition C.3, the restricted Riemannian exponential map at the identity coincides with the Lie group exponential* $\operatorname{map} \exp^G : \mathfrak{g} \to G$*. (See [12] Proposition C.7.)*

(d) Let $\mathbb{R}^+$ *be the set of positive real numbers, regarded as a Lie group under multiplication. Show that* $g = t^{-2}dt^2$ *is a bi-invariant metric on* $\mathbb{R}^+$*, and the restricted Riemannian exponential map at 1 is given by* $c\partial/\partial t \mapsto e^c$*.*

Exercise 3.6.3 ([12] 7-13)**.** *Let* G *be a Lie group with a bi-invariant metric* g*. Show that the following formula holds whenever* X, Y, Z *are left-invariant vector fields on* G*:*

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]$$

(see Problem 3.6.2.)

Exercise 3.6.4 ([12] 8-17)**.** *Let* G *be a Lie group with a bi-invariant metric* g*.*

(a) Suppose X *and* Y *are orthonormal elements of* $\operatorname{Lie}(G)$*. Show that* $\sec(X_p, Y_p) = \frac{1}{4}|[X, Y]|^2$ *for each* $p \in G$*, and conclude that the sectional curvatures of* (G, g) *are all nonnegative.*

(b) Show that every Lie subgroup of G *is totally geodesic in* G*.*

(c) Now suppose G *is connected. Show that* G *is flat if and only if it is abelian.*

Exercise 3.6.5 ([12] 8-12)**.** *Suppose* $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ *is a Riemannian submersion, and* $(\widetilde{M}, \widetilde{g})$ *has all sectional curvatures bounded below by a constant* c*. Use O'Neill's formula to show that the sectional curvatures of* (M, g) *are bounded below by the same constant.*

Exercise 3.6.6 ([12] 8-13)**.** *Let* $p : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ *be the Riemannian submersion described in Example 2.30. In this problem, we identify* $\mathbb{C}^{n+1}$ *with* $\mathbb{R}^{2n+2}$ *by means of coordinates* $(x^1, y^1, \ldots, x^{n+1}, y^{n+1})$ *defined by* $z^j = x^j + iy^j$*.*

(a) Show that the vector field

$$S = x^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial x^j}$$

on $\mathbb{C}^{n+1}$ *is tangent to* $\mathbb{S}^{2n+1}$ *and spans the vertical space* V_z *at each point* $z \in \mathbb{S}^{2n+1}$*. (The implicit summation here is from* 1 *to* $n+1$*.)*

(b) Show that for all horizontal vector fields W, Z *on* $\mathbb{S}^{2n+1}$*,*

$$[W, Z]^V = -d\omega(W, Z)S = 2\langle W, JZ\rangle S$$

where ω *is the 1 -form on* $\mathbb{C}^{n+1}$ *given by*

$$\omega = S^{\mathrm{b}} = \sum_j x^j dy^j - y^j dx^j$$

and $J : T\mathbb{C}^{n+1} \to T\mathbb{C}^{n+1}$ *is the real-linear orthogonal map given by*

$$J\left(a^j \frac{\partial}{\partial x^j} + b^j \frac{\partial}{\partial y^j}\right) = a^j \frac{\partial}{\partial y^j} - b^j \frac{\partial}{\partial x^j}$$

(This is just multiplication by $i=\sqrt{-1}$ in complex coordinates. Notice that $J \circ J=-\mathrm{Id}$.)

(c) Using O'Neill's formula (Problem 7-14), show that the curvature tensor of $\mathbb{CP}^n$ satisfies

$$\begin{aligned}\mathrm{Rm}(w,x,y,z)=&\langle\widetilde{w},\widetilde{z}\rangle\langle\widetilde{x},\widetilde{y}\rangle-\langle\widetilde{w},\widetilde{y}\rangle\langle\widetilde{x},\widetilde{z}\rangle\\&-2\langle\widetilde{w},J\widetilde{x}\rangle\langle\widetilde{y},J\widetilde{z}\rangle-\langle\widetilde{w},J\widetilde{y}\rangle\langle\widetilde{x},J\widetilde{z}\rangle\\&+\langle\widetilde{w},J\widetilde{z}\rangle\langle\widetilde{x},J\widetilde{y}\rangle\end{aligned}$$

for every $q\in\mathbb{CP}^n$ and $w,x,y,z\in T_q\mathbb{CP}^n$, where $\widetilde{w},\widetilde{x},\widetilde{y},\widetilde{z}$ are horizontal lifts of w,x,y,z to an arbitrary point $\widetilde{q}\in p^{-1}(q)\subseteq\mathbb{S}^{2n+1}$.

(d) Using the notation of part (c), show that for orthonormal vectors $w,x\in T_q\mathbb{CP}^n$, the sectional curvature of the plane spanned by $\{w,x\}$ is

$$\sec(w,x)=1+3\langle\widetilde{w},J\widetilde{x}\rangle^2$$

(e) Show that for $n\geq 2$, the sectional curvatures at each point of $\mathbb{CP}^n$ take on all values between 1 and 4 , inclusive, and conclude that $\mathbb{CP}^n$ is not frame-homogeneous.

(f) Compute the Gaussian curvature of $\mathbb{CP}^1$.

Exercise 3.6.7. *([12] Problem 4-6) Let M be a smooth manifold and let ∇ be a connection in TM. Define a map $\tau:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ by*

$$\tau(X,Y)=\nabla_XY-\nabla_YX-[X,Y].$$

*(a) Show that τ is a $(1,2)$-tensor field, called the **torsion tensor of** ∇.*

(b) We say that ∇ is symmetric if its torsion vanishes identically. Show that ∇ is symmetric if and only if its connection coefficients with respect to every coordinate frame are symmetric: $\Gamma_{ij}^k=\Gamma_{ji}^k$. [Warning: They might not be symmetric with respect to other frames.]

(c) Show that ∇ is symmetric if and only if the covariant Hessian ∇^2u of every smooth function $u\in C^\infty(M)$ is a symmetric 2-tensor field.

(d) Show that the Euclidean connection $\bar{\nabla}$ on $\mathbb{R}^n$ is symmetric.

