Symplectic Geometry

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*Reading on Symplectic Geometry with Prof. Tang Xiang; based on Ana Cannas da Silva's *Lectures on Symplectic Geometry* [1].

Contents

Chapter 1

Symplectic Forms

1.1 Skew-Symmetric Bilinear Maps

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map, i.e., linear in one coordinate while fixing the other. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.

Theorem 1.1.1 (Standard Form for Skew-symmetric Bilinear Maps)**.** Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that

$$\begin{aligned}
&\Omega\left(u_i, v\right) = 0, \\
&\Omega\left(e_i, e_j\right) = 0 = \Omega\left(f_i, f_j\right), \\
&\Omega\left(e_i, f_j\right) = \delta_{ij},
\end{aligned}$$

for all i and all $v \in V$, for all i, j, and for all i, j.

Remark 1.1.2.

1. The basis in theorem is not unique, though it is traditionally also called a ”canonical” basis.
2. In matrix notation with respect to such basis, we have

$$\Omega(u, v) = [-u-] \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{array}\right] \left[\begin{array}{c} | \\ v \\ | \end{array}\right].$$

   where the symbol $\left[\begin{array}{c} | \\ v \\ | \end{array}\right]$ represents the column of coordinates of the vector v with respect to a symplectic basis $u_1, \cdots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n$ whereas $[-v-]$ represents its transpose line.
3. The dimension of the subspace $U = \{u \in V \mid \Omega(u, v) = 0$, for all $v \in V\}$ does not depend on the choice of basis. $\Longrightarrow k := \dim U$ is an invariant of (V, Ω).
4. Since $k + 2n = m = \dim V$, $\Longrightarrow n$ is an invariant of (V, Ω); $2n$ is called the **rank** of Ω.

1.2 Symplectic Vector Spaces

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map.

Definition 1.2.1. The map $\widetilde{\Omega} : V \to V^*$ is the linear map defined by $\widetilde{\Omega}(v)(u) = \Omega(v, u)$.

The kernel of $\widetilde{\Omega}$ is the subspace U above:

$$\ker(\widetilde{\Omega}) = \{v \in V \mid \forall u \in V,\, \widetilde{\Omega}(v)(u) = \Omega(v,u) = 0\} = U$$

Definition 1.2.2. A skew-symmetric bilinear map Ω is **symplectic** (or **nondegenerate**) if $\widetilde{\Omega}$ is bijective, i.e., $U = \{0\}$. The map Ω is then called a **linear symplectic structure** on V, and (V,Ω) is called a **symplectic vector space**.

The following are immediate properties of a linear symplectic structure Ω:

- **Duality**: the map $\widetilde{\Omega} : V \overset{\simeq}{\rightarrow} V^*$ is a bijection (injectivity plus indentical dimension)
- By Theorem 1.1.1, $k = \dim U = 0$, so $\dim V = 2n$ is **even**.
- By Theorem 1.1.1, a symplectic vector space (V,Ω) has a basis $e_1,\ldots,e_n,f_1,\ldots,f_n$ satisfying

  $$\Omega\left(e_i, f_j\right) = \delta_{ij} \quad \text{and} \quad \Omega\left(e_i, e_j\right) = 0 = \Omega\left(f_i, f_j\right).$$

  Such a basis is called a **symplectic basis** of (V,Ω). We have

  $$\Omega(u,v) = [-u-]\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}\begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

  Not all subspaces W of a symplectic vector space (V,Ω) look the same:

- A subspace W is called **symplectic** if $\Omega|_W$ is nondegenerate. For instance, the span of e_1, f_1 is symplectic.
- A subspace W is called **isotropic** if $\Omega|_W \equiv 0$. For instance, the span of e_1, e_2 is isotropic.

Homework 1 describes subspaces W of (V,Ω) in terms of the relation between W and W^Ω.

The **prototype of a symplectic vector space** is $\left(\mathbb{R}^{2n}, \Omega_0\right)$ with Ω_0 such that the basis

$$\begin{aligned} e_1 &= (1,0,\ldots,0), \quad \ldots, \quad e_n = (0,\ldots,0,\overbrace{1}^{n},0,\ldots,0), \\ f_1 &= (0,\ldots,0,\underbrace{1}_{n+1},0,\ldots,0),\ldots, \quad f_n = (0,\ldots,0,1), \end{aligned}$$

is a symplectic basis. The map Ω_0 on other vectors is determined by its values on a basis and bilinearity.

Definition 1.2.3. A **symplectomorphism** φ between symplectic vector spaces (V,Ω) and (V',Ω') is a linear isomorphism $\varphi : V \overset{\simeq}{\rightarrow} V'$ such that $\varphi^*\Omega' = \Omega$. ($\varphi^*$ is the induced map. Recall that $f^* : h \mapsto f \circ h$ and $f_* : h \mapsto h \circ f$. Thus, $\varphi^*\Omega' = \Omega$ reads as $\left(\varphi^*\Omega'\right)(u,v) = \Omega'(\varphi(u),\varphi(v))$.) If a symplectomorphism exists, (V,Ω) and (V',Ω') are said to be **symplectomorphic**.

The relation of being symplectomorphic is clearly an equivalence relation in the set of all even-dimensional vector spaces. Furthermore, by Theorem 1.1.1, every $2n$-dimensional symplectic vector space (V,Ω) is symplectomorphic to the prototype $\left(\mathbb{R}^{2n},\Omega_0\right)$; a choice of a symplectic basis for (V,Ω) yields a symplectomorphism to $\left(\mathbb{R}^{2n},\Omega_0\right)$. Hence, nonnegative even integers classify equivalence classes for the relation of being symplectomorphic.

