Lecture Note on
Complex Analysis I

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Contents

Chapter 1

Power Series and Analytic Functions

1.1 Limsup and Liminf

The limit superior of a sequence $\{x_n\}$ in $\mathbb{R}$ is defined by

$$\overline{\lim_{n\to\infty}}\, x_n = \limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right) = \inf_{n\geq 0}\left(\sup_{m\geq n} x_m\right)$$

The limit inferior of a sequence $\{x_n\}$ in $\mathbb{R}$ is defined by

$$\underline{\lim}_{n\to\infty}\, x_n = \liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right) = \sup_{n\geq 0}\left(\inf_{m\geq n} x_m\right)$$

When $\{x_n\}$ has no upper bound, we say $\overline{\lim}_{n\to\infty} x_n = +\infty$; when $\{x_n\}$ has no lower bound, we say $\underline{\lim}_{n\to\infty} x_n = -\infty$.

Theorem 1.1.1. Let $H = \overline{\lim}\, x_n$. Then

(a) When H is finite, there are infinitely many x_n falling in the interval $(H-\varepsilon, H+\varepsilon)$ for any $\varepsilon > 0$, while there are only finitely many (or even zero) x_n falling in $(H+\varepsilon, +\infty)$.

(b) When $H = +\infty$, for any $N > 0$, there are infinitely many x_n such that $x_n > N$.

(c) When $H = -\infty$, $\lim x_n = -\infty$.

Proof.

(a) $-\infty < H < +\infty$: the statement will be proved if we show that for any $\varepsilon > 0$ there are infinitely many terms x_n greater than $H-\varepsilon$ and only finitely many terms x_n greater than $H+\varepsilon$. We show the first part: BWOC, suppose there is some $\varepsilon_0 > 0$ s.t. there are only finitely many x_n greater than $H-\varepsilon_0$, say $x_{n_1}, \cdots, x_{n_k}$. Thus, $x_n \leq H-\varepsilon_0$ for all $n > n_k$. Therefore, for all $n > n_k$, the supremums have

$$\beta_n = \sup_{m\geq n} x_m = \sup\{x_n, x_{n+1}, \cdots\} \leq H-\varepsilon_0$$

Thus,

$$H = \overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \beta_n \leq H-\varepsilon_0$$

which is a contradiction. We show the second part: let $\beta_n = \sup_{m\geq n} x_m$. Since $\lim_{n\to\infty}\beta_n = H$, $\forall\varepsilon$, $\exists N\in\mathbb{N}$ s.t. $|\beta_n - H| < \varepsilon$, i.e., $H-\varepsilon < \beta_n < H+\varepsilon$. Since β_n is supremum of $\{x_n, x_{n+1}, \cdots\}$, we see when $n > N$,

$$\forall k\in\mathbb{N},\ x_{n+k} \leq \beta_n \leq H+\varepsilon$$

Thus, those x_n with $x_n > H+\varepsilon$ must have $n \leq N$, which shows that there are only finitely many x_n satisfying $x_n > H+\varepsilon$.

(b) That's because $H = +\infty$ when $\{x_n\}$ has no upper bound by definition.

(c) When $H = -\infty$, for any $G > 0$, there exists n_0, when $n > n_0$, $x_{n+1} \leq \beta_n \leq -G$, so $\lim x_n = -\infty$.

□

We have a liminf counterpart of the above theorem:

Theorem 1.1.2. Let $h = \underline{\lim}\, x_n$. Then

(a) When h is finite, there are infinitely many x_n falling in the interval $(h-\varepsilon, h+\varepsilon)$ for any $\varepsilon > 0$, while there are only finitely many (or even zero) x_n falling in $(-\infty, h-\varepsilon)$.

(b) When $h = -\infty$, for any $N > 0$, there are infinitely many x_n such that $x_n < -N$.

(c) When $h = +\infty$, $\lim x_n = +\infty$.

Another useful theorem is

Theorem 1.1.3. For limsup H and liminf h of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ with limit H and H is the largest among all limits of convergent subsequences of $\{x_n\}$; there also exists a subsequence $\{x_{n_k}\}$ with limit h and h is the smallest among all limits of convergent subsequences of $\{x_n\}$;

Corollary 1.1.4. $\lim x_n = A$ (finite or infinite) iff $\overline{\lim}\, x_n = \underline{\lim}\, x_n = A$.

Example 1.1.5. $a_n = n + (-1)^n n\,(n = 1, 2, 3, \cdots)$. It only has two subsequences with limit (including ∞): a_{2k} and $a_{2k+1}(k = 1, 2, 3, \cdots)$. The limits are respectively $+\infty$ and 0, so

$$\overline{\lim_{n\to\infty}}\, a_n = +\infty,\ \underline{\lim}_{n\to\infty}\, a_n = 0$$

Example 1.1.6. $a_n = \cos\frac{n}{4}\pi\,(n = 0, 1, 2, \cdots)$. Since $-1 \leqslant \cos\frac{n}{4}\pi \leqslant 1$, when $n = 8k(k = 1, 2, \cdots)$, $a_{8k} \to 1(k\to\infty)$; when $n = 4(2k+1)$, $(k = 1, 2, 3, \cdots)$, $a_{4(2k+1)} \to -1(k\to\infty)$. Thus,

$$\overline{\lim_{n\to\infty}}\, a_n = 1,\ \underline{\lim}_{n\to\infty}\, a_n = -1$$

Proposition 1.1.7. suppose $\lim_{n\to\infty} x_n = x, -\infty < x < 0$. Then

$$\overline{\lim_{n\to\infty}}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty}\, y_n;$$
$$\underline{\lim}_{n\to\infty}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}}\, y_n.$$

suppose $\lim_{n\to\infty} x_n = x, 0 < x < \infty$. Then

$$\overline{\lim_{n\to\infty}}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}}\, y_n;$$
$$\underline{\lim}_{n\to\infty}\,(x_n y_n) = \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty}\, y_n.$$

Proof. We prove the first two equations. The others are similar. $\lim x_n = x, -\infty < x < 0$, so for any given $\varepsilon(0 < \varepsilon < -x)$, there exists positive integer N_1 such that for all $n > N_1$,

$$x - \varepsilon < x_n < x + \varepsilon < 0.$$

Let $\overline{\lim}\, y_n = H, \underline{\lim}\, y_n = h$. Then for the above $\varepsilon(0 < \varepsilon < -x)$, there exists N_2 such that for all $n > N_2$,

$$h - \varepsilon < y_n < H + \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$\min\{(x-\varepsilon)(H+\varepsilon), (x+\varepsilon)(H+\varepsilon)\} < x_n y_n < \max\{(x-\varepsilon)(h-\varepsilon), (x+\varepsilon)(h-\varepsilon)\},$$

Thus,

$$\begin{aligned}\underline{\lim}_{n\to\infty} (x_n y_n) &\geqslant \min\{(x-\varepsilon)(H+\varepsilon), (x+\varepsilon)(H+\varepsilon)\},\\ \overline{\lim}_{n\to\infty} (x_n y_n) &\leqslant \max\{(x-\varepsilon)(h-\varepsilon), (x+\varepsilon)(h-\varepsilon)\},\end{aligned}$$

By arbitrariness of ε, we get

$$\begin{aligned}\underline{\lim}_{n\to\infty} (x_n y_n) &\geqslant xH = \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n,\\ \overline{\lim}_{n\to\infty} (x_n y_n) &\leqslant xh = \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n.\end{aligned}$$

Since

$$\begin{aligned}\underline{\lim}_{n\to\infty} y_n &= \underline{\lim}_{n\to\infty} \left[\frac{1}{x_n} \cdot (x_n y_n)\right] \geqslant \lim_{n\to\infty} \frac{1}{x_n} \cdot \overline{\lim}_{n\to\infty} (x_n y_n),\\ \overline{\lim}_{n\to\infty} y_n &= \overline{\lim}_{n\to\infty} \left[\frac{1}{x_n} \cdot (x_n y_n)\right] \leqslant \lim_{n\to\infty} \frac{1}{x_n} \cdot \underline{\lim}_{n\to\infty} (x_n y_n),\end{aligned}$$

we have

$$\begin{aligned}\underline{\lim}_{n\to\infty} (x_n y_n) &\leqslant \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n,\\ \overline{\lim}_{n\to\infty} (x_n y_n) &\geqslant \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n.\end{aligned}$$

Combine the last four equations to get

$$\begin{aligned}\overline{\lim}_{n\to\infty} (x_n, y_n) &= \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n\\ \underline{\lim}_{n\to\infty} (x_n y_n) &= \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n.\end{aligned}$$

□

Proposition 1.1.8.

$$\overline{\lim}_{n\to\infty} (c x_n) = \begin{cases} c\,\overline{\lim}_{n\to\infty} x_n & c > 0 \\ c\,\underline{\lim}_{n\to\infty} x_n & c < 0 \end{cases}$$

Similarly,

$$\underline{\lim}_{n\to\infty} (c x_n) = \begin{cases} c\,\underline{\lim}_{n\to\infty} x_n & c > 0 \\ c\,\overline{\lim}_{n\to\infty} x_n & c < 0 \end{cases}$$

In particular, c can be -1.

Proof. This is due to the similar result from supremum and infimum. For example,

$$\overline{\lim_{n\to\infty}}(cx_n) = \lim_{n\to\infty}\sup_{m\geq n}(cx_m) = \begin{cases} \lim_{n\to\infty}(c\cdot\sup_{m\geq n}x_m) = c\lim_{n\to\infty}\sup_{m\geq n}x_m & c>0 \\ \lim_{n\to\infty}(c\cdot\inf_{m\geq n}x_m) = c\lim_{n\to\infty}\inf_{m\geq n}x_m & c<0 \end{cases}$$

□

In fact, once we have $\underline{\lim}\, x_n = -\overline{\lim}(-x_n)$, we only need to state limsup half in the following observations (for 1.1.9 we have $\underline{\lim}(x_n+y_n) \geq \underline{\lim}\, x_n + \underline{\lim}\, y_n$; for 1.1.10 we have $\underline{\lim}(x_ny_n) \leq (\underline{\lim}\, x_n)(\underline{\lim}\, y_n)$). We omit the proofs.

Proposition 1.1.9. If $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers, then

$$\limsup_{n\to\infty}(x_n+y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$

provided the sum on the right is well defined (i.e., excluding the case where one summand is ∞ and the other is $-\infty$). If one of the sequences converges then the equality holds (with the same proviso).

Proposition 1.1.10. If $\{x_n\}$ and $\{y_n\}$ are two sequences of positive real numbers, then

$$\limsup_{n\to\infty}(x_ny_n) \leq \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right),$$

provided the product on the right is well defined (i.e., excluding the case where one factor is 0 and the other is ∞). If one of the sequences converges then the equality holds (with the same proviso).

1.2 Series

1.2.1 Comparison Test

Suppose we have two *nonngegative* series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$, and they have the following relationship:

$$\exists c>0, \text{ s.t. } u_n \leq cv_n \quad n=k,k+1,k+2,\cdots$$

for some k, i.e., each term of the first series is dominated by the second after $(k-1)$-th term. Since partial sum sequence of nonnegative series converges iff the sequence is bounded, one observes

- $\sum v_n$ converges $\Rightarrow \sum u_n$ converges;
- $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges.

We have the functional version: for real functions $0 \leq f_n(x) \leq g_n(x)$,

- $\sum g_n(x)$ pointwise/uniformly converges $\Rightarrow \sum f_n(x)$ pointwise/uniformly converges;
- $\sum f_n(x)$ diverges $\Rightarrow \sum g_n(x)$ diverges.

where the pointwise one follows immediately from the number series version and the uniform one follows from Cauchy criterion for uniform convergence Let $\varepsilon>0$, then exists $N\in\mathbb{N}$ such that for any $m,n\in\mathbb{N}$ if $N\leq m\leq n$ then

$$\left|\sum_{k=m}^{n} f_k\right| \leq \left|\sum_{k=m}^{n} g_k\right| < \varepsilon$$

Then $\sum f_n$ is uniformly convergent.

1.2.2 Series of Complex Numbers

Let $\{z_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$, then the series $\sum_{n=0}^{\infty} z_n$ converges to z iff the sequence of partial sums $\{S_N(z_n)\} = \left\{\sum_{n=0}^{N} z_n\right\}$ converges to z, i.e.,

$$\forall \varepsilon > 0,\ \exists K \in \mathbb{N},\ \text{s.t.}\ \forall N > K:\ |S_N(z_n) - z| < \varepsilon.$$

We say the seires $\sum z_n$ converges absolutely if $\sum |z_n|$ converges. Note that $\{|z_n|\}$ is a nonnegative sequence and $\{S_N(|z_n|)\} = \left\{\sum_{n=0}^{N} |z_n|\right\}$ is a monotone (nonstrictly) increasing sequence, so boundedness of $\{S_N(|z_n|)\}$ is a sufficient condition for convergence of $\sum |z_n|$. Two basic facts are presented first:

Theorem 1.2.1. If the series $\sum_{n=0}^{\infty} z_n$ converges, then $z_n \to 0$ as $n \to \infty$.

Proof. Since $\lim_{N\to\infty} S_N(z_n) = z$ for some z, we get $\lim z_n = \lim(S_N(z_n) - S_{N-1}(z_n)) = z - z = 0$. □

Theorem 1.2.2. If $\sum z_n$ converges absolutely, $\sum z_n$ converges.

Proof. $\sum z_n$ converges absolutely, so for any $\varepsilon > 0$ there is K s.t. $M, N > K$ (WLOG, $M > N$) implies

$$\varepsilon > |S_M(|z_n|) - S_N(|z_n|)| = ||z_{M+1}| + \cdots + |z_N|| \geq |z_{M+1} + \cdots + z_N| = |S_M(z_n) - S_N(z_n)|$$

so $\{S_N(z_n)\}$ is Cauchy and thus converges by completeness of $\mathbb{C}$. □

Complex series can relate to real series in one way:

Theorem 1.2.3. Let $z_n = a_n + ib_n$ ($n = 1, 2, \cdots$), where a_n and b_n are real numbers. Then the series $\sum_{n=0}^{\infty} z_n$ converges to $z = a + ib$ for real numbers a and b iff $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = b$.

Proof. Apply [3] Proposition 3.5 to the sequence $S_N(z_n) = A_n + iB_N = \left(\sum_{n=0}^{N} a_n\right) + i\left(\sum_{n=0}^{N} b_n\right)$. □

Example 1.2.4. Consider the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{i}{2^n}\right)$. $\sum \frac{1}{n}$ diverges. Thus, even though $\sum \frac{1}{2^n}$ converges, the whole series diverges.

1.2.3 Sequences of Complex Functions

Consider a series of functions $\{f_n(z)\}_{n=0}^{\infty}$ commonly defined on a set $E \subseteq \mathbb{C}$. Several notions of convergence are defined:

1. **Pointwise convergence (PC)**: $\forall \varepsilon > 0, \forall z \in E, \exists N(\varepsilon, z) \in \mathbb{N}$, s.t. $\forall n > N:\ |f_n(z) - f(z)| < \varepsilon$;
2. **Uniform convergence (UC)**: $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$, s.t. $\forall z \in E, \forall n > N:\ |f_n(z) - f(z)| < \varepsilon$;

2′. [Equivalent definition of uniform convergence]: $\sup\{|f_n(z) - f(z)| : z \in E\} \to 0$ as $n \to \infty$.

