# Lecture Note on
# Ordinary Differential Equations

Anthony Hong*

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# Chapter 1

# A Theoretical Introduction

The laws of the universe are written in the language of mathematics, said Galileo Galilei, gazing in awe the beautiful mathematical equations. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**. For example, each of the following is a differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\sin x, \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+\frac{g}{l}\sin\varphi=0, \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

where the first one is a second order linear constant-coefficient ode, the second is a nonlinear ode, as $\varphi$ is the dependent variable with respect to the independent one $t$, and the third is a partial differential equation, or pde in short. There are couple of more notions about ode:

1. Carried in its most general form, an ode is written as

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right)=0 \tag{1}$$

   and if plugging in the function $y=\varphi(y)$ makes the above equation an tautology, meaning that it always stands, we then call $y=\varphi(x)$ the **(explicit) solution** of the ode (1).

2. Likewise, the **implicit solution** of the form

$$\Phi(x,y)=0$$

   also solves the ode in a sense that it determines a solution $y=\varphi(x)$. It is usually a loss of information or difficulty to write explicitly that requires one to write the implicit form instead.

3. Often when we integrate a derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ we find that there is a constant $C$ along with it (an immediate consequence of the Fundamental Theorem of Calculus):

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}=y(x)+C, C\in\mathbb{R}.$$

   As $C$ varies, we can get multiple solutions, sometimes in a geometric sense when graphed on plane, called integral curves or solution curves. The solution to equation (1) of the following form

$$y=\varphi(x, C_1, \cdots, C_n)$$

   determines a class of solution due to the varying constants and is therefore called the **general solution**. Of course, this one is written in explicit form, and it can also be expressed in implicit form, which is sometimes called **general integral**.

4. Correspondingly, we have **particular solution** when we actually settle down the constants in a general solution, according to the **initial value condition (IC)** or **boundary value condition (BC)** given. Differential equation along with initial data or boundary data is as a whole called **initial value problem (Cauchy problem)** or **boundary value problem (Dirichlet problem)**. For example, let's say we have the following initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y, y \geqslant 0, y(0) = 1$$

Then one can easily obtain a general solution

$$y(x, C) = Ce^x.$$

Plugging in initial data, we determine the constant and particular solution then

$$Ce^0 = C = 1 \Rightarrow y(x) = e^x$$

5. If a system of equation

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}x} = \vec{f}(\vec{y}, x) = \vec{f}(\vec{y})$$

We call such system **autonomous** Namely, the RHS of the system does not contain the independent variable.

This lecture note is written for a quick review, so I won't delve into the technical proof of many theorems, specifically the theorem of existence and uniqueness of initial value problem solution and of course many of the theorems in nonlinear ode. One may see the proof in Teschl's Ordinary Differential Equations. I shall state the theorem now.

Consider a first order ode of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2}$$

where $f(x, y)$ is a continuous function over the rectangle

$$R : |x - x_0| \leqslant a, |y - y_0| \leqslant b$$

The function $f(x, y)$ satisfies **Lipschitz condition** with respect to $y$ over $R$, if

$$\exists L > 0, \textbf{s.t.} \forall (x, y_1), (x, y_2) \in R, |f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

**Theorem 1.0.1** (The Uniqueness and Existence Theorem). *If $f(x, y)$ is continuous over the above rectangle and satisfies the Lipschitz condition, then the equation (2) has and only has one solution $y = \varphi(x)$ defined over the interval $|x - x_0| < h$, where*

$$h = \min{(a, \frac{b}{M})}, M = \max_{(x,y)\in R} |f(x, y)|$$

*and $\varphi$ is continuous and satisfies the initial value condition*

$$\varphi(x_0) = y_0$$

**Scholium.** *Due to the difficulty to verify the Lipschitz condition, we often use the continuous derivative of $f(x, y)$ with respect to $y$ to replace it. In fact, if $\frac{\partial f}{\partial y}$ exists and is continuous over $R$, it has to be bounded, say $|\frac{\partial f}{\partial y}| \leqslant L$, then by Lagrange mean value theorem we have*

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| =& \left| \frac{\partial f(x, y_2 + \theta(y_1 - y_2))}{\partial y} \right| |y_1 - y_2| \\ \leqslant& L|y_1 - y_2| \end{aligned}$$

*where $(x, y_1), (x, y_2) \in R, 0 < \theta < 1$. Conversely, however, Lipschitz condition cannot guarantee the existence of a continuous partial derivative, which makes the latter a stronger condition. For example, $f(x, y) = |y|$ satisfies the Lipschitz condition at any region but has no derivative at $y = 0$.*

For implicitly defined ode, we have its counterpart, which is a consequence of implicit function theorem and the above existence and uniqueness theorem.

**Theorem 1.0.2.** *For the equation*

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \tag{3}$$

*suppose in a neighborhood $O$ of the point $(x_0, y_0, y_0')$, where $y_0' = \frac{dy}{dx}|_{(x_0,y_0)}$,*

1. *$F(x, y, y')$ is continuous for all variables and has continuous partial derivatives over them;*
2. *$F(x_0, y_0, y_0') = 0$;*
3. *$\frac{\partial F(x_0,y_0,y_0')}{\partial y'} \neq 0$,*

*then the equation (3) has and only has one solution*

$$y = y(x), |x - x_0| \leqslant h$$

*satisfying*

$$y(x_0) = y_0, y'(x_0) = y_0'$$

*for some sufficiently small positive $h$.*

**Scholium.** *A geometric interpretation of the above theorem: for any pair, or point, $(x_0, y_0, y_0')$ for which $F(x_0, y_0, y_0') = 0$, the equation (3) has only one integral curve (the solution) along the given direction $y_0'$ that passes through the point $(x_0, y_0)$.*

Last word, this "lecture note" is not for beginners but for those aiming for something more detailed than merely a chart summarizing formulas of solution. For this purpose, I won't include physical manifestation as an integral part, if not necessary for facilitating understanding substantially.

# Chapter 2

# First Order ODE

The general form of a first-order differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y) \tag{1}$$

where $x$ and $y$ are real variables, and $F$ is a real-valued function of $x$ and $y$. We will examine several basics methods to solve them.

## 2.1 Method of Separation of Variables

Equations of the following form are called **separable**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\varphi(y),$$

where $f(x)$ and $\varphi(y)$ are continuous functions of $x$ and $y$ respectively. Separation allows us to integrate the functions separately, returning the functions to their antiderivative forms. If $\varphi(y) \neq 0$, then

$$\begin{aligned} \frac{dy}{\varphi(y)} &= f(x)dx \\ \Rightarrow \int \frac{\mathrm{d}y}{\varphi(y)} &= \int f(x)\,\mathrm{d}x \end{aligned}$$

**Example 2.1.1.** Solve the ode

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(-a+bx)}{x(d-cy)}(x \geqslant 0, y \geqslant 0, a, b, c, d \in \mathbb{R})$$

*solution*:

$$\begin{aligned} \text{LHS} &= \frac{y}{d-cy}\frac{-a+bx}{x} = \varphi(y)f(x) \\ &\Rightarrow \int \frac{\mathrm{d}y}{\frac{y}{d-cy}} = \int \frac{-a+bx}{x}\,\mathrm{d}x \\ &\Rightarrow \int \frac{d-cy}{y}\,\mathrm{d}y = \int \frac{-a+bx}{x}\,\mathrm{d}x \\ &\Rightarrow d\int \frac{\mathrm{d}y}{y} - \int c\,\mathrm{d}y = b\int \mathrm{d}x - a\int \frac{\mathrm{d}x}{x} \\ &\Rightarrow d\ln|y| - cy + C_1 = bx - a\ln|x| + C_2 \end{aligned}$$

where $C_1$ and $C_2$ are two constants. As $x, y \geqslant 0$, we have $|x| = x, |y| = y$, and

$$\widetilde{C} e^{d \ln y + a \ln x - cy - bx} = e^0 = 1$$

or

$$\left(e^{\ln y}\right)^d \cdot \left(e^{\ln x}\right)^a \cdot e^{-cy-bx} = y^d \cdot x^a \cdot e^{-cy-bx} = \widetilde{C},$$

where we use $\widetilde{C}$ to denote the constant however many times for which it is incorporated with other constants.

### 2.1.1 A Classic Example: Logistic Growth

The link gives a nice review about the derivation of the logistic differential equation that we shall look at

**Example 2.1.2.**

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{N_m}\right), N(t_0) = N_0, N(t) \geqslant 0$$

*solution*: We use method of separation of variables.

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= rN\left(\frac{N_m - N}{N_m}\right) \\ \frac{\mathrm{d}N}{N(N_m - N)} &= \frac{r\ \mathrm{d}t}{N_m} \\ \int \frac{\mathrm{d}N}{N(N_m - N)} &= \int \frac{r\ \mathrm{d}t}{N_m} \end{aligned}$$

Let

$$\int \frac{\mathrm{d}N}{N(N_m - N)} = \int \left(\frac{A}{N_m - N} + \frac{B}{N}\right)\ \mathrm{d}N$$

we have

$$\begin{cases} A - B = 0 \\ BN_m = 1 \end{cases}$$

or

$$\begin{cases} A = B \\ B = \frac{1}{N_m} \end{cases}$$

Thus,

$$\begin{aligned} &\int \frac{r\ \mathrm{d}t}{N_m} = \frac{r}{N_m} t + C_2 \\ =&\int \frac{\mathrm{d}N}{N(N_m - N)} = \int \frac{1}{N_m\left(\frac{1}{N_m - N} + \frac{1}{N}\right)}\ \mathrm{d}N \\ =&\frac{1}{N_m}\left(\ln|N| - \ln|N_m - N|\right) + C_1 \\ &\text{or } |N/(N_m - N)| = \widetilde{C} e^{rt} \end{aligned}$$

Notice that the population $N(t) \geqslant 0$ with environmental capacity $N_m \geqslant N(t)$, we have $|N/(N_m - N)| = N/(N_m - N)$. Hence,

$$N(t) = \frac{N_m}{1 + \widetilde{C} e^{-rt}}$$

Pugging in the IVC $N(t_0) = N_0$, we have

$$N(t) = \frac{N_m}{1 + \left(\frac{N_m}{N_0} - 1\right) e^{r(t_0 - t)}}$$

### 2.1.2 Families of Separable First Order ODE

We introduce two families of ode that can be converted into a separable type.
The first family is called the **homogeneous equation** in terms of the RHS being a homogeneous function. Namely, for equation (1), we have $F(ax, ay) = F(x, y) \forall a$. For example, $F(x, y) = \frac{6y}{x}$ is a homogeneous function, and $\mathrm{d}y/\mathrm{d}x = \frac{6y}{x}$ is a homogeneous equation. We solve it by doing a variable transformation $u = \frac{y}{x}$, or $y = ux$. We then have

$$\begin{aligned}
\frac{\mathrm{d}ux}{\mathrm{d}x} &= F(x, y) \\
u + x\frac{\mathrm{d}u}{\mathrm{d}x} &= F(x, ux) = F(1, u) \\
x\frac{\mathrm{d}u}{\mathrm{d}x} &= F(1, u) - u \\
\frac{\mathrm{d}u}{F(1, u) - u} &= \frac{\mathrm{d}x}{x}
\end{aligned}$$

which becomes a separable type.

**Example 2.1.3.** Solve the ode

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \tan\frac{y}{x}$$

*solution*:

$$\begin{aligned}
\int \frac{\mathrm{d}u}{F(1, u) - u} &= \int \frac{\mathrm{d}x}{x} \\
\int \frac{\mathrm{d}u}{u + \tan u - u} &= \int \frac{\mathrm{d}x}{x} \\
\ln|\sin u| &= \ln|x| + \widetilde{C} \\
\sin u = \sin\frac{y}{x} &= \pm e^{\widetilde{C}} x
\end{aligned}$$

The equation also has $\tan u = 0$, or $\sin u = 0$, as a solution. Notice that $c = \pm e^{\widetilde{C}}$ has its range $(0, +\infty) \cup (-\infty, 0) = \mathbb{R} - \{0\}$, so the general solution is then

$$\sin\frac{y}{x} = cx, c \in \mathbb{R}$$

The second family is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \tag{2}$$

Before solving the ode, we first give some useful tricks about the fraction

**lemma**:

L1 if $\frac{a}{b}=\frac{c}{d}$, then
$$\frac{a+b}{b}=\frac{c+d}{d} \tag{3}$$
Trivially true.

L2 if $\frac{a}{b}=\frac{c}{d}$, then
$$\frac{a-b}{b}=\frac{c-d}{d} \tag{4}$$
Trivially true.

L3 if $\frac{a}{b}=\frac{c}{d}$, then
$$\frac{a+b}{a-b}=\frac{c+d}{c-d}$$
Divide (3) by (4).

L4 if $\frac{a}{b}=\frac{c}{d}$, then
$$\frac{a}{b}=\frac{c}{d}=\frac{a\pm c}{b\pm d}$$
Let $\frac{a}{b}=\frac{c}{d}=k$ and directly verify it.