1.3 Symplectic Manifolds

Let ω be a de Rham 2-form on a manifold M, that is, for each $p \in M$, the map $\omega_p : T_pM \times T_pM \rightarrow \mathbb{R}$ is skew-symmetric bilinear on the tangent space to M at p, and ω_p varies smoothly in p. We say that ω is closed if it satisfies the differential equation $d\omega = 0$, where d is the de Rham differential (i.e., exterior derivative).

Definition 1.3.1. The 2-form ω is symplectic if ω is closed and ω_p is symplectic for all $p \in M$.

If ω is symplectic, then $\dim T_pM = \dim M$ must be even.

Definition 1.3.2. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Example 1.3.3. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic as can be easily checked, and the set

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_n}\right)_p \right\}$$

is a symplectic basis of T_pM.

Example 1.3.4. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, \ldots, z_n$. The form

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is symplectic. In fact, this form equals that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}, z_k = x_k + iy_k$.

Example 1.3.5. Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to S^2 at p may then be identified with vectors orthogonal to p. The standard symplectic form on S^2 is induced by the inner and exterior products:

$$\omega_p(u, v) := \langle p, u \times v \rangle, \quad \text{for } u, v \in T_pS^2 = \{p\}^{\perp}.$$

This form is closed because it is of top degree; it is nondegenerate because $\langle p, u \times v \rangle \neq 0$ when $u \neq 0$ and we take, for instance, $v = u \times p$.

1.4 Symplectomorphisms

Definition 1.4.1. Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a **symplectomorphism** if $\varphi^*\omega_2 = \omega_1$ (Recall that, by definition of pullback, at tangent vectors $u, v \in T_pM_1$, we have $(\varphi^*\omega_2)_p (u, v) = (\omega_2)_{\varphi(p)} (d\varphi_p(u), d\varphi_p(v))$.)

We would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem (proved in Lecture 8 and stated below) takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional symplectic manifold looks locally like $\left(\mathbb{R}^{2n}, \omega_0\right)$. More precisely, any symplectic manifold $\left(M^{2n}, \omega\right)$ is locally symplectomorphic to $\left(\mathbb{R}^{2n}, \omega_0\right)$.

Theorem 1.4.2 (Darboux)**.** Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on $\mathcal{U}$

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ as in Darboux theorem is called a **Darboux chart**. By Darboux theorem, the **prototype of a local piece of a** $2n$**-dimensional symplectic manifold** is $M = \mathbb{R}^{2n}$, with linear coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and with symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

1.5 Homework 1: Symplectic Linear Algebra

Given a linear subspace Y of a symplectic vector space (V, Ω), its **symplectic orthogonal** Y^Ω is the linear subspace defined by

$$Y^\Omega := \{v \in V \mid \Omega(v, u) = 0 \text{ for all } u \in Y\}.$$

Exercise 1.5.1. Show that $\dim Y + \dim Y^\Omega = \dim V$. Hint: What is the kernel and image of the map

$$\begin{array}{rcl} V & \longrightarrow & Y^* = \operatorname{Hom}(Y, \mathbb{R}) \quad ? \\ v & \longmapsto & \Omega(v, \cdot)|_Y \end{array}$$

Exercise 1.5.2. Show that $\left(Y^\Omega\right)^\Omega = Y$.

Exercise 1.5.3. Show that, if Y and W are subspaces, then

$$Y \subseteq W \Longleftrightarrow W^\Omega \subseteq Y^\Omega.$$

Exercise 1.5.4. Show that: Y is **symplectic** (i.e., $\Omega|_{Y \times Y}$ is nondegenerate) $\Longleftrightarrow$ $Y \cap Y^\Omega = \{0\}$ $\Longleftrightarrow$ $V = Y \oplus Y^\Omega$.

Exercise 1.5.5. We call Y **isotropic** when $Y \subseteq Y^\Omega$ (i.e., $\Omega|_{Y \times Y} \equiv 0$). Show that, if Y is isotropic, then $\dim Y \leq \frac{1}{2} \dim V$.

Exercise 1.5.6. We call Y **coisotropic** when $Y^\Omega \subseteq Y$. Check that every codimension 1 subspace Y is coisotropic.

Exercise 1.5.7. An isotropic subspace Y of (V, Ω) is called **lagrangian** when $\dim Y = \frac{1}{2} \dim V$. Check that:

$$Y \text{ is lagrangian } \Longleftrightarrow Y \text{ is isotropic and coisotropic } \Longleftrightarrow Y = Y^\Omega.$$

Exercise 1.5.8. Show that, if Y is a lagrangian subspace of (V, Ω), then any basis $e_1, \ldots, e_n$ of Y can be extended to a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of (V, Ω). Hint: Choose f_1 in W^Ω, where W is the linear span of $\{e_2, \ldots, e_n\}$.

Exercise 1.5.9. Show that, if Y is a lagrangian subspace, (V, Ω) is symplectomorphic to the space $(Y \oplus Y^*, \Omega_0)$, where Ω_0 is determined by the formula

$$\Omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v).$$

In fact, for any vector space E, the direct sum $V = E \oplus E^*$ has a canonical symplectic structure determined by the formula above. If $e_1, \ldots, e_n$ is a basis of E, and $f_1, \ldots, f_n$ is the dual basis, then $e_1 \oplus 0, \ldots, e_n \oplus 0, 0 \oplus f_1, \ldots, 0 \oplus f_n$ is a symplectic basis for V.

Chapter 2

Symplectic Form on the Cotangent Bundle

Chapter 3

Lagrangian Submanifolds

Chapter 4

Generating Functions

Chapter 5

Appendix

Bibliography

[1] da Silva, Ana Cannas. *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics 1764 (January 2006). https://people.math.ethz.ch/~acannas/Papers/lsg.pdf