Lecture Note on
Complex Analysis

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Contents

Chapter 1

Power Series and Analytic Functions

1.1 Limsup and Liminf

The limit superior of a sequence $\{x_n\}$ in $\mathbb{R}$ is defined by

$$\overline{\lim_{n\to\infty}}\, x_n = \limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right) = \inf_{n\geq 0}\left(\sup_{m\geq n} x_m\right)$$

The limit inferior of a sequence $\{x_n\}$ in $\mathbb{R}$ is defined by

$$\underline{\lim}_{n\to\infty}\, x_n = \liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right) = \sup_{n\geq 0}\left(\inf_{m\geq n} x_m\right)$$

When $\{x_n\}$ has no upper bound, we say $\overline{\lim}_{n\to\infty} x_n = +\infty$; when $\{x_n\}$ has no lower bound, we say $\underline{\lim}_{n\to\infty} x_n = -\infty$.

Theorem 1.1.1. Let $H = \overline{\lim}\, x_n$. Then

(a) When H is finite, there are infinitely many x_n falling in the interval $(H-\varepsilon, H+\varepsilon)$ for any $\varepsilon > 0$, while there are only finitely many (or even zero) x_n falling in $(H+\varepsilon, +\infty)$.

(b) When $H = +\infty$, for any $N > 0$, there are infinitely many x_n such that $x_n > N$.

(c) When $H = -\infty$, $\lim x_n = -\infty$.

Proof.

(a) $-\infty < H < +\infty$: the statement will be proved if we show that for any $\varepsilon > 0$ there are infinitely many terms x_n greater than $H - \varepsilon$ and only finitely many terms x_n greater than $H + \varepsilon$. We show the first part: BWOC, suppose there is some $\varepsilon_0 > 0$ s.t. there are only finitely many x_n greater than $H - \varepsilon_0$, say $x_{n_1}, \cdots, x_{n_k}$. Thus, $x_n \leq H - \varepsilon_0$ for all $n > n_k$. Therefore, for all $n > n_k$, the supremums have

$$\beta_n = \sup_{m\geq n} x_m = \sup\{x_n, x_{n+1}, \cdots\} \leq H - \varepsilon_0$$

Thus,

$$H = \overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \beta_n \leq H - \varepsilon_0$$

which is a contradiction. We show the second part: let $\beta_n = \sup_{m \geq n} x_m$. Since $\lim_{n\to\infty} \beta_n = H$, $\forall \varepsilon$, $\exists N \in \mathbb{N}$ s.t. $|\beta_n - H| < \varepsilon$, i.e., $H - \varepsilon < \beta_n < H + \varepsilon$. Since β_n is supremum of $\{x_n, x_{n+1}, \cdots\}$, we see when $n > N$,

$$\forall k \in \mathbb{N},\, x_{n+k} \leq \beta_n \leq H + \varepsilon$$

Thus, those x_n with $x_n > H + \varepsilon$ must have $n \leq N$, which shows that there are only finitely many x_n satisfying $x_n > H + \varepsilon$.

(b) That's because $H = +\infty$ when $\{x_n\}$ has no upper bound by definition.

(c) When $H = -\infty$, for any $G > 0$, there exists n_0, when $n > n_0$, $x_{n+1} \leq \beta_n \leq -G$, so $\lim x_n = -\infty$.

□

We have a liminf counterpart of the above theorem:

Theorem 1.1.2. Let $h = \underline{\lim}\, x_n$. Then

(a) When h is finite, there are infinitely many x_n falling in the interval $(h - \varepsilon, h + \varepsilon)$ for any $\varepsilon > 0$, while there are only finitely many (or even zero) x_n falling in $(-\infty, h - \varepsilon)$.

(b) When $h = -\infty$, for any $N > 0$, there are infinitely many x_n such that $x_n < -N$.

(c) When $h = +\infty$, $\lim x_n = +\infty$.

Another useful theorem is

Theorem 1.1.3. For limsup H and liminf h of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ with limit H and H is the largest among all limits of convergent subsequences of $\{x_n\}$; there also exists a subsequence $\{x_{n_k}\}$ with limit h and h is the smallest among all limits of convergent subsequences of $\{x_n\}$;

Corollary 1.1.4. $\lim x_n = A$ (finite or infinite) iff $\overline{\lim}\, x_n = \underline{\lim}\, x_n = A$.

Example 1.1.5. $a_n = n + (-1)^n n\, (n = 1, 2, 3, \cdots)$. It only has two subsequences with limit (including ∞): a_{2k} and $a_{2k+1} (k = 1, 2, 3, \cdots)$. The limits are respectively $+\infty$ and 0, so

$$\overline{\lim_{n\to\infty}}\, a_n = +\infty,\ \underline{\lim_{n\to\infty}}\, a_n = 0$$

Example 1.1.6. $a_n = \cos \frac{n}{4}\pi\, (n = 0, 1, 2, \cdots)$. Since $-1 \leqslant \cos \frac{n}{4}\pi \leqslant 1$, when $n = 8k (k = 1, 2, \cdots)$, $a_{8k} \to 1 (k \to \infty)$; when $n = 4(2k + 1)$, $(k = 1, 2, 3, \cdots)$, $a_{4(2k+1)} \to -1 (k \to \infty)$. Thus,

$$\overline{\lim_{n\to\infty}}\, a_n = 1,\ \underline{\lim_{n\to\infty}}\, a_n = -1$$

Proposition 1.1.7. suppose $\lim_{n\to\infty} x_n = x, -\infty < x < 0$. Then

$$\overline{\lim_{n\to\infty}}\, (x_n y_n) = \lim_{n\to\infty} x_n \cdot \underline{\lim_{n\to\infty}}\, y_n;$$
$$\underline{\lim_{n\to\infty}}\, (x_n y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}}\, y_n.$$

suppose $\lim_{n\to\infty} x_n = x, 0 < x < \infty$. Then

$$\overline{\lim_{n\to\infty}}\, (x_n y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}}\, y_n;$$
$$\underline{\lim_{n\to\infty}}\, (x_n y_n) = \lim_{n\to\infty} x_n \cdot \underline{\lim_{n\to\infty}}\, y_n.$$

Proof. We prove the first two equations. The others are similar. $\lim x_n = x, -\infty < x < 0$, so for any given $\varepsilon(0 < \varepsilon < -x)$, there exists positive integer N_1 such that for all $n > N_1$,

$$x - \varepsilon < x_n < x + \varepsilon < 0.$$

Let $\overline{\lim}\, y_n = H, \underline{\lim}\, y_n = h$. Then for the above $\varepsilon(0 < \varepsilon < -x)$, there exists N_2 such that for all $n > N_2$,

$$h - \varepsilon < y_n < H + \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$\min\{(x-\varepsilon)(H+\varepsilon), (x+\varepsilon)(H+\varepsilon)\} < x_n y_n < \max\{(x-\varepsilon)(h-\varepsilon), (x+\varepsilon)(h-\varepsilon)\},$$

Thus,

$$\underline{\lim}_{n\to\infty} (x_n y_n) \geqslant \min\{(x-\varepsilon)(H+\varepsilon), (x+\varepsilon)(H+\varepsilon)\},$$
$$\overline{\lim}_{n\to\infty} (x_n y_n) \leqslant \max\{(x-\varepsilon)(h-\varepsilon), (x+\varepsilon)(h-\varepsilon)\},$$

By arbitrariness of ε, we get

$$\underline{\lim}_{n\to\infty} (x_n y_n) \geqslant xH = \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n,$$
$$\overline{\lim}_{n\to\infty} (x_n y_n) \leqslant xh = \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n.$$

Since

$$\underline{\lim}_{n\to\infty} y_n = \underline{\lim}_{n\to\infty} \left[\frac{1}{x_n} \cdot (x_n y_n)\right] \geqslant \lim_{n\to\infty} \frac{1}{x_n} \cdot \overline{\lim}_{n\to\infty} (x_n y_n),$$
$$\overline{\lim}_{n\to\infty} y_n = \overline{\lim}_{n\to\infty} \left[\frac{1}{x_n} \cdot (x_n y_n)\right] \leqslant \lim_{n\to\infty} \frac{1}{x_n} \cdot \underline{\lim}_{n\to\infty} (x_n y_n),$$

we have

$$\underline{\lim}_{n\to\infty} (x_n y_n) \leqslant \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n,$$
$$\overline{\lim}_{n\to\infty} (x_n y_n) \geqslant \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n.$$

Combine the last four equations to get

$$\overline{\lim}_{n\to\infty} (x_n, y_n) = \lim_{n\to\infty} x_n \cdot \underline{\lim}_{n\to\infty} y_n$$
$$\underline{\lim}_{n\to\infty} (x_n y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} y_n.$$

□

Proposition 1.1.8.

$$\overline{\lim}_{n\to\infty} (cx_n) = \begin{cases} c\,\overline{\lim}_{n\to\infty} x_n & c > 0 \\ c\,\underline{\lim}_{n\to\infty} x_n & c < 0 \end{cases}$$

Similarly,

$$\underline{\lim}_{n\to\infty} (cx_n) = \begin{cases} c\,\underline{\lim}_{n\to\infty} x_n & c > 0 \\ c\,\overline{\lim}_{n\to\infty} x_n & c < 0 \end{cases}$$

In particular, c can be -1.

Proof. This is due to the similar result from supremum and infimum. For example,

$$\varlimsup_{n\to\infty}(cx_n) = \lim_{n\to\infty}\sup_{m\geq n}(cx_m) = \begin{cases} \lim_{n\to\infty}(c\cdot\sup_{m\geq n}x_m) = c\lim_{n\to\infty}\sup_{m\geq n}x_m & c>0 \\ \lim_{n\to\infty}(c\cdot\inf_{m\geq n}x_m) = c\lim_{n\to\infty}\inf_{m\geq n}x_m & c<0 \end{cases}$$

□

In fact, once we have $\varliminf x_n = -\varlimsup(-x_n)$, we only need to state limsup half in the following observations (for 1.1.9 we have $\varliminf(x_n+y_n) \geq \varliminf x_n + \varliminf y_n$; for 1.1.10 we have $\varliminf(x_n y_n) \leq (\varliminf x_n)(\varliminf y_n)$). We omit the proofs.

Proposition 1.1.9. If $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers, then

$$\limsup_{n\to\infty}(x_n+y_n) \leq \limsup_{n\to\infty}x_n + \limsup_{n\to\infty}y_n,$$

provided the sum on the right is well defined (i.e., excluding the case where one summand is ∞ and the other is $-\infty$). If one of the sequences converges then the equality holds (with the same proviso).

Proposition 1.1.10. If $\{x_n\}$ and $\{y_n\}$ are two sequences of positive real numbers, then

$$\limsup_{n\to\infty}(x_n y_n) \leq \left(\limsup_{n\to\infty}x_n\right)\left(\limsup_{n\to\infty}y_n\right),$$

provided the product on the right is well defined (i.e., excluding the case where one factor is 0 and the other is ∞). If one of the sequences converges then the equality holds (with the same proviso).

1.2 Series

1.2.1 Comparison Test

Suppose we have two *nonngegative* series $\sum_{n=0}^{\infty}u_n$ and $\sum_{n=0}^{\infty}v_n$, and they have the following relationship:

$$\exists c>0, \text{ s.t. } u_n \leq cv_n \quad n=k,k+1,k+2,\cdots$$

for some k, i.e., each term of the first series is dominated by the second after $(k-1)$-th term. Since partial sum sequence of nonnegative series converges iff the sequence is bounded, one observes

- $\sum v_n$ converges $\Rightarrow \sum u_n$ converges;
- $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges.

We have the functional version: for real functions $0 \leq f_n(x) \leq g_n(x)$,

- $\sum g_n(x)$ pointwise/uniformly converges $\Rightarrow \sum f_n(x)$ pointwise/uniformly converges;
- $\sum f_n(x)$ diverges $\Rightarrow \sum g_n(x)$ diverges.

where the pointwise one follows immediately from the number series version and the uniform one follows from Cauchy criterion for uniform convergence Let $\varepsilon>0$, then exists $N\in\mathbb{N}$ such that for any $m,n\in\mathbb{N}$ if $N\leq m\leq n$ then

$$\left|\sum_{k=m}^{n}f_k\right| \leq \left|\sum_{k=m}^{n}g_k\right| < \varepsilon$$

Then $\sum f_n$ is uniformly convergent.

1.2.2 Series of Complex Numbers

Let $\{z_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$, then the series $\sum_{n=0}^{\infty} z_n$ converges to z iff the sequence of partial sums $\{S_N(z_n)\} = \left\{\sum_{n=0}^{N} z_n\right\}$ converges to z, i.e.,

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \text{ s.t. } \forall N > K : |S_N(z_n) - z| < \varepsilon.$$

We say the seires $\sum z_n$ converges absolutely if $\sum |z_n|$ converges. Note that $\{|z_n|\}$ is a nonnegative sequence and $\{S_N(|z_n|)\} = \left\{\sum_{n=0}^{N} |z_n|\right\}$ is a monotone (nonstrictly) increasing sequence, so boundedness of $\{S_N(|z_n|)\}$ is a sufficient condition for convergence of $\sum |z_n|$. Two basic facts are presented first:

Theorem 1.2.1. If the series $\sum_{n=0}^{\infty} z_n$ converges, then $z_n \to 0$ as $n \to \infty$.

Proof. Since $\lim_{N\to\infty} S_N(z_n) = z$ for some z, we get $\lim z_n = \lim(S_N(z_n) - S_{N-1}(z_n)) = z - z = 0$. $\square$

Theorem 1.2.2. If $\sum z_n$ converges absolutely, $\sum z_n$ converges.

Proof. $\sum z_n$ converges absolutely, so for any $\varepsilon > 0$ there is K s.t. $M, N > K$ (WLOG, $M > N$) implies

$$\varepsilon > |S_M(|z_n|) - S_N(|z_n|)| = ||z_{M+1}| + \cdots + |z_N|| \geq |z_{M+1} + \cdots + z_N| = |S_M(z_n) - S_N(z_n)|$$

so $\{S_N(z_n)\}$ is Cauchy and thus converges by completeness of $\mathbb{C}$. $\square$

Complex series can relate to real series in one way:

Theorem 1.2.3. Let $z_n = a_n + ib_n$ $(n = 1, 2, \cdots)$, where a_n and b_n are real numbers. Then the series $\sum_{n=0}^{\infty} z_n$ converges to $z = a + ib$ for real numbers a and b iff $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = b$.

Proof. Apply [3] Proposition 3.5 to the sequence $S_N(z_n) = A_n + iB_N = \left(\sum_{n=0}^{N} a_n\right) + i\left(\sum_{n=0}^{N} b_n\right)$. $\square$

Example 1.2.4. Consider the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{i}{2^n}\right)$. $\sum \frac{1}{n}$ diverges. Thus, even though $\sum \frac{1}{2^n}$ converges, the whole series diverges.

1.2.3 Sequences of Complex Functions

Consider a series of functions $\{f_n(z)\}_{n=0}^{\infty}$ commonly defined on a set $E \subseteq \mathbb{C}$. Several notions of convergence are defined:

1. **Pointwise convergence (PC)**: $\forall \varepsilon > 0, \forall z \in E, \exists N(\varepsilon, z) \in \mathbb{N}$, s.t. $\forall n > N : |f_n(z) - f(z)| < \varepsilon$;
2. **Uniform convergence (UC)**: $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$, s.t. $\forall z \in E, \forall n > N : |f_n(z) - f(z)| < \varepsilon$;

2′. [Equivalent definition of uniform convergence]: $\sup\{|f_n(z) - f(z)| : z \in E\} \to 0$ as $n \to \infty$.

3. **Absolute convergence (AC)**: pointwise convergence of $\{|f_n(z)|\}_{n=0}^{\infty}$.
4. **Local uniform convergence (LUC)**: $\forall z \in E$, there is a neighborhood U of z in E such that the sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges uniformly.
5. **Compact convergence (CC)**: For each compact set $K \subseteq E$, the sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges uniformly.

It turns out that for functions on $\mathbb{C}$ (and in fact on a large class of reasonably nice spaces), the last two notions are equivalent.

Proposition 1.2.5. Let E be an open set in $\mathbb{C}$. Then the sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges locally uniformly in E iff it converges compactly in E.

Proposition 1.2.6. Sequence $\{f_n(z)\}_{n=0}^{\infty}$ converges compactly in $B(a,R)$ iff it converges in $\overline{B}(a,r)$ for every $0<r<R$.

Proof. Each closed disk is compact. Each compact set is closed and bounded and is thus contained in some closed disk in $B(a,R)$. ☐

Example 1.2.7. A simple example, with E the open unit disk, is provided by the sequence $f_n(z)=z^n$. We notice that $\sup\{|z^n| : z \in B(0,1)\} = \sup\{|z|^n : |z| \in [0,1)\} = \sup[0,1) = 1$. Thus, $\{z^n\}$ does not uniformly converge. However, if $0<r_0<1$ then this sequence converges uniformly to 0 in the disk $|z|<r_0$, and so it converges locally uniformly to 0 in the disk $|z|<1$.

1.2.4 Series of Complex Functions

Consider a series of functions $\{f_n(z)\}_{n=0}^{\infty}$ commonly defined on a set $E \subseteq \mathbb{C}$ and the sequence of partial sums $\{S_N(f_n(z))\}$. We say the series converges pointwise/absolutely/uniformly/locally uniformly/compactly if the sequence $\{S_N(f_n(z))\}$ does so. We restate them:

1. **Pointwise convergence (PC)**: $\forall \varepsilon>0, \forall z \in E, \exists K(\varepsilon, z) \in \mathbb{N}$, s.t. $\forall N>K:\ |S_N(f_n(z))(z)-f(z)|<\varepsilon$;
2. **Uniform convergence (UC)**: $\forall \varepsilon>0, \exists K(\varepsilon) \in \mathbb{N}$, s.t. $\forall z \in E, \forall N>K:\ |S_N(f_n(z))(z)-f(z)|<\varepsilon$;
3. **Absolute convergence (AC)**: pointwise convergence of $\sum_{n=0}^{\infty}|f_n(z)|$, i.e., $\{S_N(|f_n(z)|)\}_{N=1}^{\infty}$.
4. **Local uniform convergence (LUC)**: $\forall z \in E$, there is a neighborhood U of z in E such that the sequence $\{S_N(f_n(z))\}_{N=1}^{\infty}$ converges uniformly.
5. **Compact convergence (CC)**: For each compact set $K \subseteq E$, the sequence $\{S_N(f_n(z))\}_{N=1}^{\infty}$ converges uniformly.

To prove uniform convergence, one usually strengthens the inequality by finding $P_n(z)$ and Q_n such that

$$|S_N(f_n(z))-f(z)| \le P_n(z) \le Q_n$$

and then finding K for which it is true under $N>K$ by $Q_n<\varepsilon$. To show a series of functions is not uniform convergent, one proves the negation, which is obtained by switching existential and universal quantifiers and negating the statement:

$$\exists \varepsilon>0, \forall K \in \mathbb{N}, \exists z_0 \in E, \exists N_0>K \text{ s.t. } |S_{N_0}(f_n(z))-f(z_0)|>\varepsilon$$

Just like pointwise Cauchy and uniform Cauchy for sequence of functions, we have pointwise Cauchy and uniform Cauchy for complex series. We will use the last one.

Theorem 1.2.8. [Cauchy criterion for uniform convergence]

The series $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on E iff $\forall \varepsilon>0$, $\exists N(\varepsilon) \in \mathbb{N}$, such that $\forall z \in E$,

$$|f_{n+1}(z)+\cdots+f_{n+p}(z)|<\varepsilon \quad (p=1,2,\cdots)$$

Theorem 1.2.9. [Weierstrass M-test]

Suppose that $\{f_n(z)\}_{n=0}^{\infty}$ is a sequence of complex-valued functions defined on a set E, and that there is a sequence of non-negative numbers M_n satisfying the conditions

- $|f_n(z)| \le M_n$ for all $n \ge 0$ and all $z \in E$;
- $\sum_{n=0}^{\infty} M_n$ converges.

Then the series $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely and uniformly on E.

Proof. Convergence is absolute by comparison test. It is also uniform by applying Cauchy criterion for uniform convergence to the following

$$|f_n(z)+\cdots+f_{n+p}(z)| \leq |f_n(z)|+\cdots+|f_{n+p}(z)| \leq M_n+\cdots+M_{n+p}$$

□

Weierstrass M-test is often used in combination with the **uniform limit theorem** (see, e.g., [2] Theorem 21.6). Together they say that if, in addition to the above conditions, the functions f_n are continuous on E, then the series converges to a continuous function $f(z)$. A natural question of concern is about the convergence of the termwise differentiation and integration of the sum and that of the limit function. We showed in [3] Corollary 5.33 for the integration of a convergent sequence of complex function, so $S_N(f_n)(z) \rightrightarrows f(z)$ implies $\int_\gamma f(z)\,dz = \lim_{N\to\infty}\int_\gamma S_N(f_n)(z)\,dz$. Counterpart for differentiation needs theorem 1.2.6, which translates in terms of series as

Proposition 1.2.10. Series $\sum_{n=0}^\infty f_n(z)$ converges compactly in $B(a,R)$, i.e., converges uniformly for each compact set in $B(a,R)$, iff it converges in $\overline{B}(a,r)$ for every $0<r<R$.

Example 1.2.11. [Geometric series] $\sum_{n=0}^\infty z^n$.

Consider $\sum_{n=0}^\infty |z|^n$. Since $1-|z|^{N+1} = (1-|z|)(1+|z|+\cdots+|z|^N)$, we have $\sum_{n=0}^N |z|^n = \frac{1-|z|^{N+1}}{1-|z|}$. If $|z|<1$, then the series converges absolutely. Replacing $|z|$ with z in the above argument, one obtains the limit function $\frac{1}{1-z}$. The convergence is also compact in the disk $|z|<1$: for every closed disk $\overline{B}(0,r)$ with $0<r<1$ the series converges uniformly by applying Weierstrass M-test to $|z^n| \leq r^n$. If $|z|>1$, then $\lim |z|^n = \infty$ and the series diverges.

However, the series does not converge uniformly on $B(0,1)$. This is a simple consequence of the fact that each function $S_N(z^n) = \sum_{n=0}^N z^n$ is bounded while the limit function $f(z) = \frac{1}{1-z}$ is not. Hence each function $S_N - f$ is unbounded, that is, the sup-norm of $S_k - S$ is infinite, in particular the sequence of the sup-norms does not converge to zero. This last assertion is equivalent to the fact that $\{S_N\}$ does not converge uniformly to f.

Theorem 1.2.12. [Termwize Differentiation of Series]

Suppose we have

(a) a sequence of functions $f_n(z)$ $(n=1,2,\ldots)$ that are analytic in the region D;

(b) the series $\sum_{n=1}^\infty f_n(z)$ converge compactly to the function $f(z)$ inside D: $f(z) = \sum_{n=1}^\infty f_n(z)$.

Then

(1) the function $f(z)$ is analytic in the region D;

(2) for all $z\in D$ and $p=1,2,\ldots$, we have $f^{(p)}(z) = \sum_{n=1}^\infty f_n^{(p)}(z)$.

(3) The series $\sum_{n=1}^\infty f_n^{(p)}(z)$ converges compactly to $f^{(p)}(z)$ in D.

Proof. The third is left as an exercise. We prove the other two.

(1) Let z_0 be any point in D, then there exists $\rho>0$, such that the closed disk $\bar{K}: |z-z_0| \leqslant \rho$ is completely contained within D. If C is any contour within the disk $K: |z-z_0|<\rho$, then by Cauchy's integral theorem we have

$$\int_C f_n(z)\mathrm{d}z = 0, \quad n=1,2,\ldots,$$

Since the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on $\bar{K}$, and $f_n(z)$ is continuous, by uniform limit theorem, we know that $f(z)$ is continuous on $\bar{K}$. From [3] Corollary 5.33, we have

$$\int_C f(z)\mathrm{d}z = \sum_{n=1}^{\infty} \int_C f_n(z)\mathrm{d}z = 0,$$

Thus, by Morera's theorem, we know that $f(z)$ is analytic within K, that is, $f(z)$ is analytic at the point z_0. Since z_0 was arbitrary, $f(z)$ is analytic in the region D.

(2) Let z_0 be any point in D, then there exists $\rho > 0$, such that the closed disk $\bar{K} : |z - z_0| \leqslant \rho$ is completely contained within D, and the boundary of $\bar{K}$ is the circular path $\Gamma : |z - z_0| = \rho$. Hence, by [3] Theorem 8.5, we have

$$f^{(p)}(z_0) = \frac{p!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta, \quad (p = 1, 2, \ldots),$$
$$f_n^{(p)}(z_0) = \frac{p!}{2\pi i} \int_\Gamma \frac{f_n(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta,$$

On Γ, by condition (b) we know the series

$$\frac{f(\zeta)}{(\zeta - z_0)^{p+1}} = \sum_{n=1}^{\infty} \frac{f_n(\zeta)}{(\zeta - z_0)^{p+1}}$$

converges uniformly. Thus, by [3] Corollary 5.33, we get

$$\int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta = \sum_{n=1}^{\infty} \int_\Gamma \frac{f_n(\zeta)}{(\zeta - z_0)^{p+1}}\mathrm{d}\zeta,$$

Multiplying both sides by $\frac{p!}{2\pi i}$, we obtain the desired result:

$$f^{(p)}(z_0) = \sum_{n=1}^{\infty} f_n^{(p)}(z_0) \quad (p = 1, 2, \cdots).$$

□

1.3 Power Series

A power series about a is an infinite series of the form $\sum_{n=0}^{\infty} c_n(z - a)^n$. If a power series converges to a function f in a given region, we shall say that the series represents f in that region. It is important, however, to distinguish the series from the function it represents. For example, as shown in the last section, the series $\sum_{n=0}^{\infty} z^n$ represents the function $\frac{1}{1-z}$ in the disk $|z| < 1$. However, the series is not "equal" to the function, in a formal sense, even though we can write $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. The same function is represented by other power series; for example, it is represented by the series $\sum_{n=0}^{\infty} 2^{-n-1}(z + 1)^n$ in the larger disk $|z + 1| < 2$, as the reader will easily verify. A power series is best thought of as a formal sum, uniquely determined once its center and its coefficients have been specified.

1.3.1 Power Series Representation of Analytic Function

Our final goal of this subsection is to show that a power series is analytic and that an analytic function can be represented by power series. We first continue studying the convergence of series.

Proposition 1.3.1. [Abel's Theorem] If the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges on a point $z_1(\neq a)$, then it converges absolutely and compactly in the open disk with center a and radius $|z_1 - a|$, i.e., $K : |z - a| < |z_1 - a|$. The series diverges for $|z - a| > |z_1 - a|$ instead.

Proof. Let z be any point in K. Since $\sum_0^\infty c_n (z_1-a)^n$ converges, each of its term must be bounded: $\exists M>0$ such that

$$|c_n (z_1-a)^n| \leq M \quad (n=0,1,2,\cdots),$$

Therefore,

$$|c_n(z-a)^n| = \left|c_n (z_1-a)^n \left(\frac{z-a}{z_1-a}\right)^n\right| \leq M \left|\frac{z-a}{z_1-a}\right|^n,$$

Since $\forall z \in K$, $|z-a|<|z_1-a| \Rightarrow \left|\frac{z-a}{z_1-a}\right|<1 \Rightarrow$ the geometric series

$$\sum_{n=0}^{\infty} M \left|\frac{z-a}{z_1-a}\right|^n$$

converges for each $z \in K$. Thus $\sum c_n(z-a)^n$ converges absolutely in K. Besides, for any closed disk $\overline{K}_\rho$ in K, $\overline{K}_\rho : |z-a| \leq \rho \,(0<\rho<|z_1-a|)$, we have

$$|c_n(z-a)^n| \leq M \left|\frac{z-a}{z_1-a}\right|^n \leq M\left(\frac{\rho}{|z_1-a|}\right)^n,$$

By convergence of the last geometric series, apply Weierstrass M-test to see $\sum_0^\infty c_n(z-a)^n$ converges uniformly in $\overline{K}_\rho$. Then use 1.2.10. The fact that The series diverges for $|z-a|>|z_1-a|$ instead is proved by way of contradiction. □

If a power series has no such $z_1 \neq a$ for which it converges, then the series only converges at $z=a$. For example

$$1+z+2^2z^2+\cdots+n^nz^n+\cdots$$

only converges at $z=0$.

The power series can also converge (pointwise) for all z. For example

$$1+z+\frac{z^2}{2^2}+\cdots+\frac{z^n}{n^n}+\cdots.$$

For any fixed z, after some n, it has $\frac{|z|}{n}<\frac{1}{2}$. Thus, $\left|\frac{z^n}{n^n}\right|<\left(\frac{1}{2}\right)^n$ shows that it is dominated by a convergent geometric series for every z. The convergence is absolute and compact.

By Abel's theorem, if the power series does not fall into the above two cases, then the power series converges for at least $|z-a|<|z_1-a|$ for each z_1 on which it converges (pointwise). Then we can let R be the supremum of $|z_1-a|$ ranging over all z_1 on which it converges (pointwise) and see that the series converges for $|z-a|<R$ and diverges for $|z-a|>R$. Obviously, a number R making the series "converge for $|z-a|<R$ and diverge for $|z-a|>R$" is unique.

Definition 1.3.2. Let $\sum_{n=0}^\infty c_n (z-a)^n$ be a power series. We define the **radius of convergence** of the series R as the unique number such that the series converges in $|z-a|<R$ and diverges in $|z-a|>R$. If $R>0$ then the series converges absolutely and compactly in the disk $|z-a|<R$; if $R<\infty$ then the series diverges at each point of the region $|z-a|>R$. R does not give information for situation on the circle $|z-a|=R$.

Presently we shall obtain a general expression for the radius of convergence of a power series in terms of its coefficients.

Theorem 1.3.3 (Cauchy-Hadamard Theorem)**.** Consider a power series $\sum_{n=0}^\infty c_n(z-a)^n$. Then the radius of convergence R is given by

$$R=\left(\limsup_{n\to\infty} |c_n|^{1/n}\right)^{-1}. \tag{1.1}$$

That is, it satisfies

(a) The series converges absolutely for $|z-a|<R$;

(b) The series diverges for $|z-a|>R$;

(c) The series converges for every closed disk $|z-a|<r$ where $r<R$.

Proof. Assume $a=0$. Assume $0<R<\infty$ (the edge cases $R=0$ and $R=\infty$ are left as an exercise). Due to 1.1.1 (a),

$$\forall \varepsilon>0,\ \exists N \text{ s.t. } n>N \Rightarrow \frac{1}{R}-\varepsilon<|c_n|^{1/n}<\frac{1}{R}+\varepsilon. \tag{1.2}$$

So $|c_n|<\left(\frac{1}{R}+\varepsilon\right)^n$ for $n>N$. Let $z\in B(0,R)$, i.e., $|z|<R$, we have $|z|\left(\frac{1}{R}+\varepsilon\right)<1$ for some fixed $\varepsilon>0$ chosen small enough. That implies that for $n>N$ (for some large enough N as a function of ε),

$$\sum_{n=N}^{\infty}|c_nz^n|<\sum_{n=N}^{\infty}\left[\left(\frac{1}{R}+\varepsilon\right)|z|\right]^n,$$

so the series is dominated by a convergent geometric series, and hence converges. For (b), when $|z|>R$, we evoke the other side of (1.2): $|c_n|>\left(\frac{1}{R}-\varepsilon\right)^n$ for $n>N$. Besides, $|z|\left(\frac{1}{R}-\varepsilon\right)>1$ for some small enough fixed $\varepsilon>0$. Thus

$$\sum_{n=N}^{\infty}|c_nz^n|>\sum_{n=N}^{\infty}\left[\left(\frac{1}{R}-\varepsilon\right)|z|\right]^n$$

Then

$$\left(\frac{1}{R}-\varepsilon\right)^n<|c_n|$$

so the power series diverges as the geometric series diverges.