Equation (2) has three situations:

1. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$. By L4, we immediately have $\frac{\mathrm{d}y}{\mathrm{d}x} = k$ and $y = kx + C$
2. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k \neq \frac{c_1}{c_2}$.
   Let $u = a_2x + b_2y$ then $\frac{a_1x+b_1y}{a_2x+b_2y} = k$ gives that $a_1x + b_1y = ku$. Then
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} = \frac{ku + c_1}{u + c_2}$$
   and thus
   $$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{d(a_2x + b_2y)}{\mathrm{d}x} = a_2 + b_2\frac{\mathrm{d}y}{\mathrm{d}x} = a_2 + b_2\left(\frac{ku + c_1}{u + c_2}\right) = a_2 + b_2k + b_2\frac{c_1 - c_2k}{u + c_2}$$
   which is a separable form.
3. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}, \exists c_i \neq 0$ then
   $$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$
   represents two instersecting lines on the Oxy plane. Let the intersection point be $(\alpha, \beta)$ and set
   $$\begin{cases} X = x = \alpha \\ Y = y - \beta \end{cases}$$
   which yields
   $$\begin{cases} a_1X + b_1Y = 0 \\ a_2X + b_2Y = 0 \end{cases}$$
   Thus,
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1X + b_1Y}{a_2X + b_2Y} = g\left(\frac{Y}{X}\right)$$
   which becomes a homogeneous equation we've discussed. Notice that if $c_1 = c_2 = 0$, we may substitute $u = y/x$ directly. Besides, equations of the form
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$$
   can be solved by the above method too.

Other families include
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by + c)$$
where we may use substitution $u = ax + by + c$
$$\frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = f(xy)$$
where we may use substitution $u = xy$.

**Example 2.1.4.** solve
$$(x + y)dy + (x - y)dx = 0$$

*solution*:

$$
\begin{aligned}
(x+y)dy &= (y-x)dx \\
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x+y}{x+y} &= \frac{-x/x+y/x}{x/x+y/x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-1+y/x}{1+y/x} = g\left(\frac{y}{x}\right) \\
u = \frac{y}{x} \Rightarrow \int \frac{\mathrm{d}u}{\frac{-u^2-1}{u+1}} &= \int \frac{\mathrm{d}x}{x}
\end{aligned}
$$

Since

$$-\int \frac{u+1}{u^2+1}\,\mathrm{d}u = -\int \frac{u}{u^2+1}\,\mathrm{d}u - \int \frac{\mathrm{d}u}{u^2+1} = -\frac{1}{2}\ln|u^2+1| - \arctan u + C_1$$

and

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C_2$$

we have the implicit general solution

$$\ln\sqrt{x^2+y^2} + \arctan\frac{y}{x} = \widetilde{C}$$

**Exercise (1)**

*Solve the following odes*

1. $$x\frac{dy}{dx} - y + \sqrt{x^2-y^2}$$

   *Hint: divide by* $\sqrt{x^2} = \operatorname{sgn}(x)x$.

2. $$\frac{dy}{dx} = (x+y)^2$$

   *Hint: let* $u = x+y$.

3. $$\frac{dy}{dx} = \frac{1}{(x+y)^2}$$

   *Hint: let* $u = x+y$

**Exercise (2)**

*Solve the following odes*

1. $\frac{dy}{dx} = (x+1)^2 + (4y+1)^2 + 8xy + 1$.
2. $y(1+x^2y^2)\ dx = x\ dy$.
3. $\frac{y}{x}\frac{dy}{dx} = \frac{2+x^2y^2}{x-x^2y^2}$.
   *Hint: the second and the third are of the family* $\frac{x}{y}\frac{dy}{dx} = f(xy)$.

## 2.2 Linear ODE and Method of Variation of Parameters

Equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x), \tag{5}$$

where $P(x)$ and $Q(x)$ are continuous functions over $x$, is called **first order linear ode**. When $Q(x) \neq 0$, we call it **nonhomogeneous**, otherwise **homogeneous**. Notice that the previously defined homogeneity refers to the LHS being a homogeneous function, while the homogeneity there refers to the fact that the equation is written as a function of derivatives of the dependent variable $y$ (including 0-order derivative), altogether equaling $0$ rather than some function of $x$. Namely, the most general form

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right) = 0$$

becomes

$$F\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right) = 0$$

rather than

$$F\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right) = f(x)$$

We observe that the homogeneous first order linear ode is of a separable type. For

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y$$

We have

$$\int \frac{\mathrm{d}y}{y} = \int P(x)\ \mathrm{d}x$$
$$y(x) = C \cdot \exp\left(\int P(x)\ \mathrm{d}x\right)$$

We now conjecture that the nonhomogeneous one has a solution like

$$y(x) = C(x) \cdot \exp\left(\int P(x)\ \mathrm{d}x\right)$$

Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{C(x)\cdot \exp\left(\int P(x)\,\mathrm{d}x\right)}{\mathrm{d}x}\\ &= \frac{\mathrm{d}C(x)}{\mathrm{d}x}\exp\left(\int P(x)\,\mathrm{d}x\right) + \exp\left(\int P(x)\,\mathrm{d}x\right) C(x)P(x)\\ &= \frac{\mathrm{d}C(x)}{\mathrm{d}x}\exp\left(\int P(x)\,\mathrm{d}x\right) + P(x)y\end{aligned}$$

Since

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y + Q(x)$$

we have

$$P(x)y + Q(x) = \frac{\mathrm{d}C(x)}{\mathrm{d}x}\exp\left(\int P(x)\,\mathrm{d}x\right) + P(x)y$$

and

$$\begin{aligned}Q(x) &= \frac{\mathrm{d}C(x)}{\mathrm{d}x}\exp\left(\int P(x)\,\mathrm{d}x\right)\\ \Rightarrow \frac{\mathrm{d}C(x)}{\mathrm{d}x} &= Q(x)\exp\left(-\int P(x)\,\mathrm{d}x\right)\\ \Rightarrow C(x) &= \int \left[Q(x)\exp\left(-\int P(x)\,\mathrm{d}x\right)\right]\,\mathrm{d}x + \widetilde{C}\end{aligned}$$

The solution to the equation (5) is then

$$y(x) = \left\{\int \left[Q(x)\exp\left(-\int P(x)\,\mathrm{d}x\right)\right]\,\mathrm{d}x + \widetilde{C}\right\}\exp\left(\int P(x)\,\mathrm{d}x\right) \tag{6}$$

Since we change the constant $C$ by $C(x)$, we call the above method **method of variation of parameters**. Later on, we will see this method be applied to higher order ode. Note that when calculating the integrals, we let all constants to be 0 and add $\widetilde{C}$ instead.

**Example 2.2.1.** Solve ode

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - ny = e^x(x+1)^n,$$

where $n$ is a constant.
*solution*: the ode is a first order linear ode, as it can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\frac{n}{n+1})y + e^x(x+1)^n$$

Let $P(x) = \frac{n}{x+1}$ and $Q(x) = e^x(x+1)^n$. Since

$$\begin{aligned}\exp\left(\int P(x)\,\mathrm{d}x\right) &= e^{n\ln|x+1|} = |x+1|^n = \begin{cases}\mathrm{sgn}(x+1)(x+1)^n, n = 2k+1\\ (x+1)^n, n = 2k\end{cases}\\ &= (-1)^{n-1}\mathrm{sgn}(x+1)(x+1)^n\end{aligned}$$

we have

$$
\begin{aligned}
y(x) &= \left\{\int\left[Q(x)\exp\left(-\int P(x)\ \mathrm{d}x\right)\right]\ \mathrm{d}x\widetilde{C}\right\}\exp\left(\int P(x)\ \mathrm{d}x\right)\\
&= \left\{\int\left[(-1)^{n-1}\mathrm{sgn}(x+1)e^x(x+1)^n\frac{1}{(x+1)^n}\right]\ \mathrm{d}x\widetilde{C}\right\}(-1)^{n-1}\mathrm{sgn}(x+1)(x+1)^n\\
&=[(-1)^{n-1}\mathrm{sgn}(x+1)]^2(x+1)^n\left(\int e^x\ \mathrm{d}x+\widetilde{C}\right)\\
&=(x+1)^n(e^x+\widetilde{C})
\end{aligned}
$$

**Example 2.2.2.** This one is a bit different. One can regard $x$ as a function of $y$ to solve it.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{2x-y^2}$$

Let

$$\frac{\mathrm{d}x(y)}{\mathrm{d}y}=\left(\frac{2}{y}\right)x-y, P(y)=\frac{2}{y}, Q(y)=-y$$

then

$$
\begin{aligned}
x &= \left[\int Q(y)\exp\left(-\int P(y)\ \mathrm{d}y\right)\ \mathrm{d}y+\widetilde{C}\right]\exp\left(-\int P(y)\ \mathrm{d}y\right)\\
&=\left[\int -ye^{-2\ln|y|}\ \mathrm{d}y+\widetilde{C}\right]e^{2\ln|y|}\\
&=(-\ln|y|+\widetilde{C})y^2
\end{aligned}
$$

**Example 2.2.3.** Solve the ode

$$\frac{\mathrm{d}y}{\mathrm{d}x}=6\frac{y}{x}-xy^2 \tag{7}$$

We often call the ode of the following form

$$\frac{\mathrm{d}y}{\mathrm{d}x}=P(x)y+Q(x)y^n$$

the **n-th order Bernoulli equation**, where $n\neq 0,1$ and $P(x),Q(x)$ are continuous functions over $x$. If $y\neq 0$, we multiply the two sides by $y^{-n}$:

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}=y^{1-n}P(x)+Q(x)$$

Let $z=y^{1-n}$ be the substitution, then

$$\frac{\mathrm{d}z}{\mathrm{d}x}=(1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}=(1-n)[y^{1-n}P(x)+Q(x)]=(1-n)zP(x)+(1-n)Q(x)$$

which is a nonhomogeneous first order linear ode, solvable by method of variation of parameter. We now solve the equation (7). First observe that $y=0$ is a solution, and then *solving equation (7)*

| Calculation | Explanation |
|---|---|
| $\frac{\mathrm{d}y}{\mathrm{d}x} = 6\frac{y}{x} - xy^2$ | Given |
| $y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} = 6y^{-2}\frac{y}{x} - x$ | multiply two sides by $z = y^{-2}$ |
| $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}(y^{-1})}{\mathrm{d}x} = -\frac{1}{y^2}\left(6\frac{y}{x} - xy^2\right)$ | plugging in |
| $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{-6}{x}z + x$ | |

Let $P(x) = -6/x$ and $Q(x) = x$. Fro the above first order linear ode,

$$\begin{aligned}
z &= \left[\int Q(x)\text{exp}\left(-\int P(x)\ \mathrm{d}x\right)\ \mathrm{d}x + \widetilde{C}\right]\text{exp}\left(-\int P(x)\ \mathrm{d}x\right) \\
&= \left(\int xe^{6\ln|x|}\ \mathrm{d}x + \widetilde{C}\right)e^{6\ln|x|} \\
&= \left(\frac{x^8}{8} + \widetilde{C}\right)\frac{1}{x^6} \\
&= \frac{x^2}{8} + \frac{\widetilde{C}}{x^6} = \frac{1}{y}, \text{or } y = \frac{8x^6}{x^8 + \widetilde{C}}
\end{aligned}$$

**Exercise (3)**

*Solve the following odes*

1.
$$\frac{dy}{dx} = \frac{y}{x + y^3}$$

2.
$$\frac{ds}{dt} = -s\cos t + \frac{1}{2}\sin 2t$$

3.
$$\frac{dy}{dx} = \frac{x^4 + y^3}{xy^2}$$

4.
$$\frac{dy}{dx} = \frac{ay}{x} + \frac{x+1}{x}$$

*Hint: The second to last is a Bernoulli eq. For the last one, discuss cases $a = 0$, $a = 1$, and $a \neq 0, 1$.*

## 2.3 Exact Equations and Integrating factor

### 2.3.1 Exact Equations

Consider a general first order ode

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$$

We may write it as $dy - f(x,y)dx = 0$ and compare it with $M(x,y)dx + N(x,y)dy = 0$. **Exact equation** is an ode where the LHS of the above equation is exactly the total derivative of a bivariate function $u(x,y)$. Namely,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = M(x,y)dx + N(x,y)dy = 0 \tag{8}$$

So the general solution of the ode is $u(x,y) = C$. In particular, we need to find a function $u(x,y)$ such that, by (8),

$$\begin{aligned}
&\frac{\partial u}{\partial x} = M(x,y), \frac{\partial u}{\partial y} = N(x,y) \\
\Rightarrow &\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \overset{\text{clairaut's thm}}{=\!=\!=\!=} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x}
\end{aligned}$$