Solution. (a) To show τ is a $(1,2)$-tensor field, we will need to use tensor characterization lemma. That is, we need to show T is multilinear over $C^\infty(M)$. $\tau(X+Y,Z)=\tau(X,Z)+\tau(Y,Z)$ is clear. Let $f\in C^\infty(M)$.

$$\begin{aligned}\tau(fX,Y)&=\nabla_{fX}Y-\nabla_YfX-[fX,Y]\\&\xlongequal{\text{[LeeSM] prop.8.28}}\nabla_{fX}Y-\nabla_YfX-(f[X,Y]-(Yf)X)\\&\xlongequal{\nabla\text{ a connection}}f\nabla_XY-(f\nabla_YX+(Yf)X)-(f[X,Y]-(Yf)X)\\&=f\nabla_XY-f\nabla_YX-f[X,Y]\\&=fT(X,Y)\end{aligned}$$

(b) Eq (3.3) gives

$$\begin{aligned}\nabla_XY-\nabla_YX&=\left(X(Y^k)+X^iY^j\Gamma_{ij}^k\right)\partial_k-\left(Y(X^k)+Y^jX^i\Gamma_{ji}^k\right)\partial_k\\&\xlongequal{\text{[LeeSM] eq.(8.9)}}[X,Y]+\left(X^iY^j\Gamma_{ij}^k-Y^jX^i\Gamma_{ji}^k\right)\partial_k\\&=[X,Y]+X^iY^j(\Gamma_{ij}^k-\Gamma_{ji}^k)\partial_k\end{aligned}$$

Thus, for $\tau(X,Y)$ to be zero for all X and Y, we must have $\Gamma_{ij}^k=\Gamma_{ji}^k$. To verify this, note that the torsion tensor τ is determined by

$$\tau(\partial_i,\partial_j)=\nabla_{\partial_i}\partial_j-\nabla_{\partial_j}\partial_i-[\partial_i,\partial_j]=(\Gamma_{ij}^k-\Gamma_{ji}^k)\partial_k.$$

(where $[\partial_i, \partial_j] = 0$ is by [LeeSM] eq.(8.10); note that the result thus relies on this particular frame). Since the torsion tensor vanishes if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$, we conclude that:

$$\tau(X,Y) = 0 \iff \Gamma_{ij}^k = \Gamma_{ji}^k.$$

(c)

This is easy: for any smooth function u on M,

$$\begin{aligned}\nabla^2 u(Y,X) - \nabla^2 u(X,Y) &= Y(Xu) - (\nabla_Y X)u - X(Yu) + (\nabla_X Y)u \\ &= [Y,X]u + (\nabla_X Y - \nabla_X Y)u \\ &= \tau(X,Y)u\end{aligned}$$

♦

Exercise 3.6.8. *([12] Problem 4-9) Let M be a smooth manifold, and let ∇^0 and ∇^1 be two connections on TM.*

(a) Show that ∇^0 and ∇^1 have the same torsion (3.6.7) if and only if their difference tensor is symmetric, i.e., $D(X,Y) = D(Y,X)$ for all X and Y.

(b) Show that ∇^0 and ∇^1 determine the same geodesics if and only if their difference tensor is antisymmetric, i.e., $D(X,Y) = -D(Y,X)$ for all X and Y.

Exercise 3.6.9 ([12] 6-1)**.** *Suppose M is a nonempty connected Riemannian 1-manifold. Show that if M is noncompact, then it is isometric to an open interval in $\mathbb{R}$ with the Euclidean metric, while if it is compact, it is isometric to a circle $\mathbb{S}^1(R) = \{x \in \mathbb{R}^2 : |x| = R\}$ with its induced metric for some $R > 0$, using the following steps.*

(a) Let $\gamma : I \to M$ be any maximal unit-speed geodesic. Show that its image is open and closed, and therefore γ is surjective.

(b) Show that if γ is injective, then it is an isometry between I with its Euclidean metric and M.

(c) Now suppose $\gamma(t_1) = \gamma(t_2)$ for some $t_1 \neq t_2$. In case $\gamma'(t_1) = \gamma'(t_2)$, show that γ is periodic, and descends to a global isometry from an appropriate circle to M.

(d) It remains only to rule out the case $\gamma(t_1) = \gamma(t_2)$ and $\gamma'(t_1) = -\gamma'(t_2)$. If this occurs, let $t_0 = (t_1 + t_2)/2$, and define geodesics α and β by

$$\alpha(t) = \gamma(t_0 + t), \quad \beta(t) = \gamma(t_0 - t).$$

Use uniqueness of geodesics to conclude that $\alpha \equiv \beta$ on their common domain, and show that this contradicts the fact that γ is injective on some neighborhood of t_0.

Exercise 3.6.10 ([12] 6-4)**.** *Previously, we have started with a Riemannian metric and used it to define the Riemannian distance function. This problem shows how to go back the other way: the distance function determines the Riemannian metric. Let (M,g) be a connected Riemannian manifold.*

(a) Show that if $\gamma : (-\varepsilon, \varepsilon) \to M$ is any smooth curve, then

$$|\gamma'(0)|_g = \lim_{t \searrow 0} \frac{d_g(\gamma(0), \gamma(t))}{t}$$

(b) Show that if g and $\widetilde{g}$ are two Riemannian metrics on M such that $d_g(p,q) = d_{\widetilde{g}}(p,q)$ for all $p, q \in M$, then $g = \widetilde{g}$.

Exercise 3.6.11 ([12] 6-12)**.** *Let (M, g) be a connected Riemannian manifold.*

(a) Suppose there exists $\delta > 0$ such that for each $p \in M$, every maximal unit-speed geodesic starting at p is defined at least on an interval of the form $(-\delta, \delta)$. Prove that M is complete.

(b) Prove that if M has positive or infinite injectivity radius, then it is complete.

(c) Prove that if M is homogeneous, then it is complete.

(d) Give an example of a complete, connected Riemannian manifold that has zero injectivity radius.

Exercise 3.6.12 ([12] 6-13)**.** *Let G be a connected compact Lie group. Show that the Lie group exponential map of G is surjective. [Hint: Use Problem 3.6.2.]*

Exercise 3.6.13 ([12] 6-16)**.** *Suppose (M, g) is a complete, connected Riemannian manifold with positive or infinite injectivity radius.*

(a) Let $\rho \in (0, \infty]$ denote the injectivity radius of M, and define $T^\rho M$ to be the subset of TM consisting of vectors of length less than ρ, and D^ρ to be the subset $\{(p, q) : d_g(p, q) < \rho\} \subseteq M \times M$. Define $E : T^\rho M \to D^\rho$ by $E(x, v) = (x, \exp_x v)$. Prove that E is a diffeomorphism.

(b) Use part (a) to prove that if B is a topological space and $F, G : B \to M$ are continuous maps such that $d_g(F(x), G(x)) < \operatorname{inj}(M)$ for all $x \in B$, then F and G are homotopic.