For (c), one chooses ρ between r and R and then evoke Weierstrass M-test for $|c_nz^n|<\left(\frac{r}{\rho}\right)^n$. □

Theorem 1.3.4 (d'Alambert Test). If $\sum c_n(z-a)^n$ is a given power series with radius of convergence R, then

$$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

if this limit exists.

Proof. See [1] proposition 1.4. □

Example 1.3.5. Find radius of convergence of $\sum_{n=0}^{\infty}(3+4i)^n(z-i)^{2n}$

Solution. Notice that the coefficients of the odd terms are 0, so we cannot apply the formula directly. Let

$$f_n(z)=(3+4i)^n(z-i)^{2n}$$

Then

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right|=\lim_{n\to\infty}\left|\frac{(3+4i)^{n+1}(z-i)^{2n+2}}{(3+4i)^n(z-i)^{2n}}\right|=\lim_{n\to\infty}|(3+4i)(z-i)^2|=5|z-i|^2$$

When $5|z-i|^2<1$, i.e., $|z-i|<\frac{\sqrt{5}}{5}$, the power series is absolutely convergent. When $5|z-i|^2>1$, i.e., $|z-i|>\frac{\sqrt{5}}{5}$, the power series diverges. Thus $R=\frac{\sqrt{5}}{5}$.

Definition 1.3.6. Function $f(z)$ is said to be **representable by power series** in U if $\forall B(a,r)\subseteq U$, there corresponds some power series $\sum_{n=0}^{\infty}c_n(z-a)^n$ that converges in $B(a,r)$ and equals $f(z)$.

Theorem 1.3.7 (Power series is analytic and can be differentiated termwize)**.** Power series $\sum_{n=0}^{\infty} c_n(z-a)^n$, denoted as $f(z)$, is analytic on $B(a,R)=\{z:|z-a|<R\}$, where $R=$ radius of convergence. Besides, for $z\in B(a,R)$,

$$f'(z)=\sum_{n=0}^{\infty} nc_n(z-a)^{n-1}$$

Thus, if f is representable by power series in an open set $U\subseteq\mathbb{C}$, then $f\in H(U):=$ the set of all analytic functions on U and derivative is given above.

Proof. We can assume $a=0$ becasue we can apply chain rule with $g(z)=z-a$ to $f(z)=\sum c_n z^n$ on each $z\in B(0,R)$, and R is defined regardless of a. We write

$$f(z)=\sum_{n=0}^{\infty} c_n z^n=\underbrace{\sum_{n=0}^{N} c_n z^n}_{S_N(z)}+\underbrace{\sum_{n=N+1}^{\infty} c_n z^n}_{E_N(z)},\quad g(z)=\sum_{n=1}^{\infty} nc_n z^{n-1}.$$

The claim is that f is differentiable on $B(0,R)$ and its derivative is the power series g. Since $\lim n^{1/n}=\lim e^{\log n^{\frac{1}{n}}}=1$, it is easy to see that $f(z)$ and $g(z)$ have the same radius of convergence by using 1.1.7. Fix z_0 with $|z_0|<r<R$. We wish to show that $\frac{f(z_0+h)-f(z_0)}{h}$ converges to $g(z_0)$ as $h\to 0$. Observe that

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)=&\left(\frac{S_N(z_0+h)-S_N(z_0)}{h}-S_N'(z_0)\right)\\&+\left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right)+\left(S_N'(z_0)-g(z_0)\right)\end{aligned}$$

The first term converges to 0 for $h\to 0$ for any fixed N, because $S_N(z)$ is a polynomial. To bound the second term, fix some $\varepsilon>0$, and note that, if we assume that not only $|z_0|<r$ but also $|z_0+h|<r$ (an assumption that's clearly satisfied for h close enough to 0) then

$$\begin{aligned}\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right|&\le\sum_{n=N+1}^{\infty}|a_n|\left|\frac{(z_0+h)^n-z_0^n}{h}\right|\\&=\sum_{n=N+1}^{\infty}|a_n|\left|\frac{h\sum_{k=0}^{n-1}h^k(z_0+h)^{n-1-k}}{h}\right|\\&\le\sum_{n=N+1}^{\infty}|a_n|\,nr^{n-1},\end{aligned}$$

where we use the algebraic identity

$$a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1}\right).$$

The last expression in this chain of inequalities is the tail of an absolutely convergent series, so can be made $<\varepsilon$ be taking N large enough (before taking the limit as $h\to 0$).

Third, when choosing N also make sure it is chosen so that $|S_N'(z_0)-g(z_0)|<\varepsilon$, which of course is possible since $S_N'(z_0)\to g(z_0)$ as $N\to\infty$. Finally, having thus chosen N, we get that

$$\limsup_{h\to 0}\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right|\le 0+\varepsilon+\varepsilon=2\varepsilon.$$

Since ε was an arbitrary positive number, this shows that $\frac{f(z_0+h)-f(z_0)}{h}\to g(z_0)$ as $h\to 0$, as claimed. $\square$

Corollary 1.3.8. Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ have radius of convergence $R > 0$. Then by applying the theorem to f', $f'' = (f')'$, $\cdots$, the function f is infinitely differentiable on $B(a, R)$ and its k-th derivative is given by a power seires with the same radius of convergence

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1)c_n(z-a)^{n-k}$$

for all $k \geq 1$ and $|z-a| < R$. In particular, $f^{(n)}(a) = n!c_n$, or $c_n = \frac{1}{n!}f^{(n)}(a)$.

We now show the converse.

Theorem 1.3.9. Let $f \in H(U)$ where $U \subseteq \mathbb{C}$ is open. Then f is representable by power series in U. That is, for any $\overline{B}(a, r_0) \subseteq U$, f has a power series expansion at a

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

that is convergent for all $z \in B(a, r_0)$, where $c_n = f^{(n)}(a)/n!$.

Proof. The idea is that Cauchy's integral formula ([3] Corollary 7.22) gives us a representation of $f(z)$ as a weighted "sum" (=an integral, which is a limit of sums) of functions of the form $z \mapsto (\xi - z)^{-1}$. Each such function has a power series expansion since it is, more or less, a geometric series, so the sum also has a power series expansion. Let $r < r_0$. Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(\xi)}{\xi - z} d\xi, \quad z \in B(a, r)$$

We write

$$\frac{1}{\xi - z} = \frac{1}{(\xi - a) - (z - a)} = \frac{1}{\xi - a} \cdot \frac{1}{1 - \left(\frac{z-a}{\xi-a}\right)} = \frac{1}{\xi - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n$$

where $\xi \in \partial B(a, r_0)$. Since $z \in B(a, r)$, we see $\left|\frac{z-a}{\xi-a}\right| = \frac{|z-a|}{r_0} < \frac{r}{r_0} < 1$, so the geometric series converges uniformly for $\xi \in \partial B(a, r_0)$. Now,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\partial B(a,r)} f(\xi)\frac{1}{\xi - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n d\xi \\
&= \frac{1}{2\pi i}\int_{\partial B(a,r)} \lim_{N\to\infty}\sum_{n=0}^{N}\frac{f(\xi)}{\xi-a}\left(\frac{z-a}{\xi-a}\right)^n d\xi \\
&\overset{\text{unif conv.+linearity of int}}{=\!=\!=\!=} \lim_{N\to\infty}\frac{1}{2\pi i}\sum_{n=0}^{N}\int_{\partial B(a,r)}\frac{f(\xi)}{\xi-a}\left(\frac{z-a}{\xi-a}\right)^n d\xi \\
&= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{\partial B(a,r)}\frac{f(\xi)}{(\xi-a)^{n+1}}d\xi\right)}_{\text{only depends on } a \text{, called } c_n}(z-a)^n \\
&= \sum_{n=0}^{\infty} c_n(z-a)^n, \quad z \in B(a, r), r < r_0
\end{aligned}$$

We can let $r \to r_0$ since c_n does not depend on r ($c_n = f^{(n)}(a)/n!$ by previous result).

□

Remark 1.3.10. We have a new proof showing that an analytic function is infinitely differentiable due to the corollary 1.3.8. Above theorem also gives a new proof of the n-th derivative of analytic function aside [3] Theorem 8.5.

1.3.2 Power Series on $|z-a|=R$

1.3.3 Operations of Power Series

Addition

Let $\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$ and $\sum_{n=0}^{\infty} b_n\left(z-z_0\right)^n$ be two power series with the same center. Suppose the series $\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$ has a positive radius of convergence R_1 and the series $\sum_{n=0}^{\infty} b_n\left(z-z_0\right)^n$ has a positive radius of convergence R_2.

Exercise 1.3.11. Show that $\sum_{n=0}^{\infty}(a_n+b_n)(z-z_0)^n$ has $R \geq \min\{R_1, R_2\}$.

Multiplication

The **Cauchy product** of the two series is by definition the power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ whose n-th coefficient is given by $c_n=\sum_{k=0}^n a_k b_{n-k}$. It arises when one forms all products $a_j\left(z-z_0\right)^j b_k\left(z-z_0\right)^k$, adds for each n the ones with $j+k=n$, and sums the resulting terms.

Proposition 1.3.12. Their Cauchy product converges in the disk $\left|z-z_0\right|<\min\left\{R_1, R_2\right\}$ to the product of the functions represented by the two original series.

Proof. We can assume without loss of generality that $z_0=0$. Suppose $|z|<\min\left\{R_1, R_2\right\}$. For N a positive integer we have

$$
\begin{aligned}
&\left(\sum_{j=0}^N a_j z^j\right)\left(\sum_{k=0}^N b_k z^k\right)-\sum_{n=0}^N c_n z^n \\
&=\sum_{0 \leq j, k \leq N} a_j b_k z^{j+k}-\sum_{n=0}^N \sum_{j+k=n} a_j b_k z^{j+k} \\
&=\sum_{\substack{0 \leq j, k \leq N \\ j+k>N}} a_j b_k z^{j+k} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left|\left(\sum_{j=0}^N a_j z^j\right)\left(\sum_{k=0}^N b_k z^k\right)-\sum_{n=0}^N c_n z^n\right| \\
\leq & \sum_{\substack{j \leq j, k \leq N \\ j+k>N}}\left|a_j b_k z^{j+k}\right| \\
\leq & \sum_{\frac{N}{2}<\max\{j, k\} \leq N}\left|a_j b_k z^{j+k}\right| \\
\leq & \left(\sum_{j>\frac{N}{2}}\left|a_j z^j\right|\right)\left(\sum_{k=0}^N\left|b_k z^k\right|\right)+\left(\sum_{j=0}^N\left|a_j z^j\right|\right)\left(\sum_{k>\frac{N}{2}}\left|b_k z^k\right|\right) \\
\leq & \left(\sum_{j>\frac{N}{2}}\left|a_j z^j\right|\right)\left(\sum_{k=0}^{\infty}\left|b_k z^k\right|\right)+\left(\sum_{j=0}^{\infty}\left|a_j z^j\right|\right)\left(\sum_{k>\frac{N}{2}}\left|b_k z^k\right|\right) .
\end{aligned}
$$

The last expression tends to 0 as $N \longrightarrow \infty$, because both series $\sum_{j=0}^{\infty}\left|a_j z^j\right|$ and $\sum_{k=0}^{\infty}\left|b_k z^k\right|$ converge. In view of the preceding inequality, therefore, we can conclude that

$$\sum_{n=0}^{\infty} c_n z^n=\left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{k=0}^{\infty} b_k z^k\right)$$

as desired. □

Division

Suppose the power series $\sum_{n=0}^{\infty} b_n\left(z-z_0\right)^n$ and $\sum_{n=0}^{\infty} c_n\left(z-z_0\right)^n$ have positive radii of convergence and so represent holomorphic functions g and h, respectively, in disks with center z_0. Suppose also that $g\left(z_0\right)=b_0 \neq 0$. The quotient $f=h/g$ is then holomorphic in some disk with center z_0. Then f is represented in that disk by a power series $\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$, how does one find the coefficients a_n in terms of the coefficients b_n and c_n?

A method that always works in principle uses the Cauchy product, according to which

$$c_n=\sum_{k=0}^{n} a_k b_{n-k}, \quad n=0,1, \ldots$$

From this we can conclude that $a_0=c_0/b_0$ and

$$a_n=\frac{1}{b_0}\left(c_n-\sum_{k=0}^{n-1} a_k b_{n-k}\right), \quad n=1,2, \ldots$$

The last equality expresses a_n in terms of $c_n, b_0, \ldots, b_n$ and $a_0, \ldots, a_{n-1}$, enabling one to determine the coefficients a_n recursively starting from the initial value $a_0=c_0/b_0$.

Exercise 1.3.13. Use the scheme above to determine the power series with center 0 representing the function $f(z)=1/\left(1+z+z^2\right)$ near 0 . What is the radius of convergence of the series?

Chapter 2

Zeros and Residues

The fact that every holomorphic function is locally the sum of a convergent power series has a large number of interesting consequences. A few of these are developed in this chapter.

2.1 Isolated Zeros

We will see that zeros of non-vanishing analytic functions are isolated and analytic functions that agree locally actually agree globally. We recall that $x \in X$ is a limit point of A if for $\forall \varepsilon > 0,\ B(x,\varepsilon) \cap (A - \{x\}) \neq \varnothing$, and it is easy to show that for metric space (X, d), x is a limit point of A if and only if there exists a sequence $\{x_i\}$ in A such that $x_i \to x$ ($\Leftarrow$ is because each ball $B(x,\varepsilon)$ of x contain infinitely many x_i as along as $i > N(\varepsilon)$; for $\Rightarrow$, see [3] Remark 3.39 to find such sequence). We call a point x in A an **isolated point** if it is not a limit point of A. Thus, $x \in A$ is an isolated point if there is some $B(x,\varepsilon)$ intersecting no other points in A, or equivalently, there exists no sequence $\{x_i\}$ in A converging to x.

Theorem 2.1.1. Suppose U is a region (an open and connected subset), $f \in H(U)$, and

$$Z(f) = \{a \in U : f(a) = 0\}$$

Then either

(i) $Z(f) = U$, or

(ii) $Z(f)$ has no limit point in U.

In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z-a)^m g(z) \quad (z \in U), \tag{2.1}$$

where $g \in H(U)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

Definition 2.1.2. The integer m is called the **order** or **multiplicity** of the zero which f has at the point a. Clearly, $Z(f) = U$ if and only if f is identically 0 in U. We call $Z(f)$ the **zero set** of f. Analogous results hold of course for the set of α-points of f (α-level set), i.e., the zero set of $f - \alpha$, where α is any complex number.

Proof. Let $A = (Z(f))^{acc} \cap U$ be the set of all limit points of $Z(f)$ in U. Since f is continuous and for each $a \in A$, $\exists \{z_i\} \in Z(f)$ s.t. $z_i \to a$ we have $f(a) = f(\lim z_i) = \lim f(z_i) = \lim 0 = 0$. Thus, $A \subset Z(f)$.

Fix $a \in Z(f)$, and choose $r > 0$ so that $B(a, r) \subset U$. By 1.3.9,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad (z \in B(a, r)) \tag{2.2}$$

There are now two possibilities. Either

(a) all c_n are 0 , in which case $B(a, r) \subset A$; or

(b) there is a smallest integer m (necessarily positive, since $f(a) = c_0 = 0$) such that $c_m \neq 0$.

In case (b), define

$$g(z) = \begin{cases} (z-a)^{-m} f(z), & z \in U - \{a\}, \\ c_m, & z = a. \end{cases}$$

Then (2.1) holds and $g(a) = c_m \neq 0$. We are left with showing $g \in H(U)$ to complete the proof. It is clear that $g \in H(U - \{a\})$, so we need to show it is complex differentiable at a. In fact,

$$\begin{aligned} g(z) &= (z-a)^{-m} f(z) \\ &= (z-a)^{-m} \sum_{n=m}^{\infty} c_n (z-a)^n \\ &= \sum_{n=m}^{\infty} c_n (z-a)^{n-m} \\ &= \sum_{n=0}^{\infty} c_{n+m} (z-a)^n \quad (z \in B(a, r) \setminus \{a\}) \end{aligned}$$

This is also true for $z = a$: $g(a) = c_m$ and $\sum_{n=0}^{\infty} c_{n+m}(a-a)^n = c_{0+m} = c_m$. As $g(z) = \sum_{n=0}^{\infty} c_{n+m}(z-a)^n$ is a power series representation for $z \in B(a, r)$, it follows that $g \in H(B(a, r))$. In particular, g is analytic at $z = a$, so $g \in H(U)$. Moreover, since $g(a) \neq 0$, the continuity of g shows that there is a neighborhood of a in which g has no zero. f is nonzero in the same neighborhood by (2.1), so a is an isolated point of $Z(f)$.

Therefore, if $a \in A = (Z(f))^{acc} \cap U$ (recall a is point in $Z(f)$), then case (b) cannot occur. Thus $a \in A \Rightarrow$ case (a):$B(a, r) \subset A$, which implies that A is open. If $B = U - A$, it is clear from the definition of A as a set of limit points that B is open. Thus U is the union of the disjoint open sets A and B. Since U is connected, we have either $A = U$, in which case $Z(f) = U$ (so (i) and thus (a)), or $A = \varnothing$ (which is case (ii) and note that $A = \varnothing$ implies that $B(a, r) \subset A$ is impossible and thus (b) rather than (a) must be the case). Besides, in case $A = \varnothing$, $Z(f)$ has at most finitely many points in each compact subset of U, and since U is σ-compact, $Z(f)$ is at most countable. $\square$

Note: The theorem fails if we drop the assumption that U is connected: If $U = U_0 \cup U_1$, and U_0 and U_1 are disjoint open sets, put $f = 0$ in U_0 and $f = 1$ in U_1. Then $Z(f) = U_0 \neq U$. Each $z \in U_0$ is a limit point of U_0 and is in U.

Corollary 2.1.3. f and g are holomorphic functions in a region U. If there is some sequence $\{x_i\}$ in U s.t. $f(x_i) = g(x_i)$ and $x_i \to x$ for a point $x \in U$, then $f = g$ on U. Thus, if $f(z) = g(z)$ for all z in some set A which has a limit point x in U, then $f = g$ on U. In particular, A can be an open set or the trace of a path in U.

Proof. Apply previous theorem to $f - g$. Note that $(f-g)(x) = (f-g)(\lim x_i) = \lim(f-g)(x_i) = \lim 0 = 0$ implies that $x \in Z(f-g)$. Since $x_i \to x$, x is a limit point. Previous theorem then says it has to be the case that $Z(f-g) = U$. $\square$

This is the result we alluded to when we define e^z: if we would have another $g \in H(\mathbb{C})$ with $g(x) = e^x$ for $x \in \mathbb{R}$, then in fact $g(z) = e^z$ $\forall z \in \mathbb{C}$.

2.2 Isolated Singularities

2.2.1 Classification of Isolated Singularities

Definition 2.2.1. If $a \in U$ and $f \in H(U - \{a\})$, then f is said to have an **isolated singularity** at the point a. If f can be defined at a so that the extended function is holomorphic in U, the singularity is said to be **removable**.

Theorem 2.2.2. [Criterion for removable singularity] Suppose $f \in H(U - \{a\})$ and f is bounded in $B'(a, r) := \{z : 0 < |z - a| < r\}$, for some $r > 0$. Then f has a removable singularity at a.

Remark 2.2.3. We previously had a similar result, but there we assumed f is continuous in U instead of being bounded.

Proof. Define

$$h(z) = \begin{cases} (z-a)^2 f(z), & z \in U - \{a\}, \\ 0, & z = a. \end{cases}$$

h is evidently differentiable at $U - \{a\}$, and

$$\frac{h(z) - h(a)}{z - a} = (z - a)f(z) \to 0$$

as $z \to a$ due to boundedness of f near a. Thus $h \in H(U)$ with $h'(a) = 0$. Thus we can represent h by a power series in $B(a, r) \subset U$:

$$h(z) = \sum_{n=2}^{\infty} c_n (z - a)^n \quad (z \in B(a, r)).$$

Notice that the first two coefficients are zero becasue

$$c_n = \frac{h^{(n)}(a)}{n!}, \text{ and } h(a) = h'(a) = 0$$

We obtain the desired holomorphic extension of f by setting $f(a) = c_2$, for then

$$\sum_{n=0}^{\infty} c_{n+2}(z - a)^n \quad (z \in B(a, r))$$

is a power series representation of f at a: 1. the power series has the same radius of convergence as the one representing h; 2. the power series equals f for $z \in B(a, r)$ because $(z - a)^{-2}h(z)$ agrees with this for $z \neq a$, and both sides equal c_2 for $z = a$ after setting $f(a) = c_2$. □

We note that boundedness of f is only used for showing that $\lim_{z\to a}(z - a)f(z) = 0$. Therefore, we have the following criterion:

Theorem 2.2.4. [Riemann's Criterion on Removable Singularity] Let $U \subset \mathbb{C}$ be an open subset of the complex plane, $a \in U$ a point of U and f holomorphic on $U \backslash \{a\}$. The following are equivalent:

(a) f has a removable singularity at a, i.e., f is holomorphically extendable over a.

(b) f is continuously extendable over a.

(c) There exists a neighborhood of a on which f is bounded.

(d) $\lim_{z\to a}(z - a)f(z) = 0$.

Proof. The direction $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ is clear. $(d) \Rightarrow (c)$ is shown in the proof of the above theorem 2.2.2. Also note that $(b) \Rightarrow (a)$ can be proved by [3] Corollary 8.16. □

We introduce two other isolated singularities and claim that together with removable singularity they are the all isolated singularities.

Theorem 2.2.5. If $a \in U$ and $f \in H(U - \{a\})$, then one of the following three cases must occur:

(i) f has a removable singularity at a.

(ii) There are complex numbers $c_1, \ldots, c_m$, where m is a positive integer and $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

has a removable singularity at a.

(iii) If $r > 0$ and $B(a, r) \subset U$, then $f(B'(a, r))$ is dense in the plane.

In case (ii), f is said to have a **pole of order** m at a. The function

$$\sum_{k=1}^{m} c_k (z-a)^{-k}$$

a polynomial in $(z-a)^{-1}$, is called the **principal part of** f **at** a. It is clear in this situation that $|f(z)| \to \infty$ as $z \to a$. Coefficient c_{-1} is called **residue** of f at a, denoted as $\operatorname{Res}(f; a)$. In case (iii), f is said to have an **essential singularity at** a. A statement equivalent to (iii) is that for any complex number $w \in \mathbb{C}$ there exists a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$.

Proof. Suppose (c) fails. Then we must have some $w \in \mathbb{C}$ and $r > 0$ such that $w \notin \overline{f(B'(a, r))}$; so there is a neighborhood $B(w, \delta)$ of w such that $B(w, \delta) \cap f(B'(a, r)) = \varnothing$, i.e., $|f(z) - w| > \delta$ for $z \in B'(a, r)$. Write $B = B(a, r)$, $B' = B'(a, r)$, and define

$$g(z) = \frac{1}{f(z) - w}, \quad z \in B'$$

Clearly, $g \in H(B')$ and $|g(z)| < \frac{1}{\delta}$. By 2.2.2, g has a removable singularity at $z = a$, so g extends to $g \in H(B)$. If $g(a) \neq 0$, then this means

$$0 \neq |g(a)| = \lim_{\substack{z \to a \\ z \in B'}} |g(z)| = \lim_{\substack{z \to a \\ z \in B'}} \frac{1}{|f(z) - w|}$$

and thus $|f(z) - w|$ has to stay bounded in B', and so does $|f|$. Thus again by 2.2.2, f has a removable singularity at a. (a) holds.

The other case is $g(a) = 0$. We will show that this implies (b). Obviously, g is not identically zero in the connected open set B, so we may write

$$g(z) = (z-a)^m g_1(z), \quad z \in B,$$

for some $m \geq 1$ and $g_1 \in H(B)$ with $g_1(a) \neq 0$ by 2.1.1. Also, g_1 has no zero in B' as $g(z) = \frac{1}{f(z)-w}$ in B'. We define $h = 1/g_1$ in B and then $h \in H(B)$ with h having no zero in B. Now

$$f(z) - w = \frac{1}{g(z)} = (z-a)^{-m} h(z), \quad z \in B'$$

We expand the holomorphic h into power series: $h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$, $z \in B$ with $0 \neq h(a) = b_0$. Thus we get

$$\begin{aligned}
f(z)-w=(z-a)^{-m}h(z)&=\sum_{n=0}^{\infty}b_n(z-a)^{n-m}\\
&=\frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\cdots+\frac{b_{m-1}}{z-a}+\sum_{n=0}^{\infty}b_{n+m}(z-a)^n\\
\Rightarrow f(z)-\frac{b_0}{(z-a)^m}-\frac{b_1}{(z-a)^{m-1}}-\cdots-\frac{b_{m-1}}{z-a}&=G(z):=w+\sum_{n=0}^{\infty}b_{n+m}(z-a)^n
\end{aligned}$$

where $G(z)$ is analytic in B. Thus (b) holds with $c_{-m}=b_0,\cdots,c_{-1}=b_{m-1}$. □

We put following observations without proof.

Proposition 2.2.6.

(a) If both $\lim_{z\to a}f(z)$ and $\lim_{z\to a}\frac{1}{f(z)}$ exist, then a is a removable singularity of both f and $\frac{1}{f}$.

(b) If $\lim_{z\to a}f(z)$ exists but $\lim_{z\to a}\frac{1}{f(z)}$ does not exist (in fact $\lim_{z\to a}|1/f(z)|=\infty$), then a is a zero of f and a pole of $\frac{1}{f}$.

(c) if $\lim_{z\to a}f(z)$ does not exist (in fact $\lim_{z\to a}|f(z)|=\infty$) but $\lim_{z\to a}\frac{1}{f(z)}$ exists, then a is a pole of f and a zero of $\frac{1}{f}$.

(d) If neither $\lim_{z\to a}f(z)$ nor $\lim_{z\to a}\frac{1}{f(z)}$ exists, then a is an essential singularity of both f and $\frac{1}{f}$.

2.2.2 Residue Theorem for One Pole

Suppose now U is open and convex and $f\in H(U-\{a\})$ has a pole at $z=a$. Then we can write

$$f(z)=\sum_{k=1}^{m}c_{-k}(z-a)^{-k}+g(z)$$

for some $g\in H(U)$. Thus, if γ is a closed piecewise C^1 curve in $U-\{a\}$, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz=\frac{1}{2\pi i}\int_\gamma\frac{c_{-1}}{z-a}\,dz=\frac{1}{2\pi i}\int_\gamma\frac{\operatorname{Res}(f;a)}{z-a}\,dz=\operatorname{Res}(f;a)n_\gamma(a)$$

where we used the fact that each $z\mapsto(z-a)^{-k}$, $k>1$ has a primitive $\frac{1}{1-k}(z-a)^{1-k}$ in a neighborhood of γ^* (since $\operatorname{dist}(a,\gamma^*)>0$) and so their integrals over γ vanish. We also invoked the Cauchy theorem to see $g\in H(U)\Rightarrow\int_\gamma g=0$. We will generalize this residue theorem later after we get to the "global Cauchy theorem."

Example 2.2.7. We calculate the integral

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx\quad 0<a<1$$

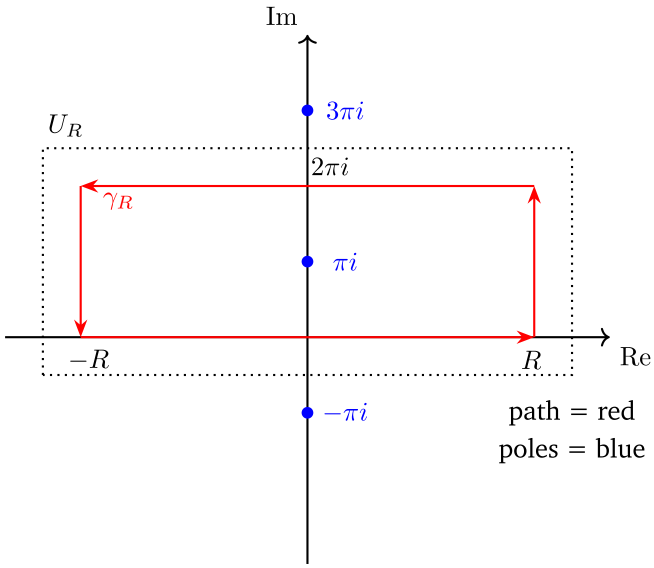
We will show that its value is $\frac{\pi}{\sin(\pi a)}$.

Solution. Let

$$f(z)=\frac{e^{az}}{1+e^z}$$

Step 1 (find poles): Notice that e^{az} is entire and that $1+e^z=0\Leftrightarrow e^{i\theta}=e^{i\pi}\Leftrightarrow\theta=2k'\pi+\pi\,(k=0,\pm1,\cdots)=k\pi\,(k=\pm1,\cdots)$, so the poles are $z=i\theta=k\pi i=\cdots,-3\pi i,-\pi i,\pi i,3\pi i,\cdots$

Step 2 (choose path and U): We consider the path γ_R in the picture, and we choose an open convex set U_R with $\gamma_R^* \subseteq U$ (for instance, U_R can be a small flattening of the box).



Step 3 (apply residue theorem): Now, f is analytic in $U_R \setminus \{\pi i\}$. We apply our toy residue theorem given just before this example to see

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \mathrm{Res}(f;\pi i)$$

where the winding number $n_{\gamma_R}(\pi i)$ is arguably just 1 as shown in the picture (we will develop tools to systematically justify computation of winding numbers later). Heuristically, we guess the order of the pole πi is 1 (so then it would be that $f(z) = \mathrm{Res}(f;\pi i)(z-\pi i)^{-1} + g(z)$ for analytic g, so $(z-\pi i)f(z) = \mathrm{Res}(f;\pi i) + (z-\pi i)g(z) \to \mathrm{Res}(f;\pi i)$ as $z \to \pi i$). We calculate

$$\begin{aligned}(z-\pi i)f(z) &= (z-\pi i)\frac{e^{az}}{1+e^z} = e^{az}\left(\frac{e^z - e^{\pi i}}{z-\pi i}\right)^{-1} \\ &\xrightarrow{z\to 2\pi i} e^{a\pi i}\left(\frac{d}{dz}e^z\right)\bigg|_{z=\pi i} = e^{a\pi i}e^{\pi i} = -e^{a\pi i}\end{aligned}$$

Thus, indeed, as the limit exists, we must have

$$\mathrm{Res}(f;\pi i) = -e^{a\pi i}$$

Thus,

$$\int_{\gamma_R} f(z)\,dz = -2\pi i e^{2\pi i}$$

Step 4 (calculate the original integral): We then relate this result to the original real integral. Let

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx =: \lim_{R\to\infty} I_R$$

The integral of f over the top line of the rectangle with orientation from right to left is along $\eta_R(t) =$

$-t+2\pi i,\ t\in[-R,R]$, so

$$\begin{aligned}
\int_{\eta_R} f(z)\,dz &= \int_{-R}^{R} f(\eta_R(t))\eta_R'(t)dt \\
&= -\int_{-R}^{R} \frac{e^{a(-t+2\pi i)}}{1+e^{-t+2\pi i}}dt \\
&= -e^{2\pi i a}\int_{-R}^{R} \frac{e^{-at}}{1+e^{-t}}dt \\
&\overset{\substack{x=-t\\ dx=-dt}}{=\!=\!=} -e^{2\pi i a}\int_{-R}^{R} \frac{e^{ax}}{1+e^{x}}dx \qquad \left(-\int_{R}^{-R}=\int_{-R}^{R}\right) \\
&= -e^{2\pi i a} I_R
\end{aligned}$$

Thus,

$$\int_{\gamma_R} f = (1-e^{2\pi i a})I_R + \int_{\text{left vertical}} f + \int_{\text{right vertical}} f$$

Notice, for instance, right vertical is parametrized by $t\to R+it,\ t\in[0,2\pi]$. Then,

$$\begin{aligned}
\left|\int_{\text{right vertical}} f(z)\,dz\right| &= \left|\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{Rtit}} i dt\right| \\
&\le \int_0^{2\pi} \frac{e^{aR}}{e^R-1}dt \qquad \left(e^R-1\ge \frac{e^R}{2} \text{ for large } R\right) \\
&\le Ce^{(a-1)R} \xrightarrow{R\to\infty} 0 \qquad (a<1 \text{ so } a-1<0)
\end{aligned}$$

Similarly,

$$\left|\int_{\text{left vertical}} f(z)\,dz\right| \xrightarrow{R\to\infty} 0$$

Therefore, by noticing that $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, we have

$$\begin{aligned}
-2\pi i e^{2\pi i} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = (1-e^{2\pi i a}I) \\
\Rightarrow I &= -2\pi i \frac{e^{a\pi i}}{1-e^{2\pi i a}} = \frac{2\pi i}{e^{\pi i a}-e^{-\pi i a}} = \frac{\pi}{\sin(\pi a)}
\end{aligned}$$

We prove a useful formula to calculate residue of f at a pole.