Thus $\partial M/\partial y = \partial N/\partial x$ is the necessary condition for the equation $M(x,y)dx + N(x,y)dy$ to be an exact equation. We now prove that it is also the sufficient condition for the equation to be exact. In other words, we use the following process to find $u(x,y)$. We start with $\partial u/\partial x = M(x,y)$, regarding $y$ as a parameter

$$\begin{aligned}
\frac{\partial u}{\partial x} &= M(x,y) \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= M(x,y) \\
u(x,y) &= \int \mathrm{d}u = \int M(x,y)\,\mathrm{d}x + \varphi(y) \\
\frac{\partial u}{\partial y} &= \frac{\partial \int M(x,y)\,\mathrm{d}x}{\partial y} + \frac{\partial \varphi(y)}{\partial y} = N(x,y) \\
\frac{\partial \varphi(y)}{\partial y} &= \frac{\mathrm{d}\varphi(y)}{\mathrm{d}y} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,\mathrm{d}x
\end{aligned}$$

We now prove that the RHS of the above equation is independent of $x$:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,\mathrm{d}x\right] &= \frac{\partial N}{\partial x} - \frac{\partial}{\partial x \partial y}\int M\,\mathrm{d}x \\
&= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int M\,\mathrm{d}x\right] \\
&= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0
\end{aligned}$$

Thus, the RHS is a function of $y$. Integrate it:

$$\int \mathrm{d}\varphi(y)\,\mathrm{d}y\,\mathrm{d}y = \varphi(y) = \int N - \frac{\partial}{\partial y}\int M\,\mathrm{d}x\,\mathrm{d}y$$

Plugging into $u(x,y) = \int M\,\mathrm{d}x + \varphi(y)$:

$$u(x,y) = \int M(x,y)\,\mathrm{d}x + \int\left[N - \frac{\partial}{\partial y}\int M(x,y)\,\mathrm{d}x\right]\,\mathrm{d}y$$

Therefore, the general solution of the exact equation is

$$\int M(x,y)\,\mathrm{d}x + \int \left[N - \frac{\partial}{\partial y}\int M(x,y)\,\mathrm{d}x\right]\,\mathrm{d}y = C \tag{9}$$

**Example 2.3.1.**

$$2(3xy^2 + 2x^3)\,\mathrm{d}x + 3(2x^2y + y^2)\,\mathrm{d}y = 0$$

*solution*:

$$M(x,y) = 6xy^2 + 4x^3, N(x,y) = 6x^2y + 3y^2$$

Thus, the necessary and sufficient condition for it being exact $\partial M/\partial y = \partial N/\partial x (= 12xy)$ is thus satisfied. Then

$$u(x,y) = \int 6xy^2 + 4x^3\,\mathrm{d}x + \varphi(y) = 3x^2y^2 + x^4 + \varphi(y)$$

and

$$\frac{\partial u}{\partial y} = 6x^2y + \frac{\mathrm{d}\varphi(y)}{\mathrm{d}y} = 3(2x^2y + y^2) \Rightarrow \varphi'(y) = 3y^2, \varphi(y) = y^3 + C$$

Thus, the general solution is

$$3x^2y^2 + x^4 + y^3 = C$$

**Example 2.3.2.**

$$\left[\frac{y^2}{(x-y)^2} - \frac{1}{x}\right]\,\mathrm{d}x + \left[\frac{1}{y} - \frac{x^2}{(x-y)^2}\right]\,\mathrm{d}y = 0$$

Notice that the equation can be written as two parts:

$$\left[\frac{y^2}{(x-y)^2} - \frac{1}{x}\right]\,\mathrm{d}x + \left[\frac{1}{y} - \frac{x^2}{(x-y)^2}\right]\,\mathrm{d}y = \frac{y^2\,\mathrm{d}x - x^2\,\mathrm{d}y}{(x-y)^2} + \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{xy}$$

By the procedures to find $u$, it is easy to see that

$$\begin{aligned} u_1 &= -\frac{xy}{x-y} \text{ for } \mathrm{d}u_1 = \frac{y^2\,\mathrm{d}x - x^2\,\mathrm{d}y}{(x-y)^2} \\ u_2 &= \ln\left|\frac{y}{x}\right| \text{ for } \mathrm{d}u_2 = \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{xy} \end{aligned}$$

Since $d$ as the differentiation is linear (in fact, differentiation is defined as a linear transformation, see the standard text Rudin's PMA), we see that $d(u_1 + u_2) = 0$ and the general solution of the original ode is therefore

$$\ln\left|\frac{y}{x}\right| - \frac{xy}{x-y} = C$$

In fact, the linearity of the differentiation as an operator implies the superposition of differential equation.

**some common total derivatives:**

- $$x\ \mathrm{d}y + y\ \mathrm{d}x = d(xy)$$
- $$\frac{y\ \mathrm{d}x - x\ \mathrm{d}y}{y^2} = \mathrm{d}\left(\frac{x}{y}\right)$$
- $$\frac{-y\ \mathrm{d}x + x\ \mathrm{d}y}{x^2} = \mathrm{d}\left(\frac{y}{x}\right)$$
- $$\frac{y\ \mathrm{d}x - x\ \mathrm{d}y}{xy} = \mathrm{d}\left(\ln\left|\frac{x}{y}\right|\right)$$
- $$\frac{y\ \mathrm{d}x - x\ \mathrm{d}y}{x^2+y^2} = \mathrm{d}\left(\arctan\frac{x}{y}\right)$$
- $$\frac{y\ \mathrm{d}x - x\ \mathrm{d}y}{x^2-y^2} = \frac{1}{2}\,\mathrm{d}\left(\ln\left|\frac{x-y}{x+y}\right|\right)$$

**Exercise (4)**

*Solve the following odes*

1. $$(x^2+y)\ dx + (x-2y)\ dy = 0$$
2. $$\left(\cos x + \frac{1}{y}\right)\ dx + \left(\frac{1}{y} - \frac{x}{y^2}\right)\ dy = 0$$

   *Hint: for the the second one, find out a common total derivative from the above list.*

### 2.3.2 Integrating Factor

The exact equations can be solved by direct integration, so transforming a non-exact equation to an exact one becomes crucial for solving certain types of equations.

If there exists a function $\mu = \mu(x,y) \neq 0$ such that

$$\mu M(x,y)\ \mathrm{d}x + \mu N(x,y)\ \mathrm{d}y = 0 \tag{10}$$

becomes an exact equation, namely theres is a function $v$ such that $\mu M\ \mathrm{d}x + \mu N\ \mathrm{d}y = \ \mathrm{d}v$, we call the function $\mu$ the **Integrating factor**. The general solution of the equation is then $v(x,y) = C$.

There may exist multiple integrating factors for a single equation. For example,

$$ydx - xdy = 0$$

has factors $1/x^2$, $1/y^2$, $1/xy$, and $1/(x^2 \pm y^2)$. I can be proved that if there is a solution for the equation, then there exists an integrating factor, and such is not unique.

Since the necessary and sufficient condition of exactness of an equation $M\ \mathrm{d}x + N\ \mathrm{d}y = 0$ is $\partial M/\partial y = \partial N/\partial x$, we have

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

for (10) to be exact. Namely,

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu$$

which is a first order partial differential equation that can be more difficult to solve. However, sometimes finding a particular solution is much easier under some special conditions. For example, the equation

$$M(x,y)\ \mathrm{d}x + N(x,y)\ \mathrm{d}y = 0 \tag{11}$$

has an integrating factor

$$\exp\left(\int \psi(x)\ \mathrm{d}x\right)$$

when

$$\psi(x) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N$$

is a only a function of $x$.

**Exercise (5)**

*Use the integrating factor to solve the first order linear ode (5)*

Likewise, (11) has an integrating factor

$$\exp\left(\int \varphi(y)\ \mathrm{d}y\right)$$

when

$$\varphi(y) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/-M$$

is only a function of $y$.

**Example 2.3.3.** solve the ode

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} + \sqrt{1+\left(\frac{x}{y}\right)^2}, (y>0)$$

*solution*:
The equation can be written as, by multiplying $y$ at both sides,

$$x\ \mathrm{d}x + y\ \mathrm{d}y = \sqrt{x^2+y^2}\ \mathrm{d}x$$

or

$$\frac{1}{2}\ \mathrm{d}(x^2+y^2) = \sqrt{x^2+y^2}\ \mathrm{d}x$$

The equation therefore has the integrating factor

$$\mu = \frac{1}{\sqrt{x^2+y^2}}$$

Then

$$\begin{aligned} \frac{\mathrm{d}(x^2+y^2)}{2\sqrt{x^2+y^2}} &= \mathrm{d}x \\ \int \frac{\mathrm{d}(x^2+y^2)}{2\sqrt{x^2+y^2}} &= \int \mathrm{d}x \\ \sqrt{x^2+y^2} &= x + C \Rightarrow y^2 = C(C+2x) \end{aligned}$$

**Exercise (6)**

*1.*

$$(e^x + 3y^2)\,dx + 2xy\,dy = 0$$

*2.*

$$y\,dx - x\,dy = (x^2+y^2)\,dx$$

*3.*

$$[x\cos(x+y) + \sin(x+y)]\,dx + x\cos(x+y)\,dy = 0$$

*4.*

$$x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$$

*Hint: for the last one, use the method of undetermined coefficients for the integrating factor $\mu = x^m y^n$. Plugging it into the necessary and sufficient condition of exact equation.*

## 2.4 First Order Implicit ODE and Its Parametrization

The common form of a first order implicit ode is often written as

$$F(x, y, y') = 0$$

If we can solve the derivative $y' = f(x,y)$ from the above equation, we can use the methods in previous sections (separation, exact euqatio, ...).

If not, we may parametrize the equation in the following four types:

1.

$$y = f\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

introduce the parameter $p = dy/dx$, then

$$y = f(x, p)$$

differentiate both sides:

$$p = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial f(x,p)}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}x}$$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial p}$ are functions about $x$ and $p$, the above equation is a first order ode about $x$ and $p$.

**Example 2.4.1.** solve

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+2x\frac{\mathrm{d}y}{\mathrm{d}x}-y=0$$

*solution*: First notice that $y=0$ is a solution. Then observe that the equation is of the form $y=f(x,dy/dx)$ where we can introduce $p=y'$

$$y=\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+2x\frac{\mathrm{d}y}{\mathrm{d}x}=p^3+2xp$$

and differentiate it:

$$p=3p^2\frac{\mathrm{d}p}{\mathrm{d}x}+2x\frac{\mathrm{d}p}{\mathrm{d}x}+2p$$

or

$$3p^2\ \mathrm{d}p+2x\ \mathrm{d}p+p\ \mathrm{d}x=0$$

When $p\neq 0$, we notice that $p$ can act as an integrating factor. Multiply it,

$$(3p^3+2xp)\ \mathrm{d}p+p^2\ \mathrm{d}x=0$$

Integrating it, we have

$$\frac{3p^4}{4}+xp^2=c\Rightarrow x=\frac{c-\frac{3}{4}p^4}{p^2},y=p^3+\frac{2\left(c-\frac{3}{4}p^4\right)}{p}$$

2.

$$x=f\left(y,\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

The same as the first type. For $p=dy/dx$

$$\frac{1}{p}=\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}y} \tag{12}$$

We reexamine example 2.4.1:

**Example 2.4.2.** Solve the equation by the above method

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+2x\frac{\mathrm{d}y}{\mathrm{d}x}-y=0$$

*solution*:
We need to express $x$ in terms of $y$ and $dy/dx$. It is easy to see

$$x=\frac{y-p^3}{2p}$$

for $p=y'\neq 0$. Then by formula (12)

$$\frac{1}{p}=\frac{1}{2p}+\left(\frac{-y}{2p^2}-p\right)\frac{\mathrm{d}p}{\mathrm{d}y}$$

and

$$\begin{aligned}
&p\ \mathrm{d}y+(2p^3+y)\ \mathrm{d}p=0\\
&p\ \mathrm{d}y+y\ \mathrm{d}p+2p^3\ \mathrm{d}p=0\\
&\mathrm{d}(yp)+2p^3\ \mathrm{d}p=0\\
&\int\mathrm{d}(yp)+2p^3\ \mathrm{d}p=yp+\frac{p^4}{2}=C\\
&y=\frac{c-p^4}{2p},x=\frac{y-p^3}{2p}=\frac{c-3p^4}{4p^2}
\end{aligned}$$

which is the parametrized general solution. Of course, $y = 0$ is still a solution of the equation.