Exercise 3.6.14 ([12] 6-17)**.** *Suppose (M, g) is a connected Riemannian manifold. A **closed geodesic** in M is a nonconstant geodesic segment $\gamma : [a, b] \to M$ such that $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$. Show that if M is a compact and connected, then every nontrivial free homotopy class in M is represented by a closed geodesic that has minimum length among all admissible loops in the given free homotopy class. [Hint: Use Prop. 3.5.33 to show that the given free homotopy class is represented by a geodesic loop, i.e., a geodesic whose starting and ending points are the same. Show that the lengths of such loops have a positive greatest lower bound; then choose a sequence of geodesic loops whose lengths approach that lower bound, and show that a subsequence converges uniformly to a geodesic loop whose length is equal to the lower bound. Use Problem 3.6.13 to show that the limiting curve is in the given free homotopy class, and apply the first variation formula to show that the limiting curve is in fact a closed geodesic.]*

Exercise 3.6.15 ([12] 6-18)**.** *A connected Riemannian manifold (M, g) is said to be k-**point homogeneous** if for any two ordered k-tuples $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ of points in M such that $d_g(p_i, p_j) = d_g(q_i, q_j)$ for all i, j, there is an isometry $\varphi : M \to M$ such that $\varphi(p_i) = q_i$ for $i = 1, \ldots, k$. Show that (M, g) is 2-point homogeneous if and only if it is isotropic. [Hint: Assuming that M is isotropic, first show that it is homogeneous by considering the midpoint of a geodesic segment joining sufficiently nearby points $p, q \in M$, and then use the result of Problem 3.6.11 (c) to show that it is complete.]*

Exercise 3.6.16 ([12] 6-19)**.** *Prove that every Riemannian symmetric space is homogeneous. [Hint: Proceed as in Problem 3.6.15.]*

Exercise 3.6.17 ([12] 5-22)**.** *A smooth vector field X on a Riemannian manifold is called a **Killing vector field** if the Lie derivative $\mathcal{L}_X g$ of the metric with respect to X vanishes. By Proposition 1.2.12 (g) (which is an analogue of [11, Theorem 9.42]), this is equivalent to the requirement that the metric be invariant under the flow of X. Prove that X is a Killing vector field if and only if the covariant 2-tensor field $(\nabla X)^\flat$ is antisymmetric. [Hint: Proposition 1.2.12 (d).]*

Exercise 3.6.18 ([12] 6-24)**.** *Let (M, g) be a Riemannian manifold.*

(a) Prove that a Killing vector field that is normal to a geodesic at one point is normal everywhere along the geodesic.

(b) Prove that if a Killing vector field vanishes at a point p, then it is tangent to geodesic spheres centered at p.

(c) Prove that a Killing vector field on an odd-dimensional manifold cannot have an isolated zero.

Exercise 3.6.19 ([12] 10-24)**.** *Let (M, g) be a complete Riemannian manifold and $p \in M$. Show that* $\operatorname{inj}(p)$ *is equal to the radius of the largest open ball in T_pM on which $\exp_p$ is injective.*

Chapter 4

Laplace Operators and Harmonic Forms

4.1 Basic Examples

[20] chapter 1 Basic Examples

Because the theory of the Laplacian on a Riemannian manifold involves some technical preliminaries, we begin by examining some simple examples. In fact, considering the Laplacian and the associated heat flow on just S^1 and $\mathbf{R}$ highlights essential differences between the Laplacian on a compact and on a noncompact manifold.

First, recall that if $T : V \to V$ is a symmetric, nonnegative linear transformation of a finite dimensional inner product space V, then there exists an orthonormal basis of eigenvectors of V with eigenvalues $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. The set $\{\lambda_i\}$ is called the spectrum of T, denoted $\sigma(T)$. Note that $\lambda \in \sigma(T)$ if any only if there is some nonzero vector v that solves the system of linear equations $(T - \lambda I)v = 0$, i.e., $\ker(T - \lambda I) \neq 0$. Also, the system $Ax = 0$ has nonzero solutions if and only if the matrix A is singular, i.e., $\det(A) = 0$. Thus, we can equivalently write

$$\lambda \notin \sigma(T) \iff \ker(T - \lambda I) = 0 \iff \det(T - \lambda I) \neq 0 \iff (T - \lambda I)^{-1} \text{ exists}$$

This eigenvector decomposition of V generalizes to the infinite dimensional case where V is a Hilbert space and T is a compact operator, i.e. an operator such that if $\{v_i\}$ is a bounded sequence in V, then $\{Tv_i\}$ has a convergent subsequence. (For example, any projection onto a finite dimensional subspace is compact, and in fact any compact operator is the norm limit of such finite rank operators.) In this case, the spectral theorem for compact operators says that V again has an orthonormal basis of eigenvectors for T, each eigenspace has only finite multiplicity, and the only (finite or infinite) accumulation point for the set of eigenvalues is zero. In particular, since the absolute values of the eigenvalues are bounded, the operator T is itself bounded. Remember that in infinite dimensions a linear operator may well be unbounded, or equivalently discontinuous.

The spectral theorem for compact operators is an easy generalization of the finite dimensional situation. We want to show that this eigenvector decomposition holds for certain unbounded differential operators on compact manifolds. The space V will be some Hilbert space of functions or forms on the manifold. We remark that unbounded operators are only defined on a dense subset of a Hilbert space, and in general one must be very careful to define the domains of such operators and their adjoints correctly. The domains of definition of our unbounded operators are rather easy to construct on compact manifolds, but noncompact manifolds are more difficult to treat.

4.2 Hilbert Spaces Associated to a Compact Riemannian Manifold

[1] chapter 2 section 1

4.3 Heat Kernel

[20] chapter 3

4.4 Problems

4.4.1 Vector and Tensor Fields Along Curves

Let M be a smooth manifold with or without boundary. Given a smooth curve $\gamma : I \to M$, a **vector field along** γ is a continuous map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$; it is a **smooth vector field along** γ if it is smooth as a map from I to TM. We let $\mathfrak{X}(\gamma)$ denote the set of all smooth vector fields along γ. It is a real vector space under pointwise vector addition and multiplication by constants, and it is a module over $C^\infty(I)$ with multiplication defined pointwise:

$$(fX)(t) = f(t)X(t). \tag{4.1}$$

The most obvious example of a vector field along a smooth curve γ is the curve's velocity: $\gamma'(t) \in T_{\gamma(t)}M$ for each t, and its coordinate expression

$$\gamma'(t) = \dot{\gamma}^1(t) \left.\frac{\partial}{\partial x^1}\right|_{\gamma(t)} + \cdots + \dot{\gamma}^n(t) \left.\frac{\partial}{\partial x^n}\right|_{\gamma(t)}$$

shows that it is smooth.