Proposition 2.2.8. If $f\in H(U\setminus\{a\})$ has a pole of order n at a, then

$$\operatorname{Res}(f;a) = \lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-a)^n f(z).$$

Proof. For $f\in H(U\setminus\{a\})$ with a pole a of n-th order, we can write

$$f(z) = \sum_{k=1}^{n} c_{-k}(z-a)^{-k} + g(z)$$

for some $g(z)\in H(U)$. Then g is representable by a power series in U. That is, for any $\overline{B}(a,r)\subseteq U$, g has a power series expansion at a

$$g(z) = \sum_{k=0}^{\infty} c_k\,(z-a)^k$$

that is convergent for all $z \in B(a,r)$. Thus,

$$f(z)=\sum_{k=1}^{n} c_{-k}(z-a)^{-k}+g(z)=\sum_{k=1}^{n} c_{-k}(z-a)^{-k}+\sum_{k=0}^{\infty} c_k (z-a)^k=\frac{\overbrace{\sum_{k=0}^{\infty} c_{k-n}(z-a)^k}^{\varphi(z)}}{(z-a)^n} \tag{2.3}$$

where $\varphi(z)$ is a power series with the same radius of convergence as g, i.e., r, since changing finitely many elements of a sequence does not affect its limsup. Thus $\varphi(z)$ is analytic on $B(a,r)$ as shown in class, and its $(n-1)$-th derivative at a is given by

$$\varphi^{(n-1)}(a)=\frac{(n-1)!}{2\pi i}\int_{\partial B}\frac{\varphi(z)}{(z-a)^n}\,dz \tag{2.4}$$

by noting that the winding number of $\partial B(a,r)$ around a is 1. This computation is shown during the proof of converse of analyticity of power series, but it can also be inferred from [3] Theorem 8.5. Now we note that equation (2.3) gives $\varphi(z)=f(z)(z-a)^n$ whenever $z\neq a$. Therefore,

$$\varphi^{(n-1)}(a)=\lim_{z\to a}\varphi^{(n-1)}(z)=\lim_{z\to a}\left(\frac{d}{dz}\right)^{n-1}(z-a)^n f(z) \tag{2.5}$$

The residue is computed as

$$\begin{aligned}\operatorname{Res}(f;a)&=\frac{1}{2\pi i}\int_{\partial B} f(z)\,dz=\frac{1}{2\pi i}\int_{\partial B}\frac{\varphi(z)}{(z-a)^n}\,dz\\ &\overset{(2.4)}{=\!=}\frac{1}{2\pi i}\left(\frac{2\pi i}{(n-1)!}\varphi^{(n-1)}(a)\right)\\ &\overset{(2.5)}{=\!=}\lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-a)^n f(z)\end{aligned}$$

□

Example 2.2.9. Calculate $\operatorname{Res}(f;0)$ for

$$f(z):=\frac{\sinh(z)e^z}{z^5},$$

where $\sinh(z):=(e^z-e^{-z})/2$.

Solution. We now apply this formula to $f(z)=\sinh(z)e^z/z^5$. Since e^z is entire, we see $\sinh(z)=\frac{e^z-e^{-z}}{2}$ is also entire. Therefore, $a=0$ is a pole of order 5 for f. Thus,

$$\begin{aligned}\operatorname{Res}(f;0)&=\lim_{z\to 0}\frac{1}{4!}\left(\frac{d}{dz}\right)^4 z^5 f(z)\frac{1}{24}\lim_{z\to 0}\left(\frac{d}{dz}\right)^4\sinh(z)e^z\\ &=\frac{1}{24}\lim_{z\to 0}\frac{d^4\left(\frac{e^{2z}}{2}-\frac{1}{2}\right)}{dz^4}=\frac{1}{24}\lim_{z\to 0}8e^{2z}=\frac{1}{3}\cdot 1=\frac{1}{3}\end{aligned}$$

There is a rather elementary way to see this, if we recall the power series expansions of $\sinh(z)$ and e^z at 0:

$$\sinh(z)=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots$$

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots$$

Then,

$$\operatorname{Res}(f;0)=c_{-1}=\operatorname{coeff}([z(z^3/3!)+z(z^3/6)]/z^5)=\frac{1}{3}$$

2.3 Global Cauchy Theorem

Let $\gamma : [a, b] \to U$ be a piecewise C^1 closed path, where U is convex and open. Suppose $z_0 \in \mathbb{C} \backslash U$. Now $f(z) := \frac{1}{z - z_0}$ is analytic in U, and so by Cauchy theorem on convex set,

$$\begin{aligned} n_\gamma(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} \\ &= \frac{1}{2\pi i} \int_\gamma f(z) dz = 0 \end{aligned}$$

Thus,

$$n_\gamma(z_0) \quad \forall z_0 \in \mathbb{C} \setminus U$$

Could it be that the validity of Cauchy is not so much about the convexity of U, but rather about this property of γ itself? Is it true that if $\gamma : [a, b] \to U$ is a closed piecewise C^1 path with $n_\gamma(z) = 0 \quad \forall z \in \mathbb{C} \backslash U$, then for all $f \in H(U)$ we have $\int_\gamma f = 0$ even if U is just assumed open? Yes!

And even more is true: we can use formal sums of paths (called cycles) and not just individual paths. To this end, we first quickly study the following concepts of chains and cycles.

2.3.1 Chains and Cycles

Suppose $\gamma_1, \ldots, \gamma_n$ are paths in the plane, and put $K = \gamma_1^* \cup \cdots \cup \gamma_n^*$. Each γ_i induces a linear functional $\widetilde{\gamma}_i$ on the vector space $C(K)$, by the formula

$$\widetilde{\gamma}_i(f) = \int_{\gamma_i} f(z) dz$$

Define

$$\widetilde{\Gamma} = \widetilde{\gamma}_1 + \cdots + \widetilde{\gamma}_n.$$

Explicitly, $\widetilde{\Gamma}(f) = \widetilde{\gamma}_1(f) + \cdots + \widetilde{\gamma}_n(f)$ for all $f \in C(K)$. The above relation suggests that we introduce a "formal sum"

$$\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n = \sum_{i=1}^n \gamma_i$$

and define

$$\int_\Gamma f(z) dz = \widetilde{\Gamma}(f)$$

Then $\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ is merely an abbreviation for the statement

$$\int_\Gamma f(z) dz = \sum_{i=1}^n \int_{\gamma_i} f(z) dz \quad (f \in C(K)).$$

Note that this equation serves as the definition of its left side. The objects Γ so defined are called **chains**. If each γ_j in $\Gamma = \sum_{i=1}^n \gamma_i$ is a closed path, then Γ is called a **cycle**. If each γ_j in $\Gamma = \sum_{i=1}^n \gamma_i$ is a path in some open set U, we say that Γ is a **chain** in U. If $\Gamma = \sum_{i=1}^n \gamma_i$ holds, we define

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*$$

If Γ is a cycle and $\alpha \notin \Gamma^*$, we define the index of α with respect to Γ by

$$\operatorname{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - \alpha},$$

Obviously, $\Gamma = \sum_{i=1}^{n} \gamma_i$ implies

$$\operatorname{Ind}_\Gamma(\alpha) = \sum_{i=1}^{n} \operatorname{Ind}_{\gamma_i}(\alpha).$$

If each γ_i in $\Gamma = \sum_{i=1}^{n} \gamma_i$ is replaced by its opposite path $\overleftarrow{\gamma_i}$, the resulting chain will be denoted by $-\Gamma$. Then

$$\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz \quad \left(f \in C\left(\Gamma^*\right)\right).$$

In particular, $\operatorname{Ind}_{-\Gamma}(\alpha) = -\operatorname{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$. Chains can be added and subtracted in the obvious way, by adding or subtracting the corresponding functionals: The statement $\Gamma = \Gamma_1 + \Gamma_2$ means

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

for every $f \in C\left(\Gamma_1^* \cup \Gamma_2^*\right)$. Finally, note that a chain may be represented as a sum of paths in many ways. To say that

$$\gamma_1 \dot{+} \cdots \dot{+} \gamma_n = \delta_1 \dot{+} \cdots \dot{+} \delta_k$$

means simply that

$$\sum_i \int_{\gamma_i} f(z)dz = \sum_j \int_{\delta_j} f(z)dz$$

for every f that is continuous on $\gamma_1^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_k^*$. In particular, a cycle may very well be represented as a sum of paths that are not closed.

2.3.2 Global Cauchy Theorem

We will use the following lemma for proof of the global Cauchy theorem.

Lemma 2.3.1. If $f \in H(U)$ and g is defined in $U \times U$ by

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

then g is continuous in $U \times U$.

Proof. The only points $(z,w) \in U \times U$ at which the continuity of g is possibly in doubt have $z = w$.

Fix $a \in U$. Fix $\varepsilon > 0$. There exists $r > 0$ such that $B(a,r) \subset U$ and $|f'(\xi) - f'(a)| < \varepsilon$ for all $\xi \in B(a,r)$. If z and w are in $B(a,r)$ and if

$$\gamma(t) = [z,w](t) = (1-t)z + tw,$$

then $\gamma(t) \in B(a,r)$ for $0 \leq t \leq 1$. When $(w,z) \neq (a,a)$, by [3] Corollary 5.45,

$$\begin{aligned} g(z,w) - g(a,a) &= \frac{f(z)-f(w)}{z-w} - f'(a) = -\frac{1}{z-w}\int_\gamma f'(\xi)d\xi - f'(a) \\ &= \frac{1}{w-z}\int_0^1 f'(\gamma(t))\gamma'(t)dt - f'(a) = \int_0^1 [f'(\gamma(t)) - f'(a)]dt \end{aligned}$$

and this equation is also true for the case $(w,z) = (a,a)$ where $\gamma(t) = a$ for $0 \leq t \leq 1$. The absolute value of the integrand is $< \varepsilon$, for every t. Thus $|g(z,w) - g(a,a)| < \varepsilon$. This proves that g is continuous at (a,a). $\square$

Theorem 2.3.2. [Global Cauchy Theorem] Suppose $f \in H(U)$, where $U \subseteq \mathbb{C}$ is an open set. If Γ is a cycle in U that satisfies

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \quad \text{for every } \alpha \text{ not in } U, \tag{2.6}$$

then

$$f(z) \cdot \operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw \quad \text{for } z \in U - \Gamma^* \tag{2.7}$$

and

$$\int_\Gamma f(z) dz = 0. \tag{2.8}$$

If Γ_0 and Γ_1 are cycles in U such that

$$\operatorname{Ind}_{\Gamma_0}(\alpha) = \operatorname{Ind}_{\Gamma_1}(\alpha) \quad \text{for every } \alpha \text{ not in } U, \tag{2.9}$$

then

$$\int_{\Gamma_0} f(z) dz = \int_{\Gamma_1} f(z) dz \tag{2.10}$$

Proof. The function g defined in $U \times U$ by

$$g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

is continuous in $U \times U$ by the lemma, so $h(z)$ defined by the integral

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z,w) dw \quad (z \in U)$$

is well-defined. Since $\int_\Gamma f'(z) dw = 0$ and $\frac{1}{2\pi i} \int_\Gamma \frac{f(w)-f(z)}{w-z} dw = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)dw}{w-z} - f(z) \operatorname{Ind}_\Gamma(z)$, the formula (2.7) is equivalent to the assertion that

$$h(z) = 0 \quad (z \in U - \Gamma^*) \tag{2.11}$$

To prove equation (2.11), let us first show h is continuous on U. Fix $z_0 \in U$ and an arbitrary sequence $\{z_n\} \in U, z_n \to z_0$. As $g : U \times U \to \mathbb{C}$ is continuous, it is uniformly continues on compact sets of $U \times U$. Choose r s.t. $\bar{B}(z_0, r) \subset U$. Fix $\varepsilon > 0$. Now, y is uniformly continuous on $\bar{B}(z_0, r) \times \Gamma^* \Rightarrow \exists \delta < r$ s.t. whenever $(z_1, w_1), (z_2, w_2) \in \bar{B}(z_0, r) \times \Gamma^*$ satisfy $|(z_1, w_1) - (z_2, w_2)| < \delta$ then $|g(z_1, w_1) - g(z_2, w_2)| < \varepsilon$. Now, choose N s.t. $z_n \in B(z_0, \delta)\ \forall n > N$. Then $\forall w \in \Gamma^*$ we have for all $n > N$ that

$$|g(z_n, w) - g(z_0, w)| < \varepsilon$$

as $(z_n, w), (z_0, w) \in B(z_0, \delta) \times \Gamma^* \subset \overline{B}(z, r) \times \Gamma^*$ and $|(z_n, w) - (z_0, w)| = |z_n - z_0| < \delta$. Therefore,

$$g_n(w) := g(z_n, w) \rightrightarrows g(z_0, w)$$

uniformly for $w \in \Gamma^*$. Then

$$\lim_{n\to\infty} h(z_n) = \frac{1}{2\pi i} \lim_{n\to\infty} \int_\Gamma g_n(w) dw = \frac{1}{2\pi i} \int_\Gamma \lim_{n\to\infty} g_n(w) dw = \frac{1}{2\pi i} \int_\Gamma g(z_0, w) dw = h(z_0)$$

This shows that h is continuous at z_0. Since z_0 is an arbitrary point in U, h is continuous on U.

To further show that $h \in H(U)$, we recall Morera's theorem ([3] Corollary 8.14) that a function $U \to \mathbb{C}$ is analytic on an open set U if its integral over any triangle $\partial\triangle$ in U is zero. Now,

$$\int_{\partial\triangle} h(z)\, dz = \int_{\partial\triangle} \left(\frac{1}{2\pi i} \int_\Gamma g(z,w) dw \right) dz \overset{\text{Fubini}}{=\!=} \frac{1}{2\pi i} \int_\Gamma \left(\int_{\partial\triangle} g(z,w) dw \right) dw$$

Now, with $w \in \Gamma^*$ fixed, the function $z \to g(z, w)$ is obviously analytic in $U \setminus \{w\}$ and continuous in U. Hence, we can either use Cauchy for triangles that allow a special point ([3] Theorem 6.3), or conclude that $z \to g(z, w)$ is in $H(U)$ as the singularity at $z = w$ must be removable by continuity. In either way, we have $\int_{\partial\triangle} g(z, w)dw = 0$. Thus, $\int_{\partial\triangle} h(z)\, dz = 0$ and Morera's theorem gives $h \in H(U)$.

Next, we define function

$$p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi \quad (z \in \mathbb{C} \setminus \Gamma^*)$$

and show its analyticity. This is more straightforward than h. In fact, recall how we proved that an analytic function can be developed into a power series by writing it via Cauchy's integral formula and expanding $\frac{1}{\xi - z}$ into a power series (see 1.3.9). The same strategy can be applied to p.

Fix $z_0 \in \mathbb{C} \setminus \Gamma^*$ and choose $\delta > 0$ such that $B(z_0, 2\delta) \subseteq \mathbb{C} \setminus \Gamma^*$. Write

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0}\frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}.$$

which uniformly converges for $\xi \in \Gamma^*$ and $z \in B(z_0, \delta)$ because $B(z_0, 2\delta) \subseteq \mathbb{C} \setminus \Gamma^* \Rightarrow |\xi - z_0| > 2\delta$ and $z \in B(z_0, \delta) \Rightarrow |z - z_0| < \delta$, so $\left|\frac{z - z_0}{\xi - z_0}\right| \leq \frac{\delta}{2\delta} = \frac{1}{2}$. Then

$$p(z) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi}_{c_n}(z - z_0)^n \quad (z \in B(z_0, \delta))$$

Thus, $p \in H(B(z_0, \delta))$. Since $z_0 \in \mathbb{C} \setminus \Gamma^*$ was arbitrary, p is analytic in $\mathbb{C} \setminus \Gamma^*$.

Now, we have analytic functions

$$h(z) = \frac{1}{2\pi i}\int_\Gamma g(z, w)dw \quad (z \in U)$$
$$p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi \quad (z \in \mathbb{C} \setminus \Gamma^*)$$

We glue them to get an entire function, which will exploit the assumption on Γ we still didn't use, i.e., $n_\Gamma(\alpha) = 0$, $\forall \alpha \in \mathbb{C} \setminus U$.

Let $\Omega := \{z \in \mathbb{C} \mid \Gamma^* : n_\Gamma(z) = 0\}$. Our assumption on Γ implies that $\mathbb{C} \setminus U \subset \Omega$, and so $\mathbb{C} = U \cup \Omega$.

If $z \in U \cap \Omega$, then both h and p are defined, and in fact, we have

$$h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi - f(z)\overbrace{n_\Gamma(z)}^{=0} = p(z).$$

So we can define $\varphi \in H(\mathbb{C})$ by setting

$$\varphi(z) := \begin{cases} h(z), & z \in U, \\ p(z), & z \in \Omega. \end{cases}$$

This is well-defined, since in $U \cap \Omega$ we have $h(z) = p(z)$.

Notice that Ω is open: $\Omega = n_\Gamma^{-1}B(0, 1)$ as $n_\gamma : \mathbb{C} \setminus \Gamma^* \to \mathbb{C}$ is $\mathbb{Z}$-valued and continuous. It is then clear that $\varphi \in H(\mathbb{C})$, as for each $z \in \mathbb{C}$ there is an open neighborhood in which φ equals to h or p.

Our final step is to show $h \equiv 0$.

We will apply Liouville's theorem to φ. Notice that Γ^* is bounded and so $\Gamma^* \subset D$ for some closed disk D, Thus $\mathbb{C}\backslash D \subset \mathbb{C}\backslash\Gamma^*$ and for big $|z|$ we must be in the unbounded connected open set $\mathbb{C} \setminus D$ where $n_\Gamma(z) = 0$. So for big $|z|$ we have $z \in \Omega$ and so

$$|\varphi(z)| = |p(z)| = \left|\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi\right| \leq \frac{1}{2\pi}\frac{\|f\|_\infty}{\operatorname{dist}(z,\Gamma^*)}$$
$$\left(\xi \in \Gamma \Rightarrow \operatorname{dist}(z,\Gamma^*) \leq |\xi - z| \Rightarrow \sup_{\xi\in\Gamma^*}\left|\frac{f(\xi)}{\xi - z}\right| \leq \frac{\sup_{\xi\in\Gamma^*|f(\xi)|}}{\operatorname{dist}(z,\Gamma^*)} = \frac{\|f\|_\infty}{\operatorname{dist}(z,\Gamma^*)}\right)$$

This implies φ is bounded in $\mathbb{C}$ (compare to the argument in the proof of the fundamental theorem of algebra) and

$$\varphi(z) \to 0 \text{ as } |z| \to \infty$$

By Liouville's theorem ([3] Corollary 8.10), φ is constant and the constant must be 0. So $h(z) = \varphi(z) = 0$ for $z \in U$, and $h(z) = 0$ for $z \in U\backslash\Gamma^*$, so (2.7) is verified (notice that $n_\Gamma(z)$ only makes sense in $U \setminus \Gamma^*$, not in whole U). To deduce (2.8) from (2.7), we pick $z_0 \in U\backslash\Gamma^*$ and define $F(z) = (z - z_0) f(z)$. Then $F \in H(U)$ and they apply to F to give

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z - z_0}dz = \overbrace{F(z_0)}^{=0}\, n_\Gamma(z_0) = 0.$$

Finally, the path deformation claim follows from applying (2.8) to $\Gamma := \Gamma_1 - \Gamma_0$. □

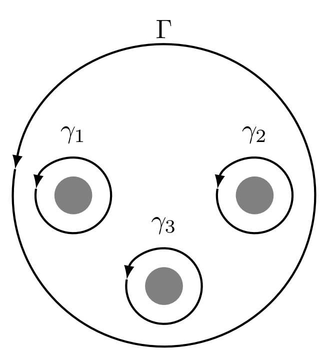
Remark 2.3.3.
(a) If γ is a closed path in a convex region Ω and if $\alpha \notin \Omega$, an application of Cauchy's theorem on convex set to $f(z) = (z - \alpha)^{-1}$ shows that $\operatorname{Ind}_\gamma(\alpha) = 0$. Assumption on Γ in global version is therefore satisfied by every cycle in Ω if Ω is convex. This shows that global version generalizes Cauchy's theorem and formula on convex set.

(b) The path deformation part of the above theorem shows under what circumstances integration over one cycle can be replaced by integration over another, without changing the value of the integral. For example, let U be the plane with three disjoint closed discs D_i removed, i.e., $U = \mathbb{C} \setminus (D_1 \cup D_2 \cup D_3)$. If $\Gamma, \gamma_1, \gamma_2, \gamma_3$ are positively oriented circles in Ω such that Γ surrounds $D_1 \cup D_2 \cup D_3$ and γ_i surrounds D_i but not D_j for $j \neq i$, then

$$\forall \alpha \in \mathbb{C} \setminus U, \operatorname{Ind}_\Gamma(\alpha) = \operatorname{Ind}_{\gamma_1+\gamma_2+\gamma_3}(\alpha).$$

and for every $f \in H(\Omega)$.

$$\int_\Gamma f(z)dz = \sum_{i=1}^{3}\int_{\gamma_i} f(z)\, dz$$



(c) In order to apply global Cauchy, it is desirable to have a reasonably efficient method of finding the index of a point with respect to a closed path. The following theorem does this for all paths that occur in practice. It says, essentially, that the index increases by 1 when the path is crossed "from right to left." If we recall that $\mathrm{Ind}_\gamma(\alpha) = 0$ if α is in the unbounded component of the complement W of γ^*, we can then successively determine $\mathrm{Ind}_\gamma(\alpha)$ in the other components of W, provided that W has only finitely many components and that γ traverses no arc more than once.

Theorem 2.3.4. Suppose γ is a closed path in the plane, with parameter interval $[\alpha, \beta]$. Suppose $\alpha < u < v < \beta, a$ and b are complex numbers, $|b| = r > 0$, and

(i) $\gamma(u) = a - b, \gamma(v) = a + b$,

(ii) $|\gamma(s) - a| < r$ if and only if $u < s < v$,

(iii) $|\gamma(s) - a| = r$ if and only if $s = u$ or $s = v$.

Assume furthermore that $D(a; r) - \gamma^*$ is the union of two regions, $D_+$and D_-, labeled so that $a + bi \in \bar{D}_+$and $a - bi \in \bar{D}_-$. Then

$$\mathrm{Ind}_\gamma(z) = 1 + \mathrm{Ind}_\gamma(w)$$

if $x \in D_+$and $w \in D_-$. As $\gamma(t)$ traverses $D(a; r)$ from $a - b$ to $a + b, D_-$is "on the right" and $D_+$is "on the left" of the path.

Proof. See [5] Theorem 10.37 □

2.3.3 Homotopy

We introduce a concept in algebraic topology that is also related to Cauchy's theorem. First, to be clearer on the terminology, we would say *curves* are continuous but not necessarily differentiable while *paths* are assumed to be piecewise C^1.

Suppose $\gamma_0, \gamma_1 : I \to U$ are closed curves in a topological space X. We say that γ_0 and γ_1 are U**-homotopic** if there is a continuous map $H : I \times I \to U$ such that

$$H(s, 0) = \gamma_0(s), \quad H(s, 1) = \gamma_1(s), \quad H(0, t) = H(1, t)$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s, t)$. Then H defines a one-parameter family of closed curves γ_t in X, which connects γ_0 and γ_1. Intuitively, this means that γ_0 can be continuously deformed to γ_1, within X.

If γ_0 is U-homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is null-homotopic in U. If U is connected and if every closed curve in U is null-homotopic, U is said to be simply connected.

For example, every convex region Ω is simply connected. To see this, let γ_0 be a closed curve in Ω, fix $z_1 \in \Omega$, and define **straight-line homotopy**

$$H(s, t) = (1 - t)\gamma_0(s) + tz_1 \quad (0 \le s \le 1, 0 \le t \le 1)$$

Theorem 2.3.6 will show that (2.9) in the global Cauchy holds whenever Γ_0 and Γ_1 are U-homotopic closed paths. As a special case of this, note that condition (2.6) of 2.3.2 holds for every closed path Γ in U if U is simply connected, since constant paths necessarily have zero index.

Lemma 2.3.5. Let γ_0 and γ_1 be closed paths with parameter interval $[0, 1]$ and let α be a complex number. If

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \quad (0 \le s \le 1) \tag{2.12}$$

then

$$\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_0}(\alpha).$$

Proof. Condition (2.12) implies that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$. Hence one can define $\gamma(t) = (\gamma_1(t) - \alpha) / (\gamma_0(t) - \alpha)$. Since

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \Rightarrow \left|\frac{\gamma_0 - \gamma_1}{\gamma_0 - \alpha}\right| = \left|\frac{\gamma_0 - \alpha - (\gamma_1 - \alpha)}{\gamma_0 - \alpha}\right| = |1 - \gamma| < 1$$

we see $\gamma^* \in B(1,1)$, which implies that $\mathrm{Ind}_\gamma(0) = 0$. The following relationship between the paths and their derivatives can also be easily calculated.

$$\frac{\gamma'}{\gamma} = \frac{\gamma_1'}{\gamma_1 - \alpha} - \frac{\gamma_0'}{\gamma_0 - \alpha}$$

Integrating the above identity over $[0,1]$ and writing them in path integral forms will give the desired result:

$$\underbrace{\int_\gamma \frac{1}{z}\,dz}_{\mathrm{Ind}_\gamma(0)=0} = \underbrace{\int_{\gamma_1} \frac{1}{z-\alpha}\,dz}_{\mathrm{Ind}_{\gamma_1}(\alpha)} - \underbrace{\int_{\gamma_0} \frac{1}{z-\alpha}\,dz}_{\mathrm{Ind}_{\gamma_0}(\alpha)}$$

□

Theorem 2.3.6. If Γ_0 and Γ_1 are U-homotopic closed paths in an open connected set U, and if $\alpha \notin U$, then

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha)$$

Proof. By definition, there is a continuous $H : I^2 \to \Omega$ such that

$$H(s,0) = \Gamma_0(s), \quad H(s,1) = \Gamma_1(s), \quad H(0,t) = H(1,t).$$

Since I^2 is compact, so is $H\left(I^2\right)$. Since α is not in the closed set $H(I^2)$, there exists $\varepsilon > 0$ such that $B(\alpha, 2\varepsilon) \cap H(I^2) = \varnothing$, i.e.,

$$|\alpha - H(s,t)| \geq 2\varepsilon \quad \forall (s,t) \in I^2. \tag{2.13}$$

Since H is continuous on compact set and is thus uniformly continuous, there is a positive integer n such that:

$$|H(s,t) - H(s',t')| < \varepsilon \quad \text{if} \quad |s - s'| + |t - t'| \leq 1/n. \tag{2.14}$$

(Note that $\sqrt{|s-s'|^{1/2} + |t-t'|^{1/2}} \leq |s-s'| + |t-t'|$)

Define polygonal closed paths $\gamma_0, \ldots, \gamma_n$ by

$$k = 0, 1, \cdots, n: \quad \gamma_k(s) = H\left(\frac{i}{n}, \frac{k}{n}\right)(1 - (i - ns)) + H\left(\frac{i-1}{n}, \frac{k}{n}\right)(i - ns) \tag{2.15}$$

if $0 \leq i - ns \leq 1$ $\left(\text{i.e., } \frac{i-1}{n} \leq s \leq \frac{i}{n}\right)$ and $i = 1, \ldots, n$. By (2.14) and (2.15), for $k \in [n], s \in [0,1]$,

$$(*): \left|\gamma_k(s) - H\left(s, \frac{k}{n}\right)\right| \leq (ns + 1 - i)\underbrace{\left|H\left(\frac{i}{n}, \frac{k}{n}\right) - H\left(s, \frac{k}{n}\right)\right|}_{<\varepsilon} + (i - ns)\underbrace{\left|H\left(\frac{i-1}{n}, \frac{k}{n}\right) - H\left(s, \frac{k}{n}\right)\right|}_{<\varepsilon} < \varepsilon$$

(Note that $(ns + 1 - i) + (i - ns) = 1$ when using the triangle inequality)

In particular, taking $k = 0$ and $k = n$,

$$|\gamma_0(s) - \Gamma_0(s)| < \varepsilon, \quad |\gamma_n(s) - \Gamma_1(s)| < \varepsilon.$$

By $(*)$ and (2.13),

$$|\alpha - \gamma_k(s)| > 2\varepsilon - \varepsilon = \varepsilon \quad (k \in [n]; s \in [0,1]).$$

On the other hand, (2.14) and (2.15) also imply that for $k \in [n]; s \in [0,1]$,

$$|\gamma_{k-1}(s) - \gamma_k(s)| \le (ns+1-i)\underbrace{\left|H\left(\frac{i}{n},\frac{k-1}{n}\right) - H\left(\frac{i}{n},\frac{k}{n}\right)\right|}_{<\varepsilon} + (i-ns)\underbrace{\left|H\left(\frac{i-1}{n},\frac{k-1}{n}\right) - H\left(\frac{i-1}{n},\frac{k}{n}\right)\right|}_{<\varepsilon} < \varepsilon.$$

Now it follows from the last three inequalities, and $n+2$ applications of Lemma 2.3.5 that α has the same index with respect to each of the paths $\Gamma_0, \gamma_0, \gamma_1, \ldots, \gamma_n, \Gamma_1$. This proves the theorem. □

Remark 2.3.7.

1. If $\Gamma_t(s) = H(s,t)$ in the preceding proof, then each Γ_t is a closed *curve*, but not necessarily a *path*, since H is not assumed to be differentiable. The paths γ_k were introduced for this reason. Another (and perhaps more satisfactory) way to circumvent this difficulty is to extend the definition of index to closed curves.
2. As promised, this theorem of sufficiency of path deformation and global cauchy show that any integral of analytic function along γ in a simply-connected set U is zero. We shall see the converse is also true, i.e., this property can be used as definition of simply-connectedness (of a set in complex plane).

Theorem 2.3.8. Let $U \subset \mathbb{C}$ be open and connected. Then the following are equivalent.

(a) U is homeomorphic to $B(0,1)$ (i.e. there is a continuous bijection $\psi : U \to B(0,1)$ s.t. ψ^{-1} is also continuous).