3.
$$F\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{13}$$

Let
$$p = \frac{\mathrm{d}y}{\mathrm{d}x} = y'$$

From a geometric viewpoint, $F(x, p) = 0$ represents a curve on the Oxp plane. Parametrize it properly,

$$\begin{cases} x = \varphi(t) \\ p = \psi(t) \end{cases}$$

where $t$ is the parameter. Notice that on any integral curve solved from the equation (13), the relation $\mathrm{d}y = p\ \mathrm{d}x$ is always true. Plugging the parametrization into the equation (13), we have

$$\mathrm{d}y = \psi(t)\varphi'(t)\ \mathrm{d}t$$

Integrating both sides, we have
$$y = \int \psi(t)\varphi'(t)\ \mathrm{d}t + c$$

so the general solution of the equation is

$$\begin{cases} x = \varphi(t) \\ y = \int \psi(t)\varphi'(t)\ \mathrm{d}t + c \end{cases}$$

**Example 2.4.3.** solve the ode
$$x^3 + y'^3 - 3xy' = 0,$$

where $y' = dy/dx$. Let $y' = p = tx$, then

$$\begin{cases} x = \frac{3t}{1+t^3} \\ p = \frac{3t^2}{1+t^3} \end{cases}$$

Thus,
$$\mathrm{d}y = p\ \mathrm{d}x = \frac{3t^2}{1+t^3}\left(\frac{3t}{1+t^3}\right)' dt = \frac{9(1-2t^3)t^2}{(1+t^3)^3}\ \mathrm{d}t$$

Integrating it, we have
$$y = \int \frac{9(1-2t^3)t^2}{(1+t^3)^3}\ \mathrm{d}t = \frac{3}{2}\frac{1+4t^3}{(1+t^3)^2} + c$$

The general solution in its parametrized form is thus

$$\begin{cases} x = \frac{3t}{1+t^3} \\ y = \frac{3}{2}\frac{1+4t^3}{(1+t^3)^2} + c \end{cases}$$

4.
$$F\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0$$

Let $p = y'$, and then
$$\begin{cases} y = \varphi(t) \\ p = \psi(t) \end{cases}$$

As $\mathrm{d}y = p\,\mathrm{d}x \Rightarrow \varphi'(t)\,\mathrm{d}t = \psi(t)\,\mathrm{d}x$, we have

$$\mathrm{d}x = \frac{\varphi'(t)}{\psi(t)}\,\mathrm{d}t$$

and

$$x = \int \frac{\varphi'(t)}{\psi(t)}\,\mathrm{d}t + c$$

Similarly, the general solution of this equation is

$$\begin{cases} x = \int \frac{\varphi'(t)}{\psi(t)}\,\mathrm{d}t + c \\ y = \varphi(t) \end{cases}$$

Besides, it is easy to see that $y = k$ is also a solution to $F(y, 0) = 0$ and is thus a solution to the equation.

**Example 2.4.4.** solve the ode

$$y^2(1 - y') = (2 - y')^2$$

*solution*:
let $2 - y' = yt$, or $y' = 2 - yt$, thus

$$y^2(1 - y') = y^2(yt - 1) = (yt)^2$$

and

$$y = \frac{1}{t} + t, y' = 1 - t^2$$

Thus,

$$\mathrm{d}x = \frac{\mathrm{d}y}{y'} = -\frac{1}{t^2}\,\mathrm{d}t$$

Integrating it we have $x = 1/t + c$ and

$$y = x + \frac{1}{x - c} - c$$

. Also notice that $F(y, y' = 0) = 0$ also has solution $y = \pm 2$.

**Exercise (7)**

*solve the following odes:*

1. $$xy'^3 = 1 + y'$$

2. $$y = y'^2 e^{y'}$$

3. $$y(1 + y'^2) = 2a$$

   *(a is a constant)*

4. $$x^2 + y'^2 = 1$$

# Chapter 3

# Linear ODE of Higher Order

We will discuss differential equations of second or higher order in the following chapters, proceeded as follows: Chapter 3 will be attributed to the linear differential equations and system of linear differential equations. Chapter 4 will present some methods for solving general ode of higher order, in particular the order reduction and the power series solution. Chapter 5 involves a brief introduction to some notions and qualitative understanding in nonlinear ode and dynamical system.

## 3.1 General Theory of Linear Differential Equations

### 3.1.1 Introduction

A linear ode of $n$-th order is of the form

$$\sum_{i=0}^{n} a_i(t)\frac{\mathrm{d}^i x}{\mathrm{d}t^i} = a_n(t)\frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \cdots + a_1(t)\frac{\mathrm{d}x}{\mathrm{d}t} + a_0(t)x = f(t) \tag{14}$$

where $a_i(t)$ and $f(t)$ are continuous functions over the interval $[a,b]$. The corresponding homogeneous linear ode of $n$-th order is

$$\sum_{i=0}^{n} a_i(t)\frac{\mathrm{d}^i x}{\mathrm{d}t^i} = a_n(t)\frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \cdots + a_1(t)\frac{\mathrm{d}x}{\mathrm{d}t} + a_0(t)x = 0 \tag{15}$$

and we hence say the the equation (14) is nonhomogeneous. We also have the theorem of existence and uniquess of solution for equation (14):

**Theorem 3.1.1.** *If $a_i(t)$ and $f(t)$ are continuous functions over $[a,b]$, then for any $t_0 \in [a,b]$ and initial conditions $x_0, x_0^{(1)}, \cdots, x_0^{(n-1)}$, the equation (14) exists the only solution $x = \varphi(t)$ defined on the interval $[a,b]$ that satisfies the initial conditions:*

$$\varphi(t_0) = x_0, \frac{d\varphi(t_0)}{dt} = x_0^{(1)}, \cdots, \frac{d^{n-1}\varphi(t_0)}{dt^{n-1}} = x_0^{(n-1)}$$

**Scholium.** *We notice that this is the existence and uniqueness theorem for linear ode of $n$-th order but not general ode of $n$-th order, unlike the counterpart for first order ode. Indeed, Theorem 1.0.1 is for first order ode that is not necessarily linear. The proof of the theorem 3.1.1 is postponed to section 3.3 where we can show that the system of $n$ linear odes is equivalent to $n$-th order linear ode.*

### 3.1.2 The Structure of Solution

We fist consider the homogeneous equation (15) and observe the following principle:

**Theorem 3.1.2** (Superposition Principle)**.** *If $x_1(t), \cdots, x_k(t)$ are $k$ solutions of the equation (15), then the linear combination of them $\sum_{i=1}^k c_i x_i(t)$ is still a solution of (15).*

**Scholium.** *The theorem is immediate from the fact that $(cu)' = cu'$ and $(\sum u_i)' = \sum u'$.*

Before discussing the general solution of the euqation (15), we first give some concepts.

**Definition 3.1.3** (Linear (in)dependence)**.** For functions $x_1(t), \ldots, x_k(t)$ defined on $[a, b]$, if there exists constants $c_1, \ldots, c_k$ not all zero such that

$$\forall t \in [a, b], \sum_{i=1}^{k} c_i x_i(t) \equiv 0$$

we call these functions **linearly dependent**; otherwise **linearly independent**.

For example, functions $\cos t$ and $\sin t$ are linearly independent for any intervals, while functions $\cos^2 t$ and $\sin^2 t - 1$ are linearly dependent over any intervals. Functions $1, t, t^2, \ldots, t^n$ are linearly independent over any intervals, because it is not true that

$$c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n \equiv 0$$

holds for every $t$ in a nondiscrete interval, or equivalently, that every $t$ in a nondiscrete interval is a zero of the above equation.Due to the Fundamental Theorem of Algebra, the above equation at most have $n$ distinct real roots, so the functions are linearly independent if the interval considered contains points other than the roots (which must happen when the interval is of the form $(a, b)$ or $[a, b]$).

**Definition 3.1.4** (Wronskian Determinant)**.** Consider functions $k-1$ times differentiable $x_1(t), \ldots, x_k(t)$ defined on $[a, b]$. Define their **Wronskian Determinant** to be

$$W(t) \equiv W[x_1(t), \ldots, x_k(t)] \equiv \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_k(t) \\ x_1'(t) & x_2'(t) & \cdots & x_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(k-1)}(t) & x_2^{(k-1)}(t) & \cdots & x_k^{(k-1)}(t) \end{vmatrix}$$

**Theorem 3.1.5.** *If functions $x_1(t), x_2(t), \cdots, x_n(t)$ are linearly dependent on the interval $[a, b]$, then the Wronskian det on the interval is $0$.*

*Proof.* Since the functions $x_1(t), x_2(t), \cdots, x_n(t)$ are linearly dependent on the interval $[a, b]$, by definition, there exists not-all-zero coefficients $c_i$ such that

$$\sum_{i=1}^{n} c_i x_1(t) = 0, \forall t \in [a, b]$$

We differentiate it $n-1$ times:

$$\begin{cases} c_1 x_1'(t) + c_2 x_2'(t) + \cdots + c_n x_n'(t) = 0, \\ c_1 x_1''(t) + c_2 x_2''(t) + \cdots + c_n x_n''(t) = 0, \\ \cdots\cdots \\ c_1 x_1^{(n-1)}(t) + c_2 x_2^{(n-1)}(t) + \cdots + c_n x_n^{(n-1)}(t) = 0. \end{cases} \quad t \in [a, b]$$

Notice that we shall regard the above system of linear equations as equations of unknowns "$c_1, c_2, \cdots, c_n$" with coefficient matrix

$$\boldsymbol{A}(t)=\begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{pmatrix}, t \in [a,b]$$

Namely,

$$\boldsymbol{A}(t)\vec{c}=\vec{0} \tag{16}$$

By theory in Linear Algebra, for every fixed $t_0 \in [a,b]$, we have that $\vec{c}(t_0)$, as the solution of the system (16) for $t_0$, is nonzero, implies that the system (16) for $t_0$ has nonzero solution and thus determinant of coefficient matrix euqaling zero. Namely,

$$W(t)=\det(\boldsymbol{A}(t))=0, \forall t \in [a,b]$$

□

**Scholium.** *However, unlike linear algebra, the converse of this theorem is not true. See the following example.*

**Example 3.1.6.** Define two functions

$$x_1(t)=\begin{cases} t^2, -1 \leqslant t < 0 \\ 0, 0 \leqslant t \leqslant 1 \end{cases}$$

and

$$x_2(t)=\begin{cases} 0, -1 \leqslant t < 0 \\ t^2, 0 \leqslant t \leqslant 1 \end{cases}$$

over the interval $[a,b]$. Their Wronskian det is

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{cases} \begin{vmatrix} t^2 & 0 \\ 2t & 0 \end{vmatrix} = 0, -1 \leqslant t < 0 \\ \\ \begin{vmatrix} 0 & t^2 \\ 0 & 2t \end{vmatrix} = 0, 0 \leqslant t \leqslant 1 \end{cases}$$

However, the two functions are linearly independent, verified by solving $c_1x_1(t)+c_2x_2(t)=0$ separately for two intervals.

However, if we add an additional condition to the above theorem 3.1.5 that

$$x_1(t), x_2(t), \cdots, x_n(t)$$

are solutions of homogeneous linear ode (15), we have the following result, whose contrapostive is the converse of the theorem 3.1.5.

**Theorem 3.1.7.** *If the ode (15) has solutions $x_1(t), x_2(t), \cdots, x_n(t)$ linearly dependent on the interval $[a,b]$, then $W(t) \neq 0$ $\forall t \in [a,b]$.*

*Proof.* We prove by contradiction. Suppose there is a $t_0 \in [a,b]$ such that $W(t_0)=0$. Consider thesystem of homogeneous linear equations with respect to $c_1, c_2, \cdots, c_n$:

$$\begin{cases} c_1x_1'(t_0)+c_2x_2'(t_0)+\cdots+c_nx_n'(t_0)=0, \\ c_1x_1''(t_0)+c_2x_2''(t_0)+\cdots+c_nx_n''(t_0)=0, \\ \cdots\cdots \\ c_1x_1^{(n-1)}(t_0)+c_2x_2^{(n-1)}(t_0)+\cdots+c_nx_n^{(n-1)}(t_0)=0. \end{cases}$$

where its det of coefficient matrix $W(t_0) = 0$. Thus, the system has nonzero solution $\vec{c} = (c_1, \cdots, c_n)^T$. Now we use these constants to construct a function

$$x(t) \equiv \sum_{i=1}^{n} c_i x_i(t), t \in [a, b]$$

By the principle of superposition, the function $x(t)$ is a solution of the ode (15). Notice that the system above, line by line, implies that

$$x(t_0) = 0, x'(t_0) = 0, \cdots, x^{(n-1)}(t_0) = 0 \tag{17}$$

In other words, we can say that $x(t)$ is the solution of the ode (15) that satisfies the initial condition (17). However, $x = 0$ is a trivial solution of ode (15) satisfying the initial condition (17) as well. By theorem of existence and uniqueness (theorem 3.1.1), the solution of ode (15) is nonetheless unique. Namely,

$$x(t) = \sum_{i=1}^{n} c_i x_i(t) \equiv 0, t \in [a, b]$$

Since $c_i$'s are not all zero, which contradicts to the linear independence of the solutions $x_i(t)$. □

According to theorem 3.1.5 and theorem 3.1.7, we see that the Wronskian determinant composed of the n solutions of the homogeneous linear ode (15) is $0$ (when solutions are linearly dependent) or nonzero for $t \in [a, b]$ (when the solutions are linear independent).