3. **Absolute convergence (AC)**: pointwise convergence of $\{|f_n(z)|\}_{n=0}^{\infty}$.
4. **Local uniform convergence (LUC)**: $\forall z \in E$, there is a neighborhood U of z in E such that the sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges uniformly.
5. **Compact convergence (CC)**: For each compact set $K \subseteq E$, the sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges uniformly.

It turns out that for functions on $\mathbb{C}$ (and in fact on a large class of reasonably nice spaces), the last two notions are equivalent.

Proposition 1.2.5. Let E be an open set in $\mathbb{C}$. Then the sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges locally uniformly in E iff it converges compactly in E.

Proposition 1.2.6. Sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges compactly in $B(a,R)$ iff it converges in $\overline{B}(a,r)$ for every $0<r<R$.

Proof. Each closed disk is compact. Each compact set is closed and bounded and is thus contained in some closed disk in $B(a,R)$. □

Example 1.2.7. A simple example, with E the open unit disk, is provided by the sequence $f_n(z)=z^n$. We notice that $\sup\{|z^n| : z\in B(0,1)\}=\sup\{|z|^n : |z|\in[0,1)\}=\sup[0,1)=1$. Thus, $\{z^n\}$ does not uniformly converge. However, if $0<r_0<1$ then this sequence converges uniformly to 0 in the disk $|z|<r_0$, and so it converges locally uniformly to 0 in the disk $|z|<1$.

1.2.4 Series of Complex Functions

Consider a series of functions $\{f_n(z)\}_{n=0}^{\infty}$ commonly defined on a set $E\subseteq\mathbb{C}$ and the sequence of partial sums $\{S_N(f_n(z))\}$. We say the series converges pointwise/absolutely/uniformly/locally uniformly/compactly if the sequence $\{S_N(f_n(z))\}$ does so. We restate them:

1. **Pointwise convergence (PC)**: $\forall\varepsilon>0,\forall z\in E,\exists K(\varepsilon,z)\in\mathbb{N}$, s.t. $\forall N>K : |S_N(f_n(z))(z)-f(z)|<\varepsilon$;
2. **Uniform convergence (UC)**: $\forall\varepsilon>0,\exists K(\varepsilon)\in\mathbb{N}$, s.t. $\forall z\in E,\forall N>K : |S_N(f_n(z))(z)-f(z)|<\varepsilon$;
3. **Absolute convergence (AC)**: pointwise convergence of $\sum_{n=0}^{\infty}|f_n(z)|$, i.e., $\{S_N(|f_n(z)|)\}_{N=1}^{\infty}$.
4. **Local uniform convergence (LUC)**: $\forall z\in E$, there is a neighborhood U of z in E such that the sequence $\{S_N(f_n(z))\}_{N=1}^{\infty}$ converges uniformly.
5. **Compact convergence (CC)**: For each compact set $K\subseteq E$, the sequence $\{S_N(f_n(z))\}_{N=1}^{\infty}$ converges uniformly.

To prove uniform convergence, one usually strengthens the inequality by finding $P_n(z)$ and Q_n such that

$$|S_N(f_n(z))-f(z)|\le P_n(z)\le Q_n$$

and then finding K for which it is true under $N>K$ by $Q_n<\varepsilon$. To show a series of functions is not uniform convergent, one proves the negation, which is obtained by switching existential and universal quantifiers and negating the statement:

$$\exists\varepsilon>0,\forall K\in\mathbb{N},\exists z_0\in E,\exists N_0>K \text{ s.t. } |S_{N_0}(f_n(z))-f(z_0)|>\varepsilon$$

Just like pointwise Cauchy and uniform Cauchy for sequence of functions, we have pointwise Cauchy and uniform Cauchy for complex series. We will use the last one.

Theorem 1.2.8. [Cauchy criterion for uniform convergence]

The series $\sum_{n=0}^{\infty}f_n(z)$ converges uniformly on E iff $\forall\varepsilon>0$, $\exists N(\varepsilon)\in\mathbb{N}$, such that $\forall z\in E$,

$$|f_{n+1}(z)+\cdots+f_{n+p}(z)|<\varepsilon\quad(p=1,2,\cdots)$$

Theorem 1.2.9. [Weierstrass M-test]

Suppose that $\{f_n(z)\}_{n=0}^{\infty}$ is a sequence of complex-valued functions defined on a set E, and that there is a sequence of non-negative numbers M_n satisfying the conditions

- $|f_n(z)|\le M_n$ for all $n\ge 0$ and all $z\in E$;
- $\sum_{n=0}^{\infty}M_n$ converges.

Then the series $\sum_{n=0}^{\infty}f_n(z)$ converges absolutely and uniformly on E.

Proof. Convergence is absolute by comparison test. It is also uniform by applying Cauchy criterion for uniform convergence to the following

$$|f_n(z) + \cdots + f_{n+p}(z)| \le |f_n(z)| + \cdots + |f_{n+p}(z)| \le M_n + \cdots + M_{n+p}$$

□

Weierstrass M-test is often used in combination with the **uniform limit theorem** (see, e.g., [2] Theorem 21.6). Together they say that if, in addition to the above conditions, the functions f_n are continuous on E, then the series converges to a continuous function $f(z)$. A natural question of concern is about the convergence of the termwise differentiation and integration of the sum and that of the limit function. We showed in [3] Corollary 5.33 for the integration of a convergent sequence of complex function, so $S_N(f_n)(z) \rightrightarrows f(z)$ implies $\int_\gamma f(z)dz = \lim_{N\to\infty} \int_\gamma S_N(f_n)(z)dz$. Counterpart for differentiation needs theorem 1.2.6, which translates in terms of series as

Proposition 1.2.10. Series $\sum_{n=0}^{\infty} f_n(z)$ converges compactly in $B(a, R)$, i.e., converges uniformly for each compact set in $B(a, R)$, iff it converges in $\overline{B}(a, r)$ for every $0 < r < R$.

Example 1.2.11. [Geometric series] $\sum_{n=0}^{\infty} z^n$.

Consider $\sum_{n=0}^{\infty} |z|^n$. Since $1 - |z|^{N+1} = (1 - |z|)(1 + |z| + \cdots + |z|^N)$, we have $\sum_{n=0}^{N} |z|^n = \frac{1-|z|^{N+1}}{1-|z|}$. If $|z| < 1$, then the series converges absolutely. Replacing $|z|$ with z in the above argument, one obtains the limit function $\frac{1}{1-z}$. The convergence is also compact in the disk $|z| < 1$: for every closed disk $\overline{B}(0, r)$ with $0 < r < 1$ the series converges uniformly by applying Weierstrass M-test to $|z^n| \le r^n$. If $|z| > 1$, then $\lim |z|^n = \infty$ and the series diverges.

However, the series does not converge uniformly on $B(0, 1)$. This is a simple consequence of the fact that each function $S_N(z^n) = \sum_{n=0}^{N} z^n$ is bounded while the limit function $f(z) = \frac{1}{1-z}$ is not. Hence each function $S_N - f$ is unbounded, that is, the sup-norm of $S_k - S$ is infinite, in particular the sequence of the sup-norms does not converge to zero. This last assertion is equivalent to the fact that $\{S_N\}$ does not converge uniformly to f.

Theorem 1.2.12. [Termwize Differentiation of Series]

Suppose we have

(a) a sequence of functions $f_n(z)$ $(n = 1, 2, \ldots)$ that are analytic in the region D;

(b) the series $\sum_{n=1}^{\infty} f_n(z)$ converge compactly to the function $f(z)$ inside D: $f(z) = \sum_{n=1}^{\infty} f_n(z)$.

Then

(1) the function $f(z)$ is analytic in the region D;

(2) for all $z \in D$ and $p = 1, 2, \ldots$, we have $f^{(p)}(z) = \sum_{n=1}^{\infty} f_n^{(p)}(z)$.

(3) The series $\sum_{n=1}^{\infty} f_n^{(p)}(z)$ converges compactly to $f^{(p)}(z)$ in D.

Proof. The third is left as an exercise. We prove the other two.

(1) Let z_0 be any point in D, then there exists $\rho > 0$, such that the closed disk $\bar{K} : |z - z_0| \leqslant \rho$ is completely contained within D. If C is any contour within the disk $K : |z - z_0| < \rho$, then by Cauchy's integral theorem we have

$$\int_C f_n(z)\mathrm{d}z = 0, \quad n = 1, 2, \ldots,$$

Since the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on $\bar{K}$, and $f_n(z)$ is continuous, by uniform limit theorem, we know that $f(z)$ is continuous on $\bar{K}$. From [3] Corollary 5.33, we have

$$\int_C f(z)\mathrm{d}z = \sum_{n=1}^{\infty} \int_C f_n(z)\mathrm{d}z = 0,$$

Thus, by Morera's theorem, we know that $f(z)$ is analytic within K, that is, $f(z)$ is analytic at the point z_0. Since z_0 was arbitrary, $f(z)$ is analytic in the region D.

(2) Let z_0 be any point in D, then there exists $\rho > 0$, such that the closed disk $\bar{K} : |z - z_0| \leqslant \rho$ is completely contained within D, and the boundary of $\bar{K}$ is the circular path $\Gamma : |z - z_0| = \rho$. Hence, by [3] Theorem 8.5, we have

$$f^{(p)}(z_0) = \frac{p!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta, \quad (p = 1, 2, \ldots),$$
$$f_n^{(p)}(z_0) = \frac{p!}{2\pi i} \int_\Gamma \frac{f_n(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta,$$

On Γ, by condition (b) we know the series

$$\frac{f(\zeta)}{(\zeta - z_0)^{p+1}} = \sum_{n=1}^{\infty} \frac{f_n(\zeta)}{(\zeta - z_0)^{p+1}}$$

converges uniformly. Thus, by [3] Corollary 5.33, we get

$$\int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta = \sum_{n=1}^{\infty} \int_\Gamma \frac{f_n(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta,$$

Multiplying both sides by $\frac{p!}{2\pi i}$, we obtain the desired result:

$$f^{(p)}(z_0) = \sum_{n=1}^{\infty} f_n^{(p)}(z_0) \quad (p = 1, 2, \cdots).$$

□

1.3 Power Series

A power series about a is an infinite series of the form $\sum_{n=0}^{\infty} c_n(z - a)^n$. If a power series converges to a function f in a given region, we shall say that the series represents f in that region. It is important, however, to distinguish the series from the function it represents. For example, as shown in the last section, the series $\sum_{n=0}^{\infty} z^n$ represents the function $\frac{1}{1-z}$ in the disk $|z| < 1$. However, the series is not "equal" to the function, in a formal sense, even though we can write $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. The same function is represented by other power series; for example, it is represented by the series $\sum_{n=0}^{\infty} 2^{-n-1}(z + 1)^n$ in the larger disk $|z + 1| < 2$, as the reader will easily verify. A power series is best thought of as a formal sum, uniquely determined once its center and its coefficients have been specified.

1.3.1 Power Series Representation of Analytic Function

Our final goal of this subsection is to show that a power series is analytic and that an analytic function can be represented by power series. We first continue studying the convergence of series.

Proposition 1.3.1. [Abel's Theorem] If the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges on a point $z_1(\neq a)$, then it converges absolutely and compactly in the open disk with center a and radius $|z_1 - a|$, i.e., $K : |z - a| < |z_1 - a|$. The series diverges for $|z - a| > |z_1 - a|$ instead.

Proof. Let z be any point in K. Since $\sum_0^\infty c_n\left(z_1-a\right)^n$ converges, each of its term must be bounded: $\exists M>0$ such that

$$\left|c_n\left(z_1-a\right)^n\right| \leq M \quad(n=0,1,2,\cdots),$$

Therefore,

$$\left|c_n(z-a)^n\right|=\left|c_n\left(z_1-a\right)^n\left(\frac{z-a}{z_1-a}\right)^n\right| \leq M\left|\frac{z-a}{z_1-a}\right|^n,$$

Since $\forall z \in K$, $|z-a|<\left|z_1-a\right| \Rightarrow\left|\frac{z-a}{z_1-a}\right|<1 \Rightarrow$ the geometric series

$$\sum_{n=0}^{\infty} M\left|\frac{z-a}{z_1-a}\right|^n$$

converges for each $z \in K$. Thus $\sum c_n(z-a)^n$ converges absolutely in K. Besides, for any closed disk $\overline{K}_\rho$ in K, $\overline{K}_\rho:|z-a| \leq \rho\left(0<\rho<\left|z_1-a\right|\right)$, we have

$$\left|c_n(z-a)^n\right| \leq M\left|\frac{z-a}{z_1-a}\right|^n \leq M\left(\frac{\rho}{\left|z_1-a\right|}\right)^n,$$

By convergence of the last geometric series, apply Weierstrass M-test to see $\sum_0^\infty c_n(z-a)^n$ converges uniformly in $\overline{K}_\rho$. Then use 1.2.10. The fact that The series diverges for $|z-a|>\left|z_1-a\right|$ instead is proved by way of contradiction. □

If a power series has no such $z_1 \neq a$ for which it converges, then the series only converges at $z=a$. For example

$$1+z+2^2 z^2+\cdots+n^n z^n+\cdots$$

only converges at $z=0$.

The power series can also converge (pointwise) for all z. For example

$$1+z+\frac{z^2}{2^2}+\cdots+\frac{z^n}{n^n}+\cdots.$$

For any fixed z, after some n, it has $\frac{|z|}{n}<\frac{1}{2}$. Thus, $\left|\frac{z^n}{n^n}\right|<\left(\frac{1}{2}\right)^n$ shows that it is dominated by a convergent geometric series for every z. The convergence is absolute and compact.

By Abel's theorem, if the power series does not fall into the above two cases, then the power series converges for at least $|z-a|<\left|z_1-a\right|$ for each z_1 on which it converges (pointwise). Then we can let R be the supremum of $\left|z_1-a\right|$ ranging over all z_1 on which it converges (pointwise) and see that the series converges for $|z-a|<R$ and diverges for $|z-a|>R$. Obviously, a number R making the series "converge for $|z-a|<R$ and diverge for $|z-a|>R$" is unique.

Definition 1.3.2. Let $\sum_{n=0}^{\infty} c_n\left(z-a\right)^n$ be a power series. We define the **radius of convergence** of the series R as the unique number such that the series converges in $|z-a|<R$ and diverges in $|z-a|>R$. If $R>0$ then the series converges absolutely and compactly in the disk $|z-a|<R$; if $R<\infty$ then the series diverges at each point of the region $|z-a|>R$. R does not give information for situation on the circle $|z-a|=R$.

Presently we shall obtain a general expression for the radius of convergence of a power series in terms of its coefficients.