(b) U is simply connected;

(c) $\int_\gamma f(z)dz = 0 \quad \forall f \in H(U)$ and every closed path $\gamma : [a,b] \to U$;

(d) Every $f \in H(U)$ has a primitive;

(e) If $f, 1/f \in H(U)$ (i.e., f is analytic and non-vanishing), then $f = e^g$ for some $g \in H(U)$ ("f has a holomorphic logarithm g in U");

(f) If f in a non-vanishing analytic function, then $f = \varphi^2$ for some $\varphi \in H(U)$ ("f has a holomorphic square root φ in U").

Proof.

$(a) \Rightarrow (b)$: that's basically because $B(0,1)$ is simply-connected and homeomorphism preserves null-homotopy (in fact the whole fundamental group). Let $\psi : U \to B(0,1)$ be the homeomorphism. For the closed curve γ in U, the map $H(s,t) = \psi^{-1}((1-t)0 + t\psi(\gamma(s))) = \psi^{-1}(t\psi(\gamma(s)))$ defines a homotopy between constant map $c_{\psi^{-1}(0)}$ and curve $\gamma(s)$.

$(b) \Rightarrow (c)$: consequence of 2.3.6 and 2.3.2.

$(c) \Rightarrow (d)$: The proof resembles that for Cauchy's theorem in a convex set ([3] Theorem 6.10). Fix $a \in U$ arbitrary, and set

$$g(z) := \int_{\gamma_z} f(w)\mathrm{d}w$$

where γ_z is any path (i.e., piecewise C^1 curve) connecting a to z inside of U. Note: the fact that we can always select a piecewise C^1 curve is not guaranteed by the definition of connectness (which only gives a continuous curve) - however, it is a well-known result from basic topology that in an open, connected set we can always connect two points with a finite union of line segments (a polygonal path).

Also note that g is well-defined: if $\tilde{\gamma}_z$ is some other path connecting a to z in U, then $\gamma_z - \tilde{\gamma}_z$ is a closed path in U and by the given assumption (c),

$$\int_{\gamma_z - \tilde{\gamma}_z} f(w)\mathrm{d}w = 0 \Rightarrow \int_{\gamma_z} f(w)\mathrm{d}w = \int_{\tilde{\gamma}_z} f(w)\mathrm{d}w$$

Choose now $r > 0$ such that $B(z,r) \subset U$. Consider $h \in \mathbb{C}$ with $|h| < r$ so that $z + h \in B(z,r)$. Let $\eta = [z, z+h]$ be the line segment path connecting z to $z+h$ inside of $B(z,r) \subset U$. Then $\gamma + \eta$ is a path connecting a to $z+h$, and due to invariance of path in the definition of $g(z+h)$ as we just showed, we have

$$g(z+h) = \int_{\gamma+\eta} f(w)\mathrm{d}w = \int_{\gamma} f(w)\mathrm{d}w + \int_{\eta} f(w)\mathrm{d}w$$

and so

$$g(z+h) - g(z) = \int_{\eta} f(w)\mathrm{d}w.$$

Since

$$\frac{1}{h}\int_{\eta} f(z)\mathrm{d}w = \frac{1}{(z+h)-z}\int_{[z,z+h]} f(z)\mathrm{d}w = f(z),$$

and $\mathrm{length}(\eta) = |z+h-z| = |h|$, we apply [3] Corollary 5.31 to see

$$\left|\frac{g(z+h)-g(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{\eta} f(w) - f(z)\mathrm{d}w\right| \le \|f - f(z)\|_{L^\infty([z,z+h])} \le \varepsilon$$

where the last inequality holds for sufficiently small $|h|$ due to continuity of f. Thus,

$$g'(z) = \lim_{h\to 0}\frac{g(z+h)-g(z)}{h} = f(z)$$

$(d) \Rightarrow (e)$: The identity $f = e^g$, if it were to hold for some $g \in H(U)$, implies $f'(z) = g'(z)e^{g(z)} = g'(z)f(z)$ so that $g'(z) = f'(z)/f(z)$. So we want a primitive of the analytic function f'/f (recall f has no zeros in U by assumption). Let $z_0 \in U$ be a fixed point and c_0 is a complex number with $e^{c_0} = f(z_0)$ (recall e^z obtains all values except 0 and $f(z_0) \neq 0$). Since $f' \in H(U)$ and $1/f \in H(U)$, assumption (d) gives a primitive g of f'/f. We can assume $g(z_0) = c_0$ (otherwise take $\tilde{g} = g + c_0 - g(z_0)$).

We claim that g in turn satisfies $f = e^g$. Motivated by the midterm Q2, we study $G(z) := e^{g(z)}/f(z)$. Now, $g'(z) = f'(z)/f(z)$ gives

$$G'(z) = g'(z)e^{g(z)}/f(z) - e^{g(z)}f(z)^{-2}f'(z) = e^{g(z)}f(z)^{-2}f'(z) - e^{g(z)}f(z)^{-2}f'(z) = 0.$$

This implies, as U is connected, that $G(z) = e^{g(z)}/f(z) = C$ for some constant C, and so $e^{g(z)} = Cf(z)$ in U. Now, as $e^{g(z_0)} = e^{c_0} = f(z_0)$ we must have $C = 1$, and so we are done.

$(e) \Rightarrow (f)$: Use (e) to write $f = e^g$ with $g \in H(U)$. Define $\varphi = e^{g/2}$. Then $\varphi^2 = e^g = f$.

$(f) \Rightarrow (a)$: If $U = \mathbb{C}$, the homeomorphism is just directly given (without using (f)) by $z \mapsto \frac{z}{1+|z|}$. If the open connected set U is not the whole $\mathbb{C}$, there actually exists a holomorphic homeomorphism $U \to B(0,1)$ (a conformal mapping). This is the Riemann Mapping Theorem. This implication is thus proved as soon as we later prove Riemann mapping theorem (using only (f) and nothing else about simply connected domains).

$\square$

2.4 Residue Theorem

Definition 2.4.1. A function f is said to be **meromorphic** in an open set U if there is a set $A \subset U$ such that

(a) A has no limit point in U,

(b) $f \in H(U - A)$,

(c) f has a pole at each point of A.

Note that the possibility $A = \varnothing$ is not excluded. Thus every $f \in H(U)$ is meromorphic in U.

Note also that (a) implies that no compact subset of U contains infinitely many points of A, and that A is therefore at most countable.

If f and A are as above, if $a \in A$, and if

$$Q(z) = \sum_{k=1}^{m} c_k (z-a)^{-k}$$

is the principal part of f at a, as defined in Theorem 2.2.5 (i.e., if $f - Q$ has a removable singularity at a), then the number c_1 is called the residue of f at a :

$$c_1 = \operatorname{Res}(f; a).$$

Theorem 2.4.2. [The Residue Theorem] Suppose f is a meromorphic function in U. Let A be the set of points in U at which f has poles. If Γ is a cycle in $U - A$ such that

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \quad \text{for all} \quad \alpha \notin U$$

then

$$\frac{1}{2\pi i} \int_\Gamma f(z) dz = \sum_{a \in A} \operatorname{Res}(f; a) \operatorname{Ind}_\Gamma(a)$$

Proof. We will argue the sum on the RHS, though formally infinite, is actually finite. Let

$$B := \{a \in A : n_\Gamma(a) \neq 0\}.$$

Let $\Omega := \mathbb{C} \backslash \Gamma^*$. Denote the components of Ω by $\mathcal{F} = \cup V$. Each V is connected. The components are disjoint. Every connected set $E \subset \Omega$ is containted in exactly one $V \in \mathcal{F}$. Choose a disk D s.t. $\Gamma^* \subset D$ (possible as Γ^* is compact) and notice $\mathbb{C} \setminus D \subset \Omega$. So there exists a unique $V_0 \in \mathcal{F}$ s.t. $\mathbb{C} \setminus D \subset V_0$ (as $\mathbb{C} \setminus D \subset \Omega$ and $\mathbb{C} \setminus D$ is connected). So $\forall V \in \mathcal{F}^* = \mathcal{F} - \{V_0\}$, we have $V \subset \mathbb{C} \setminus V_0 \subset D$. Notice that $n_\Gamma(z) = 0$ in V_0 as $V_0 \subset \Omega$ unbounded, connected. Thus, $B \subset \bigcup_{V \in \mathcal{F}^*} V \subset D$ is bounded.

If $|B|$ is not finite, we can choose $a_1, a_2, \ldots \in B$ s.t, $a_i \neq a_j$ $(i \neq j)$. As $\bar{B}$ is compact, $\exists$ subseq. $a_{i_k} \to a \in \bar{B}$. Now $a \in A^{acc}$ clearly, but also $a \in U$. Indeed, if $a \notin U$ then $n_\Gamma(a) = 0$ bs assumption. But as n_Γ is continuous and $\mathbb{Z}$-valued, this forces $n_\Gamma(a_{i_k}) = 0$ for k large, contradicting to $a_{ik} \in B$. So $a \in U \cap A^{acc}$, contradicting to our assumption that $U \cap A^{acc} = \varnothing$. Thus $|B|$ is finite and

$$\sum_{a \in A} \operatorname{Res}(f; a) n_\Gamma(a) = \sum_{a \in B} \operatorname{Res}(f; a) n_\Gamma(a) < \infty$$

Write $B = \{a_1, \cdots, a_n\}$. Let $Q_1, \cdots, Q_n$ be the principal parts of f at $a_1, \cdots, a_n$. Set

$$g := f - \sum_{i=1}^{n} Q_i.$$

Put $U_0 := U\backslash(A\backslash B)$. As A has no accumulation point in U, this means $\forall z \in U_0$, $\exists r$ s.t. $B(z,r) \in U$ contains no other points of A than possibly z. Thus $B(z,r) \subset U_0$ and U_0 is open.

If $z \in U_0 \cap A$, then $z \in B = \{a_1, \ldots, a_n\}$, and g has a removable singularity at z. If $z \in U_0\backslash A, g$ is obvious complex differentiable, so $g \in H\left(U_0\right)$. We apply the Global Cauchy to $g \in H(U_0)$:

$$\int_\Gamma g(z)\,dz = 0$$

as Γ is a cycle in U_0 with the property that

$$n_\Gamma(z) = 0 \quad \forall z \in \mathbb{C} \setminus U_0$$

Indeed, if $z \in \mathbb{C} \setminus U_0$, then either $z \in \mathbb{C} \setminus U$ and $n_\Gamma(z) = 0$ by the assumption of the theorem, or $z \in A \setminus B$ where also $n_\Gamma(z) = 0$ by the definition of B. Hence,

$$0 = \int_\Gamma f(z)dz - \sum_{i=1}^{n} \int_\Gamma Q_i(z)dz = \int_\Gamma f(z)dz - \sum_{i=1}^{n} \operatorname{Res}\left(f; a_i\right) 2\pi i\, n_\Gamma\left(a_i\right)$$

yielding

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{i=1}^{n} \operatorname{Res}\left(f; a_i\right) n_\Gamma\left(a_i\right) = \sum_{a\in A} \operatorname{Res}(f; a) n_\Gamma(a)$$

as desired. □

We will use L'Hôpital's rule in the following example:

Proposition 2.4.3. [L'Hôpital's rule] Let $U \subset \mathbb{C}$ be open, let $z \in \mathbb{C}$, and let $f, g : U \to \mathbb{C}$ be complex differentiable at z, with $g'(z) \neq 0$. Assume moreover that $f(z) = 0 = g(z)$. Then

$$\lim_{\substack{w\to z\\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(w)}{g(w)} = \frac{f'(z)}{g'(z)}$$

Proof. By [3] Proposition 4.7, we may write

$$\lim_{\substack{w\to z\\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(w)}{g(w)} = \lim_{\substack{w\to z\\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(z) + \left(f'(z) + \varepsilon_f(w)\right)(w-z)}{g(z) + \left(g'(z) + \varepsilon_g(w)\right)(w-z)},$$

where $\varepsilon_f(w) \to 0$ and $\varepsilon_g(w) \to 0$ as $w \to z$. Moreover, recalling that $f(z) = 0 = g(z)$, the above limit simplifies to

$$\lim_{\substack{w\to z\\ w\in\mathbb{C}\backslash\{z\}}} \frac{f(w)}{g(w)} = \lim_{\substack{w\to z\\ w\in\mathbb{C}\backslash\{z\}}} \frac{f'(z) + \varepsilon_f(w)}{g'(z) + \varepsilon_g(w)} = \frac{f'(z)}{g'(z)},$$

as claimed. Notice that here diving with $g'(z)$ makes sense as $g'(z) \neq 0$. Moreover, notice that by (b) this implies that in a small neighborhood of z we have $g(w) \neq g(z) = 0$ for $w \neq z$, so also the expression $f(w)/g(w)$ above makes sense for $w \neq z$ close to z. □

Example 2.4.4. We calculate the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Solution. We want to have a path γ_R such that above equals to

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz$$

where $f(z) = 1/(1+z^4)$. We let $\gamma_R = [-R, R] \star \sigma_R$, $\sigma_R(t) = Re^{it}$, $t \in [0, \pi]$. It works simply becasue of the usual estimate

$$\left| \int_{\sigma_R} f(z)\, dz \right| \leq \frac{CR}{R^4} = \frac{C}{R^3} \xrightarrow{R\to\infty} 0.$$

Now $1 + z^4 = 0$ has solutions $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$, all with order 1. Only poles $z_1 = e^{i\pi/4}$ and $z_2 = e^{i3\pi/4}$ are inside γ_R. By the residue theorem and formula 2.2.8,

$$\begin{aligned} int_{\gamma_R} f(z)\, dz &= 2\pi i(\mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2)) \\ &= 2\pi i \left(\lim_{z\to z_1} (z - z_1) f(z) + \lim_{z \to z_2} (z - z_2) f(z) \right) \end{aligned}$$

Instead of writing $z^4 + 1 = (z - z_1) \cdots (z - z_4)$, it is convenient to use L'Hôpital's rule instead:

$$\lim_{z\to z_1} (z - z_1) f(z) = \lim_{z \to z_1} \frac{z - z_1}{1 + z^4} \overset{\text{L'Hôpital}}{=\!=\!=} \lim_{z\to z_1} \frac{1}{4z^3} = \frac{1}{4(e^{i\pi/4})^3} = \frac{1}{4} e^{-i3\pi/4}.$$

Similarly,

$$\lim_{z \to z_2} (z - z_2) f(z) = \frac{1}{4} e^{-i\pi/4}.$$

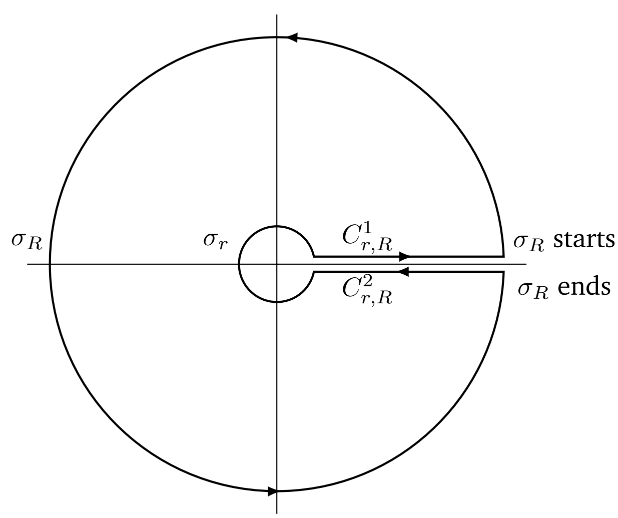
Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{2\pi i}{4} \left(e^{-i3\pi/4} + e^{-i\pi/4} \right) = \frac{\pi i}{2} \left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) = -\frac{\pi i}{2} \frac{2i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Example 2.4.5. Calculate the integral

$$I := \int_0^\infty \frac{x^{1/3}}{1+x^2} dx.$$

Solution. We need a keyhole contour line $\gamma_{r,R} := C^1_{r,R} \star \sigma_R \star C^2_{r,R} \star \sigma_r$



We define a branch of argument $\widetilde{\mathrm{Arg}} z \in (0, 2\pi)$ on $\mathbb{C} \setminus [0, \infty)$ in the natural way, i.e., $\widetilde{\mathrm{Arg}}_v(z)$ defined in [3] Theorem 2.57 with $v = 1$ here. Then we define

$$g(z) = |z|^{1/3} e^{i\widetilde{\mathrm{Arg}} z/3}$$

and define

$$f(z) = \frac{g(z)}{1+z^2}, \quad z \in \mathbb{C} \setminus [0, \infty), z \neq \pm i.$$

we first calculate $\int_\gamma f(z)dz$ using residues.

Clearly, g is analytic in a neighborhood (fattening) of the area enclosed by $\gamma_{r,R}$, and $n_{\gamma_{r,R}}(z) = 0$ for all z outside of this fattening (as z will be in the unbounded component of $\mathbb{C} \setminus \gamma_{r,R}^*$). Residue theorem gives

$$\int_{\gamma_{r,R}} f(z)\, dz = 2\pi i(\mathrm{Res}(f;i) + \mathrm{Res}(f;-i))$$

where it is easy to see by an argument of homotopy that $n_{\gamma_{r,R}}(i) = n_{\gamma_{r,R}}(-i) = 1$. Now,

$$\begin{aligned}
\mathrm{Res}(f;i) &= \lim_{z\to i} \frac{g(z)}{z+i} = \frac{e^{i\frac{\pi}{6}}}{2i} = \frac{1}{2}e^{i\left(\frac{\pi}{6}-\frac{\pi}{2}\right)} = \frac{e^{-i\frac{\pi}{3}}}{2} \\
\mathrm{Res}(f;-i) &= \lim_{2\to -i} \frac{g(z)}{z-i} = \frac{e^{i\frac{\pi}{2}}}{-2i} = -\frac{1}{2}
\end{aligned}$$

Then,

$$\begin{aligned}
\int_{\gamma_{r,R}} f(z)\, dz &= 2\pi i \cdot \frac{1}{2}\left(e^{-i\frac{\pi}{3}} - 1\right) \\
&= \pi i \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i - 1\right) \\
&= \pi i \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \\
&= -\pi i e^{i\frac{\pi}{3}}
\end{aligned}$$

Now, we relate this complex integral to the original real integral

$$\begin{aligned}
&\left|\int_{\sigma_R} f(z)dz\right| \leqslant C\frac{R^{1/3}R}{R^2} = CR^{-2/3} \to 0, \quad R\to\infty \\
&\left|\int_{\sigma_r} f(z)dz\right| \leqslant Cr \to 0, \quad r \to 0 \\
&\int_{c^1_{r,R}} f(z)dz \to I \text{ as } r\to 0, \quad R\to\infty \\
&\int_{c^2_{r,R}} f(z)dz \to -e^{i\frac{2\pi}{3}} I, \quad r\to 0, R\to\infty
\end{aligned}$$

(Since $\widetilde{\mathrm{Arg}} z \to 2\pi$ here, and we travel with opposite direction.)

Then

$$\left(1 - e^{i\frac{2\pi}{3}}\right) I = \lim_{\substack{r\to 0\\ R\to\infty}} \int_{\gamma_{r,R}} f(z)dz = -\pi i e^{i\frac{\pi}{3}}$$

Thus,

$$\begin{aligned}
I &= \frac{-\pi i e^{i\frac{\pi}{3}}}{1 - e^{i\frac{i\pi}{3}}} = \frac{\pi i}{e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}} \\
&= \frac{\pi i}{2i\sin\frac{\pi}{3}} = \frac{\pi}{2}\frac{1}{\sin\frac{\pi}{3}} \\
&= \frac{\pi}{2}\frac{2}{\sqrt{3}} = \frac{\pi}{\sqrt{3}}.
\end{aligned}$$

We conclude this chapter with Rouché's theorem.

Theorem 2.4.6. Suppose γ is a closed path in an open connected set U, such that $\operatorname{Ind}_\gamma(\alpha) = 0$ for every α not in U. Suppose also that $\operatorname{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in U - \gamma^*$. Let $U_1 = \{z \in \mathbb{C} \setminus \gamma^* : \operatorname{Ind}_\gamma(\alpha) = 1\} \subset U$. For any $f \in H(U)$ let N_f be the number of zeros of f in U_1, counted according to their multiplicities.

(a) If $f \in H(U)$ and f has no zeros on γ^* then

$$N_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \operatorname{Ind}_\Gamma(0) \tag{2.16}$$

where $\Gamma = f \circ \gamma$.

(b) (Rouché's theorem) If also $g \in H(U)$ and

$$|f(z) - g(z)| < |f(z)| \quad \text{for all } z \in \gamma^* \tag{2.17}$$

then $N_g = N_f$.

Part (b) is usually called Rouché's theorem. It says that two holomorphic functions have the same number of zeros in U_1 if they are close together on the boundary of U_1, as specified by (2.17).

Proof. Put $\varphi = f'/f$. Due to Theorem 2.1.1, if $a \in U$ and f has a zero of order $m = m(a)$ at a, then $f(z) = (z-a)^m h(z)$, where h and $1/h$ are holomorphic in some neighborhood V of a. In $V - \{a\}$,

$$\varphi(z) = \frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1}h(z) + (z-a)^m h'(z)}{(z-a)^m h(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)}.$$

The first term $\frac{m}{z-a}$ corresponds to the $c_{-1}(z-a)^{-1}$ term of $\varphi(z)$, and $\frac{h'(z)}{h(z)} \in H(V)$ corresponds to $\sum_{n=0}^{\infty} c_n(z-a)^n$ for the remaining terms of $\varphi(z)$. Thus φ is a meromorphic function in U (the set A in the definition of meromorphic function is Z_f which by Theorem 2.1.1 has not limit point in U and each $a \in A$ is a first order pole of φ). Besides,

$$\operatorname{Res}(\varphi; a) = m(a)$$

Let $A = \{a \in U_1 : f(a) = 0\} = U_1 \cap Z_f$. Now apply Residue theorem to meromorphic function φ and γ, a cycle in $U \setminus Z_f$ with $n_\gamma(z) = 0$, $\forall z \in \mathbb{C} \setminus U$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{a \in Z_f} \operatorname{Res}(\varphi; a)\operatorname{Ind}_\gamma(a) = \sum_{a \in U_1 \cap Z_f} \operatorname{Res}(\varphi; a) = \sum_{a \in A} m(a) = N_f.$$

This proves one half of (2.16). The other half is a matter of direct computation: supposing $\gamma : [a, b] \to \mathbb{C}$,

$$\begin{aligned}\operatorname{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_a^b \frac{\Gamma'(s)}{\Gamma(s)}ds \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)ds = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.\end{aligned}$$

Next, (2.17) gives $|g(z)| \geq |f(z)| - |f(z) - g(z)| > 0$, $\forall z \in \gamma^*$, so g has no zero on γ^*. Hence (2.16) holds with g in place of f. Put $\Gamma_0 = g \circ \gamma$. In order to apply Lemma 2.3.5 to Γ_0 and $\Gamma_1 = f \circ \gamma$ to get $\operatorname{Ind}_{\Gamma_0}(0) = \operatorname{Ind}_{\Gamma_1}(0)$, we need verify that

$$|\Gamma_0(s) - \Gamma_1(s)| = |g(\gamma(s)) - f(\gamma(s))| < |f(\gamma(s))| = |\underbrace{0}_{\alpha \text{ in the lemma}} - \Gamma_1(s)|$$

where the inequality is due to (2.17). Then it follows from (2.16) that

$$N_g = \operatorname{Ind}_{\Gamma_0}(0) = \operatorname{Ind}_{\Gamma_1}(0) = N_f$$

□

Remark 2.4.7. For a set of more complete results, see [1] p.123 Argument Principle

Example 2.4.8. How many roots does the polynomial $z \mapsto z^5 + 3z^2 + 1$ have in the annulus $A = \{1 < |z| < 2\}$?

Solution. The strategy is to let g be the original function $g(z) = z^5 + 3z^2 + 1$ and choose f to be the dominant part on the circle $|z| = 1$.

Define $f_1(z) = 3z^2$. Then, on the circle $|z| = 1$, we have

$$|f_1(z) - g(z)| = |z^5 + 1| < |z|^5 + 1 = 2 < 3 = 3|z|^2 = |f_1(z)|$$

By Rouché's theorem, we have

$$|\{a \in B(0,1) : g(a) = 0\}| = |\{a \in B(0,1) : f_1(a) = 0\}| = 2$$

since $z \mapsto 3z^2$ has a zero of order 2 at the origin.

Define $f_2(z) = z^5$. For $|z| = 2$ we have

$$|f_2(z) - g(z)| = |3z^2 + 1| \leq 3 \cdot 2^2 + 1 = 13 < 32 = 2^5 = |z|^5 = |f_2(z)|$$

By Rouché's theorem, we have

$$|\{a \in B(0,2) : g(a) = 0\}| = |\{a \in B(0,2) : f_2(a) = 0\}| = 5$$

since $z \mapsto z^5$ has a zero of order 5 at the origin.

Therefore, there are $5 - 2 = 3$ zeros in annulus A.

Example 2.4.9. Use Rouche's theorem to prove that all the zeros of the polynomial

$$z^n + c_{n-1} z^{n-1} + \cdots + c_0$$

lie in the open ball with center 0 and radius

$$\sqrt{1 + |c_{n-1}|^2 + \cdots + |c_0|^2}.$$

Solution. Let

$$P(z) := z^n + c_{n-1} z^{n-1} + \cdots + c_0$$

and

$$R := \sqrt{1 + |c_{n-1}|^2 + \cdots + |c_0|^2}.$$

If all of the coefficients c_i are zero, then $R = 1$, and the result is trivial as the only zero of $P(z) = z^n$ is at $z = 0 \in B(0,1)$. So we may assume $R > 1$. We will apply Rouche, and the dominating term we choose is $f(z) := z^n$. In the notation of Rouche, let $g(z) := P(z)$. For $|z| = R$ by Cauchy-Schwarz (i.e. $|w_1 \cdot w_2| \leq |w_1| |w_2|, w_1, w_2 \in \mathbb{R}^d$, where $w_1 \cdot w_2$ is the dot product in $\mathbb{R}^d$) we have

$$\begin{aligned}
|f(z) - g(z)| &= \left|c_{n-1} z^{n-1} + \cdots + c_0\right| \\
&\leq \sum_{k=0}^{n-1} |c_k| R^k \\
&\leq \left(\sum_{k=0}^{n-1} |c_k|^2\right)^{1/2} \left(\sum_{k=0}^{n-1} R^{2k}\right)^{1/2} = \left(R^2 - 1\right)^{1/2} \left(\sum_{k=0}^{n-1} R^{2k}\right)^{1/2}.
\end{aligned}$$

In the last identity we used the definition of R. By the formula for a geometric sum

$$\sum_{k=0}^{n-1} t^k = \frac{1-t^n}{1-t}, \quad t \neq 1,$$

we have (as $R^2 > 1$) that

$$\sum_{k=0}^{n-1} R^{2k} = \frac{1-R^{2n}}{1-R^2} = \frac{R^{2n}-1}{R^2-1}.$$

Thus, we obtain

$$|f(z)-g(z)| \leq \left(R^{2n}-1\right)^{1/2} < R^n = |f(z)|^n$$

on $|z| = R$. By Rouche f and $g = P$ have the same number of zeros inside $B(0, R)$, which is clearly n for $f(z) = z^n$ (a zero of multiplicity n at $z = 0$). But as P is a polynomial of order n, it has exactly n roots, and so all of its roots are in $B(0, R)$. We are done.

Example 2.4.10. We use Rouché's theorem to prove the open mapping theorem:
Suppose $f \in H(U)$ is non-constant and $U \subset \mathbb{C}$ is open and connected. Then f is open, that is, it maps open sets to open sets.

Proof. Let $V \subset U$ be open. Consider $w_0 \in fV$, that is, $w_0 = f(z_0)$ for $z_0 \in V$. As $f - w_0$ is an analytic function that is not identically zero (as f is non-constant) in the open, connected U, we know that its zero set in U, that is, the set $\{z \in U : f(z) = w_0\}$, does not have accumulation points in U. Using this we choose $r > 0$ so that $\bar{B}(z_0, r) \subset V$ and $f(z) \neq w_0$ for $z \in \bar{B}(z_0, r) \setminus \{z_0\}$, in particular, if $|z - z_0| = r$. As the continuous function $|f - w_0|$ attains its minimum on the compact set $|z - z_0| = r$ we find $\epsilon > 0$ so that $|f(z) - w_0| \geq \epsilon$ for all z with $|z - z_0| = r$.

Consider w with $|w - w_0| < \epsilon$. We show that then $w \in fV$, showing that fV is open as desired. We do this by showing that $f - w$ has a zero in $B(z_0, r) \subset V$. Indeed, we have

$$|f(z) - w - (f(z) - w_0)| = |w - w_0| < \epsilon \leq |f(z) - w_0|$$

on the circle $|z - z_0| = r$. By Rouche's theorem $f - w$ and $f - w_0$ have the same number of zeros in $B(z_0, r)$, and we know that $f - w_0$ has exactly one zero in that ball, namely z_0. So, indeed, $f - w$ has a zero in $B(z_0, r)$ and we are done. $\square$

Chapter 3

Space of Analytic Functions

3.1 Riemann Sphere

Let the symbol ∞ be called the point at infinity and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the extended complex plane and is put with the topology of one-point compactification. We for $r > 0$ define

$$B'(\infty, r) = \{z \in \mathbb{C} : |z| > 1/r\}$$
$$B(\infty, r) = B'(\infty, r) \cup \{\infty\}$$

The topology is then defined as this: we declare $U \subseteq \overline{\mathbb{C}}$ to be open iff U is a union of some $B(a, r)$ with $a \in \overline{\mathbb{C}}$ and $r > 0$ on $\overline{\mathbb{C}} \setminus \{\infty\}$. This gives the usual topology on $\mathbb{C}$. We will show that $\overline{\mathbb{C}}$ is homeomorphic to the unit sphere $\mathbb{S}^2$, so $\overline{\mathbb{C}}$ is also called **Riemann sphere**. First, think of $\mathbb{C}$ as the embedded set $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\} = \{x \in \mathbb{R}^3 : x_3 = 0\} \subseteq \mathbb{R}^3$. Let S be $\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - 1)^2 = 1\} = \partial B((0, 0, 1), 1)$. Then $\mathbb{S}^2 \approx \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where the homeomorphism is **stereographic projection**: in Cartesian coordinates (x, y, z) on the sphere and (X, Y) on the plane, the projection and its inverse are given by the formulas

$$(X, Y) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right)$$
$$(x, y, z) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2}\right)$$

The name "projection" comes from this: if one picks a point on the sphere and draw a line passing through the point and the North pole, then the line will intersect with a point on the plane. When the point one picks is the North pole then the line intersects with the extended plane $\overline{\mathbb{C}}$ at the infinity point ∞.

Behavior of functions at ∞: if f is holomorphic in $B'(\infty, r)$, we say f has an **isolated singularity at** ∞. The type of this singularity (removable/pole/essential) is *by definition* the same as that of

$$z \mapsto f(1/z) \quad z \in B'(0, 1/r)$$

at $z = 0$. In particular, if f is bounded in $B'(\infty, r)$, then $\exists \lim_{z\to\infty} f(z) \in \mathbb{C}$ and setting $f(\infty) = \lim_{z\to\infty} f(z)$ gives a function defined on $B(\infty, r)$, which we call holomorphic (apply known result to $z \mapsto f(1/z)$). Similarly for poles and essential singularities: f has a pole of order m at ∞ if $z \mapsto f(1/z)$ has a pole of order m at 0.

3.2 Conformal Mappings

Definition 3.2.1. Let $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}$ open be a holomorphic function. We say f is **conformal at** $z \in U$ if $f'(z) = 0$, we say f is **conformal in** U if $f'(z) = 0\ \forall z \in U$.