Suppose $\gamma : I \to M$ is a smooth curve and $\widetilde{V}$ is a smooth vector field on an open subset of M containing the image of γ. Define $V : I \to TM$ by setting $V(t) = \widetilde{V}_{\gamma(t)}$ for each $t \in I$. Since V is equal to the composition $\widetilde{V} \circ \gamma$, it is smooth. A smooth vector field along γ is said to be **extendible** if there exists a smooth vector field $\widetilde{V}$ on a neighborhood of the image of γ that is related to V in this way (Fig.4.1).

Not every vector field along a curve need be extendible; for example, if $\gamma(t_1) = \gamma(t_2)$ but $\gamma'(t_1) \neq \gamma'(t_2)$ (Fig.4.2), then γ' is not extendible. Even if γ is injective, its velocity need not be extendible, as the next example shows.

Example 4.4.1 (Figure-eight Curve). Let $\gamma : (-\pi, \pi) \to \mathbb{R}^2$ be $\gamma(t) = (\sin 2t, \sin t)$. Then

$$\gamma'(t) = (2\cos 2t,\ \cos t).$$

Since $\cos t = 0 \iff t = \pm\frac{\pi}{2}$ (where $\cos 2t = -1 \neq 0$) and $\cos 2t = 0 \iff t \in \{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}$ (where $\cos t \neq 0$), the two components of $\gamma'(t)$ never vanish simultaneously. Hence $\gamma'(t) \neq 0$ for all t, so γ is a smooth immersion.

Injectivity. Suppose $\gamma(t_1) = \gamma(t_2)$. Then $\sin t_1 = \sin t_2$ and $\sin 2t_1 = \sin 2t_2$. From $\sin t_1 = \sin t_2$ we have $t_2 = t_1 + 2k\pi$ or $t_2 = \pi - t_1 + 2k\pi$. On $(-\pi, \pi)$ the first gives $t_2 = t_1$. In the second case,

$$\sin 2t_2 = \sin(2\pi - 2t_1) = -\sin 2t_1,$$

so equality $\sin 2t_2 = \sin 2t_1$ forces $\sin 2t_1 = 0$, i.e. $t_1 \in \{0, \pm\frac{\pi}{2}, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}$. The only possibility in $(-\pi, \pi)$ yielding $t_2 \neq t_1$ would be $t_2 = \pi - t_1$, but then $t_2 \in \{\pi,\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \frac{\pi}{4}\}$; among these, the only case mapping to the same point with $t_2 \in (-\pi, \pi)$ is $t_2 = t_1$. Hence γ is injective on $(-\pi, \pi)$.

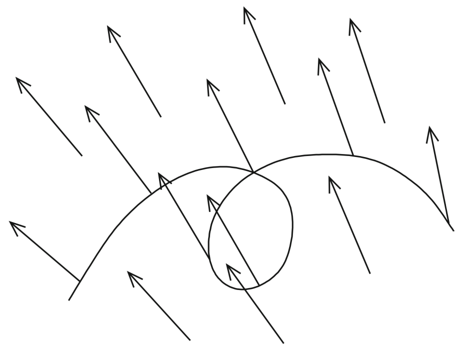

Figure 4.1: Extendible vector field

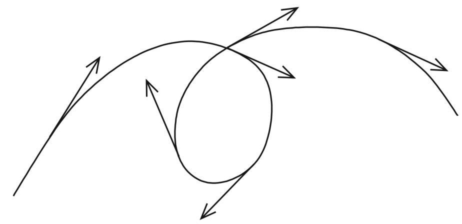

Figure 4.2: Nonextendible vector field

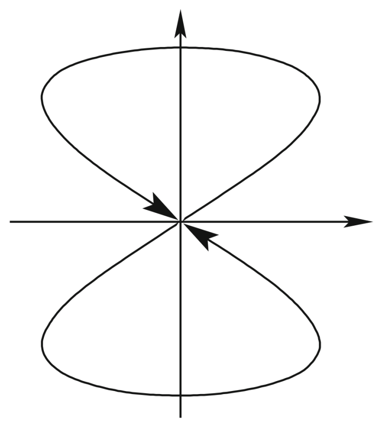

Figure 4.3: The image of the figure eight curve

Non-extendibility of the velocity field. Assume there exists a smooth vector field $V : \mathbb{R}^2 \to T\mathbb{R}^2 \cong \mathbb{R}^2$ with $V(\gamma(t)) = \gamma'(t)$ for all $t \in (-\pi, \pi)$. Then, by continuity of V,

$$\lim_{t \to 0} V(\gamma(t)) = V(0,0) = \lim_{s \to \pi^-} V(\gamma(s)).$$

But

$$\lim_{t \to 0} \gamma'(t) = (2,1), \qquad \lim_{s \to \pi^-} \gamma'(s) = (2,-1),$$

since $\cos t \to 1$ and $\cos 2t \to 1$ as $t \to 0$, while $\cos t \to -1$ and $\cos 2t \to 1$ as $s \to \pi^-$. These two limits are different, a contradiction. Therefore no smooth V extending the velocity along γ exists.

Thus γ is an injective smooth immersion whose velocity field is not extendible to a smooth vector field on $\mathbb{R}^2$. ♣

More generally, a **tensor field along** γ is a continuous map σ from I to some tensor bundle $T^{(k,l)}TM$ such that $\sigma(t) \in T^{(k,l)}\left(T_{\gamma(t)}M\right)$ for each $t \in I$. It is a **smooth tensor field along** γ if it is smooth as a map from I to $T^{(k,l)}TM$, and it is extendible if there is a smooth tensor field $\widetilde{\sigma}$ on a neighborhood of $\gamma(I)$ such that $\sigma = \widetilde{\sigma} \circ \gamma$.

4.4.2 Continuous Derivatives Along Curves

Here is the promised interpretation of a connection as a way to take derivatives of vector fields along curves.