Theorem 1.3.3 (Cauchy-Hadamard Theorem)**.** Consider a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$. Then the radius of convergence R is given by

$$R=\left(\limsup_{n \rightarrow \infty}\left|c_n\right|^{1/n}\right)^{-1}. \tag{1.1}$$

That is, it satisfies

(a) The series converges absolutely for $|z-a|<R$;

(b) The series diverges for $|z-a|>R$;

(c) The series converges for every closed disk $|z-a|<r$ where $r<R$.

Proof. Assume $a=0$. Assume $0<R<\infty$ (the edge cases $R=0$ and $R=\infty$ are left as an exercise). Due to 1.1.1 (a),

$$\forall \varepsilon>0, \exists N \text{ s.t. } n>N \Rightarrow \frac{1}{R}-\varepsilon<\left|c_n\right|^{1/n}<\frac{1}{R}+\varepsilon. \tag{1.2}$$

So $|c_n|<\left(\frac{1}{R}+\varepsilon\right)^n$ for $n>N$. Let $z\in B(0,R)$, i.e., $|z|<R$, we have $|z|\left(\frac{1}{R}+\varepsilon\right)<1$ for some fixed $\varepsilon>0$ chosen small enough. That implies that for $n>N$ (for some large enough N as a function of ε),

$$\sum_{n=N}^{\infty}\left|c_n z^n\right|<\sum_{n=N}^{\infty}\left[\left(\frac{1}{R}+\varepsilon\right)|z|\right]^n,$$

so the series is dominated by a convergent geometric series, and hence converges. For (b), when $|z|>R$, we evoke the other side of (1.2): $|c_n|>\left(\frac{1}{R}-\varepsilon\right)^n$ for $n>N$. Besides, $|z|\left(\frac{1}{R}-\varepsilon\right)>1$ for some small enough fixed $\varepsilon>0$. Thus

$$\sum_{n=N}^{\infty}\left|c_n z^n\right|>\sum_{n=N}^{\infty}\left[\left(\frac{1}{R}-\varepsilon\right)|z|\right]^n$$

Then

$$\left(\frac{1}{R}-\varepsilon\right)^n<|c_n|$$

so the power series diverges as the geometric series diverges.

For (c), one chooses ρ between r and R and then evoke Weierstrass M-test for $|c_n z^n|<\left(\frac{r}{\rho}\right)^n$. □

Theorem 1.3.4 (d'Alambert Test). If $\sum c_n(z-a)^n$ is a given power series with radius of convergence R, then

$$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

if this limit exists.

Proof. See [1] proposition 1.4. □

Example 1.3.5. Find radius of convergence of $\sum_{n=0}^{\infty}(3+4i)^n(z-i)^{2n}$

Solution. Notice that the coefficients of the odd terms are 0, so we cannot apply the formula directly. Let

$$f_n(z)=(3+4i)^n(z-i)^{2n}$$

Then

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right|=\lim_{n\to\infty}\left|\frac{(3+4i)^{n+1}(z-i)^{2n+2}}{(3+4i)^n(z-i)^{2n}}\right|=\lim_{n\to\infty}\left|(3+4i)(z-i)^2\right|=5|z-i|^2$$

When $5|z-i|^2<1$, i.e., $|z-i|<\frac{\sqrt{5}}{5}$, the power series is absolutely convergent. When $5|z-i|^2>1$, i.e., $|z-i|>\frac{\sqrt{5}}{5}$, the power series diverges. Thus $R=\frac{\sqrt{5}}{5}$.

Definition 1.3.6. Function $f(z)$ is said to be **representable by power series** in U if $\forall B(a,r)\subseteq U$, there corresponds some power series $\sum_{n=0}^{\infty}c_n(z-a)^n$ that converges in $B(a,r)$ and equals $f(z)$.

Theorem 1.3.7 (Power series is analytic and can be differentiated termwize)**.** Power series $\sum_{n=0}^{\infty} c_n(z-a)^n$, denoted as $f(z)$, is analytic on $B(a,R)=\{z:|z-a|<R\}$, where $R=$ radius of convergence. Besides, for $z\in B(a,R)$,

$$f'(z)=\sum_{n=0}^{\infty} nc_n(z-a)^{n-1}$$

Thus, if f is representable by power series in an open set $U\subseteq\mathbb{C}$, then $f\in H(U):=$ the set of all analytic functions on U and derivative is given above.

Proof. We can assume $a=0$ becasue we can apply chain rule with $g(z)=z-a$ to $f(z)=\sum c_n z^n$ on each $z\in B(0,R)$, and R is defined regardless of a. We write

$$f(z)=\sum_{n=0}^{\infty} c_n z^n=\underbrace{\sum_{n=0}^{N} c_n z^n}_{S_N(z)}+\underbrace{\sum_{n=N+1}^{\infty} c_n z^n}_{E_N(z)}, \quad g(z)=\sum_{n=1}^{\infty} nc_n z^{n-1}.$$

The claim is that f is differentiable on $B(0,R)$ and its derivative is the power series g. Since $\lim n^{1/n}=\lim e^{\log n^{\frac{1}{n}}}=1$, it is easy to see that $f(z)$ and $g(z)$ have the same radius of convergence by using 1.1.7. Fix z_0 with $|z_0|<r<R$. We wish to show that $\frac{f(z_0+h)-f(z_0)}{h}$ converges to $g(z_0)$ as $h\to 0$. Observe that

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)=&\left(\frac{S_N(z_0+h)-S_N(z_0)}{h}-S_N'(z_0)\right)\\&+\left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right)+\left(S_N'(z_0)-g(z_0)\right)\end{aligned}$$

The first term converges to 0 for $h\to 0$ for any fixed N, because $S_N(z)$ is a polynomial. To bound the second term, fix some $\varepsilon>0$, and note that, if we assume that not only $|z_0|<r$ but also $|z_0+h|<r$ (an assumption that's clearly satisfied for h close enough to 0) then

$$\begin{aligned}\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| &\leq \sum_{n=N+1}^{\infty}|a_n|\left|\frac{(z_0+h)^n-z_0^n}{h}\right|\\ &=\sum_{n=N+1}^{\infty}|a_n|\left|\frac{h\sum_{k=0}^{n-1} h^k(z_0+h)^{n-1-k}}{h}\right|\\ &\leq \sum_{n=N+1}^{\infty}|a_n| n r^{n-1},\end{aligned}$$

where we use the algebraic identity

$$a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1}\right).$$

The last expression in this chain of inequalities is the tail of an absolutely convergent series, so can be made $<\varepsilon$ be taking N large enough (before taking the limit as $h\to 0$).

Third, when choosing N also make sure it is chosen so that $|S_N'(z_0)-g(z_0)|<\varepsilon$, which of course is possible since $S_N'(z_0)\to g(z_0)$ as $N\to\infty$. Finally, having thus chosen N, we get that

$$\limsup_{h\to 0}\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right|\leq 0+\varepsilon+\varepsilon=2\varepsilon.$$

Since ε was an arbitrary positive number, this shows that $\frac{f(z_0+h)-f(z_0)}{h}\to g(z_0)$ as $h\to 0$, as claimed. $\square$

Corollary 1.3.8. Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ have radius of convergence $R > 0$. Then by applying the theorem to f', $f'' = (f')'$, $\cdots$, the function f is infinitely differentiable on $B(a,R)$ and its k-th derivative is given by a power seires with the same radius of convergence

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1)c_n(z-a)^{n-k}$$

for all $k \geq 1$ and $|z-a| < R$. In particular, $f^{(n)}(a) = n!c_n$, or $c_n = \frac{1}{n!}f^{(n)}(a)$.

We now show the converse.

Theorem 1.3.9. Let $f \in H(U)$ where $U \subseteq \mathbb{C}$ is open. Then f is representable by power series in U. That is, for any $\overline{B}(a, r_0) \subseteq U$, f has a power series expansion at a

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z-a)^n$$

that is convergent for all $z \in B\,(a, r_0)$, where $c_n = f^{(n)}\,(a)\,/n!$.

Proof. The idea is that Cauchy's integral formula ([3] Corollary 7.22) gives us a representation of $f(z)$ as a weighted "sum" (=an integral, which is a limit of sums) of functions of the form $z \mapsto (\xi - z)^{-1}$. Each such function has a power series expansion since it is, more or less, a geometric series, so the sum also has a power series expansion. Let $r < r_0$. Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(\xi)}{\xi - z}d\xi, \quad z \in B(a,r)$$

We write

$$\frac{1}{\xi - z} = \frac{1}{(\xi-a)-(z-a)} = \frac{1}{\xi-a}\cdot\frac{1}{1-\left(\frac{z-a}{\xi-a}\right)} = \frac{1}{\xi-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n$$

where $\xi \in \partial B(a, r_0)$. Since $z \in B(a,r)$, we see $\left|\frac{z-a}{\xi-a}\right| = \frac{|z-a|}{r_0} < \frac{r}{r_0} < 1$, so the geometric series converges uniformly for $\xi \in \partial B(a, r_0)$. Now,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\partial B(a,r)} f(\xi)\frac{1}{\xi-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n d\xi \\
&= \frac{1}{2\pi i}\int_{\partial B(a,r)} \lim_{N\to\infty}\sum_{n=0}^{N}\frac{f(\xi)}{\xi-a}\left(\frac{z-a}{\xi-a}\right)^n d\xi \\
&\overset{\text{unif conv.+linearity of int}}{=\!=\!=\!=\!=} \lim_{N\to\infty}\frac{1}{2\pi i}\sum_{n=0}^{N}\int_{\partial B(a,r)}\frac{f(\xi)}{\xi-a}\left(\frac{z-a}{\xi-a}\right)^n d\xi \\
&= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{\partial B(a,r)}\frac{f(\xi)}{(\xi-a)^{n+1}}d\xi\right)}_{\text{only depends on } a\text{, called } c_n}(z-a)^n \\
&= \sum_{n=0}^{\infty} c_n(z-a)^n, \quad z \in B(a,r), r < r_0
\end{aligned}$$

We can let $r \to r_0$ since c_n does not depend on r ($c_n = f^{(n)}(a)/n!$ by previous result).

$\square$

Remark 1.3.10. We have a new proof showing that an analytic function is infinitely differentiable due to the corollary 1.3.8. Above theorem also gives a new proof of the n-th derivative of analytic function aside [3] Theorem 8.5.

1.3.2 Power Series on $|z-a|=R$

1.3.3 Operations of Power Series

Addition

Let $\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$ and $\sum_{n=0}^{\infty} b_n\left(z-z_0\right)^n$ be two power series with the same center. Suppose the series $\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$ has a positive radius of convergence R_1 and the series $\sum_{n=0}^{\infty} b_n\left(z-z_0\right)^n$ has a positive radius of convergence R_2.

Exercise 1.3.11. Show that $\sum_{n=0}^{\infty}(a_n+b_n)(z-z_0)^n$ has $R \geq \min\{R_1, R_2\}$.

Multiplication

The **Cauchy product** of the two series is by definition the power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ whose n-th coefficient is given by $c_n=\sum_{k=0}^n a_k b_{n-k}$. It arises when one forms all products $a_j\left(z-z_0\right)^j b_k\left(z-z_0\right)^k$, adds for each n the ones with $j+k=n$, and sums the resulting terms.

Proposition 1.3.12. Their Cauchy product converges in the disk $\left|z-z_0\right|<\min \left\{R_1, R_2\right\}$ to the product of the functions represented by the two original series.

Proof. We can assume without loss of generality that $z_0=0$. Suppose $|z|<\min \left\{R_1, R_2\right\}$. For N a positive integer we have

$$
\begin{aligned}
& \left(\sum_{j=0}^N a_j z^j\right)\left(\sum_{k=0}^N b_k z^k\right)-\sum_{n=0}^N c_n z^n \\
= & \sum_{0 \leq j, k \leq N} a_j b_k z^{j+k}-\sum_{n=0}^N \sum_{j+k=n} a_j b_k z^{j+k} \\
= & \sum_{\substack{0 \leq j, k \leq N \\ j+k>N}} a_j b_k z^{j+k} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left|\left(\sum_{j=0}^N a_j z^j\right)\left(\sum_{k=0}^N b_k z^k\right)-\sum_{n=0}^N c_n z^n\right| \\
\leq & \sum_{\substack{j \leq j, k \leq N \\ j+k>N}}\left|a_j b_k z^{j+k}\right| \\
\leq & \sum_{\frac{N}{2}<\max \{j, k\} \leq N}\left|a_j b_k z^{j+k}\right| \\
\leq & \left(\sum_{j>\frac{N}{2}}\left|a_j z^j\right|\right)\left(\sum_{k=0}^N\left|b_k z^k\right|\right)+\left(\sum_{j=0}^N\left|a_j z^j\right|\right)\left(\sum_{k>\frac{N}{2}}\left|b_k z^k\right|\right) \\
\leq & \left(\sum_{j>\frac{N}{2}}\left|a_j z^j\right|\right)\left(\sum_{k=0}^{\infty}\left|b_k z^k\right|\right)+\left(\sum_{j=0}^{\infty}\left|a_j z^j\right|\right)\left(\sum_{k>\frac{N}{2}}\left|b_k z^k\right|\right) .
\end{aligned}
$$

The last expression tends to 0 as $N \longrightarrow \infty$, because both series $\sum_{j=0}^{\infty}\left|a_j z^j\right|$ and $\sum_{k=0}^{\infty}\left|b_k z^k\right|$ converge. In view of the preceding inequality, therefore, we can conclude that

$$\sum_{n=0}^{\infty} c_n z^n=\left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{k=0}^{\infty} b_k z^k\right)$$

as desired. □

Division

Suppose the power series $\sum_{n=0}^{\infty} b_n\left(z-z_0\right)^n$ and $\sum_{n=0}^{\infty} c_n\left(z-z_0\right)^n$ have positive radii of convergence and so represent holomorphic functions g and h, respectively, in disks with center z_0. Suppose also that $g\left(z_0\right)= b_0 \neq 0$. The quotient $f=h/g$ is then holomorphic in some disk with center z_0. Then f is represented in that disk by a power series $\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$, how does one find the coefficients a_n in terms of the coefficients b_n and c_n?

A method that always works in principle uses the Cauchy product, according to which

$$c_n=\sum_{k=0}^n a_k b_{n-k}, \quad n=0,1,\ldots$$

From this we can conclude that $a_0=c_0/b_0$ and

$$a_n=\frac{1}{b_0}\left(c_n-\sum_{k=0}^{n-1} a_k b_{n-k}\right), \quad n=1,2,\ldots$$

The last equality expresses a_n in terms of $c_n, b_0, \ldots, b_n$ and $a_0, \ldots, a_{n-1}$, enabling one to determine the coefficients a_n recursively starting from the initial value $a_0=c_0/b_0$.

Exercise 1.3.13. Use the scheme above to determine the power series with center 0 representing the function $f(z)=1/\left(1+z+z^2\right)$ near 0 . What is the radius of convergence of the series?

Chapter 2

Zeros and Residues

The fact that every holomorphic function is locally the sum of a convergent power series has a large number of interesting consequences. A few of these are developed in this chapter.