We give a geometric interpretation of conformality. Let γ_0 and γ_1 be two paths that go through $z_0 = \gamma_0(t_0) = \gamma_1(t_1)$ with $\gamma_0'(t_0) \neq 0$ and $\gamma_1'(t_1) \neq 0$. Suppose

$$\gamma_0'(t_0) = r_0 e^{i\varphi_0} \gamma_1'(t_0) = r_1 e^{i\varphi_1}$$

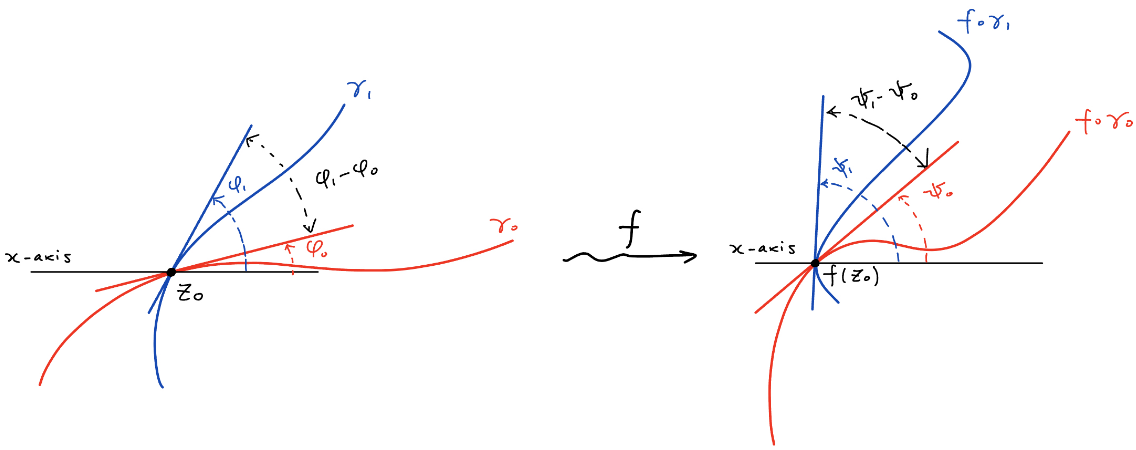


Figure 3.1: Conformal mapping preserves the angle.

Let f be conformal, then by [3] Lemma 5.44 we see

$$(f \circ \gamma_0)'(t_0) = f'(z_0)\gamma_0'(t_0) = \rho e^{i\omega} r_0 e^{i\varphi_0} = \rho r_0 e^{i(\omega+\varphi_0)} = \rho r_0 e^{i\psi_0}$$
$$(f \circ \gamma_1)'(t_1) = f'(z_0)\gamma_1'(t_1) = \rho e^{i\omega} r_1 e^{i\varphi_1} = \rho r_1 e^{i(\omega+\varphi_1)} = \rho r_1 e^{i\psi_1}$$

Then we observe that multiplication by $f'(z_0) = \rho e^{i\omega}$ stretches and rotates but preserves the original angle between $\gamma_0'(t_0)$ and $\gamma_1'(t_1)$, that is $\varphi_1 - \varphi_0 = \psi_1 - \psi_0$. In particular, for any rays L', L'' starting at z_0, the angle between the images fL', fL'' at $f(z_0)$ is the same as that made by L' and L''.

The **conformal mapping problem** between two regions $U, V \subseteq \mathbb{C}$ is the problem on the *existence* and *explicit construction* of a conformal bijection $f : U \to V$. Riemann Mapping Theorem will tell us about the *existence* and will be proved soon. First, however, we look at the relationship between $f' \neq 0$ and injectivity and then at the *explicit constructions* of cnformal bijections between geometrically simple regions (balls, sectors, half-planes, $\cdots$).

3.2.1 Relationship between nonzero derivative and injectivity

We will first show that an analytic injection automatically has nonzero derivative. Thus, looking for a conformal bijection $f : U \to V$ between regions U and V is the same as finding an analytic bijection $f : U \to V$. We also see that $f^{-1} : V \to U$ is automatically analytic too (but the converse of "nonzero derivative $\implies$ injectivity" is not true: $f(z) = e^z$ satisfies $f'(z) = e^z \neq 0$ for any $z \in \mathbb{C}$, but f is not injective in the whole $\mathbb{C}$). Nonetheless, $f'(z) \neq 0$ implies local injectivity.

Theorem 3.2.2. Suppose $U \subseteq \mathbb{C}$ is open. $z_0 \in U$, and $f \in H(U)$, $f'(z_0) \neq 0$. Then there is a neighborhood $V \subseteq U$ of z_0 such that

(a) f is injective in V. (local injectivity)

(b) $W = f(V)$ is open. (Open Mapping Theorem (as a consequence))

(c) $f^{-1} : W \to V$ is analytic. (holomorphic inverse)

Proof. Recall from the Lemma 2.3.1 where we built a function g continuous on $U \times U$,

$$g(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

So there is a neighborhood V of z_0 such that for $z_1, z_2 \in V$,

$$g(z_1, z_2) - \underbrace{g(z_0, z_0)}_{f'(z_0)} < \frac{1}{2}|f'(z_0)|$$

Thus, for $z_1, z_2 \in V$,

$$\left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \geq |f'(z_0)| - \left|\underbrace{\frac{f(z_1) - f(z_2)}{z_1 - z_2}}_{g(z_1,z_2)} - \underbrace{f'(z_0)}_{g(z_0,z_0)}\right| \geq \frac{1}{2}|f'(z_0)|$$

so

$$(*): \quad |f(z_1) - f(z_2)| \geq \frac{1}{2}|f'(z_0)||z_1 - z_2| \quad \forall z_1, z_2 \in V$$

In particular, $z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2)$, so (a) holds. We prove (b) in a manner similar to Example 2.4.10. We note that $(*)$ implies $f'(z) \neq 0$ for any $z \in V$. Pick an arbitrary $w_0 = f(z) \in fV$, $a \in V$. Then pick $v > 0$ such that $B(a, 2r) \subset V$ (V open). Then $(*)$ gives that

$$|f(z) - \underbrace{w_0}_{f(a)}| \geq \frac{1}{2}|f'(z_0)|r =: \varepsilon \neq 0 \quad \forall z \in \partial B(a, r)$$

We claim that $B(w_0, \varepsilon) \subset fV \Rightarrow fV$ is open. Indeed, for any $w \in B(w_0, \varepsilon)$, we have

$$|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < \varepsilon \leq |f(z) - w_0| \quad \forall z \in \partial B(a, r)$$

Then Rouché's theorem shows that $f - w$ and $f - w_0$ have the same number of zeros inside $B(a, r)$. As $f - w_0$ has one, there is some $b \in B(a, r)$ such that $w = f(b) \in f(\underbrace{B(a, r)}_{\subset V}) \subset fV$. So fV is open. (b) is proved.

For (c), we consider $f^{-1} : W \to V$ where $W := fV$. Let $w_1 \in W$ where $w_1 = f(z_1)$, $z_1 \in V$. For $w = f(z) \in w$ we notice that

$$\frac{f^{-1}(w) - f^{-1}(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{\frac{f(z)-f(z_1)}{z-z_1}}$$

Here,

- $(*) \Rightarrow |z - z_1| \leq \frac{2}{f'(z_0)|}|f(z) - f(z_1)| = \frac{2}{f'(z_0)}|w - w_1| \Rightarrow w \to w_1$ implies $z \to z_1$.
- $f'(z_1) \neq 0$ as $z_1 \in V$. So

$$(f^{-1})'(w_1) = \lim_{w \to w_1} \frac{f'(w) - f'(w_1)}{w - w_1} = \frac{1}{f'(z_1)} = \frac{1}{f'(f^{-1}(w_1))}$$

□

After showing the following shaper form of Open Mapping Theorem, we will prove the converse that injectivity implies $f' \neq 0$.

Theorem 3.2.3. Suppose U is a region, $f \in H(U), f$ is not constant, $z_0 \in U$, and $w_0 = f(z_0)$. Let m be the order of the zero which the function $f - w_0$ has at z_0. Then there exists a neighborhood V of z_0, $V \subset U$, and there exists $\varphi \in H(V)$, such that

(a) $f(z) = w_0 + [\varphi(z)]^m$ for all $z \in V$,

(b) φ' has no zero in V and φ is an invertible mapping of V onto a disc $B(0, r)$.

Proof. Let $V_0 = B(z_0, r_0) \subset U$ such that $f(z) \neq w_0$ for any $z \in V_0 \setminus \{z_0\}$ (we can do this because zeros of $f - w_0$ are isolated). We write

$$f(z) - w_0 = (z - z_0)^m g(z)$$

for $g \in H(V_0)$, g non-vanishing. Since $V_0 = B(z_0, r_0)$ is obviously homeomorphic to $B(0, 1)$, we by Theorem 2.3.8 see that g has a holomorphic lagarithm h in V_0, that is, we can write $g = e^h$ for some $h \in H(V_0)$. We define $\varphi(z) = (z - z_0)e^{h(z)/m}$, $z \in V_0$. Then

$$\begin{aligned} z \in V_0: \quad w_0 + [\varphi(z)]^m &= w_0 + (z - z_0)^m e^{h(z)} \\ &= w_0 + (z - z_0)^m g(z) \\ &= f(z) \end{aligned}$$

Thus, (a) holds.

Now $\varphi(z_0) = 0$, $\varphi'(z) = e^{h(z)/m} + (z - z_0)e^{h(z)/m} \cdot h'(z)/m$, so $\varphi'(z_0) = e^{h(z_0)/m} \neq 0$. By Theorem 3.2.2, φ is injective on some neighborhood V_1 of z_0 contained in V_0, $\varphi' \neq 0$ on V_1, and $0 = \varphi(z_0) \in \varphi(V_1)$, where $\varphi(V_1)$ is open. Thus, we choose $r > 0$ such that $B(0, r) \subset \varphi(V_1)$. Define $V = \varphi^{-1}(B(0, r)) \subset V_1 \subset V_0$. Then $\varphi : V \to B(0, r)$ in invertible. □

Corollary 3.2.4. Suppose U is a region, $f \in H(U)$, and f is injective on U. Then $f'(z) \neq 0$ for every $z \in U$, and the inverse f^{-1} is holomorphic.

Proof. If $f'(z_0) \neq 0$ for some $z_0 \in U$, write $f(z) = f(z_0) + [\varphi(z)]^m$ for $z \in V$, where m, φ, V are the same as in previous theorem. If $m = 1$, we would get $f'(z_0) = \varphi'(z)$ and thus $0 = f'(z_0) = \varphi'(z_0)$ (contradiction since $\varphi'(z_0) \neq 0$ as a result of previous theorem). So $m > 1$, but then f is not injective (since each $w \neq 0$ equals z^m for precisely m distinct z). Therefore, $f' \neq 0$. Analyticity of f^{-1} follows Theorem 3.2.2. □

3.2.2 Möbius Mappings

Definition 3.2.5. Rational functions of the form

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$, are called **Möbius mappings**, or **linear fractional transformations**.

It is convenient to regard f as a mapping from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ with

$$f\left(-\frac{d}{c}\right) := \infty$$

(notice that $ad - bc \neq 0$ guarantees that $z \mapsto az + b$ does not vanish at $z = -\frac{d}{c}$) and

$$f(\infty) = \begin{cases} \frac{a}{c}, & \text{if } c \neq 0 \\ \infty, & \text{if } c = 0 \end{cases}$$

Thus, f is meromorphic in $\overline{\mathbb{C}}$ with a pole at $z = -\frac{d}{c}$. Notice that f is analytic at ∞ if $c \neq 0$.

Lemma 3.2.6. A Möbius mapping is a bijection $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ whose inverse is a Möbius mapping.

Proof. One can directly verify that the inverse is given by the following Möbius mapping:

$$g(z) = \frac{dz - b}{-cz + a}$$

□

Lemma 3.2.7. The composition of Möbius mappings is Möbius. Möbius mappings are conformal in $\overline{\mathbb{C}}$. Here,

(1) When $f(\infty) \in \mathbb{C}$ (i.e., $f(\infty) \neq \infty$), we say f is **conformal at** ∞ if $z \mapsto f(1/z)$ (with $0 \mapsto f(\infty)$) is conformal at 0.

(2) When $f(\infty) = \infty$, we say f is **conformal at** ∞ if $z \mapsto \frac{1}{f(1/z)}$ (with $0 \mapsto 0$) is conformal at 0.

(3) When $f(z_0) = \infty$, we say f is **conformal at** z_0 if $z \mapsto \frac{1}{f(z)}$ (with $z_0 \mapsto 0$) is conformal at z_0.

Proof. See Math 5022 Homework 1. □

Our first explicit construction problem is to find a Möbius mapping that maps given three points as we want.

There is a particular invariance that all Möbius mappings have (they preserve the so-called cross ratio). Let

$$f(z) = \frac{az + b}{cz + d}$$

We choose first $z_2 \in \overline{\mathbb{C}}$. Now,

$$f(z) - f(z_2) = \frac{az + b}{cz + d} - \frac{az_2 + b}{cz_2 + d} = \frac{(ad - bc)(z - z_2)}{(cz + d)(cz_2 + d)}$$

Let then $z_3 \in \overline{\mathbb{C}} \backslash \{z_2\}$. Similarly,

$$f(z) - f(z_3) = \frac{(ad - bc)(z - z_3)}{(cz + d)(cz_3 + d)}$$

So,

$$(*): \quad \frac{f(z) - f(z_2)}{f(z) - f(z_3)} = \lambda \frac{z - z_2}{z - z_3}$$

with

$$\lambda = \lambda(z_2, z_3) = \frac{cz_3 + d}{cz_2 + d}$$

Notic that λ is independent of z. Apply $(*)$ twith $z = z_4$ to get

$$\frac{f(z_4) - f(z_2)}{f(z_4) - f(z_3)} = \lambda \frac{z_4 - z_2}{z_4 - z_3}.$$

Therefore,

$$\frac{(f(z) - f(z_2))(f(z_3) - f(z_4))}{(f(z) - f(z_3))(f(z_2) - f(z_4))} = \frac{(z - z_2)(z_3 - z_4)}{(z - z_3)(z_2 - z_4)}$$

This gives an invariance (preservation of cross ratio, defined below) for all Möbius mappings.

Definition 3.2.8. For complex numbers $a_1, \cdots, a_4$, where $a_i \neq a_j$ if $i \neq j$, their cross ratio is

$$(a_1, a_2, a_3, a_4) = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_3)(a_2 - a_4)}$$

(To help memorize: numerator have indices in order and in denominator the leftmost and rightmost indices are the same as those of the numerator).

We thus have proved that

Lemma 3.2.9. f Möbius, then $(f(z), f(z_2), f(z_3), f(z_4)) = (z, z_2, z_3, z_4)$.

Remark 3.2.10. If $a_i = \infty$ is in the cross ratio, we interpret it as a limit $a_i \to \infty$. For example,

$$(\infty, a_2, a_3, a_4) := \lim_{a_1 \to \infty} \frac{(1 - a_2/a_1)(a_3 - a_4)}{(1 - a_3/a_1)(a_2 - a_4)} = \frac{a_3 - a_4}{a_2 - a_4}$$

The cross ratio is a practical tool to find a Möbius mapping that maps distinct points z_2, z_3, z_4 to given w_2, w_3, w_4.

Example 3.2.11. Find the Möbius map that satisfies $f(0) = 1, f(1) = 2, f(2) = 3$. The answer is obvious, $f(z) = z + 1$, but let's still see how to obtain it vie the cross ratio. We want to find f such that

$$(z, 0, 1, 2) = (f(z), 1, 2, 3)$$

This the same as

$$\frac{-z}{(z-1)(-2)} = \frac{(f(z)-1)(-1)}{(f(z)-2)(-2)} \iff \frac{z}{2(z-1)} = \frac{f(z)-1}{2(f(z)-2)} \iff f(z) = z + 1$$

Our next explicit construction problem concerns finding the Möbius mapping that converts circles and lines.

Lemma 3.2.12. Let $a > 0$ and $w_1 \in \mathbb{C} \backslash \{0\}$, and define

$$F := \left\{ w \in \mathbb{C} : \frac{|w|}{|w - w_1|} = a \right\}.$$

Then F is a line if $a = 1$, and is a circle otherwise. Conversely, every circle not centered at 0 and not going through 0, or any line not going through 0, can be written in above form.

Proof. Exercise. $\square$

Corollary 3.2.13. Let $w_1, w_2 \in \mathbb{C}$, $w_1 \neq w_2$, and $a > 0$. Let

$$F := \left\{ w \in \mathbb{C} : \frac{|w - w_1|}{|w - w_2|} = a \right\}.$$

Then F is a circle if $a \neq 1$, and a line if $a = 1$. Conversely, every line and every circle is of this form.

Remark 3.2.14. In our language, lines L contain ∞.

Corollary 3.2.15. Every Möbius mapping f maps a circle to a circle or a line. The same holds for lines.

Proof. All circles and lines are of the form

$$F := \left\{ z \in \mathbb{C} : \frac{|z - z_1|}{|z - z_2|} = a \right\}$$

for some $z_1 \neq z_2$, $a > 0$. Recall that

$$(*): \quad \frac{f(z) - f(z_1)}{f(z) - f(z_2)} = \lambda \frac{z - z_1}{z - z_2}$$

for a constant λ depending on z_1, z_2. Thus,

$$f(F) = \left\{ f(z) : \frac{|z - z_1|}{|z - z_2|} = a \right\} \overset{(*)}{=\!=} \left\{ f(z) : \frac{|f(z) - f(z_1)|}{|f(z) - f(z_2)|} = a|\lambda| \right\} = \left\{ w : \frac{|w - f(z_1)|}{|w - f(z_2)|} = a|\lambda| \right\}.$$

is a line or a circle. $\square$

Now, every line L can be mapped with a Moöbius mapping into a given circle S. Simply choose $z_1, z_2, z_3 \in L$ (one can be ∞) and map them to some $w_1, w_2, w_3 \in S$. Then f^{-1} maps S to L, where f^{-1} is still Möbius. One finds f with the cross ratio.

Recall the following topological fact: let E be connected in topological space X and $A \subset X$. If E meets A, $X \backslash A$, then it must mee ∂A.

We ask how Möbius mapping f maps balls and planes. For instance, suppose it maps $|z| = 1$ to the real axis. Then $f(B(0,1))$ is connected, and by above topological fact lies completely in $H_1 = \{\mathrm{Im}(z) > 0\}$ or $H_2 = \{\mathrm{Im}(z) < 0\}$. Same is true for $f(\mathbb{C}\backslash\overline{B}(0,1))$. Thus, as f is bijective, $f(B(0,1))$ is either H or H_2 (to figure out which of them, just check where $f(0)$ is mapped to). If one gets an f that maps to H_2 and wants to get H_1 instead then just apply a rotation (multiplicaiton with -1).

If we are asked to find a map from half-plane to ball, one can do the inverse problem and then find inverse of mobius mapping. Note that applying inversion (composition with $1/z$) can map outside of the circle to its inside and vice versa.

To solve more complicated problems, combine with e^z, $\log z$, z^n, $z^{1/n}$, $\cdots$ For instance, e^z maps horizontal strips to sectors or half planes (see [3]). z^α maps sectors to sectors.

Example 3.2.16. Find a conformal bijection $\mathbb{C}\backslash(-\infty, 0] \to B(0,1)$.

Solution. First use $z = re^{i\theta} \mapsto z^{1/2} = r^{1/2}e^{i\theta/2}$ to map $\mathbb{C}\backslash(-\infty, 0]$ to $H = \{\mathrm{Re}(z) > 0\}$. Then find Möbius mapping g for $H \to B(0,1)$.

Choose for example $i, 0, -i$ from $\{x = 0\}$ and map them to $1, i, -1$. We compute the cross ratio

$$(g(z), 1, i, -1) = (z, i, 0, -i)$$

$$\frac{(g(z)-1)(i+1)}{(g(z)-i)(1+1)} = \frac{(z-i)(0+1)}{(z-0)(i+i)}$$

$$\iff g(z) = i\frac{1-z}{1+z}$$

As $g(1) = 0$, it turns out that we are lucky and g works (it maps to the inside of the circle; if that's not the case, we need to rotate $\{\mathrm{Re}(z) > 0\}$ to $\{\mathrm{Re}(z) < 0\}$).

Therefore, the composition if the desired conformal mapping

$$f(z) = g(e^{1/2}) = i\frac{1-\sqrt{z}}{1+\sqrt{z}}$$

3.3 Schwarz Lemma and Automorphisms on Unit Disk

We do a review of Maximum modulus principle first.

Let Ω be any subset of $\mathbb{C}$ and suppose α is in the interior of Ω. We can, therefore, choose a positive number ρ such that $B(\alpha;\rho) \subset \Omega$; it readily follows that there is a point ξ in Ω with $|\xi| > |\alpha|$. To state this another way, if α is a point in Ω with $|\alpha| \geq |\xi|$ for each ξ in the set Ω ($\neg(\exists \xi \in \Omega)$ such that $|\xi| > |\alpha|$), then α is not an iterior point and belongs to $\partial\Omega$.

Theorem 3.3.1 (Maximum Modulus Theorem-First Version)**.** If f is analytic in a region G and a is a point in G with $|f(a)| \geq |f(z)|$ for all z in G then f must be a constant function.

Proof. Let $\Omega = f(G)$ and put $\alpha = f(a)$. From the hypothesis we have that $|\alpha| \geq |\xi|$ for each ξ in Ω; as in the discussion preceding the theorem α is in $\partial\Omega \cap \Omega$. In particular, the set Ω cannot be open (because then $\Omega \cap \partial\Omega = \varnothing$). Hence the Open Mapping Theorem says that f must be constant. ☐

Theorem 3.3.2 (Maximum Modulus Theorem-Second Version)**.** Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on G^- which is analytic in G. Then

$$\max\left\{|f(z)| : z \in G^-\right\} = \max\{|f(z)| : z \in \partial G\}.$$

Proof. Since G is bounded there is a point $a \in G^-$such that $|f(a)| \geq |f(z)|$ for all z in G^-. If f is a constant function the conclusion is trivial; if f is not constant then the result follows from first version. □

Note that in the second version we did not assume that G is connected as in the first version. Do you understand how first version puts the finishing touches on the proof of the second? Or, could the assumption of connectedness in first version be dropped?

Let $G = \left\{z = x + iy : -\frac{1}{2}\pi < y < \frac{1}{2}\pi\right\}$ and put $f(z) = \exp[\exp z]$. Then f is continuous on $G^-$and analytic on G. If $z \in \partial G$ then $z = x \pm \frac{1}{2}\pi i$ so $|f(z)| = |\exp(\pm ie^x)| = 1$. However, as x goes to infinity through the real numbers, $f(x) \to \infty$. This does not contradict the Maximum Modulus Theorem because G is not bounded.

In light of the above example it is impossible to drop the assumption of the boundedness of G in the second version.

The following theorem is stated without full proof in [3]. We complete it now.

Theorem 3.3.3 (Schwarz Lemma)**.** Let $\mathbb{D} = \{z \mid |z| < 1\}$ be the unit disk and let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function on it, i.e., $\forall z \in \mathbb{D}$, $|f(z)| \leq 1$. Also suppose $f(0) = 0$. Then

(i) $\forall z \in \mathbb{D}$, $|f(z)| \leq |z|$.

(ii) If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ then f is a rotation, i.e., $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$.

(iii) $|f'(0)| \leq 1$, and if the equality holds, then f is a rotation like (ii).

Proof. Let

$$g(z) := \begin{cases} f(z)/z, & z \neq 0, \\ f'(0), & z = 0, \end{cases}$$

Notice that this is analytic in $\mathbb{D}$, since

$$\lim_{z\to 0}\frac{f(z)}{z} = \lim_{z\to 0}\frac{f(z) - \overbrace{f(0)}^{=0}}{z - 0} = f'(0).$$

so $z = 0$ must be a removable singularity of $\frac{f(z)}{z}$ and assigning the value $f'(0)$ at $z = 0$ makes $\frac{f(z)}{z}$ analytic.

Now if $r < 1$ we have for $|z| = r$ that

$$|g(z)| = \frac{\overbrace{|f(z)|}^{<1}}{r} \leq \frac{1}{r}.$$

Applying the maximum modulus principle to $g \in H(\bar{B}(0, r))$, where $B(0, r)$ is open, connected and bounded, we get

$$\max_{z\in\bar{B}(0,r)}|g(z)| = \max_{|z|=r}|g(z)| \leqslant \frac{1}{r}.$$

Letting $r \to 1$ gives $|g(z)| \leqslant 1$ and so $\forall z \in \mathbb{D}$, $|f(z)| \leqslant |z|$. If we have equality for same $0 \neq z_0 \in \mathbb{D}$, then g attains its maximum in the interior of $\mathbb{D}$. Maximum modulus principle implies that g is constant, so $f(z) = cz$ for some constant c. Since $|z_0| = |f(z_0)| = |c||z_0|$ we see $|c| = 1 \Rightarrow c = e^{i\theta}$. (i) and (ii) are then proved.

Finally, if $|f'(0)| = 1$ this means $|g(0)| = 1$ so again g reaches its maximum at the interior point $0 \in \mathbb{D}$, and is so a constant. This proves (iii). □

A conformal bijection $U \to U$ is called an **automorphism** of U.

What is in $\mathrm{Aut}(\mathbb{D})$ - the automorphism group of $\mathbb{D}$? Obviously $z \mapsto z$ and all rotations $z \mapsto e^{i\theta}z$. But recall also Q4 of 5021/HW1 that for $\alpha \in \mathbb{C}$, $|\alpha| < 1$, the mapping

$$z \mapsto \frac{\alpha - z}{1 - \bar{\alpha}z}$$

maps $\mathbb{D}$ to $\mathbb{D}$. It is clearly Möbius, so we can write its inverse

$$z \mapsto \frac{z - \alpha}{\bar{\alpha}z - 1} = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

so it is its own inverse. Also, from formula of derivative in Q1 of 5022/HW1, one can easily check that

$$(\psi_\alpha)'(0) = \frac{|\alpha|^2 - 1}{(-\bar{\alpha}z + 1)^2}(0) = |\alpha|^2 - 1$$

$$(\psi_\alpha)'(\alpha) = \frac{|\alpha|^2 - 1}{(-\bar{\alpha}z + 1)^2}(\alpha) = (|\alpha|^2 - 1)^{-1}$$

These mappings $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ thus satisfy

- $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$,
- $\psi_\alpha(\alpha) = 0, \psi_\alpha(0) = \alpha$,
- $\psi_\alpha^{-1} = \psi_\alpha$.
- $(\psi_\alpha)'(0) = |\alpha|^2 - 1,\ (\psi_\alpha)'(\alpha) = (|\alpha|^2 - 1)^{-1}$.

What's also interesting is that rotations of these Möbius mappings exhaust all of $\mathbb{D}$:

Theorem 3.3.4. If $f \in \mathrm{Aut}(\mathbb{D})$, then $f(z) = e^{i\theta}\psi_\alpha(z)$ for $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$.

Proof. $\exists!\alpha \in \mathbb{D}$ s.t. $f(\alpha) = 0$ as $f \in \mathrm{Aut}(\mathbb{D})$. Define $g := f \circ \psi_\alpha \in \mathrm{Aut}(\mathbb{D})$. Then $g(0) = f(\alpha) = 0$ and Schwarz lemma gives $|g(z)| \le |z|\ \forall z \in \mathbb{D}$, Also $g^{-1} = \psi_\alpha^{-1} \circ f^{-1} \in \mathrm{Aut}(\mathbb{D})$ satisfies $g^{-1}(0) = 0$ so again $\left|g^{-1}(2)\right| \le |z|\ \forall z \in \mathbb{D}$. Thus

$$|z| = |g^{-1}(g(z))| \le |g(z)| \le |z|$$

which implies that $|g(z)| = |z|\ \forall z \in \mathbb{D}$. Schwarz lemma $\Rightarrow g(z) = e^{i\theta}z$. Then

$$f(z) = g\left(\psi_\alpha^{-1}(z)\right) = g\left(\psi_\alpha(z)\right) = e^{i\theta}\psi_\alpha(z)$$

□

Corollary 3.3.5. If $f \in \mathrm{Aut}(\mathbb{D})$ with $f(0) = 0$, then f is a rotation.

Proof. $f(z) = e^{i\theta}\psi_\alpha(z)$ and $0 = f(0) = e^{i\theta}\alpha \Rightarrow \alpha = 0$, so $f(z) = e^{i\theta}\psi_0(z) = -e^{i\theta}z$. □

Notice that if you want $f \in \mathrm{Aut}(\mathbb{D})$ with $f(\alpha) = \beta$ for given $\alpha, \beta \in D$, just set $f = \psi_\beta \circ \psi_\alpha$.

Using this one can calculate

$$\mathrm{Aut}(\mathbb{H}^+) = \left\{\frac{az + b}{cz + d} \mid ad - bc = 1\right\}$$

for $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ by using a conformal mapping $f : H \to \mathbb{D}$. See other references for more details.

3.4 Space of Continuous and Analytic Functions

This section belongs to classical results in a real analysis course. We proceed by [1].

If G is an open set in $\mathbb{C}$ and (Ω, d) is a complete metric space then designate by $C(G,\Omega)$ the set of all continuous functions from G to Ω.

The set $C(G,\Omega)$ is never empty since it always contains the constant functions. However, it is possible that $C(G,\Omega)$ contains only the constant functions. For example, suppose that G is connected and $\Omega = \mathbb{N} = \{1, 2, \ldots\}$. If f is in $C(G,\Omega)$ then $f(G)$ must be connected in Ω and, hence, must reduce to a point. However, our principal concern will be when Ω is either $\mathbb{C}$ or $\mathbb{C}_\infty$.

To put a metric on $C(G,\Omega)$ we must first prove a fact about open subsets of $\mathbb{C}$.

Proposition 3.4.1. If G is open in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions:

(a) $K_n \subset \operatorname{int} K_{n+1}$;

(b) $K \subset G$ and K compact implies $K \subset K_n$ for some n;

(c) Every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$.

If $G = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact and $K_n \subset \operatorname{int} K_{n+1}$, define

$$\rho_n(f,g) = \sup\{d(f(z),g(z)) : z \in K_n\}$$

for all functions f and g in $C(G,\Omega)$. Also define

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}; \tag{3.1}$$

since $t(1+t)^{-1} \le 1$ for all $t \ge 0$, the series above is dominated by $\sum \left(\frac{1}{2}\right)^n$ and must converge. It will be shown that ρ is a metric for $C(G,\Omega)$. To do this the following lemma, whose proof is left as an exercise, is needed.

Lemma 3.4.2. If (S,d) is a metric space then

$$\mu(s,t) = \frac{d(s,t)}{1+d(s,t)}$$

is also a metric on S. A set is open in (S,d) iff it is open in (S,μ); a sequence is a Cauchy sequence in (S,d) iff it is a Cauchy sequence in (S,μ).

Proposition 3.4.3. $(C(G,\Omega),\rho)$ is a metric space.

Proof. It is clear that $\rho(f,g) = \rho(g,f)$. Also, since each ρ_n satisfies the triangle inequality, the preceding lemma can be used to show that ρ satisfies the triangle inequality. Finally, the fact that $G = \bigcup_{n=1}^{\infty} K_n$ gives that $f = g$ whenever $\rho(f,g) = 0$ □

The next lemma concerns subsets of $C(G,\Omega) \times C(G,\Omega)$ and is very useful because it gives insight into the behavior of the metric ρ. Those who know the appropriate definitions will recognize that this lemma says that two uniformities are equivalent.