Theorem 4.4.2 (Covariant Derivative Along a Curve)**.** *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For each smooth curve $\gamma : I \to M$, the connection determines a unique operator*

$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

called the ***covariant derivative along*** *γ, satisfying the following properties:*

(i) linearity $\mathbb{R}$:

$$D_t(aV + bW) = aD_tV + bD_tW \quad \text{for } a, b \in \mathbb{R}.$$

(ii) Leibniz rule:

$$D_t(fV) = f'V + fD_tV \quad \text{for } f \in C^\infty(I).$$

(iii) If $V \in \mathfrak{X}(\gamma)$ is extendible, then for every extension $\widetilde{V}$ of V,

$$\left(D_tV\right)(t) = \nabla_{\gamma'(t)}\widetilde{V}$$

Remark 4.4.3. There are analogous operators on the space of C^∞ tensor fields of any type along γ. For example, the above explains $T^{(1,0)}TM = TM$ case. Another of peculiarity is $T^{(0,0)}TM = M \times \mathbb{R}$ which as explained in prop. 3.1.13 gives rise to smooth functions along the curve γ, i.e., $f : \operatorname{Im}(\gamma) \to \mathbb{R}$. Then analogusly, we have $D_t(af + bg) = aD_tf + bD_tg; D_t(fg) = f'g + fD_tg$; and for extension $\widetilde{f} \in C^\infty(U)$ where $\operatorname{Im}(\gamma) \subseteq U$, we from an extension X with $X_p = \gamma'(t)$ get $\left(D_tf\right)(t) = \nabla_{\gamma'(t)}\widetilde{f} = \left(\nabla_X\widetilde{f}\right)_p = (X\widetilde{f})_p = X_p\widetilde{f} = \gamma'(t)(\widetilde{f}) \xlongequal{\text{[11]p.69}} (\widetilde{f} \circ \gamma)'(t) = (f \circ \gamma)'(t) = \frac{d}{dt}(f \circ \gamma)(t)$. ♠

Proof. For simplicity, we prove the theorem for the case of vector fields along γ; the proof for arbitrary tensor fields is essentially identical except for notation.

First we show uniqueness. Suppose D_t is such an operator, and let $t_0 \in I$ be arbitrary. An argument similar to that of Lemma 3.1.8 shows that the value of D_tV at t_0 depends only on the values of V in any interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ containing t_0. (If t_0 is an endpoint of I, extend γ to a slightly bigger open interval, prove the lemma there, and then restrict back to I. If M has nonempty boundary, we can do this after first embedding M into a smooth manifold $\widetilde{M}$ without boundary and extending ∇ arbitrarily to a connection on $\widetilde{M}$.) Choose smooth coordinates $\left(x^i\right)$ for M in a neighborhood of $\gamma\left(t_0\right)$, and write

$$V(t) = V^j(t)\partial_j\big|_{\gamma(t)}$$

for t near t_0, where $V^1, \ldots, V^n$ are smooth real-valued functions defined on some neighborhood of t_0 in I. Note that ∂_j is a coordinate vector field on U and we still denote its restriction $\partial_j|_\gamma$ on the curve γ by ∂_j, so $\partial_j = \partial_j|_\gamma \in \mathfrak{X}(\gamma)$ has a natural extension $\partial_j \in \mathfrak{X}(U)$.

$$\begin{aligned}
\left(D_tV\right)(t) &= D_t\left(V^j\partial_j\right)(t) = \left[\dot{V}^j\partial_j + V^jD_t\left(\partial_j\right)\right](t) \\
&\xlongequal{(4.1)} \dot{V}^j(t)\partial_j(t) + V^j(t)D_t\left(\partial_j\right)(t) \\
&\xlongequal{\text{(iii)}} \dot{V}^j(t)\left.\partial_j\right|_{\gamma(t)} + V^j(t)\nabla_{\gamma'(t)}\partial_j \\
&\xlongequal{(*)} \left(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma^k_{ij}(\gamma(t))\right)\left.\partial_k\right|_{\gamma(t)}.
\end{aligned} \tag{4.2}$$

We explain the last step now: Let X be a vector field on a nieghborhood of the point $p=\gamma(t)$ such that $X_p=\gamma'(t)=v\in T_pM$, and $\nabla_{\gamma'(t)}(\partial_j)=(\nabla_X(\partial_j))_p$. Now $\nabla_X(\partial_j)=X^i\Gamma_{ij}^k\partial_k$ and

$$(*):\qquad \begin{aligned}\nabla_{\gamma'(t)}\partial_j &= (\nabla_X(\partial_j))_{\gamma(t)}=\left(X^i\Gamma_{ij}^k\partial_k\right)_{\gamma(t)}=X^i(\gamma(t))\Gamma_{ij}^k(\gamma(t))\left.\partial_k\right|_{\gamma(t)}\\ &=\left(X_{\gamma(t)}\right)^i\Gamma_{ij}^k(\gamma(t))\left.\partial_k\right|_{\gamma(t)}=\dot\gamma^i(t)\Gamma_{ij}^k(\gamma(t))\left.\partial_k\right|_{\gamma(t)}\end{aligned}$$

Equation (4.2) shows that such an operator is unique if it exists. For existence, if $\gamma(I)$ is contained in a single chart, we can define D_tV by (4.2); the easy verification that it satisfies the requisite properties is left as an exercise. In the general case, we can cover $\gamma(I)$ with coordinate charts and define D_tV by this formula in each chart, and uniqueness implies that the various definitions agree whenever two or more charts overlap. ■

Apart from its use in proving existence of the covariant derivative along a curve, (4.2) also gives a practical formula for computing such covariant derivatives in coordinates.

Now we can further improve proposition 3.1.10 by showing that ∇_vY actually depends only on the values of Y along any curve through p whose velocity is v.

Proposition 4.4.4. *Let M be a smooth manifold with or without boundary, let ∇ be a connection in TM, and let $p\in M$ and $v\in T_pM$. Suppose Y and $\widetilde{Y}$ are two smooth vector fields that agree at points in the image of some smooth curve $\gamma:I\to M$ such that $\gamma(t_0)=p$ and $\gamma'(t_0)=v$. Then $\nabla_vY=\nabla_v\widetilde{Y}$.*

Proof. We can define a smooth vector field Z along γ by $Z(t)=Y_{\gamma(t)}=\widetilde{Y}_{\gamma(t)}$. Since both Y and $\widetilde{Y}$ are extensions of Z, it follows from condition (iii) in above theorem that both ∇_vY and $\nabla_v\widetilde{Y}$ are equal to $D_tZ(t_0)$. ■

4.4.3 Geodesics

Armed with the notion of covariant differentiation along curves, we can now define acceleration and geodesics.

Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth curve $\gamma:I\to M$, we define the **acceleration of** γ to be the vector field $D_t\gamma'$ along γ. A smooth curve γ is called a **geodesic** (with respect to ∇) if its acceleration is zero: $D_t\gamma'\equiv 0$. In terms of smooth coordinates $\left(x^i\right)$, if we write the component functions of γ as $\gamma(t)=\left(x^1(t),\ldots,x^n(t)\right)$, then it follows from (4.2) that γ is a geodesic if and only if its component functions satisfy the following **geodesic equation**:

$$\ddot{x}^k(t)+\dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k(x(t))=0, \tag{4.3}$$

where we use $x(t)$ as an abbreviation for the n-tuple of component functions $\left(x^1(t),\ldots,x^n(t)\right)$. This is a system of second-order ordinary differential equations (ODEs) for the real-valued functions $x^1,\ldots,x^n$. The next theorem uses ODE theory to prove existence and uniqueness of geodesics with suitable initial conditions. (Because difficulties can arise when a geodesic starts on the boundary or later hits the boundary, we state and prove this theorem only for manifolds without boundary.)

Theorem 4.4.5 (Existence and Uniqueness of Geodesics). *Let M be a smooth manifold and ∇ a connection in TM. For every $p\in M, w\in T_pM$, and $t_0\in\mathbb{R}$, there exist an open interval $I\subseteq\mathbb{R}$ containing t_0 and a geodesic $\gamma:I\to M$ satisfying $\gamma(t_0)=p$ and $\gamma'(t_0)=w$. Any two such geodesics agree on their common domain.*

Proof. Let $\left(x^i\right)$ be smooth coordinates on some neighborhood U of p. A smooth curve in U, written as $\gamma(t)=\left(x^1(t),\ldots,x^n(t)\right)$, is a geodesic if and only if its component functions satisfy (4.3). The standard trick for proving existence and uniqueness for such a second-order system is to introduce auxiliary variables $v^i = \dot{x}^i$ to convert it to the following equivalent first-order system in twice the number of variables:

$$\begin{aligned} \dot{x}^k(t) &= v^k(t), \\ \dot{v}^k(t) &= -v^i(t)v^j(t)\Gamma_{ij}^k(x(t)). \end{aligned} \tag{4.4}$$

Treating $\left(x^1, \ldots, x^n, v^1, \ldots, v^n\right)$ as coordinates on $U \times \mathbb{R}^n$, we can recognize (4.4) as the equations for the flow of the vector field $G \in \mathfrak{X}\left(U \times \mathbb{R}^n\right)$ given by

$$G_{(x,v)} = v^k \left.\frac{\partial}{\partial x^k}\right|_{(x,v)} - v^i v^j \Gamma_{ij}^k(x) \left.\frac{\partial}{\partial v^k}\right|_{(x,v)}. \tag{4.5}$$

By the fundamental theorem on flows 1.2.8, for each $(p, w) \in U \times \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, there exist an open interval I_0 containing t_0 and a unique smooth solution $\zeta : I_0 \to U \times \mathbb{R}^n$ to this system satisfying the initial condition $\zeta\left(t_0\right) = (p, w)$. If we write the component functions of ζ as $\zeta(t) = \left(x^i(t), v^i(t)\right)$, then we can easily check that the curve $\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$ in U satisfies the existence claim of the theorem.

To prove the uniqueness claim, suppose $\gamma, \widetilde{\gamma} : I \to M$ are both geodesics defined on some open interval with $\gamma\left(t_0\right) = \widetilde{\gamma}\left(t_0\right)$ and $\gamma'\left(t_0\right) = \widetilde{\gamma}'\left(t_0\right)$. In any local coordinates around $\gamma\left(t_0\right)$, we can define smooth curves $\zeta, \widetilde{\zeta} : \left(t_0 - \varepsilon, t_0 + \varepsilon\right) \to U \times \mathbb{R}^n$ as above. These curves both satisfy the same initial value problem for the system (4.4), so by the uniqueness of ODE solutions, they agree on $\left(t_0 - \varepsilon, t_0 + \varepsilon\right)$ for some $\varepsilon > 0$. Suppose for the sake of contradiction that $\gamma(b) \neq \widetilde{\gamma}(b)$ for some $b \in I$. First suppose $b > t_0$, and let β be the infimum of numbers $b \in I$ such that $b > t_0$ and $\gamma(b) \neq \widetilde{\gamma}(b)$ (Fig.4.4).

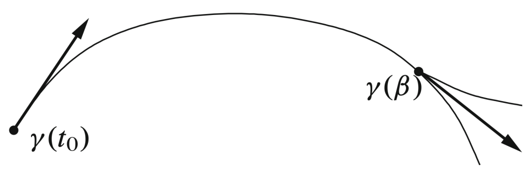

Figure 4.4: Uniqueness of geodesics

Then $\beta \in I$, and by continuity, $\gamma(\beta) = \widetilde{\gamma}(\beta)$ and $\gamma'(\beta) = \widetilde{\gamma}'(\beta)$. Applying local uniqueness in a neighborhood of β, we conclude that γ and $\widetilde{\gamma}$ agree on a neighborhood of β, which contradicts our choice of β. Arguing similarly to the left of t_0, we conclude that $\gamma \equiv \widetilde{\gamma}$ on all of I. ∎

A geodesic $\gamma : I \to M$ is said to be **maximal** if it cannot be extended to a geodesic on a larger interval, that is, if there does not exist a geodesic $\widetilde{\gamma} : \widetilde{I} \to M$ defined on an interval $\widetilde{I}$ properly containing I and satisfying $\left.\widetilde{\gamma}\right|_I = \gamma$. A **geodesic segment** is a geodesic whose domain is a compact interval.

Corollary 4.4.6. *Let M be a smooth manifold and let ∇ be a connection in TM. For each $p \in M$ and $v \in T_pM$, there is a unique maximal geodesic $\gamma : I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, defined on some open interval I containing 0 .*

Proof. Given $p \in M$ and $v \in T_pM$, let I be the union of all open intervals containing 0 on which there is a geodesic with the given initial conditions. By Theorem 4.4.5 , all such geodesics agree where they overlap, so they define a geodesic $\gamma : I \to M$, which is obviously the unique maximal geodesic with the given initial conditions. ∎

The unique maximal geodesic γ with $\gamma(0)=p$ and $\gamma'(0)=v$ is often called simply the **geodesic with initial point** p **and initial velocity** v, and is denoted by γ_v. (For simplicity, we do not specify the initial point p in the notation; it can implicitly be recovered from v by $p=\pi(v)$, where $\pi: TM \rightarrow M$ is the natural projection.)