2.1 Isolated Zeros

We will see that zeros of non-vanishing analytic functions are isolated and analytic functions that agree locally actually agree globally. We recall that $x \in X$ is a limit point of A if for $\forall \varepsilon > 0$, $B(x, \varepsilon) \cap (A - \{x\}) \neq \varnothing$, and it is easy to show that for metric space (X, d), x is a limit point of A if and only if there exists a sequence $\{x_i\}$ in A such that $x_i \to x$ ($\Leftarrow$ is because each ball $B(x, \varepsilon)$ of x contain infinitely many x_i as along as $i > N(\varepsilon)$; for $\Rightarrow$, see [3] Remark 3.39 to find such sequence). We call a point x in A an **isolated point** if it is not a limit point of A. Thus, $x \in A$ is an isolated point if there is some $B(x, \varepsilon)$ intersecting no other points in A, or equivalently, there exists no sequence $\{x_i\}$ in A converging to x.

Theorem 2.1.1. Suppose U is a region (an open and connected subset), $f \in H(U)$, and

$$Z(f) = \{a \in U : f(a) = 0\}$$

Then either

(i) $Z(f) = U$, or

(ii) $Z(f)$ has no limit point in U.

In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z - a)^m g(z) \quad (z \in U), \tag{2.1}$$

where $g \in H(U)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

Definition 2.1.2. The integer m is called the **order** or **multiplicity** of the zero which f has at the point a. Clearly, $Z(f) = U$ if and only if f is identically 0 in U. We call $Z(f)$ the **zero set** of f. Analogous results hold of course for the set of α-points of f (α-level set), i.e., the zero set of $f - \alpha$, where α is any complex number.

Proof. Let $A = (Z(f))^{acc} \cap U$ be the set of all limit points of $Z(f)$ in U. Since f is continuous and for each $a \in A$, $\exists \{z_i\} \in Z(f)$ s.t. $z_i \to a$ we have $f(a) = f(\lim z_i) = \lim f(z_i) = \lim 0 = 0$. Thus, $A \subset Z(f)$.

Fix $a \in Z(f)$, and choose $r > 0$ so that $B(a,r) \subset U$. By 1.3.9,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad (z \in B(a,r)) \tag{2.2}$$

There are now two possibilities. Either

(a) all c_n are 0 , in which case $B(a,r) \subset A$; or

(b) there is a smallest integer m (necessarily positive, since $f(a) = c_0 = 0$) such that $c_m \neq 0$.

In case (b), define

$$g(z) = \begin{cases} (z-a)^{-m} f(z), & z \in U - \{a\}, \\ c_m, & z = a. \end{cases}$$

Then (2.1) holds and $g(a) = c_m \neq 0$. We are left with showing $g \in H(U)$ to complete the proof. It is clear that $g \in H(U - \{a\})$, so we need to show it is complex differentiable at a. In fact,

$$\begin{aligned} g(z) &= (z-a)^{-m} f(z) \\ &= (z-a)^{-m} \sum_{n=m}^{\infty} c_n (z-a)^n \\ &= \sum_{n=m}^{\infty} c_n (z-a)^{n-m} \\ &= \sum_{n=0}^{\infty} c_{n+m} (z-a)^n \quad (z \in B(a,r) \setminus \{a\}) \end{aligned}$$

This is also true for $z = a$: $g(a) = c_m$ and $\sum_{n=0}^{\infty} c_{n+m}(a-a)^n = c_{0+m} = c_m$. As $g(z) = \sum_{n=0}^{\infty} c_{n+m}(z-a)^n$ is a power series representation for $z \in B(a,r)$, it follows that $g \in H(B(a,r))$. In particular, g is analytic at $z = a$, so $g \in H(U)$. Moreover, since $g(a) \neq 0$, the continuity of g shows that there is a neighborhood of a in which g has no zero. f is nonzero in the same neighborhood by (2.1), so a is an isolated point of $Z(f)$.

Therefore, if $a \in A = (Z(f))^{acc} \cap U$ (recall a is point in $Z(f)$), then case (b) cannot occur. Thus $a \in A \Rightarrow$ case (a):$B(a,r) \subset A$, which implies that A is open. If $B = U - A$, it is clear from the definition of A as a set of limit points that B is open. Thus U is the union of the disjoint open sets A and B. Since U is connected, we have either $A = U$, in which case $Z(f) = U$ (so (i) and thus (a)), or $A = \varnothing$ (which is case (ii) and note that $A = \varnothing$ implies that $B(a,r) \subset A$ is impossible and thus (b) rather than (a) must be the case). Besides, in case $A = \varnothing$, $Z(f)$ has at most finitely many points in each compact subset of U, and since U is σ-compact, $Z(f)$ is at most countable. □

Note: The theorem fails if we drop the assumption that U is connected: If $U = U_0 \cup U_1$, and U_0 and U_1 are disjoint open sets, put $f = 0$ in U_0 and $f = 1$ in U_1. Then $Z(f) = U_0 \neq U$. Each $z \in U_0$ is a limit point of U_0 and is in U.

Corollary 2.1.3. f and g are holomorphic functions in a region U. If there is some sequence $\{x_i\}$ in U s.t. $f(x_i) = g(x_i)$ and $x_i \to x$ for a point $x \in U$, then $f = g$ on U. Thus, if $f(z) = g(z)$ for all z in some set A which has a limit point x in U, then $f = g$ on U. In particular, A can be an open set or the trace of a path in U.

Proof. Apply previous theorem to $f - g$. Note that $(f-g)(x) = (f-g)(\lim x_i) = \lim (f-g)(x_i) = \lim 0 = 0$ implies that $x \in Z(f-g)$. Since $x_i \to x$, x is a limit point. Previous theorem then says it has to be the case that $Z(f-g) = U$. □

This is the result we alluded to when we define e^z: if we would have another $g \in H(\mathbb{C})$ with $g(x) = e^x$ for $x \in \mathbb{R}$, then in fact $g(z) = e^z$ $\forall z \in \mathbb{C}$.

2.2 Isolated Singularities

2.2.1 Classification of Isolated Singularities

Definition 2.2.1. If $a \in U$ and $f \in H(U - \{a\})$, then f is said to have an **isolated singularity** at the point a. If f can be defined at a so that the extended function is holomorphic in U, the singularity is said to be **removable**.

Theorem 2.2.2. [Criterion for removable singularity] Suppose $f \in H(U - \{a\})$ and f is bounded in $B'(a, r) := \{z : 0 < |z - a| < r\}$, for some $r > 0$. Then f has a removable singularity at a.

Remark 2.2.3. We previously had a similar result, but there we assumed f is continuous in U instead of being bounded.

Proof. Define

$$h(z) = \begin{cases} (z-a)^2 f(z), & z \in U - \{a\}, \\ 0, & z = a. \end{cases}$$

h is evidently differentiable at $U - \{a\}$, and

$$\frac{h(z) - h(a)}{z - a} = (z - a)f(z) \to 0$$

as $z \to a$ due to boundedness of f near a. Thus $h \in H(U)$ with $h'(a) = 0$. Thus we can represent h by a power series in $B(a, r) \subset U$:

$$h(z) = \sum_{n=2}^{\infty} c_n (z-a)^n \quad (z \in B(a, r)).$$

Notice that the first two coefficients are zero becasue

$$c_n = \frac{h^{(n)}(a)}{n!}, \text{ and } h(a) = h'(a) = 0$$

We obtain the desired holomorphic extension of f by setting $f(a) = c_2$, for then

$$\sum_{n=0}^{\infty} c_{n+2}(z-a)^n \quad (z \in B(a, r))$$

is a power series representation of f at a: 1. the power series has the same radius of convergence as the one representing h; 2. the power series equals f for $z \in B(a, r)$ because $(z - a)^{-2}h(z)$ agrees with this for $z \neq a$, and both sides equal c_2 for $z = a$ after setting $f(a) = c_2$. □

We note that boundedness of f is only used for showing that $\lim_{z\to a}(z - a)f(z) = 0$. Therefore, we have the following criterion:

Theorem 2.2.4. [Riemann's Criterion on Removable Singularity] Let $U \subset \mathbb{C}$ be an open subset of the complex plane, $a \in U$ a point of U and f holomorphic on $U \backslash \{a\}$. The following are equivalent:

(a) f has a removable singularity at a, i.e., f is holomorphically extendable over a.

(b) f is continuously extendable over a.

(c) There exists a neighborhood of a on which f is bounded.

(d) $\lim_{z\to a}(z - a)f(z) = 0$.

Proof. The direction $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ is clear. $(d) \Rightarrow (c)$ is shown in the proof of the above theorem 2.2.2. Also note that $(b) \Rightarrow (a)$ can be proved by [3] Corollary 8.16. □

We introduce two other isolated singularities and claim that together with removable singularity they are the all isolated singularities.

Theorem 2.2.5. If $a \in U$ and $f \in H(U - \{a\})$, then one of the following three cases must occur:

(i) f has a removable singularity at a.

(ii) There are complex numbers $c_1, \ldots, c_m$, where m is a positive integer and $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

has a removable singularity at a.

(iii) If $r > 0$ and $B(a,r) \subset U$, then $f(B'(a,r))$ is dense in the plane.

In case (ii), f is said to have a **pole of order** m at a. The function

$$\sum_{k=1}^{m} c_k (z-a)^{-k}$$

a polynomial in $(z-a)^{-1}$, is called the **principal part of** f **at** a. It is clear in this situation that $|f(z)| \to \infty$ as $z \to a$. Coefficient c_{-1} is called **residue** of f at a, denoted as $\operatorname{Res}(f; a)$. In case (iii), f is said to have an **essential singularity at** a. A statement equivalent to (iii) is that for any complex number $w \in \mathbb{C}$ there exists a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$.

Proof. Suppose (c) fails. Then we must have some $w \in \mathbb{C}$ and $r > 0$ such that $w \notin \overline{f(B'(a,r))}$; so there is a neighborhood $B(w,\delta)$ of w such that $B(w,\delta) \cap f(B'(a,r)) = \varnothing$, i.e., $|f(z) - w| > \delta$ for $z \in B'(a,r)$. Write $B = B(a,r)$, $B' = B'(a,r)$, and define

$$g(z) = \frac{1}{f(z) - w}, \quad z \in B'$$

Clearly, $g \in H(B')$ and $|g(z)| < \frac{1}{\delta}$. By 2.2.2, g has a removable singularity at $z = a$, so g extends to $g \in H(B)$. If $g(a) \neq 0$, then this means

$$0 \neq |g(a)| = \lim_{\substack{z \to a \\ z \in B'}} |g(z)| = \lim_{\substack{z \to a \\ z \in B'}} \frac{1}{|f(z) - w|}$$

and thus $|f(z) - w|$ has to stay bounded in B', and so does $|f|$. Thus again by 2.2.2, f has a removable singularity at a. (a) holds.

The other case is $g(a) = 0$. We will show that this implies (b). Obviously, g is not identically zero in the connected open set B, so we may write

$$g(z) = (z-a)^m g_1(z), \quad z \in B,$$

for some $m \geq 1$ and $g_1 \in H(B)$ with $g_1(a) \neq 0$ by 2.1.1. Also, g_1 has no zero in B' as $g(z) = \frac{1}{f(z) - w}$ in B'. We define $h = 1/g_1$ in B and then $h \in H(B)$ with h having no zero in B. Now

$$f(z) - w = \frac{1}{g(z)} = (z-a)^{-m} h(z), \quad z \in B'$$

We expand the holomorphic h into power series: $h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$, $z \in B$ with $0 \neq h(a) = b_0$. Thus we get

$$\begin{aligned}
f(z)-w=(z-a)^{-m}h(z) &= \sum_{n=0}^{\infty} b_n(z-a)^{n-m}\\
&= \frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\cdots+\frac{b_{m-1}}{z-a}+\sum_{n=0}^{\infty} b_{n+m}(z-a)^n\\
\Rightarrow f(z)-\frac{b_0}{(z-a)^m}-\frac{b_1}{(z-a)^{m-1}}-\cdots-\frac{b_{m-1}}{z-a} &= G(z):=w+\sum_{n=0}^{\infty} b_{n+m}(z-a)^n
\end{aligned}$$

where $G(z)$ is analytic in B. Thus (b) holds with $c_{-m}=b_0,\cdots,c_{-1}=b_{m-1}$. □

We put following observations without proof.

Proposition 2.2.6.

(a) If both $\lim_{z\to a} f(z)$ and $\lim_{z\to a}\frac{1}{f(z)}$ exist, then a is a removable singularity of both f and $\frac{1}{f}$.

(b) If $\lim_{z\to a} f(z)$ exists but $\lim_{z\to a}\frac{1}{f(z)}$ does not exist (in fact $\lim_{z\to a}|1/f(z)|=\infty$), then a is a zero of f and a pole of $\frac{1}{f}$.

(c) if $\lim_{z\to a} f(z)$ does not exist (in fact $\lim_{z\to a}|f(z)|=\infty$) but $\lim_{z\to a}\frac{1}{f(z)}$ exists, then a is a pole of f and a zero of $\frac{1}{f}$.

(d) If neither $\lim_{z\to a} f(z)$ nor $\lim_{z\to a}\frac{1}{f(z)}$ exists, then a is an essential singularity of both f and $\frac{1}{f}$.

2.2.2 Residue Theorem for One Pole

Suppose now U is open and convex and $f\in H(U-\{a\})$ has a pole at $z=a$. Then we can write

$$f(z)=\sum_{k=1}^{m} c_{-k}(z-a)^{-k}+g(z)$$

for some $g\in H(U)$. Thus, if γ is a closed piecewise C^1 curve in $U-\{a\}$, then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz=\frac{1}{2\pi i}\int_\gamma \frac{c_{-1}}{z-a}dz=\frac{1}{2\pi i}\int_\gamma \frac{\operatorname{Res}(f;a)}{z-a}dz=\operatorname{Res}(f;a)n_\gamma(a)$$

where we used the fact that each $z\mapsto (z-a)^{-k}$, $k>1$ has a primitive $\frac{1}{1-k}(z-a)^{1-k}$ in a neighborhood of γ^* (since $\operatorname{dist}(a,\gamma^*)>0$) and so their integrals over γ vanish. We also invoked the Cauchy theorem to see $g\in H(U)\Rightarrow\int_\gamma g=0$. We will generalize this residue theorem later after we get to the "global Cauchy theorem."

Example 2.2.7. We calculate the integral

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx \quad 0<a<1$$

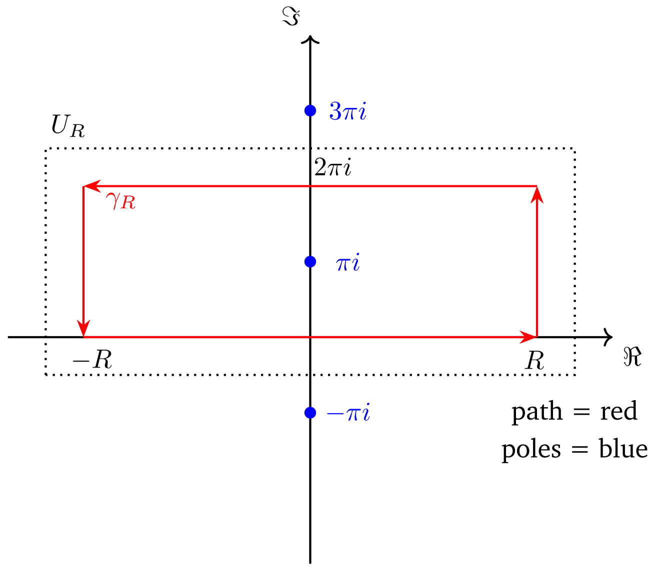
We will show that its value is $\frac{\pi}{\sin(\pi a)}$.