According to the theorem of eixstence and uniqueness, there uniquely eixts the solutions

$$x_1(t), x_2(t), \cdots, x_n(t)$$

determined by the n batches of initial conditions respectively

$$\begin{cases} x_1(t_0) = 1, x_1'(t_0) = 0, \cdots, & x_1^{(n-1)}(t_0) = 0; \\ x_2(t_0) = 0, x_2'(t_0) = 1, \cdots, & x_2^{(n-1)}(t_0) = 0; \\ \cdots\cdots & \\ x_n(t_0) = 0, x_n'(t_0) = 0, \cdots, & x_n^{(n-1)}(t_0) = 1. \end{cases}$$

Since

$$W[x_1(t)], x_2(t), \cdots, x_n(t)|_{t=t_0} = \begin{vmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_n(t_0) \\ x_1'(t_0) & x_2'(t_0) & \cdots & x_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t_0) & x_2^{(n-1)}(t_0) & \cdots & x_n^{(n-1)}(t_0) \end{vmatrix} = \det(I) = 1 \neq 0,$$

by theorem 3.1.5, the n solutions

$$x_1(t), x_2(t), \cdots, x_n(t)$$

are linearly independent. Therefore, we have the following theorems:

**Theorem 3.1.8.** *Equation (15), homogeneous linear ode of order n, has n linearly independent solutions.*

**Theorem 3.1.9** (Strucutre of General Solution)**.** *If $x_1(t), x_2(t), \cdots, x_n(t)$ are n linearly independent solutions of the equation (15), then the general solution of equation (15) can be expressed as*

$$x = \sum_{i=1}^{n} c_i x_i(t), \tag{18}$$

*where $c_i$ are any constants. Besides, the general solution includes all the solutions of the equation.*

*Proof.* First, by the principle of superposition, we know (18) is the solution of equation (15), consisting of n abitrary constants. We point out that these constants are independent of each other. In fact, by regarding $c_i$ as variables and $x_i(t)$ as constants, we have

$$\begin{aligned}\frac{\partial x}{\partial c_i} =& \frac{\partial x(c_1x_1(t),\cdots,c_nx_n(t))}{\partial c_i} = \frac{\partial x(\xi_1,\cdots,\xi_n)}{\partial c_i}\\ =& \sum_{j=1}^{n}\frac{\partial x}{\partial \xi_j}\frac{\partial \xi_j}{\partial c_i} = \sum_{j=1}^{n}\frac{\partial(\xi_1+\cdots+\xi_n)}{\partial \xi_j}\frac{\partial(c_jx_j(t))}{\partial c_i}\\ =& 1\cdot 0+\cdots+1\cdot x_i(t)+\cdots+1\cdot 0 = x_i(t),\end{aligned}$$

we have

$$\begin{vmatrix} \frac{\partial x}{\partial c_1} & \frac{\partial x}{\partial c_2} & \cdots & \frac{\partial x}{\partial x_n} \\ \frac{\partial x'}{\partial c_1} & \frac{\partial x'}{\partial c_2} & \cdots & \frac{\partial x'}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^{(n-1)}}{\partial c_1} & \frac{\partial x^{(n-1)}}{\partial c_2} & \cdots & \frac{\partial x^{(n-1)}}{\partial x_n} \end{vmatrix} = W[x_1(t),x_2(t),\cdots,x_n(t)] \neq 0, (t\in[a,b])$$

Therefore, (18) is the general solution of the equation (15). Now, we have proved that (18) constains all the solutions. By the solution is uniquely determined by the initial condition, which means that we only need to show: for any given initial condition

$$x(t_0) = x_0, x'(t_0) = x'_0, \cdots, x^{(n-1)}(t_0) = x_0^{(n-1)} \tag{19}$$

we can determine the value of constants $c_i$ to let (18) satisfies (19).

We let (18) satisfies the condition (19) to get the following system of linear equation with respect to $c_1,\cdots,c_n$:

$$\begin{cases} c_1x_1(t_0)+c_2x_2(t_0)+\cdots+c_nx_n(t_0) = x_0,\\ c_1x'_1(t_0)+c_2x'_2(t_0)+\cdots+c_nx'_n(t_0) = x'_0,\\ \cdots\cdots\\ c_1x_1^{(n-1)}(t_0)+c_2x_2^{(n-1)}(t_0)+\cdots+c_nx_n^{(n-1)}(t_0) = x_0^{(n-1)}.\end{cases}$$

whose coefficient det is $W(t_0)\neq 0$ by theorem 3.1.7. By the theory of Linear Algebra, the above system of linear equation has the only solution $\widetilde{c_1},\cdots,\widetilde{c_n}$. Thus, we let the expression (18) take $\widetilde{c_1},\cdots,\widetilde{c_n}$ as constants, and it then satisfies condition (19). □

**Corollary 3.1.10.** *All the solutions of the homogeneous linear ode of order n form an n-dimensional vector space.*

**Exercise (8)**

1. *Let $x(t)$ and $y(t)$ be two continuous functions on the interval $t\in[a,b]$. Prove that if*

$$\frac{x(t)}{y(t)} \neq \textit{constant} \vee \frac{y(t)}{x(t)} \neq \textit{constant}$$

*on the interval $[a,b]$, then $x(t)$ and $y(t)$ are linearly independent on the interval $[a,b]$ (Hint: BWOC).*

### 3.1.3 Method of Variation of Parameters

We first point out two obvious relationships between equation (14) and equation (15):

1. if $x_1(t)$ is a solution of equation (14), and $x_2(t)$ is a solution of euqation (15), then $x_1(t) + x_2(t)$ is still a solution of equation (14).
2. the difference between any solution of equation (14) is a solution of equation (15).

**Theorem 3.1.11.** *Let $x_1(t), \cdots, x_n(t)$ be a basis of solution set of equation (15) and let $x_p(t)$ be a solution of equation (14). Then the general solution of equation (14) can be written as*

$$x(t) = x_p(t) + \sum_{i=1}^{n} c_i x_i(t) = x_p(t) + x_c(t)$$

*where $x_c(t)$ is called **complementary solution** of nonhomogeneous ode (14) and $y_p(t)$ is called a **particular solution** of (14).*

**Method of variation of parameters** requires one to first know the basis (a set of linearly independent solutions) of equation (15) and then helps one to get a solution (and thus the general solution) of equation (16). It is not a particularly efficient method, and we shall only present the result in order 2. Consider

$$ay'' + by' + cy = f$$

first find a fundamental pair $\{y1, y2\}$ of solutions to the corresponding homogeneous equation

$$ay'' + by' + cy = 0$$

Then set

$$y = y_1 c_1 + y_2 c_2 \tag{20}$$

assuming that $c_1 = c_1(x)$ and $c_2 = c_2(x)$ are unknown functions whose derivatives satisfy the system

$$\begin{cases} y_1 c_1' + y_2 c_2' = 0 \\ y_1' c_1' + y_2' c_2' = f/a \end{cases}$$

Solve the system for $c_1'$ and $c_2'$; integrate to get the formulas for $u$ and $v$ , and plug the results back into (20). That formula for $y$ is your solution.

**Example 3.1.12.** Solve the equation

$$x'' + x = \frac{1}{\cos t}$$

where the fundamental pair $\{\sin t, \cos t\}$ is given. We follow the above recipe:
let

$$x(t) = c_1(t)\cos t + c_1(t)\sin t$$

and solve

$$\begin{cases} \cos t c_1' + \sin t c_2' = 0 \\ (\cos t)' c_1' + (\sin t)' c_2' = \frac{1}{\cos t} \end{cases}$$

or

$$\begin{cases} \cos t c_1' + \sin t c_2' = 0 \\ -\sin t c_1' + \cos t c_2' = \frac{1}{\cos t} \end{cases}$$

to get

$$\begin{cases} c_1'(t) = -\frac{\sin t}{\cos t} \\ c_2'(t) = 1 \end{cases} \Rightarrow \begin{cases} c_1(t) = \ln|\cos t| + \gamma_1 \\ c_2'(t) = t + \gamma_2 \end{cases}$$

Therefore, the general solution of the ode is

$$x = \gamma_1 \cos t + \gamma_2 \sin t + \cos t \ln|\cos t| + t \sin t$$

where $\gamma_i$ are any constants.

For more detials and examples, one may visit the link.

**Exercise (9)**

*1. Given fundamental pairs $\{x_1, x_2\}$, solve the following odes:*

*(1)*

$$x'' - x = \cos t, x_1 = e^t, x_2 = e^{-t}$$

*(2)*

$$x'' + \frac{t}{1-t}x' - \frac{1}{1-t}x = t - 1, x_1 = t, x_2 = e^t$$

*(3)*

$$t^2 x'' - 4tx' + 6x = 36\frac{\ln t}{t}, x_1 = t^2, x_2 = t^3$$

*(4)*

$$t^2 x'' - 3tx' = 8x = 18t^2 \sin(\ln t), x_1 = t^2 \cos(2\ln t), x_2 = t^2 \sin 2\ln t$$

*2. let $x_i(t)(i = 1, 2, \cdots, n)$ be any n solutions of the the homogeneous linear ode (15), and let $W(t)$ be their Wronskian determinant. Prove that $W(t)$ satisfies the following first order linear ode:*

$$W' + a_1(t)W = 0$$

*and thus*

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} a_1(s)\, ds\right) t_0, t \in (a, b)$$

## 3.2 Constant Coefficient Linear Differential Equations

In this section, we focus on solving linear ode with constant coefficients:

$$\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i x}{\mathrm{d}t^i} = a_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = f(t) \tag{21}$$

with its corresponding homogeneous equation

$$\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i x}{\mathrm{d}t^i} = a_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = 0 \tag{22}$$

where the coefficients $a_0, a_1, \cdots, a_n$ are real constants with $a_n \neq 0$. Besides, we introduce the **differential operator**

$$L := \sum_{i=0}^{n} a_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} = a_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0$$

as a handful notation.

By the theorem on structure of general solution of linear ode, we see that we need to find n linearly independent solutions of equation (22) first. The tool we will look at for this constant-coefficient homogeneous linear ode is called the **Euler Characteristic Equation**.

### 3.2.1 The Characteristic Equation

Review the first order homogeneous linear ode

$$\frac{\mathrm{d}x}{\mathrm{d}t} + ax = 0$$

we know that it has solution of the form $x = e^{-at}$, and the general solution of it is just $x = ce^{-at}$ with constant $c$. This inspires us to find exponential solutions

$$x = e^{\lambda t} \tag{23}$$

for equation (22), where $\lambda$ is an undetermined coefficient.

Notice that

$$L[e^{\lambda t}] \equiv \sum_{i=0}^{n} a_i \frac{\mathrm{d}^i e^{\lambda t}}{\mathrm{d}t^i} = \sum_{i=0}^{n} a_i \lambda^i e^{\lambda t} \equiv F(\lambda) e^{\lambda t} \tag{24}$$

where $F(\lambda) := \sum_{i=0}^{n} a_i \lambda^i$ is an n-th order polynomial of $\lambda$. Since $e^{\lambda t} > 0$, we see that
(23) solves (22) $\Leftrightarrow F(\lambda) = 0 \Leftrightarrow \lambda$ is the root of the algebraic equation

$$F(\lambda) \equiv \sum_{i=0}^{n} a_i \lambda^i = a_n \lambda^n + \cdots + a_1 \lambda + a_0 = 0 \tag{24}$$

which is called the **Characteristic Equation** or **Auxiliary Equation**, whose roots are called **Characteristic Roots**.