4.4.4 Parallel Transport

Another construction involving covariant differentiation along curves that will be useful later is called parallel transport. Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. A smooth vector or tensor field V along a smooth curve γ is said to be parallel along γ (with respect to ∇) if $D_t V \equiv 0$ (Fig.4.5). Thus a geodesic can be characterized as a curve whose velocity vector field is parallel along the curve.

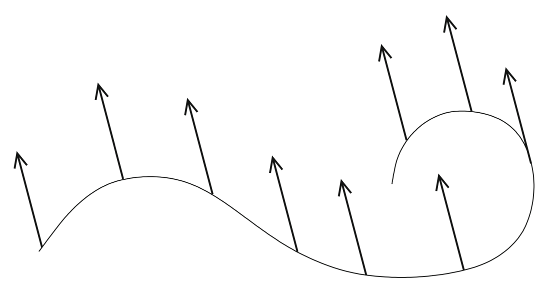

Figure 4.5: A parallel vector field along a curve

The fundamental fact about parallel vector and tensor fields along curves is that every tangent vector or tensor at any point on a curve can be uniquely extended to a parallel field along the entire curve. Before we prove this claim, let us examine what the equation of parallelism looks like in coordinates. Given a smooth curve γ with a local coordinate representation $\gamma(t)=\left(\gamma^1(t), \ldots, \gamma^n(t)\right)$, formula (4.2) shows that a vector field V is parallel along γ if and only if

$$\dot{V}^k(t)=-V^j(t) \dot{\gamma}^i(t) \Gamma_{ij}^k(\gamma(t)), \quad k=1, \ldots, n, \tag{4.6}$$

with analogous expressions based on Proposition 3.1.17 for tensor fields of other types. In each case, this is a system of first-order linear ordinary differential equations for the unknown coefficients of the vector or tensor field-in the vector case, the functions $\left(V^1(t), \ldots, V^n(t)\right)$. The usual ODE theorem guarantees the existence and uniqueness of a solution for a short time, given any initial values at $t=t_0$; but since the equation is linear, we can actually show much more: there exists a unique solution on the entire parameter interval.

Theorem 4.4.7 (Existence, Uniqueness, and Smoothness for Linear ODEs). *Let $I \subseteq \mathbb{R}$ be an open interval, and for $1 \leq j, k \leq n$, let $A_j^k: I \rightarrow \mathbb{R}$ be smooth functions. For all $t_0 \in I$ and every initial vector $\left(c^1, \ldots, c^n\right) \in \mathbb{R}^n$, the linear initial value problem*

$$\begin{aligned} \dot{V}^k(t) &= A_j^k(t) V^j(t) \\ V^k\left(t_0\right) &= c^k \end{aligned}$$

has a unique smooth solution on all of I, and the solution depends smoothly on $(t, c) \in I \times \mathbb{R}^n$.

Theorem 4.4.8 (Existence and Uniqueness of Parallel Transport). *Suppose M is a smooth manifold with or without boundary, and ∇ is a connection in TM. Given a smooth curve $\gamma: I \rightarrow M, t_0 \in I$, and a vector $v \in T_{\gamma(t_0)} M$ or tensor $v \in T^{(k, l)}\left(T_{\gamma(t)} M\right)$, there exists a unique parallel vector or tensor field V along γ such that $V\left(t_0\right)=v$.*

Proof. As in the proof of Theorem 4.4.2, we carry out the proof for vector fields. The case of tensor fields differs only in notation.

First suppose $\gamma(I)$ is contained in a single coordinate chart. Then V is parallel along γ if and only if its components satisfy the linear system of ODEs (4.6). Theorem 4.4.7 guarantees the existence and uniqueness of a solution on all of I with any initial condition $V(t_0) = v$.

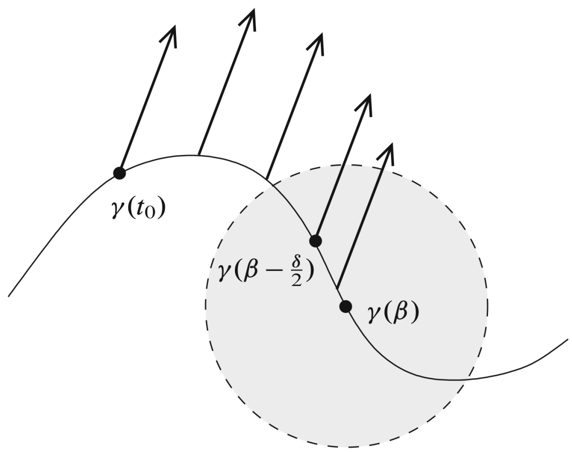

Figure 4.6: Existence and uniqueness of parallel transports

Now suppose $\gamma(I)$ is not covered by a single chart. Let β denote the supremum of all $b > t_0$ for which a unique parallel transport exists on $[t_0, b]$. (The argument for $t < t_0$ is similar.) We know that $\beta > t_0$, since for b close enough to $t_0, \gamma([t_0, b])$ is contained in a single chart and the above argument applies. Then a unique parallel transport V exists on $[t_0, \beta)$ (Fig.4.6). If β is equal to sup I, we are done. If not, choose smooth coordinates on an open set containing $\gamma(\beta - \delta, \beta + \delta)$ for some positive δ. Then there exists a unique parallel vector field $\widetilde{V}$ on $(\beta - \delta, \beta + \delta)$ satisfying the initial condition $\widetilde{V}(\beta - \delta/2) = V(\beta - \delta/2)$. By uniqueness, $V = \widetilde{V}$ on their common domain, and therefore $\widetilde{V}$ is a parallel extension of V past β, which is a contradiction. ∎

The vector or tensor field whose existence and uniqueness are proved in Theorem 4.4.8 is called the **parallel transport of** v **along** γ. For each $t_0, t_1 \in I$, we define a map

$$P^{\gamma}_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M,$$

called the **parallel transport map**, by setting $P^{\gamma}_{t_0 t_1}(v) = V(t_1)$ for each $v \in T_{\gamma(t_0)}M$, where V is the parallel transport of v along γ. This map is linear, because the equation of parallelism is linear. It is in fact an isomorphism, because $P^{\gamma}_{t_1 t_0}$ is an inverse for it.