Lemma 3.4.4. Let the metric ρ be defined as eq. (3.1). If $\epsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subset G$ such that for f and g in $C(G,\Omega)$,

$$\sup\{d(f(z),g(z)) : z \in K\} < \delta \Rightarrow \rho(f,g) < \epsilon.$$

Conversely, if $\delta > 0$ and a compact set K are given, there is an $\epsilon > 0$ such that for f and g in $C(G, \Omega)$,

$$\rho(f, g) < \epsilon \Rightarrow \sup\{d(f(z), g(z)) : z \in K\} < \delta.$$

Proposition 3.4.5.
(a) A set $\mathcal{O} \subset (C(G, \Omega), \rho)$ is open iff for each f in $\mathcal{O}$ there is a compact set K and $a\delta > 0$ such that

$$\mathcal{O} \supset \{g : d(f(z), g(z)) < \delta, z \in K\}$$

(b) A sequence $\{f_n\}$ in $(C(G, \Omega), \rho)$ converges to f iff $\{f_n\}$ converges to f uniformly on all compact subsets of G.

Henceforward, whenever we consider $C(G, \Omega)$ as a metric space it will be assumed that the metric ρ is given by formula (3.1) for some sequence $\{K_n\}$ of compact sets such that $K_n \subset \operatorname{int} K_{n+1}$ and $G = \bigcup_{n=1}^{\infty} K_n$. Actually, the requirement that $K_n \subset \operatorname{int} K_{n+1}$ can be dropped and the above results will remain valid. However, to show this requires some extra effort (e.g., the Baire Category Theorem) which, though interesting, would be a detour.

Nothing done so far has used the assumption that Ω is complete. However, if Ω is not complete then $C(G, \Omega)$ is not complete. In fact, if $\{\omega_n\}$ is a non-convergent Cauchy sequence in Ω and $f_n(z) = \omega_n$ for all z in G, then $\{f_n\}$ is a non-convergent Cauchy sequence in $C(G, \Omega)$. However, we are assuming that Ω is complete and this gives the following.

Proposition 3.4.6. $C(G, \Omega)$ is a complete metric space.

Definition 3.4.7. A set $\mathscr{F} \subset C(G, \Omega)$ is **normal** if each sequence in $\mathscr{F}$ has a subsequence which converges to a function f in $C(G, \Omega)$.

This of course looks like the definition of sequentially compact subsets, but the limit of the subsequence is not required to be in the set $\mathscr{F}$. The next proof is left to the reader.

Proposition 3.4.8. A set $\mathscr{F} \subset C(G, \Omega)$ is normal iff its closure is compact.

Proposition 3.4.9. A set $\mathscr{F} \subset C(G, \Omega)$ is normal iff for every compact set $K \subset G$ and $\delta > 0$ there are functions $f_1, \ldots, f_n$ in $\mathscr{F}$ such that for f in $\mathscr{F}$ there is at least one $k, 1 \leq k \leq n$, with

$$\sup\left\{d\left(f(z), f_k(z)\right) : z \in K\right\} < \delta.$$

This section concludes by presenting the Arzela-Ascoli Theorem. Although its proof is not overly complicated it is a deep result which has proved extremely useful in many areas of analysis. Before stating the theorem a few results of a more general nature are needed.

Let (X_n, d_n) be a metric space for each $n \geq 1$ and let $X = \prod_{n=1}^{\infty} X_n$ be their cartesian product. That is, $X = \{\xi = \{x_n\} : x_n \in X_n \text{ for each } n \geq 1\}$. For $\xi = \{x_n\}$ and $\eta = \{y_n\}$ in X define

$$d(\xi, \eta) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n\left(x_n, y_n\right)}{1 + d_n\left(x_n, y_n\right)}.$$

Proposition 3.4.10. $(\prod_{n=1}^{\infty} X_n, d)$, where d is defined above, is a metric space. If $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in $X = \prod_{n=1}^{\infty} X_n$ then $\xi^k \to \xi = \{x_n\}$ iff $x_n^k \to x_n$ for each n. Also, if each (X_n, d_n) is compact then X is compact.

The following definition plays a central role in the Arzela-Ascoli Theorem.

Definition 3.4.11. A set $\mathscr{F} \subset C(G, \Omega)$ is equicontinuous at a point z_0 in G iff for every $\epsilon > 0$ there is a $\delta > 0$ such that for $|z - z_0| < \delta$,

$$d(f(z), f(z_0)) < \epsilon$$

for every f in $\mathscr{F}$.$\mathscr{F}$ is equicontinuous over a set $E \subset G$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for z and z' in E and $|z - z'| < \delta$,

$$d(f(z), f(z')) < \epsilon$$

for all f in $\mathscr{F}$.

Notice that if $\mathscr{F}$ consists of a single function f then the statement that $\mathscr{F}$ is equicontinuous at z_0 is only the statement that f is continuous at z_0. The important thing about equicontinuity is that the same δ will work for all the functions in $\mathscr{F}$. Also, for $\mathscr{F} = \{f\}$ to be equicontinuous over E is to require that f is uniformly continuous on E. For a larger family $\mathscr{F}$ to be equicontinuous there must be uniform uniform continuity.

Because of this analogy with continuity and uniform continuity the following proposition should not come as a surprise.

Proposition 3.4.12. Suppose $\mathscr{F} \subset C(G, \Omega)$ is equicontinuous at each point of G; then $\mathscr{F}$ is equicontinuous over each compact subset of G.

Proof. Let $K \subset G$ be compact and fix $\epsilon > 0$. Then for each w in K there is a $\delta_w > 0$ such that

$$d(f(w'), f(w)) < \frac{1}{2}\epsilon$$

for all f in $\mathscr{F}$ whenever $|w - w'| < \delta_w$. Now $\{B(w; \delta_w) : w \in K\}$ forms an open cover of K; by Lebesgue's Covering Lemma there is a $\delta > 0$ such that for each z in $K, B(z; \delta)$ is contained in one of the sets of this cover. So if z and z' are in K and $|z - z'| < \delta$ there is a w in K with $z' \in B(z; \delta) \subset B(w; \delta_w)$. That is, $|z - w| < \delta_w$ and $|z' - w| < \delta_w$. This gives $d(f(z), f(w)) < \frac{1}{2}\epsilon$ and $d(f(z'), f(w)) < \frac{1}{2}\epsilon$; so that $d(f(z), f(z')) < \epsilon$ and $\mathscr{F}$ is equicontinuous over K. □

Theorem 3.4.13 (Arzela-Ascoli Theorem)**.** A set $\mathscr{F} \subset C(G, \Omega)$ is normal iff the following two conditions are satisfied:

(a) for each z in $G, \{f(z) : f \in \mathscr{F}\}$ has compact closure in Ω;

(b) $\mathscr{F}$ is equicontinuous at each point of G.

Proof. First assume that $\mathscr{F}$ is normal. Notice that for each z in G the map of $C(G, \Omega) \to \Omega$ defined by $f \to f(z)$ is continuous; since $\mathscr{F}^-$is compact its image is compact in Ω and (a) follows. To show (b) fix a point z_0 in G and let $\epsilon > 0$. If $R > 0$ is chosen so that $K = \bar{B}(z_0; R) \subset G$ then K is compact and Proposition 3.4.9 implies there are functions $f_1, \ldots, f_n$ in $\mathscr{F}$ such that for each f in $\mathscr{F}$ there is at least one f_k with

$$(*) : \qquad \sup\{d(f(z), f_k(z)) : z \in K\} < \frac{\epsilon}{3}.$$

But since each f_k is continuous there is a $\delta, 0 < \delta < R$, such that $|z - z_0| < \delta$ implies that

$$d(f_k(z), f_k(z_0)) < \frac{\epsilon}{3}$$

for $1 \leq k \leq n$. Therefore, if $|z - z_0| < \delta, f \in \mathscr{F}$, and k is chosen so that $(*)$ holds, then

$$d(f(z), f(z_0)) \leq d(f(z), f_k(z)) + d(f_k(z), f_k(z_0)) + d(f_k(z_0), f(z_0)) < \epsilon$$

That is, $\mathscr{F}$ is equicontinuous at z_0. Now suppose $\mathscr{F}$ satisfies conditions (a) and (b); it must be shown that $\mathscr{F}$ is normal. Let $\{z_n\}$ be the sequence of all points in G with rational real and imaginary parts (so for z in G and $\delta > 0$ there is a z_n with $|z - z_n| < \delta$). For each $n \geq 1$ let

$$X_n = \{f(z_n) : f \in \mathscr{F}\}^- \subset \Omega;$$

from part (a), (X_n, d) is a compact metric space. Thus, by Proposition 3.4.10, $X = \prod_{n=1}^{\infty} X_n$ is a compact metric space. For f in $\mathscr{F}$ define $\tilde{f}$ in X by

$$\tilde{f} = \{f(z_1), f(z_2), \ldots\}.$$

Let $\{f_k\}$ be a sequence in $\mathscr{F}$; so $\left\{\tilde{f}_k\right\}$ is a sequence in the compact metric space X. Thus there is a ξ in X and a subsequence of $\left\{\tilde{f}_k\right\}$ which converges to ξ. For the sake of convenient notation, assume that $\xi = \lim \tilde{f}_k$. Again from Proposition 3.4.10,

$$(**) : \qquad \lim_{k\to\infty} f_k(z_n) = \omega_n$$

where $\xi = \{\omega_n\}$. It will be shown that $\{f_k\}$ converges to a function f in $C(G, \Omega)$. By $(**)$ this function f will have to satisfy $f(z_n) = \omega_n$. The importance of $(**)$ is that it imposes control over the behavior of $\{f_k\}$ on a dense subset of G. We will use the fact that $\{f_k\}$ is equicontinuous to spread this control to the rest of G.

To find the function f and show that $\{f_k\}$ converges to f it suffices to show that $\{f_k\}$ is a Cauchy sequence. So let K be a compact set in G and let $\epsilon > 0$; by Lemma 3.4.5(b) it suffices to find an integer J such that for $k, j \geq J$,

$$(***) : \qquad \sup\{d(f_k(z), f_j(z)) : z \in K\} < \epsilon.$$

Since K is compact $R = d(K, \partial G) > 0$. Let $K_1 = \left\{z : d(z, K) \leq \frac{1}{2}R\right\}$; then K_1 is compact and $K \subset$ int $K_1 \subset K_1 \subset G$. Since $\mathscr{F}$ is equicontinuous at each point of G it is equicontinuous on K_1 by Proposition 3.4.11. So choose δ, $0 < \delta < \frac{1}{2}R$, such that

$$(1) : \qquad d(f(z), f(z')) < \frac{\epsilon}{3}$$

for all f in $\mathscr{F}$ whenever z and z' are in K_1 with $|z - z'| < \delta$. Now let D be the collection of points in $\{z_n\}$ which are also points in K_1; that is

$$D = \{z_n : z_n \in K_1\}$$

If $z \in K$ then there is a z_n with $|z - z_n| < \delta$; but $\delta < \frac{1}{2}R$ gives that $d(z_n, K) < \frac{1}{2}R$, or that $z_n \in K_1$. Hence $\{B(w; \delta) : w \in D\}$ is an open cover of K. Let $w_1, \ldots, w_n \in D$ such that

$$K \subset \bigcup_{i=1}^{n} B(w_i; \delta).$$

Since $\lim_{k\to\infty} f_k(w_i)$ exists for $1 \leq i \leq n$ (by $(**)$) there is an integer J such that for $j, k \geq J$

$$(2) : \qquad d(f_k(w_i), f_j(w_i)) < \frac{\epsilon}{3}$$

for $i = 1, \ldots, n$. Let z be an arbitrary point in K and let w_i be such that $|w_i - z| < \delta$. If k and j are larger than J then (1) and (2) give

$$d(f_k(z), f_j(z)) \leq d(f_k(z), f_k(w_i)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z)) < \epsilon.$$

Since z was arbitrary this establishes $(***)$. □

Let G be an open subset of the complex plane. If $H(G)$ is the collection of analytic functions on G, we can consider $H(G)$ as a subset of $C(G, \mathbb{C})$. We use $H(G)$ to denote the analytic functions on G rather than $A(G)$ because it is a universal practice to let $A(G)$ denote the collection of continuous functions $f : G^{-} \to \mathbb{C}$ that are analytic in G. Thus $A(G) \neq H(G)$.

The first question to ask about $H(G)$ is: Is $H(G)$ closed in $C(G, \mathbb{C})$? The next result answers this question positively and also says that the function $f \to f'$ is continuous from $H(G)$ into $H(G)$.

Theorem 3.4.14. If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G, \mathbb{C})$ such that $f_n \to f$ then f is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

Proof. We will show that f is analytic by applying Morera's Theorem. So let T be a triangle contained inside a disk $D \subset G$. Since T is compact, $\{f_n\}$ converges to f uniformly over T. Hence $\int_T f = \lim \int_T f_n = 0$ since each f_n is analytic. Thus f must be analytic in every disk $D \subset G$; but this gives that f is analytic in G.

To show that $f_n^{(k)} \to f^{(k)}$, let $D = \bar{B}(a; r) \subset G$; then there is a number $R > r$ such that $\bar{B}(a; R) \subset G$. If γ is the circle $|z - a| = R$ then Cauchy's Integral Formula gives

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f_n(w) - f(w)}{(w - z)^{k+1}} dw.$$

for z in D. Using Cauchy's Estimate,

$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| \leq \frac{k! M_n R}{(R - r)^{k+1}} \text{ for } |z - a| \leq r,$$

where $M_n = \sup\{|f_n(w) - f(w)| : |w - a| = R\}$. But since $f_n \to f, \lim M_n = 0$. Hence, it follows from above equation that $f_n^{(k)} \to f^{(k)}$ uniformly on $\bar{B}(a; r)$. Now if K is an arbitrary compact subset of G and $0 < r < d(K, \partial G)$ then there are $a_1, \ldots, a_n$ in K such that $K \subset \bigcup_{j=1}^n B(a_j; r)$. Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $B(a_j; r)$, the convergence is uniform on K. □

We will always assume that the metric on $H(G)$ is the metric which it inherits as a subset of $C(G, \mathbb{C})$. The next result follows because $C(G, \mathbb{C})$ is complete.

Corollary 3.4.15. $H(G)$ is a complete metric space.

Corollary 3.4.16. If $f_n : G \to \mathbb{C}$ is analytic and $\sum_{n=1}^\infty f_n(z)$ converges uniformly on compact sets to $f(z)$ then

$$f^{(k)}(z) = \sum_{n=1}^\infty f_n^{(k)}(z).$$

It should be pointed out that the above theorem has no analogue in the theory of functions of a real variable. For example it is easy to convince oneself by drawing pictures that the absolute value function can be obtained as the uniform limit of a sequence of differentiable functions. Also, it can be shown (using a Theorem of Weierstrass) that a continuous nowhere differentiable function on $[0, 1]$ is the limit of a sequence of polynomials. Surely this is the most emphatic contradiction of the corresponding theorem for Real Variables. A contradiction in another direction is furnished by the following. Let $f_n(x) = \frac{1}{n} x^n$ for $0 \leq x \leq 1$. Then $0 = u - \lim f_n$; however the sequence of derivatives $\{f_n'\}$ does not converge uniformly on $[0, 1]$.

3.5 Riemman Mapping Theorem

Definition 3.5.1. A set $\mathscr{F} \subset H(G)$ is **locally bounded**if for each point a in G there are constants M and $r > 0$ such that for all f in $\mathscr{F}$,

$$|f(z)| \leq M, \text{ for } |z - a| < r.$$

Alternately, $\mathscr{F}$ is locally bounded if there is an $r > 0$ such that

$$\sup\{|f(z)| : |z-a| < r, f \in \mathscr{F}\} < \infty.$$

That is, $\mathscr{F}$ is locally bounded if about each point a in G there is a disk on which $\mathscr{F}$ is uniformly bounded. This immediately extends to the requirement that $\mathscr{F}$ be uniformly bounded on compact sets in G.

Lemma 3.5.2. A set $\mathscr{F}$ in $H(G)$ is locall bounded iff for each compact set $K \subset G$ there is a constant M such that

$$|f(z)| \leq M$$

for all f in $\mathscr{F}$ and z in K.

Theorem 3.5.3 (Montel's Theorem)**.** A family $\mathscr{F}$ in $H(G)$ is normal iff $\mathscr{F}$ is locally bounded.

Proof. Suppose $\mathscr{F}$ is normal but fails to be locally bounded; then there is a compact set $K \subset G$ such that $\sup\{|f(z)| : z \in K, f \in \mathscr{F}\} = \infty$. That is, there is a sequence $\{f_n\}$ in $\mathscr{F}$ such that $\sup\{|f_n(z)| : z \in K\} \geq n$. Since $\mathscr{F}$ is normal there is a function f in $H(G)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$. But this gives that $\sup\{|f_{n_k}(z) - f(z)| : z \in K\} \to 0$ as $k \to \infty$. If $|f(z)| \leq M$ for z in K,

$$n_k \leq \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + M$$

since the right hand side converges to M, this is a contradiction. Now suppose $\mathscr{F}$ is locally bounded; the Arzela-Ascoli Theorem will be used to show that $\mathscr{F}$ is normal. Since condition (a) of Arzela-Ascoli Theorem is clearly satisfied, we must show that $\mathscr{F}$ is equicontinuous at each point of G. Fix a point a in G and $\epsilon > 0$; from the hypothesis there is an $r > 0$ and $M > 0$ such that $\bar{B}(a;r) \subset G$ and $|f(z)| \leq M$ for all z in $\bar{B}(a;r)$ and for all f in $\mathscr{F}$. Let $|z-a| < \frac{1}{2}r$ and $f \in \mathscr{F}$; then using Cauchy's Formula with $\gamma(t) = a + re^{it}, 0 \leq t \leq 2\pi$,

$$\begin{aligned} |f(a) - f(z)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)} dw\right| \\ &\leq \frac{2M}{r}|a-z| \end{aligned}$$

Letting $\delta < \min\left\{\frac{1}{2}r, \frac{r}{4M}\epsilon\right\}$ it follows that $|a-z| < \delta$ gives $|f(a) - f(z)| < \epsilon$ for all f in $\mathscr{F}$. $\square$

Corollary 3.5.4. A set $\mathscr{F} \subset H(G)$ is compact iff it is closed and locally bounded.

Theorem 3.5.5 (Riemann Mapping Theorem)**.** $U \subset \mathbb{C}$ a simply connected set, $U \neq \mathbb{C}$. Then there is a conformal bijeciton $U \to \mathbb{D} := B(0,1)$.

Proof. Let

$$\Sigma = \{\psi : U \to B(0,1) \mid \psi \in H(U) \text{ injective}\}$$

We need to show that there is a surjective $\psi \in \Sigma$.

Step I: we claim that $\sigma \neq \varnothing$.
Let $w_0 \in \mathbb{C} \setminus U$ (As $U \neq \mathbb{C}$). Then $z \mapsto z - w_0$, $z \in U$, is a non-vanishing element of $H(U)$, so by simply-connectedness (Theorem 2.3.8) there is a $\psi \in H(U)$ such that $\varphi^2(z) = z - w_0,\ \forall z \in U$. Notice that φ is injective. Indeed, $\varphi(z_1) = \varphi(z_2) \to z_1 - w_0 = \varphi(z_1)^2 = \varphi(z_2)^2 = z_2 - w_0$. Also,

$$(*): \quad \varphi(z_1) = -\varphi(z_2) \Rightarrow z_1 - w_0 = \varphi(z_1)^2 = (-\varphi(z_2))^2 = \varphi(z_2)^2 = z_2 - w_0.$$

By open mapping theorem, φ is open, and so $\varphi(U) \subset \mathbb{C}$ is open. Choose a ball $\overline{B(a,r)} \subset \varphi(U)$, $0 < r < |a|$. Now $\varphi(U) \cap \overline{B(-a,r)} = \varnothing$: if there were some $w = \varphi(z)$, $z \in U$, $|w+a| \leq r$, then we would have $|-w-a| = |w+a| \leq r$. This implies that $-w \in \overline{B(a,r)} \subset \varphi(U)$. Thus, $-w = \varphi(\widetilde{z})$ for some $\widetilde{z} \in U$. But then $\varphi(\widetilde{z}) = -\varphi(z) \Rightarrow \widetilde{z} = z$ (by $(*)$) $\Rightarrow w = -w \Rightarrow w = 0$. Contradiction (as $0 \notin \overline{B(-a,r)}$ by condition $r < |a|$.)

Define

$$\psi(z) := \frac{r}{\varphi(z)+a}, \quad z \in U.$$

By $\varphi(U) \cap \overline{B(-a,r)} = \varnothing$, we see that $|\varphi(z)+a| > r$ for all $a \in U$, so ψ mapsto $\mathbb{D}$. Also, ψ is injective: $\psi(z_1) = \psi(z_2) \Rightarrow \varphi(z_1) = \varphi(z_2) \Rightarrow z_1 = z_2$. Therefore, $\psi \in \Sigma$.

Step II: Our second claim is that if we let $z_0 \in U$ be fixed and define

$$\eta := \sup_{\psi \in \Sigma} |\psi'(z_0)|$$

Then $\eta < \infty$ and there exsits $\psi \in \Sigma$ such that $|\psi'(z_0)| = \eta$.
To see $\eta < \infty$, notice that by Cauchy estimate, if we choose some ball $\overline{B(z_0,r)} \subset U$, we get

$$|\psi'(z_0)| \leq \frac{\|\psi\|_{L^\infty(\overline{B(z_0,r)})}}{r} \leq \frac{1}{r} \quad \forall \psi \in \Sigma$$

simply becasue $\psi : \mathbb{D} \to \mathbb{D}$. Thus, $\eta \leq \frac{1}{r} \leq \infty$. By properties of supremum, there is $\psi_n \in \Sigma$ such that $|\psi_n'(z_0)| \to \eta$ as $n \to \infty$. We hope that $\{\psi_n\}$ has a limit in some sense. We use a normal family argument. By Montel's theorem, as $\{\psi_n\}$ is clearly uniformly bounded (they map to $\mathbb{D}$ after all), $\{\psi_n\}$ is a normal family. For simplicity, define the subsequence that converges uniformly in compact subsets of U still by $\{\psi_n\}$. Let $h \in H(U)$ be the limit. Also, $\psi_n' \to h'$ uniformly in compact subsets, so

$$|h'(z_0)| = \lim_{n\to\infty} |\psi_n'(z_0)| = \eta$$

h is injective by previous result, as it is not constant. It is not constant because $\eta > 0$ ($\Sigma \neq \varnothing$ and injectivity and analiticity imply nonzero derivative) and $|h'(z_0)| = \eta$. Now, h also maps to $\mathbb{D}$:

(1) $|h(z)| = \lim \underbrace{|\psi_n(z)|}_{\leq 1} \leq 1$.

(2) if $|h(z)| = 1$ for some $z \in U$, then the fact that $|h|$ obtains its maximum on the open connected U implies h is a constant.

Therefore, $h \in \Sigma$ and $|h'(z_0)| = \eta$. Step II is then complete.

Step III: any $h \in \Sigma$ with $|h'(z_0)| = \eta$ is always a surjection (This step will finish the proof due to Step II.)
We will show this by showing that if $\psi \in \Sigma$ is not subjective (ie. $\psi U \neq B(0,1)$), then $\exists \psi_1 \in \Sigma$ with $|\psi_1'(z_0)| > |\psi'(z_0)|$. This obviously implies step III. So fix $\psi \in \Sigma$ with $\psi U \neq B(0,1)$. Choose $\alpha \in B(0,1) \backslash \psi U$. We use the automorphism of the disk $\mathbb{D} = B(0,1)$

$$\varphi_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha} z},$$

t which has $\varphi_\alpha(\alpha) = 0$, $\varphi_\alpha(0) = \alpha$, and $\varphi_\alpha^{-1} = \varphi_\alpha$. Notice that $\varphi_\alpha \circ \psi \in \Sigma$ has no zero, so $\varphi_\alpha \circ \psi = g^2$ for some $g \in H(U)$ due to simply-connectedness. Note that g is injecitve:

$$\begin{aligned} g(z_1) = g(z_2) &\implies \varphi_\alpha \circ \psi(z_1) = g^2(z_1) = g^2(z_2) = \varphi_\alpha \circ \psi(z_2) \\ &\implies z_1 = z_2 \text{ (becasue } \varphi_\alpha \circ \psi \text{ injective)} \end{aligned}$$

In fact, $g \in \Sigma$ as

$$|g(z)| = |g(z)^2|^{1/2} = \underbrace{|\varphi_\alpha \circ \psi(z)|}_{<1}{}^{1/2} < 1.$$

Let $\beta = g(z_0) \in \mathbb{D}$, and let $\psi_1 := \varphi_\beta \circ g \in \Sigma$. We use Schwarz's lemma to prove that

$$|\psi_1'(z_0)| > |\psi'(z_0)|.$$

We figure out how to write the original ψ in terms of ψ_1. Define $s(z) = z^2$. Now,

$$\psi = \varphi_\alpha \circ \overbrace{s \circ g}^{g^2 = \varphi_\alpha \circ \psi} = \underbrace{\varphi_\alpha \circ s \circ \varphi_\beta}_{=:F} \circ \underbrace{\varphi_\beta \circ g}_{=\psi_1} = F \circ \psi_1.$$

where $F : \mathbb{D} \to \mathbb{D}$. We will show $|F'(0)| < 1$ using Schwartz. Notice $F : \mathbb{D} \to \mathbb{D}$ is analytic with

$$\begin{aligned} F(0) &= \varphi_\alpha\left(\beta^2\right) \\ &= \varphi_\alpha\left(g(z_0)^2\right) \\ &= \varphi_\alpha(\varphi_\alpha(\psi(z_0))) \\ &= \psi(z_0) =: \gamma \end{aligned}$$

As we want $0 \mapsto 0$ to be able to use Schwarz, we will use $\varphi_\alpha \circ F$, since $\varphi_\gamma(F(0)) = \varphi_\gamma(\gamma) = 0$. Apply Schwarz to $\varphi_\gamma \circ F$ to obtain

$$|(\varphi_\gamma \circ F)'(0)| < 1$$

as otherwise it would be a rotation, which is not possible as it is not injecitve due to s. Chain rule gives

$$(\varphi_\gamma \circ F)'(0) = \varphi_\gamma'(\gamma) F'(0)$$

where $|\varphi_\gamma'(\gamma)| = \frac{1}{1-|\gamma|^2} \geq 1$ and so $|F'(0)| < \frac{1}{|\varphi_\gamma'(\gamma)|} \leq 1$. So this Schwartz trickery gave us $|F'(0)| < 1$. Recall $\psi = F \circ \psi_1$, and so

$$\begin{aligned} \psi'(z_0) &= F'(\psi_1(z_0))\psi_1'(z_0) \\ &= F'(0)\psi_1'(z_0), \end{aligned}$$

since $\psi_1(z_0) = \varphi_\beta(\beta) = 0$. So

$$|\psi'(z_0)| = \underbrace{|F'(0)|}_{<1} |\psi_1'(z_0)| < |\psi_1'(z_0)|.$$

Notice this requires us to know $\psi_1'(z_0) \neq 0$, but this follows as $\psi \in \Sigma$ is an analytic injection. □

Notice that the proof only used that

In U we haveif $f \in H(U)$ non-vanishing, $f = g^2$ for some $g \in H(U)$.

This then completes the proof of $(f) \implies (a)$ part of the Theorem 2.3.8. But for $U \neq \mathbb{C}$, Riemann mapping theorem is much stronger than $(f) \implies (a)$.

3.6 The Phragmén-Lindelöff Type Results

We recall that the maximum modulus principle fails without assuming U to be bounded.

Example 3.6.1.
(1) $U = \{\mathrm{Re}(z) > 0\}$, $f(z) = e^z$. Then $|f(z)| = e^{\mathrm{Re}(z)} = e^0 = 1$ on ∂U but $|f(x)| = e^x \to \infty$ as $x \to \infty$ inside the half-plane.

(2) $U = \left\{-\frac{\pi}{4} < \operatorname{Arg} z < \frac{\pi}{4}\right\}$, $f(z) = e^{z^2}$. Then $f\left(re^{\pm i\frac{\pi}{4}}\right) = e^{r^2(\pm i)}$ so $|f| = 0$ on the boundary ∂U of the sector U. still $f(x) = e^{x^2} \to \infty$ as $x \to \infty$ inside the sector.

What if we impose some growth restriction on f?

Theorem 3.6.2. Let $\alpha \geq \frac{1}{2}$ and put
$$U = \left\{ z : |\operatorname{Arg} z| < \frac{\pi}{2\alpha} \right\}.$$

Suppose that f is analytic on U and continuous on $\overline{U}$ and there is a constant M such that $|f(z)| \leq M$ on ∂U. If there are positive constants C, c and $\beta < \alpha$ such that
$$(*): \quad |f(z)| \leq C \exp\left(c|z|^{\beta}\right)$$
for all $z \in U$, then $|f(z)| \leq M$ for all z in U.

Remark 3.6.3. If we suppose some reasonable growth condition inside U, then maximum modulus holds even in the unbounded sector U. $\beta < \alpha$ cannot be dropped: in the examples above, $\alpha = \beta = 1$ and $\alpha = \beta = 2$ make them fail.

proof of the theorem. Fix $\varepsilon > 0$ and $\gamma \in (\beta, \alpha)$. Define $F_{\varepsilon}(z) = F(z) = \exp\left(-\varepsilon z^{\gamma}\right) f(z)$. Here, $z^{\gamma} = r^{\gamma} e^{i\gamma\theta}$ when $z = re^{i\theta} \in \overline{U} \iff r \geq 0$ and $|\theta| \leq \frac{\pi}{2\alpha}$. Notice that $|\gamma\theta| \leq \frac{\gamma}{\alpha}\frac{\pi}{2} < \frac{\pi}{2}$, so z^{γ} is analytic, and also $\cos(\gamma\theta) > 0$. Now
$$|F(z)| = \exp\left(\operatorname{Re}(-\varepsilon z^{\gamma})\right)|f(z)| = \exp\left(-\varepsilon r^{\gamma}\cos(\gamma\theta)\right)|f(z)| \leq |f(z)| \quad \forall z \in \overline{U}$$

In particular, $|F(z)| \leq M$ on ∂U. For $z \in U$, we use the growth assumption $(*)$ and above equation to see
$$\begin{aligned} |F(z)| &\leq C\exp(cr^{\beta} - \varepsilon r^{\gamma}\cos(\gamma\theta)) \leq C\exp\left(cr^{\beta} - \varepsilon r^{\gamma}\cos\left(\frac{\gamma\pi}{2\alpha}\right)\right) \\ &= C\exp\left(r^{\gamma}\left(\underbrace{\frac{c}{r^{\gamma-\beta}}}_{\overset{r\to\infty}{\longrightarrow}0} - \varepsilon\underbrace{\cos\left(\frac{\gamma\pi}{2\alpha}\right)}_{>0}\right)\right). \end{aligned}$$

As $\gamma > \beta$, clearly, for $|z| = r$ large enough uniformly on θ the negative power dominates and the whole term goes very small, so $|F(z)| \leq M$ on U. By maximum modulus principle applied to F on a compact part K of the sector (green area in Fig. 3.2)

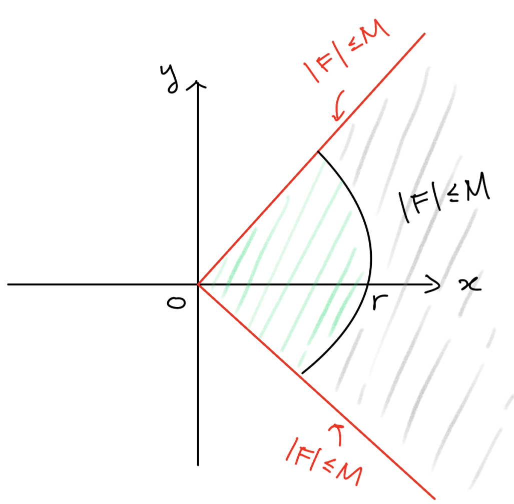


Figure 3.2: A compact part of the sector.