According to the fundamental theorem of algebra, every nth-degree polynomial has n zeros, though not necessarily distinct and not necessarily real. We first check the easiest situation:

### 3.2.2 Distinct Roots (Real and Complex)

Suppose that we have solved the characteristic equation with n distinct real roots

$$\lambda_1, \lambda_2, \cdots, \lambda_n$$

Then the functions

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}$$

are solutions of equation (22) and are linearly independent over $t \in [a, b]$ because

$$\begin{aligned}
W(t) = &\begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_n t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \cdots & \lambda_n e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 t} & \lambda_2^{n-1} e^{\lambda_2 t} & \cdots & \lambda_n^{n-1} e^{\lambda_n t} \end{vmatrix} = e^{\lambda_1 + \cdot + \lambda_n} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} \\
=& e^{\lambda_1 + \cdot + \lambda_n} \det \boldsymbol{Vandermonde}_{n \times n}(\lambda) (\text{where } V_{i,j} = \lambda_i^{j-1}) \\
=& e^{\lambda_1 + \cdot + \lambda_n} \prod_{1 \leqslant j < i \leqslant n} (\lambda_i - \lambda_j) \\
\neq& 0 \text{ due to distinctness of roots and exp's positivity}
\end{aligned}$$

We further divide the situation into two subcases:

1. All roots are real, then

$$e^{\lambda_1 t}, \cdots, e^{\lambda_n t}$$

are n linearly independent real solutions of equation (22) $L[x] = 0$, and the general solution is

$$x = c_1 e^{\lambda_1 t} + \cdots + c_n e^{\lambda_n t}$$

2. There are complex roots among $\lambda_i$, then the complex roots appear in pairs:
   if $\lambda_1 = \alpha + i\beta$ is a characteristic root, then its conjugate $\lambda_2 = \alpha - i\beta$ is also a characteristic root. the equation (22) $L[x] = 0$ hence has two complex solutions:

$$\begin{aligned} e^{\lambda_1 t} &= e^{\alpha + i\beta} = e^{\alpha t}(\cos\beta t + i\sin\beta t) \\ e^{\lambda_2 t} &= e^{\alpha - i\beta} = e^{\alpha t}(\cos\beta t - i\sin\beta t) \end{aligned}$$

   According to the following lemma, we see that their real and imaginary parts are also solutions of $L[x] = 0$, and we then also get two real solutions

$$e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t$$

**Lemma 3.2.1.** *If all the coefficients $a_i(t)$ in the equation (15) are real-valued functions and $z(t) = \varphi(t) + i\psi(t)$ is a complex solution of the equation, then the real part and imaginary part of $z(t)$ and the conjugate $\bar{z}(t)$ are all solutions of the equation.*
*Furthermore, if the equation*

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1(t)\frac{dx}{dt} + a_0(t)x = f(t) = u(t) + iv(t)$$

*has complex solution $x = R(t) + iI(t)$, then $R(t)$ and $I(t)$ solves the following equations respectively:*

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1(t)\frac{dx}{dt} + a_0(t)x = u(t)$$

*and*

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1(t)\frac{dx}{dt} + a_0(t)x = v(t)$$

### 3.2.3 Polynomial Differential Operators

Before examining the case where there are repeated roots in the characteristic equation, we shall learn some basic facts about the polynomial differential operator

$$L \equiv \sum_{i=0}^{n} a_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} = a_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0$$

We also denote $D = \frac{\mathrm{d}}{\mathrm{d}t}$ and thus $D^n = \frac{\mathrm{d}^n}{\mathrm{d}t^n}$. Accordingly, we have

$$L \equiv \sum_{i=0}^{n} a_i D^i = a_n D^n + \cdots + a_1 D + a_0 \tag{25}$$

It turns out that it is useful to think of the right-hand side in (25) as a (formal) n-th degree polynomial in the "variable" $D$; it is a polynomial differential operator. For example, A first-degree polynomial operator with leading coefficient $1$ has the form $D - a$, where $a$ is a real number. It operates on a function $x = x(t)$ to produce

$$(D - a)x = Dx - ax = x' - ax$$

The important fact about such operators is that any two of them commute:

$$(D - a)(D - b)x = (D - b)(D - a)x$$

for any twice differentiable function $x = x(t)$. The proof of the formula in (7) is the following computation:

$$\begin{aligned}(D-a)(D-b)x &= (D-a)(x'-bx)\\ &= Dx' - ax' - bDx + abx\\ &= x'' - ax' - bx' + abx\\ &= D(x'-ax) - b(x'-ax)\\ &= (D-b)(D-a)x\end{aligned}$$

We see from the proof that

$$(D-a)(D-b) = D^2 - (a+b)D + ab$$

Similarly, it can be shown by induction on the number of factors that an operator product of the form

$$(D-a_1)(D-a_2)\cdots(D-a_n)$$

expands–by multiplying out and collecting coefficients–in the same way as does an ordinary product of algebraic polynomials, regarding $D$ as variables in polynomials. Consequently, the algebra of polynomial differential operators closely resembles the algebra of ordinary real polynomials.

### 3.2.4 Repeated Roots (Real and Complex)

Let us now consider the possibility that the characteristic equation

$$F(\lambda) \equiv \sum_{i=0}^{n} a_i\lambda^i = a_n\lambda^n + \cdots + a_1\lambda + a_0 = 0 \tag{26}$$

has repeated roots. For example, suppose that (26) has only two distinct roots, $\lambda_0$ of multiplicity 1 and $\lambda_1$ of multiplicity $k = n-1 > 1$. Then (after dividing by $a_n$) (26) can be rewritten in the form

$$(\lambda-\lambda_1)^k(\lambda-\lambda_0) = 0$$

Similarly, the corresponding operator $L$ can be written as

$$L = (D-\lambda_1)^k(D-\lambda_0)$$

because we notice that

$$F(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$$

is a the polynomial sharing the same coefficients with

$$L = a_nD^n + \cdots + a_1D + a_0$$

The two polynomials naturally have the same factorization.
Two solutions of the differential equation $L[x] = 0$ are certainly $x_0 = e^{\lambda_0 t}$ and $x_1 = e^{\lambda_1 t}$. This is, however, not sufficient; we need $k+1$ linearly independent solutions in order to construct a general solution, because the equation is of order $k+1$. To find the missing $k-1$ solutions, we note that

$$L[x] = (D-\lambda_0)[(D-\lambda_1)^k x] = 0$$

Consequently, every solution of the k-th order equation

$$(D-\lambda_1)^k x = 0 \tag{27}$$

will also be a solution of the original equation $L[x] = 0$. Hence our problem is reduced to that of finding a general solution of the differential equation in (27). The fact that $x_1 = e^{\lambda_1 t}$ is one solution of (27) suggests that we try the substitution

$$x(t) = u(t)x_1(t) = u(t)e^{\lambda_1 t}$$

where $u(t)$ is a function yet to be determined. Observe that

$$\begin{aligned}(D-\lambda_1)[ue^{\lambda_1 t}] =& D(ue^{\lambda_1 t}) - \lambda_1 ue^{\lambda_1 t}\\ =&(Du)e^{\lambda_1 t} + uD(e^{\lambda_1 t}) - \lambda_1 ue^{\lambda_1 t}\\ =&(Du)e^{\lambda_1 t}\end{aligned}$$

Upon $k$ applications of this fact, it follows that

$$(D-\lambda_1)^k[ue^{\lambda_1 t}] = (D^k u)e^{\lambda_1 t}$$

for any sufficiently differentiable function $u(t)$. Hence $x = ue^{\lambda_1 t}$ will be a solution of equation (27) if and only if $D^k u = u^{(k)} = 0$. But this is so if and only if

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_k t^{k-1}$$

a polynomial of degree at most $k-1$. Hence our desired solution of equation (27) is

$$x(t) = ue^{\lambda_1 t} = (c_1 + c_2 t + c_3 t^2 + \cdots + c_k t^{k-1})e^{\lambda_1 t}$$

In particular, we see here the additional solutions $te^{\lambda_1 t}$, $t^2 e^{\lambda_1 t}$, $\cdots$, $t^{k-1}e^{\lambda_1 t}$ of the original ode $L[x] = 0$.

The preceding analysis can be carried out with the operator $D - \lambda_1$ replaced with an arbitrary polynomial operator. When this is done, the result is a proof of the following theorem.

**Theorem 3.2.2.** *If the characteristic equation in (26) has a repeated root $\lambda$ of multiplicity $k$, then the part of a general soution of the differential equation in (22) corresponding to $\lambda$ is of the form*

$$(c_1 + c_2 t + c_3 t^2 + \cdots + c_k t^{k-1})e^{\lambda t}$$

We may notice that the $k$ functions

$$e^{\lambda t}, te^{\lambda t}, \cdots, t^{k-1}e^{\lambda t}$$

are linearly independent on $\mathbb{R}$. Thus a root of multiplicity $k$ corresponds to $k$ linearly independent solutions of the differential equation. And the end of this section we will prove that $k_i$ solutions contributed by $i$ roots each of multiplicity $k_i$ are linearly independent (Lemma 3.2.4).

**Example 3.2.3.** Find a general solution of the fifth-order differential equation regarding the function $y = y(x)$:

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

*solution*:
The characteristic equation inspirational

$$9\lambda^5 - 6\lambda^4 + \lambda^3 = \lambda^3(9\lambda^2 - 6\lambda + 1) = \lambda^3(3\lambda - 1)^2 = 0$$

It has triple root $\lambda = 0$ and the double root $\lambda = \frac{1}{3}$. The triple root $\lambda = 0$ contributes

$$c_1 e^{0x} + c_2 xe^{0x} + c_3 x^3 e^{0x} = c_1 + c_2 x + c_3 x^2$$

to the solution, while the double root $\lambda = \frac{1}{3}$ contributes $c_4 e^{x/3} + c_5 xe^{x/3}$. Hence a general solution of the given differential equation is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{x/3} + c_5 xe^{x/3}$$

**Example 3.2.4.** Solve the equation

$$(D^3+1)x=0$$

*solution*:
Since $-1=1\cdot e^{-i\pi}$ $(r=1,\theta=-\pi)$ we see that the characteristic equation $\lambda^3+1=0$ has solution

$$\lambda_k=\sqrt[n]{r}\exp\left(i\frac{\theta+2k\pi}{n}\right)=\sqrt[3]{1}\exp\left(i\frac{-\pi+2k\pi}{3}\right)=e^{i\frac{\pi}{3}},e^{i\pi},e^{i\frac{5\pi}{3}},\ k=1,2,3$$

It other words,

$$\lambda_1=\frac{1}{2}+i\frac{\sqrt{3}}{2},\lambda_2=-1,\lambda_3=\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

The general solution is then $x=c_1e^{-t}+e^{t/2}[c_2\cos(\sqrt{3}t/2)+c_3\sin(\sqrt{3}t/2)]$

**Example 3.2.5.** The characteristic equation of the differential equation

$$y^{(3)}+y'-10y=0$$

is the cubic equation

$$r^3+r-10=0$$

To find the roots of the above polynomial, we fist observe that for a third order polynomial, which by FTA has 3 roots, of the form

$$P(x)=(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$$

hints us that the roots of the polynomial have to be the factors of the constant term. We may call this observation theorem of elementary algebra, or **TEA** (I don't know if there is a standard name for it).
By TEA, we see that the only possible rational roots are the factors $\pm1$, $\pm2$, $\pm5$, and $\pm10$ of the constant term $10$. By trial and error (if not by inspection) we discover the root $2$. We then divide the polynomial by $r-2$ to get

$$r^3+r-10=(r-2)(r^2+2r+5)=(r-2)[(r+1)^2+4]=0$$

We get the roots

$$r_1=2,r_2=-1+2i,r_3=-1-2i$$

Therefore, the general solution is given by

$$y(x)=c_1e^2x+c_2e^{-1t}\cos 2t+c_3e^{-1t}\sin 2t$$

For the case where the characteristic equation (26) has **complex repeated roots**, say $\lambda=\alpha+i\beta$ is a root of multiplicity $k$, then $\bar{\lambda}=\alpha-i\beta$ is also a root of multiplicity $k$. Similar to the unrepeated complex case, we can derive $2k$ real solutions

$$e^{\alpha t}\cos\beta t,te^{\alpha t}\cos\beta t,\cdots,t^{k-1}e^{\alpha t}\cos\beta t$$

and

$$e^{\alpha t}\sin\beta t,te^{\alpha t}\sin\beta t,\cdots,t^{k-1}e^{\alpha t}\sin\beta t$$

**Example 3.2.6.** The roots of the characteristic equation of a certain differential equation are $3,-5,0,0,0,0,-5,2\pm 3i$, and $2\pm3i$. Write a general solution of this homogeneous differential equation.
*solution*:
The solution can be read directly from the list of roots. It is

$$\begin{aligned}y(x)=&c_1+c_2x+c_3x^2+c_4x^3+c_5x^{3x}+c_6e^{-5x}+c_7xe^{-5x}\\&e^{2x}(c_8\cos 3x+c_9\sin 3x)+xe^{2x}(c_{10}\cos 3x+c_{11}\sin 3x)\end{aligned}$$

**Lemma 3.2.7.** *Suppose the characteristic equation (26) has roots $\lambda_1, \cdots, \lambda_m$ of multiplicity $k_1, \cdots, k_m$ respectively ($k_i \geqslant 1$, and unrepeated root $\lambda_j$ has $k_j = 1$), and $k_1 + \cdots + k_m = n$, $\lambda_i \neq \lambda_j$ when $i \neq j$, then the equation $L[x] = 0$ has solutions*

$$\begin{cases} e^{\lambda_1 t}, te^{\lambda_1 t} \cdots, t^{k_1 - 1} e^{\lambda_1 t} \\ e^{\lambda_2 t}, te^{\lambda_2 t} \cdots, t^{k_1 - 1} e^{\lambda_2 t} \\ \cdots\cdots \\ e^{\lambda_m t}, te^{\lambda_1 t} \cdots, t^{k_m - 1} e^{\lambda_m t} \end{cases}$$