Solution. Let

$$f(z)=\frac{e^{az}}{1+e^z}$$

Step 1 (find poles): Notice that e^{az} is entire and that $1+e^z=0\Leftrightarrow e^{i\theta}=e^{i\pi}\Leftrightarrow\theta=2k'\pi+\pi\,(k=0,\pm1,\cdots)=k\pi\,(k=\pm1,\cdots)$, so the poles are $z=i\theta=k\pi i=\cdots,-3\pi i,-\pi i,\pi i,3\pi i,\cdots$

Step 2 (choose path and U): We consider the path γ_R in the picture, and we choose an open convex set U_R with $\gamma_R^* \subseteq U$ (for instance, U_R can be a small flattening of the box).



Step 3 (apply residue theorem): Now, f is analytic in $U_R \setminus \{\pi i\}$. We apply our toy residue theorem given just before this example to see

$$\int_{\gamma_R} f(z)dz = 2\pi i \mathrm{Res}(f;\pi i)$$

where the winding number $n_{\gamma_R}(\pi i)$ is arguably just 1 as shown in the picture (we will develop tools to systematically justify computation of winding numbers later). Heuristically, we guess the order of the pole πi is 1 (so then it would be that $f(z) = \mathrm{Res}(f;\pi i)(z-\pi i)^{-1} + g(z)$ for analytic g, so $(z-\pi i)f(z) = \mathrm{Res}(f;\pi i) + (z-\pi i)g(z) \to \mathrm{Res}(f;\pi i)$ as $z \to \pi i$). We calculate

$$\begin{aligned}(z-\pi i)f(z) = (z-\pi i)\frac{e^{az}}{1+e^z} = e^{az}\left(\frac{e^z - e^{\pi i}}{z-\pi i}\right)^{-1} \\ \xrightarrow{z\to 2\pi i} e^{a\pi i}\left(\frac{d}{dz}e^z\right)\Big|_{z=\pi i} = e^{a\pi i}e^{\pi i} = -e^{a\pi i}\end{aligned}$$

Thus, indeed, as the limit exists, we must have

$$\mathrm{Res}(f;\pi i) = -e^{a\pi i}$$

Thus,

$$\int_{\gamma_R} f(z)dz = -2\pi i e^{2\pi i}$$

Step 4 (calculate the original integral): We then relate this result to the original real integral. Let

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx =: \lim_{R\to\infty} I_R$$

The integral of f over the top line of the rectangle with orientation from right to left is along $\eta_R(t) =$

$-t+2\pi i,\ t\in[-R,R]$, so

$$\begin{aligned}
\int_{\eta_R} f(z)dz &= \int_{-R}^{R} f(\eta_R(t))\eta_R'(t)dt\\
&= -\int_{-R}^{R}\frac{e^{a(-t+2\pi i)}}{1+e^{-t+2\pi i}}dt\\
&= -e^{2\pi i a}\int_{-R}^{R}\frac{e^{-at}}{1+e^{-t}}dt\\
&\xlongequal[dx=-dt]{x=-t} -e^{2\pi i a}\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}dx \qquad \left(-\int_{R}^{-R}=\int_{-R}^{R}\right)\\
&= -e^{2\pi i a}I_R
\end{aligned}$$

Thus,

$$\int_{\gamma_R} f = (1-e^{2\pi i a})I_R + \int_{\text{left vertical}} f + \int_{\text{right vertical}} f$$

Notice, for instance, right vertical is parametrized by $t\to R+it,\ t\in[0,2\pi]$. Then,

$$\begin{aligned}
\left|\int_{\text{right vertical}} f(z)dz\right| &= \left|\int_0^{2\pi}\frac{e^{a(R+it)}}{1+e^{Rtit}}idt\right|\\
&\le \int_0^{2\pi}\frac{e^{aR}}{e^R-1}dt \qquad \left(e^R-1\ge\frac{e^R}{2}\text{ for large }R\right)\\
&\le Ce^{(a-1)R}\xrightarrow{R\to\infty}0 \qquad (a<1\text{ so }a-1<0)
\end{aligned}$$

Similarly,

$$\left|\int_{\text{left vertical}} f(z)dz\right|\xrightarrow{R\to\infty}0$$

Therefore, by noticing that $\sin x=\frac{e^{ix}-e^{-ix}}{2i}$, we have

$$\begin{aligned}
-2\pi i e^{2\pi i} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz = (1-e^{2\pi i a}I)\\
\Rightarrow I &= -2\pi i\frac{e^{a\pi i}}{1-e^{2\pi i a}} = \frac{2\pi i}{e^{\pi i a}-e^{-\pi i a}} = \frac{\pi}{\sin(\pi a)}
\end{aligned}$$

We prove a useful formula to calculate residue of f at a pole.

Proposition 2.2.8. If $f\in H(U\setminus\{a\})$ has a pole of order n at a, then

$$\operatorname{Res}(f;a)=\lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-a)^n f(z).$$

Proof. For $f\in H(U\setminus\{a\})$ with a pole a of n-th order, we can write

$$f(z)=\sum_{k=1}^{n}c_{-k}(z-a)^{-k}+g(z)$$

for some $g(z)\in H(U)$. Then g is representable by a power series in U. That is, for any $\overline{B}(a,r)\subseteq U$, g has a power series expansion at a

$$g(z)=\sum_{k=0}^{\infty}c_k\,(z-a)^k$$

that is convergent for all $z \in B(a,r)$. Thus,

$$f(z)=\sum_{k=1}^{n} c_{-k}(z-a)^{-k}+g(z)=\sum_{k=1}^{n} c_{-k}(z-a)^{-k}+\sum_{k=0}^{\infty} c_k\,(z-a)^k=\frac{\overbrace{\sum_{k=0}^{\infty} c_{k-n}\,(z-a)^k}^{\varphi(z)}}{(z-a)^n} \tag{2.3}$$

where $\varphi(z)$ is a power series with the same radius of convergence as g, i.e., r, since changing finitely many elements of a sequence does not affect its limsup. Thus $\varphi(z)$ is analytic on $B(a,r)$ as shown in class, and its $(n-1)$-th derivative at a is given by

$$\varphi^{(n-1)}(a)=\frac{(n-1)!}{2\pi i}\int_{\partial B}\frac{\varphi(z)}{(z-a)^n}dz \tag{2.4}$$

by noting that the winding number of $\partial B(a,r)$ around a is 1. This computation is shown during the proof of converse of analyticity of power series, but it can also be inferred from [3] Theorem 8.5. Now we note that equation (2.3) gives $\varphi(z)=f(z)(z-a)^n$ whenever $z\neq a$. Therefore,

$$\varphi^{(n-1)}(a)=\lim_{z\to a}\varphi^{(n-1)}(z)=\lim_{z\to a}\left(\frac{d}{dz}\right)^{n-1}(z-a)^n f(z) \tag{2.5}$$

The residue is computed as

$$\begin{aligned}
\operatorname{Res}(f;a)&=\frac{1}{2\pi i}\int_{\partial B} f(z)dz=\frac{1}{2\pi i}\int_{\partial B}\frac{\varphi(z)}{(z-a)^n}dz\\
&\overset{(2.4)}{=\!=}\frac{1}{2\pi i}\left(\frac{2\pi i}{(n-1)!}\varphi^{(n-1)}(a)\right)\\
&\overset{(2.5)}{=\!=}\lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-a)^n f(z)
\end{aligned}$$

□

Example 2.2.9. Calculate $\operatorname{Res}(f;0)$ for

$$f(z):=\frac{\sinh(z)e^z}{z^5},$$

where $\sinh(z):=(e^z-e^{-z})/2$.

Solution. We now apply this formula to $f(z)=\sinh(z)e^z/z^5$. Since e^z is entire, we see $\sinh(z)=\frac{e^z-e^{-z}}{2}$ is also entire. Therefore, $a=0$ is a pole of order 5 for f. Thus,

$$\begin{aligned}
\operatorname{Res}(f;0)&=\lim_{z\to 0}\frac{1}{4!}\left(\frac{d}{dz}\right)^4 z^5 f(z)\frac{1}{24}\lim_{z\to 0}\left(\frac{d}{dz}\right)^4\sinh(z)e^z\\
&=\frac{1}{24}\lim_{z\to 0}\frac{d^4\left(\frac{e^{2z}}{2}-\frac{1}{2}\right)}{dz^4}=\frac{1}{24}\lim_{z\to 0}8e^{2z}=\frac{1}{3}\cdot 1=\frac{1}{3}
\end{aligned}$$

There is a rather elementary way to see this, if we recall the power series expansions of $\sinh(z)$ and e^z at 0:

$$\sinh(z)=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots$$

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots$$

Then,

$$\operatorname{Res}(f;0)=c_{-1}=\operatorname{coeff}([z(z^3/3!)+z(z^3/6)]/z^5)=\frac{1}{3}$$

2.3 Global Cauchy Theorem

Let $\gamma : [a,b] \to U$ be a piecewise C^1 closed path, where U is convex and open. Suppose $z_0 \in \mathbb{C} \backslash U$. Now $f(z) := \frac{1}{z-z_0}$ is analytic in U, and so by Cauchy theorem on convex set,

$$\begin{aligned} n_\gamma(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-z_0} \\ &= \frac{1}{2\pi i} \int_\gamma f(z) dz = 0 \end{aligned}$$

Thus,

$$n_\gamma(z_0) \quad \forall z_0 \in \mathbb{C} \setminus U$$

Could it be that the validity of Cauchy is not so much about the convexity of U, but rather about this property of γ itself? Is it true that if $\gamma : [a,b] \to U$ is a closed piecewise C^1 path with $n_\gamma(z) = 0 \quad \forall z \in \mathbb{C} \backslash U$, then for all $f \in H(U)$ we have $\int_\gamma f = 0$ even if U is just assumed open? Yes!

And even more is true: we can use formal sums of paths (called cycles) and not just individual paths. To this end, we first quickly study the following concepts of chains and cycles.

2.3.1 Chains and Cycles

Suppose $\gamma_1, \ldots, \gamma_n$ are paths in the plane, and put $K = \gamma_1^* \cup \cdots \cup \gamma_n^*$. Each γ_i induces a linear functional $\widetilde{\gamma}_i$ on the vector space $C(K)$, by the formula

$$\widetilde{\gamma}_i(f) = \int_{\gamma_i} f(z) dz$$

Define

$$\widetilde{\Gamma} = \widetilde{\gamma}_1 + \cdots + \widetilde{\gamma}_n.$$

Explicitly, $\widetilde{\Gamma}(f) = \widetilde{\gamma}_1(f) + \cdots + \widetilde{\gamma}_n(f)$ for all $f \in C(K)$. The above relation suggests that we introduce a "formal sum"

$$\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n = \sum_{i=1}^n \gamma_i$$

and define

$$\int_\Gamma f(z) dz = \widetilde{\Gamma}(f)$$

Then $\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ is merely an abbreviation for the statement

$$\int_\Gamma f(z) dz = \sum_{i=1}^n \int_{\gamma_i} f(z) dz \quad (f \in C(K)).$$

Note that this equation serves as the definition of its left side. The objects Γ so defined are called **chains**. If each γ_j in $\Gamma = \sum_{i=1}^n \gamma_i$ is a closed path, then Γ is called a **cycle**. If each γ_j in $\Gamma = \sum_{i=1}^n \gamma_i$ is a path in some open set U, we say that Γ is a **chain** in U. If $\Gamma = \sum_{i=1}^n \gamma_i$ holds, we define

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*$$

If Γ is a cycle and $\alpha \notin \Gamma^*$, we define the index of α with respect to Γ by

$$\operatorname{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-\alpha},$$

Obviously, $\Gamma = \sum_{i=1}^n \gamma_i$ implies

$$\operatorname{Ind}_\Gamma(\alpha) = \sum_{i=1}^n \operatorname{Ind}_{\gamma_i}(\alpha).$$

If each γ_i in $\Gamma = \sum_{i=1}^n \gamma_i$ is replaced by its opposite path $\overleftarrow{\gamma_i}$, the resulting chain will be denoted by $-\Gamma$. Then

$$\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz \quad \left(f \in C\left(\Gamma^*\right)\right).$$

In particular, $\operatorname{Ind}_{-\Gamma}(\alpha) = -\operatorname{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$. Chains can be added and subtracted in the obvious way, by adding or subtracting the corresponding functionals: The statement $\Gamma = \Gamma_1 + \Gamma_2$ means

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

for every $f \in C\left(\Gamma_1^* \cup \Gamma_2^*\right)$. Finally, note that a chain may be represented as a sum of paths in many ways. To say that

$$\gamma_1 \dot{+} \cdots \dot{+} \gamma_n = \delta_1 \dot{+} \cdots \dot{+} \delta_k$$

means simply that

$$\sum_i \int_{\gamma_i} f(z)dz = \sum_j \int_{\delta_j} f(z)dz$$

for every f that is continuous on $\gamma_1^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_k^*$. In particular, a cycle may very well be represented as a sum of paths that are not closed.

2.3.2 Global Cauchy Theorem

We will use the following lemma for proof of the global Cauchy theorem.

Lemma 2.3.1. If $f \in H(U)$ and g is defined in $U \times U$ by

$$g(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

then g is continuous in $U \times U$.

Proof. The only points $(z, w) \in U \times U$ at which the continuity of g is possibly in doubt have $z = w$.