Then since all of the boundary of K have $|F| \leq M$, we get

$$|F| \leq \max_{z\in K}\{|F(z)|\} = \max_{z\in \partial K}\{|F(z)|\} \leq M.$$

Evoking proposition 3.4.1, we have $|F| \leq M$ on $\overline{U}$. Finally,

$$|f(z)| \leq |\exp\left(\varepsilon z^{\gamma}\right) F(z)| \leq M \exp\left(\varepsilon r^{\gamma}\cos(\gamma\theta)\right) \xrightarrow{\varepsilon\to 0} M$$

□

The idea of this Phragmén-Lindelöff type proof is to modify f by some h_ε, that is, to form $F_\varepsilon = h_\varepsilon f$. And h_ε is chosen so that,

(i) The boundary behavior moves to F_ε, e.g., $|F_\varepsilon| \leq |f|$.

(ii) Function F_ε vanishes fast enough as we go towards the unbounded parts of U. This allows one to apply maximum modulus principle in a bounded set.

(iii) We can move the result obtained from F_ε to f, e.g., $1/|h_\varepsilon(z)| \to 1$ as $\varepsilon \to 0$

The following version is used to show the Riesze-Thorin interpolation theorem later and is left as an exercise.

Proposition 3.6.4. Let $U := \left\{-\frac{\pi}{2\alpha} < \operatorname{Arg} z < \frac{\pi}{2\alpha}\right\}$, $\alpha > 1/2$. Suppose $f \in H(U) \cap C(\overline{U})$ and $|f| \leq M$ on ∂U. Suppose we have the following a priori estimate: for all $\delta > 0$ we have for all $z \in U$ that

$$|f(z)| \lesssim_\delta e^{\delta|z|^\alpha}$$

with the notation meaning that for some $C_\delta < \infty$ we have $|f(z)| \leq C_\delta e^{\delta|z|^\alpha}$, $z \in U$. Then

$$|f(z)| \leq M, \qquad z \in U.$$

Proof. See Math5022 HW3 Ex1. □

There are versions in other type of regions as well - in particular, in some strips.

Theorem 3.6.5 (Hadamard three-lines theorem)**.** Let $f(z)$ be a function on the strip

$$U = \{x + iy : a < x < b\},$$

holomorphic in the strip and continuous on the closure of the strip. Suppose $|f(z)| \leq B\, \forall z \in \overline{U}$. If

$$M(x) = \sup_y |f(x+iy)|$$

then $\log M(x)$ is a convex function on $[a, b]$. In other words, if $x = (1-t)a + tb$ with $0 \leq t \leq 1$, then

$$\log M(x) \leq (1-t)\log M(a) + t\log M(b),$$

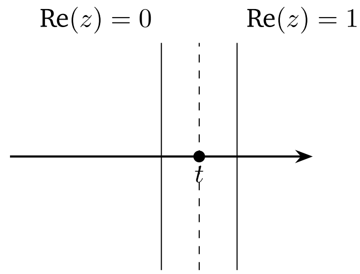
or

$$M(x) \leq M(a)^{1-t} M(b)^t.$$

or

$$M(x) \leq M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}}$$

In particular, $|f| \leq B$ can be replaced by $|f| \leq \max(M(a), M(b))$



Proof. After an affine transformation in the coordinate z, we can assume that $a = 0$, $b = 1$. Then $x = (1-t)a + tb = t \in [0,1]$ and we need to show

$$M(x) \leq M(0)^{1-x} M(1)^x,\ 0 < x < 1.$$

We do the special case $B = 1$ first: $|f(z)| \leq 1$ on $\partial U \implies |f| \leq 1$ on $\overline{U}$.

This is the usual Phragmén-Lindelöff strategy. Given $\varepsilon > 0$, define

$$F_\varepsilon(z) = F(z) = \frac{f(z)}{1+\varepsilon z},\ z \in \overline{U}$$

As $|\mathrm{Re}(z)| \leq |z|$, we have $|1+\varepsilon z| \geq |\mathrm{Re}(1+\varepsilon z)| = 1 + \varepsilon x \geq 1$, so $|F(z)| \leq |f(z)|$ for $z \in \overline{U}$. In particular, $|F(z)| \leq 1$ on ∂U. As $|1+\varepsilon z| \geq |\operatorname{Im}(1+\varepsilon z)| = \varepsilon |y|$, we see

$$|F(z)| \leq \frac{|f(z)|}{\varepsilon |y|} \leqslant \frac{1}{\varepsilon |y|}.$$

Thus now $|F| \leq 1$ for $|y|$ larger then some number c uniformly on x. By maximum modulus principle applied to the rectangular region bounded by $\partial\{|y| \geq c\}$ and $\partial\{x \in [0,1]\}$, we see $|F| \leq 1$ on $\overline{U}$ after letting c go to infinity. Let $\varepsilon \to 0$ to get $|f| \leq 1$ on $\overline{U}$.

The general case reduces to this, but the reduction is non-trivial. First observe that we can assume $M(0), M(1) > 0$ (note that $M(x) = \sup_y |f(x+yi)| \geq 0$) because suppose we have proved the statement for the case $M(0), M(1) > 0$ and now for some f we have say $M(0) = 0$, then we define $g = f + \varepsilon$ and apply the statment to g:

$$\sup_y |f| - \varepsilon \leq \sup_y |g| \overset{\text{statement}}{\leq} (M(0)+\varepsilon)^{1-x}(M(1)+\varepsilon)^x$$

where the last step is by noticing that for any $x \in [0,1]$,

$$\sup_y |g| = \sup_y |f(x+yi)+\varepsilon| \leq \sup_y |f(x+yi)| + \varepsilon = M(x) + \varepsilon.$$

Now $M(0) = 0$, so

$$\sup_y |f| \leq \varepsilon + \varepsilon^{1-x}(M(1)+\varepsilon)^x \overset{\varepsilon \to 0}{\longrightarrow} 0$$

implying that $f = 0$, which means the statement is trivially true.

Assume $M(0), M(1) > 0$. Define $g(z) = M(0)^{1-z} M(1)^z$ for $z \in \mathbb{C}$. Here for any $M \neq 0$ a function of the type $z \mapsto M^z := e^{z \log M}$ is clearly entire, and so g is entire. We have

$$\begin{aligned} |g(z)| &= |M(0)^{1-z}||M(1)^z| \\ &= e^{(1-x)\log M(0)} e^{x \log M(1)} \\ &= M(0)^{1-x} M(1)^x \\ &= |g(x)| \end{aligned}$$

is independent of y. In particular, $|g| \geq \min\{M(0), M(1)\} > 0$, implying $1/g$ is bounded. So $f/g \in C(\overline{U}) \cap H(U)$ is bounded and when $x = 0$ we have $|g(iy)| = g(0) = M(0)$ and when $x = 1$ we have $|g(1+iy)| = g(1) = M(1)$. This implies $|f/g| \leq 1$ on ∂U, and then our special case $B = 1$ above implies $|f/g| \leq 1$ on $\overline{U}$. Thus,

$$|f(z)| \leq |g(z)| = M(0)^{1-x} M(1)^x, \quad z \in \overline{U}.$$

□

3.7 The Riesz interpolation theorem

We copy the section 1 and 2 from second chapter of [7].

3.7.1 Motivation from Fourier Analysis

An initial problem considered was that of formulating an L^p analog of the basic L^2 Parseval relation for functions on $[0, 2\pi]$. This theorem states that if $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta} d\theta$ denotes the Fourier coefficients of a function f in $L^2([0, 2\pi])$, usually written as

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \tag{3.2}$$

then the following fundamental identity holds:

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta. \tag{3.3}$$

Conversely, if $\{a_n\}$ is a sequence for which the left-hand side of (3.3) is finite, then there exists a unique f in $L^2([0, 2\pi])$ so that both (3.2) and (3.3) hold. Notice, in particular, if $f \in L^2([0, 2\pi])$, then its Fourier coefficients $\{a_n\}$ belong to $L^2(\mathbb{Z}) = \ell^2(\mathbb{Z})$.[1] The question that arose was: is there an analog of this result for L^p when $p \neq 2$?

Here an important dichotomy between the case $p > 2$ and $p < 2$ occurs. In the first case, when $f \in L^p([0, 2\pi])$, although f is automatically in $L^2([0, 2\pi])$, examples show that no better conclusion than $\sum |a_n|^2 < \infty$ is possible. On the other hand, when $p < 2$ one can see that essentially there can be no better conclusion than $\sum |a_n|^q < \infty$, with q the dual exponent of p. Analogous restrictions must be envisaged when the roles of f and $\{a_n\}$ are reversed. In fact, what does hold is the Hausdorff-Young inequality:

$$\left(\sum |a_n|^q\right)^{1/q} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^p d\theta\right)^{1/p}, \tag{3.4}$$

and its ”dual”

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^q d\theta\right)^{1/q} \leq \left(\sum |a_n|^p\right)^{1/p} \tag{3.5}$$

both valid when $1 \leq p \leq 2$ and $1/p + 1/q = 1$. (The case $q = \infty$ corresponds to the usual L^∞ norm.) These may be viewed as intermediate results, between the case $p = 2$ corresponding to Parseval's theorem, and its ”trivial” case $p = 1$ and $q = \infty$.

A few words about how the inequalities (3.4) and (3.5) were first attacked are in order, because they contain a useful insight about L^p spaces: often, the simplest case arises when p (or its dual) is an even integer. Indeed, when, for example $q = 4$, a function belonging to L^4 is the same as its square belonging to L^2, and this sometimes allows reduction to the easier situation when $p = 2$. To see how this works in the present situation, let us take $q = 4$ (and $p = 4/3$) in (3.4). With f given in L^p, we denote by $\mathcal{F}$ the convolution of f with itself,

$$\mathcal{F}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta - \varphi)f(\varphi)d\varphi$$

By the multiplicative property of Fourier coefficients of convolutions we have

$$\mathcal{F}(\theta) \sim \sum_{n=-\infty}^{\infty} a_n^2 e^{in\theta},$$

with $\{a_n\}$ the Fourier coefficients of f. Parseval's identity applied to $\mathcal{F}$ then yields

$$\sum |a_n|^4 = \frac{1}{2\pi}\int_0^{2\pi} |\mathcal{F}(\theta)|^2 d\theta,$$

and Young's inequality for convolutions gives

$$\|\mathcal{F}\|_{L^2} \leq \|f\|_{L^{4/3}}^2,$$

proving (3.4) when $p = 4/3$ and $q = 4$. Once the case $q = 4$ has been established, the cases corresponding to $q = 2k$, where k is a positive integer, can be handled in a similar way. However the general situation, $2 \leq q \leq \infty$, corresponding to $1 \leq p \leq 2$, involves further ideas.

In contrast to the above ingenious but special argument, in turns out that there is a general principle of great interest that underlies such inequalities, which in fact leads to direct and abstract proofs of both (3.4) and (3.5). This is the M. Riesz interpolation theorem. Stated succinctly, it asserts that whenever a linear operator satisfies a pair of inequalities (like (**??**) for $p = 2$ and $p = 1$), then automatically the operator satisfies the corresponding inequalities for the intermediate exponents: here all p for $1 \leq p \leq 2$, and q with $1/p + 1/q = 1$. The formulation and proof of this general theorem will be our first task in the next section.

3.7.2 The Riesz interpolation theorem

Suppose (p_0, q_0) and (p_1, q_1) are two pairs of indices with $1 \leq p_j, q_j \leq \infty$, and assume that

$$\begin{cases} \|T(f)\|_{L^{q_0}} \leq M_0\|f\|_{L^{p_0}}, & \forall f \in L^{p_0} \\ \|T(f)\|_{L^{q_1}} \leq M_1\|f\|_{L^{p_1}}, & \forall f \in L^{q_0} \end{cases}$$

where T is a linear operator. That is, T maps $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$ boundedly in the above sense. Does it follow that $T : L^p \to L^q$ for some intermediate p, q, i.e.,

$$\|T(f)\|_{L^q} \leq M(M_0, M_1)\|f\|_{L^p}, \quad \text{for other pairs } (p, q)?$$

Yes, and the Marcinkiewicz interpolation theorem (see Math5051) applies but in the case where
(1) $p_0 = q_0$ and $p_1 = q_1$.
(2) T is sublinear instead of linear.
(3) there is an even weaker assumption than T mapping $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$ boundedly to conclude $T : L^p \to L^q$.
(4) there isn't a good bound resulted for $M(M_0, M_1)$.

Thus, (2) and (3) are positive things while (1) and (4) are restrictions. We will give Riesz's answer to the question.

The precise statement of the theorem requires that we fix some notation. Let (X, μ) and (Y, ν) be a pair of measure spaces. We shall abbreviate the L^p norm on (X, μ) by writing $\|f\|_{L^p} = \|f\|_{L^p(X,\mu)}$, and similarly for the L^q norm for functions on $(Y, d\nu)$. We will also consider the space $L^{p_0} + L^{p_1}$ that consists of functions on (X, μ) that can be written as $f_0 + f_1$, with $f_j \in L^{p_j}(X, \mu)$, with a similar definition for $L^{q_0} + L^{q_1}$.

Theorem 3.7.1 (Riesz interpolation theorem)**.** Suppose T is a linear mapping from $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$. Assume that T is bounded from L^{p_0} to L^{q_0} and from L^{p_1} to L^{q_1}

$$\begin{cases} \|T(f)\|_{L^{q_0}} \leq M_0\|f\|_{L^{p_0}}, \\ \|T(f)\|_{L^{q_1}} \leq M_1\|f\|_{L^{p_1}}. \end{cases}$$

Then T is bounded from L^p to L^q,

$$(*): \quad \|T(f)\|_{L^q} \leq M\|f\|_{L^p},$$

whenever the pair (p, q) can be written as

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for some t with $0 \leq t \leq 1$. Moreover, the bound M satisfies $M \leq M_0^{1-t} M_1^t$.

Proof. We begin by establishing the inequality when f is a simple function, $f = \sum a_k \chi_{E_k}$ where sets E_k are disjoint and of finite measure. We can assume $\|f\|_{L^p} = 1$ because if we proved $(*)$ for this case, i.e.,

$$\|T(f)\|_{L^q} \leq M,$$

we then apply the result to the function $f/\|f\|_{L^p}$ for any $f \in L^p$ to get

$$\|T(f/\|f\|_{L^p}) = \|_{L^q} \leq M,$$

Lemma 5.1.14 and remark 5.1.15 assert that

$$\|Tf\|_{L^q} = \sup_{\substack{\|g\|_{L^{q'}}=1 \\ g \text{ simple}}} \left| \int (Tf) g d\nu \right|$$

so we only need to show for any g simple and $\|g\|_{L^{q'}} = 1$,

$$\left| \int (Tf) g d\nu \right| \leq M\|f\|_{L^p}\|g\|_{L^{q'}}$$

For now, we also assume that $p < \infty$ and $q > 1$. Suppose $f \in L^p$ is simple with $\|f\|_{L^p} = 1$, and define

$$f_z = |f|^{\gamma(z)} \frac{f}{|f|} \quad \text{where} \quad \gamma(z) = p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right),$$

and

$$g_z = |g|^{\delta(z)} \frac{g}{|g|} \quad \text{where} \quad \delta(z) = q'\left(\frac{1-z}{q_0'} + \frac{z}{q_1'}\right),$$

with q', q_0' and q_1' denoting the duals of q, q_0, and q_1 respectively. Then, we note that $f_t = f$. We also observe that if $\mathrm{Re}(z) = 0$, i.e., $z = yi$, then

$$\begin{aligned} |f_z|^{p_0} \left| |f|^{\gamma(z)} \right|^{p_0} &= \left| e^{\gamma(yi) \log |f|} \right|^{p_0} \\ &= \left| e^{p\left(\frac{1-yi}{p_0} + \frac{yi}{p_1}\right) \log |f|} \right|^{p_0} \\ &= \left| e^{p\left(\frac{1}{p_0} + i\left(\frac{yi}{p_1} - \frac{y}{p_0}\right)\right) \log |f|} \right|^{p_0} \\ &= e^{p \cdot \frac{1}{p_0} \cdot \log |f| \cdot p_0} = |f|^p \end{aligned}$$

and consequently $\|f\|_{L^p} = 1$ implies that

$$\|f_z\|_{L^{p_0}} = \left(\int |f_z|^{p_0}\right)^{1/p_0} = \left(\int |f|^p\right)^{1/p_0} = (\|f\|_{L^p})^{p/p_0} = 1$$

One can also compute that $\mathrm{Re}(z) = 1$ results in $\|f_z\|_{L^{p_1}} = 1$, and there are analogous results for g_z. We summarize them below.
- $f_t = f$ and

$$\begin{cases} \|f_z\|_{L^{p_0}} = 1 & \text{if } \mathrm{Re}(z) = 0 \\ \|f_z\|_{L^{p_1}} = 1 & \text{if } \mathrm{Re}(z) = 1. \end{cases}$$

- $g_t = g$ and

$$\begin{cases} \|g_z\|_{L^{p_0'}} = 1 & \text{if } \mathrm{Re}(z) = 0 \\ \|g_z\|_{L^{p_1'}} = 1 & \text{if } \mathrm{Re}(z) = 1. \end{cases}$$

The trick now is to consider

$$\Phi(z) = \int (Tf_z)\, g_z d\nu$$

Since f is a finite sum, $f = \sum a_k \chi_{E_k}$ where the sets E_k are disjoint and of finite measure, then f_z is also simple with

$$f_z = \sum |a_k|^{\gamma(z)} \frac{a_k}{|a_k|} \chi_{E_k}.$$

Since $g = \sum b_j \chi_{F_j}$ is also simple, then

$$g_z = \sum |b_j|^{\delta(z)} \frac{b_j}{|b_j|} \chi_{F_j}.$$

With the above notation, we find

$$\Phi(z) = \sum_{j,k} |a_k|^{\gamma(z)} |b_j|^{\delta(z)} \frac{a_k}{|a_k|} \frac{b_j}{|b_j|} \left(\int T(\chi_{E_k}) \chi_{F_j} d\nu\right),$$

so that the function Φ is a holomorphic function in the strip $0 < \mathrm{Re}(z) < 1$ that is bounded and continuous in its closure. After an application of Hölder's inequality and using the fact that T is bounded on L^{p_0} with bound M_0, we find that if $\mathrm{Re}(z) = 0$, then

$$|\Phi(z)| \le \|Tf_z\|_{L^{q_0}} \|g_z\|_{L^{q_0'}} \le M_0 \|f_z\|_{L^{p_0}} = M_0.$$

Similarly we find $|\Phi(z)| \le M_1$ on the line $\mathrm{Re}(z) = 1$. Therefore, by the Hadamard three-lines theorem 3.6.5, we conclude that Φ is bounded by $M_0^{1-t} M_1^t$ on the line $\mathrm{Re}(z) = t$. Since $\Phi(t) = \int (Tf) g d\nu$, this gives the desired result, at least when f is simple.

In general, when $f \in L^p$ with $1 \le p < \infty$, we may choose a sequence $\{f_n\}$ of simple functions in L^p so that $\|f_n - f\|_{L^p} \to 0$ (as in Exercise 6 in Chapter 1 of [7]). Since $\|T(f_n)\|_{L^q} \le M \|f_n\|_{L^p}$, we find that $T(f_n)$ is a Cauchy sequence in L^q and if we can show that $\lim_{n\to\infty} T(f_n) = T(f)$ almost everywhere, it would follow that we also have $\|T(f)\|_{L^q} \le M\|f\|_{L^p}$.

To do this, write $f = f^U + f^L$, where $f^U(x) = f(x)$ if $|f(x)| \ge 1$ and 0 elsewhere, while $f^L(x) = f(x)$ if $|f(x)| < 1$ and 0 elsewhere. Similarly, set $f_n = f_n^U + f_n^L$. Now assume that $p_0 \le p_1$ (the case $p_0 \ge p_1$ is parallel). Then $p_0 \le p \le p_1$, and since $f \in L^p$, it follows that $f^U \in L^{p_0}$ and $f^L \in L^{p_1}$. Moreover, since $f_n \to f$ in the L^p norm, then $f_n^U \to f^U$ in the L^{p_0} norm and $f_n^L \to f^L$ in the L^{p_1} norm. By hypothesis, then

$T\left(f_n^U\right) \rightarrow T\left(f^U\right)$ in L^{q_0} and $T\left(f_n^L\right) \rightarrow T\left(f^L\right)$ in L^{q_1}, and selecting appropriate subsequences we see that $T\left(f_n\right)=T\left(f_n^U\right)+T\left(f_n^L\right)$ converges to $T(f)$ almost everywhere, which establishes the claim.

It remains to consider the cases $q=1$ and $p=\infty$. In the latter case then necessarily $p_0=p_1=\infty$, and the hypotheses $\|T(f)\|_{L^{q_0}} \leq M_0\|f\|_{L^\infty}$ and $\|T(f)\|_{L^{q_1}} \leq M_1\|f\|_{L^\infty}$ imply the conclusion

$$\|T(f)\|_{L^q} \leq M_0^{1-t} M_1^t\|f\|_{L^\infty}$$

by Hölder's inequality (as in Exercise 20 in Chapter 1 of [7]). Finally if $p<\infty$ and $q=1$, then $q_0=q_1=1$, then we may take $g_z=g$ for all z, and argue as in the case when $q>1$. This completes the proof of the theorem. □

We shall now describe a slightly different but useful way of stating the essence of the theorem. Here we assume that our linear operator T is initially defined on simple functions of X, mapping these to functions on Y that are integrable on sets of finite measure. We then ask: for which (p, q) is the operator of **type** (p, q), in the sense that there is a bound M so that

$$\|T(f)\|_{L^q} \leq M\|f\|_{L^p}, \quad \text{whenever } f \text{ is simple?} \tag{3.6}$$

In this formulation of the question, the useful role of simple functions is that they are at once common to all the L^p spaces. Moreover, if (3.6) holds then T has a unique extension to all of L^p, with the same bound M in (3.6), as long as either $p<\infty$; or $p=\infty$ in the case X has finite measure. This is a consequence of the density of the simple functions in L^p, and the extension argument in Proposition 5.4 of Chapter 1 of [7].

With these remarks in mind, we define the **Riesz diagram** of T to consist of all all points in the unit square $\{(x, y): 0 \leq x \leq 1,0 \leq y \leq 1\}$ that arise when we set $x=1/p$ and $y=1/q$ whenever T is of type (p, q). We then also define $M_{x, y}$ as the least M for which (8) holds when $x=1/p$ and $y=1/q$.

Corollary 3.7.2. With T as before:
(a) The Riesz diagram of T is a convex set.
(b) $\log M_{x, y}$ is a convex function on this set.

Conclusion (a) means that if $\left(x_0, y_0\right)=\left(1/p_0, 1/q_0\right)$ and $\left(x_1, y_1\right)=\left(1/p_1, 1/q_1\right)$ are points in the Riesz diagram of T, then so is the line segment joining them. This is an immediate consequence of Riesz interpolation. Similarly the convexity of the function $\log M_{x, y}$ is its convexity on each line segment, and this follows from the conclusion $M \leq M_0^{1-t} M_1^t$ guaranteed also by Riesz interpolation.

In view of this corollary, the theorem is often referred to as the "Riesz convexity theorem."

Example 3.7.3. The first application of Riesz interpolation is the Hausdorff-Young inequality (3.4). Here X is $[0,2\pi]$ with the normalized Lebesgue measure $d\theta/(2\pi)$, and $Y=\mathbb{Z}$ with its usual counting measure. The mapping T is defined by $T(f)=\{a_n\}$, with

$$a_n=\frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta.$$

Corollary 3.7.4. If $1 \leq p \leq 2$ and $1/p+1/q=1$, then

$$\|T(f)\|_{L^q(\mathbb{Z})} \leq\|f\|_{L^p([0,2\pi])}.$$

Note that since $L^2([0,2\pi]) \subset L^1([0,2\pi])$ and $L^2(\mathbb{Z}) \subset L^\infty(\mathbb{Z})$ we have $L^2([0,2\pi])+L^1([0,2\pi])=L^1([0,2\pi])$, and also $L^2(\mathbb{Z})+L^\infty(\mathbb{Z})=L^\infty(\mathbb{Z})$.

The inequality for $p_0=q_0=2$ is a consequence of Parseval's identity, while the one for $p_1=1, q_1=\infty$ follows from the observation that for all n,

$$|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi}|f(\theta)| d\theta$$

Thus Riesz's theorem guarantees the conclusion when $1/p = \frac{(1-t)}{2} + t$, $1/q = \frac{(1-t)}{2}$ for any t with $0 \leq t \leq 1$. This gives all p with $1 \leq p \leq 2$, and q related to p by $1/p + 1/q = 1$.

Example 3.7.5. We next come to the dual Hausdorff-Young inequality (3.5). Here we define the operator T' mapping functions on $\mathbb{Z}$ to functions on $[0, 2\pi]$ by

$$T'\left(\{a_n\}\right) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

Notice that since $L^p(\mathbb{Z}) \subset L^2(\mathbb{Z})$ when $p \leq 2$, then the above is a welldefined function on $L^2([0, 2\pi])$ when $\{a_n\} \in L^p(\mathbb{Z})$, by the unitary character of Parseval's identity.

Corollary 3.7.6. If $1 \leq p \leq 2$ and $1/p + 1/q = 1$, then

$$\left\|T'\left(\{a_n\}\right)\right\|_{L^q([0,2\pi])} \leq \left\|\{a_n\}\right\|_{L^p(\mathbb{Z})}.$$

The proof is parallel to that of the previous corollary. The case $p_0 = q_0 = 2$ is, as has already been mentioned, a consequence of Parseval's identity, while the case $p_1 = 1$ and $q_1 = \infty$ follows directly from the fact that

$$\left|\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right| \leq \sum_{n=-\infty}^{\infty} |a_n|$$

3.7.3 Fourier Transform and Paley-Wiener Theorem

We recall that the **Fourier transform (FT)** $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ of a function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \xi} dx, \ \xi \in \mathbb{R}^d. \tag{3.7}$$

The **inversion** is

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \ x \in \mathbb{R}^d. \tag{3.8}$$

the most elegant and useful formulations of Fourier inversion are in terms of the L^2 theory, or in its greatest generality stated in the language of distributions. We are satisfied by the following results. See Stein's third book p.86 for proofs.

Proposition 3.7.7. Suppose $f \in L^1(\mathbb{R}^d)$. Then $\widehat{f}$ defined by (3.7) is continuous and bounded in $\mathbb{R}^d$.

Proposition 3.7.8. Suppose $f \in L^1(\mathbb{R}^d)$ and assume also that $\widehat{f} \in L^1(\mathbb{R}^d)$. Then the inversion formula (3.8) holds for almost every x.

Corollary 3.7.9. Suppose $\widehat{f}(\xi) = 0$ for all ξ. Then $f = 0$ a.e.

We consider the analog of Hausdorff-Young for the Fourier transform. Here the setting is $\mathbb{R}^d$ and the L^p spaces are taken with respect to the usual Lebesgue measure. We initially define the Fourier transform (denoted here by T) on simple functions by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Then clearly, $\|\mathcal{F}(f)\|_{L^\infty} \leq \|f\|_{L^1}$, and $\mathcal{F}$ has an extension (by Proposition 5.4 in Chapter 1 for instance) to $L^1\left(\mathbb{R}^d\right)$ for which this inequality continues to hold. Also, $\mathcal{F}$ has an extension to $L^2\left(\mathbb{R}^d\right)$ as a unitary mapping. (This is essentially the content of Plancherel's theorem. See Section 1, Chapter 5 in Book III.) Thus in particular $\|\mathcal{F}(f)\|_{L^2} \leq \|f\|_{L^2}$, for f simple. The same arguments as before then prove:

Corollary 3.7.10 (Hausdorff-Young)**.** If $1 \le p \le 2$ and $1/p + 1/q = 1$, then the Fourier transform $\mathcal{F}$ defines a linear mapping $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ has a unique extension to a bounded map $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$, with $\|T(f)\|_{L^q} \le \|f\|_{L^p}$.

Proof. $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ with $\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)}$ becasue $\forall \xi$,

$$|\widehat{f}(\xi)| = \left| \int_{\mathbb{R}^d} f(x) e^{2\pi i x\xi} dx \right| \le \int |f(x)e^{2\pi i x \xi}| dx = \|f\|_{L^1(\mathbb{R}^d)}$$

Also, $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with $\|\widehat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R}^d)}$ by Plancherel. We will use Riesz interpolation theorem with $p_0 = 1$, $q_0 = \infty$, $M_0 = 1$ and $p_1 = q_1 = 2$, $M_1 = 1$.

Fix $1 \le p \le 2$. Notice that then $q \ge 2$ implying that $1/q \le 1/2$ and thus $t := 2/q \le 1$. We deifne

$$p_t = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

as in Riesz's theorem. With the t we defined, we have

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} = \frac{1-2/q}{1} + \frac{2/q}{2} = 1 - \frac{2}{q} + \frac{1}{q} = 1 - \frac{1}{q} = \frac{1}{p}$$

so $p_t = p$. Similar computation gives $q_t = q$.

Riesz's theorem gives

$$\|\widehat{f}\|_{L^q} = \|\widehat{f}\|_{L^{q_t}} \le M_0^{1-t} M_1^t \|f\|_{L^{p_t}} = \|f\|_p$$

$\square$

Chapter 4

Harmonic Functions

chapter 2 of Stein also contains more topics including hardy space

Chapter 5

Appendix

We copy the first chapter of [7] to smoothen our discussion.

5.1 L^p Spaces and Banach Spaces

Let $(X, \mathcal{A})$ be a measurable space and μ a measure on it. The measure μ is called a **σ-finite measure**, if it satisfies one of the four following equivalent criteria:
1. the set X can be covered with at most countably many measurable sets with finite measure. This means that there are sets $A_1, A_2, \ldots \in \mathcal{A}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ that satisfy $\bigcup_{n \in \mathbb{N}} A_n = X$.
2. the set X can be covered with at most countably many measurable disjoint sets with finite measure. This means that there are sets $B_1, B_2, \ldots \in \mathcal{A}$ with $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$ and $B_i \cap B_j = \varnothing$ for $i \neq j$ that satisfy $\bigcup_{n \in \mathbb{N}} B_n = X$.
3. the set X can be covered with a monotone sequence of measurable sets with finite measure. This means that there are sets $C_1, C_2, \ldots \in \mathcal{A}$ with $C_1 \subset C_2 \subset \cdots$ and $\mu(C_n) < \infty$ for all $n \in \mathbb{N}$ that satisfy $\bigcup_{n \in \mathbb{N}} C_n = X$.
4. there exists a strictly positive measurable function f whose integral is finite. This means that $f(x) > 0$ for all $x \in X$ and $\int f(x) \mu(\mathrm{d}x) < \infty$.

If μ is a σ-finite measure, the measure space $(X, \mathcal{A}, \mu)$ is called a **σ-finite measure space**.

Throughout this section $(X, \mathcal{F}, \mu)$ denotes a σ-finite measure space: X denotes the underlying space, $\mathcal{F}$ the σ-algebra of measurable sets, and μ the measure. If $1 \leq p < \infty$, the space $L^p(X, \mathcal{F}, \mu)$ consists of all comple-xvalued measurable functions on X that satisfy

$$\int_X |f(x)|^p d\mu(x) < \infty. \tag{5.1}$$

To simplify the notation, we write $L^p(X, \mu)$, or $L^p(X)$, or simply L^p when the underlying measure space has been specified. Then, if $f \in L^p(X, \mathcal{F}, \mu)$ we define the L^p norm of f by

$$\|f\|_{L^p(X, \mathcal{F}, \mu)} = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}.$$

We also abbreviate this to $\|f\|_{L^p(X)}, \|f\|_{L^p}$, or $\|f\|_p$. When $p = 1$ the space $L^1(X, \mathcal{F}, \mu)$ consists of all integrable functions on X, and we have shown in Chapter 6 of Book III, that L^1 together with $\|\cdot\|_{L^1}$ is a complete normed vector space. Also, the case $p = 2$ warrants special attention: it is a Hilbert space.