*and we need to show that these solutions are linearly independent.*

*Proof.* BWOC (abbr. of "by way of contradiction"), suppose these functions are linearly dependent. Then,

$$\sum_{r=1}^{m} (A_0^{(r)} + A_1^{(r)} t + \cdots + A_{k_r - 1}^{(r)} t^{k_r - 1}) e^{\lambda_r t} := \sum_{r=1}^{m} P_r(t) e^{\lambda_r t} = 0 \tag{28}$$

where $A_j^{(r)}$ are constants that are not all zero. WLOG, we let the polynomial $P_m(t)$ have at least a non-zero coefficient. Hence $P_m(t) \neq 0$. Divide eq. (28) by $e^{\lambda_1 t}$ and differentiate $k_1$ times with respect to $t$ to get

$$\sum_{r=1}^{m} D^{k_1} [P_r(t) e^{(\lambda_r - \lambda_1)t}] = \sum_{r=2}^{m} Q_r(t) e^{(\lambda_r - \lambda_1)t} = 0 \tag{29}$$

where $Q_r(t) = (\lambda_r - \lambda_1)^{k_1} P_r(t) + S_r(t)$, with $S_r(t)$ a polynomial or a lower order than $P_r(t)$, so that $P_r(t)$ and $Q_r(t)$ have the same order and $Q_m(t) \neq 0$. Eq. (29) and eq. (28) are similar but has less terms. If we do the same process as eq. (28) to eq. (29) (i.e. divide eq. (29) by $e^{(\lambda_2 - \lambda_1)t}$ and differentiate it by $k_2$ times), then we will get an equation with fewer items. If we keep doing this, after $m - 1$ times, we will get the following equation

$$R_m(t) e^{(\lambda_m - \lambda_{m-1})t} = 0$$

which is impossible by $e^x > 0$, $R_m(t) \neq 0$ and the fact that $R_m(t)$ and $P_m(t)$ have the same order. It is not hard to calculate that

$$R_m(t) = (\lambda_m - \lambda_1)^{k_1} (\lambda_m - \lambda_2)^{k_2} \cdots (\lambda_m - \lambda_{m-1})^{k_{m-1}} P_m(t) + W_m(t)$$

with $W_m(t)$ a polynomial or lower order than $P_m(t)$. □

We ended this subsection with a notable example.

**Example 3.2.8** (Euler–Cauchy Equation). We call the ode of the following form

$$(x^n D^n + a_1 x^n - 1 D^{n-1} + \cdots + a_{n-1} x D + a_n) y = \sum_{i=1}^{n} a_{n-1} D^i y = 0$$

the **Euler–Cauchy Equation**. Notice that we can always assume $a_0 = 1$ becasue otherwise we can divide the whole eq. by $a_0$. This eq. is solvable by method of substitution to transform it into a homogeneous linear ode. In fact, introduce*

$$x = e^t, t = \ln x$$

One may see the procedure on wikipedia for second order. We only give the conclusion below. The characteristic equation

$$K(K-1) \cdots (K - n + 1) + a_1 K(K-1) \cdots (K - n + 2) + \cdots + a_n = 0$$

*if $x < 0$ then use $x = -e^t$ to get the same result. For convenience, we assume $x > 0$ and plug in $t = \ln |x|$ at the end.

determines the roots. For each root $K = K_0$ of multiplicity $m$, we have $m$ solutions of the equation

$$x^{K_0}, x^{K_0}\ln|x|, x^{K_0}\ln^2|x|, \cdots, x^{K_0}\ln^{m-1}|x|$$

If $K = K_0 = \alpha + i\beta$, then we get $2m$ real solutions

$$x^{\alpha}\cos(\beta\ln|x|), x^{\alpha}\ln|x|\cos(\beta\ln|x|), \cdots, x^{\alpha}\ln^{m-1}|x|\cos(\beta\ln|x|)$$

$$x^{\alpha}\sin(\beta\ln|x|), x^{\alpha}\ln|x|\sin(\beta\ln|x|), \cdots, x^{\alpha}\ln^{m-1}|x|\sin(\beta\ln|x|)$$

### 3.2.5 Nonhomogeneous Linear ODE: Method of Undetermined Coefficients

We have discussed in subsection 3.1.3 for solving nonhomogeneous linear ode by first solving its homogeneous counterpart and then using method of variation of parameters to find a particular solution. However, there are some special forms of nonhomogeneous linear ode which we can solve directly in a neater way. We will talk about the method of undetermined coefficients in this subsection and Laplace Transform method in the next subsection.

Given a nonhomogeneous linear ode of $n$-th order.

$$L[x] := \frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_1\frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1}\frac{\mathrm{d}x}{\mathrm{d}t} + a_n x = f(t) \tag{30}$$

whose characteristic equation is

$$F(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

We by the type of $f(t)$ consider two types that can be applied with the **Method of Undetermined Coefficients**.

I . Polynomial & exponential form: $f(t) = (b_0t^m + b_1t^{m-1} + \cdots + b_{m-1}t + b_m)e^{\lambda_0 t}$

II . Triangular form: $f(t) = [A(t)\cos\beta t + B(t)\cos\beta t]e^{\alpha t}$

Type I:
For $f(t) = (b_0t^m + b_1t^{m-1} + \cdots + b_{m-1}t + b_m)e^{\lambda_0 t}$, where $\lambda_0$ can be $0$, we have a particular solution of the form

$$\widetilde{x} = t^k(B_0t^m + B_1t^{m-1} + \cdots + B_{m-1} + B_m)e^{\lambda_0 t} \tag{31}$$

where $k$ is the multiplicity of the root $\lambda_0$ of the characteristic equation $F(\lambda) = 0$ (when $\lambda_0$ is a distinct root, take $k = 1$; when $\lambda_0$ is not a root of the characteristic equation, take $k = 0$), and $B_0, B_1, \cdots, B_m$ are coefficients to be determined (plugging the corresponded form back into the original equation). For the case $\lambda_0 \neq 0$, we can do the transformation $x = ye^{\lambda_0 t}$ and transform the solution back after solving the transformed equation.

**Example 3.2.9.** Solve the ode

$$(D^2 - 2D - 3)x = 3t + 1$$

The first step is always finding the general solution of the homogeneous counterpart

$$(D^2 - 2D - 3)x = 0$$

The characteristic eq. $F(\lambda) = \lambda^2 - 2\lambda - 3 = 0$ has solutions $\lambda_1 = 3, \lambda_2 = -1$. Hence the general solution of the homogeneous eq. is $x_c = c_1e^{3t} + c_2e^{-t}$. We then need to find a particular solution of the original. Notice that $\lambda_0 = 0$ is not a root of the characteristic eq. Thus, we set a particular solution a polynomial of the same order as $f(t) = 3t + 1$

$$x_p = B_0t + B_1$$

Plugging it into the original results in

$$-2B_0 - 3B_0t - 3B_1 = 3t + 1 \Rightarrow -3B_0 = 3, -2B_0 - 3B_1 = 1 \Rightarrow B_0 = -1B_1 = \frac{1}{3}$$

Thus, $x_p = -t + \frac{1}{3}$ and the general solution of the ode $(D^2 - 2D - 3)x = 3t + 1$ is

$$x(t) = x_c + x_p = c_1e^{3t} + c_2e^{-t} - t + \frac{1}{3}$$

**Example 3.2.10.** Solve the ode

$$(D^2 - 2D - 3)x = e^{-2t}$$

We from example 3.2.9 get the general solution of the corresponding homogeneous eq.

$$x_c = c_1e^{3t} + c_2e^{-t}$$

We now seek a particular solution of the equation. Since $f(t) = e^{-2t}$ and $\lambda_0 = -2$ is not a root of the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$, we see that a particular solution is $\widetilde{x} = Ae^{-2t}$, which, plugged into the original eq, results in

$$4Ae^{-2t} + 4Ae^{-2t} - 3Ae^{-2t} = e^{-2t} \Rightarrow 4A + 4A - 3A = 1 \Rightarrow A = 1/5, x_p = \frac{1}{5}e^{-2t}$$

The general solution is then

$$x(t) = x_c + x_p = c_1e^{3t} + c_2e^{-t} + \frac{1}{5}e^{-2t}$$

**Example 3.2.11.** Solve the ode

$$(D^3 + 3D^2 + 3D + 1)x = e^{-t}(t - 5)$$

Its characteristic equation $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$ has $\lambda_{1,2,3} = -1$ as a root of multiplicity 3. The general solution of the homogeneous equation is then

$$x_c = (c_1 + c_2t + c_3t^2)e^{-t}$$

A particular solution of the equation is given by

$$\widetilde{x} = t^3(A + Bt)e^{-t}$$

which, plugged into the original eq, results in

$$(6A + 24Bt)e^{-t} = e^{-t}(t - 5) \Rightarrow A = -\frac{5}{6}, B = \frac{1}{24} \Rightarrow x_p = \frac{1}{24}t^3(t - 20)e^{-t}$$

The general solution is then

$$x(t) = x_c + x_p = (c_1 + c_2t + c_3t^2)e^{-t} + \frac{1}{24}t^3(t - 20)e^{-t}$$

Type II:
For $f(t) = [A(t)\cos\beta t + B(t)\cos\beta t]e^{\alpha t}$, where $\alpha, \beta$ are constants, $A(t), B(t)$ are real-coefficient polynomials with their highest order denoted as $m$, we have particular solution

$$\widetilde{x} = t^k[P(t)\cos\beta t + Q(t)\sin\beta t]e^{\alpha t} \tag{32}$$

where $k$ is the multiplicity of the root $\alpha + i\beta$ of the characteristic equqation $F(\lambda) = 0$, and $P(t), Q(t)$ are real-coefficient polynomials of order lower than $m$ to be determined by plugging in. Note that if $\alpha + i\beta$ is a root of a polynomial then the conjugate $\alpha + i\beta$ is also a root (Complex Conjugate Root Theorem).

**Example 3.2.12.** Solve the ode

$$(D^2+4D+4)x=\cos 2t$$

Its characteristic equation $\lambda^2+4\lambda+4=(\lambda+2)^2=0$ has $\lambda_{1,2}=-2$ as a root of multiplicity 2. The general solution of the homogeneous equation is then

$$x_c=(c_1+c_2t)e^{-2t}$$

Notice that

$$A(t)=1, B(t)=0, m=0, \beta=2, \alpha=0$$

Since $\pm 2i$ is not a root of $F(\lambda)=0$, A particular solution of the equation is then given by

$$\widetilde{x}=A\cos 2t+B\cos 2t$$

which, plugged into the original eq, results in

$$8B\cos 2t-8A\sin 2t=\cos 2t \Rightarrow A=0, B=\frac{1}{8}\Rightarrow x_p=\frac{1}{8}\sin 2t$$

The general solution is then

$$x(t)=x_c+x_p=(c_1+c_2t)e^{-2t}+\frac{1}{8}\sin 2t$$

**Example 3.2.13.** Determine the appropriate form for a particular solution of the fifth-order equation

$$(D-2)^3(D^2+9)x=t^2e^{2t}+t\sin 3t$$

Its characteristic equation $(\lambda-2)^3(\lambda^2+9)=0$ has roots $\lambda=2,2,2,+3i,-3i$. The general solution of the homogeneous equation is then

$$x_c=(c_1+c_2t+c_3t^2)e^{2t}+c_4\cos 3t+c_5\sin 3t$$

We get a particular solution of the form

$$\widetilde{x}=t^3[At^2+Bt+C]e^{2t}+t[(Dt+E)\cos 3t+(Ft+G)\sin 3t]$$

**Exercise (10)**

1. *Solve the following odes:*

*(1)*

$$x''+x=\sin t-\cos 2t$$

*(2)*

$$x''-4x'+4x=e^t+e^{2t}+1$$

*(3)*

$$x''-2x'+2x=te^t\cos t$$

### 3.2.6 Nonhomogeneous Linear ODE: Laplace Transform

**Definition 3.2.14** (Laplace Transform)**.** Given a function $f(t)$ defined for all $t \geq 0$, the **Laplace transform** of $f$ is the function $F$ defined as follows:

$$F : D \subseteq \mathbb{C} \to \mathbb{C}$$
$$s \mapsto F(s) = \mathscr{L}[f(t)](s) := \int_0^{+\infty} f(t)e^{-st}\, \mathrm{d}t$$

where $D$ is the domain of the transform where the integral converges for $s \in D$.