It is also useful to extend the parallel transport operation to curves that are merely piecewise smooth. Given an admissible curve $\gamma : [a, b] \to M$, a map $V : [a, b] \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for each t is called a **piecewise smooth vector field along** γ if V is continuous and there is an admissible partition $(a_0, \ldots, a_k)$ for γ such that V is smooth on each subinterval $[a_{i-1}, a_i]$. We will call any such partition an **admissible partition for** V. A piecewise smooth vector field V along γ is said to be **parallel along** γ if $D_t V = 0$ wherever V is smooth.

Corollary 4.4.9 (Parallel Transport Along Piecewise Smooth Curves)**.** *Suppose M is a smooth manifold with or without boundary, and ∇ is a connection in TM. Given an admissible curve $\gamma : [a, b] \to M$ and a vector $v \in T_{\gamma(t_0)}M$ or tensor $v \in T^{(k,l)}\left(T_{\gamma(t)}M\right)$, there exists a unique piecewise smooth parallel vector or tensor field V along γ such that $V(a) = v$, and V is smooth wherever γ is.*

Proof. Let $(a_0, \ldots, a_k)$ be an admissible partition for γ. First define $V|_{[a_0,a_1]}$ to be the parallel transport of v along the first smooth segment $\gamma|_{[a_0,a_1]}$; then define $V|_{[a_1,a_2]}$ to be the parallel transport of $V(a_1)$ along the next smooth segment $\gamma|_{[a_1,a_2]}$; and continue by induction. ■

Here is an extremely useful tool for working with parallel transport. Given any basis $(b_1, \ldots, b_n)$ for $T_{\gamma(t_0)}M$, we can parallel transport the vectors b_i along γ, thus obtaining an n-tuple of parallel vector fields $(E_1, \ldots, E_n)$ along γ. Because each parallel transport map is an isomorphism, the vectors $(E_i(t))$ form a basis for $T_{\gamma(t)}M$ at each point $\gamma(t)$. Such an n-tuple of vector fields along γ is called a **parallel frame along** γ. Every smooth (or piecewise smooth) vector field along γ can be expressed in terms of such a frame as $V(t) = V^i(t)E_i(t)$, and then the properties of covariant derivatives along curves, together with the fact that the E_i 's are parallel, imply

$$D_t V(t) = \dot{V}^i(t)E_i(t) \tag{4.7}$$

wherever V and γ are smooth. This means that a vector field is parallel along γ if and only if its component functions with respect to the frame (E_i) are constants.

The parallel transport map is the means by which a connection "connects" nearby tangent spaces. The next theorem and its corollary show that parallel transport determines covariant differentiation along curves, and thereby the connection itself.

Theorem 4.4.10 (Parallel Transport Determines Covariant Differentiation)**.** *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose $\gamma : I \to M$ is a smooth curve and V is a smooth vector field along γ. For each $t_0 \in I$,*

$$D_t V(t_0) = \lim_{t_1 \to t_0} \frac{P^{\gamma}_{t_1 t_0} V(t_1) - V(t_0)}{t_1 - t_0}. \tag{4.8}$$

Proof. Let (E_i) be a parallel frame along γ, and write $V(t) = V^i(t)E_i(t)$ for $t \in I$. On the one hand, (4.7) shows that $D_t V(t_0) = \dot{V}^i(t_0) E_i(t_0)$.

On the other hand, for every fixed $t_1 \in I$, the parallel transport of the vector $V(t_1)$ along γ is the constant-coefficient vector field $W(t) = V^i(t_1) E_i(t)$ along γ, so $P^{\gamma}_{t_1 t_0} V(t_1) = V^i(t_1) E_i(t_0)$. Inserting these formulas into (4.8) and taking the limit as $t_1 \to t_0$, we conclude that the right-hand side is also equal to $\dot{V}^i(t_0) E_i(t_0)$. ■

Corollary 4.4.11 (Parallel Transport Determines the Connection)**.** *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose X and Y are smooth vector fields on M. For every $p \in M$,*

$$\nabla_X Y\big|_p = \lim_{h \to 0} \frac{P^{\gamma}_{h0} Y_{\gamma(h)} - Y_p}{h} \tag{4.9}$$

where $\gamma : I \to M$ is any smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = X_p$.

Proof. Given $p \in M$ and a smooth curve γ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$, let $V(t)$ denote the vector field along γ determined by Y, so $V(t) = Y_{\gamma(t)}$. By property (iii) of Theorem 4.4.2, $\nabla_X Y\big|_p$ is equal to $D_t V(0)$, so the result follows from Theorem 4.4.10. ■

A smooth vector or tensor field on M is said to be **parallel** (with respect to ∇) if it is parallel along every smooth curve in M.

Proposition 4.4.12. *Suppose M is a smooth manifold with or without boundary, ∇ is a connection in TM, and A is a smooth vector or tensor field on M. Then A is parallel on M if and only if $\nabla A \equiv 0$.*

Proof. [12] Problem 4-12. ■

Although Theorem 4.4.8 showed that it is always possible to extend a vector at a point to a parallel vector field along any given curve, it may not be possible in general to extend it to a parallel vector field on an open subset of the manifold. The impossibility of finding such extensions is intimately connected with the phenomenon of curvature, which will occupy a major portion of our attention in the second half of the book.

Proposition 4.4.13 (Properties of Pullback Connections)**.** *Suppose M and $\widetilde{M}$ are smooth manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is a diffeomorphism. Let $\widetilde{\nabla}$ be a connection in $T\widetilde{M}$ and let $\nabla = \varphi^*\widetilde{\nabla}$ be the pullback connection in TM. Suppose $\gamma : I \to M$ is a smooth curve.*

(a) *φ takes covariant derivatives along curves to covariant derivatives along curves: if V is a smooth vector field along γ, then*
$$d\varphi \circ D_t V = \widetilde{D}_t(d\varphi \circ V),$$
where D_t is covariant differentiation along γ with respect to ∇, and $\widetilde{D}_t$ is covariant differentiation along $\varphi \circ \gamma$ with respect to $\widetilde{\nabla}$.

(b) *φ takes geodesics to geodesics: if γ is a ∇-geodesic in M, then $\varphi \circ \gamma$ is a $\widetilde{\nabla}$-geodesic in $\widetilde{M}$.*

(c) *φ takes parallel transport to parallel transport: for every $t_0, t_1 \in I$,*
$$d\varphi_{\gamma(t_1)} \circ P^{\gamma}_{t_0 t_1} = P^{\varphi\circ\gamma}_{t_0 t_1} \circ d\varphi_{\gamma(t_0)}.$$

Proof. Exercise. ■

Chapter 5

Curvature and Topology

Chapter 6

Appendix

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