We note here that we encounter the same technical point that we already discussed in Book III. The problem is that $\|f\|_{L^p} = 0$ does not imply that $f = 0$, but merely $f = 0$ almost everywhere (for the measure μ). Therefore, the precise definition of L^p requires introducing the equivalence relation, in which f and g are equivalent if $f = g$ a.e. Then, L^p consists of all equivalence classes of functions which satisfy (5.1). However, in practice there is little risk of error by thinking of elements in L^p as functions rather than equivalence classes of functions.

The following are some common examples of L^p spaces.
(a) The case $X = \mathbb{R}^d$ and μ equals Lebesgue measure is often used in practice. There, we have

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}.$$

(b) Also, one can take $X = \mathbb{Z}$, and μ equal to the counting measure. Then, we get the "discrete" version of the L^p spaces. Measurable functions are simply sequences $f = \{a_n\}_{n\in\mathbb{Z}}$ of complex numbers, and

$$\|f\|_{L^p} = \left(\sum_{n=-\infty}^{\infty} |a_n|^p\right)^{1/p}.$$

When $p = 2$, we recover the familiar sequence space $\ell^2(\mathbb{Z})$. The spaces L^p are examples of normed vector spaces. The basic property satisfied by the norm is the triangle inequality, which we shall prove shortly.

The range of p which is of interest in most applications is $1 \leq p < \infty$, and later also $p = \infty$. There are at least two reasons why we restrict our attention to these values of p : when $0 < p < 1$, the function $\|\cdot\|_{L^p}$ does not satisfy the triangle inequality, and moreover, for such p, the space L^p has no non-trivial bounded linear functionals.

When $p = 1$ the norm $\|\cdot\|_{L^1}$ satisfies the triangle inequality, and L^1 is a complete normed vector space. When $p = 2$, this result continues to hold, although one needs the Cauchy-Schwarz inequality to prove it. In the same way, for $1 \leq p < \infty$ the proof of the triangle inequality relies on a generalized version of the Cauchy-Schwarz inequality. This is Hölder's inequality, which is also the key in the duality of the L^p spaces, as we will see in subsection 4.

The Hölder and Minkowski inequalities

If the two exponents p and q satisfy $1 \leq p, q \leq \infty$, and the relation

$$\frac{1}{p} + \frac{1}{q} = 1$$

holds, we say that p and q are conjugate or dual exponents. Here, we use the convention $1/\infty = 0$. Later, we shall sometimes use p' to denote the conjugate exponent of p. Note that $p = 2$ is self-dual, that is, $p = q = 2$; also $p = 1, \infty$ corresponds to $q = \infty, 1$ respectively.

Theorem 5.1.1 (Hölder)**.** Suppose $1 < p < \infty$ and $1 < q < \infty$ are conjugate exponents. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Note. Once we have defined L^∞ (see subsection 2) the corresponding inequality for the exponents 1 and ∞ will be seen to be essentially trivial.

The proof of the theorem relies on a simple generalized form of the arithmetic-geometric mean inequality: if $A, B \geq 0$, and $0 \leq \theta \leq 1$, then

$$A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B. \tag{5.2}$$

Note that when $\theta = 1/2$, the inequality (5.2) states the familiar fact that the geometric mean of two numbers is majorized by their arithmetic mean.

To establish (2), we observe first that we may assume $B \neq 0$, and replacing A by AB, we see that it suffices to prove that $A^\theta \leq \theta A + (1 - \theta)$. If we let $f(x) = x^\theta - \theta x - (1 - \theta)$, then $f'(x) = \theta\left(x^{\theta-1} - 1\right)$. Thus $f(x)$ increases when $0 \leq x \leq 1$ and decreases when $1 \leq x$, and we see that the continuous function f attains a maximum at $x = 1$, where $f(1) = 0$. Therefore $f(A) \leq 0$, as desired.

To prove Hölder's inequality we argue as follows. If either $\|f\|_{L^p} = 0$ or $\|f\|_{L^q} = 0$, then $fg = 0$ a.e. and the inequality is obviously verified. Therefore, we may assume that neither of these norms vanish, and after replacing f by $f/\|f\|_{L^p}$ and g by $g/\|g\|_{L^q}$, we may further assume that $\|f\|_{L^p} = \|g\|_{L^q} = 1$. We now need to prove that $\|fg\|_{L^1} \leq 1$.

If we set $A = |f(x)|^p, B = |g(x)|^q$, and $\theta = 1/p$ so that $1 - \theta = 1/q$, then (25.2) gives

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q.$$

Integrating this inequality yields $\|fg\|_{L^1} \leq 1$, and the proof of the Hölder inequality is complete.

We are now ready to prove the triangle inequality for the L^p norm.

Theorem 5.1.2 (Minkowski). If $1 \leq p < \infty$ and $f, g \in L^p$, then $f + g \in L^p$ and $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.

Proof. The case $p = 1$ is obtained by integrating $|f(x) + g(x)| \leq |f(x)| + |g(x)|$. When $p > 1$, we may begin by verifying that $f + g \in L^p$, when both f and g belong to L^p. Indeed,

$$|f(x) + g(x)|^p \leq 2^p \left(|f(x)|^p + |g(x)|^p\right),$$

as can be seen by considering separately the cases $|f(x)| \leq |g(x)|$ and $|g(x)| \leq |f(x)|$. Next we note that

$$|f(x) + g(x)|^p \leq |f(x)||f(x) + g(x)|^{p-1} + |g(x)||f(x) + g(x)|^{p-1}.$$

If q denotes the conjugate exponent of p, then $(p - 1)q = p$, so we see that $(f + g)^{p-1}$ belongs to L^q, and therefore Hölder's inequality applied to the two terms on the right-hand side of the above inequality gives

$$\|f + g\|_{L^p}^p \leq \|f\|_{L^p} \left\|(f + g)^{p-1}\right\|_{L^q} + \|g\|_{L^p} \left\|(f + g)^{p-1}\right\|_{L^q}. \tag{5.3}$$

However, using once again $(p - 1)q = p$, we get

$$\left\|(f + g)^{p-1}\right\|_{L^q} = \|f + g\|_{L^p}^{p/q}.$$

From (5.3), since $p - p/q = 1$, and because we may suppose that $\|f + g\|_{L^p} > 0$, we find

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

so the proof is finished. □

Completeness of L^p

The triangle inequality makes L^p into a metric space with distance $d(f, g) = \|f - g\|_{L^p}$. The basic analytic fact is that L^p is complete in the sense that every Cauchy sequence in the norm $\|\cdot\|_{L^p}$ converges to an element in L^p. Taking limits is a necessity in many problems, and the L^p spaces would be of little use if they were not complete. Fortunately, like L^1 and L^2, the general L^p space does satisfy this desirable property.

Theorem 5.1.3. The space $L^p(X, \mathcal{F}, \mu)$ is complete in the norm $\|\cdot\|_{L^p}$.

Proof. The argument is essentially the same as for L^1 (or L^2); see Section 2, Chapter 2 and Section 1, Chapter 4 in Book III. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in L^p, and consider a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ with the following property $\left\|f_{n_{k+1}}-f_{n_k}\right\|_{L^p} \leq 2^{-k}$ for all $k \geq 1$. We now consider the series whose convergence will be seen below

$$f(x)=f_{n_1}(x)+\sum_{k=1}^{\infty}\left(f_{n_{k+1}}(x)-f_{n_k}(x)\right)$$

and

$$g(x)=\left|f_{n_1}(x)\right|+\sum_{k=1}^{\infty}\left|f_{n_{k+1}}(x)-f_{n_k}(x)\right|,$$

and the corresponding partial sums

$$S_K(f)(x)=f_{n_1}(x)+\sum_{k=1}^{K}\left(f_{n_{k+1}}(x)-f_{n_k}(x)\right)$$

and

$$S_K(g)(x)=\left|f_{n_1}(x)\right|+\sum_{k=1}^{K}\left|f_{n_{k+1}}(x)-f_{n_k}(x)\right|.$$

The triangle inequality for L^p implies

$$\begin{aligned}
\left\|S_K(g)\right\|_{L^p} & \leq\left\|f_{n_1}\right\|_{L^p}+\sum_{k=1}^{K}\left\|f_{n_{k+1}}-f_{n_k}\right\|_{L^p} \\
& \leq\left\|f_{n_1}\right\|_{L^p}+\sum_{k=1}^{K} 2^{-k}.
\end{aligned}$$

Letting K tend to infinity, and applying the monotone convergence theorem proves that $\int g^p<\infty$, and therefore the series defining g, and hence the series defining f converges almost everywhere, and $f \in L^p$.

We now show that f is the desired limit of the sequence $\{f_n\}$. Since (by construction of the telescopic series) the $(K-1)^{\text{th}}$ partial sum of this series is precisely f_{n_K}, we find that

$$f_{n_K}(x) \rightarrow f(x) \quad \text{a.e. } x.$$

To prove that $f_{n_K} \rightarrow f$ in L^p as well, we first observe that

$$\begin{aligned}
\left|f(x)-S_K(f)(x)\right|^p & \leq\left[2 \max \left(|f(x)|,\left|S_K(f)(x)\right|\right)\right]^p \\
& \leq 2^p|f(x)|^p+2^p\left|S_K(f)(x)\right|^p \\
& \leq 2^{p+1}|g(x)|^p
\end{aligned}$$

for all K. Then, we may apply the dominated convergence theorem to get $\left\|f_{n_K}-f\right\|_{L^p} \rightarrow 0$ as K tends to infinity.

Finally, the last step of the proof consists of recalling that $\{f_n\}$ is Cauchy. Given $\epsilon>0$, there exists N so that for all $n, m>N$ we have $\left\|f_n-f_m\right\|_{L^p}<\epsilon/2$. If n_K is chosen so that $n_K>N$, and $\left\|f_{n_K}-f\right\|_{L^p}<\epsilon/2$, then the triangle inequality implies

$$\left\|f_n-f\right\|_{L^p} \leq\left\|f_n-f_{n_K}\right\|_{L^p}+\left\|f_{n_K}-f\right\|_{L^p}<\epsilon$$

whenever $n>N$. This concludes the proof of the theorem. ∎

Further remarks

We begin by looking at some possible inclusion relations between the various L^p spaces. The matter is simple if the underlying space has finite measure.

Proposition 5.1.4. If X has finite positive measure, and $p_0 \leq p_1$, then $L^{p_1}(X) \subset L^{p_0}(X)$ and

$$\frac{1}{\mu(X)^{1/p_0}}\|f\|_{L^{p_0}} \leq \frac{1}{\mu(X)^{1/p_1}}\|f\|_{L^{p_1}}.$$

We may assume that $p_1 > p_0$. Suppose $f \in L^{p_1}$, and set $F = |f|^{p_0}$, $G = 1, p = p_1/p_0 > 1$, and $1/p + 1/q = 1$, in Hölder's inequality applied to F and G. This yields

$$\|f\|_{L^{p_0}}^{p_0 0} \leq \left(\int |f|^{p_1}\right)^{p_0/p_1} \cdot \mu(X)^{1-p_0/p_1}.$$

In particular, we find that $\|f\|_{L^{p_0}} < \infty$. Moreover, by taking the p_0^{th} root of both sides of the above equation, we find that the inequality in the proposition holds.

However, as is easily seen, such inclusion does not hold when X has infinite measure. Yet, in an interesting special case the opposite inclusion does hold.

Proposition 5.1.5. If $X = \mathbb{Z}$ is equipped with counting measure, then the reverse inclusion holds, namely $L^{p_0}(\mathbb{Z}) \subset L^{p_1}(\mathbb{Z})$ if $p_0 \leq p_1$. Moreover, $\|f\|_{L^{p_1}} \leq \|f\|_{L^{p_0}}$.

Indeed, if $f = \{f(n)\}_{n\in\mathbb{Z}}$, then $\sum |f(n)|^{p_0} = \|f\|_{L^{p_0}}^{p_0}, \text{and}_{\sup_n} |f(n)| \leq \|f\|_{L^{p_0}}$. However

$$\begin{aligned}\sum |f(n)|^{p_1} &= \sum |f(n)|^{p_0}|f(n)|^{p_1-p_0} \\ &\left(\leq \sup_n |f(n)|\right)^{p_1-p_0} \|f\|_{L^{p_0}}^{p_0} \\ &\leq \|f\|_{L^{p_0}}^{p_1}\end{aligned}$$

Thus $\|f\|_{L^{p_1}} \leq \|f\|_{L^{p_0}}$.

The case $p = \infty$

Finally, we also consider the limiting case $p = \infty$. The space L^∞ will be defined as all functions that are "essentially bounded" in the following sense. We take the space $L^\infty(X, \mathcal{F}, \mu)$ to consist of all (equivalence classes of) measurable functions on X, so that there exists a positive number $0 < M < \infty$, with

$$|f(x)| \leq M \quad \text{a.e. } x.$$

Then, we define $\|f\|_{L^\infty(X,\mathcal{F},\mu)}$ to be the infimum of all possible values M satisfying the above inequality. The quantity $\|f\|_{L^\infty}$ is sometimes called the **essential-supremum** of f.

We note that with this definition, we have $|f(x)| \leq \|f\|_{L^\infty}$ for a.e. x. Indeed, if $E = \{x : |f(x)| > \|f\|_{L^\infty}\}$, and $E_n = \{x : |f(x)| > \|f\|_{L^\infty} + 1/n\}$, then we have $\mu(E_n) = 0$, and $E = \bigcup E_n$, hence $\mu(E) = 0$.

Theorem 5.1.6. The vector space L^∞ equipped with $\|\cdot\|_{L^\infty}$ is a complete vector space.

This assertion is easy to verify and is left to the reader. Moreover, Hölder's inequality continues to hold for values of p and q in the larger range $1 \leq p, q \leq \infty$, once we take $p = 1$ and $q = \infty$ as conjugate exponents, as we mentioned before.

The fact that L^∞ is a limiting case of L^p when p tends to ∞ can be understood as follows.

Proposition 5.1.7. Suppose $f \in L^\infty$ is supported on a set of finite measure. Then $f \in L^p$ for all $p < \infty$, and

$$\|f\|_{L^p} \to \|f\|_{L^\infty} \quad \text{as } p \to \infty.$$

Proof. Let E be a measurable subset of X with $\mu(E) < \infty$, and so that f vanishes in the complement of E. If $\mu(E) = 0$, then $\|f\|_{L^\infty} = \|f\|_{L^p} = 0$ and there is nothing to prove. Otherwise

$$\|f\|_{L^p} = \left(\int_E |f(x)|^p d\mu\right)^{1/p} \leq \left(\int_E \|f\|_{L^\infty}^p d\mu\right)^{1/p} \leq \|f\|_{L^\infty} \mu(E)^{1/p}.$$

Since $\mu(E)^{1/p} \to 1$ as $p \to \infty$, we find that $\limsup_{p\to\infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}$. On the other hand, given $\epsilon > 0$, we have

$$\mu\left(\{x : |f(x)| \geq \|f\|_{L^\infty} - \epsilon\}\right) \geq \delta \quad \text{for some } \delta > 0,$$

hence

$$\int_X |f|^p d\mu \geq \delta\left(\|f\|_{L^\infty} - \epsilon\right)^p.$$

Therefore $\liminf \inf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty} - \epsilon$, and since ϵ is arbitrary, we have $\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}$. Hence the limit $\lim_{p\to\infty} \|f\|_{L^p}$ exists, and equals $\|f\|_{L^\infty}$. □

Banach spaces

We introduce here a general notion which encompasses the L^p spaces as specific examples.

First, a **normed vector space** consists of an underlying vector space V over a field of scalars (the real or complex numbers), together with a norm $\|\cdot\| : V \to \mathbb{R}^+$that satisfies:
- $\|v\| = 0$ if and only if $v = 0$.
- $\|\alpha v\| = |\alpha|\|v\|$, whenever α is a scalar and $v \in V$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

The space V is said to be **complete** if whenever $\{v_n\}$ is a Cauchy sequence in V, that is, $\|v_n - v_m\| \to 0$ as $n, m \to \infty$, then there exists a $v \in V$ such that $\|v_n - v\| \to 0$ as $n \to \infty$.

A complete normed vector space is called a **Banach space**. Here again, we stress the importance of the fact that Cauchy sequences converge to a limit in the space itself, hence the space is "closed" under limiting operations.

Examples

The real numbers $\mathbb{R}$ with the usual absolute value form an initial example of a Banach space. Other easy examples are $\mathbb{R}^d$, with the Euclidean norm, and more generally a Hilbert space with its norm given in terms of its inner product. Several further relevant examples are as follows:

Example 5.1.8. The family of L^p spaces with $1 \leq p \leq \infty$ which we have just introduced are also important examples of Banach spaces. Incidentally, L^2 is the only Hilbert space in the family L^p, where $1 \leq p \leq \infty$ (Exercise 25) and this in part accounts for the special flavor of the analysis carried out in L^2 as opposed to L^1 or more generally L^p for $p \neq 2$.

Finally, observe that since the triangle inequality fails in general when $0 < p < 1$, $\|\cdot\|_{L^p}$ is not a norm on L^p for this range of p, hence it is not a Banach space.

Example 5.1.9. Another example of a Banach space is $C([0, 1])$, or more generally $C(X)$ with X a compact set in a metric space. By definition, $C(X)$ is the vector space of continuous functions on X equipped with the sup-norm $\|f\| = \sup_{x\in X} |f(x)|$. Completeness is guaranteed by the fact that the uniform limit of a sequence of continuous functions is also continuous.

Example 5.1.10. The space $\Lambda^\alpha\left(\mathbb{R}^d\right)$ of all continuous functions on $\mathbb{R}^d$ with the norm

$$\|f\|_{\Lambda^\alpha(\mathbb{R}^d)}=\sup_{x\in\mathbb{R}^d}|f(x)|+\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

is a Banach space.

The space $L_k^p\left(\mathbb{R}^d\right)$ is the subspace of $L^p\left(\mathbb{R}^d\right)$ of all functions that have weak derivatives up to order k. This space is usually referred to as a Sobolev space. A norm that turns $L_k^p\left(\mathbb{R}^d\right)$ into a Banach space is

$$\|f\|_{L_k^p(\mathbb{R}^d)}=\sum_{|\alpha|\leq k}\|\partial_x^\alpha f\|_{L^p(\mathbb{R}^d)}$$

Linear functionals and the dual of a Banach space

For the sake of simplicity, we restrict ourselves in this and the following two sections to Banach spaces over $\mathbb{R}$; the reader will find in Section 6 the slight modifications necessary to extend the results to Banach spaces over $\mathbb{C}$.

Suppose that $\mathcal{B}$ is a Banach space over $\mathbb{R}$ equipped with a norm $\|\cdot\|$. A **linear functional** is a linear mapping ℓ from $\mathcal{B}$ to $\mathbb{R}$, that is, $\ell:\mathcal{B}\to\mathbb{R}$, which satisfies

$$\ell(\alpha f+\beta g)=\alpha\ell(f)+\beta\ell(g),\quad \text{for all } \alpha,\beta\in\mathbb{R}, \text{ and } f,g\in\mathcal{B}.$$

A linear functional ℓ is **continuous** if given $\epsilon>0$ there exists $\delta>0$ so that $|\ell(f)-\ell(g)|\leq\epsilon$ whenever $\|f-g\|\leq\delta$. Also we say that a linear functional is **bounded** if there is $M>0$ with $|\ell(f)|\leq M\|f\|$ for all $f\in\mathcal{B}$. The linearity of ℓ shows that these two notions are in fact equivalent.

Proposition 5.1.11. A linear functional on a Banach space is continuous, if and only if it is bounded.

Proof. The key is to observe that ℓ is continuous if and only if ℓ is continuous at the origin.

Indeed, if ℓ is continuous, we choose $\epsilon=1$ and $g=0$ in the above definition so that $|\ell(f)|\leq 1$ whenever $\|f\|\leq\delta$, for some $\delta>0$. Hence, given any non-zero h, an element of $\mathcal{B}$, we see that $\delta h/\|h\|$ has norm equal to δ, and hence $|\ell(\delta h/\|h\|)|\leq 1$. Thus $|\ell(h)|\leq M\|h\|$ with $M=1/\delta$.

Conversely, if ℓ is bounded it is clearly continuous at the origin, hence continuous. □

The significance of continuous linear functionals in terms of closed hyperplanes in $\mathcal{B}$ is a noteworthy geometric point to which we return later on. Now we take up analytic aspects of linear functionals.

The set of all continuous linear functionals over $\mathcal{B}$ is a vector space since we may add linear functionals and multiply them by scalars:

$$\left(\ell_1+\ell_2\right)(f)=\ell_1(f)+\ell_2(f)\quad\text{and}\quad(\alpha\ell)(f)=\alpha\ell(f).$$

This vector space may be equipped with a norm as follows. The norm $\|\ell\|$ of a continuous linear functional ℓ is the infimum of all values M for which $|\ell(f)|\leq M\|f\|$ for all $f\in\mathcal{B}$. From this definition and the linearity of ℓ it is clear that

$$\|\ell\|=\sup_{\|f\|\leq 1}|\ell(f)|=\sup_{\|f\|=1}|\ell(f)|=\sup_{f\neq 0}\frac{|\ell(f)|}{\|f\|}.$$

The vector space of all continuous linear functionals on $\mathcal{B}$ equipped with $\|\cdot\|$ is called the **dual space** of $\mathcal{B}$, and is denoted by $\mathcal{B}^*$.

Theorem 5.1.12. The vector space $\mathcal{B}^*$ is a Banach space with the norm $\|\cdot\|$.

In general, given a Banach space $\mathcal{B}$, it is interesting and very useful to be able to describe its dual $\mathcal{B}^*$. This problem has an essentially complete answer in the case of the L^p spaces introduced before.

The dual space of L^p when $1 \le p < \infty$

Suppose that $1 \le p \le \infty$ and q is the conjugate exponent of p, that is, $1/p + 1/q = 1$. The key observation to make is the following: Hölder's inequality shows that every function $g \in L^q$ gives rise to a bounded linear functional on L^p by

$$\ell(f) = \int_X f(x)g(x)d\mu(x), \tag{5.4}$$

and that $\|\ell\| \le \|g\|_{L^q}$. Therefore, if we associate g to ℓ above, then we find that $L^q \subset (L^p)^*$ when $1 \le p \le \infty$. The main result in this section is to prove that when $1 \le p < \infty$, every linear functional on L^p is of the form (5.4) for some $g \in L^q$. This implies that $(L^p)^* = L^q$ whenever $1 \le p < \infty$. We remark that this result is in general not true when $p = \infty$; the dual of L^∞ contains L^1, but it is larger.

Theorem 5.1.13. Suppose $1 \le p < \infty$, and $1/p + 1/q = 1$. Then, with $\mathcal{B} = L^p$ we have

$$\mathcal{B}^* = L^q,$$

in the following sense: For every bounded linear functional ℓ on L^p there is a unique $g \in L^q$ so that

$$\ell(f) = \int_X f(x)g(x)d\mu(x), \quad \text{for all } f \in L^p.$$

Moreover, $\|\ell\|_{\mathcal{B}^*} = \|g\|_{L^q}$.

This theorem justifies the terminology whereby q is usually called the dual exponent of p.

The proof of the theorem is based on two ideas. The first, as already seen, is Hölder's inequality; to which a converse is also needed. The second is the fact that a linear functional ℓ on $L^p, 1 \le p < \infty$, leads naturally to a (signed) measure ν. Because of the continuity of ℓ the measure ν is absolutely continuous with respect to the underlying measure μ, and our desired function g is then the density function of ν in terms of μ. We begin with:

Lemma 5.1.14. Suppose $1 \le p, q \le \infty$, are conjugate exponents.

(i) If $g \in L^q$, then $\|g\|_{L^q} = \sup_{\|f\|_{L^p} \le 1} \left|\int fg\right|$.

(ii) Suppose g is integrable on all sets of finite measure, and

$$\sup_{\substack{\|f\|_{L^p} \le 1 \\ f \text{ simple}}} \left|\int fg\right| = M < \infty.$$

Then $g \in L^q$, and $\|g\|_{L^q} = M$.

For the proof of the lemma, we recall the **signum** of a real number defined by

$$\operatorname{sign}(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Proof. We start with (i). If $g = 0$, there is nothing to prove, so we may assume that g is not 0 a.e., and hence $\|g\|_{L^q} \ne 0$. By Hölder's inequality, we have that

$$\|g\|_{L^q} \ge \sup_{\|f\|_{L^p} \le 1} \left|\int fg\right|.$$

To prove the reverse inequality we consider several cases.

- First, if $q = 1$ and $p = \infty$, we may take $f(x) = \operatorname{sign} g(x)$. Then, we have $\|f\|_{L^\infty} = 1$, and clearly, $\int fg = \|g\|_{L^1}$.
- If $1 < p, q < \infty$, then we set $f(x) = |g(x)|^{q-1} \operatorname{sign} g(x)/\|g\|_{L^q}^{q-1}$. We observe that

$$\|f\|_{L^p}^p = \int |g(x)|^{p(q-1)} d\mu/\|g\|_{L^q}^{p(q-1)} = 1$$

since $p(q-1) = q$, and that

$$\int fg = \|g\|_{L^q}.$$

- Finally, if $q = \infty$ and $p = 1$, let $\epsilon > 0$, and E a set of finite positive measure, where $|g(x)| \geq \|g\|_{L^\infty} - \epsilon$. (Such a set exists by the definition of $\|g\|_{L^\infty}$ and the fact that the measure μ is σ-finite.) Then, if we take $f(x) = \chi_E(x) \operatorname{sign} g(x)/\mu(E)$, where χ_E denotes the characteristic function of the set E, we see that $\|f\|_{L^1} = 1$, and also

$$\left|\int fg\right| = \frac{1}{\mu(E)} \int_E |g| \geq \|g\|_\infty - \epsilon.$$

This completes the proof of part (i). To prove (ii) we recall, e.g. see Section 2 in Chapter 6 of Book III, that we can find a sequence $\{g_n\}$ of simple functions so that $|g_n(x)| \leq |g(x)|$ while $g_n(x) \to g(x)$ for each x. When $p > 1$ (so $q < \infty$), we take $f_n(x) = |g_n(x)|^{q-1} \operatorname{sign} g(x)/ \|g_n\|_{L^q}^{q-1}$. As before, $\|f_n\|_{L^p} = 1$. However

$$\int f_n g = \frac{\int |g_n(x)|^q}{\|g_n\|_{L^q}^{q-1}} = \|g_n\|_{L^q},$$

and this does not exceed M. By Fatou's lemma it follows that $\int |g|^q \leq M^q$, so $g \in L^q$ with $\|g\|_{L^q} \leq M$. The direction $\|g\|_{L^q} \geq M$ is of course implied by Hölder's inequality.

When $p = 1$ the argument is parallel with the above but simpler. Here we take $f_n(x) = (\operatorname{sign} g(x))\chi_{E_n}(x)$, where E_n is an increasing sequence of sets of finite measure whose union is X. The details may be left to the reader.

With the lemma established we turn to the proof of the theorem. It is simpler to consider first the case when the underlying space has finite measure. In this case, with ℓ the given functional on L^p, we can then define a set function ν by

$$\nu(E) = \ell(\chi_E),$$

where E is any measurable set. This definition makes sense because χ_E is now automatically in L^p since the space has finite measure. We observe that

$$|\nu(E)| \leq c(\mu(E))^{1/p}, \tag{5.5}$$

where c is the norm of the linear functional, taking into account the fact that $\|\chi_E\|_{L^p} = (\mu(E))^{1/p}$.

Now the linearity of ℓ clearly implies that ν is finitely-additive. Moreover, if $\{E_n\}$ is a countable collection of disjoint measurable sets, and we put $E = \bigcup_{n=1}^\infty E_n, E_N^* = \bigcup_{n=N+1}^\infty E_n$, then obviously

$$\chi_E = \chi_{E_N^*} + \sum_{n=1}^N \chi_{E_n}.$$

Thus $\nu(E) = \nu(E_N^*) + \sum_{n=1}^N \nu(E_n)$. However $\nu(E_N^*) \to 0$, as $N \to \infty$, because of (5.5) and the assumption $p < \infty$. This shows that ν is countably additive and, moreover, (5.5) also shows us that ν is absolutely continuous with respect to μ.

We can now invoke the key result about absolutely continuous measures, the Lebesgue-Radon-Nykodim theorem. (See for example Theorem 4.3, Chapter 6 in Book III.) It guarantees the existence of an integrable function g so that $\nu(E) = \int_E g d\mu$ for every measurable set E. Thus we have $\ell(\chi_E) = \int \chi_E g d\mu$. The representation $\ell(f) = \int f g d\mu$ then extends immediately to simple functions f, and by a passage to the limit, to all $f \in L^p$ since the simple functions are dense in $L^p, 1 \leq p < \infty$. Also by Lemma 4.2, we see that $\|g\|_{L^q} = \|\ell\|$. $\square$

To pass from the situation where the measure of X is finite to the general case, we use an increasing sequence $\{E_n\}$ of sets of finite measure that exhaust X, that is, $X = \bigcup_{n=1}^{\infty} E_n$. According to what we have just proved, for each n there is an integrable function g_n on E_n (which we can set to be zero in E_n^c) so that

$$\ell(f) = \int f g_n d\mu$$

whenever f is supported in E_n and $f \in L^p$. Moreover by conclusion (ii) of the lemma $\|g_n\|_{L^q} \leq \|\ell\|$.

Now it is easy to see because of above displayed equation that $g_n = g_m$ a.e. on E_m, whenever $n \geq m$. Thus $\lim_{n \to \infty} g_n(x) = g(x)$ exists for almost every x, and by Fatou's lemma, $\|g\|_{L^q} \leq \|\ell\|$. As a result we have that $\ell(f) = \int f g d\mu$ for each $f \in L^p$ supported in E_n, and then by a simple limiting argument, for all $f \in L^p$. The fact that $\|\ell\| \leq \|g\|_{L^q}$, is already contained in Hölder's inequality, and therefore the proof of the theorem is complete.

Remark 5.1.15. For Lemma above, we note that

$$\sup\left\{\left|\int fg\right| : \|f\|_{L^p} \leq 1\right\} = \sup\left\{\left|\int fg\right| : \|f\|_{L^p} = 1\right\}$$

and similarly,

$$\sup\left\{\left|\int fg\right| : \|f\|_{L^p} \leq 1,\ f \text{ simple}\right\} = \sup\left\{\left|\int fg\right| : \|f\|_{L^p} = 1,\ f \text{ simple}\right\}$$

The $\geq$ direction is trivial. The other direction is because for any g with $\|f\|_{L^p} \leq 1$, we have

$$(*): \quad \frac{1}{\|f\|_{L^p}} \geq 1 \Rightarrow \left|\int \left(\frac{f}{\|f\|_{L^p}}\right) g\right| = \frac{|\int fg|}{\|f\|_{L^p}} \geq \left|\int fg\right|.$$

Since $\|\frac{f}{\|f\|_{L^p}}\|_{L^p} = 1$, we see

$$\sup\left\{\left|\int fg\right| : \|f\|_{L^p} = 1\right\} \geq \left|\int \frac{f}{\|f\|_{L^p}} g\right| \overset{(*)}{\geq} \left|\int fg\right|,$$

establishing the reverse direction.

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