Fix $a \in U$. Fix $\varepsilon > 0$. There exists $r > 0$ such that $B(a, r) \subset U$ and $|f'(\xi) - f'(a)| < \varepsilon$ for all $\xi \in B(a, r)$. If z and w are in $B(a, r)$ and if

$$\gamma(t) = [z, w](t) = (1-t)z + tw,$$

then $\gamma(t) \in B(a, r)$ for $0 \leq t \leq 1$. When $(w, z) \neq (a, a)$, by [3] Corollary 5.45,

$$\begin{aligned} g(z, w) - g(a, a) = \frac{f(z) - f(w)}{z - w} - f'(a) &= -\frac{1}{z-w}\int_\gamma f'(\xi)d\xi - f'(a) \\ &= \frac{1}{w-z}\int_0^1 f'(\gamma(t))\gamma'(t)dt - f'(a) = \int_0^1 [f'(\gamma(t)) - f'(a)]dt \end{aligned}$$

and this equation is also true for the case $(w, z) = (a, a)$ where $\gamma(t) = a$ for $0 \leq t \leq 1$. The absolute value of the integrand is $< \varepsilon$, for every t. Thus $|g(z, w) - g(a, a)| < \varepsilon$. This proves that g is continuous at (a, a). $\square$

Theorem 2.3.2. [Global Cauchy Theorem] Suppose $f \in H(U)$, where $U \subseteq \mathbb{C}$ is an open set. If Γ is a cycle in U that satisfies

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \quad \text{for every } \alpha \text{ not in } U, \tag{2.6}$$

then

$$f(z) \cdot \operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw \quad \text{for } z \in U - \Gamma^* \tag{2.7}$$

and

$$\int_\Gamma f(z) dz = 0. \tag{2.8}$$

If Γ_0 and Γ_1 are cycles in U such that

$$\operatorname{Ind}_{\Gamma_0}(\alpha) = \operatorname{Ind}_{\Gamma_1}(\alpha) \quad \text{for every } \alpha \text{ not in } U, \tag{2.9}$$

then

$$\int_{\Gamma_0} f(z) dz = \int_{\Gamma_1} f(z) dz \tag{2.10}$$

Proof. The function g defined in $U \times U$ by

$$g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

is continuous in $U \times U$ by the lemma, so $h(z)$ defined by the integral

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z,w) dw \quad (z \in U)$$

is well-defined. Since $\int_\Gamma f'(z) dw = 0$ and $\frac{1}{2\pi i} \int_\Gamma \frac{f(w)-f(z)}{w-z} dw = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)dw}{w-z} - f(z) \operatorname{Ind}_\Gamma(z)$, the formula (2.7) is equivalent to the assertion that

$$h(z) = 0 \quad (z \in U - \Gamma^*) \tag{2.11}$$

To prove equation (2.11), let us first show h is continuous on U. Fix $z_0 \in U$ and an arbitrary sequence $\{z_n\} \in U, z_n \to z_0$. As $g : U \times U \to \mathbb{C}$ is continuous, it is uniformly continues on compact sets of $U \times U$. Choose r s.t. $\bar{B}(z_0, r) \subset U$. Fix $\varepsilon > 0$. Now, y is uniformly continuous on $\bar{B}(z_0, r) \times \Gamma^* \Rightarrow \exists \delta < r$ s.t. whenever $(z_1, w_1), (z_2, w_2) \in \bar{B}(z_0, r) \times \Gamma^*$ satisfy $|(z_1, w_1) - (z_2, w_2)| < \delta$ then $|g(z_1, w_1) - g(z_2, w_2)| < \varepsilon$. Now, choose N s.t. $z_n \in B(z_0, \delta)$ $\forall n > N$. Then $\forall w \in \Gamma^*$ we have for all $n > N$ that

$$|g(z_n, w) - g(z_0, w)| < \varepsilon$$

as $(z_n, w), (z_0, w) \in B(z_0, \delta) \times \Gamma^* \subset \overline{B}(z, r) \times \Gamma^*$ and $|(z_n, w) - (z_0, w)| = |z_n - z_0| < \delta$. Therefore,

$$g_n(w) := g(z_n, w) \rightrightarrows g(z_0, w)$$

uniformly for $w \in \Gamma^*$. Then

$$\lim_{n\to\infty} h(z_n) = \frac{1}{2\pi i} \lim_{n\to\infty} \int_\Gamma g_n(w) dw = \frac{1}{2\pi i} \int_\Gamma \lim_{n\to\infty} g_n(w) dw = \frac{1}{2\pi i} \int_\Gamma g(z_0, w) dw = h(z_0)$$

This shows that h is continuous at z_0. Since z_0 is an arbitrary point in U, h is continuous on U.

To further show that $h \in H(U)$, we recall Morera's theorem ([3] Corollary 8.14) that a function $U \to \mathbb{C}$ is analytic on an open set U if its integral over any triangle $\partial\triangle$ in U is zero. Now,

$$\int_{\partial\triangle} h(z) dz = \int_{\partial\triangle} \left(\frac{1}{2\pi i} \int_\Gamma g(z,w) dw \right) dz \xlongequal{\text{Fubini}} \frac{1}{2\pi i} \int_\Gamma \left(\int_{\partial\triangle} g(z,w) dw \right) dw$$

Now, with $w \in \Gamma^*$ fixed, the function $z \to g(z,w)$ is obviously analytic in $U \setminus \{w\}$ and continuous in U. Hence, we can either use Cauchy for triangles that allow a special point ([3] Theorem 6.3), or conclude that $z \to g(z,w)$ is in $H(U)$ as the singularity at $z = w$ must be removable by continuity. In either way, we have $\int_{\partial\triangle} g(z,w)dw = 0$. Thus, $\int_{\partial\triangle} h(z)dz = 0$ and Morera's theorem gives $h \in H(U)$.

Next, we define function

$$p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi \quad (z \in \mathbb{C} \setminus \Gamma^*)$$

and show its analyticity. This is more straightforward than h. In fact, recall how we proved that an analytic function can be developed into a power series by writing it via Cauchy's integral formula and expanding $\frac{1}{\xi - z}$ into a power series (see 1.3.9). The same strategy can be applied to p.

Fix $z_0 \in \mathbb{C} \setminus \Gamma^*$ and choose $\delta > 0$ such that $B(z_0, 2\delta) \subseteq \mathbb{C} \setminus \Gamma^*$. Write

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0}\frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}.$$

which uniformly converges for $\xi \in \Gamma^*$ and $z \in B(z_0, \delta)$ because $B(z_0, 2\delta) \subseteq \mathbb{C} \setminus \Gamma^* \Rightarrow |\xi - z_0| > 2\delta$ and $z \in B(z_0, \delta) \Rightarrow |z - z_0| < \delta$, so $\left|\frac{z - z_0}{\xi - z_0}\right| \leq \frac{\delta}{2\delta} = \frac{1}{2}$. Then

$$p(z) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi}_{c_n}(z - z_0)^n \quad (z \in B(z_0, \delta))$$

Thus, $p \in H(B(z_0, \delta))$. Since $z_0 \in \mathbb{C} \setminus \Gamma^*$ was arbitrary, p is analytic in $\mathbb{C} \setminus \Gamma^*$.

Now, we have analytic functions

$$h(z) = \frac{1}{2\pi i}\int_\Gamma g(z,w)dw \quad (z \in U)$$
$$p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi \quad (z \in \mathbb{C} \setminus \Gamma^*)$$

We glue them to get an entire function, which will exploit the assumption on Γ we still didn't use, i.e., $n_\Gamma(\alpha) = 0$, $\forall \alpha \in \mathbb{C} \setminus U$.

Let $\Omega := \{z \in \mathbb{C} \mid \Gamma^* : n_\Gamma(z) = 0\}$. Our assumption on Γ implies that $\mathbb{C} \setminus U \subset \Omega$, and so $\mathbb{C} = U \cup \Omega$.

If $z \in U \cap \Omega$, then both h and p are defined, and in fact, we have

$$h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi - f(z)\overbrace{n_\Gamma(z)}^{=0} = p(z).$$

So we can define $\varphi \in H(\mathbb{C})$ by setting

$$\varphi(z) := \begin{cases} h(z), & z \in U, \\ p(z), & z \in \Omega. \end{cases}$$

This is well-defined, since in $U \cap \Omega$ we have $h(z) = p(z)$.

Notice that Ω is open: $\Omega = n_\Gamma^{-1}B(0,1)$ as $n_\gamma : \mathbb{C} \setminus \Gamma^* \to \mathbb{C}$ is $\mathbb{Z}$-valued and continuous. It is then clear that $\varphi \in H(\mathbb{C})$, as for each $z \in \mathbb{C}$ there is an open neighborhood in which φ equals to h or p.

Our final step is to show $h \equiv 0$.

We will apply Liouville's theorem to φ. Notice that Γ^* is bounded and so $\Gamma^* \subset D$ for some closed disk D, Thus $\mathbb{C}\backslash D \subset \mathbb{C}\backslash\Gamma^*$ and for big $|z|$ we must be in the unbounded connected open set $\mathbb{C} \setminus D$ where $n_\Gamma(z) = 0$. So for big $|z|$ we have $z \in \Omega$ and so

$$|\varphi(z)| = |p(z)| = \left|\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi\right| \leq \frac{1}{2\pi}\frac{\|f\|_\infty}{\operatorname{dist}(z,\Gamma^*)}$$
$$\left(\xi \in \Gamma \Rightarrow \operatorname{dist}(z,\Gamma^*) \leq |\xi - z| \Rightarrow \sup_{\xi\in\Gamma^*}\left|\frac{f(\xi)}{\xi - z}\right| \leq \frac{\sup_{\xi\in\Gamma^*|f(\xi)|}}{\operatorname{dist}(z,\Gamma^*)} = \frac{\|f\|_\infty}{\operatorname{dist}(z,\Gamma^*)}\right)$$

This implies φ is bounded in $\mathbb{C}$ (compare to the argument in the proof of the fundamental theorem of algebra) and

$$\varphi(z) \to 0 \text{ as } |z| \to \infty$$

By Liouville's theorem ([3] Corollary 8.10), φ is constant and the constant must be 0. So $h(z) = \varphi(z) = 0$ for $z \in U$, and $h(z) = 0$ for $z \in U\backslash\Gamma^*$, so (2.7) is verified (notice that $n_\Gamma(z)$ only makes sense in $U \setminus \Gamma^*$, not in whole U). To deduce (2.8) from (2.7), we pick $z_0 \in U\backslash\Gamma^*$ and define $F(z) = (z - z_0)\, f(z)$. Then $F \in H(U)$ and they apply to F to give

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z - z_0}dz = \overbrace{F(z_0)}^{=0}\, n_\Gamma(z_0) = 0.$$

Finally, the path deformation claim follows from applying (2.8) to $\Gamma := \Gamma_1 - \Gamma_0$. □

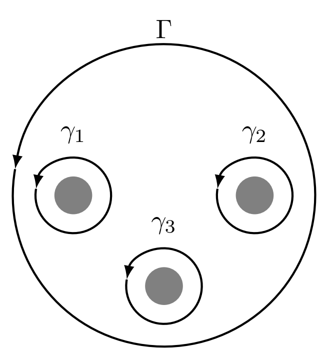
Remark 2.3.3.
(a) If γ is a closed path in a convex region Ω and if $\alpha \notin \Omega$, an application of Cauchy's theorem on convex set to $f(z) = (z - \alpha)^{-1}$ shows that $\operatorname{Ind}_\gamma(\alpha) = 0$. Assumption on Γ in global version is therefore satisfied by every cycle in Ω if Ω is convex. This shows that global version generalizes Cauchy's theorem and formula on convex set.

(b) The path deformation part of the above theorem shows under what circumstances integration over one cycle can be replaced by integration over another, without changing the value of the integral. For example, let U be the plane with three disjoint closed discs D_i removed, i.e., $U = \mathbb{C} \setminus (D_1 \cup D_2 \cup D_3)$. If $\Gamma, \gamma_1, \gamma_2, \gamma_3$ are positively oriented circles in Ω such that Γ surrounds $D_1 \cup D_2 \cup D_3$ and γ_i surrounds D_i but not D_j for $j \neq i$, then

$$\forall \alpha \in \mathbb{C} \setminus U, \operatorname{Ind}_\Gamma(\alpha) = \operatorname{Ind}_{\gamma_1+\gamma_2+\gamma_3}(\alpha).$$

and for every $f \in H(\Omega)$.

$$\int_\Gamma f(z)dz = \sum_{i=1}^{3}\int_{\gamma_i} f(z)dz$$



(c) In order to apply global Cauchy, it is desirable to have a reasonably efficient method of finding the index of a point with respect to a closed path. The following theorem does this for all paths that occur in practice. It says, essentially, that the index increases by 1 when the path is crossed "from right to left." If we recall that $\mathrm{Ind}_\gamma(\alpha) = 0$ if α is in the unbounded component of the complement W of γ^*, we can then successively determine $\mathrm{Ind}_\gamma(\alpha)$ in the other components of W, provided that W has only finitely many components and that γ traverses no arc more than once.

Theorem 2.3.4. Suppose γ is a closed path in the plane, with parameter interval $[\alpha, \beta]$. Suppose $\alpha < u < v < \beta, a$ and b are complex numbers, $|b| = r > 0$, and

(i) $\gamma(u) = a - b, \gamma(v) = a + b,$

(ii) $|\gamma(s) - a| < r$ if and only if $u < s < v,$

(iii) $|\gamma(s) - a| = r$ if and only if $s = u$ or $s = v.$

Assume furthermore that $D(a; r) - \gamma^*$ is the union of two regions, $D_+$and D_-, labeled so that $a + bi \in \bar{D}_+$and $a - bi \in \bar{D}_-$. Then

$$\mathrm{Ind}_\gamma(z) = 1 + \mathrm{Ind}_\gamma(w)$$

if $x \in D_+$and $w \in D_-$. As $\gamma(t)$ traverses $D(a; r)$ from $a - b$ to $a + b, D_-$is "on the right" and $D_+$is "on the left" of the path.

Proof. See [5] Theorem 10.37 □

2.3.3 Homotopy

We introduce a concept in algebraic topology that is also related to Cauchy's theorem. First, to be clearer on the terminology, we would say *curves* are continuous but not necessarily differentiable while *paths* are assumed to be piecewise C^1.

Suppose $\gamma_0, \gamma_1 : I \to U$ are closed curves in a topological space X. We say that γ_0 and γ_1 are U**-homotopic** if there is a continuous map $H : I \times I \to U$ such that

$$H(s, 0) = \gamma_0(s), \quad H(s, 1) = \gamma_1(s), \quad H(0, t) = H(1, t)$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s, t)$. Then H defines a one-parameter family of closed curves γ_t in X, which connects γ_0 and γ_1. Intuitively, this means that γ_0 can be continuously deformed to γ_1, within X.

If γ_0 is U-homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is null-homotopic in U. If U is connected and if every closed curve in U is null-homotopic, U is said to be simply connected.

For example, every convex region Ω is simply connected. To see this, let γ_0 be a closed curve in Ω, fix $z_1 \in \Omega$, and define **straight-line homotopy**

$$H(s, t) = (1 - t)\gamma_0(s) + tz_1 \quad (0 \le s \le 1, 0 \le t \le 1)$$

Theorem 2.3.6 will show that (2.9) in the global Cauchy holds whenever Γ_0 and Γ_1 are U-homotopic closed paths. As a special case of this, note that condition (2.6) of 2.3.2 holds for every closed path Γ in U if U is simply connected, since constant paths necessarily have zero index.