In fact, the domain $D$ always takes the form Re$(s) > \sigma$ for which $|f(t)| < Me^{\sigma t}$ with some positive constants $\sigma$ and $M$. $|\cdot|$ stands for the norm (modulus) in $\mathbb{C}$. Below is a table of frequently used Laplace transforms.

| function $f(t) = \mathscr{L}^{-1}[F(s)](t)$ | transform $\mathscr{L}[f(t)](s)$ | Domain $D$ |
|---|---|---|
| unit step: $u(t) = \mathbb{1}_{[0,+\infty)}$ | $1/s$ | $\Re(s) > 0$ |
| delayed unit step: $u_c(t) = u(t-c) = \mathbb{1}_{[c,+\infty)}$ | $e^{-cs}/s$ | $\Re(s) > 0$ |
| unit impulse (Dirac function): $\delta(t)$ | $1$ | all s |
| delayed unit impulse: $\delta_c(t) = \delta(t-c)$ | $e^{-cs}$ | $\Re(s) > 0$ |
| $t$ | $1/s^2$ | $\Re(s) > 0$ |
| $t^n (n = 0, 1, \cdots)$ | $n!/s^{n+1}$ | $\Re(s) > 0$ |
| $t^q (\Re(q) > -1)$ | $\Gamma(q+1)/s^{q+1}$ | $\Re(s) > 0$ |
| $e^{zt}$ | $1/(s-z)$ | $\Re(s) > \Re(z)$ |
| $te^{zt}$ | $1/(s-z)^2$ | $\Re(s) > \Re(z)$ |
| $t^n e^{zt} (n > -1)$ | $n!/(s-z)^{n+1}$ | $\Re(s) > \Re(z)$ |
| $\sin \omega t$ | $\omega/(s^2+\omega^2)$ | $\Re(s) > 0$ |
| $\cos \omega t$ | $s/(s^2+\omega^2)$ | $\Re(s) > 0$ |
| $\sinh \omega t$ | $\omega/(s^2-\omega^2)$ | $\Re(s) > \lvert\omega\rvert$ |
| $\cosh \omega t$ | $s/(s^2-\omega^2)$ | $\Re(s) < \lvert\omega\rvert$ |
| $t \sin \omega t$ | $2s\omega/(s^2+\omega^2)^2$ | $\Re(s) > 0$ |
| $t \cos \omega t$ | $(s^2-\omega^2)/(s^2+\omega^2)^2$ | $\Re(s) > 0$ |
| $e^{\lambda t} \sin \omega t$ | $\omega/(s-\lambda)^2+\omega^2$ | $\Re(s) > \lambda$ |
| $e^{\lambda t} \cos \omega t$ | $(s-\lambda)/(s-\lambda)^2+\omega^2$ | $\Re(s) > \lambda$ |
| $te^{\lambda t} \sin \omega t$ | $2\omega(s-\lambda)/[(s-\lambda)^2+\omega^2]^2$ | $\Re(s) > \lambda$ |
| $te^{\lambda t} \cos \omega t$ | $(s-\lambda)^2-\omega^2/[(s-\lambda)^2+\omega^2]^2$ | $\Re(s) > \lambda$ |

Now we explain how can Laplace transform help us solve initial value problem:

suppose we have an ode

$$L[x] = \sum_{i=0}^{n} a_i \frac{\mathrm{d}^i x}{\mathrm{d}t^i} = a_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 = f(t), a_n = 1$$

with initial values

$$x(0) = x_0, x'(0) = x_0', \cdots, x^{(n-1)}(0) = x_0^{(n-1)}$$

and $f(t)$ is continuous, defined on $t \in [0, \infty)$ with $|f(t)| < Me^{\sigma t}$. Denote

$$F(s) = \mathscr{L}[f(t)](s) \equiv \int_0^{+\infty} f(t)e^{-st}\, \mathrm{d}t$$
$$X(s) = \mathscr{L}[x(t)](s) \equiv \int_0^{+\infty} x(t)e^{-st}\, \mathrm{d}t$$

Then

$$\begin{aligned}
\mathscr{L}[x'(t)](s) &\equiv \int_0^{+\infty} x'(t)e^{-st}\,\mathrm{d}t = \int_0^{+\infty} e^{-st}\,\mathrm{d}(x(t)) \\
&= \left[e^{-st}x(t)\right]_0^{\infty} - \int_0^{+\infty} (-s)x(t)e^{-st}\,\mathrm{d}t \\
&= 0 - x(0) + s\int_0^{+\infty} x(t)e^{-st}\,\mathrm{d}t = sX(s) - x_0
\end{aligned}$$

By induction (assume $\mathscr{L}[x^{(k)}(t)](s) = s^k X(s) - s^{k-1}x_0 - s^{k-2}x_0' - \cdots - x_0^{(k-1)}$) we see

$$\mathscr{L}[x^{(n)}(t)](s) = s^n X(s) - \sum_{i=0}^{n-1} x_0^{(i)} s^{n-1-i}, n \geq 1$$

because

$$\begin{aligned}
\mathscr{L}[x^{(k+1)}(t)](s) &\equiv \int_0^{+\infty} x^{(k+1)}(t)e^{-st}\,\mathrm{d}t = \int_0^{+\infty} e^{-st}\,\mathrm{d}(x^{(k)}(t)) \\
&= \left[e^{-st}x^{(k)}(t)\right]_0^{\infty} + s\int_0^{+\infty} x^{(k)}(t)e^{-st}\,\mathrm{d}t \\
&= -x^{(k)}(0) + s[x^{(k)}(t)](s) \\
&= s\left(s^k X(s) - \sum_{i=0}^{k-1} x_0^{(i)} s^{k-1-i}\right) - x_0^{(k)} \\
&= s^{k+1}X(s) - \sum_{i=0}^{k} x_0^{(i)} s^{k-i}
\end{aligned}$$

Therefore, transform two sides of the equation $L[x] = f(t)$ gives

$$\begin{aligned}
\mathscr{L}\left[\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i x}{\mathrm{d}t^i}\right] =& a_0 X(s) + \sum_{i=1}^{n} a_i \left[s^{(i)}X(s) - \sum_{j=0}^{i-1} x_0^{(i)} s^{i-1-j}\right] \\
=& a_0 X(s) + a_1\left(sX(s) - x_0\right) \\
&+ a_2\left(s^2 X(s) - sx_0 - x_0'\right) \\
&+ \cdots \\
&+ a_n\left(s^n X(s) - s^{n-1}x_0 - s^{n-2}x_0' - \cdots - x_0^{(n-1)}\right) = F(s)
\end{aligned}$$

Let

$$A(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0 = \sum_{i=0}^{n} a_i s^i$$

$$B(s) = (a_1 + a_2 s + \cdots + a_n s^{n-1})x_0 + (a_2 + \cdots + a_n s^{n-2})x_0' + \cdots + a_n x_0^{(n-1)}$$

Then $L[x] = f(t)$ transforms to

$$A(s)X(s) = F(s) + B(s)$$

Since $A(s), B(s)$, and $F(s)$ are known (computable) polynomials we have

$$X(s) = \frac{F(s) + B(s)}{A(s)} \Rightarrow x(t) = \mathscr{L}^{-1}[X(s)](t) = \mathscr{L}^{-1}\left[\frac{F(s) + B(s)}{A(s)}\right](t)$$

as the solution of the original ode.

Before we explore some of the properties of the Laplace transform, we shall first see examples to which it can be applied.

**Example 3.2.15.** solve the following ivp

$$\frac{\mathrm{d}x}{\mathrm{d}t} - x = e^{2t}, \text{ i.v. } x(0) = 0$$

*solution*:

$$F(s) = \mathscr{L}[e^{2t}](s) = \frac{1}{s-2}, s > 2$$
$$A(s) = s - 1$$
$$B(s) = x_0 = 0$$

Thus

$$X(s) = \frac{F(s) + B(s)}{A(s)} = \frac{\frac{1}{s-2}}{s-1} = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}$$
$$\Rightarrow x(t) = \mathscr{L}^{-1}\left[\frac{1}{s-2} - \frac{1}{s-1}\right](t) = \mathscr{L}^{-1}\left[\frac{1}{s-2}\right](t) - \mathscr{L}^{-1}\left[\frac{1}{s-1}\right](t) = e^{2t} - e^t$$

is the solution of ivp.

**Example 3.2.16.** solve the following ivp

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{-x}, \text{ i.v. } y(1) = y'(1) = 0$$

*solution*: Notice that we need to have initial values at $x = 0$. Hence, we do the translation $z = \varphi(x) = x - 1$ or $x = \varphi^{-1}(z) = \phi(z) = z + 1$ so that

$$y(x) = y(\phi(z))$$
$$\frac{\mathrm{d}y(\phi(z))}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}\phi(z)}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot 1 = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\begin{aligned}
\frac{\mathrm{d}^2 y(\phi(z))}{\mathrm{d}z^2} &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y(\phi(z))}{\mathrm{d}z}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}\phi(z)}{\mathrm{d}z}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}x}(\phi(z))\right) \cdot \frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\phi(z)}{\mathrm{d}z}\right) \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \frac{\mathrm{d}\phi(z)}{\mathrm{d}z}\right] \cdot \frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\phi(z)}{\mathrm{d}z}\right) \\
&= \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\phi(z)}{\mathrm{d}z}\right)^2 + \frac{\mathrm{d}^2\phi(z)}{\mathrm{d}z^2} \\
&= \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \cdot 1^2 + 0 = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}
\end{aligned}$$

Therefore the ivp becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{-z-1} = e^{-1}e^{-z}, \text{ i.v. } y(0) = y'(0) = 0$$

we then perform the Laplace transform method:

$$F(s) = \mathscr{L}[e^{-1}e^{-z}](s) = \frac{1}{e} \cdot \frac{1}{s+1}, s > -1$$

$$A(s) = s^2 + 2s + 1$$
$$B(s) = (1s + 2)y_0 + y_0' = 0$$

Thus

$$Y(s) = \frac{F(s) + B(s)}{A(s)} = \frac{\frac{1}{e} \cdot \frac{1}{s+1}}{s^2 + 2x + 1} = \frac{1}{e} \frac{1}{(s+1)^3} = \frac{1}{2e} \frac{2!}{(s+1)^3}$$
$$\Rightarrow y(z) = \mathscr{L}^{-1}\left[\frac{1}{2e} \frac{2!}{(s+1)^3}\right](z) = \frac{1}{2} \cdot z^2 e^{-z-1}$$

However, one should carefully interpret this result. In particular, writing $y(x) = y(\phi(z))$ will give an incorrect answer because this notation misunderstands $y$ as a mapping. A safer way of writing this can be

$$y = g(z) = \frac{1}{2} \cdot z^2 e^{-z-1}$$

and we want to write $y$ in terms of $x$ by combining the above result with relationship between $x$ and $z$.

$$y = g(z) = g(\varphi(x)) = g(x-1) = \frac{1}{2} \cdot (x-1)^2 e^{-x}$$

which is the solution of ivp.

We then list some properties of the Laplace transform.

**some common total derivatives:**

- $$x \,\mathrm{d}y + y \,\mathrm{d}x = d(xy)$$
- $$\frac{y \,\mathrm{d}x - x \,\mathrm{d}y}{y^2} = \mathrm{d}\left(\frac{x}{y}\right)$$
- $$\frac{-y \,\mathrm{d}x + x \,\mathrm{d}y}{x^2} = \mathrm{d}\left(\frac{y}{x}\right)$$
- $$\frac{y \,\mathrm{d}x - x \,\mathrm{d}y}{xy} = \mathrm{d}\left(\ln\left|\frac{x}{y}\right|\right)$$
- $$\frac{y \,\mathrm{d}x - x \,\mathrm{d}y}{x^2 + y^2} = \mathrm{d}\left(\arctan\frac{x}{y}\right)$$
- $$\frac{y \,\mathrm{d}x - x \,\mathrm{d}y}{x^2 - y^2} = \frac{1}{2}\,\mathrm{d}\left(\ln\left|\frac{x-y}{x+y}\right|\right)$$

**Exercise (11)**

*1. Given fundamental pairs $\{x_1, x_2\}$, solve the following odes:*

*(1)*

$$x'' - x = \cos t, x_1 = e^t, x_2 = e^{-t}$$

*(2)*

$$x'' + \frac{t}{1-t}x' - \frac{1}{1-t}x = t - 1, x_1 = t, x_2 = e^t$$

*(3)*

$$t^2x'' - 4tx' + 6x = 36\frac{\ln t}{t}, x_1 = t^2, x_2 = t^3$$

*(4)*

$$t^2x'' - 3tx' = 8x = 18t^2 \sin(\ln t), x_1 = t^2 \cos(2\ln t), x_2 = t^2 \sin 2\ln t$$

*2. let $x_i(t)(i = 1, 2, \cdots, n)$ be any n solutions of the the homogeneous linear ode (15), and let $W(t)$ be their Wronskian determinant. Prove that $W(t)$ satisfies the following first order linear ode:*

$$W' + a_1(t)W = 0$$

*and thus*

$$W(t) = W(t_0)t_0 \exp\left(-\int_{t_0}^{t} a_1(s)\, ds\right), t \in (a, b)$$

## 3.3 Various Notions in the System of Linear ODE

## 3.4 General Theory of System of Linear ODE

## 3.5 System of Constant Coefficient Linear ODE

# Chapter 4

# General ODE in Higher Order

# Chapter 5

# Nonlinear ODE and Dynamical System