Lemma 2.3.5. Let γ_0 and γ_1 be closed paths with parameter interval $[0, 1]$ and let α be a complex number. If

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \quad (0 \le s \le 1) \tag{2.12}$$

then

$$\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_0}(\alpha).$$

Proof. Condition (2.12) implies that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$. Hence one can define $\gamma(t) = (\gamma_1(t) - \alpha) / (\gamma_0(t) - \alpha)$. Since

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \Rightarrow \left|\frac{\gamma_0 - \gamma_1}{\gamma_0 - \alpha}\right| = \left|\frac{\gamma_0 - \alpha - (\gamma_1 - \alpha)}{\gamma_0 - \alpha}\right| = |1 - \gamma| < 1$$

we see $\gamma^* \in B(1,1)$, which implies that $\mathrm{Ind}_\gamma(0) = 0$. The following relationship between the paths and their derivatives can also be easily calculated.

$$\frac{\gamma'}{\gamma} = \frac{\gamma_1'}{\gamma_1 - \alpha} - \frac{\gamma_0'}{\gamma_0 - \alpha}$$

Integrating the above identity over $[0,1]$ and writing them in path integral forms will give the desired result:

$$\underbrace{\int_\gamma \frac{1}{z} dz}_{\mathrm{Ind}_\gamma(0)=0} = \underbrace{\int_{\gamma_1} \frac{1}{z-\alpha} dz}_{\mathrm{Ind}_{\gamma_1}(\alpha)} - \underbrace{\int_{\gamma_0} \frac{1}{z-\alpha} dz}_{\mathrm{Ind}_{\gamma_0}(\alpha)}$$

□

Theorem 2.3.6. If Γ_0 and Γ_1 are U-homotopic closed paths in an open connected set U, and if $\alpha \notin U$, then

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha)$$

Proof. By definition, there is a continuous $H : I^2 \to \Omega$ such that

$$H(s,0) = \Gamma_0(s), \quad H(s,1) = \Gamma_1(s), \quad H(0,t) = H(1,t).$$

Since I^2 is compact, so is $H\left(I^2\right)$. Since α is not in the closed set $H(I^2)$, there exists $\varepsilon > 0$ such that $B(\alpha, 2\varepsilon) \cap H(I^2) = \varnothing$, i.e.,

$$|\alpha - H(s,t)| \geq 2\varepsilon \quad \forall (s,t) \in I^2. \tag{2.13}$$

Since H is continuous on compact set and is thus uniformly continuous, there is a positive integer n such that:

$$|H(s,t) - H(s', t')| < \varepsilon \quad \text{if} \quad |s - s'| + |t - t'| \leq 1/n. \tag{2.14}$$

(Note that $\sqrt{|s-s'|^{1/2} + |t-t'|^{1/2}} \leq |s-s'| + |t-t'|$)

Define polygonal closed paths $\gamma_0, \ldots, \gamma_n$ by

$$k = 0, 1, \cdots, n: \quad \gamma_k(s) = H\left(\frac{i}{n}, \frac{k}{n}\right)(1 - (i - ns)) + H\left(\frac{i-1}{n}, \frac{k}{n}\right)(i - ns) \tag{2.15}$$

if $0 \leq i - ns \leq 1$ $\left(\text{i.e., } \frac{i-1}{n} \leq s \leq \frac{i}{n}\right)$ and $i = 1, \ldots, n$. By (2.14) and (2.15), for $k \in [n], s \in [0,1]$,

$$(*): \left|\gamma_k(s) - H\left(s, \frac{k}{n}\right)\right| \leq (ns + 1 - i)\underbrace{\left|H\left(\frac{i}{n}, \frac{k}{n}\right) - H\left(s, \frac{k}{n}\right)\right|}_{<\varepsilon} + (i - ns)\underbrace{\left|H\left(\frac{i-1}{n}, \frac{k}{n}\right) - H\left(s, \frac{k}{n}\right)\right|}_{<\varepsilon} < \varepsilon$$

(Note that $(ns + 1 - i) + (i - ns) = 1$ when using the triangle inequality)

In particular, taking $k = 0$ and $k = n$,

$$|\gamma_0(s) - \Gamma_0(s)| < \varepsilon, \quad |\gamma_n(s) - \Gamma_1(s)| < \varepsilon.$$

By $(*)$ and (2.13),

$$|\alpha - \gamma_k(s)| > 2\varepsilon - \varepsilon = \varepsilon \quad (k \in [n]; s \in [0,1]).$$

On the other hand, (2.14) and (2.15) also imply that for $k \in [n]; s \in [0,1]$,

$$|\gamma_{k-1}(s) - \gamma_k(s)| \leq (ns+1-i) \underbrace{\left| H\left(\frac{i}{n}, \frac{k-1}{n}\right) - H\left(\frac{i}{n}, \frac{k}{n}\right) \right|}_{<\varepsilon} + (i-ns) \underbrace{\left| H\left(\frac{i-1}{n}, \frac{k-1}{n}\right) - H\left(\frac{i-1}{n}, \frac{k}{n}\right) \right|}_{<\varepsilon} < \varepsilon.$$

Now it follows from the last three inequalities, and $n+2$ applications of Lemma 2.3.5 that α has the same index with respect to each of the paths $\Gamma_0, \gamma_0, \gamma_1, \ldots, \gamma_n, \Gamma_1$. This proves the theorem. $\square$

Remark 2.3.7.

1. If $\Gamma_t(s) = H(s,t)$ in the preceding proof, then each Γ_t is a closed *curve*, but not necessarily a *path*, since H is not assumed to be differentiable. The paths γ_k were introduced for this reason. Another (and perhaps more satisfactory) way to circumvent this difficulty is to extend the definition of index to closed curves.
2. As promised, this theorem of sufficiency of path deformation and global cauchy show that any integral of analytic function along γ in a simply-connected set U is zero. We shall see the converse is also true, i.e., this property can be used as definition of simply-connectedness (of a set in complex plane).

Theorem 2.3.8. Let $U \subset \mathbb{C}$ be open and connected. Then the following are equivalent.

(a) U is homeomorphic to $B(0,1)$ (i.e. there is a continuous bijection $\psi : U \to B(0,1)$ s.t. ψ^{-1} is also continuous).

(b) U is simply connected;

(c) $\int_\gamma f(z)dz = 0 \quad \forall f \in H(U)$ and every closed path $\gamma : [a,b] \to U$;

(d) Every $f \in H(U)$ has a primitive;

(e) If $f, 1/f \in H(U)$ (i.e., f is analytic and non-vanishing), then $f = e^y$ for some $g \in H(U)$ ("f has a holomorphic logarithm g in U");

(f) If f in a non-vanishing analytic function, then $f = \varphi^2$ for some $\varphi \in H(U)$ ("f has a holomorphic square root φ in U").

Proof.

$(a) \Rightarrow (b)$: that's basically because $B(0,1)$ is simply-connected and homeomorphism preserves null-homotopy (in fact the whole fundamental group). Let $\psi : U \to B(0,1)$ be the homeomorphism. For the closed curve γ in U, the map $H(s,t) = \psi^{-1}((1-t)0 + t\psi(\gamma(s))) = \psi^{-1}(t\psi(\gamma(s)))$ defines a homotopy between constant map $c_{\psi^{-1}(0)}$ and curve $\gamma(s)$.

$(b) \Rightarrow (c)$: consequence of 2.3.6 and 2.3.2.

$(c) \Rightarrow (d)$: The proof resembles that for Cauchy's theorem in a convex set ([3] Theorem 6.10). Fix $a \in U$ arbitrary, and set

$$g(z) := \int_{\gamma_z} f(w)\mathrm{d}w$$

where γ_z is any path (i.e., piecewise C^1 curve) connecting a to z inside of U. Note: the fact that we can always select a piecewise C^1 curve is not guaranteed by the definition of connectness (which only gives a continuous curve) - however, it is a well-known result from basic topology that in an open, connected set we can always connect two points with a finite union of line segments (a polygonal path).

Also note that g is well-defined: if $\tilde{\gamma}_z$ is some other path connecting a to z in U, then $\gamma_z - \tilde{\gamma}_z$ is a closed path in U and by the given assumption (c),

$$\int_{\gamma_z - \tilde{\gamma}_z} f(w)\mathrm{d}w = 0 \Rightarrow \int_{\gamma_z} f(w)\mathrm{d}w = \int_{\tilde{\gamma}_z} f(w)\mathrm{d}w$$

Choose now $r > 0$ such that $B(z, r) \subset U$. Consider $h \in \mathbb{C}$ with $|h| < r$ so that $z + h \in B(z, r)$. Let $\eta = [z, z+h]$ be the line segment path connecting z to $z+h$ inside of $B(z, r) \subset U$. Then $\gamma + \eta$ is a path connecting a to $z + h$, and due to invariance of path in the definition of $g(z+h)$ as we just showed, we have

$$g(z+h) = \int_{\gamma+\eta} f(w)\mathrm{d}w = \int_{\gamma} f(w)\mathrm{d}w + \int_{\eta} f(w)\mathrm{d}w$$

and so

$$g(z+h) - g(z) = \int_{\eta} f(w)\mathrm{d}w.$$

Since

$$\frac{1}{h}\int_{\eta} f(z)\mathrm{d}w = \frac{1}{(z+h)-z}\int_{[z,z+h]} f(z)\mathrm{d}w = f(z),$$

and $\operatorname{length}(\eta) = |z+h-z| = |h|$, we apply [3] Corollary 5.31 to see

$$\left|\frac{g(z+h)-g(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{\eta} f(w) - f(z)\mathrm{d}w\right| \leq \|f - f(z)\|_{L^\infty([z,z+h])} \leq \varepsilon$$

where the last inequality holds for sufficiently small $|h|$ due to continuity of f. Thus,

$$g'(z) = \lim_{h\to 0}\frac{g(z+h)-g(z)}{h} = f(z)$$

$(d) \Rightarrow (e)$: The identity $f = e^g$, if it were to hold for some $g \in H(U)$, implies $f'(z) = g'(z)e^{g(z)} = g'(z)f(z)$ so that $g'(z) = f'(z)/f(z)$. So we want a primitive of the analytic function f'/f (recall f has no zeros in U by assumption). Let $z_0 \in U$ be a fixed point and c_0 is a complex number with $e^{c_0} = f(z_0)$ (recall e^z obtains all values except 0 and $f(z_0) \neq 0$). Since $f' \in H(U)$ and $1/f \in H(U)$, assumption (d) gives a primitive g of f'/f. We can assume $g(z_0) = c_0$ (otherwise take $\tilde{g} = g + c_0 - g(z_0)$).

We claim that g in turn satisfies $f = e^g$. Motivated by the midterm Q2, we study $G(z) := e^{g(z)}/f(z)$. Now, $g'(z) = f'(z)/f(z)$ gives

$$G'(z) = g'(z)e^{g(z)}/f(z) - e^{g(z)}f(z)^{-2}f'(z) = e^{g(z)}f(z)^{-2}f'(z) - e^{g(z)}f(z)^{-2}f'(z) = 0.$$

This implies, as U is connected, that $G(z) = e^{g(z)}/f(z) = C$ for some constant C, and so $e^{g(z)} = Cf(z)$ in U. Now, as $e^{g(z_0)} = e^{c_0} = f(z_0)$ we must have $C = 1$, and so we are done.

$(e) \Rightarrow (f)$: Use (e) to write $f = e^g$ with $g \in H(U)$. Define $\varphi = e^{g/2}$. Then $\varphi^2 = e^g = f$.

$(f) \Rightarrow (a)$: If $U = \mathbb{C}$, the homeomorphism is just directly given (without using (f)) by $z \mapsto \frac{z}{1+|z|}$. If the open connected set U is not the whole $\mathbb{C}$, there actually exists a holomorphic homeomorphism $U \to B(0,1)$ (a conformal mapping). This is the Riemann Mapping Theorem. This implication is thus proved as soon as we later prove Riemann mapping theorem (using only (f) and nothing else about simply connected domains).

□

2.4 Residue Theorem

Definition 2.4.1. A function f is said to be **meromorphic** in an open set U if there is a set $A \subset U$ such that

(a) A has no limit point in U,

(b) $f \in H(U - A)$,

(c) f has a pole at each point of A.

Note that the possibility $A = \varnothing$ is not excluded. Thus every $f \in H(U)$ is meromorphic in U.

Note also that (a) implies that no compact subset of U contains infinitely many points of A, and that A is therefore at most countable.

If f and A are as above, if $a \in A$, and if

$$Q(z) = \sum_{k=1}^{m} c_k (z-a)^{-k}$$

is the principal part of f at a, as defined in Theorem 2.2.5 (i.e., if $f - Q$ has a removable singularity at a), then the number c_1 is called the residue of f at a :

$$c_1 = \operatorname{Res}(f; a).$$

Theorem 2.4.2. [The Residue Theorem] Suppose f is a meromorphic function in U. Let A be the set of points in U at which f has poles. If Γ is a cycle in $U - A$ such that

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \quad \text{for all} \quad \alpha \notin U$$

then

$$\frac{1}{2\pi i} \int_\Gamma f(z) dz = \sum_{a \in A} \operatorname{Res}(f; a) \operatorname{Ind}_\Gamma(a)$$

Proof. We will argue the sum on the RHS, though formally infinite, is actually finite. Let

$$B := \{a \in A : n_\Gamma(a) \neq 0\}.$$

Let $\Omega := \mathbb{C} \backslash \Gamma^*$. Denote the components of Ω by $\mathcal{F} = \cup V$. Each V is connected. The components are disjoint. Every connected set $E \subset \Omega$ is containted in exactly one $V \in \mathcal{F}$. Choose a disk D s.t. $\Gamma^* \subset D$ (possible as Γ^* is compact) and notice $\mathbb{C} \setminus D \subset \Omega$. So there exists a unique $V_0 \in \mathcal{F}$ s.t. $\mathbb{C} \setminus D \subset V_0$ (as $\mathbb{C} \setminus D \subset \Omega$ and $\mathbb{C} \setminus D$ is connected). So $\forall V \in \mathcal{F}^* = \mathcal{F} - \{V_0\}$, we have $V \subset \mathbb{C} \setminus V_0 \subset D$. Notice that $n_\Gamma(z) = 0$ in V_0 as $V_0 \subset \Omega$ unbounded, connected. Thus, $B \subset \bigcup_{V \in \mathcal{F}^*} V \subset D$ is bounded.

If $|B|$ is not finite, we can choose $a_1, a_2, \ldots \in B$ s.t, $a_i \neq a_j$ $(i \neq j)$. As $\bar{B}$ is compact, $\exists$ subseq. $a_{i_k} \to a \in \bar{B}$. Now $a \in A^{acc}$ clearly, but also $a \in U$. Indeed, if $a \notin U$ then $n_\Gamma(a) = 0$ bs assumption. But as n_Γ is continuous and $\mathbb{Z}$-valued, this forces $n_\Gamma(a_{i_k}) = 0$ for k large, contradicting to $a_{ik} \in B$. So $a \in U \cap A^{acc}$, contradicting to our assumption that $U \cap A^{acc} = \varnothing$. Thus $|B|$ is finite and

$$\sum_{a \in A} \operatorname{Res}(f; a) n_\Gamma(a) = \sum_{a \in B} \operatorname{Res}(f; a) n_\Gamma(a) < \infty$$

Write $B = \{a_1, \cdots, a_n\}$. Let $Q_1, \cdots, Q_n$ be the principal parts of f at $a_1, \cdots, a_n$. Set

$$g := f - \sum_{i=1}^{n} Q_i.$$

Put $U_0 := U\backslash(A\backslash B)$. As A has no accumulation point in U, this means $\forall z \in U_0$, $\exists r$ s.t. $B(z,r) \in U$ contains no other points of A than possibly z. Thus $B(z,r) \subset U_0$ and U_0 is open.

If $z \in U_0 \cap A$, then $z \in B = \{a_1, \ldots, a_n\}$, and g has a removable singularity at z. If $z \in U_0\backslash A, g$ is obvious complex differentiable, so $g \in H(U_0)$. We apply the Global Cauchy to $g \in H(U_0)$:

$$\int_\Gamma g(z)dz = 0$$

as Γ is a cycle in U_0 with the property that

$$n_\Gamma(z) = 0 \quad \forall z \in \mathbb{C} \setminus U_0$$

Indeed, if $z \in \mathbb{C} \setminus U_0$, then either $z \in \mathbb{C} \setminus U$ and $n_\Gamma(z) = 0$ by the assumption of the theorem, or $z \in A \setminus B$ where also $n_\Gamma(z) = 0$ by the definition of B. Hence,

$$0 = \int_\Gamma f(z)dz - \sum_{i=1}^n \int_\Gamma Q_i(z)dz = \int_\Gamma f(z)dz - \sum_{i=1}^n \operatorname{Res}(f; a_i)\, 2\pi i\, n_\Gamma(a_i)$$

yielding

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{i=1}^n \operatorname{Res}(f; a_i)\, n_\Gamma(a_i) = \sum_{a\in A} \operatorname{Res}(f; a) n_\Gamma(a)$$

as desired. □

We will use L'Hôpital's rule in the following example:

Proposition 2.4.3. [L'Hôpital's rule] Let $U \subset \mathbb{C}$ be open, let $z \in \mathbb{C}$, and let $f, g : U \to \mathbb{C}$ be complex differentiable at z, with $g'(z) \neq 0$. Assume moreover that $f(z) = 0 = g(z)$. Then

$$\lim_{\substack{w\to z \\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(w)}{g(w)} = \frac{f'(z)}{g'(z)}$$

Proof. By [3] Proposition 4.7, we may write

$$\lim_{\substack{w\to z \\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(w)}{g(w)} = \lim_{\substack{w\to z \\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(z) + (f'(z) + \varepsilon_f(w))(w-z)}{g(z) + (g'(z) + \varepsilon_g(w))(w-z)},$$

where $\varepsilon_f(w) \to 0$ and $\varepsilon_g(w) \to 0$ as $w \to z$. Moreover, recalling that $f(z) = 0 = g(z)$, the above limit simplifies to

$$\lim_{\substack{w\to z \\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(w)}{g(w)} = \lim_{\substack{w\to z \\ w\in\mathbb{C}\backslash\{z\}}} \frac{f'(z) + \varepsilon_f(w)}{g'(z) + \varepsilon_g(w)} = \frac{f'(z)}{g'(z)},$$

as claimed. Notice that here diving with $g'(z)$ makes sense as $g'(z) \neq 0$. Moreover, notice that by (b) this implies that in a small neighborhood of z we have $g(w) \neq g(z) = 0$ for $w \neq z$, so also the expression $f(w)/g(w)$ above makes sense for $w \neq z$ close to z. □

Example 2.4.4. We calculate the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Solution. We want to have a path γ_R such that above equals to

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)dz$$

where $f(z) = 1/(1+z^4)$. We let $\gamma_R = [-R, R] \star \sigma_R$, $\sigma_R(t) = Re^{it}$, $t \in [0, \pi]$. It works simply becasue of the usual estimate

$$\left|\int_{\sigma_R} f(z)dz\right| \leq \frac{CR}{R^4} = \frac{C}{R^3} \xrightarrow{R\to\infty} 0.$$

Now $1 + z^4 = 0$ has solutions $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$, all with order 1. Only poles $z_1 = e^{i\pi/4}$ and $z_2 = e^{i3\pi/4}$ are inside γ_R. By the residue theorem and formula 2.2.8,

$$\begin{aligned} int_{\gamma_R} f(z)dz &= 2\pi i(\mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2)) \\ &= 2\pi i \left(\lim_{z\to z_1}(z-z_1)f(z) + \lim_{z\to z_2}(z-z_2)f(z)\right) \end{aligned}$$

Instead of writing $z^4 + 1 = (z - z_1)\cdots(z - z_4)$, it is convenient to use L'Hôpital's rule instead:

$$\lim_{z\to z_1}(z-z_1)f(z) = \lim_{z\to z_1}\frac{z-z_1}{1+z^4} \overset{\text{L'Hôpital}}{=\!=} \lim_{z\to z_1}\frac{1}{4z^3} = \frac{1}{4(e^{i\pi/4})^3} = \frac{1}{4}e^{-i3\pi/4}.$$

Similarly,

$$\lim_{z\to z_2}(z-z_2)f(z) = \frac{1}{4}e^{-i\pi/4}.$$

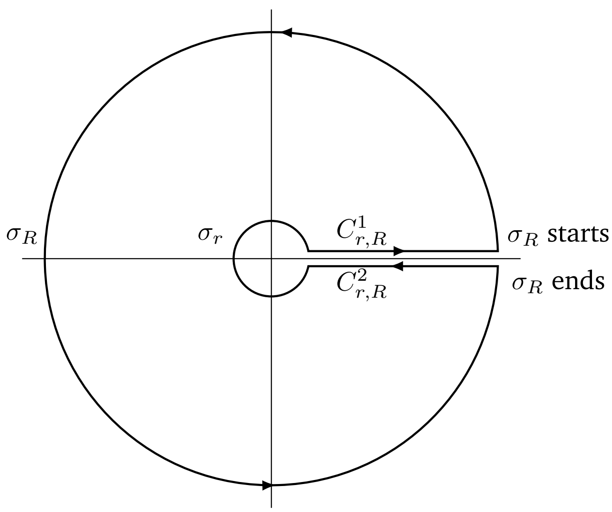
Thus,

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^4} = \frac{2\pi i}{4}\left(e^{-i3\pi/4} + e^{-i\pi/4}\right) = \frac{\pi i}{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = -\frac{\pi i}{2}\frac{2i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Example 2.4.5. Calculate the integral

$$I := \int_0^{\infty}\frac{x^{1/3}}{1+x^2}dx.$$

Solution. We need a keyhole contour line $\gamma_{r,R} := C^1_{r,R} \star \sigma_R \star C^2_{r,R} \star \sigma_r$



We define a branch of argument $\widetilde{\mathrm{Arg}} z \in (0, 2\pi)$ on $\mathbb{C} \setminus [0, \infty)$ in the natural way, i.e., $\widetilde{\mathrm{Arg}}_v(z)$ defined in [3] Theorem 2.57 with $v = 1$ here. Then we define

$$g(z) = |z|^{1/3}e^{i\widetilde{\mathrm{Arg}} z/3}$$

and define

$$f(z) = \frac{g(z)}{1+z^2}, \quad z \in \mathbb{C}\setminus[0,\infty), z \neq \pm i.$$

we first calculate $\int_\gamma f(z)dz$ using residues.

Clearly, g is analytic in a neighborhood (fattening) of the area enclosed by $\gamma_{r,R}$, and $n_{\gamma_{r,R}}(z) = 0$ for all z outside of this fattening (as z will be in the unbounded component of $\mathbb{C} \setminus \gamma^*_{r,R}$). Residue theorem gives

$$\int_{\gamma_{r,R}} f(z)dz = 2\pi i(\operatorname{Res}(f;i) + \operatorname{Res}(f;-i))$$

where it is easy to see by an argument of homotopy that $n_{\gamma_{r,R}}(i) = n_{\gamma_{r,R}}(-i) = 1$. Now,

$$\begin{aligned}\operatorname{Res}(f;i) &= \lim_{z\to i} \frac{g(z)}{z+i} = \frac{e^{i\frac{\pi}{6}}}{2i} = \frac{1}{2}e^{i\left(\frac{\pi}{6}-\frac{\pi}{2}\right)} = \frac{e^{-i\frac{\pi}{3}}}{2} \\ \operatorname{Res}(f;-i) &= \lim_{2\to -i} \frac{g(z)}{z-i} = \frac{e^{i\frac{\pi}{2}}}{-2i} = -\frac{1}{2}\end{aligned}$$

Then,

$$\begin{aligned}\int_{\gamma_{r,R}} f(z)dz &= 2\pi i \cdot \frac{1}{2}\left(e^{-i\frac{\pi}{3}} - 1\right) \\ &= \pi i\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i - 1\right) \\ &= \pi i\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -\pi i e^{i\frac{\pi}{3}}\end{aligned}$$

Now, we relate this complex integral to the original real integral

$$\begin{aligned}&\left|\int_{\sigma_R} f(z)dz\right| \leqslant C\frac{R^{1/3}R}{R^2} = CR^{-2/3} \to 0, \quad R\to\infty \\ &\left|\int_{\sigma_r} f(z)dz\right| \leqslant Cr \to 0, \quad r \to 0 \\ &\int_{c^1_{r,R}} f(z)dz \to I \text{ as } r\to 0, \quad R\to\infty \\ &\int_{c^2_{r,R}} f(z)dz \to -e^{i\frac{2\pi}{3}} I, \quad r\to 0, R\to\infty \\ &(\text{Since } \widetilde{\operatorname{Arg}} z \to 2\pi \text{ here, and we travel with opposite direction.})\end{aligned}$$

Then

$$\left(1 - e^{i\frac{2\pi}{3}}\right) I = \lim_{\substack{r\to 0\\ R\to\infty}} \int_{\gamma_{r,R}} f(z)dz = -\pi i e^{i\frac{\pi}{3}}$$

Thus,

$$\begin{aligned}I &= \frac{-\pi i e^{i\frac{\pi}{3}}}{1 - e^{\frac{i\pi}{3}}} = \frac{\pi i}{e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}} \\ &= \frac{\pi i}{2i\sin\frac{\pi}{3}} = \frac{\pi}{2}\frac{1}{\sin\frac{\pi}{3}} \\ &= \frac{\pi}{2}\frac{2}{\sqrt{3}} = \frac{\pi}{\sqrt{3}}.\end{aligned}$$

We conclude this chapter with two typical applications of the residue theorem. The first one concerns zeros of holomorphic functions, the second is the evaluation of a certain integral.

Theorem 2.4.6. Suppose γ is a closed path in an open connected set U, such that $\mathrm{Ind}_\gamma(\alpha) = 0$ for every α not in U. Suppose also that $\mathrm{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in U - \gamma^*$. Let $U_1 = \{z \in \mathbb{C} \setminus \gamma^* : \mathrm{Ind}_\gamma(\alpha) = 1\} \subset U$. For any $f \in H(U)$ let N_f be the number of zeros of f in U_1, counted according to their multiplicities.

(a) If $f \in H(U)$ and f has no zeros on γ^* then

$$N_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \mathrm{Ind}_\Gamma(0) \tag{2.16}$$

where $\Gamma = f \circ \gamma$.

(b) If also $g \in H(U)$ and

$$|f(z) - g(z)| < |f(z)| \quad \text{for all } z \in \gamma^* \tag{2.17}$$

then $N_g = N_f$.

Part (b) is usually called Rouché's theorem. It says that two holomorphic functions have the same number of zeros in U_1 if they are close together on the boundary of U_1, as specified by (2.17).

Proof. Put $\varphi = f'/f$. Due to Theorem 2.1.1, if $a \in U$ and f has a zero of order $m = m(a)$ at a, then $f(z) = (z-a)^m h(z)$, where h and $1/h$ are holomorphic in some neighborhood V of a. In $V - \{a\}$,

$$\varphi(z) = \frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1}h(z) + (z-a)^m h'(z)}{(z-a)^m h(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)}.$$

The first term $\frac{m}{z-a}$ corresponds to the $c_{-1}(z-a)^{-1}$ term of $\varphi(z)$, and $\frac{h'(z)}{h(z)} \in H(V)$ corresponds to $\sum_{n=0}^\infty c_n(z-a)^n$ for the remaining terms of $\varphi(z)$. Thus φ is a meromorphic function in U (the set A in the definition of meromorphic function is Z_f which by Theorem 2.1.1 has not limit point in U and each $a \in A$ is a first order pole of φ). Besides,

$$\mathrm{Res}(\varphi; a) = m(a)$$

Let $A = \{a \in U_1 : f(a) = 0\} = U_1 \cap Z_f$. Now apply Residue theorem to meromorphic function φ and γ, a cycle in $U \setminus Z_f$ with $n_\gamma(z) = 0$, $\forall z \in \mathbb{C} \setminus U$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{a \in Z_f} \mathrm{Res}(\varphi; a)\,\mathrm{Ind}_\gamma(a) = \sum_{a \in U_1 \cap Z_f} \mathrm{Res}(\varphi; a) = \sum_{a \in A} m(a) = N_f.$$

This proves one half of (2.16). The other half is a matter of direct computation: supposing $\gamma : [a, b] \to \mathbb{C}$,

$$\begin{aligned}\mathrm{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_a^b \frac{\Gamma'(s)}{\Gamma(s)}ds \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)ds = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.\end{aligned}$$

Next, (2.17) gives $|g(z)| \geq |f(z)| - |f(z) - g(z)| > 0$, $\forall z \in \gamma^*$, so g has no zero on γ^*. Hence (2.16) holds with g in place of f. Put $\Gamma_0 = g \circ \gamma$. In order to apply Lemma 2.3.5 to Γ_0 and $\Gamma_1 = f \circ \gamma$ to get $\mathrm{Ind}_{\Gamma_0}(0) = \mathrm{Ind}_{\Gamma_1}(0)$, we need verify that

$$|\Gamma_0(s) - \Gamma_1(s)| = |g(\gamma(s)) - f(\gamma(s))| < |f(\gamma(s))| = |\underbrace{0}_{\alpha \text{ in the lemma}} - \Gamma_1(s)|$$

where the inequality is due to (2.17). Then it follows from (2.16) that

$$N_g = \mathrm{Ind}_{\Gamma_0}(0) = \mathrm{Ind}_{\Gamma_1}(0) = N_f$$

□

Example 2.4.7. How many roots does the polynomial $z \mapsto z^5 + 3z^2 + 1$ have in the annulus $A = \{1 < |z| < 2\}$?

Solution. The strategy is to let g be the original function $g(z) = z^5 + 3z^2 + 1$ and choose f to be the dominant part on the circle $|z| = 1$.

Define $f_1(z) = 3z^2$. Then, on the circle $|z| = 1$, we have

$$|f_1(z) - g(z)| = |z^5 + 1| < |z|^5 + 1 = 2 < 3 = 3|z|^2 = |f_1(z)|$$

By Rouché's theorem, we have

$$|\{a \in B(0,1) : g(a) = 0\}| = |\{a \in B(0,1) : f_1(a) = 0\}| = 2$$

since $z \mapsto 3z^2$ has a zero of order 2 at the origin.

Define $f_2(z) = z^5$. For $|z| = 2$ we have

$$|f_2(z) - g(z)| = |3z^2 + 1| \leq 3 \cdot 2^2 + 1 = 13 < 32 = 2^5 = |z|^5 = |f_2(z)|$$

By Rouché's theorem, we have

$$|\{a \in B(0,2) : g(a) = 0\}| = |\{a \in B(0,2) : f_2(a) = 0\}| = 5$$

since $z \mapsto z^5$ has a zero of order 5 at the origin.

Therefore, there are $5 - 2 = 3$ zeros in annulus A.

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